DISTAL TRANSFORMATION GROUPS

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REFERENCES

References are given separately for each of the parts A, B, C of the thesis. For each part, the references are almost, but not quite, in alphabetical order.

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SUMMARY

The thesis consists of three parts A, B, C, part A being the longest part. The objects of interest throughout are minimal distal transformation groups, in particular those for which the phase space is a compact topological manifold. Although many of the results obtained are true for a transfromation group in which the group acting is an arbitrary topological group, there is an emphasis, particularly in the latter part of part A, on the groups of integers and of reals.

Part A is concerned mainly with a classification of those minimal distal transformation groups (X,T) for which X is a compact manifold. A refinement of the Furstenburg Structure Theorem, for such a phase space X, is proved, to show that there exists a finite sequence $\{(X_i,T)\}_{i=0}^{r}$ of transformation groups with (X_0,T) trivial, $(X_r,T) = (X,T)$, and (X_{i+1},T) an almost periodic extension of (X_i,T), where $r \leq \dim X$. An important step in the proof is the Addition Theorem: if (W,T) is any minimal distal transformation group with factor (Z,T) and factor map π, then the covering dimension of the fibres $\pi^{-1}(z)$ is constant, and:

$$\dim Z + \dim \pi^{-1}(z) = \dim W \quad \text{for all } z \in Z.$$

It is also shown in the course of the proof that the compact group associated with the extension (X_{i+1},T) is Lie, so that X_{i+1} is the total space of a fibre bundle with base X_i, homogeneous fibre, and Lie structure group. One can then use fibre bundle theory to obtain information about the structure of (X,T) when dim X is low (i.e. ≤ 3). The details of this are worked out in the second half of part A.

Since fibre results are thus available for (X,T) with X a compact manifold, one might expect such transformation groups to be less pathological than in general, and part B gives an example of this:

using the Homotopy Covering Theorem, it is shown that any two fibres of a minimal distal transformation group over a factor are homeomorphic if the corresponding two points in the factor can be connected by a path (which implies all fibres are the same up to homeomorphism if, for example, the phase space of the transformation group is a compact manifold). But it is shown that two fibres need not be homeomorphic in general. In the example given, the phase space of the transformation group is a connected metric 3-dimensional space resembling a nilmanifold, and the phase space of the factor is a solenoid. Points in the solenoid, over which the fibres belong to a fixed homeomorphism class, lie in a nowhere dense set consisting of at most countably many pathwise-connected components of the solenoid.

The "classification" of part A is a topological one in a topological category, since the Furstenburg Structure Theorem deals with topological structure. A differentiable analogue of the Furstenburg Structure Theorem might be the following: if (X',T) is minimal distal with X' a C^r manifold and T a group of C^r diffeomorphisms, then (X',T) is topologically conjugate to (X,T) (where X is a C^r manifold and T again a group of C^r diffeomorphisms) such that there exists a finite sequence $\{(X_i,T)\}_{i=0}^{r}$ of transformation groups with (X_0,T) trivial, $(X_r,T) = (X,T)$, X_i a C^r manifold, (X_{i+1},T) an almost periodic extension of (X_i,T), the factor map being C^r, and the associated fibre bundle being C^r. (One sees at once that one has to allow topological conjugacy between X and X' by considering minimal diffeomorphisms of the circle, which are always topologically, but not necessarily C^1, conjugate to rotations). Part C shows that the differentiable analogue suggested here does not hold. By part A, a minimal distal positively-oriented non-almost-periodic homeomorphism of the torus is topologically conjugate to a topological skew-product of the form:

$$(x,y) \longmapsto (x+\alpha, y+g(x)),$$

where α is uniquely determined up to its sign. In part C, a minimal distal positively oriented analytic diffeomorphism of the torus is constructed which is not topologically conjugate to any C^1 skew product, thus providing a counterexample to the suggested differentiable structure theorem. However, the construction, (which is similar to Arnold's construction of an analytic diffeomorphism of the circle which is not C^1-conjugate to a rotation) depends on the associated irrational, which has, among other things, to be Liouville. It is not clear what happens if, for example, the irrational is of bounded density, in which case, as Arnold conjectured, and Herman has shown, an analytic diffeomorphism with the irrational as rotation number is analytically conjugate to a rotation, so that Arnold's construction most certainly does not work.

A

ON THE STRUCTURE OF MINIMAL DISTAL TRANSFORMATION

GROUPS WITH TOPOLOGICAL MANIFOLDS AS PHASE SPACES

A

ON THE STRUCTURE OF MINIMAL DISTAL TRANSFORMATION GROUPS WITH TOPOLOGICAL MANIFOLDS AS PHASE SPACES

M. REES

§1 Introduction and statement of the two basic theorems

The first purpose of this paper is to show how the Furstenberg Structure Theorem for minimal distal transformation groups [2], [3] can be refined when applied to a minimal distal transformation group (X,T) for which X is a compact topological manifold. The refinement is given by the Manifold Structure Theorem 1.2, for which we need a result concerning the dimension of a factor of a minimal distal transformation group, namely the Addition Theorem 1.1. §§ 2 – 7 are devoted to proving these two basic theorems – the actual proofs are given in §§ 6 - 7. The rest of the paper is devoted to examining, in some detail, what the structure theorem tells us in the case of connected manifolds of dimension ≤ 3; an explanation of how the structure theorem gives us some sort of classification of the transformation groups is given in §9, and the results are summarized there in tabular form, using the notation in the index of § 8, which is a constant reference for the rest of the paper. Details of the results are worked out in §§10 - 13.

There is some overlap in this work with that of Bronstein [1] which will be discussed where it seems appropriate to do so.

I should like to thank my supervisor, Professor W. Parry, for considerable help, particularly in the preparation of this paper. This paper will be part of my Ph.D. thesis, and I should like to thank the S.R.C. for financial support.

We now proceed to the two basic theorems:

1.1 The Addition Theorem

Let (X,T) be a minimal distal transformation group (4.1) and let $(Y,T) <_w (X,T)$ (4.2). Then if "dim" denotes covering dimension, $\dim \pi^{-1}(y)$ is constant for $y \in Y$ and :

$$\dim Y + \dim \pi^{-1}(y) = \dim X \qquad (y \in Y),$$

with the convention that $n + \infty = \infty$ ($n = \infty$ or n an integer).

1.2 The Manifold Structure Theorem

Let (X,T) be a minimal distal transformation group (4.1) and let X be finite-dimensional with finitely many arcwise-connected components. (These hypotheses are automatically satisfied if X is a topological manifold.) Then the following conclusions hold:

(i) If $(Y,T) < (X,T)$ then Y is a topological manifold (and, in particular, X is a manifold).

(ii) (X,T) has order r, where $r \le \mathrm{Max}(1, \dim X)$ (4.10).

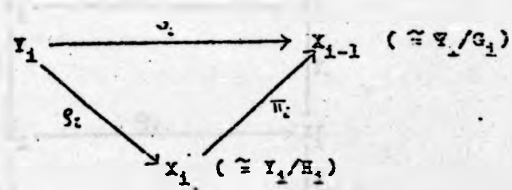
(iii) Let (X_0,T) denote the trivial transformation group and let (X_{i+1},T) denote the (unique up to isomorphism) maximal almost periodic extension of (X_i,T) in (X,T) (4.9). Then there exists a minimal distal transformation group (Y_i,T), a compact Lie group G_i and a closed subgroup H_i, such that G_i acts freely and jointly continuously on Y_i,

$$(g.y)t = g.(yt) \quad \text{for all } g \in G_i,\ y \in Y_i,\ t \in T,$$

$$\bigcap_{g \in G_i} g^{-1}H_i g = \{e\}, \text{ and the following diagram is commutative}$$

for $1 \le i \le r$:

Diagram 1.2(a)



so that $\mathcal{E}_i = (Y_i, X_i, X_{i-1}, G_i, H_i, \pi_i, \rho_i, \psi_i)$ is a fibre bundle (3.1) for $1 \le i \le r$ and the X_i's and Y_i's are manifolds.

$\dim X_{i+1} > \dim X_i$ unless $\dim X = 0$ (in which case X is finite).

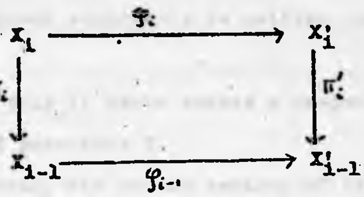
If $\dim G_i/H_i = r_i$ then $\dim G_i \le r_i(r_i+1)/2$ by a result of [10].

(iv) G_i/H_i is connected for $i \ge 2$, and G_1/H_1 is connected if and only if X is connected.

(v) (A uniqueness property.) Let $(X,T) \cong (X',T)$.

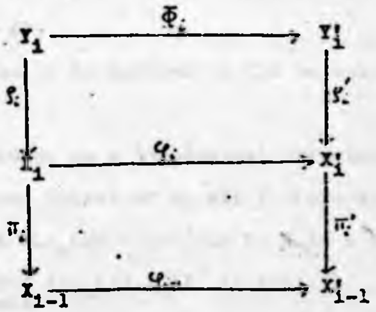
Let (X_0',T) denote the trivial transformation group and let (X_{i+1}',T) be a maximal almost periodic extension of (X_i',T) in (X',T), and let $(X_i',T) \underset{\pi_{i+1}'}{<} (X_{i+1}',T)$ so that (by 4.9) there exist T-isomorphisms φ_i $(0 \le i \le r)$ such that the following diagram is commutative:

Diagram 1.2(b).



Let Y_i', G_i', H_i', ξ_i', ν_i', β_i', $(1 \le i \le r)$ bear the same relation to X_i', π_i', as Y_i, G_i, H_i, ξ_i, ν_i, β_i $(1 \le i \le r)$ bear to X_i, π_i in (iii). Then there exist T-isomorphisms $\Phi_i : (Y_i,T) \longrightarrow (Y_i',T)$, and topological group isomorphisms $\alpha_i : G_i \longrightarrow G_i'$ carrying H_i onto H_i' such that $\Phi_i(g.y) = \alpha_i(g).\Phi_i(y)$ for all $y \in Y_i$, $g \in G_i$, and such that the following diagram commutes:

Diagram 1.2(c)



1.3 In [1] Bronstein proved, among other things, a slightly different formulation of theorem 1.2(i)-(iii) with the hypothesis that X have finitely many arcwise-connected components replaced by the hypothesis that X be locally connected; neither of these conditions on X implies the other.

Bronstein seems to use in the proof the following: if $(Y,T) \prec (X,T)$ for (X,T) minimal distal, then $\dim Y \leq \dim X$ (which, of course, follows from 1.1), but this result does not seem to be stated in [1] as either a theorem or an assumption, which is part of our justification for duplicating some of Bronstein's work.

1.4 A similar theorem to 1.2 holds if the hypothesis that X have finitely many arcwise-connected components is omitted, and the hypothesis "$T \in \mathfrak{J}$" is added, where:

$T \in \mathfrak{J}$ if and only if there exists a compact $K \subseteq T$ such that every neighbourhood of K generates T.

Roughly speaking, the second version of (1.2) is obtained by replacing the words "manifold" and "Lie group", wherever they occur, by "finite-dimensional space" and "finite-dimensional group" respectively, and omitting all reference to fibre bundles. This second version of (1.2) will not be proved here.

§2. Preliminaries on Dimension Theory

It seems helpful to list here various properties of covering dimension which will be used subsequently, particularly in the proof of the Addition Theorem 1.1 (see § 6).

Covering dimension is defined on the category of compact Hausdorff spaces [11], [12].

2.1 Covering dimension is a topological invariant.

2.2 If Y is a closed subset of X, $\dim Y \leq \dim X$.

2.3 For $x \in X$, let $\dim_x(X) = \inf\{\dim U : U \text{ is a closed neighbourhood of } x\}$.

Then $\dim X = \sup_{x \in X} \dim_x(X)$ ([11] 11.6-11.8)

2.4 $\operatorname{Max}(\dim X, \dim Y) \leq \dim X{\times}Y \leq \dim X + \dim Y$ ([11] 26.4).

2.5 $\operatorname{Dim} [0,1]^n = n$ ([12] Chapter IV).

From 2.5, 2.3, it follows that the covering dimension of a manifold is the same as the usual dimension.

2.6 If D is a partially ordered net and $(\{X_\alpha\}_{\alpha\in D}, \{\pi_{\alpha\beta}\}_{\alpha\le\beta})$ is an inverse system of compact Hausdorff spaces with inverse limit $(X, \{\pi_\alpha\}_{\alpha\in D})$ then $\dim X \le \limsup_{\alpha\in D} \dim X_\alpha$.

§3 Preliminaries on Fibre Bundles

The relevance of fibre bundles to the study of minimal distal transformation groups follows, of course, from the Furstenberg Structure Theorem (4.7). The definitions given here are considerably less general than the customary ones, but are used for simplicity.

3.1 Definition $\mathcal{B} = (Y,W,X,G,H,\pi,\rho,\nu)$ is a fibre bundle (or bundle) if:

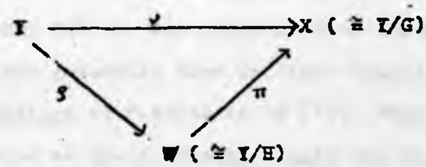
(i) Y,W,X are compact Hausdorff spaces and π, ρ, ν are continuous surjective maps.

(ii) G is compact Lie, $H \le G$ is closed and $\bigcap_{g\in G} g^{-1}Hg = \{e\}$.

(iii) G acts freely on the left of Y, the action $(g,y) \longmapsto gy$ being jointly continuous.

(iv) The following diagram commutes:

Diagram 3.1



X is called the base of the bundle, G the group of the bundle and H the isotropy subgroup.

If H is trivial, $\mathcal{B}$ is a principal bundle, and we write $\mathcal{B} = (Y,X,G,\pi)$.

3.2 The above definition of fibre bundle is essentially the same as that of [17] Chapter 1, §2, because of the following, which will be used in the proof of the Addition Theorem 1.1 (see [9] Theorem 1 in §5.4).

(i) If Y, W, X, G, H, π, ρ, ν satisfy (i)-(iv) of 3.1, then for each $y \in Y$, if $\nu(y) = x$, there exists a compact neighbourhood U of $x \in X$, and a continuous one-to-one map $\varphi : U \longrightarrow Y$ such that $\nu\circ\varphi$ = identity on U.

(ii) Let $V = \nu^{-1}(U)$. If $\lambda : V \longrightarrow G$ is defined by $\lambda(v).\varphi\nu(v) = v$, then $\lambda(g.v) = g.\lambda(v)$ for all $g \in G$, $v \in V$, λ is continuous, and $\nu \times \lambda : V \longrightarrow U \times G$ is a homeomorphism of V onto $U \times G$.

(iii) $\tilde{\lambda} : \rho(V) \longrightarrow G/H = \{Hg : g \in G\}$ is well defined by:

$\tilde{\lambda}(\rho v) = H\lambda(v)$ $(v \in V)$, and is continuous, and

$\pi \times \tilde{\lambda} : \rho(V) \longrightarrow U \times G/H$ is a homeomorphism of the neighbourhood $\rho(V)$ of $\rho(y)$ onto $U \times G/H$.

3.3 Lemma If $\mathcal{B} = (Y,W,X,G,H,\pi,\rho,\nu)$ is a fibre bundle, then

$$\dim W \leq \dim X + \dim G/H.$$

Proof By 2.2 and 2.3, it suffices to show that given $w \in W$, there exists a closed neighbourhood V of w such that:

$$\dim V \leq \dim \pi(V) + \dim G/H.$$

By 3.2, w has a neighbourhood V homeomorphic to $\pi(V) \times G/H$, so that

$$\dim V = \dim(\pi(V) \times G/H) \quad (2.1)$$
$$\leq \dim \pi(V) + \dim G/H \quad (2.4).$$

3.4 We now define three different types of isomorphism of fibre bundles. This may seem cumbersome, but for the justification see §9. 1st-isomorphism, essentially the type generally used in fibre bundle theory, is essentially the same as equivalence of bundles as in [17]. Roughly speaking, 3rd-isomorphism is necessary because we shall usually regard the base space of a bundle as the phase space of a transformation group, and shall want to consider certain transformation-group-isomorphisms of it.

Definitions Let $\mathcal{B} = (Y,W,X,G,H,\pi,\rho,\nu)$ and $\mathcal{B}' = (Y',W',X',G',H',\pi',\rho',\nu')$ be two fibre bundles.

a) $\mathcal{B}$ and $\mathcal{B}'$ are 3rd-isomorphic under (Φ,α) (write $(\Phi,\alpha) : \mathcal{B} \longrightarrow \mathcal{B}'$) if α is a topological group isomorphism of G onto G' carrying H onto H', and $\Phi : Y \longrightarrow Y'$ is a homeomorphism satisfying:

$\Phi(g\cdot y) = \alpha(g).\Phi(y)$ for all $y \in Y$, $g \in G$.

Note that Φ induces homeomorphisms of W onto W' and X onto X' (φ_1, φ_2, say) such that the following diagram commutes:

Diagram 3.4

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y' \\ \downarrow \rho & & \downarrow \rho' \\ W & \xrightarrow{\varphi_1} & W' \\ \downarrow \pi & & \downarrow \pi' \\ X & \xrightarrow{\varphi_2} & X' \end{array}$$

b) $\mathcal{B}$ and $\mathcal{B}'$ are 2nd-isomorphic under (Φ,α) if $(\Phi,\alpha) : \mathcal{B} \longrightarrow \mathcal{B}'$ is a 3rd-isomorphism, X = X', and the map φ_2 in Diagram 3.4 is the identity.

c) $\mathcal{B}$ and $\mathcal{B}'$ are 1st-isomorphic under Φ if G = G', H = H', X = X' and $(\Phi,1) : \mathcal{B} \longrightarrow \mathcal{B}'$ is a 2nd-isomorphism, where 1 denotes the identity isomorphism.

3.5 Definition The product bundle with base X, group G and isotropy subgroup H is the bundle $(X\times G, X\times G/H, X, G, H, \pi, \rho, \nu)$, where the action of G on X×G is given by:

$$g.(x,g') = (x,gg') \text{ for all } x \in X,\ g,\ g' \in G.$$

$$\pi(x, Hg) = x, \quad \nu(x, g) = x, \quad \rho(x, g) = (x, Hg).$$

3.6 Theorem (See [17] 11.6) Any bundle with base $[0,1]^I$ (where I is any index set) is 1st-isomorphic to a product bundle.

§4 Preliminaries on Transformation Groups

4.1 Definition Throughout this work, we shall be considering transformation groups (t.g.'s) where the phase space X is compact Hausdorff and T is an arbitrary topological group acting on X (on the right) such that the map $(x,t) \longmapsto xt$ is jointly continuous.

4.2 Definition If (X,T) is a factor of (Y,T) and $\pi : (Y,T) \longrightarrow (X,T)$ is the factor homomorphism, write $(X,T) <_{\pi} (Y,T)$. (The suffix π will frequently be omitted.)

4.3 Definition Given a t.g. (X,T), write E(X) for the envelopping semigroup of X. E(X) is a compact Hausdorff space when given the topology $\mathfrak{I}_p$ of pointwise convergence. Write (E(X),T) for the canonical t.g. with phase space E(X) and group T ([2] Chapter 3).

<u>4.4</u> Let (X,T) be a minimal distal t.g.. A reference for the following is [13]. (Note that [13] deals with left, rather than right, t.g.'s.)

(a) For any $x \in X$, the map π_x : $(E(X),T) \longrightarrow (X,T)$ is a T-homomorphism onto (X,T), where $\pi_x(p) = xp$.

(b) $(E(X),\mathfrak{I}_p)$ is a group in which the following maps are continuous:

$$p \longmapsto qp \quad (p,\ q \in E(X)),$$

$$p \longmapsto pt \quad (t \text{ in the image of } T \text{ in } E(X),\ p \in E(X)).$$

(c) Let σ be the weakest topology on E(X) making the map φ continuous, where $\varphi : (E(X)\times E(X),\mathfrak{I}_p\times\mathfrak{I}_p) \longrightarrow E(X)$ is given by $\varphi(p,q) = pq^{-1}$. Then $\sigma \leq \mathfrak{I}_p$.

(d) If H is a subgroup of E(X), $(E(X)/H,\mathfrak{I}_p)$ is Hausdorff if and only if H is σ-closed, where $E(X)/H = \{Hp : p \in E(X)\}$.

Define $(Hp)t = H(pt) \quad (t \in T)$.

Then $(E(X)/H,T) \prec_\rho (E(X),T)$, where $\rho(p) = Hp$.

(e) If $(Y,T) \prec_\pi (E(X),T)$, then if e is the identity of E(X), let $H = \pi^{-1}\pi(e)$. H is a σ-closed subgroup of E(X) and the following diagram commutes:

<u>Diagram 4.4</u>

$$\begin{array}{ccccc}
 & & (E(X),T) & & p \\
 & \swarrow & \downarrow & & \downarrow \\
(Y,T) & \longleftrightarrow & (E(X)/H,T) & & Hp \\
\pi(p) & \longmapsto & Hp & &
\end{array}$$

(f) E(X) can be identified with the group of T-isomorphisms of (E(X),T). For consider the map $p \longmapsto L_p$ where $L_p(q) = pq \ (q \in E(X))$.

(g) Similarly, the group of T-isomorphisms of (E(X)/H,T) can be identified with L/H, where $L = \{p \in E(X) : pH = Hp\}$ (so L is σ-closed).

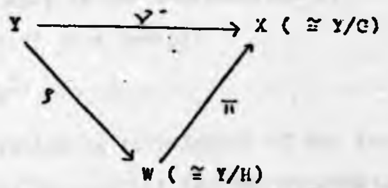
(h) For a σ-closed $H \leq E(X)$, define $\mathrm{alg}(H) = \{f \in C(E(X)) : L_p^* f = f\}$ (see (f)) so that alg(H) is a T-invariant (i.e. $tf \in \mathrm{alg}(H)$ for all $f \in \mathrm{alg}(H)$, $t \in T$, where $tf(p) = f(pt)$) C^*-subalgebra of C(E(X)).

For a T-invariant C^*-subalgebra $\mathcal{A}$ of $C(E(X))$, define $gp(\mathcal{A}) = \{p \in E(X) : L_p^* f = f \text{ for all } f \in \mathcal{A}\}$. Then $gp(\mathcal{A})$ is a σ-closed subgroup of $E(X)$.

We have $alg(gp(\mathcal{A})) = \mathcal{A}$ and $gp(alg(H)) = H$ (use Uryssohn's lemma and the Stone-weierstrass Theorem).

<u>4.5</u> <u>Definition</u> A minimal t.g. (W,T) is a <u>quotient-group-extension</u> of (X,T) if there exists a compact topological group G with closed subgroup H such that $\bigcap_{g \in G} g^{-1}Hg = \{e\}$, and a minimal t.g. (Y,T) such that G acts freely on the left of Y, the action $(g,y) \longmapsto gy$ being jointly continuous, $(gy)t = g(yt)$ for all $g \in G$, $y \in Y$, $t \in T$, and such that the following diagram commutes:

<u>Diagram 4.5</u>



In this diagram <u>and all subsequent diagrams</u>, if the objects in the diagram are phase spaces of t.g.'s with respect to a group T, and the arrows denote T-homomorphisms.

We also say (W,T) is a <u>G/H-extension of (X,T)</u>. If G is Lie, finite etc., we say (W,T) is a <u>quotient-Lie-group-extension</u> etc. of (X,T). If H is trivial, we say (W,T) is a <u>group-extension</u> of (X,T)

Note that if G is Lie, $(Y_0, W_0, X_0', G, H, \pi, \rho, \gamma)$ is a fibre bundle (3.1) for any closed $X_0 \subseteq X$ with $Y_0 = \gamma^{-1}(X_0)$, $W_0 = \pi^{-1}(X_0)$.

<u>4.6</u> Let $(X,T) <_w (W,T)$ with (W,T) minimal. The following are equivalent conditions for (W,T) to be an <u>almost periodic (a.p.) extension</u> of (X,T) ([2], [13])

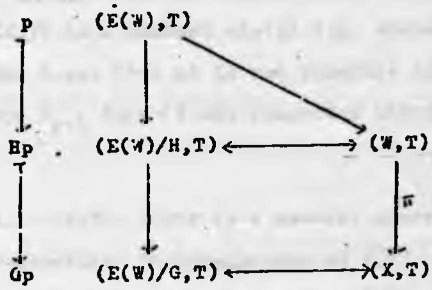
(i) Given an index ε on W, there exists an index $\delta = \delta(\varepsilon)$ on W such that $((w_1,w_2) \in \delta$ and $\pi(w_1) = \pi(w_2))$ imply $((w_1 t, w_2 t) \in \varepsilon$ for all $t \in T)$.

(ii) (W,T) is a quotient-group-extension of (X,T).

A

For (iii) and (iv), we make the additional assumption that (W,T) is distal, and choose σ-closed subgroups H, G, of $\Sigma(W)$ (4.3) such that the following diagram commutes (see 4.4b)):

<u>Diagram 4.6</u>



(iii) $N(G) \leq H$, where $N(G)$ is the intersection of the σ-closed σ-neighbourhood of the identity in G ($N(G)$ is a group).

(iv) $(G/H,\sigma) = (G/H,\mathcal{J}_p)$.

<u>4.7</u> We shall use the following formulation of the Furstenburg Structure Theorem (see [2] Chapter 15, and [3] for the elimination of the assumption of quasiseparability):

<u>Theorem</u>

(a) Let (X,T) be a minimal distal t.g.. Let $(Y,T) \underset{\neq}{<} (X,T)$. Then there exists (Z,T) with $(Y,T) \underset{\neq}{<} (Z,T) < (X,T)$ such that (Z,T) is an a.p. extension of (Y,T).

(b) If $(Y,T) < (X,T)$, then by transfinite induction on a), there exists an ordinal α and $\{(X_\beta,T): 0 \leq \beta \leq \alpha\}$, $\{\pi_{\beta\gamma}: 0 \leq \beta \leq \gamma \leq \alpha\}$ satisfying:

(i) $(X_\beta,T) \underset{\pi_{\beta\gamma}}{<} (X_\gamma,T)$, $0 \leq \beta \leq \gamma \leq \alpha$.

(ii) $\pi_{\beta\gamma} \circ \pi_{\gamma\delta} = \pi_{\beta\delta}$, $0 \leq \beta \leq \gamma \leq \delta \leq \alpha$, $\pi_{0\alpha} = \pi$.

(iii) $(X_0,T) = (Y,T)$, $(X_\alpha,T) = (X,T)$.

(iv) $(X_{\beta+1},T)$ is a <u>proper</u> a.p. extension of (X_β,T) for $\beta < \alpha$.

(v) If β is a limit ordinal, (X_β,T) is the inverse limit of $\{(X_\gamma,T)\}_{\gamma<\beta}$.

<u>4.8</u> In 4.7b), (iv) can be replaced by:

(iv)' $(X_{\beta+1},T)$ is a proper quotient-<u>Lie</u>-group-extension of (X_β,T).

This will be proved in 5.1-5.3. It was shown by Bronstein in [1]. However, a slight error in the proof led to the conclusion that one could assume that $(X_{\beta+1},T)$ was a $G_{\beta+1}/H_{\beta+1}$-extension of (X_β,T) ($\beta<\alpha$) where $G_{\beta+1}$ was either a connected Lie group or finite. This is not true: for example, if T is an arbitrary group, and (X,T) is a minimal distal t.g. where X is a Klein bottle, and (Y,T) is the trivial t.g., then it is not possible to choose $\{(X_\beta,T)\}_{0\le\beta\le\alpha}$ such that all the groups $G_{\beta+1}$ ($\beta<\alpha$) are connected Lie or finite. We omit the details.

4.9 Given a minimal t.g. (X,T), there is a natural correspondence between factors of (X,T) and T-invariant C*-subalgebras of C(X), and any two factors associated with the same subalgebra are isomorphic [2].

If (X,T) is minimal and $(Y,T) <_\pi (X,T)$, then there exists (Z,T) such that $(Y,T) <_{\pi_1} (Z,T) <_{\pi_2} (X,T)$ $(\pi_1 \cdot \pi_2 = \pi)$, (Z,T) is an a.p. extension of (Y,T), and the subalgebra of C(X) corresponding to (Z,T) is at least as large as that corresponding to any other a.p. extension of (Y,T) in (X,T). (Z,T) is called the maximal almost periodic extension of (Y,T) in (X.T) [2].

4.10 Definition Let (X,T) be minimal distal, and (Y,T) the trivial factor. If, in the transfinite induction procedure of 4.7(b) we take $(X_{\beta+1},T)$ to be the maximal a.p. extension of (X_β,T) in (X,T), then we obtain the smallest ordinal α for which there exists a system $\{(X_\beta,T): 0\le\beta\le\alpha\}$, $\{\pi_{\beta\gamma}: 0\le\beta\le\gamma\le\alpha\}$ satisfying (i)-(v) of 4.7(b). This α is called the order of (X,T).

§5 On Quotient-Group-Extensions

In this section, various results on quotient-group-extensions (see 4.5 for definition) are collected together. 5.1-5.3 contain the proof of the modified Furstenburg Structure Theorem (4.8). The main result is 5.5, which concerns the "uniqueness" of a group-extension associated with a given quotient-group-extension.

5.1 Lemma Let G be a compact topological group, and H a closed subgroup. Let $N_1 \lhd G$ with G/N_1 Lie and $HN_1 \ne H$. Then there exists $N \lhd G$ with $N \le N_1$, and $\bigcap_{g\in G} g^{-1} HN_1 g = N_1$

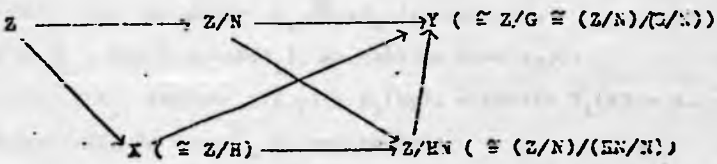
G/N Lie, $HN \neq HN_1$ and $\bigcap_{g \in G} g^{-1}HNg = N$.

<u>Proof</u> Choose $x \in (G \setminus H) \cap HN_1$, and let ρ be a finite-dimensional representation of G such that $\rho(x) \neq \rho(h)$ for any $h \in H$ (ρ exists by Urysohn's lemma and the Peter-Weyl Theorem [15] Section 33). Put $N_2 = N_1 \cap \operatorname{Ker} \rho$, and put $N = \bigcap_{g \in G} g^{-1}HN_2g$.

<u>5.2</u> <u>Lemma</u> Let (X,T) be a minimal distal t.g. and let $(Y,T) \underset{\neq}{<} (X,T)$. Then there exists (Z,T) with $(Y,T) \underset{\neq}{<} (Z,T) < (X,T)$ and (Z,T) a quotient-Lie-group-extension of (Y,T).

<u>Proof</u> By 4.7(a) we can assume (X,T) is a G/H-extension of (Y,T) for some compact topological group G. By 5.1 (with $N_1 = G$) we can find $N \triangleleft G$ with G/N Lie, $\bigcap_{g \in G} g^{-1}HNg = N$ and $HN \neq G$. Then $\bigcap_{\bar{g} \in G/N} \bar{g}HN/N\bar{g}^{-1} = \{N\}$, and we have the following commutative diagram:

<u>Diagram 5.2</u>



So (Z/HN,T) is a quotient-Lie-group-extension of (Y,T), and

$(Y,T) \underset{\neq}{<} (Z/HN,T) < (X,T)$.

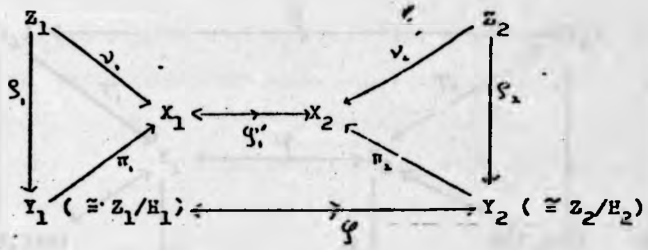
<u>5.3</u> Let (X,T) be a minimal distal t.g., and $(Y,T) <_r (X,T)$. By using 5.2 to obtain a quotient-Lie-group-extension $(X_{\beta+1},T)$ of (X_β,T), find by transfinite induction a system $\{(X_\beta,T)\}_{0 \le \beta \le \nu}$, $\{\pi_\beta\}_{0 \le \beta \le \nu}$ satisfying (i), (ii), (iii), (v) of 4.7(b) and (iv)' of 4.8. Hence 4.8 is proved.

<u>5.4</u> It follows from 5.1 that if (X,T) is minimal, and a finite a.p. extension of (Y,T), then (X,T) is a quotient-finite-group-extension of (Y,T), hence a covering of (Y,T).

<u>5.5</u> The following proposition holds without the assumption that the (Z_i,T) (i = 1, 2) be distal, but the proof of this will not be given here.

<u>Proposition</u> Let (Z_i,T) be minimal distal $(i = 1,2)$ and suppose we have the following commutative diagram:

<u>Diagram 5.5a)</u>



where G_i is (as usual) a compact topological group acting freely and continuously on Z_i, and H_i is a closed subgroup with $\bigcap_{g\in G_i} g^{-1}H_i g = \{e\}$.

Then there exists a T-isomorphism $\Phi : (Z_1,T) \longrightarrow (Z_2,T)$ and a topological group isomorphism $\alpha : G_1 \longrightarrow G_2$ carrying H_1 onto H_2 such that $\Phi(gz) = \alpha(g)\,\Phi(z)$ for all $z \in Z_1$ and $g \in G_1$, and such that diagram 5.5a) remains commutative when the arrow $Z_1 \xrightarrow{\Phi} Z_2$ is inserted.

<u>Proof</u> 1. Define $\tilde{\rho}_1 : E(Z_1) \longrightarrow E(Y_1)$ as follows (see 4.3):

For $p \in E(Z_1)$ and $y \in Y_1$, define $y(\tilde{\rho}_1 p) = \rho_1(zp)$, whenever $\rho_1(z) = y$.

Then $\tilde{\rho}_1$ is well-defined. To show $\tilde{\rho}_1$ is one-to-one:

Let $p, q \in E(Z_1)$ and suppose $\tilde{\rho}_1(p) = \tilde{\rho}_1(q)$. Then $\rho_1(zp) = \rho_1(zq)$ for all $z \in Z_1$.

Fix $z \in Z_1$. For each $g \in G_1$, there exists $h_g \in H_1$ such that $gzp = h_g gzq$ (because $\rho_1(gzp) = \rho_1(gzq)$).

i.e. $zp = (g^{-1}h_g g)zq$, i.e. $zp = kzq$, where $k \in \bigcap_{g\in G_1} g^{-1}H_1 g = \{e\}$.

i.e. $zp = zq$, and hence, since z is arbitrary, $p = q$.

$\tilde{\rho}_1 : E(Z_1) \longrightarrow E(Y_1)$ is a T-isomorphism and (clearly) a group isomorphism.

2. Let $(Y_1^{Y_1}, \mathcal{T}_p)$ denote the semigroup of (not necessarily continuous) maps from Y_1 to Y_1, with the topology $\mathcal{T}_p$ of pointwise convergence. Consider the map $Y_1^{Y_1} \longrightarrow Y_2^{Y_2}$ given by $h \longmapsto \varphi\cdot h\cdot\varphi^{-1}$. The restriction $\tilde{\varphi}$ of this map to $E(Y_1)$ is a T-isomorphism and group isomorphism onto $E(Y_2)$. By 4.4a), it is

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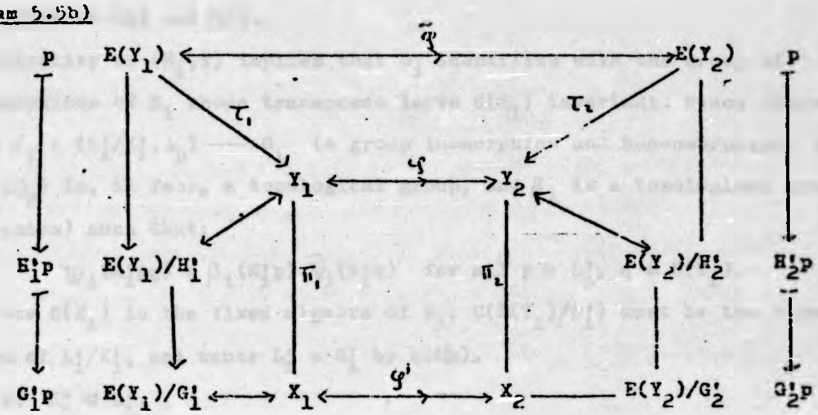
possible to find σ-closed subgroups G_i', H_i' of $E(Y_i)$, and T-homomorphisms τ_i such that the following diagram commutes:

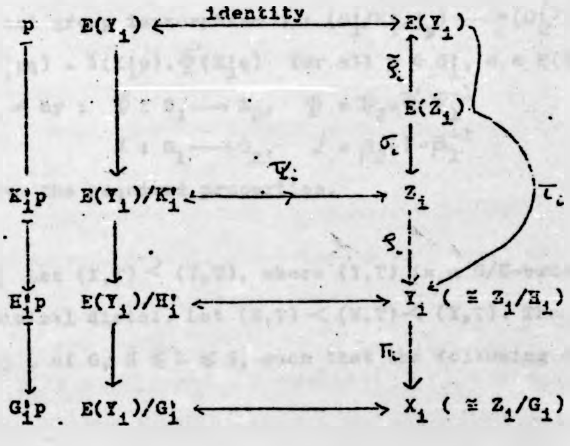
Diagram 5.5b)



3. Let $\tau_i(e) = y_i$ (e the identity of $E(Y_i)$) and choose $z_i \in Z_i$ such that $\rho_i(z_i) = y_i$. Now define $\sigma_i : E(Z_i) \longrightarrow Z_i$ by $\sigma_i(p) = z_i p$ ($p \in E(Z_i)$). Then $\rho_i \circ \sigma_i = \tau_i \circ \tilde{\rho}_i$.

Then 4.4b) implies the existence of a T-isomorphism Ψ_i and K_i' (a σ-closed subgroup of $E(Y_i)$) such that the following diagram commutes:

Diagram 5.5c)



4. Let $L_i' = \{p \in G_i' : pK_i' = K_i'p\}$. Then $(L_i'/K_i', \mathcal{J}_p)$ is a group and a compact

Hausdorff space, and identifies with the group of T-isomorphisms of $E(Y_1)/K_1'$ whose transposes leave $C(E(Y_1)/G_1')$ invariant, with the topology of pointwise convergence (4.4(g) and (h)).

Minimality of (Z_1,T) implies that G_1 identifies with the group of T-isomorphisms of Z_1 whose transposes leave $C(X_1)$ invariant. Hence there exists $\beta_1 : (L_1'/K_1', \mathfrak{I}_p) \longrightarrow G_1$ (a group isomorphism and homeomorphism, so that $(L_1'/K_1', \mathfrak{I}_p)$ is, in fact, a topological group, and β_1 is a topological group isomorphism) such that:

$$\Psi_1(K_1'pq) = \beta_1(K_1'p).\Psi_1(K_1'q) \quad \text{for all } p \in L_1',\ q \in E(Y_1).$$

Since $C(X_1)$ is the fixed algebra of G_1, $C(E(Y_1)/G_1')$ must be the fixed algebra of L_1'/K_1', and hence $L_1' = G_1'$ by 4.4(h).

i.e. $K_1' \triangleleft G_1'$.

Since $\beta_1(K_1'/H_1') = H_1$, we have $K_1' = \bigcap_{g \in G_1'} g^{-1}H_1'g$.

5. We have $\tilde{\varphi} : (E(Y_1), \mathfrak{I}_p) \longrightarrow (E(Y_2), \mathfrak{I}_p)$ is a group isomorphism and homeomorphism, where $\tilde{\varphi}(G_1') = G_2'$, $\tilde{\varphi}(H_1') = H_2'$.

Hence, since $K_1' = \bigcap_{g \in G_1'} g^{-1}H_1'g$, $\tilde{\varphi}(K_1') = K_2'$.

Then $\tilde{\varphi}$ induces a T-isomorphism $\Phi' : (E(Y_1)/K_1', T) \longrightarrow (E(Y_2)/K_2', T)$ and a topological group isomorphism $\gamma : (G_1'/K_1', \mathfrak{I}_p) \longrightarrow (G_2'/K_2', \mathfrak{I}_p)$ such that $\Phi'(K_1'pq) = \gamma(K_1'p).\Phi'(K_1'q)$ for all $p \in G_1'$, $q \in E(Y_1)$.

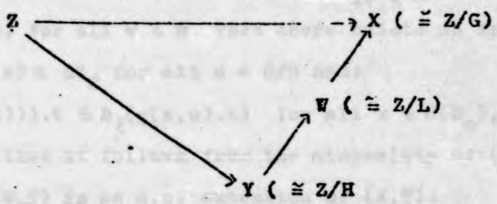
Then define Φ, α by : $\Phi : Z_1 \longrightarrow Z_2$, $\Phi = \Psi_2 \circ \Phi' \circ \Psi_1^{-1}$

$\alpha : G_1 \longrightarrow G_2$, $\alpha = \beta_2 \circ \gamma \circ \beta_1^{-1}$

Φ and α have the required properties.

5.6 Corollary Let $(X,T) \prec (Y,T)$, where (Y,T) is a G/H-extension of (X,T) and (Y,T) is minimal distal. Let $(X,T) \prec (W,T) \prec (Y,T)$. Then there exists a closed subgroup L of G, $H \leq L \leq G$, such that the following diagram commutes:

Diagram 5.6



Proof By Proposition 5.5 and 4.6, we can assume $(Y,T) = (E(Y)/H',T)$ and $(X,T) = (E(Y)/G',T)$ where G', H' are σ-closed subgroups of $E(Y)$ with $N(G') \leq H' \leq G'$ (see 4.4 and 4.6), $G = (G'/N',\mathfrak{J}_p) = (G'/N',\sigma)$, and $H = (H'/N',\mathfrak{J}_p) = (H'/N',\sigma)$, where $N' = \bigcap_{g \in G'} g^{-1}H'g$.

In this case, $(W,T) = (E(Y)/L',T)$ for some L', $H' \leq L' \leq G'$ (4.4), and we can take $L = (L'/N',\mathfrak{J}_p) = (L'/N',\sigma)$.

5.7 The following proposition will be needed in the proof of the Manifold Structure Theorem 1.2:

Proposition Let (W,T) be minimal distal and $(X,T) \prec_{\pi_2} (Y,T) \prec_{\pi_1} (W,T)$ ($\pi_2 \circ \pi_1 = \pi$) where (Y,T) is a quotient-Lie-group-extension of (X,T) and (W,T) is an a.p. N-extension of (Y,T). (i.e. $\pi_1^{-1}(y)$ has N elements for one, hence all, $y \in Y$.) Then (W,T) is a quotient-Lie-group-extension of (X,T).

Proof By 4.6, 5.5 and repeated application of 5.1, it suffices to prove (W,T) is an a.p. extension of (X,T).

Use the following standard notation: for an index ε on a uniform space Z, let $B_\varepsilon(z) = \{z' : (z,z') \in \varepsilon\}$.

The proof is analogous to that of theorem 3 in [16].

Y is a G/H-extension of X, say, where G is compact Lie. Choose any open $W_0 \subseteq W$ such that $\pi_1|W_0$ is a homeomorphism (5.4) and such that there exists a homeomorphism of the form:

$\pi_2 \times \lambda : \pi_1(W_0) \longrightarrow \pi(W_0) \times U$ where U is open in G/H (3.2(iii)).

Write $f = (\pi_2 \times \lambda)^{-1}|(\pi(W_0) \times U)$ and $g = ((\pi_2 \times \lambda) \circ \pi_1)^{-1}|(\pi(W_0) \times U)$.

Find open $U_1 \subseteq U$ and an open neighbourhood U_2 of $e \in G$ such that $U_1U_2 \subseteq U$. To complete the proof, it suffices to prove the following:

5.7.1 Suppose given an index ε on W such that $\pi_1 | B_\varepsilon(w)$ is a homeomorphism onto $\pi_1(B_\varepsilon(w))$ for all $w \in W$. Then there exists an index δ on G/H such that $B_\delta(u) \subseteq uU_2$ for all $u \in G/H$ and:

$$(g(\{x\} \times B_\delta(u))).t \subseteq B_\varepsilon(g(x,u).t) \quad \text{for all } x \in \pi(W_0),\ u \in U_1,\ t \in T.$$

For if 5.7.1 holds, then it follows from the minimality of (W,T) that 4.6(i) is satisfied, i.e. (W,T) is an a.p. extension of (X,T).

Suppose given such an ε. Choose an index ε' on Y such that $B_{\varepsilon'}(\pi_1(w.t)) \subseteq \pi_1(B_\varepsilon(w.t))$ for all $w \in W$, $t \in T$ ([16] lemma 2). Since (Y,T) is an a.p. extension of (X,T), choose an index δ' on G/H such that:

5.7.2 $f(\{x\} \times B_{\delta'}(u)).t \subseteq B_{\varepsilon'}(f(x,u).t)$ for all $x \in \pi(W_0)$, $t \in T$.

Now choose an index δ on G/H, and a connected neighbourhood $V \subseteq U_2$ of $e \in G$ such that $B_\delta(u) \subseteq uV \subseteq B_{\delta'}(u)$ for all $u \in G/H$.

Fix $x \in \pi(W_0)$ and $u \in U_1$. Then $uV \subseteq U$ and :

$$(g(\{x\} \times B_\delta(u))).t \subseteq (g(\{x\} \times uV)).t \subseteq \pi_1^{-1}((f(\{x\} \times uV).t)$$
$$\subseteq \pi_1^{-1}((f(\{x\} \times B_{\delta'}(u)).t) \subseteq \pi_1^{-1} B_{\varepsilon'}(f(x,u).t) \quad \text{(by 5.7.2)}$$
$$\subseteq \pi_1^{-1}\pi_1 B_\varepsilon(g(x,u).t) \subseteq \underbrace{B_\varepsilon(g(x,u).t) \cup U_2 \quad \cdots \cup U_N}_{N \text{ disjoint open sets}}$$

Since $g(x,u).t \in (g(\{x\} \times uV)).t$ and V is connected, $g(\{x\} \times B_\delta(u)).t \subseteq g(\{x\} \times uV).t \subseteq B_\varepsilon(g(x,u).t)$ as required.

§6 Proof of the Addition Theorem 1.1

6.1 The hypothesis is that (X,T) is minimal distal, and $(Y,T) \prec_\pi (X,T)$. Using 4.8 (twice), choose ordinals α_1, α_2 ($\alpha_1 \leq \alpha_2$) and a system $\{(X_\beta,T)_{0 \leq \beta \leq \alpha_2}, (\pi_{\beta\gamma})_{0 \leq \beta \leq \gamma \leq \alpha_2}\}$ of factors of (X,T) such that:

(i) $(X_\beta,T) \prec_{\pi_{\beta\gamma}} (X_\gamma,T)$, $\quad 0 \leq \beta \leq \gamma \leq \alpha_2$.

(ii) $\pi_{\beta\gamma} \circ \pi_{\gamma\delta} = \pi_{\beta\delta}$, $\quad 0 \leq \beta \leq \gamma \leq \delta \leq \alpha_2$, $\pi_{\alpha_1\alpha_2} = \pi$.

(iii) (X_0,T) is the trivial t.g., $(X_{\alpha_1},T) = (Y,T)$, $(X_{\alpha_2},T) = (X,T)$.

(iv) For $\beta < \alpha_2$, $(X_{\beta+1},T)$ is a $G_{\beta+1}/H_{\beta+1}$-extension of (X_β,T), where $G_{\beta+1}$ is compact Lie, and $\bigcap_{g \in G_{\beta+1}} g^{-1}H_{\beta+1}g = \{e\}$ (4.5).

(v) If β is a limit ordinal, (X_β,T) is the inverse limit of $\{(X_\gamma,T)\}_{\gamma<\beta}$.

Write $n_\beta = \dim G_\beta/H_\beta$ for β not a limit ordinal, $\beta > 0$, and $n_\beta = 0$ for β a limit ordinal.

The Addition Theorem will be proved if it can be proved that:

a) $\dim Y = \sum_{0<\beta\le\alpha_1} n_\beta$ b) $\dim \pi^{-1}(y) = \sum_{\alpha_1<\beta\le\alpha_2} n_\beta$ for all $y \in Y$

c) $\dim X = \sum_{0<\beta\le\alpha_2} n_\beta$ (where these sums are interpreted as ∞ if they do not converge).

Only b) will be proved: c) is proved in the same way as b) with Y replaced by the trivial factor and α_1 by 0, and a) is proved in the same way as b) with Y replaced by the trivial factor, α_1 by 0, X by Y and α_2 by α_1.

<u>6.2</u> <u>Proof of 6.1b)</u> Fix $y \in Y$. For $\alpha_1 < \beta \le \alpha_2$ we shall construct by transfinite induction closed sets Q_β and homeomorphisms φ_β such that:

(i) $Q_\beta \subseteq X_\beta$, $\pi_{\gamma\beta}(Q_\beta) = Q_\gamma$, $\alpha_1 \le \gamma < \beta \le \alpha_2$.

(ii) $\varphi_\beta : Q_\beta \longrightarrow \prod_{\alpha_1<\delta\le\beta} [0,1]^{n_\delta}$ is a homeomorphism (where $[0,1]^0 = \{0\}$ by definition) such that the following diagram commutes for $\gamma<\beta$:

<u>Diagram 6.2a)</u>

$$\begin{array}{ccc} Q_\beta & \xrightarrow{\ \pi_{\gamma\beta}\ } & Q_\gamma \\ \downarrow{\scriptstyle \varphi_\beta} & & \downarrow{\scriptstyle \varphi_\gamma} \\ \prod_{\alpha_1<\delta\le\beta} [0,1]^{n_\delta} & \xrightarrow{\ p_{\gamma\beta}\ } & \prod_{\alpha_1<\delta\le\gamma} [0,1]^{n_\delta} \end{array}$$

where we regard $\prod_{\alpha_1<\delta\le\beta} [0,1]^{n_\delta}$ as $\prod_{\alpha_1<\delta\le\gamma} [0,1]^{n_\delta} \times \prod_{\gamma<\delta\le\beta} [0,1]^{n_\delta}$, and $p_{\gamma\beta}$ is the natural projection.

(iii) $\dim Q_\beta = \dim \pi^{-1}_{\alpha_1\beta}(y)$

Then, since it is clear that $\dim Q_\beta = \sum_{\alpha_1<\delta\le\beta} n_\delta$ (by (ii)), 6.1b) will be proved by putting $\beta = \alpha_2$ in (iii).

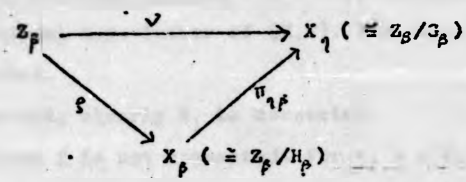
<u>Case $\beta = \alpha_1 + 1$</u> Clearly $\pi^{-1}_{\alpha_1,\alpha_1+1}(y)$ is homeomorphic to $G_{\alpha_1+1}/H_{\alpha_1+1}$. Hence find a closed subset Q_{α_1+1} of $\pi^{-1}_{\alpha_1,\alpha_1+1}(y)$ homeomorphic under some φ_{α_1+1} to $[0,1]^{n_{\alpha_1+1}}$.

Now suppose Q_γ, φ_γ have been constructed for $\alpha_1<\gamma<\beta$.

Case $\beta = \eta + 1$, some $\eta > \alpha$.

By 6.1(iv) there exists a minimal distal t.g. (Z_β, T) and continuous surjective maps ρ, ν such that the following diagram commutes:

Diagram 6.2b)



Then $(\nu^{-1}(Q_\eta), \pi_{\eta\beta}^{-1}(Q_\eta), Q_\eta, G_\beta, H_\beta, \pi_{\eta\beta}, \rho, \nu)$ is a fibre bundle (4.5,3.1) and by 3.6 there exists a homeomorphism

$$(\pi_{\eta\beta} \times \lambda_\beta) : \pi_{\eta\beta}^{-1}(Q_\eta) \longrightarrow (Q_\eta \times G_\beta/H_\beta).$$

Now let $N_\beta \subseteq G_\beta/H_\beta$ be a closed subset isomorphic to $[0,1]^{n_\beta}$ under ψ_β, say, and let Q_β be the inverse image under $(\pi_{\eta\beta} \times \lambda_\beta)$ of $(Q_\eta \times N_\beta)$.

Let $\varphi_\beta : Q_\beta \longrightarrow \prod_{\alpha<\delta\leq\beta} [0,1]^{n_\delta}$ be defined by $\varphi_\beta = (\varphi_\eta \times \psi_\beta) \circ (\pi_{\eta\beta} \times \lambda_\beta)$.

Clearly (i) and (ii) of 6.2 hold, and $\dim Q_\beta = \sum_{\alpha<\delta\leq\beta} n_\delta$.

By considering the fibre bundle

$$((\pi_{\eta\beta} \circ \nu)^{-1}(y), \pi_{\alpha\beta}^{-1}(y), \pi_{\alpha\eta}^{-1}(y), G_\beta, H_\beta, \pi_{\eta\beta}, \rho, \nu),$$ we see that

$\dim \pi_{\alpha\beta}^{-1}(y) \leq \dim \pi_{\alpha\eta}^{-1}(y) + \dim G_\beta/H_\beta$ (by 3.3) $\leq \sum_{\alpha<\delta\leq\beta} n_\delta$, and hence:

$\dim \pi_{\alpha\beta}^{-1}(y) = \sum_{\alpha<\delta\leq\beta} n_\delta$, since $Q_\beta \subseteq \pi_{\alpha\beta}^{-1}(y)$ (2.2), and so (iii) is satisfied.

Case β a limit ordinal, $\beta > \alpha$.

Define $Q_\beta = \bigcap_{\alpha<\gamma<\beta} \pi_{\gamma\beta}^{-1}(Q_\gamma)$.

Define $\varphi_\beta : Q_\beta \longrightarrow \prod_{\alpha<\delta\leq\beta}[0,1]^{n_\delta}$ by:

$$p_{\gamma\beta} \circ \varphi_\beta(z) = \varphi_\gamma \circ \pi_{\gamma\beta}(z) \quad (z \in Q_\beta) \text{ for all } \gamma < \beta.$$

Then φ_β is well-defined and a homeomorphism. (i) and (ii) of 6.2 are clearly satisfied, and clearly:

$$\sum_{\alpha<\delta\leq\beta} n_\delta = \dim Q_\beta \leq \dim \pi_{\alpha\beta}^{-1}(y).$$

But by 2.6, $\dim \pi_{\alpha,\beta}^{-1}(y) \leq \limsup_{\gamma<\beta} \dim \pi_{\alpha,\gamma}^{-1}(y) = \sum_{\alpha\leq\delta<\beta} n_\delta$, and (iii) is satisfied.

6.3 Corollary to the Addition Theorem Let (X,T) be minimal distal. Let (X_1,T) denote the maximal a.p. factor of (X,T). Then X is connected if and only if X_1 is connected.

Proof If X is connected, clearly X_1 is connected.

Conversely, suppose X is not connected. For x, y ∈ X, define x ∼ y if x and y lie in the same connected component of X. Then ∼ is a closed T-invariant equivalence relation on X, hence induces a factor (X/∼,T) of (X,T). By the Furstenburg Structure Theorem (4.7), (X/∼,T) has a non-trivial a.p. factor (Y,T), which, by the Addition Theorem 1.1, must be 0-dimensional, hence totally disconnected. But Y is a continuous image of X_1. Hence X_1 is not connected.

§7 Proof of the Manifold Structure Theorem 1.2

7.1 Throughout this section use the notation of 1.2.

First note that, since, in 1.2(i), Y, like X, has finitely many arcwise connected components and hence, like X, satisfies the hypotheses of the theorem it suffices to prove X is a manifold, which will follow from 1.2(ii) and (iii) (since each X_i is there proved to be a manifold).

(ii) will follow from (iii) ($\dim X_{i+1} > \dim X_i$) and the Addition Theorem ($\dim X_i \leq \dim X$ for all i).

In 1.2(iv), "G_1/H_1 connected if and only if X is connected" is precisely 6.3. 1.2(v) follows from Proposition 5.5.

Thus we only need to prove 1.2(iii), and that G_i/H_i is connected for $i \geq 2$, which we proceed to do.

7.2 Suppose (inductive hypothesis) that X_i, Y_i, G_i, H_i, π_i, ρ_i, ν_i have been have been constructed for $i < s \leq$ order(X) satisfying all the conditions in 1.2(iii), and with G_i/H_i connected for $2 \leq i < s$. Let (X_s,T) be the maximal a.p. extension of (X_{s-1},T) in (X,T). Let Y_s, G_s, H_s, π_s, ς_s, ν_s be such that the following diagram commutes (4.6(ii)):

Diagram 7.2

$$\begin{array}{ccc} Y_s & \xrightarrow{\nu_s} & X_{s-1}\ (\cong Y_s/G_s) \\ & {\scriptstyle \varsigma_s}\searrow \quad \nearrow{\scriptstyle \pi_s} & \\ & X_s\ (\cong Y_s/H_s) & \end{array}$$

Suppose also that $\bigcap_{g \in G_s} g^{-1}H_s g = \{e\}$.

We know that G_s is non-trivial, and it is easily seen that X_1 is not finite. Hence, to complete the proof of (iii) and (iv) of 1.2 for X_s, Y_s, G_s, H_s, π_s, ς_s, ν_s, it will suffice to prove:

(a) G_s is a Lie group (for then Y_s and X_s will be manifolds by 3.2).

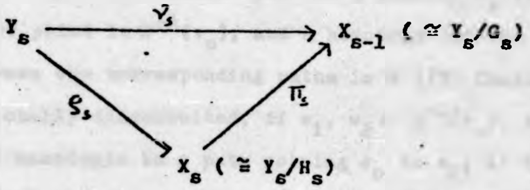
(b) G_s/H_s is connected if $s \geq 2$.

7.3 Proof of 7.2(a) If G_s is not Lie then there exists a strictly decreasing sequence $\{N_i\}$ of normal subgroups of G_s such that $H_sN_{i+1} \lneqq H_sN_i$ and G_s/N_i is Lie (5.1), where $\dim G_s/H_sN_i \leq \dim G_s/HN_{i+1} \leq \dim G_s/H_s < \infty$ by the Addition Theorem 1.1. Y_s/H_sN_i is a manifold for each i (3.2). We obtain the required contradiction to G_s not being Lie from the following lemma:

Lemma Let (W,T) be minimal distal with W having finitely many arcwise-connected components, and let $(V,T) \prec_\pi (W,T)$, with V a manifold. Then it is not possible to find a strictly increasing sequence $\{(V_n,T)\}_{n=0}^{\infty}$ of factors of (W,T) such that $(V_0,T) = (V,T)$ and (V_n,T) is a finite a.p. extension of (V,T).

Proof Suppose for contradiction that such a sequence exists. Replacing T by a syndetic subgroup, $\{(V_n,T)\}$ by a proper subsequence and W, V_n by one of the connected components of W, V_n if necessary, we can assume that W is arcwise-connected. We can also assume (W,T) is the inverse limit of $\{(V_n,T)\}$. Then (W,T) is an a.p. extension of (V,T), hence a G/H-extension of (V,T) for some compact topological group G. Fix $v_0 \in V$. Then $\pi^{-1}(v_0)$ is infinite and totally a.p. extension of (X_{s-1},T) in (X,T). Let Y_s, G_s, H_s, π_s, ς_s, ν_s be such that the following diagram commutes (4.6(ii)):

Diagram 7.2



Suppose also that $\bigcap_{g\in G_s} g^{-1}H_s g = \{e\}$.

We know that G_s is non-trivial, and it is easily seen that X_1 is not finite. Hence, to complete the proof of (iii) and (iv) of 1.2 for X_s, Y_s, G_s, H_s, π_s, ς_s, ν_s, it will suffice to prove:

(a) G_s is a Lie group (for then Y_s and X_s will be manifolds by 3.2).

(b) G_s/H_s is connected if $s \geq 2$.

7.3 Proof of 7.2(a) If G_s is not Lie then there exists a strictly decreasing sequence $\{N_i\}$ of normal subgroups of G_s such that $H_sN_{i+1} \lneq H_sN_i$ and G_s/N_i is Lie (5.1), where $\dim G_s/H_sN_i \leq \dim G_s/HN_{i+1} \leq \dim G_s/H_s < \infty$ by the Addition Theorem 1.1. Y_s/H_sN_i is a manifold for each i (3.2). We obtain the required contradiction to G_s not being Lie from the following lemma:

Lemma Let (W,T) be minimal distal with W having finitely many arcwise-connected components, and let $(V,T) \prec_\pi (W,T)$, with V a manifold. Then it is not possible to find a strictly increasing sequence $\{(V_n,T)\}_{n=0}^{\infty}$ of factors of (W,T) such that $(V_0,T) = (V,T)$ and (V_n,T) is a finite a.p. extension of (V,T).

Proof Suppose for contradiction that such a sequence exists. Replacing T by a syndetic subgroup, $\{(V_n,T)\}$ by a proper subsequence and W, V_n by one of the connected components of W, V_n if necessary, we can assume that W is arcwise-connected. We can also assume (W,T) is the inverse limit of $\{(V_n,T)\}$. Then (W,T) is an a.p. extension of (V,T), hence a G/H-extension of (V,T) for some compact topological group G. Fix $v_0 \in V$. Then $\pi^{-1}(v_0)$ is infinite and totally disconnected. Since $\pi^{-1}(v_0)$ is homeomorphic to G/H, it is also compact and perfect, hence uncountable. Since each V_n is a finite cover of V (5.4), a loop in V based at v_0 will lift to a unique path in W joining w_0 (a fixed point in $\pi^{-1}(v_0)$) to another point in $\pi^{-1}(v_0)$, and a homotopy between two loops lifts to a homotopy between the corresponding paths in W ([9] Chapter 6 Theorem 4). Since $\pi^{-1}(v_0)$ is totally disconnected, if $w_1, w_2 \in \pi^{-1}(v_0)$, a path in W joining w_0 to w_1 cannot be homotopic to a path joining w_0 to w_2, if the endpoints are restricted to $\pi^{-1}(v_0)$ and $w_1 \neq w_2$. Hence the fundamental group of V (based at v_0) is uncountable. But this is impossible since V is a compact manifold.

<u>7.4</u> <u>Proof of 7.2(b)</u> If $s \geq 2$ and G_s/H_s is not connected, then define an equivalence relation $\sim$ on X_s by:
$(x \sim y)$ if and only if $(\pi_s(x) = \pi_s(y)$ and x and y lie in the same connected component of $\pi_s^{-1}\pi_s(x))$.
Then $(X_s/\sim,T)$ is a proper finite extension of (X_{s-1},T), so that $(X_s/\sim,T)$ is (5. an a.p. extension of (X_{s-2},T) - which contradicts (X_{s-1},T) being the maximal a.p. extension of (X_{s-2},T) in (X,T). Therefore G_s/H_s must be connected.

<u>§8 Index of Notation and List of Fundamental Groups</u>

In this section we give a list of the symbols used from now on to denote the indicated standard (topological) groups and topological spaces. There follo (8.3) a table of fundamental groups which is sufficient for proof that most of the topological spaces mentionned in 8.2 are of distinct topological types.

<u>8.1</u> <u>Note</u> If X is a topological space and $\sim$ is an equivalence relation on X, $X/\sim$ will denote the space of equivalence classes with the quotient topology. For $x \in X$, [x] will denote the $\sim$-equivalence class of x; square brackets will be used without mention of the associated equivalence relation, if it is though that no confusion can arise. In particular, [x] will often denote the orbit of x under the action of some group on X.

8.2 List of symbols

A_n — Group of permutations of $\{1\ldots n\}$ wich can be written as the product of an even number of 2-cycles (so $|A_n| = (n!)/2$).

Aut(G) — The automorphism group of a topological group G.

$\mathbb{C}$ — The field of complex numbers.

D_n — $(n \geq 2)$: Dihedral group of order 2n, $\langle a, b : a^n = b^2 = 1, ab = ba^{-1}\rangle$.
As a subgroup of SO(3): the group generated by the set

$$\left\{\begin{pmatrix} \cos 2\pi r/n & \sin 2\pi r/n & 0 \\ -\sin 2\pi r/n & \cos 2\pi r/n & 0 \\ 0 & 0 & 1 \end{pmatrix} : r = 0\ldots n-1\right\} \cup \left\{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right\}$$

$GL(n,\mathbb{R})$ — Group of real n x n invertible matrices.

$GL(n,\mathbb{Z})$ — Group of matrices with integer coefficients and determinants ± 1.

K — Circle group $\{z \in \mathbb{C} : |z| = 1\}$ (group operation being multiplication).
As a subgroup of SO(3): $\left\{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R}\right\}$

As a subgroup of SU(2): $\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} : \lambda \in K\right\}$

K^n — $\underbrace{K \times \ldots \times K}_{n \text{ copies}}$ n-dimensional torus.

$K \times_s Z_2$ — Z_2 is identified with $Aut(K) = \{1, \varepsilon\}$, where $\varepsilon(k) = k^{-1}$.
$K \times_s Z_2 = \{(k,\sigma) : k \in K, \sigma = 1 \text{ or } \varepsilon\}$.
Multiplication is defined by $(k_1,\sigma_1).(k_2,\sigma_2) = (k_1 \sigma_1(k_2), \sigma_1\sigma_2)$
As a subgroup of SO(3), $K \times_s Z_2$ is the group generated by the set:

$$\left\{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R}\right\} \cup \left\{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right\}$$

As a subgroup of SO(3) $\times_s Z_2$ (10.7(iii)),

$$K \times_s Z_2 = \left\{ (k,\sigma) : k \in K \text{ as a subgroup of SO(3) and } \sigma \in Z_2 \text{ as a subgroup of Aut(SO(3))} \right\}.$$

K^3/σ — (σ is an automorphism of K^2 of order r.) This denotes the orbit space of K^3 under the free action of $\langle\sigma\rangle$ defined by: $\sigma.(k_1,k_2,k_3) = (e^{2\pi i/r}k_1, \sigma(k_2,k_3))$.

$K^3\Big/\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22}\end{pmatrix}$ — This denotes K^3/σ where σ corresponds to $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22}\end{pmatrix} \in GL(2,\mathbb{Z})$ (See 10.8.)

$(K \times S^2)/\sim$ — $\sim$ denotes the equivalence relation $(k, x) \sim (-k, -x)$ for $k \in K$ and $x \in S^2$ (see 8.1).

KB — This denotes the Klein bottle $K^2/\sim$ where $(k_1,k_2) \sim (-k_1,k_2^{-1})$.

N/Γ_n — ($n \geq 1$): N denotes the Lie group of matrices:

$$\left\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1\end{pmatrix} : x, y, z \in \mathbb{R}\right\}$$

, the group operation being matrix multiplication.

Γ_n denotes the subgroup

$$\left\{\begin{pmatrix} 1 & m_1 & m_2/n \\ 0 & 1 & m_3 \\ 0 & 0 & 1\end{pmatrix} : m_1, m_2, m_3 \in \mathbb{Z}\right\}$$

Where no confusion can arise, $[x,y,z]$ denotes the element:

$$\Gamma_n\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1\end{pmatrix} \text{ of } N/\Gamma_n.$$

$O(n)$ — Group of real orthogonal $n \times n$ matrices.

P^n — ($n \geq 2$): n-dimensional real projective plane $S^n/\sim$ where $x \sim -x$.

$\mathbb{R}$ — The field of real numbers.

S^n — ($n \geq 1$): $\left\{(x_1 \ldots x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\right\}$.

S_n — Group of permutations of $\{1\ldots n\}$ (so $|S_n| = n!$).

$SU(2)$ — $\left\{\begin{pmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda}\end{pmatrix} : \lambda, \mu \in \mathbb{C},\ |\lambda|^2 + |\mu|^2 = 1\right\}$

$(S^2 \times K)/\sim$ — $\sim$ denotes the equivalence relation $(x,k) \sim (-x,-k)$ for $x \in S^2$, $k \in K$. This space is homeomorphic to $(K \times S^2)/\sim$.

$(S^2 \times K)/\approx$ — $\approx$ denotes the equivalence relation $(x,k) \approx (-x,k^{-1})$.

W_0 $K^3/\sim$ where $\sim$ is the equivalence relation:

$(k_1,k_2,k_3) \sim (-k_1,k_2^{-1},k_3) \sim (k_1,-k_2,k_3^{-1}) \sim (-k_1,-k_2^{-1},k_3^{-1})$.

W_2 $K^3/\sim$ where $\sim$ is the equivalence relation:

$(k_1,k_2,k_3) \sim (-k_1,k_2^{-1},k_3) \sim (k_1,-k_2,k_1^2k_3^{-1}) \sim (-k_1,-k_2^{-1},k_1^2k_3^{-1})$.

$\mathbb{Z}$ Group of integers.

Z_n Cyclic group of order n $\langle a : a^n = 1 \rangle$.

As a subgroup of SO(3): the group generated by:

$$\begin{pmatrix} \cos 2\pi/n & \sin 2\pi/n & 0 \\ -\sin 2\pi/n & \cos 2\pi/n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As a subgroup of SU(2): the group generated by: $\begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}$

8.3 Table of Fundamental Groups

Space	Fundamental Group	Number of homomorphisms of fundamental group into: Z_2	Z_3	Z_4	Z_r	D_6
K^3	$\mathbb{Z}^3$	8	27			
$K^3/\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$\langle a,b,c:\ ab = b^{-1}a,\ ac = c^{-1}a,\ bc = cb\rangle$	8	3			48
$K^3/\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\langle a,b,c:\ ab = ba,\ ac = c^{-1}a,\ bc = cb\rangle$	8	9			36
$K^3/\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$	$\langle a,b,c:\ ab = ba,\ ac = c^{-1}ab,\ bc = cb\rangle$	4	9		r^2	24
$K^3/\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$\langle a,b,c:\ bc = cb,\ ba = ac^{-1},\ ca = ac^{-1}b\rangle$	2	9			
$K^3/\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$\langle a,b,c:\ bc = cb,\ ab = c^{-1}a,\ ac = ba\rangle$	4	3	8		
$K^3/\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$	$\langle a,b,c:\ bc = cb,\ ba = ac,\ ca = ab^{-1}c\rangle$	2	3			
W_0	$\langle a,b,c:\ ab = b^{-1}a,\ bc = c^{-1}b,\ ac = ca\rangle$	8	3			36
W_2	$\langle a,b,c:\ ab = cb^{-1}a,\ bc = c^{-1}b,\ ac = ca\rangle$	4	3	16		
N/Γ_1	$\langle a,b,c:\ ab = ba,\ ac = cab,\ bc = cb\rangle$	4	9		r^2	18
N/Γ_2	$\langle a,b,c:\ ab = ba,\ ac = cab^2,\ bc = cb\rangle$	8	9			30
N/Γ_3	$\langle a,b,c:\ ab = ba,\ ac = cab^3,\ bc = cb\rangle$	4	27		$r^2 x(3,r)$	
$N/\Gamma_n (n \geq 4)$	$\langle a,b,c:\ ab = ba,\ ac = cab^n,\ bc = cb\rangle$				$r^2 x(n,r)$	
$S^2 \times K$	$\mathbb{Z}$					
$(S^2 \times K)/\sim$	$\mathbb{Z}$					
$P^2 \times K$	$\mathbb{Z} \times Z_2$					
$(S^2 \times K)/\approx$	$\mathbb{Z} \times Z_2$					

Notes on table

(i) (n, r) denotes the highest common factor of n and r.

(ii) It can be shown that $S^2 \times K$ and $(S^2 \times K)/\sim$ are not homeomorphic, even though they have the same fundamental group, and similarly for $P^2 \times K$ and $(S^2 \times K)/\approx$.

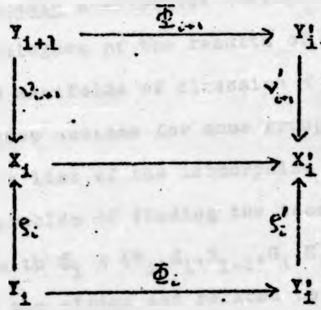
§9 On the String of a Minimal Distal Transformation Group

First we need some definitions (9.1-9.2):

9.1 Definitions A string $(\mathcal{B}_1 \dots \mathcal{B}_n)$ of bundles is a finite sequence of bundles $\mathcal{B}_i = (Y_i, X_i, X_{i-1}, G_i, H_i, \pi_i, \rho_i, \nu_i)$ $(1 \le i \le n)$ where X_0 is a one-point set. n is called the length of the string, which is denoted by $\underline{\mathcal{B}}$, say.

If $\underline{\mathcal{B}}' = (\mathcal{B}'_1 \dots \mathcal{B}'_n)$ is another string with $\mathcal{B}'_i = (Y'_i, X'_i, X'_{i-1}, G'_i, H'_i, \pi'_i, \rho'_i, \nu'_i)$ then $\underline{\mathcal{B}}$ and $\underline{\mathcal{B}}'$ are isomorphic under $(\Phi_1 \dots \Phi_n, \alpha_1 \dots \alpha_n)$ if there exist 3rd-isomorphisms (3.4) $(\Phi_i, \alpha_i): \mathcal{B}_i \longrightarrow \mathcal{B}'_i$ such that the following diagram commutes:

Diagram 9.1



9.2 Definition Let $\underline{\mathcal{B}} = (\mathcal{B}_1 \dots \mathcal{B}_r)$ be a string with:

$$\mathcal{B}_i = (Y_i, X_i, X_{i-1}, G_i, H_i, \pi_i, \rho_i, \nu_i) \quad (1 \le i \le r).$$

$\underline{\mathcal{B}}$ is n-allowable if each G_i/H_i is connected, $\dim G_i/H_i \geq 1$ and

$$n = \sum_{i=1}^{r} \dim (G_i/H_i) \qquad (= \dim X_r).$$

$\underline{\mathcal{B}}$ is allowable if $\underline{\mathcal{B}}$ is n-allowable for some n.

9.3 Use the notation of the Manifold Structure Theorem (1.2): this theorem shows that given a minimal distal t.g. (X,T) where X is a compact connected n-dimensional manifold, we can associate with it an n-allowable string $(\mathcal{B}_1 \dots \mathcal{B}_r) = \underline{\mathcal{B}}(X,T)$ where r is the <u>order</u> of (X,T) (4.10). There is some choice in the strings which can be associated with (X,T) in this way, but any two choices are <u>isomorphic as strings</u>. Moreover, if $(X,T) \cong (X',T)$, then $\underline{\mathcal{B}}(X,T) \cong \underline{\mathcal{B}}(X',T)$. Therefore we have:

<u>9.4</u> <u>Definition</u> Given a topological group T, a string $\underline{\mathcal{B}}$ is <u>admissable</u> if it is $\underline{\mathcal{B}}(X,T)$ (up to isomorphism) for some minimal distal t.g. (X,T) where X is a compact connected topological manifold.

$\underline{\mathcal{B}}$ is <u>admissable</u> if $\underline{\mathcal{B}}$ is T-admissable for some T. Clearly (9.3) admissable strings are allowable.

<u>9.5</u> Later (9.7) we give a complete list of $\mathbb{Z}$-admissable and $\mathbb{R}$-admissable n-allowable strings for $n \leq 3$, and hence obtain a coarse classification of minimal distal $\mathbb{Z}$- and $\mathbb{R}$-actions on compact connected manifolds of dimension ≤ 3. It is easy - but rather tedious, so we shall not do it - to give a complete list of the <u>admissable</u> n-allowable strings for $n \leq 3$, by using the results of §§10-11 and analogues of the results of §12. However, we list (9.6) the compact connected manifolds of dimension ≤ 3 which can be phase spaces of minimal distal group actions for some group. It is clear that such a list is a "corollary" of a list of the isomorphism classes of admissable strings.

Clearly the problem of finding the isomorphism classes of strings $\underline{\mathcal{B}} = (\mathcal{B}_1 \dots \mathcal{B}_r)$ with $\mathcal{B}_i = (Y_i, X_i, X_{i-1}, G_i, H_i, \pi_i, \rho_i, \nu_i)$ $(1 \leq i \leq r)$ is inductive on the length of the string and related to the following two problems:

(i) Find the possibilities for $\{(G_i,H_i)\}_{1 \leq i \leq r}$ up to isomorphism (§10).

(ii) Having found X_{i-1}, G_i, H_i, find the 1st, 2nd and 3rd-isomorphism classes of bundles with base X_{i-1}, group G_i and isotropy subgroup H_i. 1st-isomorphism classes are given in §11. 2nd- and 3rd-isomorphism classes are easily deduced from these.

<u>9.6</u> <u>Manifolds of dimension ≤ 3 supporting minimal distal actions of some group</u>: (for notation see §8, §10):

<u>Actions of order 1</u> (almost periodic): $K, K^2, S^2, P^2, K^3, SU(2)/Z_n$ $(n \geq 1)$,

SO(3)/D_{2n} ($n \geq 2$), SO(3)/A_4, SO(3)/S_4, SO(3)/A_5, $S^2 \times K$, $P^2 \times K$, $(S^2 \times K)/\sim$.

Actions of order 2: K^2, KB, K^3, $K^3/\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$, $K^3/\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, $K^3/\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}$, $K^3/\begin{pmatrix}0 & 1\\-1 & -1\end{pmatrix}$, $K^3/\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$, $K^3/\begin{pmatrix}0 & -1\\1 & 1\end{pmatrix}$, SU(2)/$Z_n$ ($n \geq 1$), SO(3)/D_{2n} ($n \geq 2$), N/Γ_n ($n \geq 1$), $S^2 \times K$, $P^2 \times K$, $(S^2 \times K)/\sim$, $(S^2 \times K)/\approx$.

Actions of order 3: K^3, $K^3/\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$, $K^3/\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, $K^3/\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}$, N/Γ_n ($n \geq 1$), W_c, W_z.

9.7 If $\mathcal{B} = (\mathcal{B}_1 \ldots \mathcal{B}_r)$ is an n-allowable string for $n \leq 3$ and $\mathcal{B}_i = (Y_i, X_i, X_{i-1}, G_i, H_i, \pi_i, \rho_i, \nu_i)$ and $\mathcal{B} = \mathcal{B}(X_r, T)$ for some T, then $Y_1 = G_1$ and T acts on G_1 by right multiplication on G_1 of a homomorphic image (in G_1) of T. So if T is abelian, H_1 is trivial, G_1 is abelian and $X_1 = Y_1$.

Therefore, in tables A and B we list the n-allowable strings ($n \leq 3$) of length 2 and 3 for which G_1 is abelian and H_1 is trivial, stating which of them are $\mathbb{Z}$-admissable and which are $\mathbb{R}$-admissable. Each line in the tables – except A4 – represents exactly one isomorphism class.

The tables are intended merely as a summary of information and can only be understood in conjunction with §§10 and 11.

Table A (order 2)

	dim X_2	G_1	(G_2,H_2) (§10)	$\mathcal{B}_2$ (§11)	X_2	$\mathcal{L}$-admiss §12	$\mathcal{L}$-admiss §13
A1	2	K	$(K,\{1\})$	$\mathcal{K}(K)$ 11.4	K^2 8.2	Yes	No
A2	"	"	$(K\times_s Z_2, Z_2)$	$\mathcal{K}(K,Z_2)$ 11.4	" "	"	"
A3	"	"	"	$\mathcal{K}(K,Z_2,\varepsilon)$ "	KB "	"	"
A4	3	"	$(K^2\times_s A, A)$	$\mathcal{K}(K^2,A,\alpha)$ "	$K^3/\nless$ "	9.8,12.6	"
A5	"	"	$(SO(3),K)$	$\mathcal{K}(SO(3),K)$ "	$K\times S^2$ 8.2	Yes	"
A6	"	"	$(SO(3)\times_s Z_2, K\times_s Z_2)$	$\mathcal{K}(SO(3),K,Z_2)$ "	" "	"	"
A7	"	"	"	$\mathcal{K}(SO(3),K,Z_2,\varepsilon)$ "	$(K\times S^2)/\nless$ "	"	"
A8	"	"	$(SO(3),K\times_s Z_2)$	$\mathcal{K}(SO(3),K\times_s Z_2)$ "	$K\times P^2$ "	"	"
A9	"	K^2	$(K,\{1\})$	$\mathcal{J}_o$ 11.7	K^3 "	"	Yes
A10	"	"	"	$\mathcal{J}_n$, $n\geq 1$ "	N/Γ_n, $n\geq 1$ "	"	"
A11	"	"	$(K\times_s Z_2, Z_2)$	$\mathcal{J}_o\times_s Z_2$ 11.7,11.3	K^3 "	"	No
A12	"	"	"	$\mathcal{J}_n\times_s Z_2$, $n\geq 1$; 11.7,11.3	N/Γ_n, $n\geq 1$ "	No	"
A13	"	"	"	$\mathcal{J}_o(k_1,-k_2,k_3^{-1})$ 11.7	$K^3/\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ "	Yes	Yes
A14	"	"	"	$\mathcal{J}_o(k_1,-k_2,k_1k_3^{-1})$ 11.7	$K^3/\begin{pmatrix}1&0\\1&-1\end{pmatrix}$ "	"	"

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Table B

	$(\mathfrak{B}_1,\mathfrak{B}_2)$ as in: (see table A)	(G_3,H_3)	$\mathfrak{B}_3$		X_3		Z-admiss.
B1	A1	$(K,\{1\})$	$\mathfrak{J}_0$	11.7	K^3	8.2	Yes
B2	A2	"	"	"	"	"	"
B3	A1	"	$\mathfrak{J}_n$, $n \geq 1$	"	N/Γ_n, $n\geq 1$	"	"
B4	A2	"	"	"	"	"	No
B5	A1	$(K x_s Z_2, Z_2)$	$\mathfrak{J}_0 x_s Z_2$	11.7,11.3	K^3	"	Yes
B6	A2	"	"	11.7,11.3	"	"	"
B7	A1	"	$\mathfrak{J}_n x_s Z_2$, $n\geq 1$	11.7,11.3	N/Γ_n, $n\geq 1$	"	No
B8	A2	"	"	11.7,11.3	"	"	Yes
B9	A1	"	$\mathfrak{J}_0(-k_1,k_2,k_3^{-1})$	11.7	$K^3/\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	"	"
B10	A2	"	"	"	"	"	"
B11	A1	"	$\mathfrak{J}_0(k_1,-k_2,k_3^{-1})$	"	"	"	"
B12	A2	"	"	"	"	"	"
B13	A1	"	$\mathfrak{J}_0(-k_1,k_2,k_2k_3^{-1})$	"	$K^3/\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$	"	"
B14	A2	"	"	"	"	"	"
B15	A1	"	$\mathfrak{J}_0(k_1,-k_2,k_1k_3^{-1})$	"	"	"	"
B16	A2	"	"	"	"	"	"
B17	A3	$(K,\{1\})$	$\mathcal{K}\mathfrak{B}_0$	11.9	$K^3/\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	"	"
B18	"		$\mathcal{K}\mathfrak{B}_1$	"	$K^3/\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$	"	"
B19	"	$(K x_s Z_2, Z_2)$	$\mathcal{K}\mathfrak{B}_0 x_s Z_2$	11.9,11.3	$K^3/\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	"	"
B20	"	"	$\mathcal{K}\mathfrak{B}_1 x_s Z_2$	11.9,11.3	$K^3/\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$	"	"
B21	"	"	$\mathcal{K}\mathfrak{B}_0'$	11.9	$K^3/\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	"	"
B22	"	"	$\mathcal{K}\mathfrak{B}_0([k_1,-k_2],k_3^{-1})$	11.9	W_0	"	"
B23	"	"	$\mathcal{K}\mathfrak{B}_0([k_1,-k_2],k_1^2k_3^{-1})$	"	W_2	"	"

9.8 Notes (i) None of the strings of table B are $\mathcal{K}$-admissable (§ 13).
(ii) In A4, A can be assumed to be one of the groups of 10.8, and $\sigma \in A$. For $\sigma, \tau \in A$, $\mathcal{K}(K^2,A,\sigma)$ and $\mathcal{K}(K^2,A,\tau)$ give rise to isomorphic strings if and only if there exists $\eta \in Aut(K^2)$ with $\eta A \eta^{-1} = A$ and $\eta \sigma \eta^{-1} = \tau$.
(iii) In A4, $\mathcal{K}(K^2,A,\sigma)$ gives rise to a $\mathbb{Z}$-admissable string if and only if $\langle\sigma\rangle \triangleleft A$ and $A/\langle\sigma\rangle$ is cyclic (12.6).

§10 On Connected Irreducible Pairs

In this section we give some isomorphism classes of irreducible pairs (G,H) (definitions 10.1 - 10.2) - this information is needed for finding the isomorphism classes of strings (9.1) and can doubtless be found elsewhere, but is collected together for convenient reference.

10.1 Definition An irreducible pair (G,H) consists of a compact Lie group G and a closed subgroup H such that $\bigcap_{g \in G} g^{-1}Hg = \{e\}$.

(G,H) is connected if G/H is connected.

(G,H) is group-connected if G is connected.

10.2 Definition Irreducible pairs (G_1,H_1) and (G_2,H_2) are isomorphic if there exists a topological group isomorphism of G_1 onto G_2 which maps H_1 onto H_2 (written $(G_1,H_1) \cong (G_2,H_2)$).

10.3 The following lemma gives the relationship between connected irreducible pairs and group-connected irreducible pairs. The proof is straightforward and will be omitted.

Lemma (i) If (G,H) is a connected irreducible pair, then $(G_o, H \cap G_o)$ is a group-connected irreducible pair, where G_o denotes the identity component of G.

$G_oH = HG_o = G$, and the map $Hg \longmapsto (H \cap G_o)g$ $(g \in G_o)$ defines a homeomorphism of G/H onto $G_o/(H \cap G_o)$.

(ii) If, for $h \in H$, $g \in G_o$, $\theta_h(g) = hgh^{-1}$, the map $h \longmapsto \theta_h$ is a topological group isomorphism of H into the subgroup $S(G_o, H \cap G_o)$ of $Aut(G_o)$ of automorphisms leaving $H \cap G_o$ invariant, where $Aut(G_o)$ is given the topology of pointwise convergence. So identify H with $\{\theta_h : h \in H\} \leq S(G_o, H \cap G_o)$.

(iii) Suppose, further, there exists a subgroup $S_1 = S_1(G_o, H \cap G_o)$ of $S(G_o, H \cap G_o)$ such that each element of $S(G_o, H \cap G_o)$ can be uniquely written in the form xy, where $x \in S_1$ and $y \in H \cap G_o \leq S(G_o, H \cap G_o)$ (which clearly happens if $H \cap G_o$ is trivial). Then write $A = S_1 \cap H \leq S(G_o, H \cap G_o)$. A is finite and $(G,H) = (G_o \times_s A, (H \cap G_o) \times_s A)$ where $G_o \times_s A = \{(g,\sigma) : g \in G_o, \sigma \in A\}$ and multiplication is defined by $(g_1,\sigma_1).(g_2,\sigma_2) = (g_1\sigma_1(g_2), \sigma_1\sigma_2)$.

(iv) If A_1 and A_2 are finite subgroups of $Aut(G_o)$ then $(G_o \times_s A_1, A_1) \cong (G_o \times_s A_2, A$ if and only if A_1 and A_2 are conjugate in $Aut(G_o)$.

<u>10.4</u> Detailed proof that the list of 10.6 is exhaustive will not be given, but the following facts are used:

(i) <u>If (G,H) is an irreducible pair and dim G/H ≤ n then dim G ≤ n(n+1)/2 [1C</u>

(ii) A compact connected Lie group is isomorphic to one of the form $(S \times T)/Z$ where S is semisimple compact connected, T is a torus and Z is a finite central subgroup with $S \cap Z$ and $T \cap Z$ trivial ([8] Chapter XIII Theorem 1.

(iii) Given a compact semi-simple Lie algebra $\mathfrak{g}$, there is a unique compact <u>simply-connected</u> connected Lie group G with Lie algebra $\mathfrak{g}$ (up to isomorphism), and if G_1 is another connected Lie group with Lie algebra $\mathfrak{g}$, then $G_1 \cong G/Z$ for some finite central subgroup Z of G [15].

(iv) A compact semisimple Lie algebra is a direct sum of compact simple Lie algebras, which have been completely classified [15].

(v) Any toral subgroup of a compact connected Lie group G is contained in a maximal torus, and any two maximal tori are conjugate ([8] Chapter XIII.4).

<u>10.5</u> <u>Definition</u> If G is a compact Lie group, and H is a closed subgroup, and $Z = \bigcap_{g \in G} g^{-1}Hg$, then [G,H] will denote the irreducible pair (G/Z,H/Z).

<u>10.6</u> We now list the group-connected irreducible pairs (G,H) with $\dim G/H \leq 3$, using the notation of 8.2.

(i) $(K,\{1\})$

(ii) $(K^2,\{1\})$

(iii) $(SO(3),K) \cong [SU(2),K]$. $SO(3)/K$ is homeomorphic to S^2.

(iv) $(SO(3), K x_s Z_2) \cong [SU(2),M]$ where M is the subgroup generated by the set

$$\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} : \lambda \in K\right\} \cup \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

$SO(3)/(K x_s Z_2)$ is homeomorphic to P^2.

(v) $(K^3,\{1\})$

(vi) $[SU(2),Z_n]$

(vii) $(SO(3),D_{2n}) \cong [SU(2),M_{4n}]$ $(n \geqslant 2)$ where M_{4n} is the subgroup generated by the set

$$\left\{\begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/r} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

(viii) $(SO(3),A_4)$, $(SO(3),S_4)$, $(SO(3),A_5)$. All subgroups of $SO(3)$ isomorphic to A_4, S_4, A_5 respectively are conjugate ([7] Chapter 2).

(ix) $[SU(2) \times K, Z_n']$ where $Z_n' = \left\{\left(\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \lambda^n\right) : \lambda \in K\right\}$ $(n \geqslant 0)$.

If $n = 0$, G/H is homeomorphic to $S^2 \times K$. If $n > 0$, G/H is homeomorphic to $SU(2)/Z_n$.

(x) $[SU(2) \times K, M \times \{1\}]$ where M is as in (iv). G/H is homeomorphic to $P^2 \times K$.

(xi) $[SU(2) \times K, W]$, where $W = \left\{\left(\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, 1\right) : \lambda \in K\right\} \cup \left\{\left(\begin{pmatrix} 0 & \lambda \\ -\bar{\lambda} & 0 \end{pmatrix}, -1\right) : \lambda \in\right.$

G/H is homeomorphic to $(S^2 \times K)/\sim$.

(xii) $[SU(2) \times SU(2), V_1]$, where $V_1 = \{(u,u) : u \in SU(2)\}$. G/H is homeomorphic to $SU(2)$, equivalently to S^3.

(xiii) $[SU(2) \times SU(2), V_2]$ where $V_2 = V_1 \cup \{(u, -u) : u \in SU(2)\}$. G/H is homeomorphic to $SU(2)/Z_2$, equivalently to $SO(3)$ and to P^3.

<u>10.7</u> Let (G',H') be one of the group-connected irreducible pairs of 10.6(i)-(and let $S(G',H') = \{\theta \in \mathrm{Aut}(G') : \theta(H') = H'\}$, so that $H' \leq S(G',H')$ (10.3(ii)

For each such (G',H'), we define a $S_1(G',H') \leq S(G',H')$ such that each element of $S(G',H')$ can be uniquely written in the form xy ($x \in H'$, $y \in S_1(G',H')$) and hence show that if (G,H) is a connected irreducible pair with $\dim G/H \leq 2$, then $(G,H) = (G_o x_s A, (H \cap G_o) x_s A)$, for a finite $A \leq S_1(G_o, H \cap G_o)$ (10.3(iii)). $(G_o, H \cap G_o)$ being (up to isomorphism) one of the pairs 10.6(i) - (iv).

(i) $\underline{(G_o, H \cap G_o) = (K, \{1\})}$ $S_1(K,\{1\}) = S(K,\{1\}) = Aut(K) = \{1, \varepsilon\} \cong Z_2$, where $\varepsilon(k) = k^{-1}$ ($k \in K$).

(ii) $\underline{(G_o, H \cap G_o) = (K^2, \{1\})}$ $S_1(K^2,\{1\}) = S(K^2,\{1\}) = Aut(K^2)$

$Aut(K^2) \cong GL(2,\mathbb{Z})$ (8.2) where the isomorphism is given by:

$$\sigma \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } \sigma(k_1,k_2) = (k_1^a k_2^b,\ k_1^c k_2^d).$$

(iii) and (iv) $\underline{(G_o, H \cap G_o) = (SO(3),K) \text{ or } (SO(3), K x_s Z_2)}$

All automorphisms of $SO(3)$ are inner, so that $Aut(SO(3)) \cong SO(3)$. Under this identification, $S(SO(3),K) = S(SO(3), K x_s Z_2) = K x_s Z_2 \leq SO(3)$.

(iii) $\underline{(G_o, H \cap G_o) = (SO(3),K)}$ $S_1(SO(3),K) = \{1, \varepsilon\} \cong Z_2$, where

$$\varepsilon((u_{ij})) = ((-1)^{i+j} u_{ij}).$$

ε is the inner automorphism corresponding to $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(iv) $\underline{(G_o, H \cap G_o) = (SO(3), K x_s Z_2)}$ $S_1(SO(3), K x_s Z_2)$ is trivial.

<u>10.8</u> In order to completely classify the connected irreducible pairs (G,H) for which $(G_o, H \cap G_o) = (K^2, \{1\})$, it remains to find the conjugacy classes of finite subgroups of $Aut(K^2) \cong GL(2,\mathbb{Z})$ (10.3(iv) and 10.7). I am indebted to my father, D.Rees, for finding the conjugacy classes, although he says the answer must be known. Note that:

(a) If $u \in GL(2,\mathbb{Z})$ has finite order then u must have order 1, 2, 3, 4 or 6. (Consider the minimal polynomial of u, which must have integral coefficients.)

(b) A finite subgroup of $GL(2,\mathbb{Z})$ is conjugate in $GL(2,\mathbb{R})$ to a subgroup of $O(2)$, which is, of course, isomorphic to $K x_s Z_2$ ([6] Theorem 16.9.1).

(a) and (b) imply that a non-trivial finite subgroup of $Aut(K^2)$ must be

A

isomorphic to Z_n or D_{2n} (n = 2, 3, 4 or 6). It can be shown, further, that the conjugacy classes of finite subgroups are as follows:

(i) $\langle I\rangle$ (trivial subgroup)

(ii) $\langle -I\rangle$, (iii) $\left\langle\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\rangle$, (iv) $\left\langle\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\right\rangle$ $\cong Z_2$

(v) $\left\langle\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}\right\rangle \cong Z_3$

(vi) $\left\langle\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\rangle \cong Z_4$

(vii) $\left\langle\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\right\rangle \cong Z_6$

(viii) $\left\langle -I, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\rangle$, (ix) $\left\langle -I, \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\right\rangle$ $\cong D_4$

(x) $\left\langle\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\rangle$, (xi) $\left\langle\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\right\rangle$ $\cong D_6$

(xii) $\left\langle\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\rangle \cong D_8$

(xiii) $\left\langle\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\rangle \cong D_{12}$

§11 1st-isomorphism Classes of Fibre Bundles

In order to determine the isomorphism classes of strings, it is necessary (9.5(ii)) to find the 1st-isomorphism classes (3.4) of bundles:

(a) with base K, group G and isotropy subgroup H, where (G,H) is a connected irreducible pair with dim $G/H \leq 2$. (For the possibilities for (G,H), see §10.) See 11.4.

(b) with base K^2, KB, S^2 or P^2, group G and isotropy subgroup H where $(G,H) = (K,\{1\})$ or $(K \times_s Z_2, Z_2)$. See 11.5 - 11.10.

This is just a matter of collecting together known results. 1st-isomorphism classes rather than 2nd-isomorphism classes are given (the latter would in some ways be more convenient) mainly because 1st-isomorphism is the type of isomorphism usually used in fibre bundle theory.

The notation of §8 will be used throughout this section.

I should like to thank E. César de Sá, M. Eastwood, J. Eells and D. Epstein for helpful suggestions and discussion.

11.1 We define a complete 1st-isomorphism invariant $\chi : \mathcal{C}_i \xrightarrow{\text{onto}} \mathcal{D}_i$ (i = 1, 2, 3 or 4) for each of the following classes $\mathcal{C}_i$ of principal bundles, where $\mathcal{D}_1$ is the given range space.

<u>(i)</u> ([17] 13.5) <u>$\mathcal{C}_1(X,G)$</u> is the set of principal bundles with base X (a compact connected manifold) and finite group G. Let $\mathcal{B} = (Y,X,G,\pi) \in \mathcal{C}_1(X,G)$. Fix $x_0 \in X$, $y_0 \in Y$ with $\pi(y_0) = x_0$. π is a finite covering map, hence determines (with x_0, y_0) a homomorphism $\varphi_{\mathcal{B}} : \pi_1(X) \longrightarrow G$, where $\pi_1(X)$ denotes the fundamental group of X. Two homomorphisms in $\mathrm{Hom}(\pi_1(X),G)$ are said to be equivalent if one is the composition of the other with an inner automorphism of G. <u>$\mathcal{D}_1(X,G)$</u> is the set of equivalence classes in $\mathrm{Hom}\,(\pi_1(X),G)$.
$\chi(\mathcal{B})$ is defined to be the equivalence class of $\varphi_{\mathcal{B}}$, and $\chi : \mathcal{C}_1(X,G) \longrightarrow \mathcal{D}_1(X$ thus defined is independent of the x_0, y_0 chosen for each $\mathcal{B}$.
<u>(ii)</u> ([17] 18.5) <u>$\mathcal{C}_2(G)$</u> is the set of principal bundles with base K and group G. <u>$\mathcal{D}_2(G)$</u> is the set of conjugacy classes in G/G_0, where G_0 is the component of the identity in G.

Let $\mathcal{B} = (Y,K,G,\pi) \in \mathcal{C}_2(G)$. For $\theta_1, \theta_2 \in \mathbb{R}$, let $\{\{e^{i\theta_1}, e^{i\theta_2}\}\}$ denote $\{e^{i\varphi} : \theta_1 \leq \varphi \leq \theta_2\}$.
Define $V_1 = \{\{e^{-3i\pi/4}, e^{3i\pi/4}\}\}$ and $V_2 = \{\{e^{i\pi/4}, e^{7i\pi/4}\}\}$.

<u>Diagram 11.1(a)</u>

Choose maps $\varphi_i : V_i \longrightarrow Y$ with $\pi \circ \varphi_i$ = identity (3.2) and $\varphi_1(i) = \varphi_2(i)$.
Define $g_{1,2} : V_1 \cap V_2 \longrightarrow G$ by $\varphi_1(x) = g_{1,2}(x).\varphi_2(x)$ for all $x \in V_1 \cap V_2$.
Define $\chi(\mathcal{B})$ to be the conjugacy class in G/G_0 of $G_0 g_{1,2}(-i)$.
(a) The definitions of χ on $\mathcal{C}_1(K,G)$ and on $\mathcal{C}_2(G)$, for finite G, do not quite coincide on $\mathcal{C}_1(K,G) \cap \mathcal{C}_2(G)$, but there is a natural correspondence between the two definitions, and in any case no confusion should arise.

(b) It follows from (ii) that a principal bundle with base K and group K must be a product bundle. Hence ([17] 11.4) a principal bundle with base K x [0,1] and group K must be a product bundle.

<u>(iii) and (iv)</u>

$\mathcal{C}_3$ is the set of principal bundles with base K^2 and group K.

$\mathcal{C}_4$ is the set of principal bundles with base KB and group K.

$\Phi_3 = \mathbb{Z}$, $\Phi_4 = \{0, 1\}$.

<u>(iii)</u> Let $V_1 = \{\{e^{-i\pi/4}, e^{5i\pi/4}\}\} \times K$, $V_2 = \{\{e^{3i\pi/4}, e^{9i\pi/4}\}\} \times K$,

$S = \{1\} \times K$, $T = \{-1\} \times K$. So $K^2 = V_1 \cup V_2$.

<u>(iv)</u> Recall that $KB = \{[k_1,k_2] : (k_1,k_2) \in K^2\}$ (8.2).

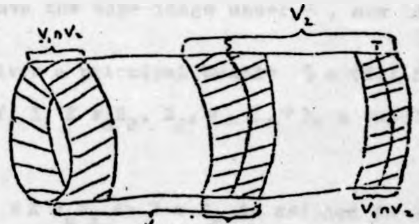
Let $V_1 = \{[k_1,k_2] : k_1 \in \{\{e^{-i\pi/4}, e^{5i\pi/8}\}\}, k_2 \in K\}$.

Let $V_2 = \{[k_1,k_2] : k_1 \in \{\{e^{3i\pi/8}, e^{5i\pi/4}\}\}, k_2 \in K\}$.

Let $S = \{[1,k] : k \in K\}$, $T = \{[i,k] : k \in K\}$.

So $KB = V_1 \cup V_2$.

<u>Diagram 11.1(b)</u>



This diagram represents how V_1 and V_2 are related for both (iii) and (iv).

If $\mathcal{B} = (Y, X, K, \pi) \in \mathcal{C}_i$ ($i = 3$ or 4, so that $X = K^2$ or KB), choose maps $\varphi_j : V_j \longrightarrow Y$ ($j = 1, 2$) with $\pi \circ \varphi_j$ = identity (see (ii)(b)) and: $\varphi_1|S = \varphi_2|S$.

Define $g_{1,2} : V_1 \cap V_2 \longrightarrow K$ by $\varphi_1(x) = g_{1,2}(x) \cdot \varphi_2(x)$, $x \in V_1 \cap V_2$.

Then $g_{1,2}|T$ is homotopic to:

(iii) $(-1,k) \longmapsto k^n$ for a unique $n \in \mathbb{Z}$.

(iv) $[i,k] \longmapsto k^n$ for a unique $n \in \mathbb{Z}$.

Define $\chi(\mathcal{B})$ by : (iii) $\chi(\mathcal{B}) = n$

(iv) $\chi(\mathcal{B}) = 0$ if n is even and 1 if n is odd.

χ is independent of the choice of φ_1, φ_2.

11.2 We "define" a 1st-isomorphism invariant χ on the class $\mathcal{C}_5(X)$ of principal bundles with group $K \times_s Z_2$ and base X, for a fixed compact Hausdorff X.

Suppose $\mathcal{B} = (Y,X,K \times_s Z_2,\nu)$. Let Y/K denote the orbit space of Y under $K \leq K \times_s Z_2$, and $\nu_2 : Y \longrightarrow Y/K$ the orbit map. Let $\mathcal{B}_1 = (Y/K,X,Z_2,\nu_1)$ and $\mathcal{B}_2 = (Y,Y/K,K,\nu_2)$, where $\nu_1 \cdot \nu_2 = \nu$.

Define $\chi(\mathcal{B}) = (\mathcal{B}_1, \mathcal{B}_2)$.

It is simple - but tedious - to give a more rigorous definition of χ making it a 1st-isomorphism invariant on $\mathcal{C}_5(X)$. But χ is not a complete invariant and does not map onto any simply defined domain. However, χ will be a help in determining the 1st-isomorphism classes of bundles in $\mathcal{C}_5(X)$.

"Non-surjectivity" For an example of how one determines when a couple $(\mathcal{B}_1, \mathcal{B}_2)$ of fibre bundles is not in the image of χ, see 11.8.

Non-completeness For an example of how one determines how many 1st-isomorphism classes in $\mathcal{C}_5(X)$ have the same image under χ, see 11.10.

11.3 Definition Given a principal bundle $\mathcal{B} = (Y,X,K,\pi)$, define $\mathcal{B} \times_s Z_2 = (Y \times Z_2,\ Y,\ X,\ K \times_s Z_2,\ Z_2,\ \pi,\ \xi,\ \nu)$, a bundle with group $K \times_s Z_2$, as follows:

the action of $K \leq K \times_s Z_2$ on $Y \times Z_2$ is defined in terms of the action of K on Y for $\mathcal{B}$ by $k.(y,1) = (k.y,1)$

$k.(y,\varepsilon) = (k^{-1}.y,\varepsilon)$ for all $y \in Y$, $k \in K$, where $Z_2 = \{1,\varepsilon\}$.

ε acts on $Y \times Z_2$ by $\varepsilon.(y,\sigma) = (y,\varepsilon\sigma)$ for all $y \in Y$, $\sigma \in Z_2$.

$\nu(y,\sigma) = \pi(y)$. $\xi(y,\sigma) = y$.

Note that if $\mathcal{B}$ is a product bundle, so is $\mathcal{B} \times_s Z_2$.

<u>11.4 1st-isomorphism classes of bundles with base K</u>

We wish to find the 1st-isomorphism classes of bundles $(Y,W,K,G,H,\overline{W},\xi,\nu)$ where (G,H) is a connected irreducible pair with $\dim G/H \leq 2$. So (10.6 - 10.8) $(G,H) = (G_0 \times_s A, (H \cap G_0) \times_s A)$, where A is a finite subgroup of $\mathrm{Aut}(G_0)$ and $(G_0, H \cap G_0) = (K,\{1\}), (K^2,\{1\}), (SO(3),K)$ or $(SO(3), K \times_s Z_2)$.

Fix $(G,H) = (G_0 \times_s A, H' \times_s A)$ and $\sigma \in A$. The 1st-isomorphism classes are given by the bundles $\mathcal{K}(G_0,H',A,\sigma) = (Y(\sigma),W(\sigma),K,G,H,\overline{W},\xi,\nu)$ ($= \mathcal{K}(G_0,A,\sigma)$ or $\mathcal{K}(G_0,H',A)$ or $\mathcal{K}(G_0,H')$ or $\mathcal{K}(G_0,A)$ or $\mathcal{K}(G_0)$, depending on which of H', A, σ are trivial) where σ runs through the A-conjugacy classes in A, and the principal G-bundle associated with $\mathcal{K}(G_0,H',A,\sigma)$, which is an element of $\mathcal{C}_2(G)$ (11.1), is mapped to σ under χ.

<u>Define $Y(\sigma)$</u> $= K \times (G_0 \times A/A')$, where $A' = \langle\sigma\rangle$ and $A/A' = \{\tau A' : \tau \in A\}$.

There is a natural left G-action on $G_0 \times A/A'$.

Let r = order of σ. There is a natural left $K \times_s Z_2$-action on K. Define left G-action on K by $(g_0,\tau).k = F_\sigma(\tau).k$ for all $g_0 \in G_0$, $\tau \in A$, $k \in K$, where $F_\sigma : A \longrightarrow K \times_s Z_2$ is some chosen homomorphism for which $F_\sigma(\sigma) = (e^{2\pi i/r},1)$ (always possible for the A's being considered).

<u>Define action of G on $Y(\sigma)$</u> by $g.(k,x) = (g.k,g.x)$ for all $g \in G$, $k \in K$, $x \in G_0 \times A/A'$.

Define $\nu : Y(\sigma) \longrightarrow K$ by $\nu((1,\tau).(k,g_0,A')) = k^r$ for all $\tau \in A$, $k \in K$, $g_0 \in G_0$. (ν is well-defined.)

<u>Define</u> $\varphi_1 : V_1 \longrightarrow Y(\sigma)$ (see 11.1(ii)) by $\varphi_1(e^{i\theta}) = (e^{i\theta/r},1,A')$ for $5\pi/4 \leq \theta \leq 11\pi/4$.

$\varphi_2 : V_2 \longrightarrow Y(\sigma)$ by $\varphi_2(e^{i\theta}) = (e^{i\theta/r},1,A')$ for $\pi/4 \leq \theta \leq 7\pi/4$.

Then $g_{1,2}(-1) = (1,\sigma)$ as required (see 11.1(ii)).

For definition of $\mathcal{K}(\sigma)$ and ρ consider the different possibilities for (G,H):

(G,H) = $(K \times_s A, A)$ where $A = \{1\}$ or $A = \{1,\varepsilon\} \cong Z_2$ where $\varepsilon(k) = k^{-1}$ $(k \in K)$.

$\mathcal{K}(K)$ and $\mathcal{K}(K,Z_2)$ are the product bundles.

$\mathcal{K}(K,Z_2,\varepsilon) = (K^2, KB, K, K \times_s Z_2, Z_2, \pi, \rho, \nu)$ where $\rho : K^2 \longrightarrow KB$ is defined by

$\rho(k_1,k_2) = [k_1,k_2]$ (see 8.1, 8.2).

(G,H) = $(K^2 \times_s A, A)$

$\mathcal{K}(K^2,A,\sigma) = (K^3 \times A/A', K^3/\sigma, K, K^2 \times_s A, A, \pi, \rho, \nu)$ where $\rho : K^3 \times A/A' \longrightarrow K^3/\sigma$ is well-defined by $\rho((1,\tau).(k_1,k_2,k_3,A')) = [k_1,k_2,k_3]$ for all $k_1, k_2, k_3 \in K$ and $\tau \in A$ (see 8.2 for definition of K^3/σ, and 8.1).

K^3/σ is homeomorphic to the unique K^3/η in the list of 8.3 for which σ is conjugate in $Aut(K^2)$ to η.

(G,H) = (SO(3),K) or $(SO(3), K \times_s Z_2)$

$\mathcal{K}(SO(3),K) = (K \times SO(3), K \times S^2, K, SO(3), K, \pi_1, \rho_1, \nu_1)$ and

$\mathcal{K}(SO(3), K \times_s Z_2) = (K \times SO(3), K \times P^2, K, SO(3), K \times_s Z_2, \pi_2, \rho_2, \nu_2)$

are product bundles where ρ_1 and ρ_2 are defined by

$\rho_1(k, (u_{ij})) = (k, (u_{11},u_{12},u_{13}))$ $\rho_1 : K \times SO(3) \longrightarrow K \times S^2$

$\rho_2(k, (u_{ij})) = (k, [(u_{11},u_{12},u_{13})])$ $\rho_2 : K \times SO(3) \longrightarrow K \times P^2$

(See 8.2 for the definition of P^2, and 8.1.)

(G,H) = $(SO(3) \times_s Z_2, K \times_s Z_2)$

$\mathcal{K}(SO(3),K,Z_2) = (K \times SO(3) \times Z_2, K \times S^2, SO(3) \times_s Z_2, K \times_s Z_2, \pi, \rho, \nu)$ is the product bundle.

$\mathcal{K}(SO(3),K,Z_2,\varepsilon) = (K \times SO(3), (K \times S^2)/\sim, K, SO(3) \times_s Z_2, K \times_s Z_2, \pi, \rho, \nu)$ has $\rho : K \times SO(3) \longrightarrow (K \times S^2)/\sim$ defined by:

$\rho(k, (u_{ij})) = [k, (u_{11},u_{12},u_{13})\omega_k]$ where, if $k = e^{i\theta}$, $\omega_k \in SO(3)$ is defined by

$$\omega_k = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

11.5 1st-isomorphism classes of bundles with base S^2

We state the results without proof. (See [17] 18.5.)

(i) Bundles with group K

The distinct 1st-isomorphism classes are given by ξ_n ($n \in \mathbb{Z}$).

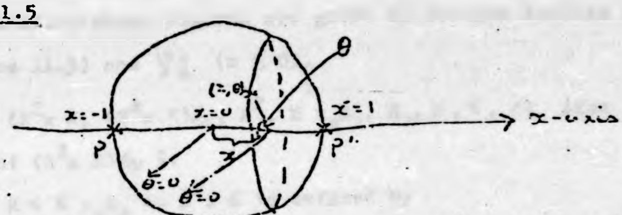
n = 0 ξ_0 is the product bundle.

Fix n > 0 $\xi_n = (SU(2)/Z_n, S^2, K, \pi_n)$.

The action of K on $SU(2)/Z_n$ is defined by:

$$k \cdot Z_n\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = Z_n \begin{pmatrix} k^{1/n} & 0 \\ 0 & k^{-1/n} \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

Diagram 11.5



S^2 is given "cylindrical coordinates", so $S^2 = ((-1,1)\times K) \cup \{P, P'\}$.

Using those coordinates, $\pi_n : SU(2)/Z_n \longrightarrow S^2$ is defined by:

$$\pi_n\left(Z_n\begin{pmatrix} \sqrt{r}e^{i\theta} & \sqrt{1-r}e^{i\varphi} \\ -\sqrt{1-r}e^{-i\varphi} & \sqrt{r}e^{-i\theta} \end{pmatrix}\right) = \begin{cases} (1-2r, e^{i(\theta-\varphi)}), & 0 < r < 1 \\ P, & r = 1 \\ P', & r = 0 \end{cases}$$

For n > 0, $\xi_{-n} = (SU(2)/Z_n, S^2, K, \pi_{-n})$ where $\pi_{-n} = \pi_n$ and the action of K on $SU(2)/Z_n$ is defined by:

$$k \cdot Z_n\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = Z_n \begin{pmatrix} k^{-1/n} & 0 \\ 0 & k^{1/n} \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

(ii) Bundles with group $K \times_s Z_2$

The distinct 1st-isomorphism classes are given by $\xi_n \times_s Z_2$, $n \geq 0$ (see 11.

11.6 1st-isomorphism classes of bundles with base P^2

We state the results without proof.

(i) Bundles with group K

There are two 1st-isomorphism classes of bundles, denoted by ρ_0 and ρ_1.

ρ_0 is the product bundle.

$\rho_1 = ((S^2 \times K)/\sim, P^2, K, \pi)$, where K acts on $(S^2 \times K)/\sim$ by

$k.[x_1, k_1] = [x_1, kk_1]$ for all $x_1 \in S^2$, $k, k_1 \in K$.

$\pi : (S^2 \times K)/\sim \longrightarrow P^2$ is defined by $\pi([x,k]) = [x]$.

(ii) Bundles with group $K \times_s Z_2$

The 1st-isomorphism classes are given by bundles denoted by $\rho_0 \times_s Z_2$, $\rho_1 \times_s Z_2$ (see 11.3) and ρ'_n $(n \geq 0)$.

$\rho'_0 = (S^2 \times K, (S^2 \times K)/\sim, P^2, K \times_s Z_2, Z_2, \pi, \rho, \nu)$ (see 8.2 for definition of $(S^2 \times K)/\sim$).

Action of $K \leq K \times_s Z_2$ on $S^2 \times K$ is defined by

$k.(y_1, k_1) = (y_1, kk_1)$ $(y_1 \in S^2, k, k_1 \in K)$.

Action of $\xi \in Z_2 \leq K \times_s Z_2$ on $S^2 \times K$ is defined by

$\xi.(x,k) = (-x, k^{-1})$.

$\nu : S^2 \times K \longrightarrow P^2$ is defined by $\nu(x,k) = [x]$ $(x \in S^2, k \in K)$.

$\rho'_n = (SU(2)/Z_{2n}, SO(3)/D_{2n}, P^2, K \times_s Z_2, Z_2, \pi'_{2n}, \rho'_{2n}, \nu'_{2n})$

Action of K on $SU(2)/Z_{2n}$ is as for $\mathcal{S}_{2n}$ (11.5), and ν'_{2n} is as π_{2n} for $\mathcal{S}_{2n}$.

Action of $\xi \in Z_2 \leq K \times_s Z_2$ on $SU(2)/Z_{2n}$ is defined by.

$$\xi . Z_{2n} \cdot \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = Z_{2n} \begin{pmatrix} u_{21} & u_{22} \\ -u_{11} & -u_{12} \end{pmatrix}$$

11.7 1st-isomorphism classes of bundles with base K^2

(i) Bundles with group K : use $\chi : \mathcal{C}_3 \longrightarrow \mathcal{Q}_3$ (11.1).

$\chi(\beta) = 0$ This gives the product bundle, denoted by $\mathcal{J}_0$.

$\chi(\beta) = n \neq 0$ A bundle with this characteristic is $\mathcal{J}_n$, defined as follows.

For $n > 0$, define $\Gamma_{-n} = \Gamma_n$ (8.2) , and let $\mathfrak{J}_n = (N/\Gamma_n, K^2, K, \pi_n)$.

Action of K on N/Γ_n is given by $e^{2\pi i t}.[x,y,z] = [x, y+t/n, z]$.

$\pi_n : N/\Gamma_n \longrightarrow K^2$ is defined by $\pi_n([x,y,z]) = (e^{2\pi i x}, e^{2\pi i z})$.

Define $\varphi_i : V_i \longrightarrow N/\Gamma_n$, $i = 1, 2$ (11.1(iii)) by

$\varphi_1(e^{2\pi i x}, e^{2\pi i z}) = [x,0,z]$, $-\frac{1}{4} \le x \le \frac{3}{4}$

$\varphi_2(e^{2\pi i x}, e^{2\pi i z}) = [x,0,z]$, $-\frac{3}{4} \le x \le \frac{1}{4}$.

(ii) <u>Bundles with group Z_2</u>: use $\chi_1 : \mathcal{C}_1(K^2, Z_2) \longrightarrow \mathcal{D}_1(K^2, Z_2)$ (11.1).

$\mathcal{D}_1(K^2, Z_2) = \mathrm{Hom}(\pi_1(K^2), Z_2) = \{\eta_1, \eta_2, \eta_3, \eta_4\}$.

$\pi_1(K^2) = \langle a, b : ab = ba \rangle$.

a and b are the homotopy classes corresponding to the paths $t \longmapsto (e^{2\pi i t}, 1)$ and $t \longmapsto (1, e^{2\pi i t})$ $(t \in [0,1])$ respectively.

We define a bundle $\mathcal{B}_1^i = (X_1^i, K^2, Z_2, \nu_1^i)$ with $\chi(\mathcal{B}_1^i) = \eta_i$ $(i = 1 \ldots 4)$.

$\eta_1(a) = \eta_1(b) = 1$: $\mathcal{B}_1^1$ is the product bundle.

$\eta_2(a) = \varepsilon$, $\eta_2(b) = 1$: $X_1^2 = K^2$, $\varepsilon.(k_1,k_2) = (-k_1,k_2)$, $\nu_1^2(k_1,k_2) = (k_1^2,k_2)$

$\eta_3(a) = 1$, $\eta_3(b) = \varepsilon$: $X_1^3 = K^2$, $\varepsilon.(k_1,k_2) = (k_1,-k_2)$, $\nu_1^3(k_1,k_2) = (k_1,k_2^2)$

$\eta_4(a) = \eta_4(b) = \varepsilon$: $X_1^4 = K^2$, $\varepsilon.(k_1,k_2) = (-k_1,-k_2)$, $\nu_1^4(k_1,k_2) = (k_1k_2, k_1^{-1}k_2)$.

(iii) <u>Bundles with group $K \times_s Z_2$</u> : use χ_5 defined on $\mathcal{C}_5(K^2)$ (11.2).

i.e. we find 1st-isomorphism classes of bundles

$\mathcal{B} = (Y, Y/Z_2, K^2, K \times_s Z_2, Z_2, \Pi, \rho, \nu_1 \circ \nu_2)$ in terms of the bundles $\mathcal{B}_1 = (X_1, K^2, Z_2, \nu_1)$ and $\mathcal{B}_2 = (Y, X_1, K, \nu_2)$. Complete proofs will not be given.

<u>$\chi(\mathcal{B}_1) = \eta_1$</u> So $\mathcal{B}_2$ has base $K^2 \times Z_2$. It can be shown that the only possibilities up to 1st-isomorphism are $\mathfrak{J}_n \times_s Z_2$, $n \geq 0$ (11.7(i), 11.3).

<u>$\chi(\mathcal{B}_1) = \eta_2, \eta_3$ or η_4</u> It can be shown that $\mathcal{B}_2$ must be $\mathfrak{J}_0$ (11.7(i)).

The action of $\varepsilon \in Z_2 \le K \times_s Z_2$ on K^3 determines $\mathcal{B}$ up to 1st-isomorphism, and can be taken to be as follows:

<u>$\chi(\mathcal{B}_1) = \eta_2$</u> $\varepsilon.(k_1,k_2,k_3) = (-k_1,k_2,k_3^{-1})$: gives bundle $\mathfrak{J}_0(-k_1,k_2,k_3^{-1})$

or $\varepsilon.(k_1,k_2,k_3) = (-k_1,k_2,k_2k_3^{-1})$: gives bundle $\mathcal{J}_0(-k_1,k_2,k_2k_3^{-1})$

$\underline{\chi(\mathcal{B}_1) = \eta_3}$: $\varepsilon.(k_1,k_2,k_3) = (k_1,-k_2,k_3^{-1})$: gives bundle $\mathcal{J}_0(k_1,-k_2,k_3^{-1})$

or $\varepsilon.(k_1,k_2,k_3) = (k_1,-k_2,k_1k_3^{-1})$: gives bundle $\mathcal{J}_0(k_1,-k_2,k_1k_3^{-1})$

$\underline{\chi(\mathcal{B}_1) = \eta_4}$: $\varepsilon.(k_1,k_2,k_3) = (-k_1,-k_2,k_3^{-1})$: gives bundle $\mathcal{J}_0(-k_1,-k_2,k_3^{-1})$

or $\varepsilon.(k_1,k_2,k_3) = (-k_1,-k_2,k_1k_2k_3^{-1})$: gives bundle $\mathcal{J}_0(-k_1,-k_2,k_1k_2k_3^{-1})$

In each case Y/Z_2 is homeomorphic to $K^3/\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $K^3/\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ respectively.

<u>11.8</u> As an example of the method used in the calculation of 11.7(iii), we show that if $\chi(\mathcal{B}_1) = \eta_2$, and $\mathcal{B}_2 = \mathcal{J}_0$, then up to 1st-isomorphism, the action of ε on K^3 must be:

$$\varepsilon.(k_1,k_2,k_3) = (-k_1,k_2,k_3^{-1}) \text{ or } (-k_1,k_2,k_2k_3^{-1}).$$

It can be shown that the action of ε must be of the form:

$$\varepsilon_f(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, e^{2\pi i\theta_3}) = (-e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, e^{2\pi i(f(\theta_1,\theta_2)-\theta_3)}),$$

where $f \in C(\mathbb{R}^2,\mathbb{R})$ has $f(\theta_1+\tfrac{1}{2},\theta_2) = f(\theta_1,\theta_2) \pmod{\mathbb{Z}} = f(\theta_1,\theta_2+1)$.

It can be shown that ε_{f_1} and ε_{f_2} are associated with 1st-isomorphic bundles if there exists $\varphi \in C(\mathbb{R}^2,\mathbb{R})$ with:

$$\varphi(\theta_1+1,\theta_2) = \varphi(\theta_1,\theta_2) \pmod{\mathbb{Z}} = \varphi(\theta_1,\theta_2+1) \text{ and :}$$

$$e^{2\pi i f_2(\theta_1,\theta_2)} = e^{2\pi i(f_1(\theta_1,\theta_2) - \varphi(\theta_1+\frac{1}{2},\theta_2) - \varphi(\theta_1,\theta_2))}$$

i.e. $f_2(\theta_1,\theta_2) = f_1(\theta_1,\theta_2) - \varphi(\theta_1+\tfrac{1}{2},\theta_2) - \varphi(\theta_1,\theta_2) \pmod{\mathbb{Z}}$.

Given f_1, we can choose a suitable $\varphi(\theta_1,\theta_2) = (f_1(\theta_1,\theta_2))/2 - \alpha\theta_2/2 + \beta/4$ ($\alpha = 0$ or 1, $\beta = 0$ or 1) so that $f_2(\theta_1,\theta_2) = 0$ or θ_2 as required.

11.9 1st-isomorphism classes of bundles with base KB

(i) Bundles with group K : use $\chi : \mathcal{C}_4 \longrightarrow \mathcal{D}_4$ (11.1).

$\chi(\mathcal{B}) = 0$ This gives the product bundle, denoted by $\mathcal{KB}_0$.

$\chi(\mathcal{B}) = 1$ This gives $\mathcal{KB}_1 = (K^3 / \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$, KB, K, π), where the action of K

is defined by $k.[k_1,k_2,k_3] = [k_1,k_2,kk_3]$ $(k_1, k_2, k_3, k \in K)$.

$\pi : K^3 / \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix} \longrightarrow KB$ is defined by $\pi([k_1,k_2,k_3]) = [k_1,k_2]$.

(ii) Bundles with group Z_2 : use $\chi : \mathcal{C}_1(KB,Z_2) \longrightarrow \mathcal{D}_1(KB,Z_2)$ (11.1).

$\mathcal{D}_1(KB,Z_2) = \mathrm{Hom}(\pi_1(KB),Z_2) = \{\eta_1,\eta_2,\eta_3,\eta_4\}$

$\pi_1(KB) = \langle a, b : ab = b^{-1}a \rangle$.

a and b are the homotopy classes corresponding to the paths $t \longmapsto [e^{\pi it},1]$ and $t \longmapsto [1,e^{2\pi it}]$ $(t \in [0,1])$ respectively.

We define a bundle $\mathcal{B}_1^i = (X_1^i,KB,Z_2,\nu_1^i)$ with $\chi(\mathcal{B}_1^i) = \eta_i$ (i = 1...4).

$\eta_1(a) = \eta_1(b) = 1$: $\mathcal{B}_1^1$ is the product bundle.

$\eta_2(a) = \varepsilon$, $\eta_2(b) = 1$: $X_1^2 = K^2$, $\varepsilon.(k_1,k_2) = (-k_1,k_2^{-1})$, $\nu_1^2(k_1,k_2) = [k_1,k_2]$

$\eta_3(a) = 1$, $\eta_3(b) = \varepsilon$: $X_1^3 = KB$, $\varepsilon.[k_1,k_2] = [k_1,-k_2]$, $\nu_1^3([k_1,k_2]) = [k_1,k_2^2]$

$\eta_4(a) = \eta_4(b) = \varepsilon$: $X_1^4 = KB$, $\varepsilon.[k_1,k_2] = [k_1,-k_2]$, $\nu_1^4([k_1,k_2]) = [k_1,-k_2^2]$

(iii) Bundles with group $K \times_s Z_2$: use χ_5 defined on $\mathcal{C}_5(KB)$ (11.2).

i.e. we find 1st-isomorphism classes of bundles

$\mathcal{B} = (Y,Y/Z_2,KB,K \times_s Z_2,Z_2,\pi,\xi,\nu_1 \circ \nu_2)$ in terms of the bundles

$\mathcal{B}_1 = (X_1,K^2,Z_2,\nu_1)$ and $\mathcal{B}_2 = (Y,X_1,K,\nu_2)$.

$\chi(\mathcal{B}_1) = \eta_1$: It can be shown that up to isomorphism, the only possibilities for $\mathcal{B}$ are $\mathcal{KB}_0 \times_s Z_2$ and $\mathcal{KB}_1 \times_s Z_2$ (11.9(i), 11.3).

$\chi(\mathcal{B}_1) = \eta_2$: It can be shown that $\mathcal{B}_2$ must be $\mathcal{T}_0$ (11.7), and then the only possibility for $\mathcal{B}$ is $\mathcal{KB}_0' = (K^3, K^3 / \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, KB, $K \times_s Z_2$, Z_2, π, ξ, ν),

A

where the action of $K \times_s Z_2$ on K^3 is given by:

$k.(k_1,k_2,k_3) = (k_1,k_2,kk_3)$, $\quad \varepsilon.(k_1,k_2,k_3) = (-k_1,k_2^{-1},k_3^{-1})$.

ν is given by $\nu(k_1,k_2,k_3) = [k_1,k_2]$.

$\chi(\mathcal{B}_1) = \eta_3$: It can be shown (11.10) that $\mathcal{B}_2$ must be $\mathcal{K}\mathcal{B}_0$ - not $\mathcal{K}\mathcal{B}_1$ - up to 1st-isomorphism. Up to 1st-isomorphism, there are two possibilities for $\mathcal{B}$, determined by two possible actions of ε on $Y = KB \times K$:

$\varepsilon.([k_1,k_2],k_3) = ([k_1,-k_2],k_3^{-1})$: gives bundle $\mathcal{K}\mathcal{B}_0([k_1,-k_2],k_3^{-1})$

or $\varepsilon.([k_1,k_2],k_3) = ([k_1,-k_2],k_1^2k_3^{-1})$: gives bundle $\mathcal{K}\mathcal{B}_0([k_1,-k_2],k_1^2k_3^{-1})$.

Y/Z_2 is homeomorphic to W_0, W_2 respectively (see 8.2, 8.3).

$\chi(\mathcal{B}_1) = \eta_4$: There is a natural correspondence of the possible bundles with those for η_3, therefore they will not be listed.

11.10 As an example of the method used in the calculation of 11.9(iii), we sketch the proof that if $\chi(\mathcal{B}_1) = \eta_3$ then $\mathcal{B}_2$ must be $\mathcal{K}\mathcal{B}_0$ up to 1st-isomorphism.

If $\mathcal{B}_2$ is $\mathcal{K}\mathcal{B}_1$, then there exists a homeomorphism $\varepsilon : K^3/\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix} \longrightarrow K^3/\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$

such that ε^2 = identity and the following diagram commutes:

Diagram 11.10

$$\begin{array}{ccc} K^3/\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix} \ni [k_1,k_2,k_3] & \overset{\varepsilon}{\longmapsto} & \varepsilon([k_1,k_2,k_3]) = [k_1',k_2',k_3'] \\ \downarrow & & \downarrow \\ KB \ni [k_1,k_2] & \longmapsto & [k_1,-k_2] = [k_1',k_2'] \end{array}$$

i.e. ε is of the form:

$$\varepsilon.[e^{2\pi ix},e^{2\pi iy},e^{2\pi iz}] = [e^{2\pi ix},e^{2\pi i(y+\frac{1}{2})},e^{2\pi i(\psi(x,y)-z)}]$$

where $\psi \in C(\mathbb{R}^2,\mathbb{R})$ and :

$\psi(x,y+1) = \psi(x,y) \mod Z$

$-2y + \psi(x+\frac{1}{2},-y) = \psi(x,y) + \frac{1}{2} \mod Z$

$\psi(x,y+\frac{1}{2}) = \psi(x,y) \mod Z$ (since ε^2 = identity).

By considering a suitable function $\psi_1(x,y) = \psi(x,y) + ax$, we can assume:

$\psi_1(x + \frac{1}{2}, -y) - 2y = \psi_1(x,y)$; $\psi_1(x,y + \frac{1}{2}) = \psi_1(x,y) + d$, some $d \in \mathbb{Z}$.

Let $\psi_1(0,0) = c$. Evaluating $\psi_1(\frac{1}{2},\frac{1}{2})$ in two different ways, we see ψ_1 cannot exist.

§12 $\mathbb{Z}$-admissability

<u>12.1</u> <u>We use the notation of 8.2 throughout this section.</u> Denote a t.g. $(X,\mathbb{Z})$ by (X,t) where t is the homeomorphism of X corresponding to $1 \in \mathbb{Z}$. In this section we prove (without full details) that the n-allowable strings ($n \leq 3$) are $\mathbb{Z}$-admissable or not as recorded in tables A and B of §9. 12.2 - 12.6 are devoted to showing that the strings for which non-$\mathbb{Z}$-admissability is claimed in tables A and B are indeed not $\mathbb{Z}$-admissable. 12.7 - 12.16 are devoted to reducing the problems of $\mathbb{Z}$-admissability of the remaining strings to problems concerning the existence of minimal group extensions of certain t.g.'s, and 12.17 - 12.18 are devoted to solving these problems.

<u>12.2</u> <u>Definition</u> For a homeomorphism φ of K^2, let $r(\varphi)$ be the unique (r_{ij}) in $GL(2,\mathbb{Z})$ such that φ is homotopic to:

$$(k_1,k_2) \longmapsto (k_1^{r_{11}}k_2^{r_{12}}, k_1^{r_{21}}k_2^{r_{22}})$$

Then $\det r(\varphi) = \pm 1$

<u>12.3</u> If (K^2,t) is minimal almost periodic, it is immediate that $\det r(t) = 1$.

<u>12.4</u> If (X,t) is a minimal distal t.g. with $\mathcal{B}(X,t)$ as in A1, A2, A3 of table A it is clear that $t : X \longrightarrow X$ must be of the form:

A1 $(k_1,k_2)t = (\alpha k_1, g(k_1)k_2)$

A2 $(k_1,k_2)t = (\alpha k_1, g(k_1)k_2^{-1})$

A3 $[k_1,k_2]t = [\alpha k_1, g(k_1)k_2]$ for all $k_1, k_2 \in K$.

In each case, $\alpha = e^{2\pi i\beta}$, where β is irraional, and $g \in C(K,K)$. For A3, $g(-k_1) = \overline{g(k_1)}$ for all $k_1 \in K$.

Therefore, if $\mathcal{B}(K^2,t)$ is as in A1, $\det r(t) = 1$, and if $\mathcal{B}(K^2,t)$ is as in A2, $\det r(t) = -1$.

<u>12.5</u> The following lemma shows that the strings of A12, B4, B7 are not

$\mathbb{Z}$-admissable, and will help prove the $\mathbb{Z}$-admissability of the strings A10, B3, B5.

<u>Lemma</u> For $n > 0$, let $\mathcal{J}_n = (N/\Gamma_n, K^2, K, \pi_n)$ be as in 11.7. Let $\varphi : K^2 \longrightarrow K^2$ be a homeomorphism.

(i) det $r(\varphi) = 1$ if and only if Φ exists as in the commutative diagram 12.5 with $\Phi(k.x) = k.(\Phi(x)$ for all $x \in N/\Gamma_n$, $k \in K$ (action of K on N/Γ_n as for $\mathcal{J}_n$).

(ii) det $r(\varphi) = -1$ if and only if Φ exists as in diagram 12.5 with $\Phi(k.x) = k^{-1}.\Phi(x)$ for all $x \in N/\Gamma_n$, $k \in K$.

<u>Diagram 12.5</u>

$$\begin{array}{ccc} N/\Gamma_n & \xrightarrow{\ \Phi\ } & N/\Gamma_n \\ \pi_n \downarrow & & \downarrow \pi_n \\ K^2 & \xrightarrow[\ \varphi\ ]{} & K^2 \end{array}$$

<u>Proof</u> It suffices to show the bundle $(N/\Gamma_n, K^2, K, \pi_n \circ \varphi)$ (where the action of K on N/Γ_n is as for $\mathcal{J}_n$) is 1st-isomorphic to $\mathcal{J}_n$ if det $r(\varphi) = 1$ and to $\mathcal{J}_{-n}$ if det $r(\varphi) = -1$. By the First Homotopy Covering Theorem ([17] 11.3) it suffices to prove this for φ of the form:

$$\varphi(k_1,k_2) = (k_1^{r_{11}}k_2^{r_{12}}, k_1^{r_{21}}k_2^{r_{22}}), \quad (r_{ij}) \in GL(2,\mathbb{Z}).$$

But this is a straightforward computation.

<u>12.6</u> <u>Lemma</u> The string of A4 corresponding to $\mathcal{K}(K^2,A,\sigma)$ is $\mathbb{Z}$-admissable only if $\langle\sigma\rangle \triangleleft A$ and $A/\langle\sigma\rangle$ is cyclic (see table A).

<u>Proof</u> Suppose $(K^3/\sigma, t)$ is a minimal distal t.g. with $\mathcal{B}(K^3/\sigma, t)$ the string of A4 corresponding to $\mathcal{K}(K^2,A,\sigma)$. Then $(K^3/\sigma, t) \prec (K^3 \times A/\langle\sigma\rangle, s)$ where s is a minimal distal homeomorphism commuting with the action of $K^2 \times_s A$ (11.4).

Fix $\tau \in A$. Write $\underline{1} = (1,1,1,\langle\sigma\rangle)$. $\underline{1}s^m \in K^3 \times \tau\langle\sigma\rangle$ for some m, by the minimality of s.

$$\begin{aligned} \underline{1}s^m &= (1,1,1,\eta\langle\sigma\rangle)s^m \quad \text{for all } \eta \in \langle\sigma\rangle \\ &= ((1,\eta)(F_\sigma(\eta^{-1}),1,1,\langle\sigma\rangle)s^m \in K^3 \times \eta\tau\langle\sigma\rangle \quad \text{(see 11.4).} \end{aligned}$$

Therefore $\tau^{-1}\eta\tau \in \langle\sigma\rangle$ for all $\eta \in \langle\sigma\rangle$, $\tau \in A$.

Therefore $\langle\sigma\rangle \triangleleft A$. Clearly $A/\langle\sigma\rangle$ must be cyclic.

<u>12.7</u> <u>Definitions</u> Let (Y,X,G,ν) be a principal bundle (3.1) and let t be a homeomorphism of X. Let <u>$Hom(Y:X,G,\nu,t)$</u> denote the set of homeomorphisms of Y such that:

(i) $(g.y)s = g.(ys)$ for all $g \in G$, $y \in Y$.

(ii) $(X,t) \prec_{\nu} (Y,s)$

Write <u>$Hom(Y:G,t)$</u> for $Hom(Y:X,G,\nu,t)$ if the definition of X, ν are clear from the context.

Let $Hom(Y:G,t) \neq \phi$. Let Y be metric, and note that all metrics on Y,G respectively (giving rise to the right topologies) are equivalent. Let $Hom(Y:G,t)$ and $C(Y,G)$ be given supremum metrics (any two such metrics on $Hom(Y:G,t)$, $C(Y,G)$ respectively are equivalent). Then $Hom(Y:G,t)$ is a complete metric space and is isomorphic (as a metric space) to:

$\{ f \in C(Y,G) : f(g.y)g = gf(y) \text{ for all } y \in Y,\ g \in G \}$ which is in turn isomorphic (as a metric space) to $C(X,G)$ if G is abelian, or if (Y,X,G,ν) is a product bundle. In the latter case, for a homeomorphism t of X, the element of $Hom(X \times G:G,t)$ corresponding to $f \in C(X,G)$ is denoted by s_f if this notation cannot give rise to confusion, where:

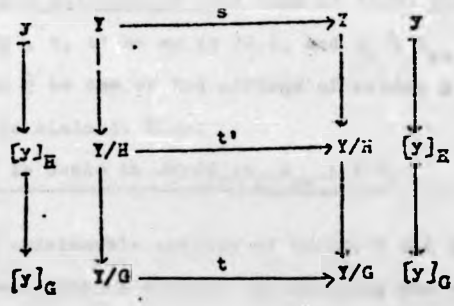
$(x,g)s_f = (xt, gf(x))$ for all $x \in X$, $g \in G$.

If $H \triangleleft G$ and $t' \in Hom(Y/H:G/H,t)$ then define:

<u>$Hom(Y:G,H,t,t')$</u> $= Hom(Y:G,t) \cap Hom(Y:H,t')$

so that if $s \in Hom(Y:G,H,t,t')$, in particular the following diagram commutes:

<u>Diagram 12.7</u>



If t is minimal, let $\underline{\mathcal{M}(Y:X,G,\nu,t)}$ (or $\underline{\mathcal{M}(Y:G,t)}$) be:
$\{ s \in \mathrm{Hom}(Y:G,t) : s \text{ is minimal} \}$.
If t, t' are minimal, let $\underline{\mathcal{M}(Y:G,H,t,t')} = \mathcal{M}(Y:G,t) \cap \mathrm{Hom}(Y:G,H,t,t')$

<u>12.8</u> <u>Definitions</u> Let $\underline{\mathcal{S}} = (\mathcal{S}_1 \ldots \mathcal{S}_r)$ be a string (9.1) with $\mathcal{S}_i = (Y_i, X_i, X_{i-1}, G_i, H_i, \pi_i, \rho_i, \nu_i)$ $(1 \leq i \leq r)$. Let t be a minimal distal homeomorphism of X_{r-1} with $\underline{\mathcal{S}}(X_{r-1},t) = (\mathcal{S}_1 \ldots \mathcal{S}_{r-1})$.

Define $\underline{\mathcal{D}(Y_r : \underline{\mathcal{S}}, t)}$ as follows: $s \in \mathcal{M}(Y_r : G_r, t)$ is in $\mathcal{D}(Y_r : \underline{\mathcal{S}}, t)$ if and only if $\underline{\mathcal{S}}(X_r, u) = \underline{\mathcal{S}}$ where u is the unique homeomorphism making the following diagram commutative:

<u>Diagram 12.8</u>

$$\begin{array}{ccc}
 & Y_r & \xrightarrow{\quad s \quad} & Y_r & \\
 & \downarrow{\scriptstyle \rho_r} & & \downarrow{\scriptstyle \rho_r} & \\
(Y_r/H_r \cong) & X_r & \xrightarrow{\quad u \quad} & X_r & (\cong Y_r/H_r) \\
 & \downarrow{\scriptstyle \pi_r} & & \downarrow{\scriptstyle \pi_r} & \\
(Y_r/G_r \cong) & X_{r-1} & \xrightarrow{\quad t \quad} & X_{r-1} & (\cong Y_r/G_r)
\end{array}$$

Let $L_r \triangleleft G_r$ and $t' \in \mathcal{M}(Y_r/L_r : G_r/L_r, t)$.
Define $\underline{\mathcal{D}(Y_r : \underline{\mathcal{S}}, t, t')} = \mathcal{D}(Y_r : \underline{\mathcal{S}}, t) \cap \mathrm{Hom}(Y_r : G_r, L_r, t, t')$.

<u>12.9</u> Using induction, $\mathbb{Z}$-admissability of strings of tables A and B is implied by the following proposition, which we shall spend the rest of the section in proving <u>using the notation of 12.8</u> (and of 12.7) <u>throughout</u>.

<u>Proposition</u> Let $\underline{\mathcal{S}}$, t, t' be as in 12.8, and $L_r = G_{r0}$, the identity component of G_r, and suppose $\underline{\mathcal{S}}$ is one of the strings of tables A and B for which $\mathbb{Z}$-admissability is claimed. Then:

$\underline{\mathcal{D}(Y_r : \underline{\mathcal{S}}, t, t') \text{ is dense in } \mathrm{Hom}(Y_r : G_r, G_{r0}, t, t')}$.

For all the $\mathbb{Z}$-admissable strings of tables A and B <u>except</u> A10, B3, B8, proof of $\mathbb{Z}$-admissability is achieved by reducing the problem to a similar problem concerning minimal extensions and strings in which the final bundle $\mathcal{B}_r$ is a product bundle with connected group and (possibly non-connected) base (see 12.17 for statement of the reduced problem). First (12.10 - 12.11) we deal with the strings of A10, B3, B8.

<u>12.10</u> <u>Lemma</u> If $\mathcal{B}$ is one of the strings A2, A3, A5 - A8, A10, A11, A13, A14, B3, B8, B22, B23, then $\mathcal{M}(Y_r:G_r,t) = \mathcal{D}(Y_r:\mathcal{B},t)$, so that $\mathcal{M}(Y_r:G_r,G_{ro},t,t') = \mathcal{D}(Y_r:\mathcal{B},t,t')$.

<u>Proof</u> Let $s \in \mathcal{M}(Y_r:G_r,t)$ and suppose $s \notin \mathcal{D}(Y_r:\mathcal{B},t)$.

We shall assume $\mathcal{B}$ is one of the strings of table A (proof is similar for B3, B8, B22, B23). So $r = 2$. If $s \notin \mathcal{D}(Y_2:\mathcal{B},t)$ then the phase space of the maximal almost periodic factor of (X_2,u), where $(X_2,u) \prec_{G_2}(Y_2,s)$, must be Y_2/L_2 where $L_2 \neq G_2$ and $H_2 \leq L_2 < G_2$, with $H_2 = L_2$ if L_2/H_2 is finite (5.5, 5.6, 5.7). In the particular cases considered, this implies $H_2 = L_2$, hence (X_2,u) is almost periodic, H_2 is trivial, G_2 is abelian and X_2 is a torus - which is not true for the strings of A2, A3, A5 - A8, A10, A11, A13, A14.

<u>12.11</u> If $\mathcal{B}$ is the string of A10, [2]6.19.2.6. implies $Hom(Y_r:G_r,G_{ro},t,t') = \mathcal{M}(Y_r:G_r,G_{ro},t,t')$ and hence by a simple argument the same is true of the strings of B3, B8. By 12.10 this implies proposition 12.9 is proved for the strings of A10, B3, B8.

<u>12.12</u> Now we need some definitions:

<u>Definitions</u> (i) If $f \in C(K,K)$, f can be uniquely written in the form

$$f(k_1) = k_1^p c e^{ih(k_1)} \quad \text{where } p \in \mathbb{Z},\ c \in K \text{ and } \int h(k_1)\,dk_1 = 0,\quad h \in C(K,\mathbb{R}).$$

<u>Define P</u> : $C(K,K) \longrightarrow C(K,K)$ by $Pf(k_1) = e^{ih(k_1)}$.

(ii) If $f \in C(K^2,K)$, f can be uniquely written in the form:

$$f(k_1,k_2) = k_1^p k_2^q e^{il(k_1) + ih(k_1,k_2)}, \text{ where } \int h(k_1,k_2)\,dk_2 = 0.$$

<u>Define P</u> : $C(K^2,K) \longrightarrow C(K^2,K)$ by $Pf(k_1,k_2) = e^{ih(k_1,k_2)}$.

(iii) For $f \in C(K,K^r)$ or $C(K^2,K^r)$, <u>define</u> $Pf = (Pf_1,\dots Pf_r)$ if $f = (f_1\dots f_r)$.

(iv) If B is a finite set then $C(K^s \times B, K^r)$ is isomorphic (as a group, with pointwise multiplication) to $C(K^s, (K^r)^B)$ under the map:

$$f \longmapsto (f_b)_{b \in B} \quad \text{where } f_b(\underline{k}) = f(\underline{k}, b) \quad (\underline{k} \in K^s,\ b \in B).$$

Using this isomorphism, define $P : C(K^s \times B, K^r) \longrightarrow C(K^s \times B, K^r)$ for $s = 1, 2$.

(v) In each case, P is a continuous group homomorphism with respect to the uniform topology, and $P^2 = P$.

12.13 Definitions (i) For X a compact Hausdorff space and G a compact group, a group C is said to be an automorphism group of (X,G) if C acts freely on X and acts as a group of automorphisms on G, both actions being on the left.

(ii) If C is an automorphism group of (X,G), let $\mathfrak{A}_C$ denote the closed subgroup of the group $C(X,G)$ (pointwise multiplication) defined by:

$\mathfrak{A}_C = \{ f \in C(X,G) : f(c.x) = c.f(x) \text{ for all } x \in X,\ c \in C \}$.

(iii) If C is an automorphism group of (X,G), define $R_c : C(X,G) \longrightarrow C(X,G)$ by $(R_c f)(x) = f(c.x)$ $(c \in C)$.

If $X = K^s \times B$ ($s = 1, 2$ and B finite) and $G = K^r$, C is said to be a P-invariant automorphism group if $R_c P = P R_c$ for all $c \in C$. If this condition is satisfied, $P(\mathfrak{A}_C) \subseteq \mathfrak{A}_C$.

12.14 Suppose $\mathfrak{S}$ is one of the strings A1 - A9, A11, A13, A14, B1, B2, B5, B6.

(i) The principal bundle defined by the action of G_{ro} on Y_r is a product bundle. Recall (§10) that we can assume $G_r = G_{ro} \times_s H_r$, where H_r is a finite subgroup of $\mathrm{Aut}(G_{ro})$ in these particular cases. Thus H_r is canonically isomorphic to G_r/G_{ro}, which acts on Y_r/G_{ro} (and commutes with t'). Therefore H_r is a finite automorphism group of $(Y_r/G_{ro}, G_{ro})$.

(ii) There exists $f_1 \in C(Y_r/G_{ro})$ such that $\mathrm{Hom}(Y_r:G_r,G_{ro},t,t') = \{ s_f : f \in \mathfrak{A}_{H_r} \cdot$ where we can take $f_1 \equiv 1$ except in the case of the string of A13, when $C(Y_r/G_{ro}, G_{ro}) = C(K^2, K)$ and we can assume $Pf_1 \equiv 1$. $\mathfrak{A}_{H_r} \cdot f_1 = \{ f_2 f_1 : f_2 \in \mathfrak{A}_{H_r} \}$.

(iii) If $\mathfrak{S}$ is one of the strings of A1 - A4, A9, A11, A13, A14, B1, B2, B5, B6, then H_r is P-invariant.

12.15 If $\mathcal{B}'$ is one of the strings of B9 - B23 then the proof of 12.9 for $\mathcal{B}'$ reduces to proving:

$$\mathcal{D}(Y_r:\mathcal{B},t,t') \cap \{s_f : f \in \mathcal{O}_C\} \quad \text{is dense in } Hom(Y_r:G_r,G_{ro},t,t') \cap \{s_f : f \in \mathcal{O}_C\}$$

where:

(i) $\mathcal{B}$ is the string of B1, B2, B5 or B6.

(ii) C is a finite P-invariant automorphism group of $(Y_r/G_{ro},G_{ro})$ such that the actions of C on Y_r/G_{ro} and G_{ro} commute with those of H_r, and CH_r acts freely on Y_r/G_{ro}. Hence CH_r is a finite P-invariant automorphism group of $(Y_r/G_{ro},G_{ro})$.

12.16 Lemma If $\mathcal{B}$ is one of the strings A1, A4, A9, B1, B2, B5, B6 and $s_f \in Hom(Y_r:G_r,G_{ro},t,t')$ then a sufficient condition for $s_f \in \mathcal{D}(Y_r:\mathcal{B},t,t')$ is that s_{Pf} be minimal.

Proof We indicate the proof only when $\mathcal{B}$ is one of B1, B2, B5, B6. For the strings B1, B2, B5, B6, we can assume $\mathcal{B}$ is B1. This follows from the fact that if $\mathcal{B}(K^2,u)$ is as in B2, B5, or B6, then $\mathcal{B}(K^2,u^2)$ is as in B1.

Hence suppose $\mathcal{B}$ is the string of B1, so that $t = t' : K^2 \longrightarrow K^2$ is of the form: $(k_1,k_2)t = (\alpha k_1, g(k_1)k_2)$, $g \in C(K,K)$,

and $(k_1,k_2,k_3)s_f = (\alpha k_1, g(k_1)k_2, f(k_1,k_2)k_3)$.

Suppose s_{Pf} is minimal. By [14] Theorem 1.1, this is true if and only if there is no continuous solution φ to the equation:

(12.16.1) $\varphi((k_1,k_2)t).(Pf(k_1,k_2))^m = \varphi(k_1,k_2)$ for any $m \in \mathbb{Z} \sim \{0\}$.

If this equation cannot hold for Pf, it also cannot hold for f, so that s_f is minimal.

Suppose $s_f \notin \mathcal{D}(Y_r:\mathcal{B},t)$. Then (5.5, 5.6, 5.7) $(Y_r,s_f) \approx (Y_3,s_f) = (K^3,s_f)$ must be an almost periodic extension of (K,α) where α denotes the homeomorphism $k_1 \longmapsto \alpha k_1$ $(k_1 \in K)$.

Hence (K^3,s_f) is a G/H-extension of (K,α) where there exists a group L with $H \triangleleft L \triangleleft G$ and $L/H \cong G/H \cong K$ (5.5, 5.6). This forces $(G,H) \approx (K^2,\{1\})$.

This means there exists a minimal homeomorphism s' of K^3 of the form: $(k_1,k_2,k_3)s' = (\alpha k_1, l(k_1)k_2, m(k_1)k_3)$ with m, $l \in C(K,K)$, and a homeomorphism

Φ of K^3 of the form:

$\Phi(k_1,k_2,k_3) = (k_1, \varphi_2(k_1)k_2, \varphi_3(k_1,k_2)k_3)$, where $\varphi_2 \in C(K,K), \varphi_3 \in C(K^2,K)$,

such that $\Phi : (K^3, s_f) \longrightarrow (K^3, s')$ is an isomorphism.

This implies $\varphi_3((k_1,k_2)t).f(k_1,k_2) = \varphi_3(k_1,k_2).m(k_1)$

and hence $(P\varphi_3)((k_1,k_2)t).Pf(k_1,k_2) = P\varphi_3(k_1,k_2)$

which contradicts (12.16.1) having no continuous solution.

So $s_f \in \mathcal{O}(Y_r : \underline{s}, t)$. Q.E.D.

12.17 The previous paragraphs, in particular 12.14 - 12.16, indicate that the proof of 12.9, except for the strings A10, B3, B3 (see 12.11) is a consequence of the following proposition, which is analogous to a result of [4].

Proposition Let t be a minimal distal homeomorphism of X. Let G be a compact connected Lie group and let C be a finite automorphism group of (X,G) (12.13) such that $c(xt) = c(xt)$ for all $c \in C$, $x \in X$. Let s_f denote the homeomorphism of $Y = X \times G$ defined by:

$$(x,g)s_f = (xt, gf(x)).$$

Then $\mathcal{O} \cap \{f : s_f \text{ is minimal}\}$ is dense in $\mathcal{O}$ where $\mathcal{O}$ is a closed subset of $C(X,G)$ of one of the following forms (see 12.12, 12.13):

(i) $\mathcal{O} = \mathcal{O}_C$

(ii) $X = K^s \times B$ for $s = 1$ or 2 and B a finite set, $G = K^r$ and

$\mathcal{O} = \mathcal{O}_C \cap \{f : f = Pf\}$ where C is P-invariant.

Proof The following are true:

(a) Given an open cover $\{V_1 \ldots V_n\}$ of G, there exists an integer p such that if $W_i \in \{V_1 \ldots V_n\}$ $(i = 1 \ldots p)$ then $G = W_1 W_2 \ldots W_p$.

(b) Given $f \in \mathcal{O}$ and $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $x_0 \in X$ is fixed and $u : F \rightarrow G$ satisfies $d(u(y), f(y)) < \delta$ for all $y \in F$ where F is a finite set with $c.F \cap F = \phi$ for all $c \in C$, then there exists $v \in \mathcal{O}$ with $v|F = u$, and $\sup_{x \in X} d(v(x), f(x)) < \varepsilon$, where d is a metric on G.

(a) is proved in [4] Proposition 2. For the proof of (b) when $\mathcal{O}$ is as in (i), see lemma 12.18. The proof of (b) when $\mathcal{O}$ is as in (ii) is omitted.

Now fix $y_0 = (x_0,1) \in Y$. For U open in Y, let

$E(U) = \{f \in \mathcal{Q} : \{y_0 s_f^n : n \geq 0\} \cap U \neq \phi\}$. Using (i) and (ii), use an argument similar to that of [3] lemma 2 to show that E(U) is dense in C(X,G). Then note that $\{f \in \mathcal{Q} : s_f \text{ is minimal}\} = \bigcap_{U \text{ open in } Y} E(U)$, hence is dense in $\mathcal{Q}$, since Y has a countable basis of open sets.

<u>12.18</u> <u>Lemma</u> (b) of 12.17 is true for $\mathcal{Q}$ as in (i) of 12.17.

<u>Proof</u> Assume without loss of generality that the metric d is C-invariant. Choose $\delta > 0$ and $S(g)$ $(g \in G)$ such that:

$\{g' : d(g,g') < \delta\} \subseteq S(g) \subseteq \{g' : d(g,g') < \varepsilon/2\}$, where S(g) is homeomorphic to $\mathbb{R}^n$ for some n.

For $y \in F$, choose U_y, an open neighbourhood of y, such that: $f(U_y) \subseteq \{g' : d(g',f(y)) < \delta\}$, and $U_y \cap c.U_y = \phi$ for $c \in C$.

If $y = c.y_1$ $(y_1 \in F)$, define $U_y = c.U_{y_1}$.

For $y \in \bigcup_{c \in C} c.F$, define $v_y(y) = c.u(y_1)$ if $y = c.y_1$ for $c \in C$, $y_1 \in F$

$v_y(x) = f(x)$, for x in the boundary of U_y,

and extend v_y to a function $v_y : U_y \longrightarrow S(f(y))$ such that $v_{cy}(c.x) = c.v_y(x)$.

Then define $v = v_y$ on U_y $(y \in C.F)$

$= f$ otherwise.

§13 <u>$\mathbb{R}$-admissability</u>

<u>13.1</u> The different types of minimal distal $\mathbb{R}$-actions on compact connected topological manifolds of dimension ≤ 3 were obtained by Bronstein [1], though not quite in the form given here.

<u>13.2</u> Clearly $\mathbb{R}$ (with the usual topology) can only act minimally on a <u>connected</u> space. Then the following lemma, quoted by Bronstein for roughly the same purpose , and easily verified, shows that all the strings in tables A and B except for A9, A10, A13, A14, are not $\mathbb{R}$-admissable.

<u>Lemma</u> [5] Let $(X,\mathbb{R})$ be a minimal <u>periodic</u> t.g. and $(Y,\mathbb{R})$ a minimal almost periodic extension of $(X,\mathbb{R})$. Then $(Y,\mathbb{R})$ is almost periodic.

(Note that an almost periodic action of $\mathbb{R}$ on K must be periodic, hence the lemma implies a distal action of $\mathbb{R}$ on a 2-dimensional manifold must be almost periodic.)

<u>13.13</u> <u>The string of A10 is $\mathbb{R}$-admissable.</u>

$\mathcal{B}(X,\mathbb{R})$ is the string of A10 if and only if $X = N/\Gamma_n$ (8.2), and the action of $\mathbb{R}$ is given by:

$$[x,y,z]t = [x+at,\, y+bt+act^2/2 + xct + g_t(x,z),\, z+ct] \quad \text{for all } x, y, z, t \in \mathbb{R},$$

where a and $c \in \mathbb{R}$ are rationally independent and the function

$(t,x,z) \longmapsto g_t(x,z)$ is jointly continuous, with:

$$g_{t+s}(x,z) = g_t(x,z) + g_s(x+at, z+ct)$$

$$g_t(x+1,z) = g_t(x,z) \pmod{\mathbb{Z}} = g_t(x,z+1) \quad \text{for all } x, z, s, t \in \mathbb{R}.$$

For proof of minimality see, for example, [2] 6.19.2.6. Proof that $\mathcal{B}(X,\mathbb{R})$ is the string of A10 is analogous to 12.10.

<u>13.4</u> We outline the proof that the string of A9 is $\mathbb{R}$-admissable. (The proofs for A13, A14 are similar.)

If $\mathcal{B}(K^3,\mathbb{R})$ is the string of A9 then the action of $\mathbb{R}$ is of the form:

$(k_1,k_2,k_3)t = (k_1e^{2\pi i at}, k_2e^{2\pi i bt}, k_3e^{2\pi i g_t(k_1,k_2)})$ for all $k_1,k_2,k_3 \in K$ and $t \in \mathbb{R}$, where a, $b \in \mathbb{R}$ are rationally independent, and if $(k_1,k_2)t = (k_1e^{2\pi i at}, k_2e^{2\pi i bt})$ then $g_{t+s}(k_1,k_2) = g_t(k_1,k_2)\cdot g_s((k_1,k_2)t)$ for all k_1, $k_2 \in K$ and t, $s \in \mathbb{R}$.

A necessary and sufficient condition that $(K^3,\mathbb{R})$ be minimal and that $\mathcal{B}(K^3,\mathbb{R})$ be the string of A9 is that there exist no continuous solution $f \in C(K^2,K)$ to the equation:

<u>(13.4.1)</u> $f(k_1e^{2\pi i at}, k_2e^{2\pi i bt}) = f(k_1,k_2)\cdot e^{2\pi i(m g_t(k_1,k_2)+\lambda t)}$ for any $m \in \mathbb{Z}\setminus\{0\}$ and $\lambda \in \mathbb{R}$.

Writing $f(k_1,k_2) = k_1^{\,p}k_2^{\,q}e^{2\pi i g(k_1,k_2)}$ ($g \in C(K^2,\mathbb{R})$), the condition becomes that there is no continuous solution $f_1 \in C(K^2,\mathbb{R})$ to the equation:

<u>(13.4.2)</u> $f_1(k_1e^{2\pi i at}, k_2e^{2\pi i bt}) = f_1(k_1,k_2) + g_t(k_1,k_2) + \mu t$ for any $\mu \in \mathbb{R}$.

Φ of K^3 of the form:

$\Phi(k_1,k_2,k_3) = (k_1, \varphi_2(k_1)k_2, \varphi_3(k_1,k_2)k_3)$, where $\varphi_2 \in C(K,K)$, $\varphi_3 \in C(K^2,K)$,

such that $\Phi : (K^3,s_f) \longrightarrow (K^3,s')$ is an isomorphism.

This implies $\varphi_3((k_1,k_2)t).f(k_1,k_2) = \varphi_3(k_1,k_2).m(k_1)$

and hence $(P\varphi_3)((k_1,k_2)t).Pf(k_1,k_2) = P\varphi_3(k_1,k_2)$

which contradicts (12.16.1) having no continuous solution.

So $s_f \in \mathcal{O}(Y_r:\Phi,t)$. Q.E.D.

<u>12.17</u> The previous paragraphs, in particular 12.14 - 12.16, indicate that the proof of 12.9, except for the strings A10, B3, B8 (see 12.11) is a consequence of the following proposition, which is analogous to a result of [4].

<u>Proposition</u> Let t be a minimal distal homeomorphism of X. Let G be a compact connected Lie group and let C be a finite automorphism group of (X,G) (12.13) such that $c(xt) = c(xt)$ for all $c \in C$, $x \in X$. Let s_f denote the homeomorphism of $Y = X \times G$ defined by:

$$(x,g)s_f = (xt, gf(x)).$$

Then <u>$\mathcal{O} \cap \{f : s_f$ is minimal$\}$ is dense in $\mathcal{O}$</u> where $\mathcal{O}$ is a closed subset of $C(X,G)$ of one of the following forms (see 12.12, 12.13):

(i) $\mathcal{O} = \mathcal{O}_C$

(ii) $X = K^s \times B$ for $s = 1$ or 2 and B a finite set, $G = K^r$ and

$\mathcal{O} = \mathcal{O}_C \cap \{f : f = Pf\}$ where C is P-invariant.

<u>Proof</u> The following are true:

(a) Given an open cover $\{V_1 \ldots V_n\}$ of G, there exists an integer p such that if $W_i \in \{V_1 \ldots V_n\}$ $(i = 1 \ldots p)$ then $G = W_1 W_2 \ldots W_p$.

(b) Given $f \in \mathcal{O}$ and $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $x_0 \in X$ is fixed and $u : F \to G$ satisfies $d(u(y), f(y)) < \delta$ for all $y \in F$ where F is a finite set with $c.F \cap F = \phi$ for all $c \in C$, then there exists $v \in \mathcal{O}$ with $v|F = u$, and $\sup_{x \in X} d(v(x), f(x)) < \varepsilon$, where d is a metric on G.

(a) is proved in [4] Proposition 2. For the proof of (b) when $\mathcal{O}$ is as in (i), see lemma 12.18. The proof of (b) when $\mathcal{O}$ is as in (ii) is omitted.

Now fix $y_0 = (x_0,1) \in Y$. For U open in Y, let

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12.18 Lemma (b) of 12.17 is true for $\mathcal{Q}$ as in (i) of 12.17.

Proof Assume without loss of generality that the metric d is C-invariant. Choose $\delta > 0$ and $S(g)$ $(g \in G)$ such that:

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§13 $\mathbb{R}$-admissability

13.1 The different types of minimal distal $\mathbb{R}$-actions on compact connected topological manifolds of dimension ≤ 3 were obtained by Bronstein [1], though not quite in the form given here.

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Lemma [5] Let $(X,\mathbb{R})$ be a minimal periodic t.g. and $(Y,\mathbb{R})$ a minimal almost periodic extension of $(X,\mathbb{R})$. Then $(Y,\mathbb{R})$ is almost periodic.

12.15 If $\mathcal{B}'$ is one of the strings of B9 - B23 then the proof of 12.9 for $\mathcal{B}'$ reduces to proving:

$\mathcal{D}(Y_r:\mathcal{B},t,t') \cap \{s_f: f \in \mathcal{O}_c\}$ is dense in $Hom(Y_r:G_r,G_{ro},t,t') \cap \{s_f: f \in \mathcal{O}_C\}$

where:

(i) $\mathcal{B}$ is the string of B1, B2, B5 or B6.

(ii) C is a finite P-invariant automorphism group of $(Y_r/G_{ro},G_{ro})$ such that the actions of C on Y_r/G_{ro} and G_{ro} commute with those of H_r, and CH_r acts freely on Y_r/G_{ro}. Hence CH_r is a finite P-invariant automorphism group of $(Y_r/G_{ro},G_{ro})$.

12.16 Lemma If $\mathcal{B}$ is one of the strings A1, A4, A9, B1, B2, B5, B6 and $s_f \in Hom(Y_r:G_r,G_{ro},t,t')$ then a sufficient condition for $s_f \in \mathcal{D}(Y_r:\mathcal{B},t,t')$ is that s_{Pf} be minimal.

Proof We indicate the proof only when $\mathcal{B}$ is one of B1, B2, B5, B6. For the strings B1, B2, B5, B6, we can assume $\mathcal{B}$ is B1. This follows from the fact that if $\mathcal{B}(K^2,u)$ is as in B2, B5, or B6, then $\mathcal{B}(K^2,u^2)$ is as in B1.

Hence suppose $\mathcal{B}$ is the string of B1, so that $t = t' : K^2 \longrightarrow K^2$ is of the form: $(k_1,k_2)t = (\alpha k_1, g(k_1)k_2)$, $g \in C(K,K)$.

and $(k_1,k_2,k_3)s_f = (\alpha k_1, g(k_1)k_2, f(k_1,k_2)k_3)$.

Suppose s_{Pf} is minimal. By [14] Theorem 1.1, this is true if and only if there is no continuous solution φ to the equation:

(12.16.1) $\varphi((k_1,k_2)t).(Pf(k_1,k_2))^m = \varphi(k_1,k_2)$ for any $m \in \mathbb{Z} \sim \{0\}$.

If this equation cannot hold for Pf, it also cannot hold for f, so that s_f is minimal.

Suppose $s_f \notin \mathcal{D}(Y_r:\mathcal{B},t)$. Then (5.5, 5.6, 5.7) $(Y_r,s_f) \to (Y_3,s_f) = (K^3,s_f)$ must be an almost periodic extension of (K,α) where α denotes the homeomorphism $k_1 \longmapsto \alpha k_1$ $(k_1 \in K)$.

Hence (K^3,s_f) is a G/H-extension of (K,α) where there exists a group L with $H \triangleleft L \triangleleft G$ and $L/H \cong G/H \cong K$ (5.5, 5.6). This forces $(G,H) = (K^2,\{1\})$.

This means there exists a minimal homeomorphism s' of K^3 of the form: $(k_1,k_2,k_3)s' = (\alpha k_1, l(k_1)k_2, m(k_1)k_3)$ with m, $l \in C(K,K)$, and a homeomorphism

Φ of K^3 of the form:

$\Phi(k_1,k_2,k_3) = (k_1, \varphi_2(k_1)k_2, \varphi_3(k_1,k_2)k_3)$, where $\varphi_2 \in C(K,K)$, $\varphi_3 \in C(K^2,K)$,

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This implies $\varphi_3((k_1,k_2)t).f(k_1,k_2) = \varphi_3(k_1,k_2).m(k_1)$

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which contradicts (12.16.1) having no continuous solution.

So $s_f \in \mathcal{O}(Y_r : \underline{\dot{s}}, t)$. Q.E.D.

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<u>Proposition</u> Let t be a minimal distal homeomorphism of X. Let G be a compact connected Lie group and let C be a finite automorphism group of (X,G) (12.15) such that $c(xt) = c(xt)$ for all $c \in C$, $x \in X$. Let s_f denote the homeomorphism of $Y = X \times G$ defined by:

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<u>Proof</u> The following are true:

(a) Given an open cover $\{V_1 \dots V_n\}$ of G, there exists an integer p such that if $W_i \in \{V_1 \dots V_n\}$ $(i = 1 \dots p)$ then $G = W_1 W_2 \dots W_p$.

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$v_y(x) = f(x)$, for x in the boundary of U_y,
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(Note that an almost periodic action of $\mathbb{R}$ on K must be periodic, hence the lemma implies a distal action of $\mathbb{R}$ on a 2-dimensional manifold must be almost periodic.)

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$\mathcal{B}(X,\mathbb{R})$ is the string of A10 if and only if $X = N/\Gamma_n$ (8.2), and the action of $\mathbb{R}$ is given by:

$[x,y,z]t = [x+at, y+bt+act^2/2 + xct + g_t(x,z), z+ct]$ for all x, y, z, $t \in \mathbb{R}$, where a and $c \in \mathbb{R}$ are rationally independent and the function

$(t,x,z) \longmapsto g_t(x,z)$ is jointly continuous, with:

$g_{t+s}(x,z) = g_t(x,z) + g_s(x+at, z+ct)$

$g_t(x+1,z) = g_t(x,z) \pmod{\mathbb{Z}} = g_t(x,z+1)$ for all x, z, s, $t \in \mathbb{R}$.

For proof of minimality see, for example, [2] 6.19.2.6. Proof that $\mathcal{B}(X,\mathbb{R})$ is the string of A10 is analogous to 12.10.

13.4 We outline the proof that the string of A9 is $\mathbb{R}$-admissable. (The proofs for A13, A14 are similar.)

If $\mathcal{B}(K^3,\mathbb{R})$ is the string of A9 then the action of $\mathbb{R}$ is of the form:

$(k_1,k_2,k_3)t = (k_1 e^{2\pi iat}, k_2 e^{2\pi ibt}, k_3 e^{2\pi i g_t(k_1,k_2)})$ for all $k_1,k_2,k_3 \in K$ and $t \in \mathbb{R}$, where a, $b \in \mathbb{R}$ are rationally independent, and if $(k_1,k_2)t = (k_1 e^{2\pi iat}, k_2 e^{2\pi ibt})$ then $g_{t+s}(k_1,k_2) = g_t(k_1,k_2)\cdot g_s((k_1,k_2)t)$ for all k_1, $k_2 \in K$ and t, $s \in \mathbb{R}$.

A necessary and sufficient condition that $(K^3,\mathbb{R})$ be minimal and that $\mathcal{B}(K^3,\mathbb{R})$ be the string of A9 is that there exist no continuous solution $f \in C(K^2,K)$ to the equation:

(13.4.1) $\quad f(k_1 e^{2\pi iat}, k_2 e^{2\pi ibt}) = f(k_1,k_2)\cdot e^{2\pi i(m g_t(k_1,k_2)+\lambda t)}$ for any $m \in \mathbb{Z}\setminus\{0\}$ and $\lambda \in \mathbb{R}$.

Writing $f(k_1,k_2) = k_1^p k_2^q e^{2\pi i g(k_1,k_2)}$ $(g \in C(K^2,\mathbb{R}))$, the condition becomes that there is no continuous solution $f_1 \in C(K^2,\mathbb{R})$ to the equation:

(13.4.2) $\quad f_1(k_1 e^{2\pi iat}, k_2 e^{2\pi ibt}) = f_1(k_1,k_2) + g_t(k_1,k_2) + \mu t$ for any $\mu \in \mathbb{R}$.

Let $g_t(k_1,k_2) = \int_0^t h((k_1,k_2)u)\, du$.

By choosing h with suitable Fourier coefficients, we can ensure that there is no continuous solution f_1 to (13.4.2).

§ 14. Appendix.

In this appendix we give details of results which were omitted from §§ 1 - 13 for the purpose of brevity, since those sections were submitted for publication:

(i) We prove that the assumption of distality in proposition 5.5 is unnecessary (14.1 - 14.2).

(ii) We give the general "finite-dimensional" version of theorem 1.2 (see 1.4) with such details of the proof as seem necessary (14.3 - 14.12). Note that the assumption "$T \in \mathfrak{J}$ " (1.4) is not necessary after all.

(iii) We show that in theorem 1.2, the hypothesis that X have finitely many arcwise-connected components can be replaced by the hypothesis that X be locally connected (14.13 - 14.14) (see 1.3).

14.1 For the proof of the more general version of proposition 5.5, we need the following facts about distal extensions. A reference is [2].

For a group T, there exists a universal minimal set (I,T) such that $(I, \mathfrak{J}_p)$ is a compact Hausdorff topological semigroup with dense subgroup T, where $\mathfrak{J}_p$ denotes the topology on I, the identity of T is an idempotent of I, I = uI has no non-trivial ideals and:

$$q \longmapsto pq \quad (p, q \in I) \qquad q \longmapsto qt \quad (q \in I, \ t \in T)$$

are $\mathfrak{J}_p$-continuous.

If (X,T) is a minimal t.g. then there exist universal minimal distal and almost periodic extensions of (X,T) denoted by (X^*,T) and $(X^\#,T)$ respectively.

$$(X,T) \prec (X^\#,T) \prec (X^*,T) \prec (I,T).$$

(X,T) can be regarded as $\{[p]_X : p \in I\}$, where $[p]_X$ is the $\sim_X$-equivalence class of $p \in I$, where $\sim_X$ is a closed T-invariant equivalence relation on I.

Write $G_X = \{g \in G : [g]_X = [u]\}$ where G is the subgroup Iu of I. Then $G_{X^*} \triangleleft G_X$, and $G_{X^\#} \triangleleft G_X$. Now let (X,T) be fixed.

(a) If (Y,T) is a distal minimal extension of (X,T) with $(X,T) <_\pi (Y,T)$, then $g \longmapsto [gp]_Y$ maps G_X onto $\pi^{-1}\pi([p]_Y)$.

Hence $(G_X/G_Y, \mathcal{J}_p)$ is homeomorphic to $(\pi^{-1}\pi([u]_Y), \mathcal{J}_p)$.

(b) There exists a topology $\sigma \leq \mathcal{J}_p$ on G (σ would be called the $\tau(C(X^*))$-topology in [2]) such that each of the following maps $(G_X,\sigma) \longrightarrow (G_X,\sigma)$ is continuous ([2] 11.17):

$p \longmapsto qp \qquad p \longmapsto pq \qquad (p, q \in G_X)$.

$(G_X/G_{X^*},\sigma)$ is compact T_1.

(c) For a σ-closed H, $G_{X^*} \leq H \leq G_X$, define:

$alg(H) = \{f \in C(X^*) : f(hp) = f(p) \text{ for all } h \in H, p \in I\}$.

Then alg(H) is a T-invariant C*-subalgebra of C(X*) containing C(X).

For a T-invariant C*-subalgebra $\mathcal{A}$, $C(X) \subseteq \mathcal{A} \subseteq C(X^*)$, define

$gp(\mathcal{A}) = \{h \in G_X : f(hp) = f(p) \text{ for all } f \in \mathcal{A}, p \in I\}$.

Then $gp(\mathcal{A})$ is a σ-closed subgroup of G_X.

$alg(gp(\mathcal{A})) = \mathcal{A}$ and $gp(alg(H)) = H$ ([2] Ch.13).

(d) For a σ-closed H, $G_{X^*} \leq H \leq G_X$, $G_{X^\#} \leq H$ if and only if $(G_X/H,\sigma) = (G_X/H, \mathcal{J}_p)$. For a σ-closed H, $G_{X^*} \leq H \triangleleft G_X$, $G_{X^\#} < H$ if and only if multiplication in G_X/H is $\mathcal{J}_p$-continuous in each variable. In this case, the left-action of $(G_X/H, \mathcal{J}_p)$ on $(I/H, \mathcal{J}_p)$ is continuous in each variable, where $I/H = \{Hp : p \in I\}$. Since $(I/H, \mathcal{J}_p)$ is compact Hausdorff by (c), this implies the left-action is jointly continuous ([18]).

<u>14.2</u> <u>Proposition</u> Proposition 5.5 is true without the assumption that the (Z_i,T) be distal.

<u>Proof</u> As in 5.5, we construct $\tilde{\varphi} : E(Y_1) \longrightarrow E(Y_2)$. Let J_1 be a minimal ideal of $E(Y_1)$, and $J_2 = \tilde{\varphi}(J_1)$. In a similar manner to 5.5, we can make (Z_i,T) a factor of (J_i,T) so that the following diagram

commutes:

Diagram 14.2

$$\begin{array}{ccc}
J_1 & \xrightarrow{\tilde{\varphi}} & J_2 \\
\downarrow & & \downarrow \\
Z_1 & & Z_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\varphi} & Y_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\varphi'} & X_2
\end{array}$$

Using the notation established in 14.1, $G_{J_i} \lhd G$ ([2] 11.19 - 11.21), and $G_{X_i^{\#}} \lhd G_{X_i}$, so $N_i = \overline{G_{J_i} G_{X_i^{\#}}}^{\sigma}$ is a normal σ-closed subgroup of G_{Y_i} contained in G_{Z_i}.

Write $G_i' = G_{X_i}/N_i$, $H_i' = G_{Y_i}/N_i$, $K_i' = G_{Z_i}/N_i$, $(W_i,T) = (J_i/N_i,T)$. Then $(G_i', \mathcal{J}_p) = (G_i', \sigma)$.

Since (W_i,T) is the maximal a.p. extension of (X_i,T) in (J_i,T), $\tilde{\varphi}$ induces an isomorphism of (W_1,T) onto (W_2,T). Now proceed much as in 5.5.

<u>14.3</u> The statement of the general "finite-dimensional" version of theorem 1.2 is obtained from the statement of theorem 1.2 as follows:

Replace the hypothesis that X have finitely many <u>arcwise-connected</u> components by the hypothesis that X have finitely many <u>connected</u> components. Omit the sentence "These hypotheses....topological manifold". Omit conclusion (i). In conclusion (iii), omit the words "so that $\mathcal{B}_i = (Y_i, X_i, X_{i-1}, G_i, N_i, \pi_i, \rho_i, \gamma_i)$ is a fibre bundle (3.1) for $1 \leq i \leq r$".

Replace the words "manifold" and "Lie group", wherever they occur in the statement of the theorem, by "finite-dimensional space" and finite-dimensional group" respectively.

The proof of the new version follows the lines of the proof of 1.2

once we have proved the following:

<u>14.4</u> <u>Proposition</u> Let $(X,T) <_{\pi_1} (Y,T) <_{\pi_2} (Z,T)$ $(\pi_1 \circ \pi_2 = \pi)$ where (Z,T) is minimal, $\pi_1^{-1}(x)$ is connected $(x \in X)$, (Y,T) is an a.p. extension of (X,T), and (Z,T) is a finite a.p. extension of (Y,T). Then (Z,T) is an a.p. extension of (X,T).

For the proof we need a sequence of lemmas. Proofs of the easier ones will be omitted.

<u>14.5</u> <u>Lemma</u> If $(X,T) < (Y,T) < (Z,T)$ where (Y,T) is a finite a.p. extension of (X,T) and (Z,T) is an a.p. extension of (Y,T), then (Z,T) is an a.p. extension of (X,T).

<u>14.6</u> <u>Lemma</u> For proposition 14.4, we may assume $\pi^{-1}(x)$ is connected $(x \in X)$.

<u>14.7</u> <u>Lemma</u> Let G be a compact topological group, $H \leq G$, and suppose G/H is connected. Then if G_o denotes the connected component of $1 \in G$, $G_oH = HG_o = G$.

<u>14.8</u> <u>Lemma</u> Let G be a compact connected topological group. Let A be a finite group acting freely and continuously on the compact connected Hausdorff space X such that G identifies with the orbit space under the map $\rho : X \longrightarrow G$. Suppose $\rho(x_o) = 1$. Then X can be made a topological group in such a way that x_o is the identity and ρ a group homomorphism. The group structure is the unique group structure on X making x_o the identity and ρ a group homomorphism and the maps $q \longmapsto pq$ continuous for each $p \in X$ (alternatively the maps $q \longmapsto qp$ continuous for each $p \in X$).

<u>Proof</u> G is the inverse limit of the net $(\{G_n\}_{n \in D}, \{\pi_{nm}\}_{n \leq m})$ of compact connected Lie groups. Let $\pi_n : G \longrightarrow G_n$ be the limit map. Let $\rho_n = \pi_n \circ \rho$. Then for each $x \in X$, $\rho^{-1}\rho(x) = \bigcap_{n \in D} \rho_n^{-1}\rho_n(x)$

For an index α on X, let $B_\alpha(x) = \{x' : (x,x') \in \alpha\}$.

Let $U_\alpha(x) = \bigcup_{\rho(x')=\rho(x)} B_\alpha(x')$.

Choose a symmetric index δ on X such that if $\rho(x_1) = \rho(x_2)$ and $(x_1,x_2) \in \delta\circ\delta\circ\delta$, then $x_1 = x_2$.

Choose a symmetric closed index ε on X such that $(x_1,x_2) \in \varepsilon$ implies $(ax_1,ax_2) \in \delta$ for all $a \in A$.

There exists $n_0 \in D$ such that $\rho_n^{-1}\rho_n(x) \subseteq U_\varepsilon(x)$ for all $x \in X$, $n \geqslant n_0$.

Define $\sim_n$ by $x \sim_n x'$ $(n \geqslant n_0)$ if and only if $\rho_n(x) = \rho_n(x')$ and $(x,x') \in \varepsilon$. This is a closed A-invariant equivalence relation on X. Write $X_n = X/\sim_n$. A acts freely and continuously on X_n by $a.[x]_n = [ax]_n$.

Define $\tau_n : X_n \longrightarrow G_n$ by $\tau_n([x]_n) = \rho_n(x)$.

Define $\sigma_n : X \longrightarrow X_n$ by $\sigma_n(x) = [x]_n$ and $\sigma_{nm} : X_m \longrightarrow X_n$ $(n \leq m)$ by $\sigma_{nm}([x]_m) = [x]_n$. Then the following diagram commutes $(n_0 \leq m \leq n)$:

$$\begin{array}{ccccc} X & \xrightarrow{\sigma_n} & X_n & \xrightarrow{\sigma_{mn}} & X_m \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_m} \\ G & \xrightarrow{\pi_n} & G_n & \xrightarrow{\pi_{mn}} & G_m \end{array}$$

Write $x_n = [x_0]_n$. Then $\sigma_{mn}(x_n) = x_m$ $(m \leq n)$. For each $n \geqslant n_0$, there exists a unique topological group structure on X_n making x_n the identity and τ_n a group homomorphism. Then each σ_{mn} $(m \leq n)$ is a group homomorphism. Then $(X, \{\sigma_n\})$ is the inverse limit of the net $(\{X_n\}, \{\sigma_{nm}\}_{n \leq m})$ of groups, hence X can be given a topological group structure such that each σ_n is a group homomorphism, and x_0 is the identity. Then each $\rho_n = \tau_n \circ \sigma_n$ is a group homomorphism, and ρ is a group homomorphism.

The uniqueness statement of the lemma is the "unique lifting theorem" for covering spaces (see, for instance, [19]).

14.9 Proof of proposition 14.4.

Let (X,T), (Y,T), (Z,T) be as in the statement of proposition 14.4. Use the notation of 14.1.

Let $G' = G_X/G_{X*}$, $H' = G_Y/G_{X*}$, $L' = G_Z/G_{X*}$.

Then $(G'/L', \mathfrak{J}_p)$ is connected and H'/L' is finite.

Put $N' = \bigcap_{g\in G'} g^{-1}H'g$. Then $(G'/N', \mathfrak{J}_p) = (G'/N', \sigma)$.

$N'/(N'\cap L')$ is finite. Put $M' = \bigcap_{n\in N'} n^{-1}(N'\cap L')n$. N'/M' is finite, since N'/M' acts effectively on $N'/(N'\cap L')$.

We can assume that $M' \lhd G'$, from which it will follow that $M' = \bigcap_{g\in G'} g^{-1}L'g$.

For let $R' = \{g \in G' : gM' = M'g\}$. R' is σ-closed, and since $N' \lhd G'$, R' is of finite index in G'. If necessary, replace X by X^*/R', Y by $X^*/(H'\cap R')$, and Z by $X^*/(L'\cap R')$.

Now let B_1', B_2' be the groups containing M' such that B_1'/M' and B_2'/M' are the $\mathfrak{J}_p$-connected and σ-connected components of M' in G'/M' respectively. Then $B_1' = B_2' = B'$, say (14.10), and B' is σ-closed.

Write $G = G'/M'$, $N = N'/M'$, $H = H'/M'$, $L = L'/M'$, $B = B'/M'$.

G inherits σ- and $\mathfrak{J}_p$-topologies from G'.

To prove 14.4, we only have to show the maps:

$q \longmapsto pq$ and $q \longmapsto qp$ $(p,q \in G)$ are $\mathfrak{J}_p$-continuous (14.1(d)).

$(B,\mathfrak{J}_p)$ is a finite cover of $(B/N\cap B, \mathfrak{J}_p) = (B/N\cap B, \sigma)$. 14.8 implies there exists a topological group structure on B making $1 \in B$ the identity and the natural quotient map (relative to the original group structure) $B \longrightarrow B/(N\cap L)$ a group homomorphism. The uniqueness clause of 14.8 implies that the topological group structure is the same as the original group structure. So $(B,\mathfrak{J}_p) = (B,\sigma)$ (essentially 14.1(d)- see also [2] Chs. 11-13).

$B \lhd G$. So $b \longmapsto a^{-1}ba : (B,\mathfrak{J}_p) \longrightarrow (B,\mathfrak{J}_p)$ is continuous for each $a \in L$.

$G = BL = LB$ (14.11), so $(G/L,\mathfrak{J}_p) = (B/L,\mathfrak{J}_p)$, and the maps:

$(G/L, \mathcal{J}_p) \longrightarrow (G/L, \mathcal{J}_p)$: $Lg \longmapsto Lgg'$

$(G, \mathcal{J}_p) \longrightarrow (G/L, \mathcal{J}_p)$: $g \longmapsto Lg'g$ $(g, g' \in G)$ are continuous.

Hence ([18]) the map:

$(G/L \times G, \mathcal{J}_p \times \mathcal{J}_p) \longrightarrow (G/L, \mathcal{J}_p)$: $(Lg, g') \longmapsto Lgg'$ is continuous.

Let C(G/L,G/L) denote the topological semigroup of $\mathcal{J}_p$-continuous maps of G/L into itself, where the multiplication is composition of functions and the topology is the topology of uniform convergence.

Let $G \longrightarrow C(G/L,G/L)$: $g \longmapsto \varphi_g$ be defined by $(Lg')\varphi_g = Lg'g$. Since this is a continuous injective homomorphism of G into C(G/L,G/L), multiplication in G is $\mathcal{J}_p$-continuous in each variable, as required.

<u>14.10</u> <u>Lemma</u> Let B_1', B_2' be the groups containing N' such that B_1'/N' and B_2'/N' are the $\mathcal{J}_p$-connected and σ-connected components of M' in G'/M' respectively. Then $B_1' = B_2'$.

<u>Proof</u> Clearly $B_1' \leq B_2'$ and $N'B_1' = N'B_2'$. So B_1' is of finite index in B_2'. To show $B_1' = B_2'$, it suffices to show B_1' is σ-closed.

Use the notation of 14.1. Let (W,T) = (I/M',T) where $I/M' = \{M'p : p \in I\}$. (W,T) is a distal extension of (X,T), say $(X,T) <_\varsigma (W,T)$. Let $(V,T) = (W/\sim, T)$, where $w_1 \sim w_2$ if and only if $\varsigma(w_1) = \varsigma(w_2)$ and w_1, w_2 lie in the same connected component of $\varsigma^{-1}\varsigma(w_1)$. By 14.1(a) and (c), $G_V = B_1'$, so that B_1' is σ-closed as required.

<u>14.11</u> <u>Lemma</u> BL = LB = G.

<u>Proof</u> B/N is the connected component of the identity in G/N. So (14.7) BH/N = G/N. So BH = HB = G. So BL is of finite index in G. So G/L is a finite union of cosets of B/L, which are σ-closed, hence $\mathcal{J}_p$-closed. So, since G/L is $\mathcal{J}_p$-connected, BL = G.

<u>14.12</u> In [20] an example is constructed of a minimal t.g. (X,T) with totally disconnected phase space such that (X,T) is a finite group

extension of an a.p. factor, but (X,T) is not almost periodic.

14.13 Proposition. Let (X,T) be minimal distal and let X be finite-dimensional and locally connected. Then X is a manifold.

Note. This was proved by Bronstein in [1]. As I was unable to understand the proof, I include one here.

It suffices to prove the following lemma, by analogue with § 7.

14.14 Lemma. Let (W,T) be minimal distal, with W locally connected and connected. Let $(V,T) <_{\pi} (W,T)$, with V a manifold. Then it is not possible to find a strictly increasing sequence $\{(V_n,T)\}_{n=1}^{\infty}$ such that each (V_n,T) is a finite extension of (V,T) and (W,T) the inverse limit of $\{(V_n,T)\}$.

Proof. Suppose for contradiction that (W,T) is the inverse limit of a strictly increasing sequence $\{(V_n,T)\}$ as described in the statement of the lemma. Let $U \subseteq V$ be a simply connected open set. Since each V_n is an open cover of V, by passing to the limit we can find a map $\sigma : U \longrightarrow \pi^{-1}(U)$ with $\pi \circ \sigma$ = identity. Then $\pi^{-1}(U)$ is homeomorphic to $U \times G/H$ (3.2) where (W,T) is a G/H-extension of (V,T), and hence G/H is totally disconnected, infinite and perfect. It follows that no open subset of $\pi^{-1}(U)$ is connected, contradicting the fact that W is locally connected.

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ON THE FIBRES OF A MINIMAL DISTAL EXTENSION OF A TRANSFORMATION GROUP

ON THE FIBRES OF A MINIMAL DISTAL EXTENSION OF A TRANSFORMATION GROUP

M. REES

§1. INTRODUCTION

Let (X,T) be a minimal quasi-separable transformation group, and an extension of (Y,T), with Π: (X,T) $\to$(Y,T) as the factor map. If (X,T) is an almost periodic extension of (Y,T) then all the fibres $\Pi^{-1}(y)$ ($y \in Y$) are homeomorphic to a fixed homogeneous space [1]. One might ask whether the fibres $\Pi^{-1}(y)$ ($y \in Y$) are all homeomorphic if (X,T) is a distal extension of (Y,T). The answer is, in general, no, as is shown in §6. However, if we assume that Y is arcwise-connected (or, more generally, has finitely many arcwise-connected components) then theorem 5.1 gives a positive answer.

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§2. Notation

2.1. (X,T) will denote a transformation group (t.g.) with compact Hausdorff phase space X. T will be a topological group acting on X on the right, the action being jointly continuous.

2.2. If (Y,T) is a factor of (X,T), the factor map being Π: (X,T) $\to$ (Y,T) we shall write (Y,T) $<_\pi$ (X,T).

2.3. If G is a compact topological group acting on a compact Hausdorff space Z, the action being jointly continuous, then Z/G will denote the compact Hausdorff orbit space endowed with the quotient topology.

2.4. A word about diagrams: all arrows in diagrams will denote continuous <u>surjective</u> maps, and two-ended arrows will denote homeomorphisms; if the objects in a diagram are the phase spaces of transformation groups with respect to a group T, all maps in the diagram will be assumed to denote T-homomorphisms; if G is a compact topological group acting continuously on a compact Hausdorff space Z, then $Z \to {}^{Z}/G$ will denote the orbit map.

§3. PRELIMINARIES ABOUT FIBRE BUNDLES

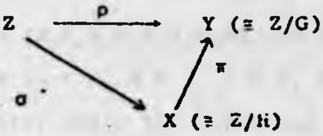
3.1. <u>Definition</u> For present purposes, a <u>fibre bundle</u> $\mathcal{B} = (Z,X,Y,G,H,\Pi,\sigma,\rho)$ satisfies

(i) X,Y,Z are compact Hausdorff spaces, and Π,σ,ρ are continuous surjective maps.

(ii) G is a compact <u>Lie</u> group with closed subgroup H, $\bigcap_{g\in G} g^{-1}Hg = \{e\}$ and G acts freely on the left of Z, the action being jointly continuous.

(iii) The following diagram is commutative.

Diagram 3.1.



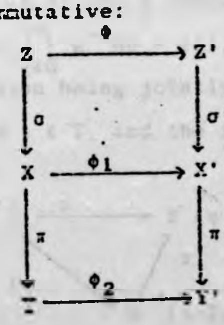
Y is called the base of the bundle, and G the group of the bundle. This definition of fibre bundle is essentially the same, for a restricted class of bundles, as that in [4] chapter 1 §2, since ([2], theorem 1 of §5.4) the free action of a compact Lie group on a compact Hausdorff space is locally trivial.

The following definition of bundle map essentially coincides, for the restricted class of bundles (3.1), with that of [4] Chapter 1, §2.

3.2. Definition Let $\mathfrak{B} = (Z,X,Y,G,H,\Pi,\sigma,\rho)$ and $\mathfrak{B}' = (Z',X',Y',G,H,\Pi',\sigma',\rho')$ be fibre bundles. Φ is a bundle map between $\mathfrak{B}$ and $\mathfrak{B}'$ (written $\Phi : \mathfrak{B} \longrightarrow \mathfrak{B}'$) if Φ is continuous, $\Phi : Z \to Z'$, and $\Phi(g.z) = g.\Phi(z)$ for all $g \in G$, $z \in Z$.

Then Φ induces maps $\phi_1 : X \to X'$ and $\phi_2 : Y \to Y'$ such that the following diagram is commutative:

Diagram 3.2.



3.3. <u>Definition.</u> If $\mathcal{B} = (Z,X,Y,G,H,\Pi,\sigma,\rho)$ and $I = [0,1]$, then $\mathcal{B} \times I$ denotes the bundle ($Z \times I$, $X \times I$, $Y \times I$, G, H, $\Pi \times$ identity, $\sigma \times$ identity, $\rho \times$ identity) where the action of G on $Z \times I$ is defined in terms of the action of G on Z by $g.(z,t) = (g.z,t)$ for all $g \in G$, $z \in Z$, $t \in I$.

3.4. We shall need:

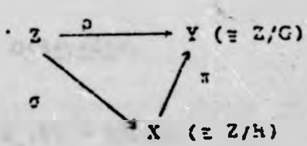
<u>Homotopy Covering Theorem</u> (Cf. [4] theorem 11.3)

Let $\mathcal{B} = (Z,X,Y,G,H,\Pi,\sigma,\rho)$ and $\mathcal{B}' = (Z',X',Y',G,H,\Pi',\sigma',\rho')$ be bundles and let $\Phi : \mathcal{B} \to \mathcal{B}'$ be a bundle map inducing $\phi: Y \to Y'$. Let $h: Y \times I \to Y'$ be a continuous map with $h(y,0) = \phi(y)$ for all $y \in Y$. Then there exists a bundle map $k: \mathcal{B} \times I \to \mathcal{B}'$ inducing $h: Y \times I \to Y'$ such that $k(z,0) = \Phi(z)$ for all $z \in Z$.

§4. <u>ON THE FURSTENBERG STRUCTURE THEOREM</u>

4.1. <u>Definition.</u> Let (X,T) be minimal and $(Y,T) <_{\pi} (X,T)$. (X,T) is a <u>quotient (Lie) group extension</u> of (Y,T) if there exist a minimal t.g. (Z,T) with $(X,T) <_{\sigma} (Z,T)$ and a compact (Lie) topological group G with closed subgroup H, $\bigcap_{g \in G} g^{-1}Hg = \{e\}$, such that G acts freely on the left of Z, the action being jointly continuous, $(gz)t = g(zt)$ for all $g \in G$, $z \in Z$, and $t \in T$, and the following diagram is commutative.

<u>Diagram 4.1.</u>



Remark. Note that if G is Lie then $(Z_o, X_o, Y_o, G, H, \Pi, \sigma, \rho)$ is a fibre bundle where Y_o is any closed subset of Y, $Z_o = \rho^{-1}(Y_o)$ and $X_o = \Pi^{-1}(Y_o)$.

4.2. Definition. Let $(Y,T) <_{\Pi} (X,T)$ with (X,T) minimal. (X,T) is a distal extension of (Y,T) if given $x_1, x_2 \in X$ with $\Pi(x_1) = \Pi(x_2)$, the existence of a net $\{t_n\} \subseteq T$ with $\lim_n x_1 t_n = \lim_n x_2 t_n$ implies $x_1 = x_2$.

4.3. Definition. (X,T) is quasi-separable if C(X) is generated by its norm-separable T-invariant subalgebras. For instance, if X is metric, or if T is separable or σ-compact, then (X,T) is quasi-separable.

4.4. The following modification of the Furstenburg structure theorem will be needed in the proof of theorem 5.1. Proof of the modification will not be given here. The proof of conclusions (i) - (v) with the word "Lie" omitted in (iv) can be found in [1] Chapters 14 and 15.

Theorem Let (X,T) be a minimal quasi-separable t.g., $(Y,T) <_{\Pi} (X,T)$, (X,T) a distal extension of (Y,T). Then there exist an ordinal α, a family $\{(X_\beta,T)\}_{0 \le \beta \le \alpha}$ of factors of (X,T) and T-homomorphisms $\{\Pi_{\beta,\gamma}\}_{0 \le \beta \le \gamma \le \alpha}$ such that

(i) $(X_\beta,T) <_{\Pi_{\beta,\gamma}} (X_\gamma,T)$, $0 \le \beta \le \gamma \le \alpha$.

(ii) $\Pi_{\beta,\gamma} \circ \Pi_{\gamma,\delta} = \Pi_{\beta,\delta}$, $0 \le \beta \le \gamma \le \delta \le \alpha$.

$\Pi_{o\alpha} = \Pi$.

(iii) $(X_o,T) = (Y,T)$, $(X_\alpha,T) = (X,T)$.

(iv) $(X_{\beta+1},T)$ is a proper quotient Lie group extension of (X_β,T) for $\beta < \alpha$.

(v) If β is a limit ordinal then (X_β,T) is the inverse limit of $\{(X_\gamma,T)\}_{0 \le \gamma < \beta}$.

B

§5. THE POSITIVE RESULT

5.1. Theorem. Let (X,T) be a minimal quasi-separable t.g., $(Y,T) <_{\pi} (X,T)$, (X,T) a distal extension of (Y,T), and let Y be arcwise-connected. Then for any $y_o, y_1 \in Y$, $\Pi^{-1}(y_o)$ and $\Pi^{-1}(y_1)$ are homeomorphic.

Proof. Let α, $\{(X_\beta,T)\}_{0 \leq \beta \leq \alpha}$ and $\{\Pi_{\beta,\gamma}\}_{0 \leq \beta \leq \gamma \leq \alpha}$ satisfy (i) - (v) of (4.4). Write $I = [0,1]$.

Choose a path $h_o : \{y_o\} \times I \to Y$ with $h_o(y_o,0) = y_o$ and $h_o(y_o,1) = y_1$.

Write $P_\beta = \Pi_{0,\beta}{}^{-1}(y_o)$ $Q_\beta = \Pi_{0,\beta}{}^{-1}(y_1)$ $R_\beta = \Pi_{0,\beta}{}^{-1}(h_o(\{y_o\} \times I))$.

Hence $P_\beta \subseteq R_\beta$, $Q_\beta \subseteq R_\beta$, $P_\alpha = \Pi^{-1}(y_o)$, $Q_\alpha = \Pi^{-1}(y_1)$.

Find by transfinite induction on β continuous maps $\{h_\beta\}_{0 \leq \beta \leq \alpha}$ such that:

(i) $h_\beta : P_\beta \times I \to R_\beta$ restricts to a homeomorphism of $P_\beta \times \{1\}$ onto Q_β, and $h_\beta(x,0) = x$ for all $x \in P_\beta$.

(ii) The following diagram commutes for $\gamma \leq \beta$:

Diagram 5.1(a)

$$\begin{array}{ccc} P_\beta \times I & \xrightarrow{h_\beta} & R_\beta \\ \Big\downarrow \Pi_{\gamma,\beta} \times \text{identity} & & \Big\downarrow \Pi_{\gamma,\beta} \\ P_\gamma \times I & \xrightarrow{h_\gamma} & R_\gamma \end{array}$$

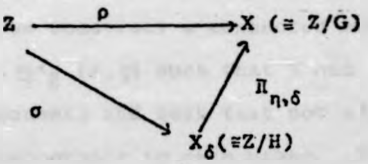
If (i) and (ii) are satisfied, h_α will restrict to a homeomorphism of $P_\alpha \times \{0\} = \Pi^{-1}(y_o) \times \{0\}$ onto $Q_\alpha = \Pi^{-1}(y_1)$ and the proof will be completed.

h_o satisfies (i) and (ii), hence it remains to assume that $\{h_\beta\}_{\beta<\delta}$ have been constructed satisfying (i) and (ii) for $\beta < \delta \leq \alpha$, and to construct h_δ . .

(1) <u>Case $\delta = \eta+1$, some η.</u>

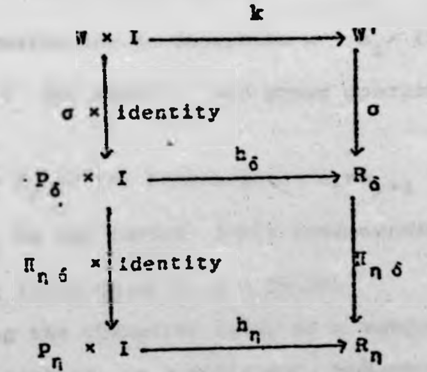
Since X_δ is a quotient Lie group extension of X_η (by 4.4(iv)) we have the following diagram for some minimal t.g. (Z,T), compact Lie group G with closed subgroup H and $\bigcap_{g\in G} g^{-1}Hg = \{e\}$:

<u>Diagram 5.1(b)</u>



By the remark of (4.1), $\mathcal{B} = (W, P_\delta, P_\eta, G, H, \Pi_{\eta\delta}, \sigma, \rho)$ and $\mathcal{B}' = (W', R_\delta, R_\eta, G, H, \Pi_{\eta\delta}, \sigma, \rho)$ are fibre bundles where $W = p^{-1}(P_\eta)$ and $W' = p^{-1}(R_\eta)$ (so $W \subseteq W'$). By the inductive hypothesis we have a map $h_\eta: P_\eta \times I \to R_\eta$ satisfying (i) and (ii). By the Homotopy Covering Theorem 3.4, we can find a bundle map $k: \mathcal{B} \times I \to \mathcal{B}'$ inducing $h_\eta: P_\eta \times I \to R_\delta$ and such that $k(w,0) = w$ for all $w \in W$. Let $h_c: P_\delta \times I \to R_\delta$ be the induced map as shown in the diagram:

<u>Diagram 5.1(c)</u>



(2) Case δ a limit ordinal

h_δ is well-defined and satisfies (i) and (ii) if we define $h_\delta(v,t)$ ($v \in P_\delta$, $t \in I$) by

$$\Pi_{\eta\delta}(h_\delta(v,t)) = h_\eta(\Pi_{\eta\delta}(v),t) \text{ for all } \eta < \delta.$$

§6. A COUNTEREXAMPLE

6.1. In this section we construct a connected minimal distal t.g. $(X,\mathcal{T})$ and a factor $(Y,\mathcal{T}) <_\Pi (X,\mathcal{T})$ such that Y has infinitely many arcwise-connected components and such that not all the fibres $\Pi^{-1}(y)$ ($y \in Y$) are homeomorphic to each other. X will be a connected metric space of covering dimension 3, and $(X,\mathcal{T})$ will be a group extension of an almost periodic t.g., so $(X,\mathcal{T})$ will be of degree 2 (Cf [1], 15.1.2). $\mathcal{T}$ will be the group Z of integers.

6.2. Definitions. Let Z, R denote the additive groups of integers and reals respectively, let $K = R/Z$ denote the circle group and Sd a fixed solenoid which is the inverse limit of the sequence

$$\cdots \xrightarrow{n_3} K_3 \xrightarrow{n_2} K_2 \xrightarrow{n_1} K_1$$

where $K_i = K$, n_i denotes the homomorphism $x \mapsto n_i x$ ($x \in K_{i+1}$), n_i being an integer ≥ 2 for each i. All group operations will be written additively.

Let $n_{r,s}: K_s \to K_r$ be the homomorphism $n_r \cdot n_{r+1} \cdots n_{s-1}$ ($r < s$).

Let $\chi_i: Sd \to K_i$ be the inverse limit homomorphism.

6.3. The following facts about Sd are known.

(i) Since Sd, being the character group of a subgroup of the reals with the discrete topology, is a continuous homomorphic image of the Bohr compactification of R, ([3] 1.8), Sd contains a dense 1-parameter subgroup Γ.

(ii) $\chi_1^{-1}(0)$ is an infinite closed subgroup of Sd. hence uncountable. Since $\Gamma \cap \chi_1^{-1}(0)$ is countable, Γ has uncountable index in Sd.

(iii) For $\beta \in \Gamma$ and $x \in \mathbb{R}$ we can uniquely define $x\beta$ satisfying

(a) $x \longmapsto x\beta$ is a continuous homomorphism of $\mathbb{R}$ onto Γ

(b) $1\beta = \beta$.

(iv) Γ is the arcwise-connected component of $0 \in$ Sd. For let $x \in$ Sd be in the arcwise-connected component of $0 \in$ Sd. Let σ: $[0,1] \to$ Sd be a path from 0 to x. σ is the unique lifting to Sd of the path $\chi_1 \circ \sigma$: $[0,1] \to K$, with the property $\sigma(0) = 0$. Now $\chi_1 \circ \sigma$ must be homotopic to the path $\tau_a(t)$: $Z + at$ for some $a \in \mathbb{R}$ where $Z + a = \chi_1(x)$; τ_a lifts to a unique path in Sd joining 0 to x. But we can assume $\Gamma = \{\gamma(t): t \in \mathbb{R}\}$ where γ is a homomorphism such that $\chi_1 \circ \gamma(t) = Z + t$ for all $t \in \mathbb{R}$, and then the path $t \to a\gamma(t)$ is the lifting of τ_a, hence $x = a\gamma(1)$, $x \in \Gamma$.

(v) For each integer $n > 0$, Sd contains at most n elements x such that $nx = 0$, since the same is true for each K_i.

6.4. Definitions of some phase spaces

Let $X = (\mathbb{R} \times Sd^2)/\sim$ where $\sim$ is the smallest equivalence relation on $\mathbb{R} \times Sd^2$ such that $(x,y,z) \sim (x+1, y+z,z)$ for all $x \in \mathbb{R}$, $y,z \in$ Sd. Let $[x,y,z]$ denote the equivalence class of (x,y,z). Then X is compact metric and can be realized as an "inverse limit of nilmanifolds".

For $i = 1,2,...$ let $X_i = (\mathbb{R} \times K^2)/\sim_i$ where $\sim_i$ is the smallest equivalence relation on X_i such that $(x,y,z)\sim_i (x+1, y+z,z)$ for all $x \in \mathbb{R}$, $y,z \in K$. Let $[x,y,z]_i$ denote the $\sim_i$-equivalence class of (x,y,z).

B

Define $\Pi_{i,j}: X_i \to X_j$ by $\Pi_{i,j}[x,y,z]_j = [x, n_{i,j}y, n_{i,j}z]$ $(i < j)$.
Define $\Pi_i: X \to X_i$ by $\Pi_i[x,y,z] = [x, \chi_i(y), \chi_i(z)]_i$

Then $(X, \{\Pi_i\})$ is the inverse limit of $(\{X_i\}, \{\Pi_{i,j}\}_{i<j})$. Moreover it can be shown that X_i is a 3-dimensional nilmanifold whose fundamental group $\Pi_1(X_i)$ is isomorphic to the multiplicative group of matrices

$$\left\{ \begin{pmatrix} 1 & m & n \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} : m,n,p \in Z \right\}.$$

6.5. Definitions of some Z-actions.

For actions of Z with phase space W, if t is the homeomorphism of W corresponding to the action of $1 \in Z$ on W, we shall denote the corresponding t.g. by (W,t).

We now define minimal distal Z-actions on X and X_i $(i = 1,2,...)$. Choose $1, \alpha, \beta_1 \in R$ to be rationally independent and let $\beta \in \Gamma$ be in the inverse image under χ_1 of $Z + \beta_1 \in K_1$ – such a β exists by considering a lifting (under χ_1) to Sd of a path in K_1 from Z to $Z + \beta_1$.

Define $t: X \to X$ by $[x,y,z]t = [x+\alpha, y+x\beta, z+\beta]$ (6.3(iii)). Then $(X_i,t_i) <_{\Pi_{i,j}} (X_j,t_j) <_{\Pi_j} (X,t)$ $(i < j)$, where $t_i: X_i \to X_i$ is of the form

$$[x,y_i,z_i]_i\, t_i = [x+\alpha,\ y_i+x\gamma_i,\ z+\gamma_i]_i$$ where $\gamma_i \in R$ is the unique element of R for which

$$\chi_i(x\beta) = Z + x\gamma_i \text{ for all } x \in R.$$

Since (X,t) is the inverse limit of $\{(X_i,t_i)\}$, to show (X,t) is minimal distal it suffices to show each (X_i,t_i) is minimal distal. Define a free action of K on X_i by

$$w.[x,y_i,z_i]_i = [x, y_i + w, z_i]_i \quad (w \in K).$$

The action of K on X_i commutes with t_i and we have

$$(X_i, t_i) \longrightarrow X_i/K \cong (K^2, s_i) \quad \text{(see (2.4))},$$

where $(x,z)s_i = (x + \alpha,\ z + \gamma_i)$, $(x, z \in K)$.

α and γ_i are rationally independent since $Z + \gamma_1 = Z + \beta$ and hence γ_i and β are rationally <u>dependent</u>. Thus (K^2, s_i) is minimal and hence (X_i, t_i) is minimal - for if not we could find a finite subgroup H of K such that X_i/H was homeomorphic to K^3, which would imply that the commutator subgroup of $\pi_1(X)$ was of finite index (see, for example [1] (6.19.2.6)).

Clearly (X_i, t_i) is also distal.

6.6. <u>Definitions</u>. Define $Y = Sd$ and $\pi : X \to Y$ by $\pi[x,y,z] = z$.
Then $(Y, s) <_\pi (X, t)$ where $zs = z + \beta$.
Then $\pi^{-1}(z) = (\mathbb{R} \times Sd)/\sim_z$ where $\sim_z$ is the smallest equivalence relation such that $(x, y) \sim_z (x + 1, y + z)$ for all $x \in \mathbb{R}$, $y \in Sd$. Let $[x, y]_z$ denote the $\sim_z$-equivalence class of $(x, y) \in \mathbb{R} \times Sd$.

6.7. <u>Proposition</u>. Let O denote the identity of $Sd = Y$. There exists $z \in Y$ such that $nz \notin \Gamma$ for any $n \in Z$, $n \neq O$. For such a z, $\pi^{-1}(z)$ and $\pi^{-1}(O)$ are not homeomorphic.

<u>Proof</u>. Let $A = \{v \in Sd : nv = O,$ some $n \in Z$, $n \neq O\}$. Then by (6.3(v)) A is countable, hence by (6.3(ii)) $\Gamma + A \neq Sd$. But $nw \in \Gamma$ for some $n \in Z$, $n \neq O$, if and only if $w \in \Gamma + A$. Hence z exists.

Note that $\pi^{-1}(w)$ is a quotient of $\mathbb{R} \times Sd$ by a discrete subgroup, so that its fundamental group $\pi_1(\pi^{-1}(w))$ is independent of the base-point, and $\mathbb{R} \times Sd$ is a covering of $\pi^{-1}(w)$, so that loops in $\pi^{-1}(w)$ based at $[0,0]_w$ lift to paths in $\mathbb{R} \times Sd$ joining $(0,0)$ to (n, nw) $(n \in Z)$.

$(0,0)$ can be joined to $(n, 0)$ for any $n \in Z$. So $\pi_1(\pi^{-1}(O)) \cong Z$.

$(0,0)$ cannot be joined to (n, nz) for any $n \in Z$, $n \neq O$ (6.3(iv)).

So $\pi_1(\pi^{-1}(z)) = 0$.

Therefore $\pi^{-1}(z)$ and $\pi^{-1}(0)$ are not homeomorphic.

§7. UNCOUNTABLY MANY HOMOTOPIC TYPES OF FIBRES

7.1 In this section we show that in the example of §6, where $(Y,T) <_\pi (X,T)$, and $X = (Sd \times \mathbb{R})/\sim$ and $Y = Sd$, the cardinality of the set of distinct homotopic (hence topological) types of the fibres $\pi^{-1}(z)$ ($z \in Sd = Y$) is the cardinality of the continuum. (In § 6 it was shown merely that the cardinality was greater than one.)

Recall that Sd is the inverse limit of the sequence:

$$\ldots K_4 \xrightarrow[n_3]{} K_3 \xrightarrow[n_2]{} K_2 \xrightarrow[n_1]{} K_1,$$

with limit maps χ_i: Sd $\longrightarrow K_i$.

We make the additional assumption that each $n_i = 2$.

The fibre $\pi^{-1}(z)$ is the compact connected abelian group $(Sd \times \mathbb{R})/N_z$ where N_z is the discrete group generated by the element $(z,1)$. Let A_z denote the character group of $(Sd \times \mathbb{R})/N_z$.

7.2. Proposition. $\pi^{-1}(z)$ and $\pi^{-1}(w)$ are homotopic if and only if A_z and A_w are isomorphic as groups.

We do not give a proof of this. It follows from

either (i) A_z is the first Cech cohomology group of $(Sd \times \mathbb{R})/N_z$ ([5]).

or (ii) It can be shown that a continuous map between two compact connected abelian groups is homotopic to a unique group homomorphism.

7.3. Description of A_z.

Fix $z \in$ Sd.

$(Sd \times \mathbb{R})/N_z$ is the inverse limit of $(\{(K_n \times \mathbb{R})/N_{n,z}\}, \{\sigma_{n,n+1}\})$ where $N_{n,z}$ is generated by $(\chi_n(z),1)$ and:

$\sigma_{n,n+1}: (K_{n+1} \times \mathbb{R})/N_{n+1,z} \longrightarrow (K_n \times \mathbb{R})/N_{n,z}$ is defined by:

$\sigma_{n,n+1}(N_{n+1,z} + (k,x)) = N_{n,z} + (k^2,x)$.

The limit map $\sigma_n : (Sd \times \mathbb{R})/N_z \longrightarrow (K_n \times \mathbb{R})/N_{n,z}$ is defined by:
$\sigma_n(N_z + (y,x)) = N_{n,z} + (\chi_n(y),x)$.

Now $\chi_n(z) = e^{2\pi i\theta_n}$ for a unique $\theta_n = \theta_n(z) \in [0,1)$. Then $\theta_{n+1} = \theta_n/2$ or $\theta_n/2 + 1/2$.

$\varphi_n : (K_n \times \mathbb{R})/N_{n,z} \longrightarrow K^2$ is a group isomorphism,

if $\varphi_n(N_{n,z} + (k,x)) = (ke^{-2\pi i x\theta_n}, e^{2\pi i x})$.

Then $(Sd \times \mathbb{R})/N_z$ is the inverse limit of:

$$K^2 \xrightarrow[\rho_{n,n+1}]{} K^2 \;\ldots\ldots\; K^2 \xrightarrow[\rho_{1,2}]{} K^2$$

where $\rho_{n,n+1} = \varphi_n \circ \sigma_{n,n+1} \circ \varphi_{n+1}^{-1}$.

If $\theta_{n+1} = \theta_n/2$ then $\rho_{n,n+1}(k_1,k_2) = (k_1^{\,2},k_2)$.
If $\theta_{n+1} = \theta_n/2 + 1/2$ then $\rho_{n,n+1}(k_1,k_2) = (k_1^{\,2}k_2,k_2)$.

Let $A_{n,z}$ be an isomorphic copy of the character group of K^2. We have:

$$A_{1,z} \xrightarrow[\rho^*_{1,2}]{} A_{2,z} \xrightarrow[\rho^*_{2,3}]{} A_{3,z} \longrightarrow \;\ldots.$$

A_z can be regarded as the inductive limit of this sequence. We shall now define $A_{n,z}$ so that $A_{n,z} \subseteq A_{n+1,z} \subseteq A_z \subseteq \mathbb{Q}^2$ and $\rho^*_{n,n+1}$ is the inclusion map.

Define $a_n = a_n(z) = 0$ if $\theta_{n+1} = \theta_n/2$
$a_n = a_n(z) = 1$ if $\theta_{n+1} = \theta_n/2 + 1/2$.

Then let $A_{n,z}$ be generated by the elements $(0,1)$ and $(1/2^{n-1}, -\sum_{i=1}^{n-1} a_{n-i}/2^i)$.

7.4 Lemma If A and B are subgroups of $\mathbb{Q}^n$ with rational span $\mathbb{Q}^n$ then any group isomorphism between A and B can be uniquely extended to a $\mathbb{Q}$-linear isomorphism of $\mathbb{Q}^n$.

We omit the proof. It follows that A_z is isomorphic to only countably many distinct groups A_w $(w \in Sd)$. Thus we only have to prove:

7.5 Proposition. If $A_z = \bigcup_{n=1}^{\infty} A_{n,z}$ where $A_{n,z} \subseteq \mathbb{Q}^2$ is generated by $(1/2^{n-1}, -\sum_{i=1}^{n-1} a_{n-i}(z)/2^i)$ as described in 7.3, then the cardinality of the set $\{A_z : z \in Sd\}$ is the cardinality of the continuum.

<u>Proof</u> (i) We claim $A_z = A_w$ if and only if $a_n(z) = a_n(w)$ for all n. For suppose $A_z = A_w$. Then for each n,

$$(1/2^{n-1}, -\sum_{i=1}^{n-1} a_{n-i}(z)/2^i) \in A_w = \bigcup_m A_{m,w}.$$

Then $(1/2^{n-1}, \sum_{i=1}^{n-1} a_{n-i}(z)/2^i) = (0,p) + q(1/2^{m-1}, -\sum_{i=1}^{m-1} a_{m-i}(w)/2^i)$ for some $p, q \in \mathbb{Z}$, $m \geq 1$. Then $q = 2^{m-n} (m \geq n)$ and:

$$p - 2^{m-n} \sum_{i=1}^{m-1} a_{m-i}(w)/2^i = -\sum_{i=1}^{n-1} a_{n-i}(z)/2^i .$$

The fractional parts are equal. So:

$$-\sum_{i=m-n+1}^{m-1} a_{m-i}(w)/2^{i-m+n} = -\sum_{i=1}^{n-1} a_{n-i}(z)/2^i .$$

So $-\sum_{i=1}^{n-1} a_{n-i}(w)/2^i = -\sum_{i=1}^{n-1} a_{n-i}(z)/2^i$.

So $a_i(z) = a_i(w)$ for $1 \leq i \leq n-1$. Hence result, since n is arbitrary.

(ii) Let $\{b_n\}_{n=1}^{\infty}$ be any sequence of 0's and 1's. We claim there exists $z \in Sd$ with $a_n(z) = b_n$ for all n.

For take $z \in \bigcap_{n=1} \chi_n^{-1}(e^{2\pi i\theta_n})$ where $\theta_1 = 0$ and θ_n is defined inductively for $n \geq 1$ by:

$$\theta_{n+1} = \theta_n/2 \quad \text{if } b_n = 0$$
$$= \theta_n/2 + 1/2 \quad \text{if } b_n = 1.$$

(i) and (ii) show that the cardinality of $\{A_z : z \in Sd\}$ is the cardinality of the set of sequences of 0's and 1's, as required.

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NON-CONJUGACY OF A MINIMAL DISTAL DIFFEOMORPHISM OF THE TORUS TO A C^1 SKEW-PRODUCT c

NON-CONJUGACY OF A MINIMAL DISTAL DIFFEOMORPHISM OF THE TORUS TO A C^1 SKEW-PRODUCT.

M. REES

Introduction

Let $Hom^+(K^n)$ denote the set of orientation-preserving homeomorphisms of the n-dimensional torus $K^n = \mathbb{R}^n/\mathbb{Z}^n$. If T is a minimal element of $Hom^+(K)$, then it is known that T is topologically conjugate to an irrational rotation of K, which is, of course, C^∞. Correspondingly, if T is a minimal distal element of $Hom^+(K^2)$, it is known (see, for instance, [4]) that T is topologically conjugate to a homeomorphism of K^2 of the form:

$T_{\alpha,g}$: $(x,y) \longmapsto (x+\alpha, y+g(x))$ where $g \in C(K,K)$ and α is irrational. In this paper, it is shown that, contrary to what happens for the circle, or for almost periodic homeomorphisms in general, there is a minimal distal C^∞ element of $Hom^+(K^2)$ which is not topologically conjugate to any C^1 homeomorphism of the form $T_{\alpha,g}$.

I should like to thank my supervisor, W.Parry, for suggesting the problem and for helpful discussion. I should like to thank the S.R.C. for financial support.

§1. Preliminaries.

1.1. If $f \in C(K^n,K^n)$, then there exists a unique element of $C(\mathbb{R}^n,\mathbb{R}^n)$, again denoted by f, such that $f(\underline{0}) \in [0,1)^n$, and the following diagram commutes:

$$\begin{array}{ccccc} x & \mathbb{R}^n & \xrightarrow{\ f\ } & \mathbb{R}^n & x \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \mathbb{Z}^n+x & K^n & \xrightarrow{\ f\ } & K^n & \mathbb{Z}^n+x \end{array}$$

For $Hom^+(K)$, this correspondence reduces to a correspondence between $Hom^+(K)$ and $\{f \in C(\mathbb{R},\mathbb{R}) : f$ is a homeomorphism, $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$, and $f(0) \in [0,1)\}$.

Note that in what follows, for all equations (inequalities) involving elements of $C(\mathbb{R}^n,\mathbb{R}^n)$ corresponding to elements of $C(K^n,K^n)$, the equality (inequality) sign denotes real equality (inequality) and not equality (inequality) mod $\mathbb{Z}^n$.

1.2. Let $Hom^+(K)$ be given the topology of uniform convergence. The rotation number function $\rho : Hom^+(K) \longrightarrow K$ is continuous.

If $q \in \mathbb{Z}$ and $f \in Hom^+(K)$, then $\rho(f) = \mathbb{Z}+(p/q)$ for some $p \in \mathbb{Z}$ if and only if there exists $x \in K$ with $f^q(x) = x$. (See, for example, [3], [1] for definition and basic properties of ρ.)

1.3. Definition. Let $f \in Hom^+(K)$ with $\rho(f) = \mathbb{Z}+(p/q)$ with p, q coprime and positive, $0 \leq p < q$. We follow [1] in defining f to be semistable forward if:

$f^q(x) \geq x + p$ for all $x \in \mathbb{R}$.

1.4. Denjoy's Theorem. (See, for example, [3].)

Let $f \in Hom^+(K)$ be C^2 and $\rho(f) = \mathbb{Z}+\alpha$, $\alpha \in [0,1)$ and irrational. Then there exists a unique $\varphi \in Hom^+(K)$ such that:

$\varphi(f(x)) = \varphi(x) + \alpha$ for all $x \in \mathbb{R}$, $\varphi(0) = 0$.

φ is called the eigenfunction of f corresponding to α. Note that, in particular, f is minimal almost periodic.

§2. Reduction of the problem.

Throughout this section, let $f \in Hom^+(K)$ be C^∞ with $\rho(f) = \mathbb{Z}+\alpha$, α irrational, $\alpha \in [0,1)$.

Let $T \in Hom^+(K^2)$ be given by:

$T(x,y) = (f(x),x+y)$.

Then (K^2,T) is distal, and the maximal almost periodic factor is

(K,f). Since (K,f) is minimal by 1.4, (K^2,T) is minimal by [2] §2.

Consider the following four statements. It will be shown that $2.4 \Rightarrow 2.3 \Rightarrow 2.2 \Rightarrow 2.1$.

2.1. If $T(x,y) = (f(x),x+y)$, then T is not conjugate to any C^1 homeomorphism of the form:

$$T_{\beta,g} : (x,y) \longmapsto (x+\beta, y+g(x)), \text{ where } \beta \in \mathbb{R} \text{ and } g \in C^1(K,K).$$

2.2. The equation:

$$x - \varphi(x) = \psi(\varphi(x)) + \chi(f(x)) - \chi(x) + \mu$$

does not hold for any $\psi \in C^1(\mathbb{R},\mathbb{R})$, $\chi \in C(\mathbb{R},\mathbb{R})$, $\mu \in \mathbb{R}$, where ψ and χ have period 1, $\int_0^1 \psi = 0$, and φ is the eigenfunction of f corresponding to α (see 1.4).

2.3. For each $\psi \in C^1(\mathbb{R},\mathbb{R})$ with period 1 and $\int_0^1 \psi = 0$, there exists a strictly increasing sequence $\{m_n\}$ of positive integers with:

(i) $$\sup_n \sup_{x \in \mathbb{R}} \left| \sum_{i=0}^{m_n - 1} \psi(x+i\alpha) \right| < \infty$$

(ii) The sequence $$\left\{ \sup_{x \in \mathbb{R}} \left| \sum_{i=0}^{m_n - 1} (f^i(x) - i\alpha) - (m_n/m_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f^i(x) - i\alpha) \right| \right\}$$

is unbounded.

2.4. There exists a constant $B > 0$, a sequence $\{q_n\}$ of positive integers with $q_{n+1} > q_n^6$ and a sequence $\{x_n\}$ of elements of $\mathbb{R}$ such that, if for each n, m_n is any multiple of q_n with $q_n \le m_n \le q_n^2$, then:

(i) $$\left| \frac{1 - e^{2\pi i r m_n \alpha}}{1 - e^{2\pi i r \alpha}} \right| \le 1 \text{ for } r \le q_n^6, \ r \text{ not a multiple of } q_n.$$

(ii) $$\left\{ (1/m_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f^i(x_n) - i\alpha) \right\} - \left\{ (1/m_n) \sum_{i=0}^{m_n - 1} (f^i(x_n) - i\alpha) \right\} \ge B/q_n.$$

2.2 $\Rightarrow$ 2.1. If T is conjugate to a C^1 homeomorphism of the form $T_{\beta,g}$, we can assume $\beta = \alpha$, and that the conjugacy is given by:

$(x,y) \longmapsto (\varphi(x),\ h(x)+y)$ where $h \in C(K,K)$ and φ is the eigenfunction of f corresponding to α.

This is essentially because the group of eigenvalues is preserved under conjugacy, and a conjugacy must give 1 - 1 correspondences between the groups of eigenfunctions, and between the groups of generalized eigenfunctions of order 2. The result follows.

2.3 $\Rightarrow$ 2.2. Suppose 2.2 does not hold, i.e. the equation of 2.2 is satisfied by some ψ, χ, μ. Replacing x by $f^i(x)$ in the equation, we obtain:

$$f^i(x) - i\alpha - \varphi(x) - \mu = \psi(\varphi(x)+i\alpha) + \chi(f^{i+1}(x)) - \chi(f^i(x))$$

Summing over i from 0 to m_n-1, we obtain:

$$\sum_{i=0}^{m_n-1} (f^i(x)-i\alpha) - m_n\varphi(x) - m_n\mu = \sum_{i=0}^{m_n-1} \psi(\varphi(x)+i\alpha) + \chi(f^{m_n}(x)) - \chi(x).$$

Then (i) and (ii) of 2.3 cannot hold simultaneously for any sequence $\{m_n\}$.

2.4 $\Rightarrow$ 2.3. Suppose 2.4 holds.

Let $\psi \in C^1(\mathbb{R},\mathbb{R})$ have period 1, and $\int_0^1 \psi = 0$. It suffices to find a sequence $\{m_n\}$ with $q_n \le m_n \le q_n^2$, m_n a multiple of q_n, such that $\{m_n/q_n\}$ is unbounded, and:

$$\sup_n \sup_{x\in\mathbb{R}} \left| \sum_{i=0}^{m_n-1} \psi(x+i\alpha) \right| < \infty .$$

Suppose $\psi(x) = \sum_{r=-\infty}^{\infty} a_r e^{2\pi i r x}$. Then $\sum |a_r|^2 r^2 < \infty$ and $a_0 = 0$.

For all $x \in \mathbb{R}$, $\displaystyle\sum_{i=0}^{m_n-1} \psi(x+i\alpha) = \sum_{r=-\infty}^{\infty} a_r \Big(\sum_{s=0}^{m_n-1} e^{2\pi i r s\alpha} \Big) e^{2\pi i r x}$.

For each r, $\displaystyle\left| \sum_{s=0}^{m_n-1} e^{2\pi i r s \alpha} \right| \le m_n$.

So $$\sum_{r=-\infty}^{\infty}|a_r|\left|\sum_{s=0}^{m_n-1} e^{2\pi i r s\alpha}\right| \le \sum'_{|r|<q_n^3} |a_r|\left|\frac{e^{2\pi i r m_n \alpha}-1}{e^{2\pi i r\alpha}-1}\right| + \sum_{|r|\ge q_n^3} r^{1/3}|a_r|$$

$$+ m_n \sum_{t=-q_n^5}^{q_n^5} |a_{tq_n}| ,$$

where $\sum'$ denotes that the r'th term is omitted if r is a multiple of q_n.

Then, by 2.4(1):

$$\left|\sum_{i=0}^{m_n-1} \psi(x+i\alpha)\right| \le \sum_{-\infty}^{\infty} r^{1/3}|a_r| + m_n \sum_{|t|\ge 1} |a_{tq_n}| .$$

$$\sum_{r=-\infty}^{\infty} r^{1/3}|a_r| \le \left\{\sum_{r=-\infty}^{\infty} r^{-4/3}\right\}^{1/2} \times \left\{\sum_{r=-\infty}^{\infty} |a_r|^2 r^2\right\}^{1/2} < \infty .$$

Thus it suffices to find a sequence $\{m_n\}$ such that:

(2.5) $\{m_n/q_n\}$ is unbounded, $q_n \le m_n < q_n^2$, m_n is a multiple of q_n and:

$$\sup_n m_n \sum_{|t|\ge 1} |a_{tq_n}| < \infty .$$

Now $$\sum_{|t|\ge 1}|a_{tq_n}| \le \left\{\sum_{|r|\ge q_n} r^2|a_r|^2\right\}^{1/2} \times \left\{\sum_{|t|\ge 1}(1/t^2 q_n^2)\right\}^{1/2} .$$

Write $C = \left\{\sum_{|t|\ge 1}(1/t^2)\right\}^{1/2}$ and $\delta(q_n) = \left\{\sum_{|r|\ge q_n} r^2|a_r|^2\right\}^{1/2}$.

Then $\delta(q_n) \to 0$ as $n \to \infty$ and $\sum_{|t|\ge 1}|a_{tq_n}| \le C\delta(q_n)/q_n$.

Now take m_n to be the greatest multiple of q_n which is not greater than $\mathrm{Min}(q_n/\delta(q_n), q_n^2)$, or take $m_n = q_n$ if q_n is too small for such a multiple to exist. Then the sequence $\{m_n\}$ satisfies (2.5), as required.

§ 3. Solution of the reduced problem.

We are now reduced to constructing a C^∞ $f \in \mathrm{Hom}^+(K)$ with $\rho(f) = \alpha$, α irrational, such that f, α satisfy the conditions of 2.4. The construction is similar to Arnold's construction [1] of a C^∞ $f \in \mathrm{Hom}^+(K)$ with irrational rotation number and eigenfunction which is not absolutely continuous.

The construction of f.

Sequences $\{f_n\}$, $\{p_n\}$, $\{q_n\}$, $\{x_n\}$ $(n \geq 1)$ will be constructed such that:

3.1. Each f_n is defined and analytic in $\{z : |\mathrm{Im}\, z| < 1\}$, $f_n(\mathbb{R}) \subseteq \mathbb{R}$, $f_n(z+1) = f_n(z) + 1$ for all z, $f_n(0) \in [0,1)$, $f_n'(x) > 1/2$ for all $x \in \mathbb{R}$, (so that $f_n|\mathbb{R} \in \mathrm{Hom}^+(K)$), $f_{n+1}|\mathbb{R} \geq f_n|\mathbb{R}$ and:

$$\sup_{|\mathrm{Im}\, z| < 1} |f_n(z) - f_{n+1}(z)| < 1/2^n.$$

3.2. p_n and q_n are coprime, $0 < p_n < q_n$, $\rho(f_n) = \mathbb{Z} + (p_n/q_n)$, $q_{n+1} > q_n^6$ and $p_{n+1}/q_{n+1} - p_n/q_n = 1/q_nq_{n+1}$.

3.3 f_n is semistable forward and has exactly one cycle i.e. exactly one finite minimal f_n-invariant set (see 1.2).

3.4 - 3.6 hold for any sequence $\{m_n\}$ of positive integers such that m_n is a multiple of q_n with $q_n \leq m_n \leq q_n^2$:

3.4
$$\left|\left|\frac{1 - e^{2\pi i r m_s p_n/q_n}}{1 - e^{2\pi i r p_n/q_n}}\right| - \left|\frac{1 - e^{2\pi i r m_s p_{n+1}/q_{n+1}}}{1 - e^{2\pi i r p_{n+1}/q_{n+1}}}\right|\right| < 1/2^n,$$

for $r \leq q_s^6$, r not a multiple of q_s, $s \leq n$.

3.5.
$$\sup_{x \in \mathbb{R}} \left|(1/m_r) \sum_{i=0}^{m_r - 1} (f_n^i(x) - ip_n/q_n) - (1/m_r) \sum_{i=0}^{m_r - 1} (f_{n+1}^i(x) - ip_{n+1}/q_{n+1})\right| < (1/2^{n+4})q_r \quad \text{for } r \leq n.$$

3.6 $\{x_n\}$ is a sequence in $\mathbb{R}$ and:

$$(1/m_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f_{n+1}^i(x_n) - ip_{n+1}/q_{n+1}) > (1/m_n) \sum_{i=0}^{m_n - 1} (f_n^i(x_n) - ip_n/q_n) + (1/4q_n)$$

Then let $f = \lim_n f_n$, $\alpha = \lim_n p_n/q_n$.

3.2 implies $|p_n/q_n - p_{n+1}/q_{n+1}| < 1/n^2q_n^2$ for sufficiently large n, hence α is irrational ([1] §1).

Taking limits in 3.4 -3.6 implies f, α satisfy 2.4, with B = 1/8 in 2.4(ii). For 3.5 implies that:

$$\left| (1/m_r) \sum_{i=0}^{m_r-1} (f_r^{\,i}(x) - ip_r/q_r) \;-\; (1/m_r) \sum_{i=0}^{m_r-1} (f^i(x) - i\alpha) \right| < 1/16q_r$$

Now use this in 3.6 with r = n and r = n+1, to get 2.4(ii) with B = 1/8.

Let p_1, q_1 be arbitrary coprime integers, $0 < p_1 < q_1$, and take any f_1 satisfying 3.1 and 3.3 with $\rho(f_1) = p_1/q_1 + \mathbb{Z}$. (Use [1] §1 lemma α to get a unique cycle for f_1.)

Suppose f_n, p_n, q_n have been chosen and define x_n, f_{n+1}, p_{n+1}, q_{n+1} as follows:

<u>Choice of x_n</u>. There are precisely q_n points in any half-open interval of $\mathbb{R}$ of length one, which correspond to the points of the unique cycle of $f_n | \mathbb{R} \in \mathrm{Hom}^+(K)$. Let y, z $\in \mathbb{R}$ correspond to points in the cycle with $y < z$, and such that if $y < w < z$, then w does not correspond to a point in the cycle. Then for each i, $f_n^{\,i}(y)$ and $f_n^{\,i}(z)$ have the same property.

Choose x_n with $y < x_n < z$ and such that:

$$0 < f_n^{\,i}(x_n) - f_n^{\,i}(y) < (1/8)(f_n^{\,i}(z) - f_n^{\,i}(y)), \quad 0 \le i \le q_n^{\,2} - 1.$$

Then if m_n is any multiple of q_n with $q_n \le m_n \le q_n^{\,2}$:

$$(3.7)\quad (1/m_n) \sum_{i=0}^{m_n-1} (f_n^{\,i}(z) - f_n^{\,i}(x_n)) > (7/8m_n) \sum_{i=0}^{m_n-1} (f_n^{\,i}(z) - f_n^{\,i}(y)) = 7/8q_n.$$

<u>Lemma</u>. $(1/s) \sum_{i=0}^{s-1} (f_n^{\,i}(x_n) - ip_n/q_n) \longrightarrow (1/q_n) \sum_{i=0}^{q_n-1} (f_n^{\,i}(z) - ip_n/q_n)$ as $s \to \infty$.

<u>Proof</u>. Clearly, it suffices to show:

$$(1/rq_n) \sum_{i=0}^{rq_n-1} (f_n^{\,i}(x_n) - ip_n/q_n) \longrightarrow (1/q_n) \sum_{i=0}^{q_n-1} (f_n^{\,i}(z) - ip_n/q_n) \quad \text{as } r \to \infty.$$

But

$$(1/rq_n) \sum_{i=0}^{rq_n-1} (f_n^{\,i}(x_n) - ip_n/q_n) = (1/q_n) \sum_{i=0}^{q_n-1} \left\{ (1/r) \sum_{s=0}^{r-1} (f_n^{\,i+sq_n}(x_n) - sp_n - ip_n/q_n) \right\}$$

So it suffices to show that for each i, $0 \le i \le q_n - 1$:

$$(1/r)\sum_{s=0}^{r-1}\left(f_n^{i+sq_n}(x_n) - sp_n - i(p_n/q_n)\right) \longrightarrow f_n^{i}(z) - ip_n/q_n \quad \text{as } r \to \infty .$$

For this it suffices to show:

$f_n^{i+sq_n}(x_n) - sp_n - ip_n/q_n \longrightarrow f_n^{i}(z) - ip_n/q_n$ as $s \to \infty$. But this follows from there being no elements of the cycle of f_n between x_n and z ([1]§1). Q.E.D.

Now choose $t_n > q_n^6$ such that:

$$\left|(1/t)\sum_{i=0}^{t-1}(f_n^{i}(x_n) - ip_n/q_n) - (1/q_n)\sum_{i=0}^{q_n-1}(f_n^{i}(z) - ip_n/q_n)\right| < 1/(8q_n)$$

for all $t \ge t_n$. Then if $t \ge t_n$:

$$(3.8)\quad (1/t)\sum_{i=0}^{t-1}(f_n^{i}(x_n) - ip_n/q_n) > (1/q_n)\sum_{i=0}^{q_n-1}(f_n^{i}(z) - ip_n/q_n) - 1/(8q_n)$$

$$= (1/m_n)\sum_{i=0}^{m_n-1}(f_n^{i}(z) - ip_n/q_n) - 1/(8q_n) > (1/m_n)\sum_{i=0}^{m_n-1}(f_n^{i}(x_n) - ip_n/q_n)$$

$$+\ 3/(4q_n) \quad \text{by (3.7)},$$

where m_n is any multiple of q_n with $q_n \le m_n \le q_n^2$.

Choice of p_{n+1}, q_{n+1}: Choose $1/2^n > \delta_n > 0$ such that if $0 < \lambda < \delta_n$, $f_{n+1}(z) = f_n(z) + \lambda$ ($|\mathrm{Im}\, z| < 1$), and f_{n+1} is semistable forward with rotation number p_{n+1}/q_{n+1}, then f_{n+1}, p_{n+1}, q_{n+1} satisfy conditions 3.4, 3.5. Choose $a, b \in \mathbb{Z}$ such that $aq_n - bp_n = 1$.

Take $q_{n+1} = b + uq_n$, $p_{n+1} = a + up_n$, for u large enough to ensure $q_{n+1} \ge t_n$, and such that $\rho(f_n + \delta_n) > p_{n+1}/q_{n+1}$. Then p_{n+1}, q_{n+1} satisfy 3.2 and 3.4.

Choice of f_{n+1}. Suppose $\rho(f_n + \lambda_n) = p_{n+1}/q_{n+1}$, where $f_n + \lambda_n$ is semistable forward. Such a λ_n exists and is unique ([1] §1).

Choose $f_{n+1}(z) = f_n(z) + \lambda_n + \varepsilon_n(z)$ such that $\rho(f_{n+1}) = p_{n+1}/q_{n+1}$, $\varepsilon_n(x) \ge 0$ for all $x \in \mathbb{R}$, f_{n+1} has a unique cycle, and ε_n is small enough to ensure 3.1 - 3.5 are satisfied ([1]§1).

Verification that 3.6 is satisfied.

$$(1/m_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f_{n+1}^{i}(x_n) - ip_{n+1}/q_{n+1}) \geqslant (1/q_{n+1}) \sum_{i=0}^{q_{n+1}-1} (f_{n+1}^{i}(x_n) - ip_{n+1}/q_{n+1})$$

(since f_{n+1} is semistable forward)

$$\geqslant (1/q_{n+1}) \sum_{i=0}^{q_{n+1}-1} (f_n^{i}(x_n) - ip_n/q_n) - (1/q_{n+1}) \sum_{i=0}^{q_{n+1}-1} (ip_{n+1}/q_{n+1} - ip_n/q_n)$$

$$> (1/m_n) \sum_{i=0}^{m_n-1} (f_n^{i}(x_n) - ip_n/q_n) + 3/(4q_n) - 1/(2q_n),$$ by 3.8, 3.2 and because $q_{n+1} \geqslant t_n$, where m_n and m_{n+1} are multiples of q_n, q_{n+1} respectively.

The construction is completed.