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Comment on "The equivalence principle in the Schwarzschild geometry" [Am. J. Phys. 62, 1037 (1994)]

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In the article "The equivalence principle in the Schwarzschild geometry" [*Am. J. Phys.* **62**, 1037-1040 (1994)], some manifestation of the equivalence principle in Schwarzschild spacetime was studied. We point out that the result for the Riemann-tensor component in this article is incorrect because only the first order terms in the metric expansion are taken into account. We show the corrected result, illustrating the importance of the second order terms in the approximate metric in computing curvature quantities.

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In Ref. 1, the authors provide a concrete example of manifestation of the equivalence principle in General Relativity, using the Schwarzschild solution. It is shown, by making a suitable coordinate transformation, that in the vicinity of a displaced Cartesian coordinate system, the Schwarzschild line element is equivalent to the line element associated with a uniformly accelerated observer in flat spacetime, except for some off-diagonal components. In order to show this equivalence, the components of the metric tensor, written in displaced Cartesian coordinates, are expanded up to first order in x/R, y/R and z/R, where R is the Schwarzschild radial coordinate of the origin of the displaced Cartesian coordinates. The result is given in Eq. (7) of Ref. 1. Under the approximation considered, the off-diagonal terms in the metric do not affect the motion along the acceleration direction.[2] In Sec. IV of Ref. 1, the authors compute the zeroth-order expression of the Riemann tensor using the results of Eq. (7), i.e., the components of the metric tensor expanded up to first order. We point out that these results are not correct.

Although the off-diagonal terms may be linked to tidal effects by showing that they contribute to the Riemann tensor and, thus, to the geodesic deviation equation, we note that, to correctly compute the zeroth-order terms of the Riemann tensor, the second order terms in x/R, y/R and z/R of the metric components are needed. The components of the Riemann tensor are mainly comprised of second derivatives of the metric. Hence, quadratic terms in x/R, y/R and z/R will contribute to the zeroth order expression of the Riemann tensor. In Ref. 1, the (incorrect) component ${R^0}_{101}$ of the Riemann tensor was computed using the linear approximation, where $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ labels the displaced coordinate system. The z-direction is parallel to the radial direction pointing from the center of the gravitational source to the origin of this displaced coordinate system, whereas the x- and y-directions are orthogonal to it. The results are given in Eq. (17) of Ref. 1.

We compute the component ${R^0}_{101}$ of the Riemann tensor using the second order approximation of the metric tensor. We first find that the nonzero components of the metric tensor up to second order in x/R, y/R and z/R are given by

$$g_{00} = -F(R) - F'(R)z - \frac{F'(R)}{2R}(x^2+y^2) - \frac{F''(R)}{2}z^2, \tag{1}$$

$$g_{11} = 1 + \frac{G(R)-1}{R^2}x^2, \tag{2}$$

$$g_{22} = 1 + \frac{G(R)-1}{R^2}y^2, \tag{3}$$

$$g_{33} = G(R) + G'(R)z + \frac{H(R)}{2R^2}(x^2+y^2) + \frac{G''(R)}{2}z^2, \tag{4}$$

$$g_{12} = \frac{G(R)-1}{R^2}xy, \tag{5}$$

$$g_{13} = \frac{G(R)-1}{R}x + \frac{1-G(R)+RG'(R)}{R^2}xz, \tag{6}$$

$$g_{23} = \frac{G(R)-1}{R}y + \frac{1-G(R)+RG'(R)}{R^2}yz, \tag{7}$$

where

$$F(R) = \frac{1}{G(R)} = 1 - \frac{2\,m}{R}, \tag{8}$$

$$H(R) = RG'(R) - 2G(R) + 2. \tag{9}$$

The correct[3] zeroth-order expression of the Riemann-tensor component $R^0{}_{101}$, computed with the metric-tensor components up to second order in x/R, y/R and z/R, is given by

$$R^0{}_{1\,0\,1} = -\frac{GM}{c^2 R^3}. \tag{10}$$

We also note that the zeroth-order expression of the Ricci scalar vanishes, as expected for the Schwarzschild solution. In summary, when computed in the first-order approximation of the metric-tensor components, the zeroth-order expressions of the curvature quantities are incorrect due to missing information from the second order coefficients in the metric. The Ricci scalar, for example, is nonvanishing in the linear approximation.

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[2] The off-diagonal terms in this case are directly related to the geodesic deviation equation – we can think of x and y as being related to the components of the separation vector between two nearby geodesics directed towards the source of gravity.

[3] Although, in Ref. 1, they do not give details to the definition of the Riemann-tensor components used, our results here would be different by, at most, a change of sign.