On a Mean Field Game formulation of Fish Stock Exploitation

Ruaraidh McPike [1] Michael Grinfield [1] Mike Heath [1]
Marie-Therese Wolfram [2]

[1]University of Strathclyde
[2]University of Warwick

National Meeting of the Portuguese Mathematical Society 2018
Stochastic Differential Equations: From Theory to Practice

10/07/2018

On a Mean Field Game formulation of Fish Stock Exploitation

- Introduction
- Game Theory and Mean Field Games
- The Mean Field Game Equations
- A Mean Field Game Formulation of Fish Stock Exploitation
- Numerical Simulations of MFG Fish Stock Exploitation Model with Regulations

Introduction

- Mean field games: stochastic differential games with large number of players ($N \to \infty$).
- Developed by Jean-Michel Lasry and Pierre-Louis Lions (2007) and by Minyi Huang, Roland Malhame and Peter Caines (2006).

Introduction

- Mean field games: stochastic differential games with large number of players ($N \to \infty$).
- Developed by Jean-Michel Lasry and Pierre-Louis Lions (2007) and by Minyi Huang, Roland Malhame and Peter Caines (2006).
- Coupled set of PDEs: **Hamilton-Jacob-Bellman** (backward in time) and **Fokker-Planck** (forward in time).

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\Delta u + H(x, \nabla u) = V(x, m)$$

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\Delta m + \nabla \cdot (m\ H_p(x, \nabla u)) = 0.$$

$$u(x, T) = G(x(T), m(x, T)),\ \ m(x, 0) = m_0(x).$$

Introduction

In application to fisheries, we arrive at a coupled system of equations of the form:

$$\frac{\partial u}{\partial t} + \frac{(u_x)^2}{2} + \frac{\sigma^2}{2} u_{xx} - ru = -F(x, N, L(N, m)), \tag{1}$$

$$\frac{\partial m}{\partial t} + (mu_x)_x - \frac{\sigma^2}{2} m_{xx} = 0, \tag{2}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN\left(1 - \frac{N}{K}\right) - L(N, m). \tag{3}$$

$$u(T, x) = G(x(T), m(T), N(T)); \quad m(0, x) = m_0(x); \quad N(0) = N_0. \tag{4}$$

Game Theory Terminology

Game theory:

- **Rational** agents aiming to maximise their **payoff**.
- They select a **strategy** from set of possible strategies.

Game Theory Terminology

Game theory:

- **Rational** agents aiming to maximise their **payoff**.
- They select a **strategy** from set of possible strategies.
- Payoff depends on their own strategy and the strategy of other players.
- **Nash Equilibrium**: no player can improve payoff by (unilaterally) altering strategy.

Game Theory Example - Prisoner's Dilemma

- Players aim to minimise time in prison.
- Strategy: choose to betray the other or remain silent.

Payoff Matrix

Mr. Red

Mr. Blue	Stay Silent	Betray
Stay Silent	Red: -1; Blue: -1	Red: 0; Blue: -10
Betray	Red: -10; Blue: 0	Red: -5; Blue: -5

Game Theory Example - Prisoner's Dilemma

- Players aim to minimise time in prison.
- Strategy: choose to betray the other or remain silent.
- Nash equilibrium: both prisoners betray the other.

Payoff Matrix

		Mr. Red	
		Stay Silent	Betray
Mr. Blue	Stay Silent	-1 -1	0 -10
	Betray	-10 0	-5 -5

Game Theory to Mean Field Games

Key Assumptions For A Mean Field Game:

- The number of players N is large.
- Individual players can be considered "small" compared to the total mass of players.

Game Theory to Mean Field Games

Key Assumptions For A Mean Field Game:

- The number of players N is large.
- Individual players can be considered "small" compared to the total mass of players.
- Players are homogeneous (they would take same action if in the same position)
- Players consider other players interchangeable (no complex interindividual strategies).

Game Theory to Mean Field Games

Mean field game approach:

- Take limit as $N \to \infty$; a continuum of agents.
- Individual agents considered infinitesmal compared to the total mass of other agents.

Game Theory to Mean Field Games

Mean field game approach:

- Take limit as $N \to \infty$; a continuum of agents.
- Individual agents considered infinitesmal compared to the total mass of other agents.
- The interaction between agents is given by a "mean field".

Mean Field Game Equations - General Setting

- Agents' state given by $x \in \Omega$ with dynamics:

$$dx_t = \alpha(x,t)dt + \sigma dW_t.$$

- $\alpha(x,t)$ is the agent's control, W_t is independent Brownian noise.

Mean Field Game Equations - General Setting

- Agents' state given by $x \in \Omega$ with dynamics:

$$dx_t = \alpha(x,t)dt + \sigma dW_t.$$

- $\alpha(x,t)$ is the agent's control, W_t is independent Brownian noise.
- The density of agents is $m(x,t)$, with the initial density given by $m(x,0) = m_0$.
- m must be a valid probability density function:

$$\int_\Omega m(x,t)\, dx = 1 \quad m(x,t) \geq 0.$$

Mean Field Game Equations - General Setting

- Agents aim to maximise:

$$\mathbb{E}\left(\int_0^T \lambda(\alpha, x) + F(x, m)\, dt + G(x(T), m(x, T))\right).$$

Mean Field Game Equations - General Setting

- Agents aim to maximise:

$$\mathbb{E}\left(\int_0^T \lambda(\alpha, x) + F(x, m)\, dt + G(x(T), m(x, T))\right).$$

- $\lambda(\alpha, x)$ depends on state and control.
- $F(x, m)$ depends on state and distribution of other agents.

Mean Field Game Equations - General Setting

- Agents aim to maximise:

$$\mathbb{E}\left(\int_0^T \lambda(\alpha, x) + F(x, m)\, dt + G(x(T), m(x, T))\right).$$

- $\lambda(\alpha, x)$ depends on state and control.
- $F(x, m)$ depends on state and distribution of other agents.
- T is the horizon time.
- $G(x(T), m(x, T))$ is a cost at horizon time T.

Mean Field Game Equations - HJB Equation

- Define the **value function** $u(x, t)$ as:

$$u(x,t) = \max_{\alpha(x,t)} \mathbb{E}\left(\int_t^T \lambda(\alpha, x) + F(x,m)\, dt + G(x(T), m(x,T))\right).$$

Mean Field Game Equations - HJB Equation

- Define the **value function** $u(x,t)$ as:

$$u(x,t) = \max_{\alpha(x,t)} \mathbb{E}\left(\int_t^T \lambda(\alpha,x) + F(x,m)\,dt + G(x(T),m(x,T))\right).$$

- Define the **Hamiltonian** $H(x,p)$ as:

$$H(x,p) = \sup_{\alpha(x,t)} (\alpha(x,t)p + \lambda(\alpha,x)).$$

Mean Field Game Equations - HJB Equation

- Define the **value function** $u(x,t)$ as:

$$u(x,t) = \max_{\alpha(x,t)} \mathbb{E}\left(\int_t^T \lambda(\alpha,x) + F(x,m)\, dt + G(x(T), m(x,T))\right).$$

- Define the **Hamiltonian** $H(x,p)$ as:

$$H(x,p) = \sup_{\alpha(x,t)} (\alpha(x,t)p + \lambda(\alpha,x)).$$

- Optimal Control Theory: $u(x,t)$ satisfies the **Hamilton-Jacobi-Bellman** (**HJB**) equation:

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\Delta u + H(x,\nabla u) = -F(x,m).$$

Mean Field Game Equations - Fokker-Planck Equation

- The optimal control $\alpha^*(x,t)$ associated with the HJB equation is given by

$$\alpha^*(x,t) = H_p(x, \nabla u),$$

where $H_p(x,p))$ is the derivative of H w.r.t. p.

Mean Field Game Equations - Fokker-Planck Equation

- The optimal control $\alpha^*(x, t)$ associated with the HJB equation is given by

$$\alpha^*(x, t) = H_p(x, \nabla u),$$

where $H_p(x, p))$ is the derivative of H w.r.t. p.

- If agents behave optimally then

$$dx_t = \alpha^*(x, t)dt + \sigma dW_t.$$

Mean Field Game Equations - Fokker-Planck Equation

- The optimal control $\alpha^*(x,t)$ associated with the HJB equation is given by

$$\alpha^*(x,t) = H_p(x, \nabla u),$$

where $H_p(x,p))$ is the derivative of H w.r.t. p.

- If agents behave optimally then

$$dx_t = \alpha^*(x,t)dt + \sigma dW_t.$$

- And the evolution of the density m of agents is given by the **Fokker-Planck** (**FP**) or Kolmogorov Forward equation:

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\Delta m + \nabla \cdot (m \; H_p(x, \nabla u)) = 0.$$

Mean Field Game Equations - MFG System

The coupled MFG system is given by the HJB equation with terminal condition for u (backwards) and the Fokker-Planck equation with initial condition for m (forwards).

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\Delta u + H(x, \nabla u) = -F(x, m). \quad \textbf{(HJB)}$$

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\Delta m + \nabla \cdot (m \; H_p(x, \nabla u)) = 0. \quad \textbf{(FP)}$$

$$u(x, T) = G(x(T), m(x, T)), \;\; m(x, 0) = m_0(x).$$

Application to Fisheries

- Fisheries: often many agents (fishermen) have access to the same resource.
- "Tragedy of the Commons" - overexploitation of the common resource (NB: similar to Prisoner's Dilemma but with many players).

MFG Formulation - Fisheries

- An agent's fishing effort $x(t)$ on a single fish stock $N(t)$ (one dimension) evolves according to the SDE

$$dx_t = \alpha(x,t)dt + \sigma dW_t,$$

where $\alpha(x,t)$ is the agent's control and W_t is their independent Brownian noise.

MFG Formulation - Fisheries

- An agent's fishing effort $x(t)$ on a single fish stock $N(t)$ (one dimension) evolves according to the SDE

$$dx_t = \alpha(x,t)dt + \sigma dW_t,$$

where $\alpha(x,t)$ is the agent's control and W_t is their independent Brownian noise.

- Each agent aims to maximise total profit given by

$$\mathbb{E}\left(\int_0^T R(x,N,m) - C(x,\alpha(x,t))\,\mathrm{d}t + G(x(T),m(T,x),N(T))\right)$$

where $R(x,N,m) = qx(t)N(t)p(t,L)$ and $C(x,\alpha) = cx + \gamma\frac{\alpha^2}{2}$.

MFG Formulation - Fisheries

- Fish stock (common resource) evolves according to:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN\left(1 - \frac{N}{K}\right) - L(m, N),$$

MFG Formulation - Fisheries

- Fish stock (common resource) evolves according to:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN\left(1 - \frac{N}{K}\right) - L(m, N),$$

- The landings $L(m, N)$ are given by:

$$L(m, N) = N(t)q\int_0^\infty x(t)m(t, x)\,\mathrm{d}x.$$

MFG Formulation - Fisheries

- Fish stock (common resource) evolves according to:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN\left(1 - \frac{N}{K}\right) - L(m, N),$$

- The landings $L(m, N)$ are given by:

$$L(m, N) = N(t)q\int_0^\infty x(t)m(t, x)\,\mathrm{d}x.$$

- The price $p(t, L)$ depends on landings $L(t)$ through a price flexibility coefficient δ.

MFG Formulation - Fisheries

The full mean field game system is then given by

$$\frac{\partial u}{\partial t} + \frac{(u_x)^2}{2} + \frac{\sigma^2}{2} u_{xx} - ru = -(qx(t)N(t)p(t,L) - cx(t)), \quad (5)$$

$$\frac{\partial m}{\partial t} + (mu_x)_x - \frac{\sigma^2}{2} m_{xx} = 0, \quad (6)$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN\left(1 - \frac{N}{K}\right) - L(N,m). \quad (7)$$

$$u(T,x) = 0; \quad m(0,x) = m_0(x); \quad N(0) = N_0. \quad (8)$$

With Neumann boundary conditions for u and m at the boundaries $x = 0$ and $x = x_{max}$.

Numerical Method for MFG Fisheries Model

The system of PDEs is solved numerically using an iterated scheme:

- Solve the HJB equation backwards in time (Euler Method), where $m(t, x) = m_0(x)$ for all t to obtain u_1.
- For u_i, obtain m_i and N_i by solving the Kolmogorov and the resource ODE forward in time given u_i.
- Obtain u_{i+1} by solving the HJB equation backwards in time given m_i and N_i.
- Repeat the last two steps until the error between iterates is below a selected tolerance.

Numerical Simulations of MFG Fisheries Model

- Numerical simulations of the MFG model can be used to compare the stock, harvest, price and fishing effort distribution over time under different economic or ecological scenarios (e.g. different levels of price flexibility or different stock growth functions).
- The potential effectiveness or impact of different regulations can be considered by adding new cost terms (usually in terms of preserving fishing stock and bringing harvest to maximum sustainable yield targets)
- Possible regulation regimes: Tax on Catch, Tax on Effort, License Fees.

Numerical Simulations of MFG Fisheries Model - Base Case Example

For our test case, we will use the parameter set:

- $x_{min} = 0$, $x_{max} = 1$,
- $t_0 = 0$, $T = 10$,
- $r = 0.05$, $q = 2$, $c = 2$,
- $p_0 = 1$, $\delta = 0.4$,
- $\gamma = 10$, $\sigma = 1$,
- $a = 1.5$, $K = 200$, $N = 70$,

with m_0 as a uniform distribution. Ecological parameters fitted to North Sea Cod stock, economic parameters used are simplified selection informed by various UK fleet costs and prices.

Numerical Simulations of MFG Fisheries Model - Base Case Example

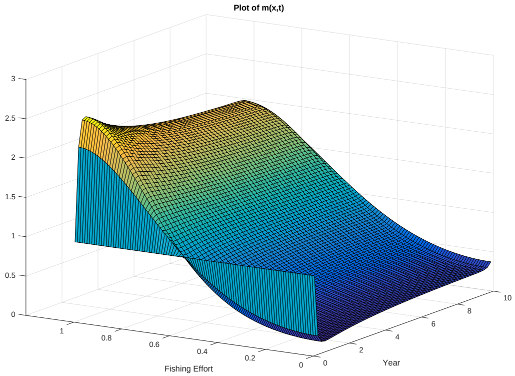

Figure: Surface plot of m with test case parameters

Numerical Simulations of MFG Fisheries Model - Base Case Example

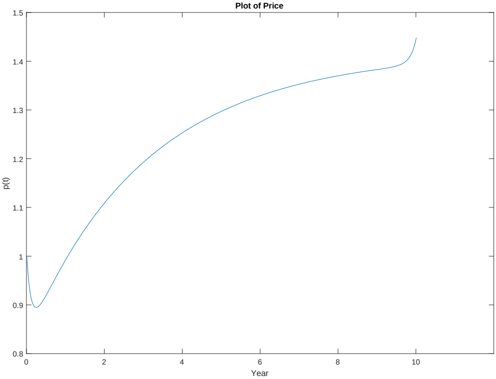

Figure: Plot of price with test case parameters

Numerical Simulations of MFG Fisheries Model - Base Case Example

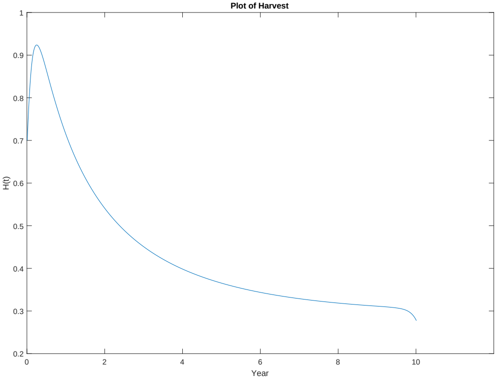

Figure: Plot of harvest with test case parameters

Numerical Simulations of MFG Fisheries Model - Base Case Example

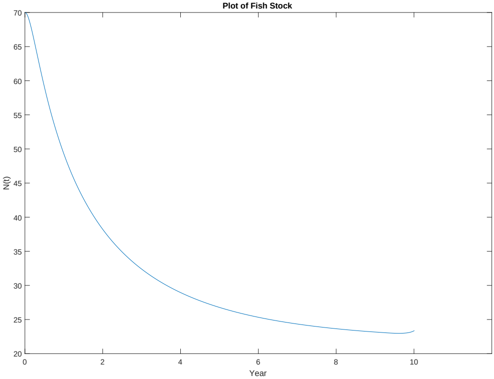

Figure: Plot of N (stock) with test case parameters, N(T) = 23.38

Numerical Simulations of MFG Fisheries Model - Regulations

Regulations are incorporated by adding cost terms to the cost function F:

- License Fees - flat fee for any fishing (above some low threshold value):

$$F(x, N, L(m, N)) = qxNp(L) - cx - C_L I_{x_L}(x), \qquad (9)$$

where $I_{x_L}(x)$ is equal to one if $x \geq x_L$ and is zero if $x < x_L$. with m_0 as a uniform distribution.

- Tax on Catch - additional fee on each unit of landings:

$$F(x, N, L(m, N)) = qxN(p(L) - C_P) - cx, \qquad (10)$$

where the parameter C_P is the additional cost per unit landed to an agent due to the regulation.

Numerical Simulations of MFG Fisheries Model - Regulations

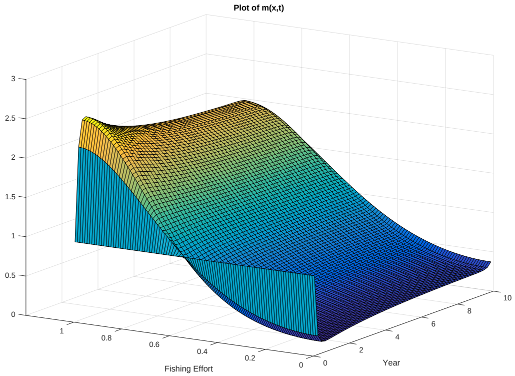

Figure: m with no regulation (Base Case parameters), $\mathrm{N(T)} = 23.38$

Numerical Simulations of MFG Fisheries Model - Regulations

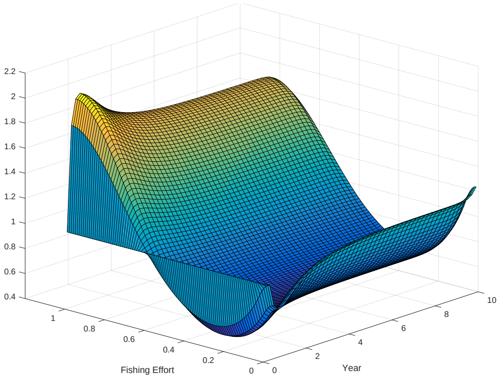

Figure: m with License Fee regulation, $\mathrm{N(T)} = 45.21$

Numerical Simulations of MFG Fisheries Model - Regulations

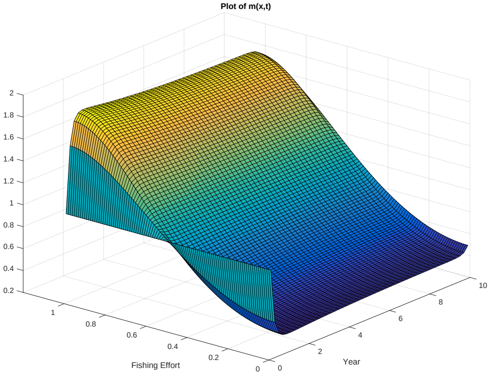

Figure: m with Tax on Catch regulation, $\mathrm{N(T)} = 32.55$

Numerical Simulations of MFG Fisheries Model - Regulations

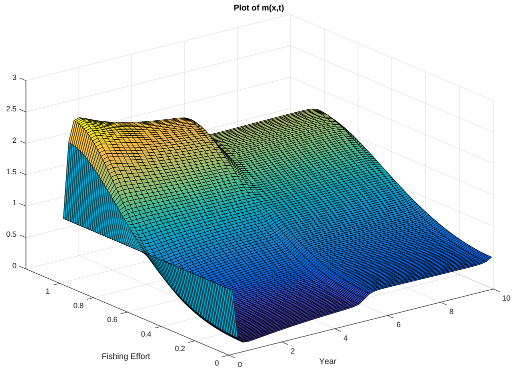

Figure: m with Time Dependent Catch on Tax Regulation, $\mathrm{N(T)} = 30.47$

Conclusions

- Mean Field Games are a tool for analysis of stochastic differential games with many players.
- Typically lead to a coupled HJB equation (backwards) and Fokker-Planck equation (forwards)
- MFG Formulation of Fish Stock Exploitation includes HJB and Fokker-Planck equations coupled with fish stock equations.
- Numerical simulations from MFG Fish Stock Exploitation model can be used to compare the impact of different regulation regimes on a scenario.