Some Aspects of the Theory of the

Propagation of Nonlinear Waves in

Unstable Media

A thesis submitted by

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Preface

This research was carried out during the period October 1971 to
October 1974 when the author was a postgraduate student at the Department
of Pnysics, University of Warwick.

The author would like to thank Dr. G. Rowlands for his help and
encouragement whilst supervising the work and would like to thank Professor
P.N. Butcher for his guidance during Dr. Rowlands' absence.   Thanks are
also due to the other members of staff and research students at the
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spent there.

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the thesis.

Finally, the author thanks his wife and his parents for their financial
and moral support during the time it has taken to complete the thesis.

Declaration

I declare that the research described in this thesis is original
except where this is specifically acknowledged in the text.   Part of the
work, described in Chapter 4 has already been published under the title -
The propagation of solitary waves in piezoelectric semiconductors -
M. Pawlik and G. Rowlands - in J. Phys. C: Solid State Phys., Vol.8,
pp.1189-1204, 1975.   I would like to thank Dr. G. Rowlands for
suggesting this problem, for his help and encouragement whilst the work
was being carried out and for his helpful revision of the manuscript.

Abstract

The behaviour of waves in stable and unstable media in the nonlinear regime is of considerable interest and relevance to many physical systems. New, tractable techniques are required to account for the possible interactions of dispersive, dissipative and modulational effects and their effects on the propagation of nonlinear waves. The reductive perturbation technique — an asymptotic expansion in multiple time and space scales — is extended to apply to wave propagation in unstable media in both one and two dimensions. It is shown that, to lowest order, the wave amplitude satisfies a form of nonlinear Schrödinger equation and the validity of this equation is established for a much wider class of systems than was previously supposed. Explicit expressions are given for determining the complex coefficients of this equation from the coefficients of the system of equations describing the original physical system.

These general methods are applied to two physical systems. A nonlinear theory of the propagation of acoustic waves in piezoelectric semiconductors is presented and an explicit solution of the relevant generalised nonlinear Schrödinger is found using a perturbation technique. This solution is found to be an envelope soliton and theoretically confirms domain propagation in piezoelectric semiconductors. A nonlinear theory of a two-stream instability in a marginally stable state is given and the wave equation is found to be a different form of the nonlinear Schrödinger equation. The nonlinear effects are found to enhance rather than suppress the instability in agreement with previously published results.

A discussion is given of the stability of inhomogeneous plasma streams in mutually perpendicular electric and magnetic fields and suggestions are made for the development of a nonlinear theory of such systems using the general techniques developed.

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Chapter 1

General aspects of nonlinear wave propagation and
the reductive perturbation technique

1.1 Introduction

Many results have been published in recent years of theoretical work on
the propagation of nonlinear waves. A progression from a simple linear
analysis to a nonlinear small amplitude analysis is the natural sequence of
events for considerations of systems which are capable of supporting wave
motion. Many examples of such systems are found in plasma physics, hydro-
dynamics, solid state physics etc.

The analysis of nonlinear wave propagation has led to interesting new
concepts such as, for example, the rediscovery of the solitary wave, the
existence of infinitely many conserved quantities, the possibility of nonlinear
instability of linearly stable systems and phenomena such as amplitude disper-
sion. We will consider these in more detail later.

The analysis of systems in the nonlinear regime is generally much more
complex than linearised analysis and many different theoretical techniques
have been developed for solving the complex nonlinear equations that arise
from such an analysis. One particularly fruitful approach has been not to
attempt to solve these equations exactly but to reduce them to a single non-
linear equation which can then be analysed in detail. This approach has proved
to be so useful because   it has been found that wide classes of nonlinear
systems can be reduced to one of three equations depending on the balance bet-
ween the nonlinearity and other effects such as dispersion and dissipation.
These three equations are: the Korteweg-de Vries equation (Korteweg and de
Vries (1895)), the Burgers equation (Burgers 1954), and the nonlinear
Schrödinger equation (Landau (1944)). For the sake of brevity we will refer
to these equations as the KdV equation, the B equation and the NLS equation
respectively. These equations can be derived in a number of different ways

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and before we discuss these methods we will consider these equations in more detail and indicate for which systems they are valid.

The KdV equation was derived by Korteweg and de Vries (1895) for the problem of long surface waves in a rectangular channel of constant depth. The general form of this equation is given by:

$$\frac{\partial u}{\partial \tau} + a U \frac{\partial u}{\partial \xi} + b \frac{\partial^3 u}{\partial \xi^3} = 0$$

where we can see that by means of the scaling $\xi \rightarrow \xi b^{1/3}$ and $u \rightarrow u/ab^{-1/3}$ that the coefficients a and b may be assumed to be of value unity. Here u represents a wave amplitude. This equation was rediscovered by: Washimi and Taniuti (1966) when considering the propagation of disturbances near the ion sound speed in a collisionless plasma of cold ions and warm electrons, Zabusky (1963) and Kruskal and Zabusky (1963) for the propagation of one dimensional acoustic waves in anharmonic crystals, Gardner and Morikawa (1960) for the long-time behaviour of disturbances propagating perpendicular to a magnetic field with a velocity near the Alfvin wave velocity in a cold plasma hydromagnetic model, Karpman (1967) for fluid flow in two dimensions around regular bodies in dispersive media.

The important characteristic of systems for which the KdV equation has been found to be valid is that they are weakly dispersive in the long wavelength limit. This in mathematical terms means that the linear dispersion relation takes the form

$$\omega = \lambda_o k \pm \alpha k^3$$

where ω and k are the frequency and wavenumber respectively, i.e. in the limit $k \rightarrow o$ the phase velocity is constant and for short wavelengths the correction is $O(k^3)$. It is obvious that for these systems that the nonlinear steepening associated with these systems is balanced by dispersive effects, i.e. wave breaking, the point at which the characteristics of the original system cross and the amplitude becomes double valued is prevented by the dispersive effects.

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This gives rise to the solitary wave or soliton originally discovered by Lord Rayleigh (1876). To see how this arises we note that if the third terms of the KdV equation is ignored then the resulting equation has the implicit solution

$$u(\xi,\tau) \; = \; u(\xi - u(\xi,\tau)\tau,o)$$

Therefore any initial disturbance will steepen in regions where $\partial u(\xi,o)/\partial \xi < 0$ and must lead to physically untenable multivalued solutions. The existence of the third term i.e. the $\partial^3 u/\partial \xi^3$ term prevents this occurring. We note that at this stage dissipative effects are not included in the discussion.

The solitary wave solutions of the KdV equation were rediscovered numerically by Zabusky and Kruskal (1965) and analytically by Gardner, Greene, Kruskal and Miura (1967). The analytic solutions were found by treating the KdV equation as an inverse scattering problem for arbitrary initial conditions and a good discussion of this method is given by Davidson (1972). By integrating the KdV equation twice, by insisting that u and its derivatives vanish as $|X| \to \infty$ and by looking for solutions as functions of X where

$$X \; = \; \xi - c\tau$$

we see that

$$u(\xi - c\tau) \; = \; 3c \; \mathrm{sech}^2 \left[\frac{1}{2}(\xi - c\tau)\sqrt{c} \right]$$

which is the solitary wave, i.e. a moving pulse where the pulse amplitude, width and velocity are proportional to c, $c^{-\frac{1}{2}}$ and c respectively. Further interesting properties of the soliton are discussed in 1.2.

The second equation, the Burgers equation we will only briefly mention as it is the least relevant to physical problems of the three equations. This equation was derived by Burgers (1954) for a simple hydrodynamical system and analysed in considerable detail in subsequent papers. This equation takes the form:

$$\frac{\partial u}{\partial t} \; + \; \alpha \, u \, \frac{\partial u}{\partial x} \; - \; \mu \, \frac{\partial^2 u}{\partial x^2} \; = \; 0$$

where we consider the last term as being a "viscous" term. This term is therefore characteristic of a dissipative system. The B equation has a solution of the form

$$u^{\prime} = u^{\prime}_{\infty} \tanh\left[-(u^{\prime}_{\infty}x^{\prime})/2\mu\right]$$

where
$$u' = u(x-\lambda t) - \lambda$$

and
$$u^{\prime}_{\infty} = u^{\prime}(x' = \infty) , \qquad x^{\prime} = x - \lambda t$$

and α has been put equal to unity.

This is a shock like solution since u' increases monotonically from $|u^{\prime}_{\infty}|$ to $u^{\prime}_{\infty}$ as x increases from $-\infty$ to $+\infty$. Therefore, the nonlinearity and dissipation balance, to prevent wave breaking and a steep but smooth wave develops and progresses. We can therefore conclude that the B equation is characteristic of weakly dissipative systems and this has been shown to be valid for a number of systems by Su and Gardner (1969). Before we consider the NLS equation we note that attempts have been made to consider both dissipative and dispersive effects and modified KdV equations have been found by Ott and Sudan (1969,1970).

The third equation, the NLS equation is the most important equation as far as this thesis is concerned and was originally proposed by Landau (1944) as being a general equation describing nonlinear systems. Since then it has been derived by a number of authors: Taniuti and Washimi (1968) for the self-trapping in intense light beams when an electromagnetic wave is trapped by the polarisation induced by itself; Chiao, Garmire and Townes (1969) for a similar system; Hocking and Stewartson (1972) and Stewartson and Stuart (1971) for a stability study of plane parallel flows in fluids; Zakharov and Shabat (1972) for general self-focussing problems; Asano, Taniuti and Yajima (1969) for the electron plasma wave in an isothermal electron fluid and for a model nonlinear Klein Gordon equation and many other authors. The common characteristic of all these systems is that now nonlinear self-modulation effects are important and hence the systems are all strongly dispersive. The general form of the NLS equation is given by

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$$i \frac{\partial \phi}{\partial \tau} + \alpha \frac{\partial^2 \phi}{\partial \xi^2} + \beta \phi + \gamma \phi |\phi|^2 = 0$$

where α, β and γ are constants, ϕ is a wave amplitude and τ and ξ are suitably chosen coordinates. This equation admits solutions which include the solitary wave, the shock and different types of periodic nonlinear waves. This equation is considered in more detail but at this point we note that Zakharov and Shabat (1972) have shown that the solitary wave for the NLS equation differs from the KdV solitary wave because even though both waves are characterised by four parameters for the NLS soliton the characteristic amplitude and velocity are independent. The two solitary waves are more fundamentally different since the NLS soliton is an envelope solitary wave, i.e. the envelope of a modulated quasi-monochromatic wave whereas the KdV soliton is a pure solitary wave.

We can conclude at this point that the KdV, the B and the NLS equations have been shown to be the equations satisfied by the amplitude in weakly dispersive, weakly dissipative and strongly dispersive systems if these effects are of equal significance to the nonlinearity of the system. When the validity of this result was established attempts were made to demonstrate their validity for more general systems using more general methods. The most important techniques used were a wave packet formalism, a Lagrangian formalism, Whithams method of slowly varying amplitudes, the multiple time scale formalism and the reductive perturbation technique (in future referred to as the RP technique). The RP technique, which belongs to a general class of multiple time scale methods will be considered in more detail in 1.2 and we will briefly describe the other techniques here.

Karpman and Kruskal (1969) using the concept of a wave packet, localised in space and with a varying phase were able to show that the NLS equation is valid for nonlinear wave modulation in dispersive systems but were unable to give explicit expressions for the coefficients. Kono and Sanuki (1972) extended this method and were able to derive the KdV equation and the NLS equation from the Vlasov equation.

Dewar (1972) showed, by deriving a Lagrangian for the slowly varying complex amplitude of an almost monochromatic electrostatic plasma wave in an unmagnetised plasma and by a technique originally proposed by Whitham (1967, 1965) that variations with respect to the amplitude of this Lagrangian lead to the NLS equation. This was again valid for the case of nearly monochromatic waves. Whitham himself, Whitham (1967) considered a Lagrangian method and used an averaging procedure to derive a single nonlinear equation. However, the result he obtained was not a NLS equation as his original method was only valid for systems where the nonlinearity exceeds the dispersion. Dewar has therefore extended the applicability of the Lagrangian method to include systems where the two effects are of equal importance.

Whitham (1967) using results of Whitham (1965) and Lighthill (1965) considered an exact uniform periodic wave train and assumed that the amplitude and wave number were slowly varying functions of space and time. By averaging over the local oscillations of the medium he was able to find an amplitude dependent frequency which as we will see later can be deduced from the NLS equation. Tam (1969,1970) has used this concept of amplitude dispersion to consider nonlinear dispersion in cold plasma waves but has' not considered modulational effects.

The multiple time and space scale method has proved to be a very powerful technique for considering nonlinear systems. This technique consists of introducing a number of differing time and space scales and by treating these as independent variables. Then, if a correct choice is made for the scaling of these variables a sequence of linear equation is obtained which describe the behaviour of the system on different time scales. Normally, to second order secular terms arise and by insisting that these terms vanish we derive the equation obeyed by the first order amplitude on the slowest time and space scales. Discussions and details of this method can be found in Sandri (1963) and Sturrock (1957) and Davidson (1972) has showed how the KdV equation may be derived for the nonlinear ion sound wave using the multiple time scale method.

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The problem with using ordinary perturbation methods for nonlinear modu-
lated waves is that they treat all interaction terms as small quantities and
do not distinguish between self and mutual interactions, and between interac-
tions due to nonlinearities or dispersion or dissipation. Therefore new
methods are required which remove secular terms which can lead to divergence
and which isolate the different interactions in order that their relative
effects can be separately identified.

The RP technique and its modifications has been the most successful
single technique which is able correctly to account for the interactions of
dispersive effects, dissipative effects and modulational effects. This
technique was originally proposed by Taniuti and Wei (1968) and Taniuti and
Yajima (1969) and is called the RP technique since it reduces a system of
nonlinear equations to a single tractable nonlinear equation using a singular
perturbation expansion. As will be seen the method is elegant and relatively
simple and is based on a multiple time and space scale method (by using
coordinate stretching). This technique correctly predicts the validity of
the KdV, the B and the NLS equation for the respective dominant interaction
for a very general system of coupled nonlinear differential equations.
This class of equations includes all the physical systems for which these
equations have previously been deduced which we have already referred to.
Although this method has proved very powerful it has not yet been applied to
unstable systems, marginally stable systems, inhomogeneous (spatially and
temporally) systems and two-dimensional systems which are strongly dispersive
i.e. where dispersion and wave modulation are also important. It is the
aim of this thesis to make some of these extensions and to demonstrate their
use on physical systems.

Before we consider the RP technique in more detail we note that if
solutions of the final nonlinear wave equation can be found then a statement
can be made about the nonlinear stability or instability of a system. The
possibility that a system may be linearly stable and nonlinearly unstable

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(or vice versa) is one of the interesting properties of the nonlinear wave
as was mentioned at the beginning of this section.   An excellent example
of this is the stability of plane parallel fluid flows as considered by
Diprima, Eckha s and Segel (1971), Stewartson and Stuart (1971) and Hocking
and Stewartson (1972) who by deriving the NLS equation were able to deduce
a nonlinear stability criterion which was the converse of the linear criterion.
We will consider such a system in subsequent Chapters but will now outline the
principles of the RP technique.

1.2   The reductive perturbation technique

In this section we consider the reductive perturbation technique as
originally proposed by Taniuti and Wei (1968) and extended by Taniuti and
Yajima (1969) and Asano, Taniuti and Yajima (1969).   The aim of Taniuti and
Wei (1968) was to show that a class of systems of nonlinear differential
equations can be reduced to a single differential equation to lowest order
in a singular perturbation expansion.   This equation was found to be either
the B equation or the KdV equation depending on the choice of a particular
parameter.   Taniuti and Wei (1968) made a number of assumptions and statements
without justification and we will start by justifying their approach.   They
wanted to account for the interaction of dispersive or dissipative effects
and nonlinear effects and needed a coordinate transformation or stretching
which accounted for this interaction.   They considered the system of equations:

$$\frac{\partial u}{\partial t} + \underline{A}(\underline{u}) \frac{\partial u}{\partial x} + \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} (\underline{H}_\alpha^{\beta} \frac{\partial}{\partial t} + \underline{K}_\alpha^{\beta} \frac{\partial}{\partial x}) \cdot \underline{u} = 0 \qquad 1.2.1$$

where $\underline{u}$ is a column vector with n components $\underline{u}_1 \ldots \underline{u}_n$ (n $\geqslant$ 2) and $\underline{A}$, $\underline{H}_\alpha^{\beta}$,
$\underline{K}_\alpha^{\beta}$ are nxn matrices, all functions of $\underline{u}$ and p $\geqslant$ 2.   All these matrices are
assumed continuous and analytic.

The dispersion relation in the long wavelength approximation for 1.2.1
is obtained by linearising around a stationary state $\underline{u}_0$.   This dispersion
relation may then be solved to give the phase velocity by using the long
wavelength approximation.   This phase velocity is then given by:

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$$\omega/k = \lambda_o + ak^{p-1} + bk^{2(p-1)} + ck^{3(p-1)} + \ldots \qquad 1.2.2$$

where ω is the frequency, k the wavenumber, λ_o is a real eigenvalue of $\underline{\underline{A}}(\underline{u}^o)$ and the a, b and c are functions of the left and right eigenvectors of $\underline{\underline{A}}(\underline{u}^o)$ (corresponding to the eigenvalue λ_o) and the matrices $\underline{\underline{H}}_\alpha^{\ \beta}(\underline{u}_o)$, $\underline{\underline{K}}_\alpha^{\ \beta}(\underline{u}_o)$.

Conversely if $\underline{u}$ is expanded as a power series in terms of a small parameter ϵ then we may write

$$\underline{u} = \underline{u}_o + \epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + \ldots\ldots$$

$$\underline{\underline{A}} = \underline{\underline{A}}_o + \epsilon \underline{\underline{A}}_1 + \epsilon^2 \underline{\underline{A}}_2 + \ldots\ldots$$

where $\underline{\underline{A}}_o = \underline{\underline{A}}(\underline{u}_o)$. This implies that the phase velocity may be expanded as

$$\lambda = \lambda_o + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \ldots\ldots \qquad 1.2.3$$

where the λ_1, λ_2 are again proportional to $\underline{\underline{A}}_o$ and its eigenvectors.

The two approximate expansions 1.2.2 and 1.2.3 may now be equated by order. We note that 1.2.2 is an expansion which represents the effect of dispersion whilst 1.2.3 represents the effect of the nonlinearity. We may therefore conclude that the interaction between the nonlinearity and the dispersive effects is greatest when the time of interaction is longest, i.e. when

$$0(k) = \epsilon^\alpha$$

where
$$\alpha = 1/(p-1) \qquad 1.2.4$$

providing $\lambda_1 \neq 0$ and $a \neq 0$. From 1.2.4 we see that ϵ^α x wavelength are of the order of unity in the frame of reference moving with the wave and so we can write

$$\xi = \epsilon^\alpha(x - \lambda_o t) \qquad 1.2.5$$

as one of the stretched coordinates. The other independent variable t of 1.2.1 must now be stretched in a consistent manner as follows: The characteristics of 1.2.1 in terms of the original variables x and t (if the nonlinearity

is ignored) are given by $dx/dt = \lambda_o$. We expect a deviation in these characteristics to order ε when the nonlinearity is included and so expect the characteristic λ_1 to be written as $\lambda_1 = d\xi/d\tau$ where τ is now a stretched time variable. This gives immediately,

$$\tau = \varepsilon^{\alpha+1} t \qquad 1.2.6$$

The two stretched coordinates 1.2.5 and 1.2.6 are now consistent with the balance between the nonlinearity and dispersion or dissipation. As we have already noted these new coordinates apply only when $\lambda_1 \neq 0$ and $a \neq 0$. If $a = 0$ but $b \neq 0$ and $\lambda_1 \neq 0$ then the nonlinearity and dispersive effects are coupled to next highest order in the dispersion i.e.

$$O(k) = \varepsilon^{\alpha}$$

where now

$$\alpha = 1/2(p-1) \qquad 1.2.7$$

The stretching 1.2.5 and 1.2.6 is now still appropriate with α now being given by 1.2.7.

We now return to the model equation 1.2.1 and make two assumptions. We assume that the constant unperturbed solution $\underline{u}_o$ exists and that the $\underline{u}$, $\underline{\underline{A}}$, $\underline{\underline{H}}_\alpha^\beta$ and $\underline{\underline{K}}_\alpha^\beta$ may be expanded as power series as follows:

$$\underline{u} = \underline{u}_o + \varepsilon \underline{u}_1 + \varepsilon^2 \underline{u}_2 + \ldots \qquad 1.2.8(a)$$

$$\underline{\underline{A}} = \underline{\underline{A}}_o + \varepsilon \underline{\underline{A}}_1 + \varepsilon^2 \underline{\underline{A}}_2 + \ldots \qquad 1.2.8(b)$$

and similarly for $\underline{\underline{H}}_\alpha^\beta$ and $\underline{\underline{K}}_\alpha^\beta$. Now since $\underline{\underline{A}}$ is a function of $\underline{u}$ we may write 1.2.8 as

$$\underline{\underline{A}} = \underline{\underline{A}}_o + \varepsilon \nabla \underline{\underline{A}}^o \cdot \underline{u}^1 + \varepsilon^2 (\nabla \underline{\underline{A}}^o \cdot \underline{u}^2 + \frac{1}{2} \nabla \underline{\underline{A}}^o : \underline{u}^1 \underline{u}^1)$$

$$+ \varepsilon^3 (\nabla \underline{\underline{A}}^o \cdot \underline{u}^3 + \nabla \nabla \underline{\underline{A}} : \underline{u}^1 \underline{u}^2 + \frac{1}{6} \nabla \nabla \nabla \underline{\underline{A}}^o : \underline{u}^1 \underline{u}^1 \underline{u}^1) + \ldots \qquad 1.2.9$$

where we have adopted the notation.

$$\nabla \underline{\underline{A}}^o \cdot \underline{u}^1 = \sum_{i=1}^{n} \left(\frac{\partial \underline{\underline{A}}}{\partial u_i} \right)_{\underline{u}=\underline{u}^o} u_i^1$$

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$$\nabla\nabla\underline{\underline{A}}:\underline{u}^1\underline{u}^1 \;=\; \sum_i \sum_j \left(\frac{\partial^2 \underline{\underline{A}}}{\partial u_i \partial u_j}\right)_{\underline{u}=\underline{u}^o} u_i^1 u_j^1$$

Similar expressions hold for higher derivatives. (This notation is used throughout the remainder of this thesis). These expansions assume that the u_1, u_2 are p times differentiable with respect to x and t. Secondly we assume that the eigenvalues of $\underline{\underline{A}}_o$ are real, that at least one non-degenerate eigenvalue λ_o exists and the eigenspace of $\underline{\underline{A}}_o$ does not have any invariant subspaces. This last assumption was made by Taniuti and Wei without explanation and they considered the case when $\underline{\underline{A}}_o$ has invariant subspaces as an exceptional case. We can now see that these conditions on $\underline{\underline{A}}$ are necessary to ensure that the dispersion relation has the form given by 1.2.2 with $\lambda_o \neq 0$ and $a \neq 0$. Their "exceptional" case arises when $a = 0$ as we shall see later.

We now introduce the stretched coordinates

$$\xi \;=\; \varepsilon^{\alpha}(x-\lambda_o t)$$

$$\tau \;=\; \varepsilon^{\alpha+1}t$$

where $\qquad \alpha \;=\; 1/(p-1)$

together with the expansions 1.2.8 and 1.2.9 into 1.2.1 to obtain

$$\sum_{i=1}^{\infty} \varepsilon^i(-\lambda_o \frac{\partial \underline{u}_i}{\partial \xi}) + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \varepsilon^{i+j}\underline{\underline{A}}_i \frac{\partial \underline{u}_j}{\partial \xi} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \varepsilon^{i+j+1}\underline{\underline{A}}_i \frac{\partial \underline{u}_j}{\partial \tau}$$

$$+ \sum_{\beta}\sum_{q=0}\sum_{j=1}\sum_{i=0} \varepsilon^{q+i+j+1} d_{qi}^{\beta}\underline{u}_j \;=\; 0 \qquad\qquad 1.2.10$$

where d_{qi}^{β} is formally defined by

$$\prod_{\alpha}^{p}(\underline{\underline{H}}_{\alpha}^{\beta} \frac{\partial}{\partial t} + \underline{\underline{K}}_{\alpha}^{\beta} \frac{\partial}{\partial x}) \;=\; \varepsilon^{p\alpha}\sum_{q=0}^{p}\sum_{k=0}^{\infty} \varepsilon^{k+q} d_{qk}^{\beta} \qquad\qquad 1.2.11$$

i.e. d_{qk}^{β} is the kth term in a power series expansion of the operator on the left hand side of 1.2.11. We now equate powers of ε in 1.2.10 to second order to obtain the following two sets of equations:

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$$(-\lambda_o \, \underline{\underline{I}} + \underline{\underline{A}}_o) \, \frac{\partial \underline{u}_1}{\partial \xi} = 0 \qquad\qquad 1.2.12$$

$$(-\lambda_o \, \underline{\underline{I}} + \underline{\underline{A}}_o) \, \frac{\partial \underline{u}_2}{\partial \xi} + \underline{\underline{A}}_o \, \frac{\partial \underline{u}_1}{\partial \tau} + (\nabla \underline{\underline{A}}^o \cdot \underline{u}_1) \, \frac{\partial \underline{u}_1}{\partial \xi}$$

$$+ \sum_{\beta}^{s} \sum_{\alpha}^{p} (-\lambda_o \, \underline{\underline{H}}_{\alpha o}^{\beta} + \underline{\underline{K}}_{\alpha o}^{p}) \, \frac{\partial^p \underline{u}_1}{\partial \xi^p} = 0 \qquad\qquad 1.2.13$$

We will now solve 1.2.12 and by means of a compatibility condition that must be satisfied by 1.2.13 (so that $\partial \underline{u}_2 / \partial \xi$ has a non-trivial solution) deduce the equation satisfied by $\underline{u}_1$. We note that it is a feature of the reductive perturbation technique that such a compatibility condition gives the required result. This is directly comparable to the technique of removing divergent terms in the multiple time scale method (discussed in 1.3) but in this formulation the compatibility condition arises naturally and is not externally imposed.

Since by assumption λ_o is an eigenvalue of $\underline{\underline{A}}_o$ we may define a left $\underline{L}$ and right $\underline{R}$ eigenvector of $\underline{\underline{A}}_o$ corresponding to this eigenvalue as:

$$(\underline{\underline{A}}_o - \lambda_o \, \underline{\underline{I}}) \, \underline{R} = 0$$

$$\underline{L}(\underline{\underline{A}}_o - \lambda_o \, \underline{\underline{I}}) = 0$$

Equation 1.2.12 may be immediately solved to give:

$$\underline{u}_1 = \underline{R}\phi(\xi,\tau) + \underline{V}(\tau) \qquad\qquad 1.2.14$$

where $\phi(\xi,\tau)$ is a scalar function of ξ and τ and $\underline{V}(\tau)$ is a vector function of τ which appears as a constant of integration and which is determined if initial conditions are imposed on $\underline{u}_1$.

Since we insist that 1.2.13 has a non-trivial solution for $\partial \underline{u}_2 / \partial \xi$, we consider 1.2.13 as the inhomogeneous form of 1.2.12 (with respect to the matrix operator $(-\lambda_o \, \underline{\underline{I}} + \underline{\underline{A}}_o)$) and so see that the inhomogeneous terms must satisfy the compatibility condition. The explicit form of this condition is readily deduced by multiplying 1.2.13 on the left by $\underline{L}$ to obtain,

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$$\underline{L}(-\lambda_0 \underline{\underline{I}} + \underline{\underline{A}}_0) \frac{\partial \underline{u}_2}{\partial \xi} + \underline{L}\, \underline{\underline{A}}_0 \frac{\partial \underline{u}_1}{\partial \tau} + \underline{L}(\nabla \underline{\underline{A}}^0 \cdot \underline{u}_1) \frac{\partial \underline{u}_1}{\partial \xi}$$

$$+ \underline{L} \sum_\alpha^s \sum_\beta^p (-\lambda_0 \underline{\underline{H}}^\beta_{\alpha o} + \underline{\underline{K}}^\beta_{\alpha o}) \frac{\partial^P \underline{u}_1}{\partial \xi^P} = 0$$

By definition the first term vanishes and leaves the equation that must be satisfied by $\underline{u}_1$. We substitute from 1.2.14 and after some manipulation obtain:

$$\frac{\partial \phi}{\partial \tau} + c_1 \phi \frac{\partial \phi}{\partial \xi} + c_2 \frac{\partial^P \phi}{\partial \xi^P} + \underline{c}_3 \frac{\partial \underline{V}}{\partial \tau} + c_4 \frac{\partial \phi}{\partial \xi} = 0 \qquad\qquad 1.2.15$$

where the constants c_1, c_2 and c_4 are given by:

$$c_1 = \frac{\underline{L}(\nabla \underline{\underline{A}}^0 \cdot \underline{R}) \underline{R}}{\lambda_0 \underline{L}\, \underline{R}}$$

$$c_2 = \frac{\underline{L} \sum_\beta^s \prod_\alpha^p (-\lambda_0 \underline{\underline{H}}^\beta_{\alpha o} + \underline{\underline{K}}^\beta_{\alpha o}) \underline{R}}{\lambda_0 \underline{L}\, \underline{R}}$$

$$c_4 = \frac{\underline{L}\,(\nabla \underline{\underline{A}}_0 \cdot \underline{V}_1) \underline{R}}{\lambda_0 \underline{L}\, \underline{R}}$$

and the vector $\underline{c}_3$ is given by:

$$\underline{c}_3 = \frac{\underline{L}}{\underline{L}\, \underline{R}}$$

Taniuti and Wei now simplify 1.2.15 by eliminating the last two terms by use of a coordinate transformation and a variable transformation which is seen to be a local Galilean transformation as follows:

$$\widetilde{\phi}(\xi,\tau) = \phi(\xi,\tau) + \underline{c}_3 \underline{V}$$

$$\eta = \xi + c_1 \underline{c}_3 \int^\tau \underline{V}(x')\,dx' - \int^\tau c_4(x')\,dx'$$

to finally obtain

$$\frac{\partial \widetilde{\phi}}{\partial \tau} + c_1 \widetilde{\phi} \frac{\partial \widetilde{\phi}}{\partial \eta} + c_2 \frac{\partial^P \widetilde{\phi}}{\partial \eta^P} = 0 \qquad\qquad 1.2.16$$

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For p = 3 this equation reduces to the KdV equation and for p = 2 the B equation. We can see that if $\tilde{\phi}$ is specified at some (η_o, τ_o) then $\tilde{\phi}$ is determined for all $\eta > \eta_o$ and $\tau > \tau_o$.

Taniuti and Wei then proceed to examine the case when $\underline{\underline{A}}_o$ has invariant subspaces. If this is the case then the matrix given by the d_{qk}^{β} also has invariant subspaces which are interchanges of the subspaces of $\underline{\underline{A}}_o$. Consequently c_2 vanishes and the method can be seen to have failed. Taniuti and Wei now assume the coordinate stretching given by 1.2.5 and 1.2.6 with the definition of α given by 1.2.7 and derive the equation 1.2.16 in the form

$$\frac{\partial \tilde{\phi}}{\partial \tau} + c_1 \tilde{\phi} \frac{\partial \tilde{\phi}}{\partial \eta} + c_2 \frac{\partial^{2p-1} \tilde{\phi}}{\partial \eta^{2p-1}} = 0$$

where the constants c_1 and c_2 are defined by expressions similar to those given above. We will not give any further consideration to this case other than point out that the analysis proceeds exactly as given above with two equations to each order for two vectors $\underline{u}^+$ and $\underline{u}^-$ where $\underline{u} = (\underline{u}^+, \underline{u}^-)^T$ and where the dimensions of $\underline{u}^+$ and hence $\underline{u}^-$ depends on the relative dimensions of the invariant subspaces of $\underline{\underline{A}}_o$. Taniuti and Wei then apply the method to the ion acoustic wave in a cold collisionless plasma and a hydrodynamic wave in an isothermal fluid. They found that the KdV equation is appropriate to the latter system whereas the B equation is appropriate to the former. They confirmed the result of Washimi and Taniuti (1966) for the ion acoustic wave, a result which had been obtained using a less general form of coordinate stretching.

The assumptions of the general RP technique presented above indicate that it is most applicable to systems in the long wavelength approximation in the presence of dispersion or dissipation. This is evident from 1.2.2, the dispersion relation which was derived in the long wavelength limit. We therefore must have a dispersion relation of the form,

$$\omega = \lambda_o k + O(k^3)$$

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for a non-dissipative system. This restriction also applied to the work
of Su and Gardner (1969) who also concluded that the KdV equation was the
appropriate equation describing nonlinear dispersive systems whilst the B
equation was appropriate for dissipative systems. Similarly we must have
a dispersion relation of the form

$$\omega = \lambda_o k + O(k^2)$$

for dissipative systems. The method is not applicable to systems having
dispersion relations of the form

$$\omega = \omega_o + \lambda_o k + O(k^2) + \ldots$$

where there is a fundamental frequency ω_o. Systems having dispersion rela-
tions of this form allow the propagation of a nearly monochromatic wave and in
addition to nonlinear, dispersive and dissipative effects, self-modulation
effects must also be considered. The next significant development in the
RP technique was given by Taniuti and Yajima (1969) and Asano, Taniuti and
Yajima (1969) who indicated how the technique may be applied to systems having
a dispersion relation which admits, in the linear approximation   a plane wave
with a characteristic oscillation frequency ω_o. Again, these authors stated
their assumptions without justification and before we discuss the method in
detail we will indicate why their choice of expansion and coordinate stretching
is appropriate.

   If the linear system admits a plane wave solution then we expect the
effect of the nonlinearity and/or dispersion to modulate the wave to produce
a nearly monochromatic wave. The envelope of this wave can then be considered
as a long wavelength wave if the deviation from a monochromatic wave is
sufficiently small. We now consider an expression similar to 1.2.2 for this
wave in the long wavelength limit as follows: Consider the group velocity of
the wave consisting of two plane waves characterised by wavenumbers and wave-
lengths k, ω and k', ω'. If the differences between these are sufficiently

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small, i.e. if $K = k'-k$ and $\Omega = \omega'-\omega$ are small then the resultant wave has an envelope of wave number K and frequency Ω subject to the dispersion relation:

$$\Omega = \lambda_o K + \frac{1}{2}\frac{\partial^2 \omega}{\partial k^2}K^2 + \frac{1}{6}\frac{\partial^3 \omega}{\partial k^2}K^3 + \ldots$$

where λ_o is the group velocity of the carrier wave. In the long wavelength limit we write the group velocity of the envelope as $\partial\Omega/\partial k = \Lambda$ and so write

$$\Lambda = \lambda_o + \frac{\partial^2 \omega}{\partial k^2}K + \frac{1}{2}\frac{\partial^3 \omega}{\partial k^3}K^2 + \ldots \qquad 1.2.17$$

We may now compare 1.2.17 with 1.2.3 in the same way as we compared 1.2.2 with 1.2.3 and conclude that providing $\lambda_1 \neq 0$ and $\partial^2\omega/\partial k^2 \neq 0$ that the choice of stretched coordinates 1.2.5 and 1.2.6 is still appropriate since for maximum interaction the nonlinearity and dispersion must be of the same order, i.e. the envelope wave is a function along the characteristic curve given by 1.2.5. However, this simple coordinate stretching must be combined with a method of accounting for the interaction of the fundamental mode with its own harmonics. The nonlinear self-interactions of the plane wave will give rise to higher and lower harmonics. These harmonics will first appear to second order in the amplitude of the plane wave as the second harmonic mode and a slow mode with no harmonic content. These then couple with the fundamental mode to give a nonlinear modulation. This nonlinear effect then only appears to third order in the amplitude. We note that now a calculation to third order will be required to determine the behaviour of the lowest order amplitude.

The combination of these arguments justifies the following choice of perturbation expansion and was originally suggested by Taniuti and Yajima (1969):

$$\underline{u} = \underline{u}_o + \sum_{\alpha=1}^{\infty}\sum_{\ell=-\infty}^{\infty}\varepsilon^\alpha \underline{u}_\ell^\alpha(\xi,\eta)\exp\left[i\ell(kx-\omega t)\right] \qquad 1.2.18$$

The validity of this expansion and subsequent modifications are the crucial

assumptions of this thesis.   We note that the amplitude is a function of the
stretched variables:

$$\xi \;=\; \epsilon(x - \lambda t)$$

$$\tau \;=\; \epsilon^2 t \qquad\qquad 1.2.19$$

with $\qquad\qquad \lambda \;=\; \partial\omega/\partial k$

(where we have now restricted the choice of systems to those having a disper-
sion relation with terms of order k^2).   However, the oscillatory part is a
function of x and t.   Therefore, as is required we have "decoupled" the rapid
oscillations of the wavemotion with slow variations of the amplitude and have
chosen the slow variables to give the correct balance between the nonlinear
self-interaction and the dispersive effects.

For any choice of system we now wish to derive an equation which describes
the modulation of the wave and following Taniuti and Yajima consider the model
system of equations:

$$\frac{\partial \underline{u}}{\partial t} + \underline{\underline{A}}(\underline{u}) \frac{\partial \underline{u}}{\partial x} + \underline{B}(\underline{u}) \;=\; 0 \qquad\qquad 1.2.20$$

where $\underline{u}$ is a column vector with n components u_1, u_2 ..... u_n and where the n x n
matrix $\underline{\underline{A}}$ and the column vector $\underline{B}$ are continuous and differentiable functions of
the $u_1's$.   We assume that there exists a constant solution $\underline{u}^o$ which satisfies

$$\underline{B}(\underline{u}^o) \;=\; 0$$

Then equation 1.2.20 linearised about $\underline{u}^o$ becomes:

$$\frac{\partial \underline{u}}{\partial t} + \underline{\underline{A}}_o \frac{\partial \underline{u}}{\partial x} + \nabla \underline{B}_o \underline{u} \;=\; 0$$

where the operator ∇ was defined following equation 1.2.9.   This linearised
equation allows a plane wave solution, i.e. varying as exp $\pm$ i(kx$-\omega$t) subject
to the dispersion relation:

$$\det \left| \mp i\omega \underline{\underline{I}} \pm ik\underline{\underline{A}}_o + \nabla \underline{B}_o \right| \;=\; 0$$

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We now assume that this dispersion relation gives at least a single real root ω for a real k. Further, we assume that $\ell\omega$ where ℓ is a positive or negative integer is not a root of this dispersion relation, i.e.

$$\det \underline{\underline{W}}_\ell \neq 0 \quad \text{for} \quad |\ell| \neq 1 \quad\quad\quad 1.2.21$$

where

$$\underline{\underline{W}}_\ell = -i\ell\omega\underline{\underline{I}} + i\ell k\underline{\underline{A}}_o + \nabla\underline{\underline{B}}_o$$

Having defined a system which is characterised by a single real frequency ω we are now able to consider the nonlinear interaction of this wave with its own harmonics. We substitute the expansion 1.2.18 into the model system 1.2.20 and change the independent variables to the stretched coordinates 1.2.19. Then equating powers of ε of the same harmonic gives an infinite set of equations of which we will require the first three, which are given as:

$$\underline{\underline{W}}_\ell \underline{u}_\ell^1 = 0 \quad\quad\quad 1.2.22$$

$$\underline{\underline{W}}_\ell \underline{u}_\ell^2 + (-\lambda\underline{\underline{I}} + \underline{\underline{A}}_o) \frac{\partial \underline{u}_\ell^1}{\partial \xi} + \nabla\underline{\underline{A}}_o \langle \underline{u}^1 \sum_{\ell'=-\infty}^{\infty} i\ell' k \underline{u}_{\ell'}^1 \exp i\ell'(kx-\omega t)\rangle_\ell$$

$$+ \frac{1}{2}\nabla\nabla\underline{\underline{B}}_o : \langle \underline{u}^1 \underline{u}^1 \rangle_\ell = 0 \quad\quad\quad 1.2.23$$

$$\underline{\underline{W}}_\ell \underline{u}_\ell^3 + (-\lambda\underline{\underline{I}} + \underline{\underline{A}}_o) \frac{\partial \underline{u}_\ell^2}{\partial \xi} + \frac{\partial \underline{u}_\ell^1}{\partial \tau} + \nabla\underline{\underline{A}}_o \langle \underline{u}^1 \frac{\partial \underline{u}^1}{\partial \xi}\rangle_\ell$$

$$+ \nabla\underline{\underline{A}}_o \langle \underline{u}^1 \sum_{\ell'} i\ell' k \underline{u}_{\ell'}^2 \exp i\ell'(kx-\omega t)\rangle_\ell + \frac{1}{2}\nabla\nabla\underline{\underline{A}}_o \langle : \underline{u}^1 \underline{u}^1 \sum_{\ell'=-\infty}^{\infty} i\ell' k \underline{u}_{\ell'}^1 \exp i\ell'(kx-\omega t)\rangle_\ell$$

$$+ \nabla\underline{\underline{A}}_o \langle . \underline{u}^2 \sum_{\ell=-\infty}^{\infty} i\ell' k \underline{u}_{\ell'}^1 \exp i\ell'(kx-\omega t)\rangle_\ell + \nabla\nabla\underline{\underline{B}}_o : \langle \underline{u}^1 \underline{u}^2 \rangle_\ell$$

$$+ \frac{1}{6}\nabla\nabla\nabla\underline{\underline{B}}_o \vdots \langle \underline{u}^1 \underline{u}^1 \underline{u}^1 \rangle_\ell = 0 \qu\quad\quad 1.2.24$$

where $\langle \ldots \rangle_\ell$ denotes the coefficient of the ℓth harmonic i.e. for any function Q

$$Q = \sum_{\ell=-\infty}^{\infty} \langle Q\rangle_\ell \exp i\ell(kx-\omega t)$$

Combining 1.2.21 and 1.2.22 gives:

$$\underline{u}_\ell^1 = \phi(\tau,\xi)\underline{R} \qquad |\ell| = 1 \qquad\qquad 1.2.25a$$

and
$$\underline{u}_\ell^1 = 0 \qquad |\ell| \neq 1 \qquad\qquad 1.2.25b$$

where $\underline{R}$ is the right eigenvector of $\underline{W}_1$ corresponding to zero eigenvalue i.e.

$$\underline{W}_1\underline{R} = 0$$

and $\phi(\tau,\xi)$ is a scalar function of the stretched coordinates which is determined to higher order. In view of 1.2.25a and 1.2.25b we see that

$$\langle \sum_\ell \sum_{\ell'} \underline{u}_\ell^1 \underline{u}_{\ell'}^1 \ell^p \ell'^{p'} \exp i(\ell+\ell')(kx-\omega t)\rangle_1 = 0$$

for any integers p and p'. Hence equation 1.2.23 for $\ell=1$ simplifies to

$$\underline{W}_1\underline{u}_1^2 + (-\lambda\underline{I} + \underline{A}_o)\underline{R}\frac{\partial\phi}{\partial\xi} = 0 \qquad\qquad 1.2.26$$

Since $\det\underline{W}_1 = 0$ this equation must satisfy a compatibility condition in order that it may be solved for $\underline{u}_1^2$. This condition is found by multiplying 1.2.26 by $\underline{L}$, the left eigenvector of $\underline{W}_1$ i.e.

$$L(-\lambda\underline{I} + \underline{A}_o)\underline{R} = 0 \qquad\qquad 1.2.27$$

where
$$\underline{L}\,\underline{W}_1 = 0 \qquad\qquad 1.2.28$$

Taniuti and Yajima show that this condition is automatically satisfied by differentiating 1.2.28 by k. In view of the definition of $\underline{W}_1$ and by multiplying the result on the left by $\underline{L}$ we can see that 1.2.27 is automatically satisfied. This condition is only satisfied if, and only if, $\lambda = \partial\omega/\partial k$ and so we could consider the compatibility condition as defining the velocity λ in the coordinate stretching 1.2.19. They then show that 1.2.26 may be solved to give:

$$\underline{u}_1^2 = \phi^{(2)}(\tau,\xi) + \underline{Z}(-\lambda\underline{I} + \underline{A}_o)\underline{R}\frac{\partial\phi}{\partial\xi} \qquad\qquad 1.2.29$$

where $\phi^{(2)}$ is another scalar function (to be determined to higher order) and $\underline{Z}$ is matrix which may be determined from the cofactors of $\det\underline{W}_1$. We will not

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go into the detailed analysis given by Taniuti,and Yajima needed to determine $\underline{Z}$ since we will present an alternative and more elegant form for 1.2.29 in Chapter 2.

For $|\ell| \neq 1$ 1.2.23 immediately gives:

$$\underline{u}_\ell^2 = 0 \quad \text{for} \quad |\ell| \geqslant 3 \qquad\qquad 1.2.30$$

since we can easily verify that

$$\langle \underline{u}^1 \underline{u}^1 \rangle_\ell = 0 \quad \text{for} \quad |\ell| \geqslant 3.$$

The remaining non-zero terms are $\underline{u}_o^2$ and $\underline{u}_2^2$ which are determined by direct matrix inversion of the $\ell = 2$ and $\ell = 0$ forms of 1.2.23, i.e.

$$\underline{u}_o^2 = -\underline{W}_o^{-1}\left[ik(\nabla\underline{A}_o \cdot \underline{R}^*)\underline{R} - ik(\nabla\underline{A}_o \cdot \underline{R})\underline{R}^* \right.$$
$$\left. + (\nabla\nabla\underline{B}^o : \underline{R}\,\underline{R}^*) \right] |\phi|^2 \qquad\qquad 1.2.31$$

and

$$\underline{u}_2^2 = -\underline{W}_2^{-1}\left[ik(\nabla\underline{A}_o \cdot \underline{R})\underline{R} + \tfrac{1}{2}\nabla\nabla\underline{B}_o : \underline{R}\,\underline{R} \right](\phi)^2 \qquad 1.2.32$$

This completes the determination of all components of the first and second order amplitudes in terms of ϕ but leaves $\underline{u}_1^2$ a function of the unknown scalar function $\phi^{(2)}$. However when we consider the third order equation which determines the equation satisfied by ϕ we find that the term depending on $\phi^{(2)}$ vanishes automatically. We therefore consider 1.2.24 for $\ell = 1$, mulitply by $\underline{L}$ on the left and substitute for $\underline{u}_1^1$, $\underline{u}_1^2$, $\underline{u}_o^2$ and $\underline{u}_2^2$ from 1.2.25a, 1.2.29, 1.2.31 and 1.2.32. The first term disappears as a result of 1.2.28 and hence eliminates the term depending on $\partial\phi^{(2)}/\partial\xi$. Again since $\langle \underline{u}^1 \underline{u}_1^2 \rangle_1 = 0$ the term depending on $\phi^{(2)}$ disappears and after some manipulation we finally obtain:

$$\alpha\,\frac{\partial\phi}{\partial\tau} + \beta\,\frac{\partial^2\phi}{\partial\xi^2} + \gamma\,\phi|\phi|^2 = 0 \qquad\qquad 1.2.33$$

where the constants α, β, γ are given by:

$$\alpha = \underline{L}\,\underline{R}$$

$$\beta = \underline{L}(-\lambda\underline{I} + \underline{A}_o)\underline{Z}(-\lambda\underline{I} + \underline{A}_o)\underline{R}$$

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$$\gamma = \underline{L}\left[ik\{2(\nabla\underline{A}_o \cdot \underline{R}^*)\underline{R}_2^2 - (\nabla\underline{A}_o \cdot \underline{R}_2^2)\underline{R}^* + (\nabla\underline{A}_o \cdot \underline{R}_o^2)\underline{R} \right.$$

$$+ (\nabla\nabla\underline{A}_o : \underline{R}\ \underline{R}^*)\underline{R} - \tfrac{1}{2}(\nabla\nabla\underline{A}_o : \underline{R}\underline{R})\underline{R}^*\}$$

$$\left. + \nabla\nabla\underline{B}_o : (\underline{R}\ \underline{R}_o^2 + \underline{R}^*\underline{R}_2^2) + \tfrac{1}{2}\nabla\nabla\nabla\underline{B}_o : \underline{R}\ \underline{R}^*\underline{R}\right]$$

where $\underline{R}_2^2$ and $\underline{R}_o^2$ are defined by:

$$\underline{R}_2^2 = -\underline{W}_2^{-1}\{ik(\nabla\underline{A}_o \cdot \underline{R})\underline{R} + \tfrac{1}{2}\nabla\nabla\underline{B}_o : \underline{R}\ \underline{R}\}$$

$$\underline{R}_o^2 = -\underline{W}_o^{-1}\{ik(\nabla\underline{A}_o \cdot \underline{R}^*)\underline{R} - ik(\nabla\underline{A}_o \cdot \underline{R})\underline{R}^* + \nabla\nabla\underline{B}_o : \underline{R}\ \underline{R}^*\}$$

Taniuti and Yajima then assume that α is pure real and that β and γ are real and finally obtain

$$i\ \frac{\partial\phi}{\partial\tau} + p\ \frac{\partial^2\phi}{\partial\xi^2} + q\ \phi|\phi|^2 = 0 \qquad\qquad 1.2.34$$

where $p = \beta/|\alpha|$ and $q = \gamma/|\alpha|$. This equation they called the nonlinear Schrödinger equation. We will show in Chapter 2 that the assumption that α is pure real is valid and will give simpler forms for the coefficients p and q. Certain solutions of 1.2.34 are known and may be written down by inspection.

If $\phi \to 0$ for $|\xi| \to \infty$ and p and q are of the same sign, then a solution of 1.2.34 is the solitary wave given by:

$$\phi = (-2\nu/q)^{\frac{1}{2}}\ \mathrm{sech}\{(-p/\nu)^{-\frac{1}{2}}\xi\}\exp(-i\nu\tau)$$

for arbitrary ν.

If $\phi \to \phi_o$, a constant for $|\xi| \to \infty$ then the solution is given by a plane wave

$$\phi = \phi_o\exp\{i(\mu\xi - E\tau)\}$$

where $\qquad E = p\mu^2 - q\phi_o^2$

for arbitrary μ. These plane wave solutions have been considered by Karpman and Krushkal (1969) who found that if p and q are of like sign that this plane

wave is unstable.   Conversely if p and q are of opposite sign then the plane
wave is stable.

A serious failing of the RP technique as presented above is that in most
physical systems the condition 1.2.21 is violated for $\ell = 0$.   If this arises
then the solutions for $\underline{u}^1_o$ i.e. $\underline{u}^1_o = 0$ and $\underline{u}^2_o$ given by 1.2.31 are not valid.
Taniuti and Yajima noted this and suggested that extraneous conditions such as
boundary conditions and subsidiary equations must be used to determine these
components.   This approach is used by Asano, Taniuti and Yajima (1969) when
they apply the RP technique to the problem of an electron plasma wave in an
isothermal electron fluid and to a nonlinear Klein-Gordon equation.   An
additional equation, Maxwell's first equation is used to bypass the difficulty
for the electron plasma wave, a problem which is in principle completely des-
cribed by the continuity equation, the momentum balance equation and Poisson's
equation.   For the Klein-Gordon equation a suitable choice of independent
variables is found to be sufficient.

We will show in Chapter 2 that the difficulty encountered above is not as
serious as was suggested by Taniuti and Yajima and present a general method for
determining the $\ell = 0$ components when $\det W_o$ vanishes.

This concludes the survey of the state of development of the RP technique
at the start of this work.   We can summarise by noting that the technique has
been shown to be applicable to systems where nonlinearities are balanced by dis-
persive or dissipative effects and where dispersive effects are of the same order
as self-modulation effects.   The NLS equation has been shown to be an equation
for the development of the lowest order amplitude in a perturbation expansion
for systems where modulation effects are important.   This equation can there-
fore be considered as the fundamental equation describing the nonlinear develop-
ment of such systems in the same sense as the B equation and the KdV equation
are fundamental equations for weakly dissipative and weakly dispersive systems
where self-modulation effects are not relevant.

We note that both Taniuti and Wei (1968) and Taniuti and Yajima (1969) use
a model equation (1.2.1) to demonstrate the RP technique.   Although this

equation describes a wide class of physical systems there are large numbers of physical systems that cannot be described by such a model equation. We must therefore enquire as to the validity of the analysis for these other systems. Further, equation 1.2.1 or even more complex model equations can describe equations of hyperbolic, elliptic or parabolic type, depending on the exact nature of the coefficients. We must therefore exercise great care in assuming that the asymptotic expansion 1.2.18 is valid for all time. If for instance, 1.2.1 is of hyperbolic type then even though the appropriate NLS equation may admit solutions valid for all time these solutions will only be valid for a certain time for the original system 1.2.1. This is a feature of all hyperbolic systems.

We have seen that the method relies heavily on the choice of stretched coordinates and asymptotic expansion. As we shall show in subsequent Chapters different stretchings and parameter orderings are required for different physical systems. There appears to be no general method of generating the correct choice of orderings and stretchings. Rather, each physical system must be individually examined and the correct ordering of parameters made such that the balance of nonlinear dispersive and dissipative effects is correct. The starting point is normally the dispersion relation. The expansion parameter ε must be chosen to give this consistent ordering and is normally defined and related to other parameters of the system by this choice.

Using the general principles outlined above we are now in a position to discuss extensions of the RP technique for systems with nonlinear wave modulation. In particular, we wish to extend the technique to systems which are unstable and marginally unstable and where dispersive and modulation effects are important. A further extension is also required to apply to systems where a spatial dimension other than the direction of wave propagation is important. These extensions and their applications to physical systems are discussed in subsequent chapters.

Before we consider these extensions we will briefly consider the interac-

tions of some of the nonlinear waves which are solutions of the KdV and NLS equations. We will consider in particular the solitary wave solutions. Zabusky and Kruskal (1965) showed, using numerical solutions of the KdV equation for weakly dispersive nonlinear media, that when two solitons interact they do so without losing their shape or identity, i.e. if two solitons of differing amplitudes (and consequently differing velocities) are well separated at some time then the faster soliton overtakes the slower soliton and after the non-linear interaction both solitons have preserved their shape and velocities. This problem was considered analytically by Zakharov and Shabat (1972) who considered the interaction of two solitons, (described by solutions of the NLS equation) when their relative velocities are small. The technique these authors used was to solve the NLS equation using the inverse scattering method of Gardner, Greene, Kruskal and Miura (1967) and derived equations describing the soliton interaction. Similarly Davidson (1972) has verified, using the same technique, the result of Zabusky and Kruskal. We can therefore conclude that soliton interaction is understood both numerically and analytically for KdV solitons in weakly dispersive systems and for weak interactions of solitons in strongly dispersive systems. Considerations of the KdV soliton as described above have the further restriction that they apply only to systems of interactin solitons which move in one direction.

Oikawa and Yajima (1973) have considered the problem of the interaction of two solitary waves in weakly dispersive media when the solitons move in opposite directions. Using an extension of the method of Benny and Luke (1964) they give an expression for the phase shift between the two solitons if their ampli-tudes remain constant. Using a variation of the RP technique they then consider the interaction of n solitons and find that the lowest order amplitudes of the solitons satisfy either the KdV equation or the B equation. These authors in a further paper, Oikawa and Yajima (1974) consider again the interac-tion of two solitary waves but in a strongly dispersive system. This interac-tion is therefore the interaction of envelope solitons which are given by

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solutions of the NLS equation. Using a further modification of the RP technique, they confirm that the two envelope solitons pull against each other as they approach, change their amplitudes, frequencies and velocities during the interaction and regain their original forms and velocities after the interaction. This again confirms the results of Zabusky and Kruskal (1965) and Zakharov and Shabat (1972) but is a more general result as it is valid for solitons moving with arbitrary relative velocities.

1.3 Description of Text

The original stimulus for the work of this thesis came from a study of the flow of plasma streams in crossed electric and magnetic fields. Preliminary work on this system showed that linear theories of the instabilities that exist in these systems - the so-called crossed field instabilities, were well documented in the literature but that a number of instabilities existed which were often not identified separately. These instabilities were classified into three types: long-wavelength, magnetron and cyclotron and striking similarities were found between the simplest of these, the long wavelength instability and a common, more tractable plasma instability, the two-stream instability. These considerations are described in Chapter 6. The system of the crossed field plasma stream was found to be dispersive, unstable and non-uniform. Therefore, any nonlinear theory of crossed field instabilities must account for the interaction of effects resulting from these three characteristics and the non-linearity. Initial considerations showed that conventional perturbation expansion methods and conventional multiple time scale expansions would be intractable for this system and that a different approach was required.

The RP method of Taniuti had proved to be an extremely powerful method for both weakly and strongly dispersive systems and an extension of this method to include dissipative or instability effects, non-uniform steady state effects and two-dimensional effects, was required if it were to be of use for the crossed field system.

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In Chapter 2 we consider the extension of the RP technique to include dissipative or instability effects. We consider a much more general system of equations than the model equation of Taniuti and his co-workers and show how to include the effects of instability or dissipation on the propagation of small amplitude waves in the nonlinear regime. The effect of the additional terms in the original system of equations is found to be minimal and merely increases the complexity of the coefficients of the Schrödinger equation that was expected. The effect of the instability is found to be the addition of an additional term to the Schrödinger equation and the change of the coefficients from being purely real to being complex. Thus the generalised NLS equation is found to be the equation describing the nonlinear evolution of the wave.

The general theory developed in 2.1 is tested and demonstrated in Chapter 4 by applying it to a well-known physical problem, i.e. the development of a nonlinear theory of the propagation of waves in piezoelectric semiconductors. Starting from the equations describing the propagation of acoustic waves in piezoelectric semiconductors, a system of three second order differential equations is deduced which completely describes the system in closed form. This system of equations is solved using the general theory and the generalised NLS equation is derived with explicit relations given for the coefficients in terms of the coefficients of the original equations. These coefficients are evaluated in the long wavelength approximation. The relative magnitudes of these coefficients in the limit of small piezoelectric coupling enable us to solve this equation using a conventional perturbation expansion and proceeding to third order. We find this solution to be a solitary wave with a small oscillation superimposed upon the general shape. Using this solution we examine the behaviour of relevant physical parameters of the system and find the solution confirms the existence of a high field domain propagating through the semiconductor. This domain, is an "envelope soliton" since it is the envelope of a rapidly oscillating wave.

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In 2.2 we consider the extension of the technique to apply to systems near a marginally stable state. A new coordinate stretching is presented which is adequate for such systems and using the algebra of the RP technique an equation of the NLS type is derived. This equation has the roles of the stretched time and space coordinates reversed compared with the generalised NLS equation derived in 2.1. Under the assumption that the coefficients of this equation are real we present its solutions using an analogy between this equation and the equation of motion of a point particle in a central field. From these solution we are able to show whether such a system is stable or unstable in the nonlinear regime, which depends on the relative signs of the coefficients of the equation.

This general theory is applied to a two-stream plasma instability in Chapter 5. The two-stream instability considered is shown to be relevant to the crossed field system in Chapter 6 and is found to have a marginal stability point. In Chapter 5 following a general discussion of two-stream instabilities the general theory is applied to the five equations describing the system in the hydrodynamic limit. The NLS equation is derived and expressions are given for the coefficients. The system is shown to be unstable in the nonlinear regime, irrespective of whether the system is stable or unstable in the linear regime. We show that this is in agreement with previous theories.

We show in Chapter 3 how the RP technique may be extended to systems where two-dimensional effects are important. We consider a simple system of equations with two space coordinates to avoid algebraic complexity. (We note that this extension automatically applies to systems having a non-uniform steady state which is the third extension necessary as mentioned above). The simple matrix algebra used in Chapters 1 and 2 now becomes more complex and the problem is reduced to one of solving systems of differential equations. The compatibility conditions of Chapters 1 and 2 now become integral relations instead of matrix relations. After considerable manipulation we show that

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the equation describing the nonlinear evolution of the system is again a generalised NLS equation and expressions for the coefficients are given as complex integrals.

Finally in Chapter 6 we consider the crossed field instability in some detail and indicate how the extensions developed in Chapters 2 and 3 may be applied to give a nonlinear theory of the long wavelength crossed field instability.

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Chapter 2

The reductive perturbation technique in weakly unstable
and marginally unstable systems

2.1. Introduction.

In Chapter 1 we considered the reductive perturbation technique as
originally presented by Taniuti and Yajima (1969) and discussed the
characteristics of systems for which the method may be used.   We now
consider how this technique may be extended to apply to weakly unstable
and marginally stable systems.   We further show that the technique may
be applied to systems governed by a much more general system of differ-
ential equations than the model equation consistently used by Taniuti
and his co-workers.   This extension is shown to be valid for weakly
unstable systems but in view of the algebraic complexity introduced by
this generalisation we return to a model equation for the extension to
marginally stable systems.   We also indicate how certain coefficients
which have to date been considered indeterminate may be determined
without assuming that they vanish identically or without resorting to
additional subsidiary conditions.   Again, in view of the algebraic
complexity that would be introduced with a general proof of this extension
only an indication of the method is given and the validity is demonstra-
ted in Chapters 4 and 5.

Initially we consider a system of nonlinear partial differential
equations which have a weakly unstable solution, that is, in the linear
approximation the plane wave solution has a complex frequency for a
real wave vector.   Further, the imaginary part of the frequency is
small so that the instability may be counterbalanced by nonlinear effects.
This allows an ordering of the stretched coordinates and the imaginary
part of the frequency which is consistent with the balance between the
nonlinear and dispersive effects.

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The equation describing the variation of the lowest order amplitude on the stretched time and space scales is found to be a nonlinear Schrödinger equation with complex coefficients. This will be called the generalised nonlinear Schrödinger equation and is of the form:

$$i \frac{\partial \phi}{\partial \tau} + \alpha \frac{\partial^2 \phi}{\partial \xi^2} + \beta \phi + \delta \phi |\phi|^2 = 0 \qquad 2.1.1$$

where β and δ are complex and α is real. General expressions are given for the coefficients in terms of coefficients of the original system of equations and its stationary solution. The imaginary part of β is found to be the growth rate of the linear theory and an effective nonlinear growth rate

$$\gamma_{eff} = \gamma(1 + \frac{\delta}{\gamma}|\phi|^2)$$

may be introduced as suggested by Whitham (1967).

The stability of the nonlinear wave can be investigated by examining the solutions of the generalised nonlinear Schrödinger equation. General solutions of this equation have not been found although some special solutions are known, and no attempt is made to discuss the general solutions of this equation here. Some of the special solutions are given in Chapter 5.

In 2.3 we consider systems that are marginally stable, that is the linear dispersion relation has a double root ω for a given wave vector k. If $D(\omega,k)$ is the dispersion relation for such a system then

$$D(\omega,k) = 0 \quad \text{and} \quad \partial D(\omega,k)/\partial \omega = 0$$

A typical dispersion relation for such a system is given by:

$$(\omega - \omega_o(s))^2 = \alpha(s) \qquad 2.1.2$$

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where s is a parameter which represents the stability or instability of the system.   The coordinate stretching as suggested by Taniuti and Yajima (1969) is not now appropriate to such systems since in a marginally stable state the group velocity $\lambda = \partial\omega/\partial k$ is not defined. An alternative stretching is suggested which is consistent with dispersion relations having this form.   The equation describing the variation of the lowest order amplitude is again found to be of the nonlinear Schrödinger type having the general form

$$i \frac{\partial\phi}{\partial\xi} = \alpha \frac{\partial^2\phi}{\partial\tau^2} - \gamma\phi - \beta\phi|\phi|^2 \qquad\qquad 2.1.3$$

where ξ and τ are suitable stretched space and time coordinates and α, β, γ are constants.   These constants are easily related to the linear growth rate and it is shown that if the linear theory shows the stationary state to be stable then the system may or may not be stable in the nonlinear system.   Similarly, a system which is unstable in the linear theory may have the linear growth enhanced or suppressed in the nonlinear regime.   The stability or instability of the nonlinear system is shown to be dependent on the relative signs of the constants γ and β.

This class of instability has recently attracted much attention and has been suggested as a possible description of turbulence in fluid systems (for example see Ruelle and Takens 1971).   The occurrence of a double root of the dispersion relation is an example of an inverted bifurcation in the sense of the Hopf bifurcation theorem.   This theorem states that in the neighbourhood of a neutral stability point at which a complex conjugate pair of roots of the linear stability problem crosses the real frequency axis there is a one parameter family of limit cycle solutions.   If this limit cycle occurs for

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values of a parameter smaller than the value required for neutral
stability then this point is called an inverted bifurcation.   In
this case the limit cycle is unstable and if the system is released
in a state within the limit cycle then it decays into the stable
state.   If the system is released outside the limit cycle then the
stable state is never reached and the system is unstable.   The inv-
erted bifurcation may be described by 2.1.2 and 2.1.3 in the following
way: consider a value of s such that $\alpha(s) > 0$ in 2.1.2.   Then the
frequency is pure real and so the system is stable as the infinitesimal
perturbations do not grow with time.   At s such that $\alpha(s) = 0$ the two
complex roots of 2.1.2 cross the real axis and for s such that $\alpha(s) < 0$
ω has a positive imaginary part and so the linear system is unstable.
This would be reflected in 2.1.3 by the constant β being positive.
If γ is positive then the linear system is unstable and the nonlinearity
enhances this instability.   If γ is negative so the linear system is
stable then the nonlinear system is now unstable.   A good discussion
of the relevance of marginally stable states and inverted bifurcations
as applied to turbulence problems is given by Mclaughlin and Martin
(1975).   Although the result 2.1.3 is derived for a simple model equa-
tion the extension to more complex model equations is evident in view
of the results deduced in 2.2.   It is also shown that the method
applies to systems where the marginally stable state is defined as a
transition from ω being pure imaginary to ω being zero.   All the
results developed in 2.3 are equally valid in this special case.   The
theory of marginally stable states as developed in 2.3 is applied to a
marginal two-stream instability problem in Chapter 5, which corresponds
to this special case.

2.2.   Weakly unstable systems.

We consider a system which can be described by a set of coupled

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nonlinear differential equations of the following form:

$$\underline{\underline{A}}'(\underline{u}) \frac{\partial^2 \underline{u}}{\partial x^2} + \underline{\underline{B}}'(\underline{u}) \frac{\partial^2 \underline{u}}{\partial t^2} + \underline{\underline{C}}'(\underline{u}) \frac{\partial \underline{u}}{\partial x} + \underline{\underline{D}}'(\underline{u}) \frac{\partial \underline{u}}{\partial t} + \underline{E}'(\underline{u}) = 0 \quad 2.2.1$$

where $\underline{u}$ is a column vector with n components u_1, u_2 .... u_n, $\underline{\underline{A}}'(\underline{u})$ .... $\underline{\underline{D}}'(\underline{u})$
are nxn matrices and $\underline{E}'(\underline{u})$ is an n component vector all being functions
of $\underline{u}$. We make few assumptions about the existence or singularity of
any of these matrices. We merely insist that one of $\underline{\underline{A}}'$ and $\underline{\underline{C}}'$ exist
and that one of $\underline{\underline{B}}'$ and $\underline{\underline{C}}'$ exist and that the vector $\underline{E}'(\underline{u})$ exists.
We assume that all matrices that exist are continuous and differentiable
functions of $\underline{u}$.

We consider a constant solution $\underline{u}_o$ of the system 2.2.1 which is
given by a solution of

$$\underline{E}'_o = \underline{E}'(\underline{u}_o) = 0 \qquad\qquad 2.2.2$$

We now look for plane wave solutions of 2.2.1 about the constant solution
$\underline{u}_o$ of the form:

$$\underline{u} = \underline{u}_o + \underline{u}_1 \exp i(kx-\omega t) + c \cdot c$$

Then 2.2.1 when linearised becomes

$$\underline{\underline{A}}'_o \frac{\partial^2 \underline{u}_1}{\partial x^2} + \underline{\underline{B}}'_o \frac{\partial^2 \underline{u}_1}{\partial t^2} + \underline{\underline{C}}'_o \frac{\partial \underline{u}_1}{\partial x} + \underline{\underline{D}}'_o \frac{\partial \underline{u}_1}{\partial t} + \nabla\underline{E}'_o \underline{u}_1 = 0 \qquad 2.2.3$$

where the $\underline{\underline{A}}_o$ .... $\underline{\underline{D}}_o$ are defined as in 2.2.2 and $\nabla\underline{E}_o$ is a matrix
whose (i,j)th component is defined by:

$$(\nabla\underline{E}'_o)_{ij} = \left(\frac{\partial \underline{E}_i}{\partial u_j}\right)_{\underline{u} = \underline{u}_o}$$

Then the system 2.2.3 admits a plane wave solution subject to the
dispersion relation:

$$\det \underline{\underline{H}} = 0$$

where

$$\underline{\underline{H}} = -k^2 \underline{\underline{A}}'_o - \omega^2 \underline{\underline{B}}'_o + ik\underline{\underline{C}}'_o - i\omega \underline{\underline{D}}'_o + \nabla \underline{\underline{E}}'_o$$

We assume that the dispersion relation gives a complex frequency ω for a real wavenumber k. Further we assume that this complex frequency ω has a small imaginary ω_i. This imaginary part ω_i is ordered as $O(\epsilon^2)$ where ϵ is the expansion parameter of the coordinate stretching. The coordinate stretching to be used is the same as was used in Chapter 1 for stable systems, i.e.

$$\tau = \epsilon^2 t \qquad\qquad 2.2.4$$

$$\xi = \epsilon(x - \lambda t)$$

where now

$$\lambda = \partial \omega_r / \partial k$$

ω_r being the real part of the frequency.

This choice of ordering can be justified as follows: we expect the final result to be a modified Schrödinger equation with additional terms to take account of the instability. If we restrict the choice of the instability to be a weak instability then we expect the nonlinearity to be of the same order as the instability in order that the two effects are comparable and interact on the same time and space scales. The nonlinear Schrödinger equation was found to be exact to third order with a nonlinear term proportional to $\phi|\phi|^2$. We consider this to be a term proportional to ϕ with an effective "growth" or "decay" rate proportional to $|\phi|^2$, i.e. of second order.

Conversely, if the nonlinearity were not present to limit the instability then the time development of the wave amplitude must be given by

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$$\frac{\partial \phi}{\partial \tau} = \omega_i \phi$$

i.e. a term proportional to ϕ must be present with a coefficient equal to the growth rate. In order that this be exact to third order leads to an ordering $\omega_i \sim O(\varepsilon^2)$. We therefore assume that the nonlinearity and instability balance each other on the same time and space scales and combine the ordering 2.2.4 with ω_i being $O(\varepsilon^2)$.

We may now derive the equation governing the behaviour of the lowest order amplitude. However, we stress that the decomposition of the complex frequency into a real and small imaginary part must be achieved before any expansion can be made. This restricts the class of problems that may be solved by this method to those having a dispersion relation which enables this analytic decomposition to be made. This implies a good knowledge of the instability mechanism and the ability to reduce a stable system to a condition of instability by modifying a parameter or set of parameters (e.g. some initial or boundary conditions). As will be seen, this is possible in the case of the acoustoelectric instability but has not yet been achieved for the crossed field instability.

This may be formally written by explicitly including these crucial parameters into the original statement of the problem, i.e. we write

$$\underline{\underline{A}}(\underline{u},p) \frac{\partial^2 \underline{u}}{\partial x^2} + \underline{\underline{B}}(\underline{u},p) \frac{\partial^2 \underline{u}}{\partial t^2} + \underline{\underline{C}}(\underline{u},p) \frac{\partial \underline{u}}{\partial x} + \underline{\underline{D}}(\underline{u},p) \frac{\partial \underline{u}}{\partial t} + \underline{\underline{E}}(\underline{u},p) = 0 \quad 2.2.5$$

where the parameter p characterises the imaginary part of the frequency which is obtained from the dispersion relation of the system 2.2.5. Assuming ω_i is $O(\varepsilon^2)$ indicates $p = \varepsilon^2 p'$ where p' is of order unity and allows an expansion of ω_i in powers of p', i.e.

$$\omega_i = \varepsilon^2 \left(\frac{\partial \omega_i}{\partial p}\right)_{p=o} p' + 0(\varepsilon^4) \qquad\qquad 2.2.6$$

This formally expresses the condition to be imposed on the dispersion relation.

We now proceed as in Chapter 1 and look for solutions of 2.2.5 of the form:

$$\underline{u} = \underline{u}_o + \sum_{\alpha=0}^{\infty} \sum_{\ell=-\infty}^{\infty} \varepsilon^\alpha \underline{u}_\ell^\alpha(\tau,\xi)\exp(i\ell(kx-\omega_r t)) \qquad 2.2.7$$

where τ and ε are given by 2.2.4 and ω_r is the real part of the frequency given by the solution of the dispersion relation for $p=0$. The amplitude $\underline{u}_\ell^\alpha(\tau,\xi)$ is now more complex as it implicitly contains the factor $\exp(\ell\omega_i\tau)$ which arises from the substitution of $\omega = \omega_r + i\varepsilon^2\omega_i$ in the harmonic part of the expansion. This dependence will not be written explicitly. The matrices $\underline{\underline{A}}(\underline{u},p) \ldots \underline{\underline{D}}(\underline{u},p)$ and the vector $\underline{E}(\underline{u},p)$ are now expanded as:

$$\underline{\underline{A}}(\underline{u},p) = \underline{\underline{A}}^o + \varepsilon \underline{\underline{A}}^o.\underline{u}^1 + \varepsilon^2(\nabla\underline{\underline{A}}^o.\underline{u}^2 + \tfrac{1}{2}\nabla\nabla\underline{\underline{A}}^o:\underline{u}^1\underline{u}^1$$

$$+ \underline{\underline{A}}^o{}_p) + \ldots\ldots \qquad\qquad 2.2.8$$

$$\underline{\underline{B}}(\underline{u},p) = \underline{\underline{B}}^o + \varepsilon \nabla.\underline{\underline{B}}^o.\underline{u}^1 +\ldots\ldots$$

etc., and

$$\underline{E}(\underline{u},p) = \varepsilon \nabla\underline{E}^o.\underline{u}^1 + \varepsilon^2(\nabla\underline{E}^o.\underline{u}^2 + \tfrac{1}{2}\nabla\nabla\underline{E}^o:\underline{u}^1\underline{u}^1)$$

$$+ \varepsilon^3(\nabla\underline{E}^o.\underline{u}^3 + \nabla\nabla\underline{E}:\underline{u}^1\underline{u}^2 + \tfrac{1}{6}\nabla\nabla\nabla\underline{E}^o:\underline{u}^1\underline{u}^1\underline{u}^1 + \nabla\underline{E}_p.\underline{u}^1)$$

$$+ \ldots\ldots\ldots \qquad\qquad 2.2.9$$

where

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$$\nabla \underline{\underline{A}}^o \cdot \underline{u}^1 = \sum_{i=1}^{n} \left(\frac{\partial \underline{\underline{A}}}{\partial u_i} \right)_{\underline{u}=\underline{u}^o, p=0} u_i^1 \qquad \text{2.2.10}$$

$$\nabla\nabla\underline{\underline{A}}:\underline{u}^1\underline{u}^1 = \sum_i \sum_j \left(\frac{\partial^2 \underline{\underline{A}}}{\partial u_i \partial u_j} \right)_{\underline{u}=\underline{u}^o, p=0} u_i^1 u_j^1 \qquad \text{2.2.11}$$

$$\underline{\underline{A}}^o_p = \left(\frac{\partial \underline{\underline{A}}}{\partial p} \right)_{\underline{u}=\underline{u}^o, p=0} p' \qquad \text{2.2.12}$$

Substituting these expansions together with 2.2.7 into 2.2.5 and equating powers of ε of the same harmonic to zero, gives, up to third order the following system of equations:

$$O(\varepsilon) \qquad\qquad \underline{\underline{W}}_\ell \underline{u}_\ell^1 = 0 \qquad\qquad \text{2.2.13}$$

$$O(\varepsilon^2) \qquad \underline{\underline{W}}_\ell \underline{u}_\ell^2 + (2i\ell k \underline{\underline{A}}^o + 2\lambda i\ell\omega_r \underline{\underline{B}}^o + \underline{\underline{C}}^o - \lambda \underline{\underline{D}}^o) \frac{\partial \underline{u}_\ell^1}{\partial \xi}$$

$$= - \nabla\underline{\underline{A}}^o \cdot \langle \sum_m \sum_n (ink)^2 \underline{u}_m^1 \underline{u}_n^1 \, P_m P_n \rangle_\ell$$

$$- \nabla\underline{\underline{B}}^o \cdot \langle \sum_m \sum_n (in\omega_r)^2 \underline{u}_m^1 \underline{u}_n^1 \, P_m P_n \rangle_\ell$$

$$- \nabla\underline{\underline{C}}^o \cdot \langle \sum_m \sum_n (ink) \underline{u}_m^1 \underline{u}_n^1 \, P_m P_n \rangle_\ell$$

$$+ \nabla\underline{\underline{D}}^o \cdot \langle \sum_m \sum_n (in\omega_r) \underline{u}_m^1 \underline{u}_n^1 \, P_m P_n \rangle_\ell$$

$$- \tfrac{1}{2} \nabla\nabla\underline{\underline{E}}^o : \langle \sum_m \sum_n \underline{u}_m^1 \underline{u}_n^1 \, P_m P_n \rangle_\ell \qquad \text{2.2.14}$$

$$O(\varepsilon^3) \qquad \underline{\underline{W}}_\ell \underline{u}_\ell^3 + (2i\ell k \underline{\underline{A}}^o + 2\lambda i\ell\omega_r \underline{\underline{B}}^o + \underline{\underline{C}}^o - \lambda \underline{\underline{D}}^o) \frac{\partial \underline{u}_\ell^2}{\partial \xi}$$

$$+ (-2i\ell\omega_r\underline{\underline{B}}^o + \underline{\underline{D}}^o) \frac{\partial \underline{u}^1}{\partial \tau}_\ell + (\underline{\underline{A}}^o + \lambda^2\underline{\underline{B}}^o) \frac{\partial^2 \underline{u}^1}{\partial \xi^2}_\ell$$

$$+ ((i\ell k)^2\underline{\underline{A}}^o_p + (i\ell\omega_r)^2\underline{\underline{B}}^o_p + (i\ell k)\underline{\underline{C}}^o_p - (i\ell\omega_r)\underline{\underline{D}}^o_p + \nabla\underline{\underline{E}}^o_p)\underline{u}^1_\ell$$

$$= -\nabla\underline{\underline{A}}^o \cdot \langle \sum_m \sum_n (2ink)\underline{u}^1_m \frac{\partial \underline{u}^1}{\partial \xi}_n P_m P_n \rangle_\ell$$

$$-\nabla\underline{\underline{A}}^o \cdot \langle \sum_m \sum_n (ink)\underline{u}^2_m \underline{u}^1_n P_m P_n \rangle_\ell$$

$$-\nabla\underline{\underline{A}}^o \cdot \langle \sum_m \sum_n (ink)^2\underline{u}^1_m \underline{u}^1_n P_m P_n \rangle_\ell$$

$$- \tfrac{1}{2}\nabla\nabla\underline{\underline{A}}^o : \langle \sum_m \sum_n \sum_q (iqk)^2\underline{u}^1_m \underline{u}^1_n \underline{u}^1_q P_m P_n P_q \rangle_\ell$$

$$- \nabla\underline{\underline{B}}^o \cdot \langle \sum_m \sum_n (ink)^2\underline{u}^1_m \underline{u}^2_n P_m P_n \rangle_\ell$$

$$- \nabla\underline{\underline{B}}^o \cdot \langle \sum_m \sum_n (ink)^2\underline{u}^2_m \underline{u}^1_n P_m P_n \rangle_\ell$$

$$- \nabla\underline{\underline{B}}^o \cdot \langle \sum_m \sum_n (2\lambda in\omega_r)\underline{u}^1_m \frac{\partial \underline{u}^1}{\partial \xi}_n P_m P_n \rangle_\ell$$

$$- \tfrac{1}{2}\nabla\nabla\underline{\underline{B}}^o : \langle \sum_m \sum_n \sum_q (iq\omega_r)^2\underline{u}^1_m \underline{u}^1_n \underline{u}^1_q P_m P_n P_q \rangle_\ell$$

$$- \nabla\underline{\underline{C}}^o \cdot \langle \sum_m \sum_n \underline{u}^1_m \frac{\partial \underline{u}^1}{\partial \xi}_n P_m P_n \rangle_\ell$$

$$- \nabla\underline{\underline{C}}^o \cdot \langle \sum_m \sum_n (ink)\underline{u}^2_m \underline{u}^1_n P_m P_n \rangle_\ell$$

$$- \nabla\underline{\underline{C}}^o \cdot \langle \sum_m \sum_n (ink)\underline{u}^1_m \underline{u}^2_n P_m P_n \rangle_\ell$$

$$- \tfrac{1}{2} \nabla\nabla\underline{\underline{C}}^o : < \sum_m \sum_n \sum_q (iqk)\underline{u}^1_m\underline{u}^1_n\underline{u}^1_q\ P_m P_n P_q >_\ell$$

$$+ \nabla\underline{D}^o \cdot < \sum_m \sum_n \lambda\underline{u}^1_m \frac{\partial\underline{u}^1}{\partial\xi}_n\ P_m P_n >_\ell$$

$$+ \nabla\underline{D}^o \cdot < \sum_m \sum_n (in\omega_r)\underline{u}^1_m\underline{u}^2_n\ P_m P_n >_\ell$$

$$+ \nabla\underline{D}^o \cdot < \sum_m \sum_n (in\omega_r)\underline{u}^2_m\underline{u}^1_n\ P_m P_n >_\ell$$

$$+ \tfrac{1}{2} \nabla\nabla\underline{\underline{D}}^o : < \sum_m \sum_n \sum_q (iq\omega_r)\underline{u}^1_m\underline{u}^1_n\underline{u}^1_q\ P_m P_n P_q >_\ell$$

$$- \nabla\nabla\underline{E}^o : < \sum_m \sum_n \underline{u}^1_m\underline{u}^2_n\ P_m P_n >_\ell$$

$$- \frac{1}{6}\nabla\nabla\nabla\underline{E}^o \vdots < \sum_m \sum_n \sum_q \underline{u}^1_m\underline{u}^1_n\underline{u}^1_q\ P_m P_n P_q >_\ell \qquad\qquad 2.2.15$$

where the notation is as used in Chapter 1.

These equations are written in simpler form as:

$$\underline{W}_\ell\underline{u}^1{}_\ell\ =\ 0$$

$$\underline{W}_\ell\underline{u}^2{}_\ell + \underline{M}_\ell \frac{\partial\underline{u}^1}{\partial\xi}_\ell\ =\ \underline{S}^1{}_\ell$$

$$\underline{W}_\ell\underline{u}^3{}_\ell + \underline{M}_\ell \frac{\partial\underline{u}^2}{\partial\xi}_\ell + \underline{N}_\ell \frac{\partial\underline{u}^1}{\partial\tau}_\ell + \underline{O}_\ell \frac{\partial^2\underline{u}^1}{\partial\xi^2}_\ell$$

$$+ \underline{P}_\ell\underline{u}^1{}_\ell\ =\ \underline{S}^2{}_\ell$$

where:

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$$\underline{\underline{W}}_{\ell} \;=\; (i\ell k)^2\underline{\underline{A}}^o \;+\; (i\ell\omega_r)^2\underline{\underline{B}}^o \;+\; (i\ell k)\underline{\underline{C}}^o \;-\; (i\ell\omega_r)\underline{\underline{D}}^o \;+\; \nabla\underline{\underline{E}}^o$$

$$\text{2.2.16}$$

$$\underline{\underline{M}}_{\ell} \;=\; (2i\ell k)\underline{\underline{A}}^o \;+\; 2\lambda i\ell\omega_r\underline{\underline{B}}^o \;+\; \underline{\underline{C}}^o \;-\; \lambda\underline{\underline{D}}^o \qquad\qquad \text{2.2.17}$$

$$\underline{\underline{N}}_{\ell} \;=\; (-2i\ell\omega_r)\underline{\underline{B}}^o \;+\; \underline{\underline{D}}^o \qquad\qquad\qquad\qquad \text{2.2.18}$$

$$\underline{\underline{O}}_{\ell} \;=\; \underline{\underline{A}}^o \;+\; \lambda^2\underline{\underline{B}}^o \qquad\qquad\qquad\qquad\qquad \text{2.2.19}$$

$$\underline{\underline{P}}_{\ell} \;=\; (i\ell k)^2\underline{\underline{A}}^o_p \;+\; (i\ell\omega_r)^2\underline{\underline{B}}^o_p \;+\; (i\ell k)\underline{\underline{C}}^o_p \;-\; (i\ell\omega_r)\underline{\underline{D}}^o_p \;+\; \nabla\underline{\underline{E}}^o_p$$

$$\text{2.2.20}$$

where $\underline{S}^1{}_{\ell}$, $\underline{S}^2{}_{\ell}$ are given by the right hand sides of 2.2.14 and 2.2.15 respectively.

We note the following relationships between these matrices:

$$\underline{\underline{M}}_{\ell} \;=\; -\frac{i}{\ell}\,\frac{\partial\underline{\underline{W}}}{\partial k}_{\ell} \qquad\qquad\qquad \text{2.2.21a}$$

$$\underline{\underline{N}}_{\ell} \;=\; -\frac{\partial\underline{\underline{M}}}{\partial\lambda}_{\ell} \qquad\qquad\qquad\qquad \text{2.2.21b}$$

$$\underline{\underline{O}}_{\ell} \;=\; -\frac{1}{2\ell}\,\frac{\partial^2\underline{\underline{W}}}{\partial k^2}_{\ell} \qquad\qquad\qquad \text{2.2.22}$$

We proceed as in Chapter 1 but with certain important differences. The real frequency ω_r is given by

$$\det\,\underline{\underline{W}}_{\pm 1} \;=\; 0$$

i.e. $\det\,\left| \pm k^2\underline{\underline{A}}^o \pm \omega^2\underline{\underline{B}}^o \pm ik\underline{\underline{B}}^o \pm i\omega\underline{\underline{D}}^o + \nabla\underline{\underline{E}}^o \right| \;=\; 0.$

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which by assumption gives a single real root ω_r. Further we assume that

$$\det \underline{\underline{W}}_\ell \neq 0 \qquad \ell \neq 1$$

This condition ensures that only a single mode exists and that all harmonics of this mode are stable. Unfortunately for most physical systems this condition is violated for $\ell = 0$ and Taniuti and Yajima (1969) resorted to subsidiary conditions to overcome this difficulty. Similarly Kako (1973) and others assume that the $\ell = 0$ components of $\underline{u}^1{}_\ell$ vanish. We will also adopt this procedure for the sake of simplicity and clarity and will indicate later how these $\ell = 0$ components may be determined. A general proof of this method has not been found but in all systems considered so far has been found to be applicable.

We therefore assume

$$\det \underline{\underline{W}}_{\pm \ell} \neq 0 \qquad \text{for } \ell \neq \pm 1$$

which from 2.2.13 gives

$$\underline{u}^1{}_\ell = 0 \qquad \ell \neq \pm 1$$

and

$$\underline{u}^1{}_1 = \underline{R}\phi(\tau, \xi) \qquad\qquad 2.2.23$$

where $\underline{R}$ is the right eigenvector of $\underline{\underline{W}}_1$ as before and evidently satisfies

$$\underline{\underline{W}}_1 \underline{R} = 0 \qquad\qquad 2.2.24$$

and where ϕ is a scalar function being the lowest order amplitude,

a function of the slow time and space variables only.

To next order 2.2.14 for $\ell = 1$ becomes

$$\underline{W}_1 \underline{u}_1^2 + \underline{\underline{M}}_1 \frac{\partial \underline{u}_1^1}{\partial \xi} = \underline{S}_1^1$$

where
$$\underline{S}_1^1 = 0.$$

Substituting for $\underline{\underline{M}}_1$ from 2.2.10 gives

$$\underline{W}_1 \underline{u}_1^2 - i \frac{\partial \underline{\underline{W}}_1}{\partial k} \frac{\partial \underline{u}_1^1}{\partial \xi} = 0$$

But differentiating 2.2.24 with respect to k gives:

$$\frac{\partial \underline{\underline{W}}_1}{\partial k} \underline{R} = - \underline{W}_1 \frac{\partial \underline{R}}{\partial k}$$

and substituting this result together with 2.2.23 gives

$$\underline{W}_1 \left(\underline{u}_1^2 + i \frac{\partial \underline{R}}{\partial k} \frac{\partial \phi}{\partial \xi} \right) = 0$$

This equation is satisfied if

$$\underline{u}_1^2 + i \frac{\partial \underline{R}}{\partial k} \frac{\partial \phi}{\partial \xi} \qquad \alpha \ \underline{R}$$

and so we may write

$$\underline{u}_1^2 = \phi^{(2)}(\tau, \xi) \underline{R} - i \frac{\partial \underline{R}}{\partial k} \frac{\partial \phi}{\partial \xi} \qquad\qquad 2.2.25$$

where $\phi^{(2)}$ is a scalar to be determined to higher order.

The other components required to second order are $\underline{u}_2^2$ and $\underline{u}_0^2$. These are in principle obtained from the $\ell = 0$ and $\ell = 2$ components of 2.2.14 i.e.

$$\underline{\underline{W}}_2 \, \underline{u}_2^2 = \underline{S}_2^1 \qquad\qquad 2.2.26$$

$$\underline{\underline{W}}_0 \, \underline{u}_2^0 = \underline{S}_0^1 \qquad\qquad 2.2.27$$

where

$$\underline{S}_0^1 = (V_+ \cdot \underline{R}^*)\underline{R}|\phi|^2 + (V_- \cdot \underline{R})\underline{R}^*|\phi|^2$$

$$- (\nabla\nabla\underline{\underline{E}}^0 : \underline{R}\ \underline{R})\}\phi|^2$$

and
$$\underline{S}_2^1 = (V_+ \cdot \underline{R})\underline{R}(\phi)^2 - \tfrac{1}{2}(\nabla\nabla\underline{\underline{E}}^0 : \underline{R}\ \underline{R})(\phi)^2$$

The operators V_+ and V_- are defined by:

$$V_+ = -\nabla\underline{\underline{A}}^0(ik)^2 - \nabla\underline{\underline{B}}^0(i\omega_r)^2 - \nabla\underline{\underline{C}}^0(ik) + \nabla\underline{\underline{D}}^0(i\omega_r)$$

$$V_- = -\nabla\underline{\underline{A}}^0(ik)^2 - \nabla\underline{\underline{B}}^0(i\omega_r)^2 + \nabla\underline{\underline{C}}^0(ik) - \nabla\underline{\underline{D}}^0(i\omega_r)$$

We again encounter difficulties since det $\underline{\underline{W}}_0$ vanishes, however, again we will assume the existence of det $\underline{\underline{W}}_0$ and indicate later how the $\ell = 0$ component of $\underline{u}_\ell^2$ may be determined. So formally:

$$\underline{u}_2^2 = \underline{\underline{W}}_2^{-1} \, \underline{S}_2^1$$

and
$$\underline{u}_0^2 = \underline{\underline{W}}_0^{-1} \, \underline{S}_0^1$$

We can therefore write

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$$\underline{W}_2 \, \underline{u}_2^2 \;=\; \underline{S}_2^1 \qquad\qquad\qquad 2.2.26$$

$$\underline{W}_0 \, \underline{u}_2^0 \;=\; \underline{S}_0^1 \qquad\qquad\qquad 2.2.27$$

where

$$\underline{S}_0^1 \;=\; (V_+ \cdot \underline{R}^*)\underline{R}|\phi|^2 \;+\; (V_- \cdot \underline{R})\underline{R}^*|\phi|^2$$

$$-\; (\nabla\nabla\underline{E}^O : \underline{R}\ \underline{R})\Big\}|\phi|^2$$

and

$$\underline{S}_2^1 \;=\; (V_+ \cdot \underline{R})\underline{R}(\phi)^2 \;-\; \tfrac{1}{2}(\nabla\nabla\underline{E}^O : \underline{R}\ \underline{R})(\phi)^2$$

The operators V_+ and V_- are defined by:

$$V_+ \;=\; -\nabla\underline{A}^O(ik)^2 \;-\; \nabla\underline{B}^O(i\omega_r)^2 \;-\; \nabla\underline{C}^O(ik) \;+\; \nabla\underline{D}^O(i\omega_r)$$

$$V_- \;=\; -\nabla\underline{A}^O(ik)^2 \;-\; \nabla\underline{B}^O(i\omega_r)^2 \;+\; \nabla\underline{C}^O(ik) \;-\; \nabla\underline{D}^O(i\omega_r)$$

We again encounter difficulties since $\det \underline{W}_0$ vanishes, however, again we will assume the existence of $\det \underline{W}_0$ and indicate later how the $\ell = 0$ component of $\underline{u}_\ell^2$ may be determined. So formally:

$$\underline{u}_2^2 \;=\; \underline{W}_2^{-1}\, \underline{S}_2^1$$

and

$$\underline{u}_0^2 \;=\; \underline{W}_0^{-1}\, \underline{S}_0^1$$

We can therefore write

$$\underline{u}_2^2 = \underline{R}_2^2(\phi)^2 \qquad\qquad 2.2.28$$

and
$$\underline{u}_0^2 = \underline{R}_0^2|\phi|^2 \qquad\qquad 2.2.29$$

where

$$\underline{R}_2^2 = \underline{W}_2^{-1}\left[(V_+.\underline{R})\underline{R} - \tfrac{1}{2}(\nabla\nabla\underline{E}^o:\underline{R}\ \underline{R})\right]$$

$$\underline{R}_0^2 = \underline{W}_o^{-1}\left[(V_+.\underline{R}^*)\underline{R} + (V_{\overline{\bullet}}.\underline{R})\underline{R}^* - (\nabla\nabla\underline{E}^o:\underline{R}^*\underline{R})\right]$$

We are now ready to determine the generalised nonlinear Schrödinger equation. We consider 2.2.15 for $\ell = 0$:

$$\underline{W}_1\underline{u}_1^3 + \underline{M}_1\frac{\partial\underline{u}_1^2}{\partial\xi} + \underline{N}_1\frac{\partial\underline{u}_1^1}{\partial\tau} + \underline{O}_1\frac{\partial^2\underline{u}_1^1}{\partial\xi^2} + \underline{P}_1\underline{u}_1^1 = \underline{S}_1^2$$

where

$$\underline{S}_1^2 = (V_-.\underline{u}_2^2)\underline{R}^*\phi^* + (VV.\underline{R}^*)\underline{u}_2^2\phi^*$$

$$(Z_+:\underline{R}\ \underline{R}^*)\underline{R}\phi|\phi|^2 + \tfrac{1}{2}(Z_-:\underline{R}\ \underline{R})\underline{R}^*\phi|\phi|^2$$

$$- (\nabla\nabla\underline{E}^o:\underline{R}^*\underline{u}_2^2)\phi^* - (\nabla\nabla\underline{E}^o:\underline{R}\ \underline{u}_0^2)\phi$$

$$- \tfrac{1}{2}(\nabla\nabla\nabla\underline{E}^o:\underline{R}\ \underline{R}\ \underline{R}^*)\phi|\phi|^2$$

where the operators V_-, V_+ are as defined before and

$$VV = -\nabla\underline{A}^o(2ik)^2 - \nabla\underline{B}^o(2i\omega_r)^2 - \nabla\underline{C}^o(2ik) + \nabla\underline{D}^o(2i\omega_r)$$

$$Z_+ = -\nabla\nabla\underline{A}^o(ik)^2 - \nabla\nabla\underline{B}^o(i\omega_r)^2 - \nabla\nabla\underline{C}^o(ik) + \nabla\nabla\underline{D}^o(i\omega_r)$$

$$Z_- = -\nabla\nabla\underline{\underline{A}}^O(ik)^2 - \nabla\nabla\underline{\underline{B}}^O(i\omega_r)^2 + \nabla\nabla\underline{\underline{C}}^O(ik) - \nabla\nabla\underline{\underline{D}}^O(i\omega_r)$$

Multiplying on the left by $\underline{L}$ and substituting for $\underline{u}_1^1$, $\underline{u}_1^2$, $\underline{u}_2^2$ and $\underline{u}_0^2$ from 2.2.23, 2.2.25, 2.2.28 and 2.2.29 gives

$$\alpha\frac{\partial\phi}{\partial\tau} + \frac{\partial^2\phi}{\partial\xi^2} + \gamma\phi + \delta\phi|\phi|^2 = 0 \qquad\qquad 2.2.30$$

where

$$\alpha = \underline{L}\,\underline{\underline{N}}_1\underline{R}$$

$$\beta = \underline{L}\,\underline{\underline{O}}_i\underline{R} - \underline{L}\,\frac{\partial\underline{\underline{W}}_1}{\partial k}\frac{\partial\underline{R}}{\partial k}$$

$$\gamma = \underline{L}\,\underline{\underline{P}}_1\underline{R}$$

$$\delta = (LV_-.\underline{R}_2^2)\underline{R}^* + (VV.\underline{R}^*)\underline{R}_2^2 + (Z_+:\underline{R}\,\underline{R}^*)\underline{R}$$

$$\qquad + \tfrac{1}{2}(Z_-:\underline{R}\,\underline{R})\underline{R}^* - (\nabla\nabla\underline{\underline{E}}^O:\underline{R}^*\underline{R}_2^2) - (\nabla\nabla\underline{\underline{E}}^O:\underline{R}\,\underline{R}_0^2)$$

$$\qquad - \tfrac{1}{2}(\nabla\nabla\nabla\underline{\underline{E}}^O:\underline{R}\,\underline{R}\,\underline{R}^*)$$

These coefficients may be put into simpler form by considering the definitions of the matrices $\underline{\underline{N}}$, $\underline{\underline{O}}$, $\underline{\underline{P}}$ and $\underline{\underline{W}}$.

Differentiating 2.2.16 i.e. the definition of $\underline{\underline{W}}$ twice with respect to k gives

$$\frac{\partial^2\underline{\underline{W}}_1}{\partial k^2} = -2\underline{\underline{A}}^O - 2\lambda^2\underline{\underline{B}}^O - \frac{\partial^2\omega_r}{\partial k^2}\left[2\omega_r\underline{\underline{B}}^O + i\,\underline{\underline{D}}^O\right]$$

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$$= -2\underline{\underline{A}}^o - 2\lambda^2\underline{\underline{B}}^o - i\frac{\partial^2\omega_r}{\partial k^2}\underline{N}_1$$

$$= -2\underline{\underline{O}}_1 - i\frac{\partial^2\omega_r}{\partial k^2}\underline{N}_1 \qquad\qquad 2.2.31$$

Now differentiating 2.2.24 twice with respect to k gives

$$\frac{\partial\underline{\underline{W}}_1}{\partial k}\underline{R} + \underline{W}_1\frac{\partial\underline{R}}{\partial k} \qquad\qquad = 0$$

i.e.

$$\frac{\partial^2\underline{\underline{W}}_1}{\partial k^2}\underline{R} + 2\frac{\partial\underline{\underline{W}}_1}{\partial k}\frac{\partial\underline{R}}{\partial k} + \underline{W}_1\frac{\partial^2\underline{R}}{\partial k^2} = 0$$

i.e.

$$\underline{L}\frac{\partial\underline{\underline{W}}_1}{\partial k}\frac{\partial\underline{R}}{\partial k} = -\tfrac{1}{2}\underline{L}\frac{\partial^2\underline{\underline{W}}_1}{\partial k^2}\underline{R}$$

Therefore premultiplying 2.2.31 by $\underline{L}$ and postmultiplying by $\underline{R}$ gives:

$$\underline{L}\frac{\partial^2\underline{\underline{W}}_1}{\partial k^2}\underline{R} = -2\underline{L}\,\underline{O}_1\underline{R} - i\frac{\partial^2\omega_r}{\partial k^2}\underline{L}\,\underline{N}_1\underline{R}$$

and substituting into the expression for β shows

$$\beta = -\tfrac{1}{2}i\frac{\partial^2\omega_r}{\partial k^2}\underline{L}\,\underline{N}\,\underline{R}$$

i.e.

$$\beta = -\tfrac{1}{2}i\frac{\partial^2\omega_r}{\partial k^2}\alpha \qquad\qquad 2.2.32$$

The coefficient γ is found to have physical significance by expanding 2.2.16 in powers of ϵ, i.e.

$$\underline{W}_1 = -k^2\underline{\underline{A}}^o - \omega_r^2\underline{\underline{B}}^o + ik\underline{\underline{C}}^o - i\omega_r\underline{\underline{D}}^o + \nabla\underline{\underline{E}}^o$$

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$$= - k^2 \underline{A}^o_p - \omega_r^2 \underline{B}^o_p + ik\underline{C}^o_p - i\omega_r\underline{D}^o_p + \nabla E^o_p$$

$$- 2i\omega_r\omega_i \ \underline{B}^o + \omega_i\underline{D}^o + \ldots$$

$$= \underline{P}_1 + \omega_i \ \underline{N}_1$$

which gives

$$\gamma = \underline{L} \ \underline{P}_1 \ \underline{R} = \omega_i \ \underline{L} \ \underline{N}_1 \ \underline{R} = \omega_i \ \alpha \qquad\qquad 2.2.33$$

Combining these results gives:

$$i \ \frac{\partial \phi}{\partial \tau} + \tfrac{1}{2} \ \frac{\partial^2 \omega_r}{\partial k^2} \ \frac{\partial^2 \phi}{\partial \xi^2} + i\omega_i \ \phi + \frac{i\delta}{\alpha} \ \phi|\phi|^2 = O \qquad\qquad 2.2.34$$

which is called the generalised nonlinear Schrödinger equation since the coefficients of ϕ and $\phi|\phi|^2$ are in general complex.

General solutions of this equation have not been found and no attempt to derive the general solution will be made in this thesis. Special solutions have been derived by assuming δ to be pure imaginary e.g. Rowlands (1974), but to date no solutions have been presented for δ being complex and having non-zero real and imaginary parts. In Chapter 4 we will see how solutions may be obtained provided a certain ordering of coefficients can be made.

We now return to the difficulties arising from the vanishing determinant of $\underline{W}_o$. As discussed previously, a general proof of how this difficulty may be overcome has not yet been derived. We will now give an indication of the technique used for real physical systems in Chapters 4 and 5.

Since det $\underline{W}_o$ vanishes in most physical systems the number of components of $\underline{u}^1_o$ which may be determined from

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$$\underline{W}_o \, \underline{u}_o^1 \;=\; 0$$

depends on the rank and degeneracy of $\underline{W}_o$. By inspection it is seen that this reduces to the properties of the matrix $\nabla \underline{E}^o$. It is supposed that a number of components of $\underline{u}_o^1$ may be deduced from the above equation or at least that an expression for a linear combination of these coefficients is derived. In the first case it is conjectured that the remaining components may be deduced from a compatibility condition as described below. In the second case it is conjectured either that a) this compatibility condition gives a further relationship between the components which enables an algebraic solution to be found with the previously deduced relationship or b) the $\ell = 0$ components that are required to higher order appear only in the combination found above.

The compatibility condition to be satisfied arises if the assumption $\underline{u}_o^1 = 0$ is not made in evaluating $\underline{S}_1^1$ in the derivation of $\underline{u}_1^2$ leading to 2.2.25. Without this assumption we obtain

$$\underline{W}_1 \, \underline{u}_1^2 + \underline{M}_1 \, \frac{\partial \underline{u}_1^1}{\partial \xi} \;=\; (V_+ \cdot \underline{u}_o^1)\underline{u}\,\phi \;-\; (\nabla\nabla\underline{E}^o : \underline{R} \, \underline{u}_o^1)\phi$$

In view of the definition of $\underline{M}_1$ then multiplying this equation by $\underline{L}$ from the left immediately gives:

$$\underline{L}\,(V_+ \cdot \underline{u}_o^1)\underline{R} \;-\; (\nabla\nabla\underline{E}^o : \underline{R}\,\underline{u}_o^1) \;=\; 0$$

which is an explicit form of the compatibility condition and gives a relationship between one or more of the components of $\underline{u}_o^1$ as required. We do however note that $\underline{u}_o^1$ is equal to a constant and can have no τ or ξ dependence. We can now assume that $\underline{u}_o^1$ is known and proceed as before.

The difficulty encountered to second order in determining $\underline{u}_o^2$ is overcome by proceeding to third order and integrating. Again we assume that 2.2.27 gives either some of the components of $\underline{u}_o^2$ or a linear combination of them. Then we consider the $\ell = 0$ component of 2.2.15, i.e.

$$\underline{W}_o \underline{u}_o^3 + \underline{M}_o \frac{\partial \underline{u}_o^2}{\partial \xi} + \underline{N}_o \frac{\partial \underline{u}_o^1}{\partial \tau} + \underline{O}_o \frac{\partial^2 \underline{u}_o^1}{\partial \xi} + \underline{P}_o \underline{u}_o^1 = \underline{S}_o^2$$

where

$$\begin{aligned}
\underline{S}_o^2 &= (\underline{Y}_+ \cdot \underline{R}^*)\underline{R}\phi^* \frac{\partial \phi}{\partial \xi} + (\underline{Y}_- \cdot \underline{R})\underline{R}^*\phi \frac{\partial \phi^*}{\partial \xi} \\[2mm]
&+ (\underline{V}_- \cdot \underline{u}_{-1}^2)\underline{R}^*\phi^* + (\underline{V}_+ \cdot \underline{u}_{-1}^2)\underline{R}\phi \\[2mm]
&+ (\underline{V}_- \cdot \underline{R})\underline{u}_{-1}^2\phi + (\underline{V}_+ \cdot \underline{R}^*)\underline{u}_1^2\phi^* \\[2mm]
&+ (\underline{Z}_+ : \underline{u}_o^1 \underline{R}^*)|\phi|^2 + (\underline{Z}_- : \underline{R}\, \underline{u}_o^1)|\phi|^2 \\[2mm]
&- (\nabla\underline{V}\underline{E}^o : \underline{R}\, \underline{u}_{-1}^2)\phi - (\nabla\underline{V}\underline{E}^o : \underline{R}^*\underline{u}_1^2)\phi^* \\[2mm]
&- (\nabla\nabla\underline{V}\underline{E}^o : \underline{R}\, \underline{R}^*\underline{u}_o^1)|\phi|^2 - \frac{1}{6}(\nabla\nabla\underline{V}\underline{E}^o : \underline{u}_o^1 \underline{u}_o^1 \underline{u}_o^1)
\end{aligned}$$

It can now be seen that provided $\underline{W}_o$ has a particular form, i.e. rows containing all zeros then by integrating the scalar equation derived from this row an explicit form for the corresponding element of $\underline{u}_o^2$ is found. This will in general necessitate the use of boundary or initial conditions.

This indicates how all the components required for the evaluation of the coefficient δ may be found. The usefulness of this approach

is demonstrated in Chapters 4 and 5 where, in view of algebraic
complexity the general method presented above is not used but the
derivation is carried out using the coefficients of the original
equations explicitly.

2.3. Marginally Stable Systems.

In view of the algebraic complexity encountered in 2.1 with the
complex model equation 2.2.1 we will consider a simpler model equation
describing a marginally stable system.   The validity of the result
obtained here for more complex model equations will be assumed without
proof.   The extension of the original result of Taniuti to weakly
unstable systems and more general systems was demonstrated in 2.1 and
the same extension would apply for marginally stable systems.

We therefore consider the model system

$$\frac{\partial \underline{u}}{\partial t} + \underline{A}'(\underline{u}) \; \frac{\partial \underline{u}}{\partial x} + \underline{B}'(\underline{u}) \;\; = \;\; 0 \qquad\qquad 2.3.1$$

where $\underline{u}$ is a column vector with n components u_1, u_2 ... u_n and $\underline{A}'(\underline{u})$
is an n x n matrix, $\underline{B}'$ is an n component vector, both being functions
of $\underline{u}$.   We assume $\underline{A}'$, $\underline{B}'$ exist, are continuous and sufficiently
differentiable.

As in 2.2 we consider a constant solution $\underline{u}_o$ which satisfies

$$\underline{B}'_o \;\; = \;\; \underline{B}'(\underline{u}_o) \;\; = \;\; 0$$

and look for plane wave solutions of 2.3.1 of the form

$$\underline{u} \;\; = \;\; \underline{u}_o + \underline{u}_1 \exp i(kx - \omega t) + c.c$$

Then using the notation of Chapter 2 we see 2.3.1 admits plane wave

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solutions subject to the dispersion relation

$$\det \underline{\underline{H}} = 0 \qquad\qquad 2.3.2$$

where
$$\underline{\underline{H}} = -i\omega\,\underline{\underline{I}} + ik\,\underline{\underline{A}}'_o + \nabla\underline{B}'_o \qquad\qquad 2.3.3$$

we now make the assumption that the dispersion relation gives a complex frequency for a real wavenumber but that this complex frequency is a double root of 2.3.2. This defines a marginally stable state. The dispersion relation must therefore satisfy the condition

$$\frac{\partial \det \underline{\underline{H}}}{\partial \omega} = 0 \qquad\qquad 2.3.4$$

We assume that the frequency ω which satisfies 2.3.2 <u>and</u> 2.3.4 may be written,

$$\omega = \omega_r + i\epsilon\omega_i + O(\epsilon^4) + \ldots$$

where we do not insist that $\omega_r \neq 0$. The method developed in this section is valid even for systems where ω is pure imaginary and the method is used in Chapter 5 for a problem where $\omega_r = 0$. We note that now ω_r is ordered $O(\epsilon)$ for the following reasons. We expect the dispersion relation to have the form:

$$\prod_{p=1}^{n=2} (\omega - a_p)\left((\omega-b)^2 + c\right) = 0$$

where the a_p, b, c are functions of k and u_o. The n-2 roots $\omega = a_1$, $\omega = a_2 \ldots$ represent stable modes such as space charge waves and the root

$$(\omega-b)^2 + c = 0 \qquad\qquad 2.3.5$$

is the marginally stable root.   Since c represents the parameters
which lead to instability it must appear in the equation for the lowest
order amplitude as a term proportional to ϕ and as this is a third
order equation it imposes a maximum order of ε^2 on c.   This implies
that ω_i must be of $O(\varepsilon)$ and ω_r of $O(1)$.   Conversely in the spirit of
the reductive perturbation technique ω_i must be at least $O(\varepsilon)$ and
hence c must be $O(\varepsilon^2)$.   We can therefore conclude that the parameter
which induces the instability is of $O(\varepsilon^2)$ and the instability is $O(\varepsilon)$.

    We must now select the appropriate coordinate stretching for this
type of system.   The stretching used in 2.2. is not appropriate here
since the group velocity λ is not now defined.   The group velocity
can be defined as

$$\lambda = \frac{\partial\omega}{\partial k} = \frac{\partial \det \underline{\underline{H}}}{\partial k} \bigg/ \frac{\partial \det \underline{\underline{H}}}{\partial \omega}$$

and since in this case $\partial \det \underline{\underline{H}}/\partial\omega = 0$ it is now apparently infinite.

    If the nonlinearity were not present then the time development of
the amplitude which reproduces the instability given by 2.3.5 is given
by:

$$\frac{\partial^2 \phi}{\partial t^2} = \omega_i^2 \phi$$

and since ϕ is $O(\varepsilon)$ and ω_i is $O(\varepsilon)$ this requires

$$\tau = \varepsilon t \qquad\qquad 2.3.6$$

Since we now no longer have a well defined group velocity and the spatial

variation must be of a lower order than the temporal variation, the choice

$$\xi = \epsilon^2 x \qquad\qquad 2.3.7$$

seems appropriate for the spatial coordinate stretching.

Having defined the marginally stable state through 2.3.2 and 2.3.4 and chosen the stretched coordinates through 2.3.6 and 2.3.7 we can now derive the equation of motion of the lowest order amplitude of the expansion. We again emphasise that the instability mechanism must be isolated in order that the controlling parameter or parameters may be ordered to give the appropriate growth rate. This is possible for a large class of two-stream instability problems, one of which is considered in Chapter 5.

As in 2.2 we formalise the knowledge of the instability mechanism by rewriting the model system of equations as:

$$\frac{\partial \underline{u}}{\partial t} + \underline{\underline{A}}(\underline{u},p) \frac{\partial \underline{u}}{\partial x} + \underline{B}(\underline{u},p) = 0 \qquad\qquad 2.3.8$$

where p characterises the imaginary part of the frequency and is of order ϵ^2.

We look for solutions of 2.3.8 of the form

$$\underline{u} = \underline{u}_o + \sum_{\alpha=0}^{\infty} \sum_{\ell=-\infty}^{\infty} \epsilon^{\alpha} \underline{u}_{\ell}^{\alpha}(\tau,\xi) \exp i\ell(kx-\omega_r t) \qquad 2.3.9$$

where τ, ξ are the stretched coordinates given by 2.3.6 and 2.3.7 respectively. The amplitude $\underline{u}_{\ell}^{1}$ again contains the factor $\exp(\omega_i \tau)$. The matrix $\underline{\underline{A}}$ and the vector $\underline{B}$ are expanded in a manner analogous to 2.2.8. Substituting these expansions together with 2.3.9 into 2.3.8 and equating powers of ϵ of the same harmonic gives, correct to third

order the system of equations:

$\epsilon = 1$:
$$\underline{\underline{W}}_\ell \underline{u}^1_\ell = 0$$

$\epsilon = 2$:
$$\frac{\partial \underline{u}^1_\ell}{\partial \tau} + \underline{\underline{W}}_\ell \underline{u}^2_\ell = -\nabla \underline{\underline{A}}^0 . <\sum_{mn} \underline{u}^1_m ink\underline{u}^1_n \ P_m P_n >_\ell$$

$$-\tfrac{1}{2}\nabla\nabla\underline{\underline{B}}^0 : <\sum_{mn} \underline{u}^1_m \underline{u}^1_n \ P_m P_n >_\ell$$

$\epsilon = 3$:
$$(i\ell k\underline{\underline{A}}^0_p + \nabla\underline{\underline{B}}_p)\underline{u}^1_\ell + \underline{\underline{A}}^0 \frac{\partial \underline{u}^1_\ell}{\partial \xi} + \frac{\partial \underline{u}^2_\ell}{\partial \tau} + \underline{\underline{W}}_\ell \underline{u}^3_\ell =$$

$$-\nabla\underline{\underline{A}}^0 <\sum_{mn} \underline{u}^2_m ink\underline{u}^1_n \ P_m P_n >_\ell - \nabla\underline{\underline{B}}^0 : <\sum_{mn} \underline{u}^1_m \underline{u}^2_n \ P_m P_n >_\ell$$

$$-\tfrac{1}{2}\nabla\nabla\underline{\underline{A}}^0 : <\sum_{mnp} \underline{u}^1_m \underline{u}^1_n ipk\underline{u}^1_p P_m P_n >_\ell$$

$$-\tfrac{1}{6}\sum\sum\underline{\underline{B}}^0 \vdots <\sum\sum\sum \underline{u}^1_m \underline{u}^1_n \underline{u}^1_p \ P_m P_n P_p >_\ell$$

These are rewritten in concise form as:

$$\underline{\underline{W}}_\ell \underline{u}^1_{\ \ell} = 0 \qquad\qquad\qquad 2.3.10$$

$$\underline{\underline{W}}_\ell \underline{u}^2_\ell + \frac{\partial \underline{u}^1_{\ \ell}}{\partial \tau} = \underline{S}^\ell_1 \qquad\qquad\qquad 2.3.11$$

$$\underline{\underline{W}}_\ell \underline{u}^3_\ell + \frac{\partial \underline{u}^2_\ell}{\partial \tau} + \underline{\underline{A}}^0 \frac{\partial \underline{u}^1_\ell}{\partial \xi} + \underline{\underline{C}}^0 \underline{u}^1_\ell = \underline{S}^\ell_2 \qquad 2.3.12$$

where
$$\underline{\underline{W}}_\ell = -i\ell\omega_r \underline{\underline{I}} + i\ell k\underline{\underline{A}}^0 + \nabla\underline{\underline{B}}^0$$

and
$$\underline{\underline{C}}^0 = i\ell k\underline{\underline{A}}^0_p + \nabla\underline{\underline{B}}_p$$

where $\underline{I}$ is the unit matrix.

We now assume

$$\det \underline{\underline{W}}_{\pm 1} = 0$$

i.e. $\qquad |-i\omega_r \underline{\underline{I}} + ik\underline{\underline{A}}^o + \nabla\underline{\underline{B}}^o| = 0$ \hfill 2.3.13

This by assumption gives a single real root ω_r. As mentioned previously this method is valid if 2.3.13 gives $\omega_r = 0$. Then 2.3.13 represents an additional condition which must be imposed on the equilibrium state $\underline{u}_o$ and the equivalent condition would be

$$\det |ik\underline{\underline{A}}^o + \nabla\underline{\underline{B}}^o| = 0 \qquad 2.3.14$$

However, a real non-zero frequency ω_r is assumed to exist since this is required in the following algebra. The condition $\omega_r = 0$ may be imposed on the final result.

Again we assume;

$$\det \underline{\underline{W}}_\ell \neq 0 \quad \text{for} \quad \ell \neq \pm 1$$

and so conclude

$$\underline{u}_1^1 = \phi(\tau,\xi)\underline{R} \qquad 2.3.15$$

$$\underline{u}^1{}_\ell = 0 \qquad |\ell| \neq 1 \qquad 2.3.16$$

where $\qquad \underline{\underline{W}}_1\underline{R} = 0$ \hfill 2.3.17

The difficulty encountered for $\ell = 0$ in 2.2 arises in this case and we assume

$$\det \underline{\underline{W}}_o \neq 0$$

Considering the second order equation for $\ell = 1$ gives

$$\underline{\underline{W}}_1 \underline{u}_1^2 + \frac{\partial \underline{u}_1^1}{\partial \tau} = \underline{S}_1^1$$

Substituting for $\underline{u}_1^1$ in this equation from 2.3.15 and 2.3.16 gives

$$\underline{\underline{W}}_1 \underline{u}_1^2 + \underline{R} \frac{\partial \phi}{\partial \tau} = 0 \qquad\qquad 2.3.18$$

Since $\det \underline{\underline{W}}_1 = 0$ a compatibility condition must be satisfied in order that $\underline{u}_1^2$ is unique.   Multiplying this equation by the left eigenvector $\underline{L}$ of $\underline{\underline{W}}$ on the left gives this condition, i.e.

$$\underline{L} . \underline{R} \frac{\partial \phi}{\partial \tau} = 0$$

i.e. the scalar product of the left and right eigenvectors of $\underline{\underline{W}}$ must vanish.   We now show that this compatibility condition is automatically satisfied.   Differentiating the identity 2.3.17 with respect to ω gives:

$$\frac{\partial}{\partial \omega} (\underline{\underline{W}}_1 \underline{R}) = \frac{\partial \underline{\underline{W}}_1}{\partial \omega} \underline{R} + \underline{\underline{W}}_1 \frac{\partial \underline{R}}{\partial \omega} = 0$$

But from the definition of $\underline{\underline{W}}_1$

$$\frac{\partial \underline{\underline{W}}_1}{\partial \omega} = -i \underline{\underline{I}}$$

and hence $\qquad\qquad \underline{R} = -i \underline{\underline{W}} \frac{\partial \underline{R}}{\partial \omega} \qquad\qquad 2.3.19$

and multiplying on the left by $\underline{L}$ gives immediately

$$\underline{L}.\underline{R} = 0$$

In view of 2.3.19, 2.3.18 may be rewritten as

$$\underline{W}_1 \left(\underline{u}_1^2 - i \frac{\partial R}{\partial \omega} \frac{\partial \phi}{\partial \tau} \right) = 0$$

which may be immediately solved to give

$$\underline{u}_1^2 = \phi^{(2)}(\tau, \xi)\underline{R} + i \frac{\partial R}{\partial \omega} \frac{\partial \phi}{\partial \tau} \qquad 2.3.20$$

where $\phi^{(2)}$ is to be determined to higher order. The remaining second order components are determined by direct solution of the $\ell = 0$, $\ell = 2$ forms of 2.3.11, i.e. for $\ell = 0$

$$\underline{W}_0 \underline{u}_0^2 = \underline{S}_1^0$$

$$= \{(\nabla\underline{A}^0.\underline{R})\underline{R}^*ik - (\nabla\underline{A}^0.\underline{R}^*)\underline{R}ik$$

$$+ \nabla\nabla\underline{B}^* : (\underline{R}\ \underline{R}^*)\}|\phi|^2$$

i.e. $\qquad \underline{u}_0^2 = \underline{W}_0^{-1} \underline{S}_1^0$

or $\qquad \underline{u}_0^2 = \underline{R}_0^2|\phi|^2 \qquad 2.3.21$

where $\qquad \underline{R}_0^2 = \underline{W}_0^{-1} \{(\nabla\underline{A}^0.\underline{R})\underline{R}^*ik - (\nabla\underline{A}^0.\underline{R}^*)\underline{R}ik + \nabla\nabla\underline{B}^0:(\underline{R}\ \underline{R}^*)\}$

and once again we have assumed $\underline{W}_0^{-1}$ exists. Similarly,

$$\underline{u}_2^2 = \underline{R}_2^2(\phi)^2 \qquad 2.3.22$$

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where
$$\underline{R}_2^2 \;=\; -\underline{W}_2^{-1}\,\left|(\nabla\underline{A}^{o}.\underline{R})\underline{R} + \tfrac{1}{2}(\nabla\nabla\underline{B}^{o}:\underline{R}\,\underline{R})\right|$$

Finally, we consider the $\ell = 1$ form of 2.3.12 and substitute for $\underline{u}^1$ from 2.3.15,16 for $\underline{u}^2$ from 2.3.20, 2.3.21 and 2.3.22 to obtain

$$\underline{W}_1\underline{u}_1^3 + \frac{\partial}{\partial\tau}\{\phi^2(\tau,\xi)\underline{R} + i\,\frac{\partial R}{\partial\omega}\}\,\frac{\partial\phi}{\partial\tau} + \underline{A}^{o}\underline{R}\,\frac{\partial\phi}{\partial\xi}$$

$$+ \underline{C}^{o}\underline{R}\phi \;=\; \underline{S}_2^1\phi|\phi|^2$$

where

$$\underline{S}_2^1 \;=\; \left[-(\nabla\underline{A}^{o}.\underline{R}_o^2)\underline{R}ik + (\nabla\underline{A}^{o}.\underline{R}_2^2)\underline{R}^*ik + \tfrac{1}{2}(\nabla\nabla\underline{A}^{o}:\underline{R}\,\underline{R})\underline{R}^*ik\right.$$

$$-(\nabla\nabla\underline{A}^{o}.\underline{R}\,\underline{R}^*)\underline{R}ik - (\nabla\nabla\underline{B}^{o}:\underline{R}\,\underline{R}_o^2) - (\nabla\nabla\underline{B}^{o}.\underline{R}^*\underline{R}_o^2)$$

$$\left.-\tfrac{1}{2}(\nabla\nabla\nabla\underline{E}^{o}:\underline{R}\,\underline{R}\,\underline{R}^*)\right]$$

Mulitplying on the left by $\underline{L}$ we see the first two terms vanish and finally we are left with

$$i\,\underline{L}\,\frac{\partial R}{\partial\omega}\,\frac{\partial^2\phi}{\partial\tau^2} + \underline{L}\,\underline{A}^{o}\underline{R}\,\frac{\partial\phi}{\partial\xi} + \underline{L}\,\underline{C}^{o}\underline{R}\phi - \underline{L}\,\underline{S}_2^1\phi|\phi|^2 \;=\; 0$$

which may be rewritten as:

$$i\,\frac{\partial\phi}{\partial\xi} \;=\; a\,\frac{\partial^2\phi}{\partial\tau^2} + b\,\phi|\phi|^2 + c\phi \;=\; 0 \qquad\qquad 2.3.23$$

where
$$a \;=\; \frac{\underline{L}\,\partial R/\partial\omega}{\underline{L}\,\underline{A}^{o}\underline{R}}$$

$$b \;=\; \frac{-i\underline{L}.\underline{S}_2^1}{\underline{L}\,\underline{A}^{o}.\underline{R}}$$

$$c = \frac{i\underline{L} \cdot \underline{\underline{C}}^o \cdot \underline{R}}{\underline{L} \cdot \underline{\underline{A}}^o \cdot \underline{R}}$$

We may now derive more general expressions for the coefficients, a and c. We consider the linear dispersion relation 2.3.2

$$\det\underline{\underline{H}} = 0$$

which may be rewritten as

$$\det\underline{\underline{H}} = D(\omega,k,p) = D(\omega_r + i\epsilon\omega_i, k, p_o + \epsilon^2 p_1) = 0$$

where we have included formally the small imaginary part ω_i of ω and the parameter p which characterises the instability. We expand $D(\omega,k,p)$ in a Taylor series around (ω_r, k, p_o) to obtain

$$D(\omega_r + i\epsilon\omega_i, k, p_o + \epsilon^2 p_1) = D(\omega_r, k, p_o) + i\epsilon\omega_i \left.\frac{\partial D}{\partial\omega}\right|_{\omega_r, k, p_o}$$

$$- \epsilon^2\omega_i^2 \left.\frac{\partial^2 D}{\partial\omega^2}\right|_{\omega_r, k, p_o} + \epsilon^2 p_1 \left.\frac{\partial D}{\partial p}\right|_{\omega_r, k, p_o} + \ldots\ldots$$

But: $D(\omega_r + i\epsilon\omega_i, k, p_o + \epsilon^2 p_1) = 0$ from the dispersion relation

$D(\omega_r, k, p_o) = 0$ from the subsidiary condition on the equilibrium

and $\left.\partial D/\partial\omega\right|_{\omega_r, k, p_o} = 0$ from the condition of marginal stability

which gives immediately

$$\omega_i = \sqrt{p_1 \, \partial D/\partial p \left| \partial^2 D/\partial\omega^2 \right.} \qquad\qquad 2.3.24$$

We now state the following two results for which a general proof has not been found:

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$$a \; \alpha \; \partial^2 D/\partial\omega^2 \qquad\qquad c \; \alpha \; p \; \partial D/\partial p$$

where the constant of proportionality is equal in both cases. (These results have been verified directly for 3 x 3 matrices of the form of $\underline{\underline{H}}$ and in the problem considered in Chapter 5). Then 2.3.24 gives

$$\omega_i \; = \; \sqrt{c/a}$$

We may now consider solutions of 2.3.23 using the analogy of a point particle moving in a potential field as suggested by Asano, Taniuti and Yajima (1969). We consider periodic boundary conditions consistent with some fundamental wavevector k and therefore only look for solutions of

$$a \frac{\partial^2 \phi}{\partial\tau^2} + b\phi|\phi|^2 + c\phi \; = \; 0 \qquad\qquad 2.3.25$$

We attempt solutions in a polar representation of the form

$$\phi(\tau) \; = \; \theta(\tau)\exp(i\psi(\tau))$$

where $\theta(\tau)$ and $\psi(\tau)$ are real. Substituting this into 2.3.25 and equating real and imaginary parts gives

$$\frac{\partial^2 \theta}{\partial\tau^2} - \theta\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{b}{a}\theta^3 + \frac{c}{a}\theta \; = \; 0 \qquad\qquad 2.3.26$$

$$\theta \frac{\partial^2 \psi}{\partial\tau^2} + 2 \frac{\partial\theta}{\partial\tau} \frac{\partial\psi}{\partial\tau} \; = \; 0 \qquad\qquad 2.3.27$$

Equation 2.3.27 may be integrated immediately to give

$$\theta^2 \frac{\partial\psi}{\partial\tau} \; = \; f \; = \; \text{constant} \qquad\qquad 2.3.28$$

If we consider θ as a "radial coordinate" and ψ as an "angular

coordinate", i.e. (θ,ψ) as a "polar coordinate" pair then 2.3.26 and 2.3.28 are seen to be equivalent to the equations of motion of a point particle in a central field, Kibble (1966), i.e. we rewrite 2.3.26 and 2.3.28 as

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \;=\; E$$

$$mr^2\dot{\theta} \;=\; J$$

where $V(r)$ is the central field, E the total energy and J the total angular momentum. The central field for 2.3.26 is given by

$$V(\theta) \;=\; \frac{b}{4a}\,\theta^4 + \frac{c}{2a}\,\theta^2 \qquad\qquad 2.3.29$$

and so the complex amplitude is considered as a "particle" moving in a two-dimensional potential well. We therefore look for solutions of (θ,ψ) as a function of τ.

We consider separate cases and represent the two-dimensional potential well in the $V(\theta)$, θ plane. The full potential function is merely obtained by rotating the curve through 180° around the $V(\theta)$ axis.

1. $b/4a > 0$, $c/2a < 0$.

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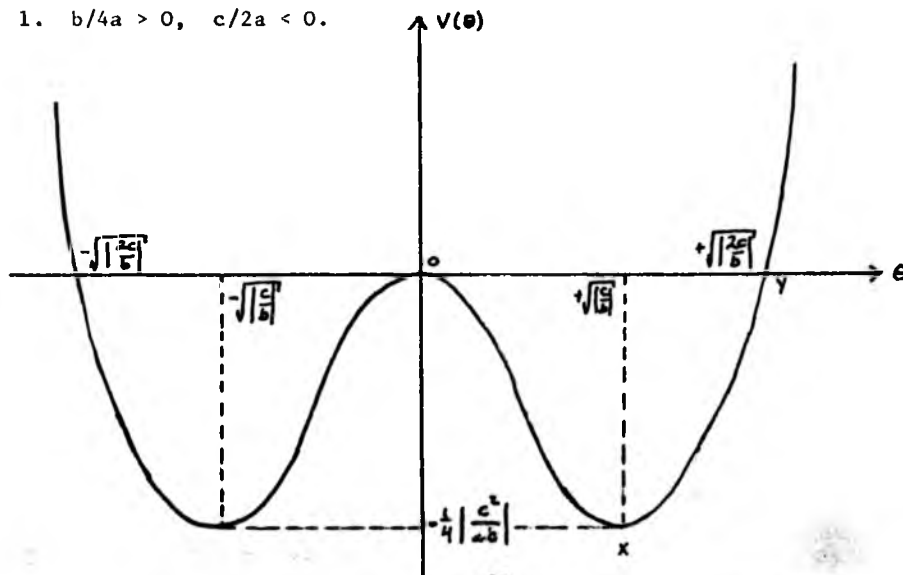
2. $b/4a < 0$, $c/2a < 0$
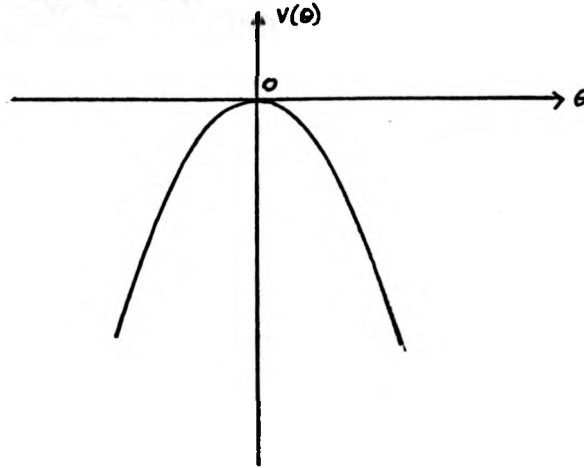
3. $C/2a > 0$, $b/4a < 0$

4. $C/2a < 0$, $b/4a > 0$

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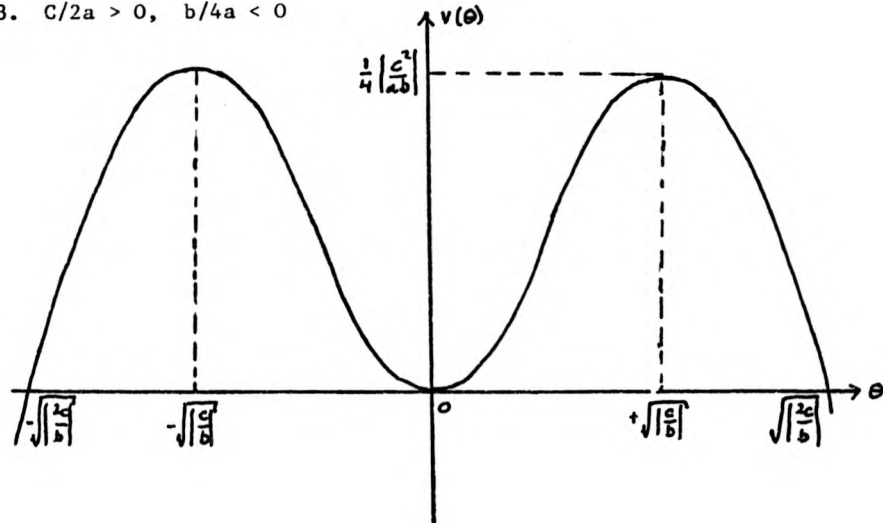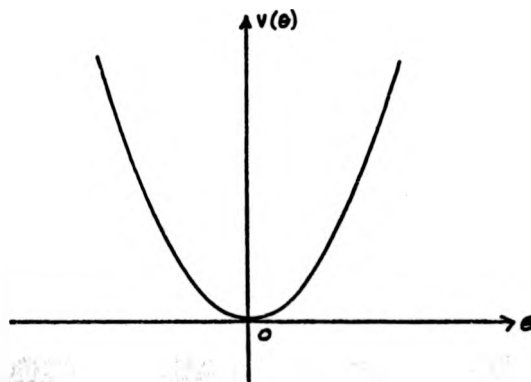
We note that the condition $c/2a < 0$ in the linear theory represents either an exponentially growing or exponentially decaying solution, i.e. stability or instability. If $c/2a > 0$ the linear solutions are oscillatory.

To analyse the nonlinear behaviour of the waves we must impose initial conditions and we will now show that the initial conditions are critical for the time evolution of the wave. From the dispersion relation 2.3.3 we have n roots $\omega_1 \ldots \omega_n$. We will consider the case when we have two modes ω_1 and ω_2 near the marginal state and suppose that the remaining modes are all oscillatory. We can therefore write for $\underline{u}(x,t)$:

$$\underline{u}(x,t) = \underline{U}_o + \epsilon \sum_{j=1}^{n} a_j \underline{R}_j(\omega_j) \exp i(kx - \omega_j t) \qquad 2.3.30$$

correct to first order, where the $\underline{R}_j$ are the right eigenvectors corresponding to the jth mode and the a_j are complex amplitudes. Since, by assumption the modes $\underline{R}_1$ and $\underline{R}_2$ are close to the marginal state we may expand these as

$$\underline{R}_1(\omega_1) = \underline{R}(\omega + i\omega_i)$$

$$= \underline{R}(\omega) + i \frac{\partial \underline{R}}{\partial \omega} \epsilon \omega_i + \ldots$$

and

$$\underline{R}_2(\omega_2) = \underline{R}(\omega - i\omega_i)$$

$$= \underline{R}(\omega) - i\epsilon \frac{\partial \underline{R}}{\partial \omega} \omega_i + \ldots$$

since the roots ω_1 and ω_2 are complex conjugates. Therefore substituting in 2.3.30 gives

$$\underline{u}(x,t) = U_o + \epsilon(a_1 e^{\omega_i t} + a_2 e^{-\omega_i t})\underline{R}(\omega)\exp i(kx-\omega t)$$

$$+ \epsilon^2(a_1 e^{\omega_i t} - a_2 e^{-\omega_i t})\omega_i\ i\ \frac{\partial R}{\partial \omega}\exp i(kx-\omega t)$$

$$+ \epsilon \sum_{j=3}^{n} a_j \underline{R}_j \exp i(kx-\omega t). \qquad\qquad 2.3.31$$

We may now compare this result with the expansion 2.3.9 where we substitute for $\underline{u}_1^1$ from 2.3.15 and $\underline{u}_1^2$ from 2.3.20, i.e.

$$\underline{u} = \underline{u}_o + \epsilon\underline{R}\phi\exp i(kx-\omega t) + \epsilon^2 i\ \frac{\partial R}{\partial \omega}\frac{\partial \phi}{\partial \tau}\exp i(kx-\omega t) \qquad 2.3.32$$

Comparing the expressions 2.3.31 and 2.3.32 for $\tau = 0$ gives

$$\phi(\tau = 0) = a_1 + a_2$$

$$\frac{\partial \phi}{\partial \tau}(\tau = 0) = (a_1 - a_2)\omega_i$$

which constitute the boundary conditions.

We therefore see that the initial conditions for ϕ and $\partial\phi|\partial\tau$ depend on the amount of stable and unstable modes in the initial value of $\underline{u}$. The constants a_1 and a_2 are in general complex but without loss of generality the initial conditions may be rewritten as

$$\phi(\tau = 0) = A$$

$$\frac{\partial \phi}{\partial \tau}(\tau = 0) = B$$

where we assume A is real and B is complex. We may now discuss the four cases separately.

2. $b/4a < 0$, $c/2a < 0$. This case corresponds to linear instability.
The potential function is a simple hill and therefore irrespective
of the initial conditions the growth of the instability is enhanced
by the nonlinearity.   The nonlinear waves do not saturate and
additional mechanisms must be introduced to obtain a finite amplitude
saturated wave.

3. $b/4a < 0$, $c/2a > 0$. This case corresponds to linear stability.
The potential function is a hill with a depression at the centre.
The development of the wave is dependent on the initial conditions.
If the amplitude is sufficiently small then we have a constant
amplitude solution with a frequency shift.   The magnitude of this
frequency shift decreases as the amplitude increases.   The solution
is given by

$$\phi = \phi_0 \exp (-i\alpha\tau)$$

where
$$\alpha = + \left|\frac{b}{a}\right| |\phi_0|^2 - \left|\frac{c}{a}\right|$$

which is valid for
$$|\phi|^2 < \frac{c}{b}$$

This frequency shift becomes zero when the amplitude becomes $\sqrt{c/b}$.
Any further increase in the amplitude makes the wave unstable.   This
is equivalent to the "particle" starting at the origin in the well and
acquiring sufficient energy to climb out of the well and be accelerated
to infinity.   This is consistent with choosing the amplitudes of the
initial wave to be

$$A = 2\alpha$$

$$B = 2i\beta\omega_i$$

or in terms of the amplitudes of the original waves

$$a_1 = \alpha + i\beta$$

$$a_2 = \alpha - i\beta$$

This indicates that initially the wave is composed of two linearly independent modes of equal amplitude but with a phase difference of π giving a small total amplitude. As this phase difference decreases with increasing amplitude the system becomes unstable after the critical amplitude $\sqrt{c/b}$ is attained.

Therefore, a wave that is stable in the linear theory becomes unstable against finite amplitude perturbations in the nonlinear theory.

4. $b/4a > 0$, $c/2a > 0$.

The potential function is an infinitely deep well and so whatever the initial nature of the wave in the nonlinear case the wave will always be stable.

1. $b/4a > 0$, $c/2a < 0$.

The potential function is a well with a small peak in its centre. Suppose initially that the wave consists of a single unstable mode. Then, the boundary conditions for this situation become

$$\phi_o = \frac{\partial \phi_o}{\partial \tau} = \alpha$$

where α is small. The solution in this case is then

$$\phi = \sqrt{\left|\frac{2c}{b}\right|} \, \text{sech}\left\{\sqrt{\left|\frac{c}{a}\right|} \, (\tau - \tau_o)\right\}$$

where τ_o is implicitly defined by

$$\alpha = \sqrt{\left|\frac{2c}{b}\right|} \text{ sech}\left\{\sqrt{\left|\frac{c}{a}\right|} \, \tau_o\right\}$$

In this case the amplitude grows with the linear growth rate and reaches the maximum value at $\tau = \tau_o$. For times longer than τ_o the amplitude decreases. This corresponds to the motion of the particle from 0 to Y through X and then a return to 0 asymptotically.

If the amplitude is larger than the critical value $\sqrt{|c/b|}$ then we again have a constant amplitude wave with an amplitude dependent frequency shift, i.e.

$$\phi = \phi_o \exp\{-i\alpha\tau\}$$

where

$$\alpha = \left|\frac{b}{a}\right| |\phi_o|^2 - \left|\frac{c}{a}\right|$$

We can therefore summarise these results as follows. If the signs of the coefficients in 2.3.25 are such that $b/4a < 0$ then regardless of whether the system is stable or unstable in the linear theory the nonlinear system always exhibits instability. If $b/4a > 0$ then once again irrespective of the stability or instability of the linear system, in the nonlinear case the system is always stable.

Even though a general expression has been given for the coefficients a,b,c we only know the relative sign of the coefficients c and a, since the sign determines the stability or instability of the linear theory. The coefficient b may, in general, be positive or negative and must be separately evaluated for each system. We may, however, suggest that the coefficient b will be a function of k and other parameters of the system. The system will therefore be stable or unstable in the nonlinear theory only for a range of wavenumbers and parameters. The two-stream instability considered in Chapter 5 gives values of the

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coefficients such that $b/4a < 0$ and so is always unstable for all wavevectors. This may not always be true if a more complete model is considered.

Finally, the discussion at the end of 2.2. regarding the derivation of the coefficients of $\underline{u}_0^\alpha$ is equally valid for the stretching suggested here. Although a proof of this is not available we demonstrate in Chapter 5 that the suggested method is successful in determining these coefficients for the system of equations considered in that Chapter.

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Chapter 3

The Reductive Perturbation Technique for
Two Dimensional Systems

3.1 Introduction

In Chapters 1 and 2 we considered one dimensional physical systems
and developed the reductive perturbation expansion method to study the
nonlinear behaviour of monochromatic waves propagating in these one
dimensional systems.   Using a model equation in Chapter 1 and a more
general equation in Chapter 2 we showed that the amplitude of this wave
must satisfy a nonlinear Schrödinger or a generalised nonlinear Schrödinger
equation if nonlinear self-interactions are to be taken into account.
The model equations 2.2.1 and 2.3.1 can be used to describe a wide class
of physical systems, e.g. the electron plasma wave in the hydrodynamic
approximation Taniuti, Asano and Yajima (1969), waves in a cold plasma
in an external magnetic field, Kako (1972), and a wide range of fluid
dynamic problems.   The extensions of the method as presented in Chapter 2
extend this class of problems to include the propagation of acoustic waves
in semiconductors (Chapter 4) and marginally stable plasma streams
(Chapter 5).

A natural extension of the reductive perturbation technique is a
modification to include the effect of two dimensions on the propagation
of plane waves.   Such an extension would broaden the class of physical
systems that could be studied using the perturbation technique and inc-
lude inhomogeneous plasma stream systems, Zhelyazkov and Rukhadze (1972),
MacFarlane and Hay (1950) and two dimensional fluid dynamics systems
Stewartson and Stuart (1971).

An attempt to do this was made by Hasegawa (1970) who considered
modulational instabilities of plasma waves in two dimensions and the

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problem of self-focussing of laser beams. Hasegawa used stretched
time and space scales in both spatial coordinates and hence assumed that
the effect of the additional spatial dimension was only significant to
second order. This excludes the majority of the two dimensional
problems described above since in these the additional spatial dimension,
normally perpendicular to the direction of wave propagation has a signifi-
cant effect to first order. The equation that Hasegawa derived was a
linear Schrödinger equation of the form:

$$i \frac{\partial \phi}{\partial \tau} + \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \frac{\partial^2 \phi}{\partial \xi^2} + a \frac{\partial^2 \phi}{\partial \eta^2} + k \phi = 0$$

where ξ is the stretched coordinate in the direction of the wave propagation
and η is the stretched coordinate perpendicular to this wave direction.
The use of the stretched variable η avoided a lot of the algebraic complex-
ity that we encounter in this chapter and essentially required only a minor
modification of the original method of Taniuti and Yajima (1969).

Stewartson and Stuart (1971) consider plane Poiseuille flow in an
incompressible viscous fluid. They found the the amplitude of a small
but finite wave satisfied a nonlinear Schrödinger equation when the
Reynolds number exceeded slightly the critical Reynolds number. This
was found to be an asymptotic solution a long time after the initial dis-
turbance. The method they used was a combination of coordinate scaling
and multiple time scale analysis. In view of the discussion in Chapter 1
and the extensions proposed in Chapter 2 i.e. an asymptotic solution for
a weakly unstable system, appears to be ideally suited to a reductive
perturbation technique in two dimensions. The method of Hasegawa is not
applicable to this system as the unperturbed state around which an expan-
sion is made is a function of the coordinate perpendicular to the direc-
tion of wave propagation.

An analogous problem in plasma physics is the so-called crossed field instability as originally considered by Macfarlane and Hay (1950). A wave with wavelength greater than some critical value in an electron plasma flowing between parallel conductors in crossed electric and magnetic fields is found to be unstable. The unperturbed steady state in this case is a function of the second spatial variable and it is believed that this additional dependence of particularly the velocity is a direct cause of the instability. (Further consideration is given to this problem in Chapter 6). Again this system would appear to be suited to an asymptotic expansion as the strength of the instability could be "controlled" by a particular choice of wavenumber.

In this chapter we extend the reductive perturbation technique to apply to systems that have two spatial coordinates x,y where x is the direction of wave propagation and y is perpendicular to this direction. The unperturbed steady state $\underline{u}_o$ around which the expansion is made is assumed to be a function of y. For the sake of clarity and algebraic simplicity we consider a system of first order equations as a model equation. In 3.2 the general properties of $\underline{U}_o$ are considered and the dispersion relation is derived by using a simple linearised theory. A nonlinear expansion is considered in 3.3 and through the general theory of linear systems of first order differential equations (Appendix) the amplitude of the wave is found to satisfy the generalised nonlinear Schrödinger equation. In the discussion of 3.4 an indication is given of how the method may be extended to apply to more complex higher order systems of equations and further consideration is given to the problems discussed above.

3.2 Solutions of the linearised systems

We consider the following systems of equations:

$$\frac{\partial \underline{u}}{\partial t} + \frac{A}{-} \frac{\partial \underline{u}}{\partial x} + \frac{B}{-} \frac{\partial \underline{u}}{\partial y} + \underline{E} = 0 \qquad\qquad 3.2.1$$

where $\underline{u} = \underline{u}(x,y,t)$ is a column vector of n components $u_1, u_2 \ldots u_n$ and $\underline{A}, \underline{B}$ are continuous nxn matrices being functions of $\underline{u}$ and $\underline{E}$ is an n component vector, also a function of $\underline{u}$. We assume that these vectors and matrices exist, are non zero, are differentiable and satisfy certain conditions as indicated later. Further, we assume that the system is unbounded in the x direction but is bounded in the y direction, i.e.

$$-\infty < x < +\infty$$

$$a \leqslant y \leqslant b.$$

where a and b are constants. In order that the problems be well posed we add the following boundary conditions

$$\underline{M} \ \underline{u}(a) \ + \ \underline{N} \ \underline{u}(b) \ = \ 0 \qquad\qquad 3.2.2$$

where $\underline{u}(a) = \underline{u}(x,y,t)$ at $y = a$ and $\underline{M}, \underline{N}$ are nxn constant matrices.

We look for a time independent steady state solution $\underline{u}^o(y)$ where $\underline{u}^o$ is defined by being a solution of

$$\underline{B}^o \ \frac{d\underline{u}^o}{dy} + \underline{E}^o \ = \ 0 \qquad\qquad 3.2.3$$

subject to the boundary conditions

$$\underline{M} \ \underline{u}^o(a) + \underline{M} \ \underline{u}^o(b) \ = \ 0 \qquad\qquad 3.2.4$$

We suppose that there exists a solution of 3.2.3 subject to 3.2.4 which can be explicitly determined.

We now look for solutions around this steady state of the form:

$$\underline{u}(x,y,t) \ = \ \underline{u}^o(y) + \underline{u}^1(y)\exp(i(kx-\omega t)) + C.C$$

Substituting this expression and linearising about $\underline{u}^o(y)$ shows that $\underline{u}^1(y)$

must satisfy the equation

$$(\underset{\sim}{W} - \omega\underline{I}) \; \underline{u}^1(y) \; = \; 0 \qquad\qquad 3.2.5$$

where $\underset{\sim}{W}$ is the matrix differential operator defined by:

$$\underset{\sim}{W} \; = \; - \; i \; \underline{\underline{B}}^o \; \frac{d}{dy} \; - \; i \; \nabla.\underline{\underline{B}}^o \; \frac{d\underline{u}^o(y)}{dy} \; + \; k\underline{\underline{A}}^o \; - \; i\nabla.\underline{\underline{E}}^o \qquad\qquad 3.2.6$$

where the same notation is used as in Chapter 2, i.e.

$$(\nabla.\underline{\underline{E}}^o)_{ij} \; = \; \left(\frac{\partial\underline{E}_i}{\partial u_j}\right)_{\underline{u}=\underline{u}^o}$$

and

$$\left(\nabla.\underline{\underline{B}}^o \; \frac{\partial\underline{u}}{\partial y}\right)_{i,j} \; = \; \underset{k}{\Sigma} \; \frac{\partial B_{ik}}{\partial u_j} \; \frac{\partial u_k}{\partial y}$$

From 3.2.2 we see that $\underline{u}^1(y)$ must satisfy the boundary conditions

$$\underline{\underline{M}} \; \underline{u}^1(a) \; + \; \underline{\underline{N}} \; \underline{u}^1(b) \; = \; 0 \qquad\qquad 3.2.7$$

We have now reduced the problem to a linearised eigenvalue problem for a matrix differential operator and suppose that 3.2.5 gives a single real eigenvalue ω_r of the operator $\underset{\sim}{W}$. This assumes that the problem is well posed and admits a non-trivial solution.

The explicit evaluation of ω_r is now reduced to the solution of a matrix equation provided solutions of 3.2.5 can be found. This is achieved as follows:

Equation 3.2.5 is a system of n first order ordinary differential equations with n linearly independent solutions $\underline{R}_j(y)$, j = 1 .... n where $\underline{R}_j(y)$ is an n component vector with components denoted by $r_{ij}(y)$, ... $r_{nj}(y)$. The general solution of 3.2.5 is given by a linear combination of these solutions as indicated in the appendix, i.e.

$$\underline{u}^1(y) = \sum_{i=1}^{n} c_i \underline{R}_i(y) \qquad\qquad 3.2.8$$

where the c_i are constants. This together with 3.2.7 gives

$$\sum_{j=1}^{n} \mathbf{c}_j \left[\underline{\underline{M}} \, \underline{R}_j(a) + \underline{\underline{N}} \, \underline{R}_j(b)\right] = 0 \qquad\qquad 3.2.9$$

We are now able to determine the dispersion relation and so deduce the eigenvalues of $\underline{\underline{W}}$. This relation is obtained by insisting that the c_i are non-trivial, a condition necessary if 3.2.8 is to represent a general solution.

This condition is seen to be

$$\det \left| \underline{\underline{M}} \, \underline{\underline{\Phi}}(a) + \underline{\underline{N}} \, \underline{\underline{\Phi}}(b) \right| = 0 \qquad\qquad 3.2.10$$

where $\Phi(y)$ is an nxn matrix, called the fundamental matrix whose (i,j)th element is given by:

$$(\Phi(y))_{ij} = r_{ij}(y) = (\underline{R}_j(y))_i$$

Equation 3.2.10 is readily derived by writing 3.2.9 in the form

$$\left[\underline{\underline{M}} \, \underline{\underline{\Phi}}(a) + \underline{\underline{N}} \, \underline{\underline{\Phi}}(b)\right] \underline{C} = 0 \qquad\qquad 3.2.11$$

where $\underline{C}$ is a column vector with components $c_1 \ldots c_n$. Solving 3.2.10 gives, by assumption, a single real eigenvalue ω_r. We now characterise the matrix $\underline{\underline{\Phi}}$ and hence the column vectors $\underline{R}_j(y)$ and $\underline{C}$ by this eigenvalue by writing $\underline{\underline{\Phi}}(\omega_r, y)$, $\underline{R}_j(\omega_r, y)$ and $\underline{C}(\omega_r)$ and write 3.2.11 as

$$\underline{\underline{D}}(\omega_r) \, \underline{C} = 0 \qquad\qquad 3.2.12$$

where $\qquad \underline{\underline{D}}(\omega_r) = \underline{\underline{M}} \, \underline{\underline{\Phi}}(\omega_r, a) + \underline{\underline{N}} \, \underline{\underline{\Phi}}(\omega_r, b)$

Since from 3.2.10 det $\underline{\underline{D}}(\omega_r) = 0$ we may solve 3.2.11 to within an arbitrary

multiplicative constant and so can write

$$\underline{C}(\omega_r) \;=\; \psi_1 \, \underline{C}^1(\omega_r)$$

We have therefore solved 3.2.5 for $\underline{u}^1(y)$ and so have deduced the eigen-value ω_r . $\underline{u}^1(y)$ is now given by

$$\underline{u}^1(y) \;=\; \sum_{i=1}^{n} c_i \underline{R}_i(y)$$

$$=\; \underline{\underline{\Phi}}(\omega_r,y) \, \underline{C}$$

$$=\; \psi_1 \underline{R}(y) \qquad\qquad 3.2.13$$

where $\underline{R}(y)$ is defined by:

$$\underline{R}(y) \;=\; \sum_{i} c_i'(\omega_r) \underline{R}_i(y)$$

Having determined the linearised dispersion relation and having solved the linearised system for a small perturbation we are now ready to consider the full nonlinear expansion. It must be emphasised that the dispersion relation 3.2.10 and hence the linearly independent solutions of 3.2.5 must in principle be known if progress is to be made.

3.3 Solutions of the nonlinear system

We again consider the model system of equations

$$\frac{\partial \underline{u}}{\partial t} + \underline{\underline{A}} \, \frac{\partial \underline{u}}{\partial x} + \underline{\underline{B}} \, \frac{\partial \underline{u}}{\partial y} + \underline{E} \;=\; 0$$

and assume the following form for $\underline{u}$:

$$\underline{u} \;=\; \underline{u}^0 + \sum_{\alpha=1}^{\infty} \sum_{\ell=-\infty}^{\infty} \varepsilon^{\alpha} \underline{u}_{\ell}^{\alpha}(\tau,\xi,y) \exp(i\ell(kx-\omega t)) \qquad 3.3.1$$

where $\qquad \xi = \epsilon(x - \lambda t)$

$$\tau = \epsilon^2 t$$

$$\lambda = \partial \omega_r / \partial k$$

and $\qquad (\underline{u}_\ell^\alpha)^* = \underline{u}_{-\ell}^\alpha$

We expand the matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$ and the vector $\underline{E}$ as follows:

$$\underline{\underline{A}} = \underline{\underline{A}}^o + \epsilon \nabla \underline{\underline{A}}^o \cdot \underline{u}^1 + \epsilon^2 (\nabla \underline{\underline{A}}^o \cdot \underline{u}^2 + \tfrac{1}{2} \nabla \nabla \underline{\underline{A}}^o : \underline{u}^1 \underline{u}^1)$$

$$+ \epsilon^3 (\nabla \underline{\underline{A}}^o \cdot \underline{u}^3 + \nabla \nabla \underline{\underline{A}}^o : \underline{u}^1 \underline{u}^2 + \tfrac{1}{6} \nabla \nabla \nabla \underline{\underline{A}}^o : \underline{u}^1 \underline{u}^1 \underline{u}^1)$$

$$+ \ldots$$

etc., and substitute these expansions together with 3.3.1 into 3.2.1. As before, equating powers of ϵ of the same harmonic to zero gives an infinite system of equations the first three of which are given by:

$O(\epsilon)$ $\qquad \underset{\sim}{\underline{W}}_\ell \underline{u}^1_\ell = 0$ $\qquad\qquad\qquad\qquad\qquad\qquad$ 3.3.2

$O(\epsilon^2)$ $\qquad \underset{\sim}{\underline{W}}_\ell \underline{u}^2_\ell + (-\lambda \underline{\underline{I}} + \underline{\underline{A}}^o) \dfrac{\partial \underline{u}^1}{\partial \xi}_\ell$

$$= - \langle \underset{mn}{\Sigma\Sigma} (\nabla \underline{\underline{B}}^o \cdot \underline{u}^1_m \dfrac{\partial \underline{u}^1}{\partial y}_n + ink \, \nabla \underline{\underline{A}}^o \cdot \underline{u}^1_n \underline{u}^1_m$$

$$+ \tfrac{1}{2} \nabla \nabla \underline{E}^o : \underline{u}^1_m \underline{u}^1_n) P_m P_n \rangle_\ell \qquad\qquad 3.3.3$$

$O(\epsilon^3)$ $\qquad \underset{\sim}{\underline{W}}_\ell \underline{u}^3_\ell + (-\lambda \underline{\underline{I}} + \underline{\underline{A}}^o) \dfrac{\partial \underline{u}^2}{\partial \xi}_\ell + \dfrac{\partial \underline{u}^1}{\partial \tau}_\ell$

$$= - \langle \underset{mn}{\Sigma\Sigma} (\nabla \underline{\underline{B}}^o \cdot \underline{u}^2_m \dfrac{\partial \underline{u}^1}{\partial y}_n + \nabla \underline{\underline{B}}^o \cdot \underline{u}^1_m \dfrac{\partial \underline{u}^2}{\partial y}_n + ink \nabla \underline{\underline{A}}^o \cdot \underline{u}^1_m \underline{u}^2_n \qquad 3.3.4$$

$$+ ink \nabla \underline{\underline{A}}^o \cdot \underline{u}^2_m \underline{u}^1_n + \nabla \underline{\underline{A}}^o \cdot \underline{u}^1_m \dfrac{\partial \underline{u}^2}{\partial \xi}_n + \nabla \underline{E}^o : \underline{u}^2_m \underline{u}^1_n) P_m P_n \rangle_\ell$$

$$- \langle \underset{mnr}{\Sigma\Sigma\Sigma} \tfrac{1}{2} (\nabla \nabla \underline{\underline{B}}^o : \underline{u}^1_m \underline{u}^1_n \dfrac{\partial \underline{u}^1}{\partial y}_r + \nabla \nabla \underline{\underline{A}}^o : \underline{u}^1_m \underline{u}^1_n irk \underline{u}^1_r) P_m P_n P_r \rangle_\ell$$

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These equations are written in more concise form as

$$\underset{\sim}{W}_\ell \underline{u}^1{}_\ell = 0$$

$$\underset{\sim}{W}_\ell \underline{u}^2{}_\ell + \underline{F}_\ell \frac{\partial \underline{u}^1}{\partial \xi}{}_\ell = \underline{S}_1{}^\ell$$

$$\underset{\sim}{W}_\ell \underline{u}^3{}_\ell + \underline{F}_\ell \frac{\partial \underline{u}^2}{\partial \xi}{}_\ell + \frac{\partial \underline{u}^1}{\partial \xi}{}_\ell = \underline{S}_2{}^\ell$$

where $\underset{\sim}{W}_\ell$ is a matrix differential operator defined by:

$$\underset{\sim}{W}_\ell = \underline{\underline{B}}^o \frac{\partial}{\partial y} + \nabla \cdot \underline{\underline{B}}^o \frac{du^o}{dy}(y) + i\ell k \underline{\underline{A}}^o + \nabla \cdot \underline{\underline{E}}^o - i\ell \omega_r \underline{\underline{I}}$$

and $\underline{F}_\ell$ is a matrix given by

$$\underline{F}_\ell = -\lambda \underline{\underline{I}} + \underline{\underline{A}}^o$$

and $\underline{S}_1{}^\ell$, $\underline{S}_2{}^\ell$ are given by the right hand sides of 3.3.3 and 3.3.4.

Initially we consider 3.3.2 for $\ell = 1$. This equation then reduces to the eigenvalue problem of section 2 i.e. equation 3.2.5. We may therefore write

$$\underline{u}^1{}_1(\tau,\xi,y) = \psi_1(\tau,\xi)\underline{R}(y) \qquad\qquad 3.3.5$$

where $\qquad\qquad \underline{R}(y) = \sum_i C_{1i}(\omega_r)\underline{R}_{1i}(y)$

and the additional subscript 1 has been added to denote that the C_i and $\underline{R}_i$ are defined for $\ell = 1$.

We now assume that for 3.3.2 with $\ell \neq 1$, $\ell\omega_r$ is not an eigenvalue of $\underline{W}_\ell$. This immediately gives:

$$\underline{u}^1{}_\ell(\tau,\xi,y) = 0 \qquad \text{for } \ell \neq 1 \qquad\qquad 3.3.6$$

This is considered to be valid even for $\ell=0$. We note that substituting

These equations are written in more concise form as

$$\underset{\sim}{W}_\ell \underline{u}^1{}_\ell = 0$$

$$\underset{\sim}{W}_\ell \underline{u}^2{}_\ell + \underline{F}_\ell \frac{\partial \underline{u}^1}{\partial \xi}{}_\ell = \underline{S}_1{}^\ell$$

$$\underset{\sim}{W}_\ell \underline{u}^3{}_\ell + \underline{F}_\ell \frac{\partial \underline{u}^2}{\partial \xi}{}_\ell + \frac{\partial \underline{u}^1}{\partial \xi}{}_\ell = \underline{S}_2{}^\ell$$

where $\underset{\sim}{W}_\ell$ is a matrix differential operator defined by:

$$\underset{\sim}{W}_\ell = \underline{\underline{B}}^o \frac{\partial}{\partial y} + \nabla . \underline{\underline{B}}^o \frac{d\underline{u}^o(y)}{dy} + i\ell k \underline{\underline{A}}^o + \nabla . \underline{\underline{E}}^o - i\ell \omega_r \underline{\underline{I}}$$

and $\underline{F}_\ell$ is a matrix given by

$$\underline{F}_\ell = -\lambda \underline{\underline{I}} + \underline{\underline{A}}^o$$

and $\underline{S}_1{}^\ell$, $\underline{S}_2{}^\ell$ are given by the right hand sides of 3.3.3 and 3.3.4.

Initially we consider 3.3.2 for $\ell = 1$. This equation then reduces to the eigenvalue problem of section 2 i.e. equation 3.2.5. We may therefore write

$$\underline{u}^1{}_1(\tau,\xi,y) = \psi_1(\tau,\xi)\underline{R}(y) \qquad\qquad 3.3.5$$

where $\qquad\qquad \underline{R}(y) = \underset{i}{\Sigma} C_{1i}(\omega_r)\underline{R}_{1i}(y)$

and the additional subscript 1 has been added to denote that the C_i and $\underline{R}_i$ are defined for $\ell = 1$.

We now assume that for 3.3.2 with $\ell \neq 1$, $\ell\omega_r$ is not an eigenvalue of $\underline{W}_\ell$. This immediately gives:

$$\underline{u}^1{}_\ell (\tau,\xi,y) = 0 \qquad \text{for } \ell \neq 1 \qquad\qquad 3.3.6$$

This is considered to be valid even for $\ell=0$. We note that substituting

3.3.1 into the boundary condition 3.2.2 gives

$$\underline{\underline{M}}\, \underline{u}_\ell^\alpha(\tau,\xi,a) + \underline{\underline{N}}\, \underline{u}_\ell^\alpha(\tau,\xi,b) = 0$$

for all α and ℓ.

We now substitute 3.3.5 and 3.3.6 into 3.3.3 for $\ell = 1$ to obtain

$$\underline{\underline{W}}_1 \underline{u}_1^2 + \underline{\underline{F}}_1 \frac{\partial \underline{u}^1}{\partial \xi} 1 = 0$$

or
$$\underline{\underline{W}}_1 \underline{u}_1^2 + \underline{\underline{F}}_1 \underline{R}(y) \frac{\partial \psi_1}{\partial \xi} = 0 \qquad\qquad 3.3.7$$

We note that

$$\underline{\underline{F}}_1 \underline{R}(y) = -i\,\frac{\partial \underline{\underline{W}}_1}{\partial k}\, \underline{R}(y) = i\,\underline{\underline{W}}_1 \frac{\partial \underline{R}}{\partial k}$$

and rewrite 3.3.7 as

$$\underline{\underline{W}}_1 \left[\underline{u}_1^2 + i\,\frac{\partial \underline{R}}{\partial k}\, \frac{\partial \psi_1}{\partial \xi} \right] = 0$$

Since the $R_{1j}(y)$ are eigenfunctions of $\underline{\underline{W}}_1$, $\underline{u}_1^2$ must be proportional to the $R_{1i}(y)$ but 3.3.7 admits a non-trivial solution only if a compatibility condition is satisfied. This compatibility condition is given in the Appendix by A.13. We put the equation 3.3.7 into the form of A.9 and replace the inhomogeneous term $\underline{f}$ by the second term of 3.3.7 to obtain

$$\int_a^b \frac{\underline{\underline{L}}}{} \left[\underline{\underline{M}}\, \underline{\Phi}(a) - \underline{\underline{N}}\, \underline{\Phi}(b) \right] \underline{\underline{B}}_o^{-1} \underline{\Phi}^{-1}(s) \underline{\underline{W}}_1(s) \frac{\partial \underline{R}(s)}{\partial k}\, ds = 0$$

where $\underline{\underline{\Phi}}$ is the fundamental matrix of the system 3.3.2. To evaluate this integral we note the following identity valid for any vector function $\underline{u}(y)$

$$\underline{\underline{W}}_i \underline{u}(y) = -i\underline{\underline{B}}_o \underline{\Phi}(y)\, \frac{d}{dy}\, (\underline{\Phi}^{-1}(y)\underline{u}(y))$$

Substitution then gives

$$- i \int_a^b \underline{L} \left[\underline{\underline{M}} \, \underline{\Phi}(a) - \underline{\underline{N}} \, \underline{\Phi}(b) \right] \frac{d}{ds} \left(\underline{\underline{\Phi}}^{-1}(s) \, \frac{\partial \underline{R}(s)}{\partial k} \right) ds = 0$$

$$- i \underline{L} (\underline{\underline{M}} \, \underline{\Phi}(a) - \underline{\underline{N}} \, \underline{\Phi}(b)) \left[\underline{\underline{\Phi}}^{-1}(s) \, \frac{\partial \underline{R}(s)}{\partial k} \right]_a^b = 0$$

i.e. $$- i \underline{L} \left[\underline{\underline{M}} \, \underline{\Phi}(a) - \underline{\underline{N}} \, \underline{\Phi}(b) \right] \left[\underline{\underline{\Phi}}^{-1}(b) \, \frac{\partial \underline{R}(b)}{\partial k} - \underline{\underline{\Phi}}^{-1}(a) \, \frac{\partial \underline{R}(a)}{\partial k} \right] = 0$$

Evaluating this expression and substituting expressions for $\underline{\Phi}(a)$, $\underline{\Phi}(b)$ and their inverses from the boundary conditions then gives

$$+ 2 i \underline{L} \left[\underline{\underline{M}} \, \frac{\partial \underline{R}(a)}{\partial k} + \underline{\underline{N}} \, \frac{\partial \underline{R}(b)}{\partial k} \right] = 0$$

Since $\underline{\underline{M}}$ and $\underline{\underline{N}}$ are constant matrices it immediately follows that

$$2 i \underline{L} \, \frac{d}{dk} \left[\underline{\underline{M}} \, \underline{R}(a) + \underline{\underline{N}} \, \underline{R}(b) \right] = 0$$

which is satisfied since the expression in brackets is zero as it is an expression of the boundary conditions.

Therefore, the compatibility condition for 3.3.7 is satisfied and 3.3.7 may immediately be solved to give:

$$\underline{u}_1^2 = \psi_2(\tau, \xi) \underline{R} - i \, \frac{\partial \underline{R}}{\partial k} \, \frac{\partial \psi_1}{\partial \xi} \qquad\qquad 3.3.8$$

where $\psi_2(\tau, \xi)$ is another function of τ, ξ analogous to ψ_1 and can be determined to higher order.

The remaining non-zero components to second order i.e. $\underline{u}_o^2$ and $\underline{u}_2^2$ are given by the direct solutions of the $\ell=0$ and $\ell=2$ forms of 3.3.3. These are formally given by

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$$\underline{u}_2{}^2 = \underline{R}_2{}^2 (\psi_1(\tau, \xi'))^2 \qquad\qquad 3.3.9$$

$$\underline{u}_o{}^2 = \underline{R}_o{}^2 |\psi_1(\tau, \xi')|^2 \qquad\qquad 3.3.10$$

These solutions are obtained by using the Green's matrix A.16 given in the Appendix, i.e. substituting 3.3.5 and 3.3.6 into the right-hand side of 3.3.3 for $\ell = 0$ and $\ell = 2$, multiplying by the appropriate Green's matrix and integrating gives

$$\underline{R}_o{}^2 = - \int_a^x \underline{\phi}_o(x) \underline{D}_o^{-1} \underline{M} \, \underline{\phi}_o(a) \underline{\phi}_o^{-1}(x^1) \underline{S}_1{}^o(x^1) \, dx^1$$

$$+ \int_x^b \underline{\phi}_o(x) \underline{D}_o^{-1} \underline{N} \, \underline{\phi}_o(b) \underline{\phi}_o^{-1}(x^1) \underline{S}_1{}^o(x^1) \, dx^1$$

where
$$\underline{S}_1{}^o = - \left[(\nabla \underline{B}^o . \underline{R}) \frac{\partial \underline{R}^*}{\partial y} + (\nabla \underline{B}^o . \underline{R}^*) \frac{\partial \underline{R}}{\partial y} + ik(\nabla \underline{A}^o . \underline{R}) \underline{R}^* \right.$$

$$\left. - ik(\nabla \underline{A}^o . \underline{R}^*) \underline{R} + \nabla \nabla \underline{E}^o : \underline{R} \, \underline{R}^* \right]$$

and
$$\underline{R}_2{}^2 = - \int_a^x \underline{\phi}_2(x) \underline{D}_2^{-1} \underline{M} \, \underline{\phi}_2(a) \underline{\phi}_2^{-1}(x^1) \underline{S}_1{}^2(x^1) \, dx^1$$

$$+ \int_x^b \underline{\phi}_2(x) \underline{D}_2^{-1} \underline{N} \, \underline{\phi}_2(b) \underline{\phi}_2^{-1}(x^1) \underline{S}_1{}^2(x^1) \, dx^1$$

where
$$\underline{S}_1{}^2 = - \left[(\nabla \underline{B}^o . \underline{R}) \frac{\partial \underline{R}}{\partial y} + (ik)(\nabla \underline{A}^o . \underline{R}) \underline{R} + \frac{1}{2}(\nabla \nabla \underline{E}^o : \underline{R} \, \underline{R}) \right]$$

In these expressions $\underline{\phi}_o$ is the fundamental matrix for the operator $\underline{W}_o$ and the matrix $\underline{D}_o$ is readily obtained from $\underline{\phi}$ and the boundary conditions. $\underline{\phi}_2$ is similarly defined for the operator $\underline{W}_2$.

Finally we consider the $\ell = 1$ form of 3.3.4:

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$$\underset{\sim}{W}_1\underline{u}_1{}^3 = -(-\lambda\underline{\underline{I}}+\underline{\underline{A}}^o)\frac{\partial\underline{u}^2}{\partial\xi}1 - \frac{\partial\underline{u}^1}{\partial\tau}1 - (\nabla\underline{\underline{B}}^o.\underline{u}^2{}_o)\frac{\partial\underline{u}^1}{\partial y}1$$

$$- (\nabla\underline{\underline{B}}^o.\underline{u}^2{}_2)\frac{\partial\underline{u}^1}{\partial y}-1 - (\nabla\underline{\underline{B}}^o.\underline{u}^1{}_1)\frac{\partial\underline{u}^2}{\partial y}o - (\nabla\underline{\underline{B}}^o.\underline{u}^1{}_{-1})\frac{\partial\underline{u}^2}{\partial y}2$$

$$- 2ik\,(\nabla\underline{\underline{A}}^o.\underline{u}^1{}_{-1})\underline{u}^2{}_2 + ik(\nabla\underline{\underline{A}}^o.\underline{u}^2{}_2)\underline{u}^1{}_{-1} - ik\,(\nabla\underline{\underline{A}}^o.\underline{u}^2{}_o)\underline{u}^1{}_1$$

$$- (\nabla\underline{\underline{A}}^o.\underline{u}^1{}_1)\frac{\partial\underline{u}^2}{\partial y}o - (\nabla\underline{\underline{A}}^o.\underline{u}^1{}_{-1})\frac{\partial\underline{u}^2}{\partial y}2 - (\nabla\nabla\underline{\underline{E}}^o:\underline{u}^2{}_o\underline{u}^1{}_1)$$

$$- (\nabla\nabla\underline{\underline{C}}^o:\underline{u}^2{}_2\underline{u}^1{}_{-1}) +$$

$$+ \nabla\nabla\underline{\underline{B}}^o:(\underline{u}^1{}_1\underline{u}^1{}_{-1}\frac{\partial\underline{u}}{\partial y}1 + \underline{u}^1{}_1\underline{u}^1{}_1\frac{\partial\underline{u}^1}{\partial y}-1) - ik\,(\nabla\nabla\underline{\underline{A}}^o:\underline{u}^1{}_1\underline{u}^1{}_{-1})\underline{u}^1{}_1$$

$$+ ik(\nabla\nabla\underline{\underline{A}}^o:\underline{u}^1{}_1\underline{u}^1{}_1)\underline{u}^1{}_{-1}$$

Substituting for $\underline{u}^2{}_2$, $\underline{u}^2{}_o$, $\underline{u}^2{}_1$ and $\underline{u}^1{}_1$ from 3.3.10, 3.3.9, 3.3.8 and 3.3.5 respectively we obtain

$$\underset{\sim}{W}_1\underline{u}_1{}^3 = i\,\frac{\partial W}{\partial k}1\,\underline{R}\,\frac{\partial\psi}{\partial\xi}2 - i\,\frac{\partial W}{\partial k}1\,\frac{\partial R}{\partial k}\frac{\partial^2\psi_1}{\partial\xi^2} - \underline{R}\,\frac{\partial\psi}{\partial\tau}1 \qquad 3.3.11$$

$$+ \left[(-\nabla\underline{\underline{B}}^o.\underline{R}^2{}_o + \nabla\nabla\underline{\underline{B}}^o:\underline{R}\,\underline{R}^*)\frac{\partial R}{\partial y} + (-\nabla\underline{\underline{B}}^o.\underline{R}^2{}_2 + \nabla\nabla\underline{\underline{B}}^o:\underline{R}\,\underline{R})\frac{\partial R^*}{\partial y}\right.$$

$$+ (-\nabla\underline{\underline{B}}^o.\underline{R} - \nabla\underline{\underline{A}}^o.R)\frac{\partial R^2}{\partial y}o + (-\nabla\underline{\underline{B}}^o.\underline{R}^* - \nabla\underline{\underline{A}}^o.\underline{R}^*)\frac{\partial R^2}{\partial y}2$$

$$+ (-ik\nabla\underline{\underline{A}}^o.\underline{R}^2{}_o + ik\nabla\nabla\underline{\underline{A}}^o:\underline{R}\,\underline{R}^*)\underline{R} + (ik\nabla\underline{\underline{A}}^o:\underline{R}^2{}_2 - ik\nabla\nabla\underline{\underline{A}}^o:\underline{R}\,\underline{R})\underline{R}^*$$

$$\left. - 2ik(\nabla\underline{\underline{A}}^o.\underline{R}^*)\underline{R}_2{}^2 - (\nabla\nabla\underline{\underline{E}}^o:\underline{R}_o{}^2\underline{R}) - (\nabla\nabla\underline{\underline{E}}^o:\underline{R}_2{}^2\underline{R}^*)\right]\psi_1|\psi_1|^2$$

This equation is again the inhomogeneous form for the operator $\underset{\sim}{W}_1$ and so

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the compatibility condition A.13 must again be satisfied where $\underline{f}$ is now the right-hand side of equation 3.3.11. We consider each term separately and apply the compatibility condition to each.

Term 3:

$$- \int_a^b \underline{L} \left[\underline{\underline{M}} \; \underline{\Phi}(a) - \underline{\underline{N}} \; \underline{\Phi}(b) \right] \underline{\underline{B}}^{o-1} \underline{\underline{\Phi}}^{-1}(s) \underline{R}(s) \; \frac{\partial \psi_1}{\partial \tau} \; ds$$

$$\equiv - \alpha \frac{\partial \psi_1}{\partial \tau} \qquad\qquad 3.3.12$$

where $\quad \alpha = \int_a^b \underline{L} \left[\underline{\underline{M}} \; \underline{\Phi}(a) - \underline{\underline{N}} \; \underline{\Phi}(b) \right] \underline{\underline{B}}^{o-1} \underline{\underline{\Phi}}^{-1}(s) \underline{R}(s) ds$

Term 1:

$$- \int_a^b \underline{L} \left[\underline{\underline{M}} \; \underline{\Phi}(a) - \underline{\underline{N}} \; \underline{\Phi}(b) \right] \underline{\underline{B}}^{o-1} \underline{\underline{\Phi}}^{-1}(s) \; \frac{\partial \underline{\underline{W}}_1}{\partial k}(s) \underline{R}(s) \; \frac{\partial \psi_2}{\partial \xi} \; ds$$

with the relation

$$\frac{\partial \underline{\underline{W}}_1}{\partial k} \; \underline{R} = \underline{\underline{W}}_1 \; \frac{\partial R}{\partial k} \qquad\qquad 3.3.13$$

this term becomes

$$+ \int_a^b \underline{L} \left[\underline{\underline{M}} \; \underline{\Phi}(a) - \underline{\underline{N}} \; \underline{\Phi}(b) \right] \underline{\underline{B}}^{o-1} \underline{\underline{\Phi}}^{-1}(s) \underline{\underline{W}}(s) \; \frac{\partial R(s)}{\partial k} \; \frac{\partial \psi_2}{\partial \xi} \; ds$$

which is identical to the compatibility condition for 3.3.7 which was found to be satisfied and to be identically zero.

Term 2:

$$= + \int_a^b \underline{L} \left[\underline{\underline{M}} \; \underline{\Phi}(a) - \underline{\underline{N}} \; \underline{\Phi}(b) \right] \underline{\underline{B}}^{o-1} \underline{\underline{\Phi}}^{-1}(s) \; \frac{\partial \underline{\underline{W}}_1}{\partial k}(s) \; \frac{\partial R(s)}{\partial k} \; \frac{\partial^2 \psi_1}{\partial \xi^2} \; ds \quad 3.3.14$$

Now differentiating equation 3.3.13 with respect to k gives

$$\frac{\partial W_1}{\partial k} \frac{\partial R}{\partial k} = -\frac{1}{2} \left(\underset{\sim}{W}_1 \frac{\partial^2 R}{\partial k^2} - i \frac{\partial^2 \omega r}{\partial k^2} \underline{R} \right)$$

Substituting this relation gives

$$-\frac{1}{2} \int_a^b \underline{L} \left[\underline{\underline{M}} \underline{\Phi}(a) - \underline{\underline{N}} \underline{\Phi}(b) \right] \underline{\underline{B}}^{0-1} \underline{\Phi}^{-1}(s) \underset{\sim}{W}_1 \frac{\partial^2 R}{\partial k^2} \frac{\partial^2 \psi_1}{\partial \xi^2} \, ds$$

$$+ \frac{1}{2} i \int_a^b \underline{L} \left[\underline{\underline{M}} \underline{\Phi}(a) - \underline{\underline{N}} \underline{\Phi}(b) \right] \underline{\underline{B}}^{0-1} \underline{\Phi}^{-1}(s) \frac{\partial^2 \omega r}{\partial k^2} \frac{\partial^2 \psi_1}{\partial \xi^2} \underline{R}(s) ds$$

By the same reasoning that led to the compatibility condition for 3.3.7 we see that the first term of this expression vanishes and the second term is given by:

$$\frac{1}{2} i \, \alpha \frac{\partial^2 \omega r}{\partial k^2} \frac{\partial^2 \psi_1}{\partial \xi^2} \qquad\qquad\qquad 3.3.15$$

Term 4:

$$\int \underline{L} \left[\underline{\underline{M}} \underline{\Phi}(a) - \underline{\underline{N}} \underline{\Phi}(b) \right] \underline{\underline{B}}^{0-1} \underline{\Phi}^{-1}(s) \underline{S}(s) \psi_1 |\psi_1|^2 \, ds = \beta' \qquad 3.3.16$$

where $\underline{S}(s)$ is the right-hand side of 3.3.11.

Combining 3.3.12, 13, 14 and 15 gives the generalised nonlinear Schrödinger equation

$$i \frac{\partial \psi_1}{\partial \tau} + \frac{1}{2} \frac{\partial^2 \omega r}{\partial k^2} \frac{\partial^2 \psi_1}{\partial \xi^2} + \beta \psi_1 |\psi_1|^2 = 0 \qquad\qquad 3.3.17$$

where $\beta = \beta^1/\alpha$. We note that it is not possible to prove generally whether β is real or complex but it will be assumed that β is in fact complex.

3.4 Discussion

We have again shown that, to lowest order, the amplitude of a wave propagating in a system described by the model equation 3.2.1 must satisfy a generalised nonlinear Schrödinger equation given by 3.3.17. The method used was an extension of the work of Taniuti, Asano and Yajima (1969) and,

despite the added complexity of a further spatial dimension, followed much the same pattern. The two first coefficients of 3.3.17 were found to be identical to those derived for other systems, as for example in Chapter 2 and the only significant difference arose in the third term i.e. the coefficient of the nonlinear term. To determine this coefficient now requires the knowledge of the fundamental matrix of the system of equations considered. This requires an explicit solution of the linearised problem to be known in order that the integration in 3.3.16 may be performed. It must be emphasised that the dependence on the additional spatial coordinate was not assumed to be weak, as was done by Hasegawa (1970) and full account was taken of this strong dependence. It is interesting to note that despite this strong dependence a total decoupling of the y dependence and the stretched x and t dependence was found, and that a nonlinear Schrödinger equation with only two independent variables τ and ξ was found. This reduces the problem of determining the stability of certain classes of two-dimensional problems to the problem of determining the stability of the nonlinear Schrödinger equation.

As in Chapter 2 we again encountered the problem of the vanishing of a determinant. In this case the matrix in question was $\underline{B}^o$ and its inverse was required for the determination of the d.c. components $\underline{u}_o^{\ 1}$ and $\underline{u}_o^{\ 2}$. For most physical systems this condition is not satisfied and the components $\underline{u}_o^{\ 1}$ and $\underline{u}_o^{\ 2}$ cannot normally be determined by the inversion of the operator $\underset{\sim}{W}_o$. In 3.3 we assumed that $\underline{u}_o^{\ 1}$ was identically zero and that det $\underline{B}^o$ did not vanish and so $\underline{u}_o^{\ 2}$ was obtained by direct inversion. The assumption that $\underline{u}_o^{\ 1} = 0$ is physically tenable since we are considering the modulation of the wave $\underline{U}_1^{\ 1} e^{i(kx-\omega t)}$ + complex conjugate. We may however speculate that, as in Chapter 2, certain components of $\underline{u}_o^{\ 1}$ may in fact be determined by direct inversion of $\underset{\sim}{W}_o$ and that the remainder are determined from the compatibility condition for the $\ell = 1$ form of the second order equation. Similarly certain coefficients of $\underline{u}_o^{\ 2}$ may be determined by direct inversion

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and the remainder determined by the integration of the $\ell = 0$ form of the third order equation. Although this has not been proved generally, physical systems where this method has not been successful have not yet been found.

The model equation 3.2.1 can be used to describe a large class of problems, e.g. the propagation of waves in a cold plasma between conducting plates with no magnetic field and the propagation of sound wave in a bounded gas. This class could be greatly extended if model equations of the form

$$A \frac{\partial^2 U}{\partial x^2} + B \frac{\partial^2 U}{\partial y^2} + C \frac{\partial^2 U}{\partial t^2} + D \frac{\partial U}{\partial x} + E \frac{\partial U}{\partial y} + F \frac{\partial U}{\partial t} + G = 0 \qquad 3.4.1$$

could be considered. Then problems such as the two-stream instability, the instability of plane Poiseuille flow etc. could be considered. This requires the extension of the work of 3.2 in the same way as the results of Chapter 2 were found by extending the original work of Taniuti and Yajima (1969). An equation of the form of 3.3.1 was not considered in 3.2 and 3.3 as this would have added considerably to the algebra and complexity of the coefficients so obtained. Differences would arise in that the operator $\underset{\sim}{W}$ would become a second order matrix differential operator. The ensuing algebra would follow a similar line with this additional complication. However, results similar to those given in the Appendix can easily be written down for second order self-adjoint matrix differential operators and it is anticipated that the following generalised nonlinear Schrödinger equation results

$$i \frac{\partial \psi 1}{\partial \tau} + \frac{1}{2} \frac{\partial^2 \omega_r}{\partial k^2} \frac{\partial^2 \psi 1}{\partial \xi^2} + i\omega_i \phi + \delta \phi |\phi|^2 = 0 \qquad 3.4.2$$

The applicability of this result can be supported by considering a simpler form of the one-dimensional system 2.2.1 without the second derivatives. The result 2.2.34 again follows with algebraically different coefficients. From these one-dimensional results we conclude that the addition of higher than first order derivatives in the model system of equations preserves the

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form of the final result, i.e. a generalised nonlinear Schrödinger equation. We therefore assume that the result 3.3.2 would follow for the system 3.3.1. As in Chapter 2 the instability mechanism for 3.3.1 would have to be well understood in order that the ordering $\omega_i \sim O(\varepsilon^2)$ could again be made.

Although the result 3.2.17 has been derived for a general model equation, in practise it would be easier to take a less general approach when attempting to solve a set of equations describing a real physical system. This is a result of the large numbers of zeros appearing in the matrices $\underline{A}$ to $\underline{G}$ for real physical systems and the resulting redundancies occurring in integrations necessary for the determination of the coefficients.

Appendix

Systems of first order differential equations

We shall consider the general properties of systems of n first order differential equations and follow John (1965). Initially we consider a first order system with an initial condition and then extend this to self-adjoint two-point boundary value problems. Suppose $\underline{\underline{A}}$ is a continuous $n \times n$ matrix of complex functions of a real variable x on an open interval of the real line I. We then consider the system

$$\frac{dy}{dx} = \underline{\underline{A}}(x)\underline{y} \qquad x \epsilon I \qquad\qquad A.1$$

with the initial condition

$$\underline{y}(x^1) = \underline{\xi} \qquad\qquad A.2$$

for any $\underline{\xi}$ and $x^1 \epsilon I$, where $\underline{y}$ is an n element column vector. Then, there exists a unique solution $\underline{\phi}$ of A.1 subject to A.2. The zero vector is always a solution of A.1 and is called the trivial solution. The set of all solutions of A.1 on I forms an n dimensional vector space over the complex field. This indicates that there exists a set of n linearly independent solutions $\underline{\phi}_1 \ldots \underline{\phi}_n$ such that every other solution of A.1 is a linear combination of this set, i.e.

$$\underline{\phi} = \sum_{i=1}^{n} C_i \underline{\phi}_i \qquad\qquad A.3$$

where the C_i are in general complex constants. The ϕ_i are called the basis or fundamental set of solutions of A.1.

We define a matrix Φ whose n columns are n linearly independent solutions of A.1. This is called the fundamental matrix and satisfies:

$$\frac{d\underline{\underline{\Phi}}}{dx} = \underline{\underline{A}}(x)\underline{\underline{\Phi}} \qquad x \epsilon I$$

We now consider the inhomogeneous problem equivalent to A.1, i.e. given a

continuous nxn matrix $\underline{\underline{A}}(x)$, a function of real x on an open interval $\underline{I}$ and given a continuous vector $\underline{b}(x)$ on $\underline{I}$ we write

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}(x)\underline{y} + \underline{b}(x) \qquad x\epsilon I \qquad\qquad A.4$$

There exists a unique solution $\underline{\phi}$ of A.4 subject to

$$\underline{\phi}(x^1) = \xi \qquad\qquad A.5$$

given any ξ and $x^1 \epsilon I$. Given the fundamental matrix $\underline{\underline{\Phi}}$ for A.1 it is possible to write down the solution of A.4 subject to A.5 i.e.

$$\underline{\psi}(x) = \underline{\psi}^1(x) + \underline{\underline{\Phi}}(x) \int_{x^1}^{x} \underline{\underline{\Phi}}^{-1}(s)\underline{b}(s)ds \qquad A.6$$

where $\underline{\psi}^1(x)$ is a solution of A.1 satisfying

$$\underline{\psi}^1(x^1) = \xi$$

Equation A.6 is derived by the method of variation of constants as follows. Suppose that a solution of the inhomogeneous problems A.4 is given by

$$\underline{\psi} = \underline{\underline{\Phi}} \; \underline{c}$$

where $\underline{c}$ is a column vector and a function of x. Differentiation then gives

$$\frac{\partial\underline{\psi}}{\partial x} = \frac{\partial\underline{\underline{\Phi}}}{\partial x} \; \underline{c} + \underline{\underline{\Phi}} \frac{\partial \underline{c}}{\partial x}$$

and substituting from A.1 and A.4,

$$\frac{\partial\underline{c}}{\partial x} = \underline{\underline{\Phi}}^{-1} \; \underline{b}$$

One integration and applying the condition A.5 leads immediately to the result A.6.

We now extend the results given above to self-adjoint problems on finite intervals of the real line. We shall in particular be interested in eigenvalue problems and therefore consider the first order eigenvalue problem

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$$\frac{\partial \underline{y}}{\partial x} - A(\underline{x})\underline{y} = \ell \underline{y} \qquad\qquad A.7$$

subject to the boundary conditions

$$\underline{U}(a,b) = \underline{\underline{M}} \, \underline{y}(a) + \underline{\underline{N}} \, \underline{y}(\underline{b}) = 0 \qquad\qquad A.8$$

where $x \ \varepsilon \ [a,b]$ and $\underline{y}$ is an n component column vector. The system A.7 and A.8 always has the trivial solution $\underline{y} = 0$. If ℓ is chosen so that A.3 has a non-trivial solution then ℓ is an eigenvalue of A.3 and these solutions are the eigenvectors.

We suppose the eigenvalues and eigenvectors of A.7 with A.8 are known and now we consider the inhomogeneous problem

$$\frac{d\underline{y}}{dx} - \underline{A}(x)\underline{y} = \underline{f} \qquad\qquad A.9$$

where we retain the boundary conditions A.8. If A.9 with $\underline{f} = 0$ has a non-trivial solution then unique solutions to A.9 exist only for a certain class of functions of $\underline{f}$ and we will now determine the condition that $\underline{f}$ must satisfy.

We suppose that the fundamental matrix for A.9 with $f = 0$ is denoted by $\underline{\underline{\Phi}}$. Then we again use the method of the variations of constants by looking for a solution of A.9 of the form $\underline{\underline{\Phi}} \, \underline{c}(x)$ to obtain:

$$\underline{\psi}(x) = \underline{\underline{\Phi}}(x) \int^{x} \underline{\underline{\Phi}}^{-1}(s)\underline{f}(s)\,ds$$

and so we can write

$$\underline{\psi}(x) = \underline{\underline{\Phi}}(x) \int_{a}^{x} \underline{\underline{\Phi}}^{-1}(s)\underline{f}(s)\,ds + \sum_{i=1}^{n} c_i \underline{\phi}_i(x) \qquad A.10$$

and $\qquad \underline{\psi}(x) = - \underline{\underline{\Phi}}(x) \int_{x}^{b} \underline{\underline{\Phi}}^{-1}(s) f(s)\,ds + \sum_{i=1}^{n} c_i \underline{\phi}_i(x) \qquad A.11$

where we have added solutions of the homogeneous to give the general solution

However, $\underline{\psi}(x)$ must satisfy the boundary conditions A.8. Multiply A.10 by $\underline{\underline{N}}$ for $x = b$ and A.11 by $\underline{\underline{M}}$ for $x = a$

$$\underline{\underline{N}}\ \underline{\psi}(b) \ = \ \underline{\underline{N}}\underline{\underline{\phi}}(b) \int_a^b \underline{\underline{\phi}}^{-1}(s)\underline{f}(s)ds + \underline{\underline{N}} \sum_{i=1}^n c_i\underline{\phi}_i(b)$$

$$\underline{\underline{M}}\ \underline{\psi}(a) \ = \ -\ \underline{\underline{M}}\ \underline{\underline{\phi}}(a) \int_a^b \underline{\underline{\phi}}^{-1}(s)\underline{f}(s)ds + \underline{\underline{M}} \sum_{i=1}^n c_i\underline{\phi}_i(a)$$

Adding these gives

$$\underline{\underline{M}}\ \underline{\psi}(a) + \underline{\underline{N}}\ \underline{\psi}(b) \ = \ -\int_a^b \left[\underline{\underline{M}}\ \underline{\underline{\phi}}(a) - \underline{\underline{N}}\ \underline{\underline{\phi}}(b) \right]\underline{\underline{\phi}}^{-1}(s)\underline{f}(s)ds$$

$$+ \sum_{i=1}^n \left[\underline{\underline{N}}c_i\phi_i(b) + \underline{\underline{M}}c_i\phi_i(a) \right]$$

$$= \ -\int_a^b \left[\underline{\underline{M}}\ \underline{\underline{\phi}}(a) - \underline{\underline{N}}\ \underline{\underline{\phi}}(b) \right]\underline{\underline{\phi}}^{-1}(s)\underline{f}(s)ds \qquad \text{A.12}$$

$$+ \underline{\underline{D}}\ \underline{C}$$

where $\underline{\underline{D}}$ and $\underline{C}$ are defined in 3.2.11 and 3.2.12. Now in view of A.8 applied to $\underline{\psi}(x)$ the L.H.S. of A.12 vanishes. Also, since $\det \underline{\underline{D}}$ vanishes, multiplying A.12 on the left by $\underline{L}$ where $\underline{L}$ is the left eigenvector of the matrix $\underline{\underline{D}}$ corresponding to zero eigenvalue we obtain the compatibility condition for the solution of A.9 as:

$$\int_a^b \underline{L} \left[\underline{\underline{M}}\ \underline{\underline{\phi}}(a) - \underline{\underline{N}}\ \underline{\underline{\phi}}(b) \right]\underline{\underline{\phi}}^{-1}(s)\underline{f}(s)ds \ = \ 0 \qquad \text{A.13}$$

We can now consider the case when the homogeneous form of A.9 has only the trivial solution, i.e. we exclude the eigenfunctions. Then, there exists a matrix $\underline{\underline{G}}(x,x^1)$ continuous for $a \leqslant x \leqslant x^1 \leqslant b$ and $a \leqslant x^1 \leqslant x \leqslant b$ such that

$$\int_a^b \underline{\underline{G}}(x,x^1)\underline{f}(x^1)dx^1 \qquad \text{A.14}$$

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is the unique solution of A.9 subject to A.8. The existence of this Green's function matrix $\underline{\underline{G}}(x,x^1)$ is verified by inspection as follows. Consider the fundamental matrix Φ for A.9 with $f = 0$ and let

$$
\underline{\underline{G}}(x,x^1) = \begin{cases} \underline{\underline{\Phi}}(x)\underline{\underline{\Phi}}^{-1}(x^1) + \underline{\underline{\Phi}}(x)\underline{\underline{J}}(x^1) & x^1 < x \\[12pt] \underline{\underline{\Phi}}(x)\underline{\underline{J}}(x^1) & x^1 > x \end{cases} \qquad \text{A.15}
$$

In order that the boundary conditions A.8 be satisfied we substitute A.14 and A.15 in A.8 to obtain:

$$
\underline{\underline{M}}\ \underline{\underline{\Phi}}(a)\underline{\underline{J}}(x) + \underline{\underline{N}}\ \underline{\underline{\Phi}}(b)\underline{\underline{\Phi}}^{-1}(x^1) + N\Phi(b)\underline{\underline{J}}(x^1) = 0
$$

which defines $\underline{\underline{J}}(x^1)$ as

$$
\underline{\underline{J}}(x^1) = - \left[\underline{\underline{M}}\ \underline{\underline{\Phi}}(a) + \underline{\underline{N}}\ \underline{\underline{\Phi}}(b) \right]^{-1} \underline{\underline{N}}\ \underline{\underline{\Phi}}(b)\underline{\underline{\Phi}}^{-1}(x^1)
$$

The Green's matrix $\underline{\underline{G}}(x,x^1)$ then becomes:

$$
\underline{\underline{G}}(x,x^1) = \begin{cases} \underline{\underline{\Phi}}(x)\underline{\underline{D}}^{-1}\underline{\underline{M}}\ \underline{\underline{\Phi}}(a)\underline{\underline{\Phi}}^{-1}(x^1) & x^1 < x \\[12pt] -\underline{\underline{\Phi}}(x)\underline{\underline{D}}^{-1}\underline{\underline{N}}\ \underline{\underline{\Phi}}(b)\underline{\underline{\Phi}}(x^1) & x^1 > x \end{cases} \qquad \text{A.16}
$$

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Chapter 4

A Nonlinear Theory of the Propagation of

Solitary Waves in Piezoelectric Semiconductors

4.1. Introduction

The interaction of an acoustic wave with mobile charge carriers in solids has been of interest for some considerable time. The term "acoustoelectric effect" used to describe this interaction was first used by Parmenter (1953) in a study of the effect in metals. The effect in semiconductors was first considered by Weinreich (1956,1957).

Interest in the interaction grew when the possibility of attenuation or amplification was realised by Hutson et al (1961). The attenuation or amplification arises due to the energy exchange between the acoustic wave and the conduction electrons. If the wave velocity is greater than the average drift velocity of the electrons then the net effect is a reduction in both the amplitude and velocity of the acoustic wave due to the absorption of energy by the electrons from the wave. Conversely, if the wave velocity is slower than the drift velocity then energy is transferred from the electrons to the wave and the wave is amplified. The simplest way to achieve this is to increase the average electron drift velocity by the application of a d.c. electric field.

The mechanism of interaction can be thought of simply as the mechanical oscillation of the atoms of the solid induced by the acoustic wave which modifies the effective potential in which the conduction electrons move, thus giving the interaction. Both long and short range potentials are affected. In metals and in non ionic materials, the principal change is in the short range part of the potential. This is called the deformation potential coupling. In ionic semiconductors and particularly in so-called piezoelectric semiconductors, the dominant

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effect is a change in the long range Coulomb interaction due to the motion of the ionic charges. This potential change can be represented as a polarisation wave in the material. This effective polarisation may be expanded as a series in the strain produced by the acoustic wave. In semiconductors with no inversion symmetry, the term proportional to the strain is nonvanishing and becomes the piezoelectric tensor of the material.

In piezoelectric materials the field associated with this interaction, called the piezoelectric field, causes the mobile charge carriers to move in such a way as to screen this field. The extent of this screening may be described by the dielectric relaxation frequency ω_c, where,

$$\omega_c = -\sigma/\epsilon$$

and σ is the conductivity. The coupling is smallest when $\omega/\omega_c \gg 1$, where ω is the frequency of the acoustic wave. Then, the carriers are unable to respond quickly enough to the field. The interaction is greatest when $\omega/\omega_c \ll 1$. This is the basic reason why this effect is dominant in semiconductors - the dielectric relaxation time is small (typically 10^{-12} sec.) because of the low conductivity. Given a typical acoustic frequency of a few MHz gives the condition for maximum interaction. The net effect is to produce bunching of the electrons and it is this effect which is responsible for the acoustoelectric amplification as will be seen in the linear theory of 4.2. Diffusion opposes the build up of these space charge waves and this effect is illustrated by considering the diffusion frequency ω_D, given by:

$$\omega_D = v_s^2/D_n$$

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where v_s is the acoustic velocity and D_n the diffusion coefficient.
If $\omega/\omega_D \gg 1$, the space charge modulation is least as then the charge
carriers diffuse away most rapidly.   So for maximum amplification
the condition $\omega/\omega_D \ll 1$ must also be satisfied.

The work of Weinreich (1956) is considered as the first recogni-
tion of the possibility of amplification in semiconductors.   The first
amplification was observed by Hutson et al (1961) who demonstrated the
effect in cadmium sulfide.   Hutson and White (1962) derived the small
signal theory for the propagation of acoustic waves and White (1962)
derived a small signal theory of the amplification process.   White's
theory agreed well with the experimental findings of Hutson et al (1961)
and most work since then has been concerned with nonlinear effects in
acoustoelectric interactions.

The first observation of a nonlinear effect was the demonstration
of current saturation by Smith (1963) and McFee (1963).   Although this
particular effect is not considered in this Chapter we briefly outline
how it arises.   The saturation is attributable to the acoustoelectric
current, i.e. the current arising from the presence of the acoustic
wave.   When the electric field across the material is increased, the
ohmic and acoustoelectric currents both rise as the acoustic waves are
amplified.   However, the acoustic gain is at the expense of the
electron system and so opposes the ohmic current.   These two effects
occur simultaneously and approximately at the same rate, giving rise
to the current saturation.

The nonlinear effect that will be considered in 4.3 concerns an
effect observed in high conductivity piezoelectric materials.   It was
observed (see for example Haydl and Quate (1966)) that if the conditions
for amplification were satisfied, i.e. a sufficiently large voltage was
applied across the sample that the current oscillates.   These oscillations

were demonstrated to be due to propagating domains of high acoustic flux and d.c. electric field. The domains were found to move with a velocity approximately equal to the velocity of sound and to move without a change of shape or size. These domains are however not the simple domains as observed by Gunn (1963) and theoretically confirmed by Butcher (1965), associated with current oscillations in semiconductors having a bulk negative differential conductivity. The equations relevant to the Gunn effect can be solved in the nonlinear regime by conventional techniques of nonlinear analysis (see for example Minorsky (1962)) and these solutions include domain-like modes of wave propagation. We show in the Appendix that the equations relevant to the acoustoelectric effect do not have a simple domain solution. We show in 4.2 that acoustoelectric domains are "envelope domains", i.e. the envelope of the wave has the shape of a domain and moves without a change of shape or size. Within this envelope the behaviour of the flux or the electric field is rapidly oscillating.

It is obvious that a simple linear theory cannot describe the creation of an acoustic domain nor demonstrate current saturation and hence a nonlinear and/or large signal analysis of the amplification of noise to form domains is required. This is a problem of great complexity and no complete theory has yet been given.

The many nonlinear theories presented to date can be roughly divided into two distinct classes. The first class consists of macroscopic theories concerned with nonlocal processes leading to the build up of acoustoelectric current, i.e. theories concerned with how acoustic waves interact a long time and a long way away from the point of generation. The acoustic waves are assumed to be well formed and propagating in a part of the semiconductor where further wave generation is not taking place. In this sense these theories can be considered asymptotic. The second class consists of theories concerned with the

local microscopic processes leading to the development of the acoustic waves themselves, i.e. theories concerned with how the acoustic waves build up from thermal noise and how they interact close to the point of generation.

The majority of published work has been concerned with the microscopic local processes and generally the assumption of a slow wave evolution is made. Examples of this are the work of Tien (1968), Butcher and Ogg (1970). Wonneberger et al (1969) and Gay and Hartnagel (1969). These results, which are valid at large amplitudes are generally in agreement with experimental work.

The macroscopic nonlocal theory has proved more difficult. Notable attempts at this, Ridly and Wilkinson (1969), Ridley (1971), have indicated a domain-like solution by including the effects of the macroscopic build up of space charge. However, Butcher et al (1971) showed that the basic assumptions of the Ridley and Wilkinson work were inconsistent with this conclusion. Thus there is no satisfactory theory of domain formation for even a single acoustic mode.

The work of this Chapter will be concerned with a nonlocal nonlinear theory of the propagation of a single acoustic wave in a piezoelectric semiconductor. The wave will be assumed to exist in the semiconductor and it's nonlinear theory will be predicted using the general theory as given in Chapter 2. In this sense the system analysed may be considered as a piezoelectric semiconductor driven at a particular frequency.

Section 4.2. outlines the formulation of the problem and summarises the well known linear theory. In 4.3 we show that the equation of motion of the envelope of the acoustic wave is the generalised nonlinear Schrödinger equation and expressions for the coefficients of this equation are given. The method used in 4.3. is not the general method as given in 2.2. but a more direct approach is taken by working with the

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matrices relevant to the problem and so avoiding algebraic complexity.
The equation is solved in 4.4 using a perturbation analysis and by
ordering the coefficients in a particular fashion and so the envelope
domain solution is obtained.   In 4.5 the coefficients are explicitly
evaluated in the long wavelength κ, small κ^2 limit and are found to
have the ordering assumed in 4.4.   An expression is given for the d.c.
electric field and is found to have the correct domain like form.
An effective linear amplification constant is deduced which shows the
correct decrease of growth rate in the presence of flux.   The results
are discussed in 4.6 and suggestions for further work are made.

4.2. Formulation and linear theory

We will consider the propagation of an acoustic wave in a piezo-
electric semiconductor in one dimension.   Although this problem should
be treated in three dimensions due to the tensor nature of the piezo-
electric equations of state for the sake of simplicity we will assume
that there is only one piezoelectric constant.   This implies that the
electric field, electric displacement and strain have only one component
in the direction of propagation.   The semiconductor is assumed to be
n type, extrinsic with electrons of charge e and mass m.   Following
White (1962) we consider the relevant equations describing the semi-
conductor.

The piezoelectric equations of state are:

$$T = cS - eE \qquad\qquad 4.2.1$$

$$D = eS + \varepsilon_D E \qquad\qquad 4.2.2$$

where ε_D is the dielectric constant at constant strain,

e is the piezoelectric constant,

c is the elastic constant at constant electric field,

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D   is the electrical displacement,

T   is the local stress,

S   is the local strain,

and   E   is the electric field.

The strain is defined in terms of the local mechanical displacement
u by:

$$S = \frac{\partial u}{\partial x} \qquad\qquad 4.2.3$$

where + x is the direction of propagation of the wave.

Differentiating 4.2.1 with respect to x gives:

$$\frac{\partial T}{\partial x} = c\,\frac{\partial S}{\partial x} - e\,\frac{\partial E}{\partial x}$$

$$= c\,\frac{\partial^2 u}{\partial x^2} - e\frac{\partial E}{\partial x} \qquad\qquad 4.2.4$$

The equation of motion of an elastic solid is given by:

$$\rho\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial T}{\partial x} \qquad\qquad 4.2.5$$

and combining 4.2.4 with 4.2.5 gives the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{c}{\rho}\,\frac{\partial^2 u}{\partial x^2} - \frac{e}{\rho}\,\frac{\partial E}{\partial x} \qquad\qquad 4.2.6$$

where $(c/\rho)^{\frac{1}{2}}$ is the velocity of sound.

These equations are now combined with, Gauss' equation

$$\frac{\partial D}{\partial x} = Q \qquad\qquad 4.2.7$$

and the charge conservation equation

$$\frac{\partial J}{\partial x} = \frac{\partial Q}{\partial t} = \frac{q \partial n_s}{\partial t} \qquad 4.2.8$$

where Q is the local space charge, q is the electronic charge and n_s is the number of electrons giving rise to this space charge. The total current density is given by

$$J = q\mu n_c E + qD_n \frac{\partial n_c}{\partial x} + \frac{\partial D}{\partial t} \qquad 4.2.9$$

where μ is the electron mobility

n_c is the number of electrons in the conduction band

and D_n is the electron diffusion constant.

We can write $n_c = n_o + fn_s$ \qquad 4.2.10

where n_o is the equilibrium number of electrons corresponding to charge neutrality. The factor f takes into account the number of electrons that may be trapped in the forbidden band which do not participate in conduction. Only a fraction f of space charge present in the conduction band is available with f = 1 for no trapping.

Equations 4.2.1-4.2.10 completely describe the problem. To obtain the linear small signal dispersion relation we look for solutions of the form:

$$u = u_1 \exp\left[i(kx-\omega t)\right]$$

$$E = E_o + E_1 \exp\left[i(kx-\omega t)\right] \qquad 4.2.11$$

$$n_c = n_o + n_1 \exp\left[i(kx-\omega t)\right]$$

where E_o is the external applied d.c. field. The only nonlinearity that occurs in the set of equations 4.2.1 to 4.2.10 is the term $q\mu n_c E$

in equation 4.2.9. In the linear theory the term $n_1 E_1$ which arises when solutions 4.2.11 are substituted into 4.2.9 is neglected. In order that 4.2.11 satisfy the above set of equations the following cubic dispersion relation is obtained:

$$\omega^3 + \omega^2 \{\mu f E_o k + i\omega_c\} - \omega \{v_s^2 k^2 + \kappa^2 k^2 v_s^2\}$$

$$- v_s^2 k^3 \mu f E_o - i\omega_c v_s^2 k^2 - \kappa^2 \mu f E_o k^3 v_s^2 = 0 \qquad 4.2.12$$

Here $\kappa^2 = e^2/\varepsilon_D c$ and is known as the electromechanical coupling constant. κ^2 is very small, typically $\sim 10^{-2}$ for commonly used piezoelectric materials such as ZnO and CdS and is used as an expansion parameter in later sections.

In the limit $\kappa^2 \equiv 0$ equation 4.2.12 has two roots:

$$\omega/k = v_s$$

and

$$\omega = \mu E_o k - i(\omega_c + k^2 D_n)$$

where

$$v_s^2 = \rho/c$$

and

$$\omega_c = q\mu n_o/\varepsilon_D$$

The second root corresponds to damped space charge waves and is not relevant to the present discussion. We now look for a solution of 4.2.12 around the first root $\omega = kv_s$ in the limit of small κ^2. Using Newton's method we obtain

$$\omega = kv_s \left(1 + \frac{\kappa^2}{2} \left[1 + \frac{\omega_c}{\omega_D \gamma^2} \left(1 + \frac{k^2 v_s^2}{\omega_c \omega_D} \right) \right] \left[1 + \frac{\omega_c^2}{\gamma^2 k^2 v_s^2} \left(1 + \frac{k^2 v_s^2}{\omega_c \omega_D} \right)^2 \right]^{-1} \right)$$

$$- i \frac{\kappa \kappa \omega_c}{2\gamma} \left[1 + \frac{\omega_c^2}{\gamma^2 k^2 v_s^2} \left(1 + \frac{k^2 v_s^2}{\omega_c \omega_D} \right)^2 \right]^{-1} \qquad 4.2.13$$

where
$$\omega_D = v_s^2 / f D_n$$

$$\gamma = 1 + \frac{\mu f E_o}{v_s} \qquad \qquad 4.2.14$$

In deriving 4.2.13 only terms to $O(\kappa^2)$ were retained. The parameter γ is important, being a measure of the ratio of the electron drift velocity to the velocity of sound. If $\gamma < 0$ (i.e. the electron drift velocity exceeds the sound velocity, then the imaginary part of ω is positive and amplification occurs. Conversely if $\gamma > 0$ the wave is attenuated.

It is the dependence of γ on the electric field through 4.2.14 that justifies applying the general theory of Chapter 2 to the full non-linear problem. If we assume that the total external d.c. field is given by:

$$E_o = - \frac{v_s}{\mu f} - \epsilon^2 E' \qquad \qquad 4.2.15$$

where ϵ is a small parameter, then 4.2.13 may be expanded as:

$$\omega = \omega_r + i\epsilon^2 \omega_j + O(\epsilon^4)$$

where
$$\omega_r = k v_s \left(1 + \frac{k^2 v_s^2}{2(\omega_c \omega_D + k^2 v_s^2)} \right) \qquad 4.2.16$$

and
$$\omega_i = - \frac{\kappa^2 k^2 v_s^2 \gamma}{2\omega_c} \left(1 + \frac{k^2 v_s^2}{\omega_c \omega_D} \right)^{-2} \qquad 4.2.17$$

If the modulus of the electric field is slightly larger than the critical value $-v_s/\mu f$ but to order ϵ^2 then the growth rate of the instability is also of order ϵ^2.

We have therefore satisfied all the conditions necessary for the validity of the general theory of Chapter 2 when considering the nonlinear

problem, i.e. the imaginary part of ω is small, $O(\epsilon^2)$ where ϵ is the magnitude of the nonlinearity. The parameter p of 2.2.5 has been found explicitly and an analytic expression for the growth rate has been obtained.

4.3. Derivation of the nonlinear Schrödinger equation

Equations 4.2.1 to 4.2.10 may be combined to give three equations in three unknowns, i.e.

$$\rho \frac{\partial^2 u}{\partial t^2} - c \frac{\partial^2 u}{\partial x^2} + e \frac{\partial E}{\partial x} = 0 \qquad\qquad 4.3.1$$

$$\mu n_o \frac{\partial E}{\partial x} + \mu f \frac{\partial}{\partial x}(n_s E) + D_n f \frac{\partial^2 n_s}{\partial x^2} - \frac{\partial n_s}{\partial t} = 0 \qquad 4.3.2$$

$$e \frac{\partial^2 u}{\partial x^2} + \epsilon_D \frac{\partial E}{\partial x} + q n_s = 0 \qquad\qquad 4.3.3$$

where all quantities are defined in 4.2.

We look for solutions of these equations of the form

$$\underline{X} = \underline{X}^o + \sum_{\ell=-\infty}^{\infty} \sum_{\alpha=1}^{\infty} \epsilon^\alpha \underline{X}_\ell^\alpha (\tau, \xi) \exp\left[i\ell(kx-\omega t)\right] \qquad 4.3.4$$

where
$$\tau = \epsilon^2 t$$

$$\xi = \epsilon(x-\lambda t)$$

and
$$\underline{X} = \begin{pmatrix} u \\ E \\ n_s \end{pmatrix}$$

The equilibrium state $\underline{X}_o$ around which the expansion is made is taken as $u = n_s = 0$ and $E = E_o$, the total external d.c. electric field. Substituting 4.3.4, expanding and equating powers of ϵ of the same harmonic to zero gives an infinite set of equations on which the first three are given by:

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$$O(\varepsilon) \qquad \underline{\underline{W}}'_\ell \, \underline{X}^1_\ell \;=\; 0$$

$$O(\varepsilon^2) \qquad \underline{\underline{W}}'_\ell \, \underline{X}^2_\ell + \underline{\underline{M}}'_\ell \frac{\partial \underline{X}^1_\ell}{\partial \xi} \;=\; \underline{S}^\ell_1(\underline{X}^1_\ell)$$

$$O(\varepsilon^3) \qquad \underline{\underline{W}}'_\ell \, \underline{X}^3_\ell + \underline{\underline{M}}'_\ell \frac{\partial \underline{X}^2_\ell}{\partial \xi} + \underline{\underline{N}}'_\ell \frac{\partial \underline{X}^1_\ell}{\partial \tau} + \underline{\underline{O}}'_\ell \frac{\partial^2 \underline{X}^1_\ell}{\partial \xi^2} \;=\; \underline{S}^\ell_2(\underline{X}^1_\ell, \underline{X}^2_\ell)$$

where

$$\underline{\underline{W}}'_\ell = \begin{pmatrix} \ell^2(ck^2 - p\omega^2) & ei\ell k & 0 \\[2ex] e(i\ell k)^2 & \varepsilon_o i\ell k & q \\[2ex] 0 & \mu n_o i\ell k & i\ell\omega + \mu fE_o i\ell k - v_s^2 k^2 \ell^2/\omega_D \end{pmatrix}$$

$$\underline{\underline{M}}'_\ell = \begin{pmatrix} 2\rho\lambda i\ell\omega - 2ci\ell k & e & 0 \\[2ex] 2ei\ell k & \varepsilon_D & 0 \\[2ex] 0 & \mu n_o \ell & \ell\mu fE_o + \lambda\ell + 2i\ell k v_s^2/\omega_D \end{pmatrix}$$

$$\underline{\underline{N}}'_\ell = \begin{pmatrix} -2\rho i\ell\omega & 0 & 0 \\[2ex] 0 & 0 & 0 \\[2ex] 0 & 0 & -1 \end{pmatrix}$$

$$\underline{\underline{O}}'_\ell = \begin{pmatrix} \lambda^2\rho - c & 0 & 0 \\[2ex] 0 & 0 & 0 \\[2ex] 0 & 0 & v_s^2/\omega_D \end{pmatrix}$$

and the source terms are given by

$$\underline{S}_1^\ell(\underline{X}_\ell^1) \;=\; -\mu f \left(<\sum_p \sum_q ipk n_{sp}^1 E_q^1 P_p P_q >_\ell \;+\; <\sum_p \sum_q n_{sp}^1 iqk E_q^1 P_p P_q >_\ell \right) \qquad 4.3.5$$

$$\underline{S}_2^\ell(\underline{X}_\ell^1,\underline{X}_\ell^2) \;=\; -\mu f \left(<\sum_p \sum_q \frac{\partial n^1}{\partial \xi} sp\, E_q^1 P_p P_q >_\ell \;+\; <\sum_p \sum_q ipk n_{sp}^2 E_q^1 P_p P_q >_\ell \right.$$

$$+\; <\sum_p \sum_q ipk n_{sp}^1 E_q^2 P_p P_q >_\ell \;+\; <\sum_p \sum_q n_{sp}^1 \frac{\partial E_q^1}{\partial \xi} P_p P_q >_\ell$$

$$\left. +\; <\sum_p \sum_q n_{sp}^2 iqk E_q^1 P_p P_q >_\ell \;+\; <\sum_p \sum_q n_{sp}^1 iqk E_q^2 P_p P_q >_\ell \right) \qquad 4.3.6$$

where $<\ldots>_\ell$ denotes the coefficient of the ℓth harmonic and P_p denotes $\exp|ip(kx-\omega t)|$.

The linear dispersion relation 4.2.13 is regained by insisting that the first order solution is non-trivial, i.e. $\underline{X}_\ell^1 \neq 0$. Thus 4.3.5 gives

$$\det \underline{W}_1' \;=\; 0$$

It can easily be seen from 4.2.14 that

$$\det \underline{W}_\ell' \;\neq\; 0$$

for $\ell \neq 1$ i.e. all harmonics of the fundamental are stable.

Now applying the condition 4.2.15 and expanding the matrices $\underline{W}', \underline{M}', \underline{N}'$ and $\underline{O}'$ gives the following set of equations

$$\underline{W}_\ell \, \underline{X}_\ell^1 \;=\; 0 \qquad\qquad 4.3.7$$

$$\underline{W}_\ell \, \underline{X}_\ell^2 + \underline{M}_\ell \frac{\partial \underline{X}_\ell^1}{\partial \xi} \;=\; \underline{S}_1^\ell(\underline{X}_\ell^1) \qquad\qquad 4.3.8$$

and

$$\underline{W}_\ell \, \underline{X}_\ell^3 + \underline{M}_\ell \frac{\partial \underline{X}_\ell^2}{\partial \xi} + \underline{N}_\ell \frac{\partial \underline{X}_\ell^1}{\partial \tau} + \underline{O}_\ell \frac{\partial^2 \underline{X}_\ell^1}{\partial \xi^2} + \underline{P}_\ell \, \underline{X}_\ell^1 \;=\; \underline{S}_2^\ell(\underline{X}_\ell^1,\underline{X}_\ell^2)$$

$$4.3.9$$

where

$$\underline{\underline{W}}_\ell = \begin{pmatrix} \ell^2(ck^2-\rho\omega_r^2) & ei\ell k & 0 \\ -ek^2\ell^2 & \varepsilon_D i\ell k & q \\ 0 & un_o i\ell k & i\ell\omega_r-v_s i\ell k-v_s^2 k^2\ell^2/\omega_D \end{pmatrix} \quad 4.3.10$$

$$\underline{\underline{M}}_\ell = \begin{pmatrix} 2\rho\lambda i\ell\omega_r-2ci\ell k & e & 0 \\ 2i\ell ke & \varepsilon_o & 0 \\ 0 & \mu n_o\ell & \ell\lambda-\ell v_s+2v_s^2 k^2\ell^2/\omega_D \end{pmatrix} \quad 4.3.11$$

$$\underline{\underline{N}}_\ell = \begin{pmatrix} -2\rho i\ell\omega_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad 4.3.12$$

$$\underline{\underline{O}}_\ell = \begin{pmatrix} \lambda^2\rho-c & 0 & 0 \\ e & 0 & 0 \\ 0 & 0 & v_s^2/\omega_D \end{pmatrix} \quad 4.3.13$$

and

$$\underline{\underline{P}}_\ell = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mu f E' ik \end{pmatrix} \quad 4.3.14$$

where $\qquad \underline{\underline{W}}'_\ell = \underline{\underline{W}}_\ell + \varepsilon^2\underline{\underline{P}}_\ell$

and $\qquad \underline{\underline{M}}'_\ell = \underline{\underline{M}}_\ell + O(\varepsilon^2)$

$\qquad\qquad \underline{\underline{N}}'_\ell = \underline{\underline{N}}_\ell + O(\varepsilon^2)$

etc.

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The ω_r and ω_i are now given by 4.2.16 and 4.2.17 and $\lambda = \partial\omega_r/\partial k$. The only additional term is the $\underline{\underline{P}}_\ell \, \underline{\underline{X}}^1_{\,\ell}$ term in the third order equation. The condition for a nontrivial solution of the first order equation is now

$$\det \underline{\underline{W}}_1 \;=\; 0$$

which gives

$$\omega_r \;=\; kv_s \left(1 + \frac{k^2 v_s^2}{2(\omega_c \omega_D + k^2 v_s^2)} \right)$$

as in 4.2.16.

Again, $\det \underline{\underline{W}}_\ell \;\neq\; 0$

for $\ell \neq 1$.

We define $\underline{\underline{X}}^1_{\,\ell} \;=\; \phi'(\tau,\xi)\underline{R}$ 4.3.15

where $\underline{R}$ is the right eigenvector of $\underline{\underline{W}}_1$ and is given as

$$\underline{R} \;=\; \begin{bmatrix} \dfrac{-eik}{(ck^2 - \rho\omega_r^2)} \\[4mm] 1 \\[4mm] \dfrac{-\mu n_o ik}{i\omega_r - v_s ik - \dfrac{v_s^2 k^2}{\omega_D}} \end{bmatrix}$$

The left eigenvector is given by

$$\underline{L} \;=\; \begin{bmatrix} \dfrac{ek^2}{(ck^2 - \rho\omega_r^2)} \\[4mm] 1 \\[4mm] \dfrac{-q}{(i\omega_r - v_s ik - v_s^2 k^2/\omega_D)} \end{bmatrix}$$

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The $\epsilon = 1$ first order equation then gives:

$$\underline{X}^1_1 = \phi'(\tau,\xi)\underline{R} \qquad\qquad 4.3.16$$

$$\underline{X}^1{}_\ell = 0 \quad \text{for } |\ell| \neq 1, 0.$$

At this stage we are unable to determine all the components of $\underline{X}^1_o$. From the first order equation for $\ell = 0$, i.e.

$$\underline{W}_o\, \underline{X}^1_o = 0$$

we see that $\qquad\qquad n^1_{so} = 0$

However the second order equation for $\ell = 1$ is

$$\underline{W}_1\, \underline{X}^2_1 + \underline{M}_1\, \underline{R}\, \frac{\partial\phi'}{\partial\xi} = \underline{S}^1_1(\underline{X}^1_1,\underline{X}^1_o)$$

where

$$\underline{S}^1_1(\underline{X}^1_1,\underline{X}^1_o) = \begin{bmatrix} 0 \\[2ex] 0 \\[2ex] -\mu f i k(E^1_o n^1_{s1} + n^1_{so} E^1_1) \end{bmatrix}$$

In order that $\underline{X}^2_1$ is unique the compatibility condition

$$\underline{L}.\underline{S}^1_1 = 0$$

must be satisfied,

since $\qquad\qquad \underline{L}.\underline{W} = 0$

and $\qquad\qquad \underline{L}.\underline{M}_1.\underline{R} = 0$

By direct evaluation and using $\quad n^1_{so} = 0$

we see $\qquad\qquad E^1_o = 0.$

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Although u_o^1 is undetermined it is not required in future calculations. The second order equation for $\ell = 1$ may now be solved to give

$$\underline{X}_1^2 = \underline{R}\phi^{(2)}(\tau,\xi) - i\,\frac{\partial R}{\partial k}\,\frac{\partial \phi}{\partial \xi}' = 0$$

where $\phi^{(2)}(\tau,\xi)$ is another scalar function which is not relevant to our solution and is eliminated later.

Now $\qquad \underline{S}_1^\ell = 0 \qquad$ for $\ell > 2$

and since $\qquad \det \underline{\underline{W}}_\ell \neq 0 \qquad$ for $\ell > 2$

it may be concluded that

$$\underline{X}_\ell^2 = 0 \qquad \text{for } \ell > 2.$$

The only other components of $\underline{X}_\ell^2$ that are required are $\underline{X}_2^2$, $\underline{X}_{-2}^2$ and $\underline{X}_o^2$

These are formally given by:

$$\underline{\underline{W}}_o \cdot \underline{X}_o^2 = \underline{S}_1^o(\underline{X}_1^1, \underline{X}_{-1}^2)$$

$$\underline{\underline{W}}_2 \cdot \underline{X}_2^2 = \underline{S}_1^2(\underline{X}_o^1, \underline{X}_1^1, \underline{X}_2^1)$$

and $\qquad\qquad \underline{X}_2^2 = (\underline{X}_{-2}^2)^*$

The source terms are explicitly given as

$$\underline{S}_1^o = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{S}_1^2 = -\mu f i k (\phi')^2 \begin{bmatrix} 0 \\ 0 \\ 2R_2R_3 \end{bmatrix}$$

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$\underline{X}_2^2$ and $\underline{X}_{-2}^2$ are given by

$$\underline{X}_2^2 = \underline{W}_2^{-1} \cdot \underline{S}_1^2$$

and

$$\underline{X}_2^2 = (\underline{X}_{-2})^*$$

i.e.

$$\underline{X}_2^2 = -\mu i k f(\phi')^2 \begin{bmatrix} (W_2^{-1})_{13} \\[2ex] (W_2^{-1})_{23} \\[2ex] (W_2^{-1})_{33} \end{bmatrix}$$

where $(W_2^{-1})_{ij}$ is the (i,j) element of $\underline{W}_2^{-1}$.

Again, not all the elements of $\underline{X}_o^2$ are immediately available, since $\det \underline{W}_o$ vanishes. As before, the third row of

$$\underline{W}_o \, \underline{X}_o^2 = \underline{S}_1^o = 0$$

gives only

$$n_{so}^2 = 0$$

The element u_o^2 is not required but determining E_o^2 is vital. E_o^2 is obtained from the third row of the $\ell = 0$ component of the third order equation

i.e.

$$\underline{W}_o \, \underline{X}_o^3 + \underline{M}_o \, \frac{\partial \underline{X}_o^2}{\partial \xi} + \underline{N}_o \, \frac{\partial \underline{X}_o^1}{\partial \tau} + \underline{O}_o \, \frac{\partial^2 \underline{X}_o^1}{\partial \xi^2} + \underline{P}_o \, \underline{X}_o^1 = \underline{S}_2^o$$

of which the third row gives:

$$\mu n_o \, \frac{\partial E_o^2}{\partial \xi} + (\mu f E_o + \lambda) \, \frac{\partial n_{so}^2}{\partial \xi} - \frac{\partial n_{so}^1}{\partial \tau} + \frac{v_s^2}{\omega_D} \, \frac{\partial^2 n_{so}^1}{\partial \xi^2} = (\underline{S}_2^o)_3$$

where

$$(\underline{S}_2^o)_3 = -\mu f \left(\frac{\partial}{\partial \xi} (E_1^1 n_{s-1}^1 + E_{-1}^1 n_{s1}^1) \right)$$

$$= -\mu f \, \frac{\partial}{\partial \xi} \{ (R_2 R_3^* + R_2^* R_3) |\phi'|^2 \}$$

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i.e.
$$\frac{\partial}{\partial \xi} \{\mu n_o E_o^2 + \mu f(R_2 R_3^* + R_2^* R_3)|\phi'|^2\} = 0$$

which gives

$$E_o^2 = \frac{-\mu f}{\mu n_o}(R_2 R_3^* + R_2^* R_3)|\phi'|^2 + g(\tau) \qquad 4.3.17$$

where $g(\tau)$ is an arbitrary function of τ and is determined by applying the initial value condition $E_o^2 = -E'$.

The nonlinear Schrödinger equation is now obtained by considering the $\ell = 1$ component of 4.3.9.

$$\underline{W}_1 X_1^3 + \underline{M}_1 \frac{\partial X_1^2}{\partial \xi} + \underline{N}_1 \frac{\partial X_1^1}{\partial \tau} + \underline{O}_1 \frac{\partial^2 X_1^1}{\partial \xi^2} + \underline{P}_1 X_1^1 = \underline{S}_2^1(\underline{X}_\ell^1, \underline{X}_\ell^2)$$

Multiplying on the left by $\underline{L}$ eliminates the first term and substituting for $\underline{X}_1^2$ in the second term and for $\underline{X}_{-2}^2$ and $\underline{X}_{-o}^2$ in the source term $\underline{S}_2^1$ gives:

$$\left[\frac{\partial L}{\partial k}\underline{W}_1 \frac{\partial R}{\partial k} + L\,\underline{O}_1 R\right]\frac{\partial^2 \phi'}{\partial \xi^2} + L\,\underline{N}_1 R\frac{\partial \phi'}{\partial \tau} + L\,\underline{P}_1 R\phi'$$

$$= R_3 L_3 (\mu fik)^2\left[2R_3^*(W_2^{-1})_{23} + 2(W_2^{-1})_{33} + \frac{1}{\mu n_o ik}(R_3 + R_3^*)\phi''|\phi''|^2\right.$$

$$\left. + \mu fik R_3 L_3 E' \phi'\right] \qquad 4.3.18$$

with the relation previously determined

$$\frac{\partial L}{\partial k}\underline{W}_1 \frac{\partial R}{\partial k} = \frac{1}{2}\underline{L}\frac{\partial^2 \omega_r}{\partial k^2}\underline{R}$$

we can now write:

$$i\frac{\partial \phi'}{\partial \tau} + X\frac{\partial^2 \phi'}{\partial \xi^2} + \beta_i \phi' + \delta\phi'|\phi'|^2 = 0 \qquad 4.3.19$$

which is the generalised nonlinear Schrödinger equation and where

$$\chi = \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} r \qquad\qquad 4.3.20$$

$$\beta_i = i\omega_i \qquad\qquad 4.3.21$$

and

$$\delta = i(\mu fk)^2 \left[-2i\rho\omega_r \frac{R_1 L_1}{R_3 L_3} - 1 \right]^{-1} \left[2R_3^*(W_2^{-1})_{23} + 2(W_2^{-1})_{33} + \frac{1}{\mu n_o ik}(R_3 + R_3^*) \right]$$

$$4.3.22$$

The coefficient χ is always real and β_i is always imaginary. A knowledge of $\phi^{\prime}$, i.e. the solution of 4.3.19 gives all physical quantities of interest, since they can all be expressed in terms of $\phi^{\prime}$.

Of particular interest is the total d.c. electric field which is given by:

$$E_{TOTAL}^{D.C} = E_{APPLIED} + \epsilon E_o^1 + \epsilon^2 E_o^2 + \ldots.$$

$$= E_o - \epsilon^2 E^{\prime} - \frac{\epsilon^2 \mu f}{\mu n_o}(R_3 + R_3^*)|\phi^{\prime}|^2 \qquad 4.3.23$$

With the appropriate $\phi^{\prime}$ this correctly predicts the high field domain. Expressions for all other A.C. and D.C. components can similarly be derived. We now look for the solution of 4.3.19 for this particular problem.

4.4. Solution of the generalised nonlinear Schrödinger equation

The nonlinear Schrödinger equation derived in 4.3 is now solved using a further perturbation expansion. An exact solution of 4.3.19 with arbitrary coefficients is not known. General solutions of the nonlinear Schrödinger equation, i.e. with β and δ real are well known, Rowlands (1974) and consist of three types of nonlinear wave, the solitary wave and the shock. Stability analyses of these solutions have also been

made, Rowlands (1974) and the techniques of that work form the basis of this analysis.

We expect the solution of the generalised equation to be similar to that obtained for the real β, real δ case and we use these solutions as a starting point about which a perturbation expansion is made. This expansion relies heavily on an ordering of the coefficients. It is difficult in general to justify the ordering assumed and the coefficients must in general be explicitly evaluated before anything can be said about their relative sizes. For this particular problem the ordering assumed in this section is found to be valid in the long wavelength small κ^2 limit. Until the coefficients are evaluated we will assume that they may be ordered as is assumed and a general notation is adopted.

Initially, we look for solutions of 4.3.19 of the form

$$\phi^- = \phi(\xi)\exp\{-i\beta_r\tau\} \qquad\qquad 4.4.1$$

where β_r is real. In terms of the new function ϕ 4.3.19 reduces to

$$\chi\,\frac{\partial^2\phi}{\partial\xi^2} + (\beta_r+i\beta_i)\phi + (\delta_r+i\delta_i)\phi|\phi|^2 = 0 \qquad\qquad 4.4.2$$

Although β_r as used here is arbitrary at this point it will be seen later that, in order that the solutions ϕ be bounded, β_r is in fact uniquely determined by the other coefficients of 4.4.2. In fact β_r itself must be expanded as a series, for which a condition is deduced such that this expansion is valid to second order. Immediately we merely assume that β_r is positive. This will in fact be verified when the coefficients are explicitly evaluated.

We suppose that β_i and δ_i are small compared with the other coefficients and introduce a small imaginary part to the wave amplitude ϕ. This imaginary part is assumed to be of the same order as the

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coefficients β_i and δ_i. We now attempt the perturbation expansion about ϕ_o which is a solution of 4.4.2 with $\beta_i = \delta_i = 0$ and ϕ real, i.e. a solution of

$$\chi \frac{\partial^2 \phi_o}{\partial \xi^2} + \beta_r \phi_o + \delta_r \phi_o^3 = 0$$

One integration gives:

$$\frac{\chi}{2} \left(\frac{\partial \phi_o}{\partial \xi}\right)^2 + \frac{\beta_r \phi_o^2}{2} + \frac{\delta_r \phi_o^4}{4} + \theta = 0 \qquad 4.4.3$$

The different classes of solution now depend on the relative signs of β_r and δ_r and the value of the constant of integration θ. For the time being we assume that β_r and δ_r are of opposite sign, which will be justified later. The only non trivial bounded solutions which have the correct asymptotic behaviour is given by choosing:

$$\theta = -\frac{1}{2} B^2 A^2$$

where

$$B^2 = \beta_r / 2\chi \qquad A^2 = -\beta_r / \delta_r \qquad 4.4.4$$

With these definitions one further integration of 4.4.3 gives

$$\phi_o = A \tanh \left[B\xi + d \right] \qquad 4.4.5$$

where d is an arbitrary constant. If β_r and δ_r are of opposite sign then

$$\phi_o = A \operatorname{sech}^2 \left[B\epsilon + d \right]$$

and the analysis proceeds in the same way.

We now assume solutions of 4.4.2 of the form:

$$\phi = (\phi_o + \eta \phi_1 + \eta^2 \phi_2 + \ldots) \exp \left| i(\eta \psi_1 + \eta^2 \psi_2 + \ldots) \right|$$

where the ϕ_i and ψ_i are real functions and η is a small expansion parameter

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later identified with κ^2. The coefficients β_i and δ_i are ordered as $O(\eta)$. Substituting into 4.4.2 and equating powers of η gives the following series of equations.

$$O(1) \qquad \chi \frac{\partial^2 \phi_o}{\partial \xi^2} + \beta_r \phi_o + \delta_r \phi_o^3 = 0 \qquad\qquad 4.4.6$$

$$O(\eta) \qquad \chi \frac{\partial^2 \phi_1}{\partial \xi^2} + \beta_r \phi_1 + 3\delta_r \phi_o^2 \phi_1 = 0 \qquad\qquad 4.4.7$$

$$\chi \phi_o \frac{\partial^2 \psi_1}{\partial \xi^2} + 2\chi \left(\frac{\partial \psi_1}{\partial \xi}\right)\left(\frac{\partial \phi_o}{\partial \xi}\right) = -\beta_i \phi_o - \delta_i \phi_o^3 \qquad\qquad 4.4.8$$

$$O(\eta^2) \qquad \chi \frac{\partial^2 \phi_2}{\partial \xi^2} + \beta_r \phi_2 + 3\delta_r \phi_o^2 \phi_2$$

$$= \chi \left(\frac{\partial \psi_1}{\partial \xi}\right)^2 \phi_o - 3\beta_i \phi_o \phi_1^2 \qquad\qquad 4.4.9$$

$$\chi \phi_o \frac{\partial^2 \psi_2}{\partial \xi^2} + 2\chi \frac{\partial \phi_o}{\partial \xi} \frac{\partial \psi_2}{\partial \xi}$$

$$= -\beta_i \phi_i - 3\delta_i \phi_o^2 \phi_1 - 2\chi \frac{\partial \phi_1}{\partial \xi} \frac{\partial \psi_1}{\partial \xi} - \chi \phi_1 \frac{\partial^2 \psi_1}{\partial \xi^2}$$

$$4.4.10$$

All high order equations have the same form, i.e. the same differential operator on the left hand side is repeated with differing right hand sides depending only on lower order solutions. In the present context only solutions to order η^2 are required.

We shall now systematically solve equations 4.4.6 to 4.4.10. The solution of 4.4.6 has already been found and is given by 4.4.5. Differentiating 4.4.6 gives:

$$\chi \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \phi_o}{\partial \xi}\right) + \beta_r \left(\frac{\partial \phi_o}{\partial \xi}\right) + 3\delta_r \phi_o^2 \left(\frac{\partial \phi_o}{\partial \xi}\right) = 0$$

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which comparing with 4.4.7 gives:

$$\phi_1 \ \alpha \ \frac{\partial \phi_0}{\partial \xi}$$

or
$$\phi_1 = G \text{sech}^2 \left[B\xi + d \right] \qquad 4.4.11$$

where G is determined by boundary conditions.

The solutions of the remaining equations 4.4.8 to 4.4.10 are readily determined using the general theory of differential equations (e.g. Moroe and Feshbach 1953). If we know one solution of a homogeneous second order differential equation then the other solution is readily obtained, i.e. if y_1 is one solution of the homogeneous equation, the other solution is given by

$$y_2(x) = y_1(x) \int^x \frac{dx'}{y_1^2(x')} \qquad 4.4.12$$

The Green's function is now readily determined and hence the solution of the inhomogeneous equation i.e.

$$y(x) = y_1(x) \int^x f(x') y_2(x') dx' - y_2(x) \int^x f(x') y_1(x') dx'$$

$$4.4.13$$

is a solution of the equation when the inhomogeneous term is given by $f(x)$.

The first equation that must be solved is 4.4.8 for ψ_1 since solving 4.4.9 and 4.4.10 depends on knowing ψ_1. The details of the solution of equations 4.4.8 and 4.4.9 are given below.

Equation 4.4.8 is readily expressed as

$$\frac{\chi}{\phi_0} \frac{\partial}{\partial \xi} \left(\phi_0^2 \frac{\partial \psi_1}{\partial \xi} \right) = - \beta_i \phi_0 - \delta_i \phi_0^3$$

and substituting for ϕ_0 gives

$$\frac{\partial}{\partial \xi}\left(\phi_o^2 \frac{\partial \psi_1}{\partial \xi}\right) = -\frac{\beta_i A^2}{\chi} \tanh^2[B\xi + d] - \frac{\delta_i A^4}{\chi} \tanh^4[B\xi + d]$$

One integration then gives:

$$A^2 \tanh^2[B\xi + d] \frac{\partial \psi_1}{\partial \xi} = \left[\frac{-\beta_i A^2}{B\chi} - \frac{\delta_i A^4}{B\chi}\right][B\xi + d]$$

$$- \left[\frac{-\beta_i A^2}{B\chi} - \frac{\delta_i A^4}{B\chi}\right]\tanh[B\xi + d] + \frac{\delta_i A^4}{B\chi} \frac{\tanh^3[B\xi + d]}{3}$$

Integrating again gives:

$$\psi_1 = \left[\frac{-\beta_i - \delta_i A^2}{B^2\chi}\right] \log|\sinh[B\xi + d]| + \frac{\delta_i A^2}{3B^2\chi} \log|\cosh[B\xi + d]|$$

$$+ \left[\frac{-\beta_i - \delta_i A^2}{B^2\chi}\right] \left[(B\xi + d)^2 - (B\xi + d)\tanh[B\xi + d]\right]$$

$$- \left[\frac{-\beta_i - \delta_i A^2}{B^2\chi}\right] \left[(B\xi + d)^2 - \log|\cosh[B\xi + d]|\right] \qquad 4.4.14$$

This corrects the result given by Pawlik and Rowlands (1975).
This solution is divergent and the divergence cannot be removed by adding
solutions of the homogeneous form of 4.4.8. However, this is not a diverg-
ence in any physical sense as ψ_1 occurs only in the argument of an expon-
ential function and so is merely a phase factor. But, the ϕ_i must all
be bounded and it is necessary that all but one of the terms in the
expression for ψ_1 vanish in order that this be satisfied. This then
gives the definition of B_r.

Before we actually solve 4.4.9 we will use it to derive this condition.
Rewrite 4.4.9 as

$$D\phi_2(\xi) = F(\xi)$$

where D is the operator

$$\chi \frac{\partial^2}{\partial \xi^2} + \beta_r + 3\delta_r \phi_o^2$$

and

$$F(\xi) = \chi \left(\frac{\partial \psi_1}{\partial \xi}\right)^2 \phi_o - 3\beta_i \phi_o \phi_1^2$$

It is easily shown that D is a linear self adjoint operator and in order that ϕ_2 be bounded the condition

$$\int_{-\infty}^{\infty} y(\xi) F(\xi) d\xi = 0$$

must be satisfied, where $y(\xi)$ is a solution of

$$Dy(\xi) = 0.$$

As we have seen already $y(\xi) = \partial \phi_o / \partial \xi$ and so

$$\int_{-\infty}^{\infty} F(\xi) \frac{\partial \phi_o}{\partial \xi} d\xi = 0$$

Substituting ψ_i in the expression for $F(\xi)$ and performing the integrals leaves nine integrals which do not identically vanish i.e. all integrals that are premultiplied by the factor $\left[-\beta_i - \delta_i A^2\right]$. Therefore,

$$(-\beta_i - \delta_i A^2) = 0$$

which defines the constant β_r introduced in 4.4.1 since

$$\frac{\beta_i}{\delta_i} = -A^2 = \frac{\beta_r}{\delta_r} \qquad\qquad 4.4.15$$

This therefore eliminates all but the second term in 4.4.14. It can generally be concluded that the greatest divergence that can be allowed in the ϕ_i is linear in ξ so that to next lowest order the ϕ_i remain bounded.

An apparent contradiction now arises with the above relationship between the coefficients of 4.4.2. Since, multiplying 4.4.2 by ϕ^* and integrating gives

$$\chi\int_{-\infty}^{\infty} \phi^* \frac{\partial^2\phi}{\partial\xi^2}\,d\xi + (\beta_r + i\beta_i)\int_{-\infty}^{\infty}\phi\phi^*d\xi + (\delta_r + i\delta_i)\int_{-\infty}^{\infty}\phi\phi^*|\phi|^2 d\xi = 0$$

which may be simplified further to give

$$\chi\int_{-\infty}^{\infty} \left|\frac{\partial\phi}{\partial\xi}\right|^2 d\xi + (\beta_r + i\beta_i)\int_{-\infty}^{\infty}|\phi|^2 d\xi + (\delta_r + i\delta_i)\int_{-\infty}^{\infty}|\phi|^4 d\xi = 0$$

where we have used $\left|\phi^* d\phi|d\xi\right|_{-\infty}^{+\infty} = 0$

Now, equating real and imaginary parts and eliminating the $\int_{-\infty}^{\infty}|\phi^4|d\xi$ term from the resulting two equations gives:

$$\chi\int_{-\infty}^{\infty} \left|\frac{\partial\phi}{\partial\xi}\right|^2 d\xi + \left(\beta_r - \frac{\delta_r\beta_i}{\delta_i}\right)\int_{-\infty}^{\infty}|\phi|^2 d\xi = 0$$

which if 4.4.15 is correct implies

$$\int_{-\infty}^{\infty} \left|\frac{\partial\phi}{\partial\xi}\right|^2 d\xi = 0$$

However, by inspection it is seen that this condition is violated. It can be seen that this arises because β_r should in fact be expanded as a series in η first. Each term in this expansion could then be determined using the compatibility conditions as has already been shown. If this is done and the expansions for ϕ included in the above analysis then the contradiction disappears. However, this is unnecessary if we work to order η^2 only, and it is sufficient to retain only β_r as defined by 4.4.15.

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We now solve 4.4.9. By inspection one of the solutions of the homogeneous form of 4.4.9 is given by

$$\phi_2 \quad \alpha \quad \text{sech}^2[B\xi + d]$$

Hence, by 4.4.12 the other solution is given by

$$\phi_2 \quad \alpha \quad \text{sech}^2[B\xi + d] \int^\epsilon \frac{1}{\text{sech}^4[Bx+d]} \, dx$$

$$= \frac{1}{4} \sinh[B\xi+d]\cosh[B\xi+d] + \frac{3}{8}\left[\tanh[B\xi+d] + [B\xi+d]\right]$$

Having obtained the two independent solutions of the homogeneous form of 4.4.9 the solution of 4.4.9 is now obtained through integration using 4.4.13. The solution of 4.4.10 follows a similar pattern.

We now summarise the solutions of equations 4.4.6-4.4.10.

$$\phi_o = A \tanh[B\xi+d]$$

$$\phi_1 = G \, \text{sech}^2[B\xi+d]$$

$$\phi_2 = -\frac{G^2}{A} \text{sech}^2[B\xi+d]\tanh[B\xi+d] - \frac{\beta_i^2 A}{9\beta_r^2} \tanh[B\xi+d]$$

$$\psi_1 = \frac{-2\beta_i}{3\beta_r} \log\left[\cosh[B\xi+d]\right]$$

$$\psi_2 = \frac{-2\beta_i}{3\beta_r} \frac{G}{A} \tanh[B\xi+d]$$

We now note that

$$\psi_2 \quad \alpha \quad \frac{\partial \psi_1}{\partial \xi}$$

and that the first term of ϕ_2 is proportional to $\partial\phi_1/\partial\xi$. Hence, in retaining terms to order η^2 only we may consider ψ_2 as the first term in

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a Taylor expansion of ψ_2 and ϕ_2 as the first term of an expansion of ϕ_1. So with this assumption we substitute the above in the expansion for ϕ and finally obtain:

$$\phi = A\left(1 - \frac{\beta_i^2 n^2}{9\beta_r^2}\right) \tanh\left[B\xi + d + \frac{nG}{AB}\right]$$

$$\times \exp\left[\frac{-2in\beta_i}{3\beta_r}\,\log\left[\cosh\left[B\xi + d + \frac{nG}{AB}\right]\right]\right] \qquad\qquad 4.4.16$$

Comparing this with the solution of the nonlinear Schrödinger equation we see the effect of generalising the coefficients to have small imaginary parts leaves the solution essentially unchanged except for a small decrease in amplitude, a physical translation and an oscillation.

4.5. Results

Having obtained the solution of 4.4.2 in a general form we can now calculate the coefficients of 4.3.19 explicitly in terms of the variables of sections 4.2 and 4.3. For algebraic simplicity we consider the long wavelength approximation in the limit of small κ^2. All coefficients of 4.3.19 are expanded in power series under the condition $\omega_c\omega_D \gg k^2 v_s^2$ and terms are retained as far as κ^4 only. These coefficients then become

$$\beta_r + i\beta_i = \frac{\mu f\kappa E'}{2} - i\,\frac{k^2\kappa^2 v_s^2 \mu f E'}{2\omega_c v_s}$$

$$\delta_r + i\delta_i = \frac{-\mu^2 f^2 k \omega_c \kappa^2}{6v_s \omega_c} + \frac{i\mu^2 f^2 k^2 \omega_c \kappa^4}{6\omega_c^2}$$

with $\qquad A^2 = \frac{3v_s \omega_c E'}{\mu f\omega_D \kappa^2} \qquad\qquad B^2 = \frac{\mu f E^1 \omega_c \omega_D}{6\kappa^2 v_s^3}$

and $\qquad \omega_r = kv_s + \frac{\kappa^2 k^3 v_s^3}{2\omega_c \omega_D}$

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$$\omega_c = \frac{\kappa^2 k^2 v_s^2 \mu f E^{\prime}}{2\omega_c v_s}$$

$$\chi = \frac{3\kappa^2 k v_s^3}{2\omega_c \omega_D}$$

The remaining coefficients of 4.4.16 are now given by:

$$\left(1 - \frac{\beta_i^2 \eta^2}{9\beta_r^2}\right) = \left(1 - \frac{\kappa^4 k^2 v_s^2}{9\omega_c^2}\right)$$

and

$$-\frac{2\eta\beta_i}{3\beta_r} = \frac{2}{3}\frac{\kappa^2 k v_s}{\omega_c}$$

where the identification of η with κ^2 has been made.

We can now calculate all variables of physical interest since we have an explicit expression for ϕ. In particular we will explicitly calculate the total d.c. electric field which will show the expected domain-like behaviour. Substituting for ϕ in 4.3.23 gives

$$E_{TOTAL}^{D.C} = E_o + 2\epsilon^2 E^1 - 3\epsilon^2 E^{\prime} \operatorname{sech}^2\left[B\xi + \frac{\kappa^2 G}{AB} + d\right] \qquad 4.5.1$$

to lowest order.

The constants G and d can now be determined by requiring that at $\tau = \epsilon = 0$ the total DC electric field is only the applied external field. The effect of the nonlinear saturation is seen to be a feedback of electric field $2\epsilon^2 E^{\prime}$ across the sample as a whole as well as a domain. The height of the pulse is directly proportional to $E^{\prime}$ i.e. proportional to the magnitude of its cause as would be expected.

The effect of the saturation on the linear gain can be deduced from the nonlinear Schrödinger equation. Multiplying the time dependent form by ϕ^* and subtracting the complex conjugate form gives:

$$i \frac{\partial |\phi|^2}{\partial \tau} + \chi \left(\frac{\partial^2 \phi}{\partial \xi^2} \phi^* - \frac{\partial^2 \phi^*}{\partial \xi^2} \phi \right) + 2i\beta_i |\phi|^2 + 2i\delta_i |\phi|^4 = 0 \qquad 4.5.2$$

The second term may be written as

$$\chi \frac{\partial J}{\partial \epsilon}$$

where
$$J = \frac{\partial \phi}{\partial \xi} \phi^* - \frac{\partial \phi^*}{\partial \xi} \phi$$

The resulting equation

$$i \frac{\partial |\phi|^2}{\partial \tau} + 2i\beta_i |\phi|^2 + 2i\delta_i |\phi|^4 = -\chi \frac{\partial J}{\partial \xi} \qquad 4.5.3$$

can easily be interpreted by considering the instantaneous energy density and flux density.

The instantaneous energy density is defined as:

$$W_{IN} = \frac{1}{2} \rho \left(\frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} c \left(\frac{\partial u}{\partial x} \right)^2 \qquad 4.5.4$$

and the instantaneous flux as

$$\Phi_{1N} = c \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \qquad 4.5.5$$

We define the total energy density W and the total flux density by averages of W_{IN} and Φ_{1N} i.e.

$$W = \langle \overline{W_{IN}} \rangle \qquad \Phi = \langle \overline{\Phi_{1N}} \rangle$$

where $\langle \overline{...} \rangle$ denotes an average which will be defined later. Substituting 4.3.4 into 4.5.4 and 4.5.5 gives to order ϵ^2

$$W_{IN} = \frac{1}{2} \rho \sum_{pq} \sum (u^1_p ip\omega u^1_q iq\omega) P_p P_q$$

$$+ \frac{1}{2} c \sum_{pq} \sum (u^1_p ipk u^1_q iqk) P_p P_q$$

$$\Phi_{1N} = c \sum_{pq} (ipku^1_p \, iq\omega u^1_q) P_p P_q$$

These expressions contain terms rapidly oscillating at the fundamental frequency and its harmonics through the factors $P_p P_q$ for $|p| \neq |q| \neq 1$ as well as terms which have variation only on the slow time and space scales through the amplitudes u^1_p and u^1_q when $|p| = |q| = 1$.

We now define the average $< \overline{\quad} >$ as the removal of all rapidly oscillating terms and keeping only the slowly varying ones, i.e.

$$W = <\overline{W_{1N}}>_{p+q = 0}$$

$$\Phi = <\overline{\Phi_{1N}}>_{p+q = 0}$$

This gives

$$\Phi = \frac{2ck\omega e^2 k^2}{(ck^2 - \rho\omega_r^2)^2} \, |\phi|^2$$

and

$$W = \frac{e^2 k^2 (\rho\omega^2 + ck^2)}{(ck^2 - \rho\omega_r^2)^2} \, |\phi|^2$$

with the relationship

$$W = \frac{(\rho\omega_r^2 + ck^2)}{2c\omega k} \, \Phi \qquad\qquad 4.5.6$$

Hence 4.5.3 may now be interpreted as the energy equation for the system with the $\partial J / \partial \epsilon$ term representing the dissipation of energy.

We can now investigate the effective growth rate i.e. the effect of the nonlinear wave saturation on the linear growth, i.e. since

$$\frac{\partial |\phi|^2}{\partial \tau} + 2\beta_i |\phi|^2 \left(1 + \frac{\delta_i}{\beta_i} |\phi|^2\right) = -i\chi \frac{\partial J}{\partial \epsilon}$$

we can write

$$\frac{\partial |\phi|^2}{\partial \tau} + \omega_i^{\text{eff}} |\phi|^2 = -i\chi \frac{\partial J}{\partial \epsilon}$$

where
$$\omega_i^{\text{eff}} = 2\beta_i \left(1 + \frac{\delta_i}{\beta_i} |\phi|^2\right) \qquad\qquad 4.5.7$$

Since β_i is ω_i, the linear growth rate, and δ_i/β_i is negative 4.5.7 shows how the growth rate is reduced in the presence of flux or energy. This is in agreement with Butcher (1971).

It is common practice to try and derive a modification of the linear gain which is valid in the nonlinear regime. We will now show that the saturation of the instability is due to a modification of the local d.c. field and the equilibrium electron density n_o.

The linear growth rate given by 4.2.14 depends on the local field through γ. As we have shown, in the small γ long wavelength limit

$$\omega_i = \frac{i k^2 \kappa^2 v_s^2 \gamma}{2\omega_c} \qquad\qquad \gamma = 1 + \frac{\mu f E_o}{v_s}$$

and using 4.3.17 we see that

$$\omega_i = 2\beta_i \left(1 + \frac{3\delta_i}{\beta_i} |\phi|^2\right)$$

which disagrees with the result 4.5.7. Hence it must be concluded that, in the nonlinear regime the linear gain formula is not valid even when the total local DC electric field is used. However, 4.5.7 is reproduced if we write

$$\omega_i = -\frac{1}{2} i k^2 \kappa^2 v_s \langle \overline{\gamma/\omega_c} \rangle \qquad\qquad 4.5.8$$

where $\langle \overline{\ldots} \rangle$ is defined as the removal of rapidly oscillating terms as before. Substituting for γ and ω_c where $\omega_c = q\mu n/\epsilon_s$ and performing the averaging reproduces 4.5.7.

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4.6. Discussion

We have shown that the amplitude of a single acoustic wave excited in a piezoelectric semiconductor under conditions such that system is amplifying, saturates. The equation obeyed by the amplitude is to lowest significant order the generalised nonlinear Schrödinger equation. Solutions to this equation were derived using a perturbation expansion based on the relative magnitude of its coefficients. This solution was found to be domain-like and an explicit expression for the electric field domain was derived.

It has therefore been shown that domain-like solutions of the governing equations, White (1962) do exist in the sense that the domain is the envelope of the nonlinear wave. It is therefore not necessary to include e.g. hot-electron effects or non-electric loss mechanisms in the governing equations (Ridley and Wilkinson 1969) to obtain the domain solution.

There are few experimental results with which the present work can be compared. Schulz and Wonneberger (1970) obtained a profile for a stationary acoustoelectric domain in GaAs. They found that the single mode domain had a lower saturation field than a multimode case and that the single mode domain had a constant saturation field. Although no comparison of theory and experiment can be made for the multimode case it has certainly been shown that the saturation field is constant. However, the theory presented in this chapter cannot claim to be directly comparable with experiment even in the single mode case since realistic boundary conditions have not been used. The only boundary conditions that have been assumed are that the amplitude tends to zero at infinity. It is well known that the boundary conditions resulting from a finite size specimen can affect the final domain shape considerably.

A comparison against other nonlinear theories can however be made. The approaches of Butcher (1971) and Tien (1968) consisted of treating all

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nonlinear terms as second order compared to the linear terms. This
is unsatisfactory as, in the nonlinear regime the nonlinear terms must
all be of the same order as the linear terms and any perturbation expan-
sion should take account of this. No attempt was made to derive domain-
like solutions but expressions for the nonlinear saturation of the gain
of time periodic waves were deduced. These are in broad agreement with
the results found here. Slechta (1972) attempted to solve the full
nonlinear equations using a many time formalism. In view of the
comments in Chapter 1 this should in principle have led to the results
found here. However, Slechta considered the problem of acoustic waves
arising from thermal vibrations and showed that certain wave-wave interac-
tions lead to saturation. This saturation is less fundamental than the
saturation mechanism indicated here since we have shown that the satura-
tion mechanism is inherent even in the case of a single acoustic wave.
Within the present formalism Slechta ignored the feedback of d.c. components
to second order and investigated the effect of higher order components only.
Thus his work can be considered incomplete.

In view of the large number of modes excited in a high field domain
a single mode theory as presented here can be considered as being totally
inadequate. In addition, the boundary conditions used were very simplistic
and again inadequate. An extension of the present work to include many
wave interactions and realistic boundary conditions is therefore required.
The problem of the interaction of modulated plane waves through nonlinear
interactions has been considered by Oikawa and Yajima (1973,1974). This
work can be extended to nonlinear interactions in strongly dissipative
systems and would lay a foundation for the study of the many wave interac-
tions during domain formation.

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Appendix

We will show that stable solutions of the set of nonlinear equations describing the acoustoelectric effect, i.e. equations 4.2.1 to 4.2.4, of domain-like behaviour do not exist in the conventional sense.

Equations 4.2.1 to 4.2.11 can be reduced to a single third order differential equation for the electric field E. Attempting solutions of the form:

$$E(y) = E(x - v_d t)$$

gives

$$\frac{v_s^2}{\omega_D} \frac{\partial^3 E}{\partial y^3} + 2vd \frac{\partial^2 E}{\partial y^2} + \frac{\sigma}{h} \frac{\partial E}{\partial y} + \mu f \frac{\partial}{\partial y} \left(E \frac{\partial E}{\partial y} \right) = 0 \qquad \text{A.1}$$

where

$$h = \left[\frac{e^2}{\rho(v_d^2 - v_s^2)} - \epsilon_D \right]$$

One integration then gives:

$$\frac{v_s^2}{\omega_D} \frac{\partial^2 E}{\partial y^2} + 2v_d \frac{\partial E}{\partial y} + \frac{\sigma}{h} E + \mu f E \frac{\partial E}{\partial y} = a \qquad \text{A.2}$$

where a is a constant of integration.

In order to see whether stable periodic solutions of A.2 exist we use standard techniques of non-linear analysis (Minorsky (1962)). Equation A.2 is an autonomous system since letting

$$Z = \frac{\partial E}{\partial y}$$

and so $$\frac{dZ}{dy} = Q(Z,E)$$

$$\frac{dE}{dy} = R(Z,E) = Z$$

we see that

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A(Z,E) and R(Z,E) have no explicit dependence on y. It is evident that

$$Q(Z,E) = -\frac{2v_d\omega_D}{v_s^2} Z - \frac{\sigma}{n} \frac{\omega_D}{v_s^2} E - \frac{\mu f E \omega_D}{v_s^2} + \frac{a\omega_D}{v_s^2}$$

Equation A.2 has one critical point (Z^O, E^O) defined by

$$R(Z^O, E^O) = Q(Z^O, E^O) = 0$$

where $\qquad Z^O = 0$

and $\qquad E_o = ha/\sigma$

In order to investigate the existence of stable periodic solutions of A.2 we use the well known phase plane analysis. If stable solutions exist then Liapunov's second theorem must be satisfied, i.e. given a differential system with a single singular point the equilibrium is asymptotically stable if it is possible to determine a function W whose Eulerian derivative W is of the sign opposite to that of L.

By inspection it can be seen that such a Liapunov function can be chosen to be:

$$L = Z - \frac{\sigma}{h\mu f} \log\left(1 + \frac{h\mu f Z}{\sigma}\right) + \frac{\mu f \omega_D}{2v_s^2}\left(E - \frac{ah}{\sigma}\right)^2$$

Then:

$$\frac{dL}{dy} = \frac{h\mu f z \frac{\partial z}{\partial y}}{(1+\frac{\mu f h}{\sigma}z)} + \frac{\mu f \omega_D}{v_s^2}\left(E - \frac{ah}{\sigma}\right)\frac{\partial E}{\partial y}$$

and substituting from A.2 gives

$$\frac{dL}{dy} = \frac{z^2}{(1+\frac{\mu f h z}{\sigma})}\left(\frac{-\mu f \omega_D h}{\sigma v_s^2}\right)\left(\frac{h\mu f a}{\sigma}\right) + 2v_d$$
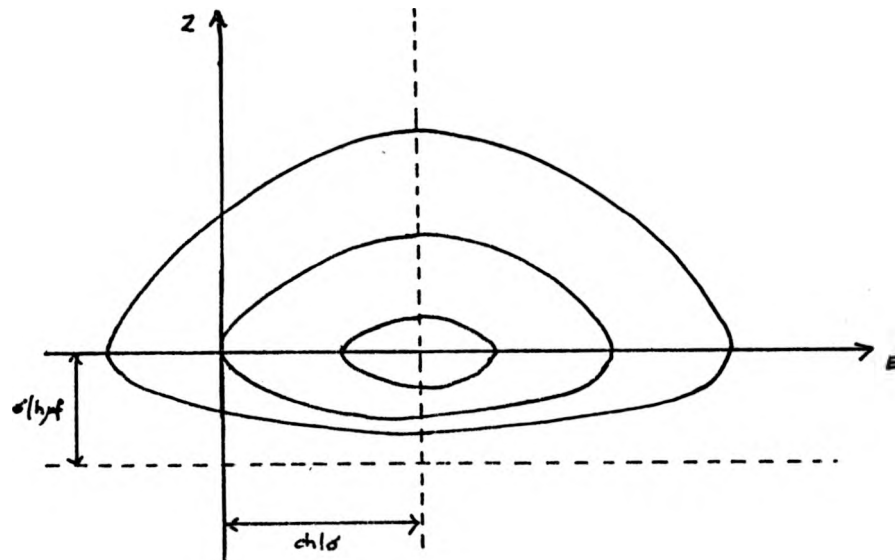
Assuming

$$1 + \frac{\mu f h}{\sigma} > 0$$

and
$$h > 0$$
$$a > 0$$

then shows that L is indeed a Liapunov function since it is definite L > 0, vanishes at the critical point and dL/dy < 0.   The phase plane diagram for the system appears as:

where the different curves correspond to different values of the constant of integration.

Since we have found a Liapunov function we may conclude that solutions of A.1 are asymptotically stable.   However these solutions are not periodic since dL/dy ≠ 0 and so the contours L = constant shown above are not trajectories of the system.   Since dL/dy < 0 any trajectory moves to decrease L and hence will always spiral to the critical point giving asymptotically stable solutions that are not domains.

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Chapter 5

A Nonlinear Theory of a Two-stream Instability

in a Marginally Stable State

5.1 Introduction

Systems of drifting and interpenetrating plasma streams have received considerable theoretical attention during the last twenty years.   There are two main reasons for this:

a.   Two-stream instabilities, as the   instabilities of these systems with two streams are normally called are some of the simplest plasma instabilities and lead to particularly simple and tractable dispersion relations.

b.   Two-stream instabilities are in principle velocity space instabilities i.e. instabilities associated with the departure of the velocity space distribution function from a Maxwellian distribution.   They can however be equally well analysed using a macroscopic or microscopic formalism.   By macroscopic we mean an electron fluid hydrodynamic description and by microscopic a description using a kinetic equation such as the Vlasov equation.   Two-stream instability analysis therefore provides a good testing ground and comparison between the two descriptions of a plasma.

Further, the analysis in the linear theory is particularly easy since the instability is electrostatic in nature i.e. associated with bunching and separation of charge and is easily extended to apply to more complex model systems.   We note however that magnetic field effects are normally excluded from the analysis but some studies of two-stream instabilities in magnetic fields have been made by Neuffield and Wright (1963).

Initially we will consider more general aspects of two-stream instabilities in terms of basic plasma theory without reference to any particular model.   The number of possible instabilities is quite large since the beams may be electron beams or ion beams, may be neutralised or current carrying, warm or cold and streaming or counterstreaming.   Following this general discussion a brief summary of previously published nonlinear theories is given.

The linear theory is well understood and for cold plasma streams was originally given by Haeff (1948). A good discussion of the effect of beam temperature on linear theory is given by Völk (1967) and a complete discussion is given by Clemmow and Dougherty (1969). Generally speaking, criteria are deduced for the stability or instability of linear waves depending on beam velocities, thermal spreads and boundary conditions.

Nonlinear theories have been concerned with three different nonlinear mechanisms depending on the exact nature of the system considered. These are the resonant particle-wave, non-resonant particle-wave and wave-wave interactions which will be considered in turn.

a. Resonant wave-particle interactions. When the thermal spread of the distribution function is such that the particle distribution overlaps the regions of the phase velocity of unstable waves resonant particle effects must be considered. These interactions occur when a wave in the system is resonant with some part of the particle distribution. This nonlinear interaction is analogous with Landau damping in the linear theory and is called nonlinear Landau damping (or growth). The damping may change to growth and so linearly stable or unstable systems may change to unstable or stable systems in the nonlinear regime. The interaction fundamentally consists of the interaction between electrons moving with nearly the same velocity as the wave. The particle therefore sees an effective electrostatic potential well and oscillates in the well whilst moving along with the wave i.e. particle trapping. Depending on the exact properties of the linear distribution function energy may either be transferred from the waves to the particles or vice versa giving either enhanced growth or suppression of the instability.

Particle wave interactions are normally modelled using the Vlasov equation and quasi-linear theory or by examining large amplitude stationary states. Numerical studies by Armstrong and Montgomery (1967) and Berk and Roberts (1967) showed that waves which are unstable in the linear theory

approach stable BGK waves in the long time limit. These nonlinear waves discovered by Bernstein, Greene and Kruskal (1957) are a direct consequence of particle trapping. If electrons are assumed to have a continuous distribution of kinetic energy then some will always be trapped in the potential wells discussed above. Bernstein et al showed that the localised potential could be made consistent with the excess trapped electron density. This nonlinear equilibrium shows electrostatic fields which are spatially periodic and are called BGK waves. A consequence of this is that there is no continuous flow of energy into higher harmonics (which are stable by appropriate choice of boundary conditions) and so allows the nonlinear state to be a single BGK wave rather than a highly turbulent state.

The quasi-linear theory of the Vlasov equation will not be considered here but will be considered in more detail later. The resonant particle wave interaction is not the dominant nonlinear process in most streaming instabilities. The reason for this is that essentially no particles have velocities near the phase velocity of unstable waves and so nonresonant interactions are more important.

b. Non-resonant wave particle interactions. If the distribution function is such that an overlap does not occur between unstable waves and the distribution function then these interactions dominate. In most forms of the two-stream instability the phase velocity of unstable waves lies between the velocities of the interpenetrating streams; the growth of the instability is not proportional to the number of resonant particles. Instead, the instability can be explained in terms of charge bunching of non-resonant particles as follows: a local in increase in charge density will induce a charge perturbation in a plasma stream passing over it. Electrons passing over this bunch will be slowed down due to the electrostatic field induced by the bunch and these electrons will therefore add to the perturbation.

We note that nonlinear BGK waves are still possible when non-resonant particle interactions are dominant but we will only discuss quasi-linear

theory here. The quasi-linear theory consists of solving the Vlasov equation by a perturbation expansion. The average distribution function, to lowest order, is assumed to have a slow temporal variation (two orders of perturbation higher). The perturbation distribution function to next order is then calculated as in linear theory. This procedure leads to the quasi-linear diffusion equations which describe the evolution of the distribution function in velocity space. Using the quasi-linear analysis for the two-stream instability shows that non-resonant particle diffusion does in general limit the growth. This however is only valid when finite temperature effects are taken into account since quasi-linear theory has no meaning for cold plasmas.

The relative unimportance of resonant wave particle interactions demonstrates the essentially hydrodynamic nature of the instability. If a fluid description is to be used then there is no mechanism for wave particle interactions, resonant or non-resonant. Therefore wave-wave interactions must be considered.

c. Wave-wave interactions. Wave-wave interactions are considered in the hydrodynamic approximation in either cold or warm plasmas. Nonlinear effects are considered by studying possible nonlinear states following linear instability and by determining whether these nonlinear states are stable. Analytic attempts at this have been made by Freidberg (1965,1967) and Knorr (1968) and will be considered in 5.3. Numerical calculations have also been made by Buneman (1959).

The nonlinear problem attempts to answer the question whether there exists a range of wavenumbers which are linearly unstable but which stabilise at large amplitudes due to nonlinear effects. We will now briefly discuss published work which attempted to answer this question. With the exception of Stringer (1964) all the theories discussed are within the framework of a warm or cold fluid model with wave-wave nonlinear interactions.

Stringer (1964), using the Vlasov equation obtained a nonlinear dispersion relation using an iterative procedure based on a WKB analysis. He applied his method both to current carrying and counterstreaming plasmas. If his results are specialised to the system that will be considered in 5.3, i.e. cold beams with equal masses and equal but opposite streaming velocities then the conclusion may be drawn that there are no linearly unstable modes which are nonlinearly stable (a further condition must be imposed to derive this result - the spatial average of the electric field is zero. This is often referred to as the short circuit case).

Freidberg (1968) looked for travelling wave solutions of the hydrodynamic equations and again derived an amplitude dependent nonlinear dispersion relation. From an analysis of this dispersion relation he deduced that with a condition of conservation of total current, i.e. so-called open circuit boundary conditions, that a number of modes did exist which were linearly unstable but nonlinearly stable. With short circuit boundary conditions Stringer's result was reproduced.

Freidberg and Armstrong (1968) in an analytic and numerical approach considered the nonlinear behaviour of a single unstable mode which was excited by an appropriate choice of boundary conditions. Again, they found that the nonlinear development of the instability was strongly dependent on the choice of boundary conditions for the electric field and the same conclusions as Stringer were reached. The modes which stabilised were found to give a single nonlinear solitary wave equilibrium. However, the system they considered consisted of two interpenetrating electron ion streams. If their method is applied to the model discussed in 5.3 then they conclude that all linearly unstable modes stay unstable in the nonlinear regime.

The main deficiency of all the above theories is that the time development of the system is not explicitly considered but the state of the system after a long time is deduced. The nonlinear theory presented in 5.3 again suffers from the same deficiency but does provide a test for the reductive perturbation

expansion against previous theories. In 5.2 we consider the linear theory for the particular choice of two-stream instability and in 5.3 consider the nonlinear theory of the system near a marginally stable point using the expansion developed in Chapter 2. In 5.4 a comparison of the result is made with the theories discussed above and suggestions are made for further work.

5.2 Linear theory

We consider two one-dimensional electron fluids, distinguished by the suffices α and β drifting with some uniform velocity in the +x direction. The electron beams are neutralised by a uniform background of ions which are assumed to have infinite mass. This restricts the validity of this analysis to high frequency oscillations only. Further we assume that both electron beams are "cold" and so ignore finite temperature effects. With these assumptions the relevant equations describing the system become, in one dimension, Continuity equation:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i v_i) = 0 \qquad i = \alpha, \beta \qquad \text{5.2.1}$$

Momentum transfer equation:

$$m\left(\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x}\right) = -eE \qquad i = \alpha, \beta \qquad \text{5.2.2}$$

which are coupled with, Poisson's equation

$$\varepsilon_o \frac{\partial E}{\partial x} = -e(n_\alpha + n_\beta - 2n_o) \qquad \text{5.2.3}$$

to give a closed system of equations. In these equations n_i is the electron density, v_i is the electron velocity, E is the self-consistent electric field, e and m the electron charge and mass, ε_o the dielectric constant and n_o the unperturbed electron density. In view of the right-hand side of 5.2.2 we see the only coupling between the two-streams is through the electric field E and see from the right-hand side of 5.2.3 that there is no unperturbed electric field.

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We now look for solutions of 5.2.1 to 5.2.3 of the following form:

$$v_i = V_{io} + V_{i1} \exp i(kx-\omega t) \qquad i = \alpha,\beta$$

$$n_i = n_o + n_{i1} \exp i(kx-\omega t) \qquad i = \alpha,\beta$$

$$E = E_1 \exp i(kx-\omega t)$$

Substituting and linearising we see that in order that these solutions satisfy 5.2.1 to 5.2.3 the following linear dispersion relation must be satisfied:

$$\frac{\omega_p^2}{(\omega-kV_{\alpha o})^2} + \frac{\omega_p^2}{(\omega-kV_{\beta o})^2} = 1 \qquad\qquad 5.2.4$$

where $\omega_p^2 = e^2 n_o/m\varepsilon_o$. We now look for solutions of 5.2.4 for a real wavevector k and complex frequency ω. To simplify the analysis we let:

$$V_{\alpha o} = U + V$$

$$V_{\beta o} = U - V$$

and by expanding 5.2.4 determine the quartic equation satisfied by ω, i.e.

$$(\omega-kU)^4 - (\omega-kU)^2 (2k^2V^2+2\omega_p^2)$$

$$+ k^4V^4 - 2k^2V^2\omega_p^2 = 0$$

which has the solutions

$$(\omega-kU)^2 = (k^2V^2+\omega_p^2) \pm \omega_p(4k^2V^2+\omega_p^2)^{\frac{1}{2}} \qquad\qquad 5.2.5$$

We see therefore that this gives four roots, two real and two complex if

$$k < \frac{\sqrt{2}\omega_p}{V} \qquad\qquad 5.2.6$$

We therefore define a critical velocity V_c by

$$V_c = \sqrt{2}\omega_p/k \qquad\qquad 5.2.7$$

and note that given a wavevector k the two-stream system is unstable if the

relative velocity of the two streams is less than the critical velocity V_c.
Conversely, for beams moving with a relative velocity V then perturbations of
a wavevector k less than a critical value, given by 5.2.6 are unstable.
However, the growth rate goes to zero as $kV \to 0$ and we must distinguish between
systems which may or may not be unstable. For a finite system where, for
example, periodic boundary conditions impose a restriction on permissible
values of k we may or may not have instability depending on the particular
choice of V. For infinite unbounded systems where an infinity of modes with
differing wavevectors k may be excited there will always be a class of unstable
modes and a class of stable modes, whatever the value of V.

We wish to consider an instability somewhat different to the velocity
induced instability introduced above and will consider an instability induced
by a small charge imbalance in two streams moving apart with equal velocities
at the critical velocity V_c. If the electron beams are counterstreaming at
the critical velocity then a small perturbation in this velocity will cause
instability and since then ω has a double root according to 5.2.5 this system
is marginally stable. We now see that if the instability is induced by a
slight imbalance of charge between the beams and ion background in a marginally
stable state then the complex frequency ω goes from pure real to pure complex,
through zero as the stability boundary is crossed. This is therefore an
example of the inverted bifurcation as discussed in Chapter 2, and we may use
the nonlinear theory as given in 2.3. We will see that this analysis is
valid for a single mode perturbation at a particular wavevector k_c and there-
fore corresponds to an infinite system driven at a particular frequency.

We look for solutions of 5.2.1 to 5.2.3 of the form:

$$V_\alpha = V_c + V_{\alpha 1} \exp i(kx - \omega t)$$

$$V_\beta = -V_c + V_{\beta 1} \exp i(kx - \omega t)$$

$$n_\alpha = n_o(1 + \Delta) + n_{\alpha 1} \exp i(kx - \omega t)$$

$$n_\beta = n_o(1 + \Delta) + n_{\beta 1} \exp i(kx - \omega t)$$

$$E = E_1 \exp i(kx - \omega t)$$

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where Δ is a small parameter.   It is easily seen that the linearised dispersion relation now becomes:

$$\frac{\omega_p^2(1+\Delta)}{(\omega-kV_c)^2} + \frac{\omega_p^2(1+\Delta)}{(\omega+kV_c)^2} = 1 \qquad\qquad 5.2.8$$

which again gives a quartic in ω:

$$\omega^4 - \omega^2(3k^2V_c^2 + \Delta V_c^2 k^2) - k^4 V_c^4 \Delta = 0 \qquad\qquad 5.2.9$$

where we have substituted for ω_p from 5.2.7.   In the limit of small Δ we can solve 5.2.9 to give:

$$\omega^2 = -\frac{2}{3}\Delta\omega_p^2 \qquad\qquad 5.2.10$$

or
$$\omega^2 = 3k^2V_c^2 + \frac{4}{3}\Delta k^2 V_c^2 \qquad\qquad 5.2.11$$

Therefore, the frequency ω for a real wavevector k is either pure real or pure imaginary.   We are only concerned with the pure imaginary solutions as the real solutions only represent space charge waves.   We note that 5.2.10 gives only one solution which is unstable, i.e. the positive square root which gives a growth rate of:

$$\omega_i = i\sqrt{\frac{2}{3}\Delta}\,\omega_p \qquad\qquad 5.2.12$$

We now have a situation where the magnitude of the instability is proportional to the square root of a small but arbitrary parameter Δ.   Therefore the ordering necessary for the validity of the general theory of Chapter 2 may now be achieved by assuming that Δ is $O(\epsilon^2)$ and hence that ω_i is $O(\epsilon)$.

We have now completed the linear theory of this particular two-stream instability and have derived an expression for the linear growth rate. Further, the conditions necessary for the validity of Chapter 2 have been satisfied.   This general theory cannot be applied directly as the equations of the system 5.2.1 to 5.2.3 cannot be put in the general form 2.3.1.   We

will however show that despite this the same result is obtained and as in Chapter 4 we will work with the matrices of the problem explicitly.

5.3   Derivation of a nonlinear Schrödinger equation

We look for solutions of 5.2.1 to 5.2.3 of the following forms

$$\underline{U} = \underline{U}^O + \sum_{\gamma=1}^{\infty} \sum_{\ell=-\infty}^{+\infty} \epsilon^{\gamma} \underline{U}^{\gamma}_{\ell}(\tau,\xi) \exp(i\ell x) \qquad\qquad 5.3.1$$

where
$$\tau = \epsilon t$$
$$\xi = \epsilon^2 x$$

and
$$\underline{U} = \begin{pmatrix} v_{\alpha} \\ v_{\beta} \\ n_{\alpha} \\ n_{\beta} \\ E \end{pmatrix}$$

The stationary state $\underline{U}^O$ around which the expansion is made is given by

$$U^O = \begin{pmatrix} +V \\ -V \\ n_o(1+\epsilon^2\Delta) \\ n_o(1+\epsilon^2\Delta) \\ 0 \end{pmatrix}$$

we do not need to specify the value of V at this stage and will see that it is given to first order.

Substituting 5.3.1 into 5.2.1 to 5.2.3 and equating the ℓth harmonic and γ^{th} order terms gives up to third order.

$$\underline{W}_{\ell} \underline{U}^1_{\ell} = 0 \qquad\qquad 5.3.2$$

$$\underline{W}_{\ell} \underline{U}^2_{\ell} + \underline{M}_{\ell} \frac{\partial \underline{U}^1}{\partial \tau}_{\ell} = \underline{S}^1_{\ell} \qquad\qquad 5.3.3$$

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$$\underline{W}_\ell\,\underline{U}^3{}_\ell + \underline{M}_\ell\,\frac{\partial \underline{U}^2}{\partial \tau}{}_\ell + \underline{N}_\ell\,\frac{\partial \underline{U}^1}{\partial \xi}{}_\ell + \underline{O}_\ell\,\underline{U}^1{}_\ell \;=\; \underline{S}^2{}_\ell \qquad\qquad 5.3.4$$

where

$$\underline{W}_\ell \;=\;
\begin{pmatrix}
V i\ell k & 0 & 0 & 0 & e/m \\
0 & -V i\ell k & 0 & 0 & e/m \\
n_o i\ell k & 0 & V i\ell k & 0 & 0 \\
0 & n_o i\ell k & 0 & -V i\ell k & 0 \\
0 & 0 & e/\varepsilon_o & e/\varepsilon_o & i\ell k
\end{pmatrix}$$

$$\underline{N}_\ell \;=\;
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\underline{M}_\ell \;=\;
\begin{pmatrix}
V & 0 & 0 & 0 & 0 \\
0 & -V & 0 & 0 & 0 \\
n_o & 0 & V & 0 & 0 \\
0 & n_o & 0 & -V & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$\underline{O}_\ell \;=\;
\begin{pmatrix}
\Omega & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
n_o \Delta i\ell k & 0 & 0 & 0 & 0 \\
0 & n_o \Delta i\ell k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\underline{S}^1{}_\ell \;=\;
\begin{pmatrix}
-\,\langle v^1_{\alpha p}\, iqk\, v^1_{\alpha q}\, P_p P_q \rangle_\ell \\[4pt]
-\,\langle v^1_{\beta p}\, iqk\, v^1_{\beta q}\, P_p P_q \rangle_\ell \\[4pt]
-\,\langle (v^1_{\alpha p}\, ipk\, n^1_{\alpha q} + n^1_{\alpha p}\, ipk\, v^1_{\alpha q})\, P_p P_q \rangle_\ell \\[4pt]
-\,\langle (v^1_{\beta p}\, ipk\, n^1_{\beta q} + n^1_{\beta p}\, ipk\, v^1_{\beta q})\, P_p P_q \rangle_\ell \\[4pt]
0
\end{pmatrix}$$

$$\underline{S}^2_{\ell} = \begin{pmatrix} - <(v^1_{\alpha p}\cdot iqkv^2_{\alpha q} + v^2_{\alpha p}iqkv^1_{\alpha q})\, P_p P_q>_{\ell} \\ - <(v^1_{\beta p}iqkv^2_{\beta q} + v^2_{\beta p}iqkv^1_{\beta q})\, P_p P_q>_{\ell} \\ - <(v^2_{\alpha p}ipkn^1_{\alpha q} + v^1_{\alpha p}ipkn^2_{\alpha q} + n^1_{\alpha p}ipkv^2_{\alpha q} + n^2_{\alpha p}ipkv^1_{\alpha q})P_p P_q>_{\ell} \\ - <(v^2_{\beta p}ipkn^1_{\beta q} + v^1_{\beta p}ipkn^2_{\beta q} + n^1_{\beta p}ipkv^2_{\beta q} + n^2_{\beta p}ipkv^1_{\beta q})P_p P_q>_{\ell} \\ 0 \end{pmatrix}$$

The same notation has been used as in Chapter 2.

We impose the condition that

$$\det \underline{W}_1 = 0 \qquad\qquad 5.3.5$$

This immediately gives

$$v^2 k^2 = 2\omega_p^2$$

or from 5.2.7 $\qquad V = V_c$

We note that 5.3.5 is only the requirement that there should be no real part of the frequency ω, i.e. had we looked for solutions proportional to $\exp i\ell(\kappa x - \omega t)$ then the matrix $\underline{W}_\ell$ would have been a function of ω_r which would have reduced to 5.3.5 under the condition $\omega_r \to 0$. Therefore the choice $V = V_c$ is consistent with the assumptions of the model. Equation 5.3.5, together with 5.3.2 gives

$$\underline{U}^1_{\ell} = \phi(\tau,\xi)\underline{R} \qquad \ell = 1 \qquad\qquad 5.3.6$$

$$\underline{U}^1_{\ell} = 0 \qquad \ell \neq 1,0$$

where $\underline{R}$ is the right eigenvector of $\underline{W}_1$. We see by direct evaluation that

$$\underline{R} = \begin{pmatrix} 1 \\ -1 \\ -n_o/V \\ -n_o/V \\ -Vik/(e/m) \end{pmatrix}$$

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and also note for completeness that the left eigenvector $\underline{L}$ is given by

$$\underline{L} = (1, 1, -V/n_o, V/n_o, \frac{V^2 ik\varepsilon_o}{n_o e})$$

We are again confronted with the difficulty of determining $\underline{U}^1_o$ since $\det \underline{\underline{W}}_o = 0$. This only gives

$$E^1_o = 0$$

and

$$n^1_{\alpha o} + n^1_{\beta o} = 0 \qquad\qquad 5.3.7$$

But considering the $\ell = 1$ component of 5.3.3 gives

$$\underline{\underline{W}} \, \underline{U}^2_1 + \underline{\underline{M}}_1 \underline{R} \frac{\partial \phi}{\partial \tau} = \underline{S}^1_1 \qquad\qquad 5.3.8$$

$$= -ik \, \dot\phi \begin{pmatrix} v^1_{\alpha o} R_1 \\[1ex] v^1_{\beta o} R_2 \\[1ex] n^1_{\alpha o} R_1 + v^1_{\alpha o} R_3 \\[1ex] n^1_{\beta o} R_2 + v^1_{\beta o} R_4 \\[1ex] 0 \end{pmatrix}$$

Multiplying 5.3.8 by $\underline{L}$ on the left gives

$$\underline{L} \cdot \underline{S}^1_1 = 0$$

since by definition $\underline{L} \cdot \underline{\underline{W}}_1 = 0$ and by inspection $\underline{L} \cdot \underline{\underline{N}}_1 \cdot \underline{R} = 0$.

Direct evaluation of this compatibility condition gives

$$- ik \, \dot\phi\{v^1_{\alpha o} R_1 L_1 + v^1_{\beta o} R_2 L_2 + n^1_{\alpha o} R_1 L_3 + v^1_{\alpha o} R_3 L_3 + n^1_{\beta o} R_2 L_4 + v^1_{\beta o} R_4 L_4\}$$

$$= 0$$

which is equivalent to

$$2(v^1_{\alpha o} - v^1_{\beta o}) - \frac{V_c}{n_o} (n^1_{\alpha o} + n^1_{\beta o}) = 0$$

This, combined with 5.3.7 gives

$$v^1_{\alpha o} - v^1_{\beta o} = 0. \qquad 5.3.9$$

In order to complete the determination of $\underline{U}^1_{\,o}$ we consider the $\ell = 0$ component of 5.3.3.   We consider each row of this equation to obtain

$$(e/m)E_o{}^2 + \frac{\partial v^1_{\alpha o}}{\partial \tau} = 0 \qquad 5.3.10a$$

$$(e/m)E_o{}^2 + \frac{\partial v^1_{\alpha o}}{\partial \tau} = 0 \qquad 5.3.10b$$

$$\frac{\partial n^1_{\alpha o}}{\partial \tau} = 0 \qquad 5.3.11a$$

$$\frac{\partial n^1_{\beta o}}{\partial \tau} = 0 \qquad 5.3.11b$$

and $$n^2_{\alpha o} + n^2_{\beta o} = 0 \qquad 5.3.12$$

From 5.3.10(a) and (b) we obtain

$$\partial/\partial \tau \ (v^1_{\alpha o} - v^1_{\beta o}) = 0$$

and one integration together with the initial conditions $v^1_{\alpha o} \ (\xi = 0) = v^1_{\beta o} \ (\xi = 0) = 0$ gives

$$v^1_{\alpha o} = v^1_{\beta o} = 0$$

Integrating 5.3.11a and b, using the initial conditions $n^1_{\alpha o}(\xi = 0) = n^1_{\beta o}(\xi = 0) = 0$ and combining this with 5.3.7 gives

$$n^1_{\alpha o} = n^1_{\beta o} = 0$$

We therefore conclude that:

$$\underline{U}^1_{\,o} = 0 \qquad 5.3.13$$

From the definition of $\underline{S}^1_{\,1}$ we now see that 5.3.8 reduces to

$$\underline{W}_1 \underline{U}_1{}^2 + \underline{M}_1 R \frac{\partial \phi}{\partial \tau} = 0$$

which may now be solved for $\underline{U}_1{}^2$.

By inspection we see

$$\underline{M}_1 R = \underline{W} \, \underline{X}$$

where

$$\underline{X} = \begin{pmatrix} 0 \\ 2/Vik \\ -n_o/V^2ik \\ 3n_o/V^2ik \\ 1/(e/m) \end{pmatrix}$$

and therefore

$$\underline{W}_1 \left(\underline{U}_1{}^2 + \underline{X} \frac{\partial \phi}{\partial \tau} \right) = 0$$

which immediately gives

$$\underline{U}_1{}^2 = \underline{R}\psi - \underline{X} \frac{\partial \phi}{\partial \tau} \qquad\qquad 5.3.14$$

where $\psi(\tau, \epsilon)$ is another scalar which may be determined to higher order. The other components of $\underline{U}$ required to second order are $\underline{U}_2{}^2$ and $\underline{U}_o{}^2$. The component $\underline{U}_2{}^2$ is obtained by direct solution of the $\ell = 2$ form of 5.3.3, i.e.

$$\underline{W}_2 \underline{U}_2{}^2 = \underline{S}_2{}^1$$

By direct evaluation we see that

$$\underline{S}_2{}^1 = -ik(\phi)^2 \begin{pmatrix} 1 \\ 1 \\ -2n_o/V \\ +2n_o/V \\ 0 \end{pmatrix}$$

and by direct inversion of $\underline{W}_2$ we find

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$$\underline{U}_2{}^2 = \begin{bmatrix} -1/V \\ +1/V \\ 2n_o/V^2 \\ 2n_o/V^2 \\ \dfrac{2in_o e}{m\varepsilon_o kV^2} \end{bmatrix} (\phi)^2 \qquad\qquad 5.3.15$$

Some of the components of $\underline{U}_o{}^2$ have already been found, i.e. 5.3.12 and 5.3.10 give

$$E_o{}^2 = 0$$

$$n^2{}_{\alpha oo} + n^2{}_{\beta o} = 0 \qquad\qquad 5.3.16$$

As in Chapter 4 we consider the $\ell = 0$ component of 5.3.4. The first two rows combined give

$$\frac{\partial}{\partial \tau} (v^2_{\alpha o} - v^2_{\beta o}) = 0$$

Integrating and using the boundary conditions $v^2_{\alpha o}(\xi = 0) = v^2_{\beta o}(\xi = 0) = 0$ gives

$$v^2_{\alpha o} - v^2_{\beta o} = 0 \qquad\qquad 5.3.17$$

Although we have not explicitly determined the components of $\underline{U}_o{}^2$ we need go no further as we shall see that when these components are required to next order they occur only in the combinations given by 5.3.16 and 5.3.17 and so these terms may then be equated to zero. We are now ready to determine the nonlinear Schrödinger equation. We require the $\ell = 1$ component of 5.3.4 and multiply on the left by $\underline{L}$ to obtain:

$$\underline{L}\,\underline{W}_1 \underline{U}_1{}^3 + \underline{L}\,\underline{M}_1 \frac{\partial \underline{U}^2}{\partial \tau}1 + \underline{L}\,\underline{N}_1 \frac{\partial \underline{U}^1}{\partial \xi}1 + \underline{L}\,\underline{O}_1 \underline{U}_1{}^1 = \underline{L}\,\underline{S}_1{}^2 \qquad 5.3.18$$

By inspection:

$$\underline{S}_1^2 \;=\; -ik \begin{pmatrix} R_1^* v_{\alpha 2}^2 \phi^* + R_1 v_{\alpha o}^2 \phi \\[4pt] R_2^* v_{\beta 2}^2 \phi^* + R_2 v_{\beta o}^2 \phi \\[4pt] R_3^* v_{\alpha 2}^2 \phi^* + R_1 n_{\alpha o}^2 \phi + R_3 v_{\alpha o}^2 \phi + R_1^* n_{\alpha 2}^2 \phi^* \\[4pt] R_4^* v_{\beta 2}^2 \phi^* + R_2 n_{\beta o}^2 \phi + R_4 v_{\beta o}^2 \phi + R_2^* n_{\beta 2}^2 \phi^* \\[4pt] 0 \end{pmatrix}$$

and so

$$\underline{L} \cdot \underline{S}_1^2 \;=\; -ik \,\{ 2(v_{\alpha 2}^2 - v_{\beta 2}^2)\phi^* + 2(v_{\alpha o}^2 - v_{\beta o}^2)\phi$$

$$- \frac{V}{n_o}(n_{\alpha o}^2 + n_{\beta o}^2)\phi - \frac{V}{n_o}(n_{\alpha 2}^2 + n_{\beta 2}^2)\phi^* \}$$

where we have substituted for $\underline{R}$ and $\underline{L}$. We can now verify that the components of $\underline{U}_o^2$ appear only in the combinations given by 5.3.16 and 5.3.17. Substituting these results and substituting for $\underline{U}_2^2$ from 5.3.15 gives

$$\underline{L} \cdot \underline{S}_1^2 \;=\; \frac{8ik}{V}\,\phi|\phi|^2$$

We now consider each of the terms on the left-hand side of 5.3.18.

$$\underline{L}\,\underline{W}_1 \underline{U}_1^3 \;=\; 0$$

From the definition of $\underline{L}$ and $\underline{W}_1$,

$$\underline{L}\,\underline{M}_1 \frac{\partial \underline{U}^2}{\partial \tau} 1 \;=\; \underline{L}\,\underline{M}_1 \left(\underline{R}\,\frac{\partial \psi}{\partial \tau} - \underline{X}\,\frac{\partial^2 \phi}{\partial \tau^2} \right)$$

$$= (L_1 R_1 + L_2 R_2 + L_3 R_3 + L_4 R_4)\frac{\partial \psi}{\partial \tau}$$

$$- (L_1 X_1 + L_2 X_2 + L_3 X_3 + L_4 X_4)\,\partial^2 \phi / \partial \tau^2$$

$$= -\frac{6}{Vik}\,\frac{\partial^2 \phi}{\partial \tau^2}$$

after substitution of 5.3.14.

$$\underline{L}\,\underline{N}_1\,\frac{\partial U^1}{\partial \xi}1 = \underline{L}\,\underline{N}_1 R\,\frac{\partial \phi}{\partial \xi}$$

$$= (VR_1L_1 - VR_2L_2 + n_oL_3R_1 + VR_3L_3 + n_oL_4R_2 - VR_4L_4 + R_5L_5)\,\frac{\partial \phi}{\partial \xi}$$

$$= 4V\,\frac{\partial \phi}{\partial \xi}$$

by direct evaluation using the definition of the matrix $\underline{N}_1$

$$\underline{L}\,\underline{O}_1 U^1{}_1 = \underline{L}\,\underline{O}_1 R\,\phi$$

$$= n_o \Delta ik\,(R_1L_3 + R_2L_4)\,\phi$$

$$= -2\Delta ikV\,\phi$$

Combining these terms and after some rearrangement gives the required equation

$$i\,\frac{\partial \phi}{\partial \xi} = a\,\frac{\partial^2 \phi}{\partial \tau^2} - b\phi|\phi|^2 - c\phi \qquad\qquad 5.3.19$$

where

$$a = \frac{3}{2V^2 k}$$

$$b = \frac{2k}{V^2}$$

$$c = \frac{1}{2}\,\Delta k$$

we note that

$$\sqrt{\frac{c}{a}} = \omega_i$$

the linear growth rate and that

$$a = \frac{1}{2}\left(\frac{1}{2}\,k\right)\left(\frac{\partial^2 \det \underline{\underline{H}}}{\partial \omega^2}\right)_{\Delta=0,\,\omega=0}$$

and

$$c = \left(\frac{1}{2}\,k\right)\Delta\left(\frac{\partial \det \underline{\underline{H}}}{\partial \Delta}\right)_{\omega=0,\,\Delta=0}$$

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where $\underline{\underline{H}}$ is the linear dispersion relation, i.e.

$$H = \frac{\omega_p^2(1+\Delta)}{(\omega+kV_c)^2} + \frac{\omega_p^2(1+\Delta)}{(\omega-kV_c)^2} - 1$$

5.4 Results and Discussion

We have derived the equation satisfied by the lowest order amplitudes and rewrite it for convenience as

$$i \frac{\partial \phi}{\partial \xi} = a \frac{\partial^2 \phi}{\partial \xi^2} - \left[\frac{b}{a} |\phi|^2 + \gamma^2\right]\phi \qquad\qquad 5.4.1$$

The coefficient γ^2 represents the linear effect due to the deviation of the system from the marginally stable state and is the square of the linear growth or attenuation rate.   The derivation of 5.3 that led to 5.4.1 did not assume that γ^2 was either positive or negative and hence 5.4.1 is valid for the two-stream instability if the linear system is stable or unstable.   It is easily seen that $b/a = 4k^2/3$ is always positive.

Following the discussion of 2.3 we may immediately conclude that if the system is stable in the linear theory then it shows instability in the nonlinear theory.   Also if the system is linearly unstable it remains nonlinearly unstable.

This result could have been deduced from the discussion in 5.1 and from the general result of Chapter 2.   In Chapter 2 (part 3) we showed that systems described by a particular set of nonlinear differential equations and having a marginally stable state lead to an equation such as 5.4.1 in the nonlinear theory.   We also suggested that in view of the general result of 2.2, and the example of Chapter 4 that 5.4.1 is valid for systems described by more general systems of differential equations (providing a marginally stable state exists).   Therefore an equation such as 5.4.1 should be valid for the two-stream instability in cold plasmas.

In 5.1 we discussed previous attempts at nonlinear theories and found that the general conclusion reached, for cold plasma streams was that the

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linearly unstable system remained unstable in the nonlinear regime when wave-wave interactions are the only available mechanisms for nonlinear interaction.

Combining these two results would immediately have given the equation 5.4.1 with the signs of the coefficients to be the same as in 5.3.19. We have therefore confirmed this result and deduced explicit forms for the coefficients.

If thermal effects are included then the possibility of nonlinear stabilisation does exist according to previous theories. We therefore suggest that the method used in 5.3 when applied to the two-stream instability for warm plasmas should confirm this. The simplest modification to the system of equations 5.2.1, 5.2.2 and 5.2.3 to include finite temperature effects would be the replacement of 5.2.2 by

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{v_s^2}{n_i} \frac{\partial n_i}{\partial x} = -\frac{e}{m} E \qquad 5.4.2$$

where v_s^2 is the sound velocity in an electron plasma. This equation is valid under the assumption of isothermal compression. The system described by 5.2.1, 5.4.2 and 5.2.3 again allows a marginally stable state.

Then an equation of the form 5.4.1 would be valid for this system where the coefficient b would be a complex function of ω, v_s, V_c and k. If there exists a range of wavenumbers for which stabilisation occurs then the sign of b will change from positive to negative as k moves into this range.

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Chapter 6

The Crossed Field Instability

6.1 Introduction

Crossed field devices, i.e. devices whose mode of operation depends on the presence of an electron beam and mutually orthogonal electric and magnetic fields have received considerable theoretical and experimental attention during the last forty years. These devices, normally called magnetrons have been important sources of microwave power for radar systems and communication systems. The magnetron is a cylindrical device having a re-entrant beam and resonant cavities around this beam to extract the power from the beam. The majority of the work during and after the second world war was concentrated on the cylindrical magnetron. The attraction of the magnetron was the fact that in principle it should provide higher sources of power than travelling wave tubes or klystrons because the beam current density can be increased without limit simply by increasing the magnetic field. However, it was soon found that magnetrons were inherently noisy and unstable devices at all levels of operation. These drawbacks were overcome to a certain extent to provide useful devices. Since magnetrons are re-entrant devices analysis of their mode of operation is difficult and during the last twenty years attention has been focussed on a linear version of the magnetron, normally called the crossed field linear amplifier. These devices consist of an electron beam flowing between parallel electrodes in crossed electric and magnetic fields. They have in turn become of some commercial and military interest as microwave amplifiers and a high level of research has been maintained in recent years.

Although the discussion above has concentrated on the microwave device aspect of crossed field electron beam flow the problem is of more general interest. Plasma confinement schemes and the possibility of crossed field interaction in the ionosphere have provided motivation for a study of

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crossed field flow in systems other than microwave devices. However, as
the original stimulus for this work came from an interest in microwave devices
and as the approximations necessary for studying these other systems differ
considerably from those necessary for microwave devices we will give no
further attention to these systems other than cite Dysthe, Misram and
Trulsen (1975) and Weng and Ma (1975) as giving good discussions of these
systems.

We will now briefly review the literature relating to crossed field
electron beam flow. Hull (1921) considered the effect of a superimposed
magnetic field on the flow of electrons between coaxial cylinders. He
predicted that below a critical anode voltage no electrons should reach the
anode and the flow would be cut off. He found a rough agreement between
his theoretical and experimental findings. Very little work was done on
crossed field flow between the time of publication of this original paper
and the early 1940s. The magnetron work done during the subsequent few
years was classified and little was published until 1951. Summaries of
this classified work were published by Buneman (1951,1957). The main
features found were that a cut-off of beam current did not occur as was pre-
dicted by Hull and that magnetrons were unstable and noisy under certain
conditions. A considerable amount of work was done on anode-cathode
configurations to utilise the amplification properties of crossed field
beams for microwave beams in magnetrons.

Sustained interest in crossed field flow led to the linear electrode
configuration and the possibility of using these linear beams as an ampli-
fying medium was first realised by Buneman (1950). As with the circular
magnetron the device was found to be highly unstable and beam turbulence was
found to set in very rapidly. Again, anomalous anode currents were found
below the theoretical anode cut-off voltage. A good summary of the early
findings of linear crossed field research is given in Okress (1961).

Theoretical work was concentrated on the problems of amplification of slow waves at the expense of beam energy and in deriving the dispersion relation for differing electrode configurations. A great deal of confusion has arisen because, as we shall see in 6.3, there are a number of different distinct instability mechanisms in crossed field flow and these were never correctly isolated and identified. The term "crossed field instability" was used to describe all these instability mechanisms. The problem of explaining the anomalous cut-off current has received less attention. The work of Mouthaan (1965) aimed at predicting this current on the basis of a diffusion theory based on the Fokker-Planck equation. Mouthaan used one of the dispersion relations derived by Buneman (1961b) and obtained a value for the anomalous current when the beam was unstable. This was found to be in rough agreement with his experimental findings. Lindsay (1960) and in many subsequent papers, attempts to explain this current by systematically integrating the equations of motion for an electron in a velocity space description. He predicts the anomalous current if the electrons on leaving the cathode have a wide distribution of thermal velocities. However, under the electric and magnetic fields normally used in linear amplifiers, the thermal velocity distribution must be considered negligible in view of the rapid particle velocities achieved on emission.

In this chapter we wish to concentrate on the problem of instabilities in crossed field flow rather than the problem of predicting the anomalous anode current. We also, as far as possible, concentrate on the plasma effects in devices rather than the interaction of the beam and complex slow wave structures. This is found to be difficult as the boundary conditions are shown to play an important role in the behaviour of the system. The electrodes are assumed to be plane parallel structures throughout the chapter.

In 6.2 we consider the possible steady state flows of a plasma in crossed electric and magnetic fields and discuss the assumptions necessary to develop the theory. With reference to microwave devices the hydrodynamic

description of the plasma is found to be appropriate. A number of different flows are found to be possible, the most likely being the plane parallel flow with a velocity gradient across the beam, called Brillouin flow. We then show how the beam may be placed between plane parallel electrodes and deduce the theoretical I-V characteristics of the system. We find that for the emitting cathode configuration that a beam in contact with the cathode and with one free surface is the only possible flow. However, for beams injected into a crossed field device the beam may be positioned anywhere in the interelectrode space by varying the electrode potentials and the beam current. This finding is significant, as we shall show later that the former beam configuration is not unstable to certain perturbations which cannot be said for the latter.

In 6.3 we consider a linear theory of the electron flow based on a conventional Fourier expansion. In 6.3.2 the dynamics of the beam are considered separately from the boundary conditions and a differential equation describing the behaviour of velocity perturbations along the electric field is derived. The choice of this variable as the dependent variable is discussed and it is found to be the most physically meaningful variable that can be considered. By an appropriate choice of parameters the velocity slip, a crucial parameter in the theory, is carried through and is not confused with other parameters of the system as has so often been done in the past.

In 6.3.3 the boundary conditions for the differential equation are considered and a general method for determining them is given, for different confining electrodes. In 6.3.4 the differential equation of 6.3.2 and the boundary conditions of 6.3.3 are combined to give the dispersion relation. Although the dispersion relation is general in the sense that solutions of the differential equation are not known the stability of the system can be determined and three separate instability mechanisms are identified. In 6.3.5 one of these instabilities, the long wavelength instability is

considered in great detail and an explicit expression for the growth rate is given.

Finally in 6.4 the long wavelength   crossed field instability is related to the two-stream instability considered in Chapter 5 and suggestions are made as to how a nonlinear theory of the instability may be developed using the general theories of Chapters 2 and 3.

6.2 The steady state

We wish to consider the possible steady state flows of an electron beam in crossed electric and magnetic fields.   Since we are particularly interested in crossed field microwave devices we will initially consider the assumptions necessary to develop a linear theory consistent with this interest.   The geometry of the system to be considered is shown below.
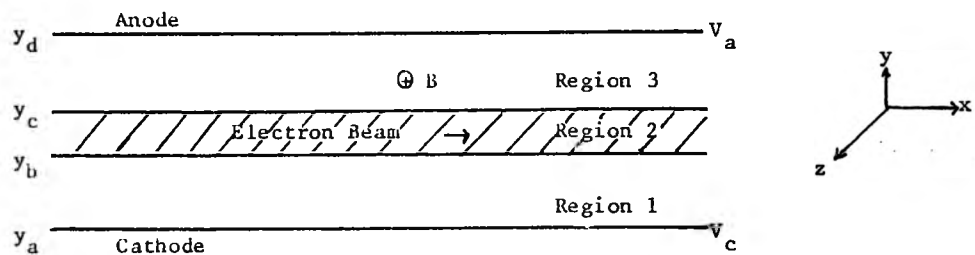

FIG. 1

The following is a list of the necessary assumptions, both for the steady state theory presented here and for the linear   theory of 6.2.

1. Relativistic effects are ignored.   Since electron velocities attained in most microwave devices correspond at most to energies of a few keV relativistic effects may be ignored. (This is consistent with 6 below).

2. The system is treated as being two-dimensional only.   All variables are assumed to be independent of the z coordinate, i.e. along the superimposed magnetic field.   Buneman (1961b) has considered the effect of variations in the z direction and found no additional effects in the steady state analysis and only small effects in the perturbation analysis.

We therefore assume no z dependence which considerably simplifies the analysis.

3.   Thermal effects.   No thermal motion of electrons is assumed and the plasma is assumed cold.   Thermally emitted electrons from a cathode are drawn into crossed field flow and the distances that electrons travel at thermal velocities are insignificant with distances travelled in significant time periods for the system, e.g. r.f periods, plasma frequencies and cyclotron frequencies.

4.   Collisions.   Given typical charge densities in microwave devices collisions between electrons are infrequent enough to be neglected.

5.   The plasma is unneutralised.   We assume that electrons are the only charged particles present in the system.   Gould (1957) has shown that if the plasma is neutralised then the instabilities discussed in 5.4 are all suppressed.

6.   The magnetic field associated with the electron motion is ignored.   It is well known that the self-magnetic field induces forces that are smaller by a factor of v^2/c^2 than forces created by static externally applied fields or space charge fields.   Buneman (1961a) has considered self-magnetic interactions on a certain class of space charge flows in cylindrical systems.   If these results are extrapolated to linear systems then there are no significant differences in the analysis. This assumption is consistent with 1 above since relativistic effects are comparable with self-magnetic effects and if one effect is neglected then the other must also be neglected.

It is evident from the above considerations that the hydrodynamic description of the electron stream is appropriate for this system.   The hydrodynamic equations for a plasma of only one species with no collisions and zero pressure gradient are given by the momentum transfer equation:

$$m \left(\frac{\partial V}{\partial t} + \underline{V}.\nabla \underline{V} \right) + e\underline{E} + e\underline{V} \wedge B = 0 , \qquad 6.2.1$$

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and the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0 \qquad\qquad 6.2.2$$

These equations may then be coupled with Maxwell's equations to give a closed set of equations:

$$\nabla \cdot \underline{E} = -\rho / \varepsilon_o \qquad\qquad 6.2.3$$

$$\nabla \cdot B = 0 \qquad\qquad 6.2.4$$

$$\nabla \wedge \underline{E} = -\partial \underline{B} / \partial t \qquad\qquad 6.2.5$$

$$\mu_o \nabla \wedge B = \varepsilon_o \frac{\partial \underline{E}}{\partial t} + \rho \underline{V} \qquad\qquad 6.2.6$$

Since self-magnetic fields are neglected 6.2.4 may be neglected and the right-hand side of 6.2.5 may be equated to zero. The electric field then becomes curl free and consequently an electrostatic potential ϕ may be used.

In the steady state we assume the following forms of the electric field, the magnetic field and the velocity.

$$\underline{B} = (0, 0, -B_o)$$

$$\underline{E} = (0, E_o(y), 0) \qquad\qquad \left.\begin{array}{c}\\\\\\\end{array}\right\} 6.2.7$$

$$\underline{V} = (V(y), 0, 0)$$

Substituting 6.2.7 into 6.2.1 immediately gives

$$V(y) = E_o(y)/B_o$$

i.e. the x directed velocity has a shear or gradient in the y direction and each electron drifts in the x direction at the local E/B velocity. In this flow the force on the electron due to the electric field is balanced by the Lorentz force. Poisson's equation 6.2.3 gives:

$$\frac{dE_o(y)}{dy} = -\rho / \varepsilon_o$$

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and if the charge density is assumed constant then:

$$E_o(y) = -(\rho/\epsilon_o)y - C_1$$

and the potential becomes:

$$\phi = \frac{\rho}{2\epsilon_o} y^2 + C_1 y + C_2$$

where C_1 and C_2 are constants. These constants may be determined by the boundary conditions at the anode and cathode but at this stage it is sufficient to note that the potential ϕ has a minimum. The velocity slip or shear Δ is given by:

$$\Delta = \frac{dV(y)}{dy} = \frac{d}{dy} (E_o(y)/B_o)$$

$$= \frac{1}{B_o} \frac{dE_o(y)}{dy} = -\frac{\rho}{B_o \epsilon_o}$$

which in terms of the plasma frequency ω_p and the cyclotron frequency ω_c is given by

$$\Delta = \omega_p^2/\omega_c \qquad\qquad 6.2.8$$

where

$$\omega_p^2 = -\frac{e\rho}{m\epsilon_o}$$

$$\omega_c = \frac{eB_o}{m}$$

We note that this solution is not unique but is a special case of a class of more general flows. Other possible space charge flows include Double Stream Flow and Benham Flow which are discussed in detail by Smol (1971). These other flows will not concern us here but it is interesting to note, as Smol (1971) has shown, that under certain conditions all these different space charge flows lead to the same relationship between beam current and applied potential.

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We now note some further consequences of the space charge flow described by 6.2.7 and 6.2.8 for electron beams between confining electrodes. This space charge flow, i.e. a stream of electrons flowing between confining electrodes is only possible if electrons have come an infinite distance (since there is no velocity component in the y direction. If the electrons leave a cathode a finite distance away then they must have left this cathode with no thermal emission energy (this is implicit in assumption 3). This implies that the electrons leave the cathode with zero velocity and that there is no component of magnetic field normal to the cathode surface. This implies that the curl of the canonical momentum or action function is zero initially,

i.e.
$$\nabla \wedge (m\underline{V}) = e\underline{B} \qquad\qquad 6.2.9$$

Gabor (1945) has shown that if the curl of the canonical momentum is zero initially then it must remain zero at all points between the emitting cathodes and in the final equilibrium distribution. This result is used in the Hamilton Jacobi formalism used by Macfarlane and Hay (1951). The y component of 6.2.9 then gives

$$\frac{\partial V(y)}{\partial y} = \frac{eB_o}{m} = \omega_c = \Delta$$

i.e.
$$\omega_p = \omega_c = \Delta \qquad\qquad 6.2.10$$

The electron beam flow described by this condition is called Brillouin flow (Brillouin (1945)) and is normally considered to be a too restrictive stationary state for a study of crossed field instabilities although in principle it is the only possible flow with a velocity gradient under the assumptions 1-6 given above.

Plasma streams where 6.2.10 is not satisfied, i.e. with unequal plasma and cyclotron frequencies and consequently through 6.2.8 with arbitrary velocity gradient are possible if some of the assumptions are not made. For instance collisions may change the canonical momentum by changing the velocity

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and hence in a dense plasma in a sufficiently high magnetic field the condition $\omega_p < \omega_c$ may be satisfied giving a small velocity gradient. Similarly if self-magnetic effects are not ignored then curl $(m\underline{V}-e\underline{A})$ is not zero initially and the conditions assumed by Gabor (1945) are then not satisfied. Therefore in the final equilibrium distribution curl $(m\underline{V}-e\underline{A}) \neq 0$ and consequently $\omega_p \neq \omega_c$. We note that $\omega_p \leqslant \omega_c$ and the equality represents a limiting case.

A number of authors, including Smol (1971) have given much consideration to the problem of launching a Brillouin team between suitable electrodes (A good bibliography of this problem is given by Smol). For our purposes it is sufficient to note that devices exhibiting an unstable Brillouin beam have been constructed and rely on creating a Brillouin beam from a specially shaped electron gun and launching this beam into a crossed field configuration.

In 6.3 it will be shown that the position of the beam relative to close or distant electrodes is critical in determining the stability or instability of a Brillouin beam (or a beam with arbitrary velocity profile). It is therefore instructive to show how a Brillouin beam may be placed in any position relative to the confining electrodes and to derive the I-V characteristics of Brillouin beams.

We suppose that the confining electrodes are perfectly conducting plates located at $y = y_a$ and $y = y_d$, being at potentials V_c and V_a respectively where V_c is positive with respect to V_a. The beam lies between the plates y_b and y_c where y_b may be coincident with y_a and y_c may be coincident with y_d. We must now solve Poisson's equation in regions 1,2,3 of Fig. 1 and match potentials and electric fields at the boundaries. Without loss of generality we assume $y_a = 0$.

In region 1,

$$\frac{d^2\phi_1}{dy^2} = 0 \qquad \text{with} \quad \phi_1(0) = \phi_c$$

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which gives

$$\phi_1(y) = a_1 y + \phi_c \qquad\qquad 6.2.11$$

for $\qquad y_b \geqslant y \geqslant 0$

In region 2,

$$\frac{d^2\phi_2(y)}{dy^2} = \rho/\epsilon_o$$

which gives

$$\phi_2(y) = \frac{m}{2e}\,\omega_c^2 y^2 + a_2 y + a_3 \qquad\qquad 6.2.12$$

for $\qquad y_b \leqslant y \leqslant y_c$

where we have used the equality of 6.2.10 to define a Brillouin beam.

In region 3,

$$\frac{d^2\phi_3(y)}{dy^2} = 0 \qquad \text{with } \phi_3(y_d) = \phi_a$$

which gives

$$\phi_3(y) = a_4(y-y_d) + \phi_a \qquad\qquad 6.2.13$$

for $\qquad y_c \leqslant y \leqslant y_d$

We must now determine the coefficients a_1-a_4. In region 2 we have one further relation, i.e. conservation of energy.

$$\frac{1}{2}\,mV^2(y) = e\phi_2(y) \qquad\qquad 6.2.14$$

where the Brillouin velocity $V(y)$ is given by:

$$V(y) = \omega_c y + c \qquad\qquad 6.2.15$$

where c is a constant.

We can now determine the I-V characteristics of this system and determine

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parameters such as the beam thickness, position between electrodes etc. for a given cathode and anode voltage and beam current. Combining 6.2.14 and 6.2.12 together with 6.2.15 gives:

$$(\omega_c y + c)^2 = \frac{2e}{m} \left[\frac{\omega_c^2 m}{2e} + a_2 y + a_3 \right]$$

i.e.
$$a_2 = \frac{m\omega_c}{e} c$$

$$a_3 = \frac{c^2 m}{2e}$$

Substituting these relations into 6.2.12 and 6.2.13 gives

$$\phi_1(y) = a_1 y + \phi_c \qquad\qquad 6.2.11a$$

$$\phi_2(y) = \frac{m}{2e} (\omega_c y + c)^2 \qquad\qquad 6.2.12a$$

$$\phi_3(y) = a_4(y - y_d) + \phi_a \qquad\qquad 6.2.13a$$

We now match potentials and electric fields at y_b and y_c:

$$a_1 y_b + \phi_c = \frac{m}{2e} (\omega_c y_b + c)^2 \qquad\qquad 6.2.16$$

$$a_4(y_c - y_d) + \phi_a = \frac{m}{2e} (\omega_c y_c + c)^2 \qquad\qquad 6.2.17$$

$$a_2 = \frac{\omega_c m}{e} (\omega_c y_b + c) \qquad\qquad 6.2.18$$

$$a_4 = \frac{\omega_c m}{e} (\omega_c y_c + c) \qquad\qquad 6.2.19$$

Substituting 6.2.18 into 6.2.16 and 6.2.19 into 6.2.17 and simplifying gives:

$$c^2 = \omega_c^2 y_b^2 + \frac{2\phi_c e}{m} \qquad\qquad 6.2.20$$

$$c^2 + 2\omega_c c y_d = \omega_c^2 y_c^2 - 2\omega_c^2 y_d y_c + \frac{2\phi_a e}{m} \qquad\qquad 6.2.21$$

and subtracting these two relations gives

$$c = \frac{\omega_c^2 y_c^2 - \omega_c^2 y_b^2 - 2\omega_c^2 y_d y_c + \frac{2e}{m} (\phi_a - \phi_c)}{2\omega_c y_d} \qquad 6.2.22$$

We may now relate the parameter c to the beam current J as follows. The beam current is defined by:

$$J = \int_{y_b}^{y_c} \omega_p V(y)\, dy$$

where ω is the beam width in the z direction. Evaluating this integral gives:

$$J = \omega_p \left\{ \frac{\omega_c}{2}(y_c^2 - y_b^2) + c(y_c - y_b) \right\} \qquad\qquad 6.2.23$$

We normalise this current with respect to the maximum beam current attainable when the beam fills the space between the electrodes. In this case $y_b = 0$, $y_c = y_b$ and $c = 0$ which gives

$$J_{MAX} = \omega_p \left\{ \frac{\omega_c y_d^2}{2} \right\}$$

and so we define

$$j = J/J_{MAX}$$

Combining 6.2.23 with 6.2.22 gives:

$$\frac{\dfrac{J}{\omega_p} - \dfrac{\omega_c y_c^2}{2} + \dfrac{\omega_c y_b^2}{2}}{(y_c - y_b)} = \frac{\omega_c^2 y_b^2 - \omega_c^2 y_c^2 + 2\omega_c^2 y_d y_c + \dfrac{2e}{m}(\phi_a - \phi_c)}{-2\omega_c y_d}$$

With the definitions

$$y_c - y_b = t$$

$$y_d = d$$

this relation may be simplified to give

$$j = \frac{2y_b}{d}\left(\frac{t}{d}\right)^2 + \left(\frac{t}{d}\right)^3 - \left(\frac{t}{d}\right)^2 + \left(\frac{t}{d}\right) + \frac{2e}{m\omega_c^2 d^2}(\phi_a - \phi_c)$$

We define a potential ϕ_k such that

$$\phi_k = \frac{m}{2e}\omega_c^2 d^2$$

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and normalise with respect to this potential

$$\psi = \phi/\phi_k$$

The physical significance of this normalising ϕ_k potential is that a single electron leaving the cathode at zero potential will just fail to reach the anode when this is at the potential ϕ_k.

With the further definition

$$\delta = t/d$$

i.e. δ represents the ratio of beam width to interelectrode space we obtain

$$j = \frac{2y_b}{d} \delta^2 + \delta^3 - \delta^2 + \delta(\psi_a - \psi_c) \qquad\qquad 6.2.24$$

We can now eliminate y_b by squaring 6.2.23, normalising and equating the result to 6.2.20 which after some algebra gives

$$\frac{y_b}{a} = \frac{1}{4\delta^2} \left[\delta(j-\delta^2) - (\delta^3 - 2\delta^2 + (2(\psi_a - \psi_c) + j)\delta - 2j \right] \; 6.2.25$$

Finally, combining 6.2.24 and 6.2.25 gives the required relation, i.e.

$$\delta^5 - 2\delta^4 - j \{2(\psi_a - \psi_c) + j\} \delta + 2j^2 + 2(\psi_a + \psi_c)\delta^2 = 0 \quad 6.2.26$$

which is a quintic equation for the beam thickness given particular values of the beam current and anode and cathode potentials. It is easy to see that this equation has at most two positive roots, i.e. physically meaningful roots.

Solutions of 6.2.26 are known if ϕ_c is set to zero. Then 6.2.26 factorises to give:

$$(\delta^2 - j)(\delta^3 - 2\delta^2 + (2\psi_a + j)\delta - 2j) = 0$$

with solutions

$$\delta^2 = j \qquad\qquad 6.2.27$$

$$\delta^3 - 2\delta^2 + (2\psi_a + j) - 2j = 0 \qquad\qquad 6.2.28$$

where we have the conditions $0 \leqslant \delta \leqslant 1$ and $j \leqslant 1$.

If 6.2.27 and 6.2.28 are both simultaneously satisfied then from 6.2.25

$$y_b = 0$$

and from 6.2.28

$$\psi_a = \sqrt{j} \, (2 - \sqrt{j})$$

This represents a beam in contact with the cathode and displaying the IV characteristic of Brillouin flow as demonstrated by Smol (1971). For this flow, condition 6.2.20 shows that c = 0 and that the velocity profile of the beam is given by:

$$V(y) = \omega_c y$$

The beam thickness varies as the square root of the beam current according to 6.2.27 and in the limit j = 1 the beam current is a maximum as would be expected with $\delta = 1$, i.e. the beam fills the inter-electrode space.

If 6.2.27 only is satisfied then the lower edge of the beam is not in contact with the cathode but this edge is at the cathode potential. This type of flow is probably non-physical. If 6.2.28 only is satisfied then the beam lies somewhere between the electrodes with the beam edge potentials lying at some value between 0 and ϕ_a. The position of the beam and its width depend on the value of ϕ_a and on the current density J. The most likely type of flow for an emitting cathode crossed field device is given by the simultaneous conditions 6.2.27 and 6.2.28 since for these c = 0 which is consistent with the assumptions of the system (particularly assumption 3). For the case when 6.2.28 only is satisfied the flow can be achieved by injecting an electron beam into a crossed field configuration.

Although we have only discussed in detail configurations which can arise under the conditions of Brillouin flow we have already noted that if any assumptions are neglected more general types of flow are possible.

We can therefore in general consider systems with the following characteristics:

1. A plasma stream flowing in a fixed direction with a velocity slip. The gradient of this velocity slip is generally given by:

$$\frac{dV(y)}{dy} \;=\; \frac{\omega_p^{\,2}}{\omega_c}$$

where the plasma frequency and cyclotron frequency are not equal. (In fact $\omega_p \ll \omega_c$ can be attained and provides an interesting exactly solvable system in the linear perturbation analysis in 6.3).

2. At least one conducting plate some distance away from the beam. The conducting plate is required to support the positive charge necessary to neutralise the negative charge of the beam. If two such electrodes exist then they fix definite potentials at some distance from the beam. The plates may be sufficiently distant compared with disturbances in the system to have negligible effect on the system. Conversely they may be comparatively near and have a profound effect on the behaviour of the system. (In a crossed field amplifier an emitting cathode and an anode capable of supporting a slow wave would be present).

Given the above two characteristics enables us to consider all the insta-bilities that have been called crossed field instabilities and to identify the parameters responsible for the instabilities.

6.3 Linear analysis of crossed field instabilities

6.3.1 Introduction

Following the discussion of 6.1 we consider the following system :
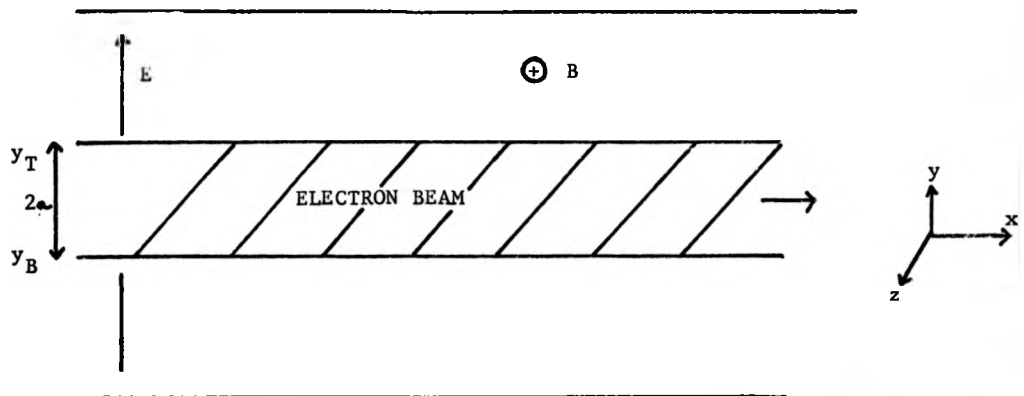
with a steady state defined by

$$\underline{E} = (0, E(y), 0)$$

$$\underline{B} = (0, 0, -B_o)$$

$$\underline{V} = (V_o(y), 0, 0)$$

where

$$dV(y)/dy = \omega_p^2/\omega_c$$

$$\rho = \rho_o = \text{constant}$$

It is convenient to consider the system from a frame of reference moving with the velocity of the centre of the beam. In this frame of reference the top and bottom edges of the beam appear to be moving in opposite directions with equal velocity.

We will consider the perturbations of the system propagating in the +x direction and consider these perturbations to be of the form

$$\phi(y) \exp^{i(kx-\omega t)}$$

since we cannot Fourier analyse in the y coordinate due to the dependence of the steady state on this coordinate.

If ω and k are real then the perturbations move in the x direction with phase velocity ω/k. However, electrons moving in the x direction will interact with the wave at a frequency $\omega-kV(y)$, i.e. at the Doppler shifted frequency. Interaction will be strongest when this frequency is zero since then the electrons stay in phase with the wave and so have a longer time to interact with it. We therefore expect some interaction when

$$\omega-kV(y) = 0$$

which is called the synchronous interaction. Similarly energy exchange is at a maximum when the Doppler frequency is equal to the cyclotron frequency, i.e.

$$\omega-kV(y) = \omega_c$$

which is called gyroresonance.

We would also expect an interaction when the wave frequency coincides with the plasma frequency ω_p. However, no resonances have been found at this frequency and this fact ultimately governs the choice of equation describing the dynamics of the beam (this point is discussed more fully in 6.3.2). (For a true Brillouin beam when $\omega_p = \omega_c$ the two resonances coincide and cannot be distinguished but since in systems when $\omega_p \neq \omega_c$ the plasma resonance is not found we assume that in the Brillouin beam the resonance is a gyroresonance rather than a plasma resonance).

Having defined the system we are to consider we now divide it into two distinct but interacting parts. The first, the internal dynamics of the beam leads to a differential equation for the perturbed variables of the system. The second, the behaviour of the velocity and electric field outside the beam provides the boundary conditions for this differential equation. Therefore, combining the two parts gives the solutions of the differential equation and hence its eigenfrequencies.

6.3.2 Electron beam dynamics

We consider equations 6.2.2 and 6.2.1 in component form and look for solutions of the form:

$$U_x = U_{xo}(y) + U_{x1}(y,t,x)$$

$$U_y = U_{y1}(y,t,x)$$

$$U_z = 0$$

$$\rho = \rho_o + \rho_1(y,t,x)$$

$$E_x = E_{x1}(y,x,t)$$

$$E_y = E_{yo}(y) + E_{y1}(y,t,x)$$

$$B_z = B_o$$

where the subscript 1 refers to perturbed quantities. We define

$$\frac{\partial U_{xo}(y)}{\partial y} = \Delta = \omega_p^2/\omega_c$$

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Substituting the assumed forms into 6.2.1, 6.2.2 and 6.2.3 and linearising gives the following equations:

$$\frac{\partial U_{x1}}{\partial t} + U_{xo} \frac{\partial U_{x1}}{\partial x} + U_{y1} \Delta = -\frac{e}{m} (E_{x1} - U_{y1} B_o)$$

$$\frac{\partial U_{y1}}{\partial t} + U_{xo} \frac{\partial U_{y1}}{\partial x} = -\frac{e}{m} (E_{y1} + U_{x1} B_o)$$

6.3.2.1

$$\frac{\partial \rho_1}{\partial t} + \rho_o \left(\frac{\partial U_{x1}}{\partial x} + \frac{\partial U_{y1}}{\partial y} \right) + U_{xo} \frac{\partial \rho_1}{\partial x} = 0$$

$$\frac{\partial E_{x1}}{\partial x} + \frac{\partial E_{y1}}{\partial y} = \rho_1 / \varepsilon$$

We now suppose that all perturbed quantities vary as exp i(kx-ωt) and define:

$$U_{x1} = U(y) \exp i (kx - \omega t)$$

$$U_{y1} = V(y) \qquad "$$

$$\rho_1 = \rho(y) \qquad "$$

$$E_{x1} = EX(y) \qquad "$$

$$E_{y1} = EY(y) \qquad "$$

Substituting these expressions into the set 6.3.2.1 gives:

$$U(-i\omega + ikU_{xo}) + V(\Delta - \omega_c) = -\frac{e}{m} EX \qquad 6.3.2.2$$

$$V(-i\omega + ikU_{xo}) + U\omega_c = -\frac{e}{m} EY \qquad 6.3.2.3$$

$$\rho(-i\omega + ikU_{xo}) + \rho_o \left(\frac{\partial V}{\partial y} + ikU \right) = 0 \qquad 6.3.2.4$$

$$ikEX + \frac{\partial EY}{\partial y} = \rho / \varepsilon \qquad 6.3.2.5$$

where we have omitted the functional dependence of all quantities for the sake of clarity.   One further equation is required to solve for the five variables U, V, EX, EY and ρ.   This relation is:

$$ikEY - \frac{\partial EX}{\partial y} = 0 \qquad 6.3.2.6$$

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which follows from the z component of 6.2.5 under the magnetostatic approxima-
tion.   We may now eliminate EX, EY, U and   to obtain a single second order
differential equation for the variation of the perturbed velocity V in the
y direction.   After some algebra we obtain:

$$\frac{\partial^2 V}{\partial y^2} + ik \frac{\partial V}{\partial y} \frac{2\Delta(-i\omega+ikU_{xo})}{\left[(-i\omega+ikU_{xo})^2+\omega_c^2\right]} \qquad\qquad 6.3.2.7$$

$$-k^2 V \left[1 + \frac{2\Delta(2\Delta-\omega_c)}{\left[(-i\omega+ikU_{xo})^2+\omega_c^2\right]} - \frac{4\Delta^2(-i\omega+ikU_{xo})^2}{\left[(-i\omega+ikU_{xo})^2+\omega_c^2\right]^2} \right] = 0$$

We have chosen V to be the dependent variable of this analysis for two
reasons.   Firstly, as we shall see in 6.3.3 when boundary conditions are
considered that only V appears in the expressions matching the electric field
at the beam edges.   Therefore, V appears to be the most physically significant
variable.   Secondly, the equation 6.3.2.7 exhibits no singularities at the
plasma frequency ω_p.   Similar differential equations to 6.3.2.7 are easily
derived for U_1, ρ and the potential ϕ and these formed the basis of the
analysis of Gould (1957), Macfarlane and Hay (1950), Dombrowski (1957) and
Knauer (1966).   These equations all exhibit singularities at the plasma
frequency ω_p.   The analysis used by these authors was unnecessarily complex
due to the presence of this additional singularity.   As mentioned in 6.3.1
no manifestation of any physical phenomena occurs at the plasma frequency ω_p
and so in this sense singularities of the differential equations for the
system at ω_p must be considered as anomalous.   Singularities of differential
equations that are not singularities of the solution are known as apparent
singularities and are discussed in Coddington and Levinson (1955).

Following Macfarlane and Hay (1950) we transform the independent variable
of 6.3.2.7 as:

$$s = (-\omega + kU_{ox})/\omega_c \qquad\qquad 6.3.2.8$$

to obtain:

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$$\frac{\partial^2 V}{\partial s^2} - \frac{2s}{(1-s^2)} \frac{\partial V}{\partial s} - \frac{\omega_c^2}{\Delta^2} \left\{ 1 + \frac{2\Delta(2\Delta-\omega_c)}{\omega_c^2(1-s^2)} + \frac{4\Delta^2 s^2}{\omega_c^2(1-s^2)^2} \right\} V = 0 \qquad 6.3.2.9$$

Following Buneman (1961a), we consider the new variable s as a variable describing spatial variations in the y direction, since, by definition it is a linear function of y through U_{ox}. The reference point for this new coordinate is the layer with $\omega = 0$ and so s measures the "distance" from the synchronous layer in normalised units of $(k\Delta/\omega_c)^{-1}$. Synchronism and gyroresonance are therefore defined by:

$$s = 0 \qquad \text{synchronism}$$

$$s = \pm 1 \qquad \text{gyroresonance}$$

One further simplification is possible and that is the elimination of the first derivate term of 6.3.2.9 by means of an integrating factor. This finally gives:

$$\frac{\Delta^2}{\omega_c^2} \frac{1}{V'} \frac{\partial^2 V'}{\partial s^2} = 1 + \frac{\Delta(3\Delta-\omega_c)}{\omega_c^2(1-s^2)} + \frac{3\Delta^2 s^2}{\omega_c^2(1-s^2)^2} \qquad 6.3.2.10$$

where $\qquad V' = V(1-s^2)^{\frac{1}{2}}$

This equation therefore describes the propagation of not the velocity perturbations but multiples of $(1-s^2)^{\frac{1}{2}}$ of it, across the beam. Again following Buneman (1961a) we note that this equation is similar to the equation describing the propagation of light in a nonlinear refractive medium. The right-hand side of 6.3.2.10 can therefore be considered as the square of an effective refractive index.

We may write the boundary conditions in terms of the variable s, i.e.

$$s_T = \omega/\omega_c + ak\Delta/\omega_c$$

$$s_B = \omega/\omega_c - ak\Delta/\omega_c$$

where a is the half width of the beam. If such s_T and s_B can be found so that the solutions of 6.3.2.10 equal the boundary conditions at s_T and s_B

simultaneously then the stability of the system may be determined from the complex frequency ω which is given by

$$\omega = -\frac{1}{2}(s_T + s_B)\omega_c \qquad 6.3.2.11$$

Finally, we note that if there is no velocity slip, i.e. $\Delta = 0$ then 6.3.2.7 reduces to:

$$\frac{d^2V}{dy^2} - k^2V = 0$$

i.e. Laplace's equation appropriate to describe incompressible flow. Similarly if the magnetic field is sufficiently high so that ω_c dominates all other frequencies then Laplace's equation is again obtained.

   Having derived the equation of motion for the beam we now turn to the boundary conditions.

6.3.3 Boundary conditions

   The boundary conditions for the electron beam are given by the exact nature of the surface perturbations of the beam density, the electric field in the space between the beam and the electrodes and the nature of the confining electrodes. Initially we will consider the surface of the perturbed beam and use the mean field approximation as originally proposed by Hahn (1939).

   We suppose the beam, due to perturbations in the flow has surface perturbations in the forms of ripples of excess charge. We must match the tangential and normal electric fields inside the beam to those outside. The tangential component of the electric field is continuous. The normal component varies by an amount proportional to the excess charge density σ at every point. We may therefore write:

$$E_{y1}^{EXT} - E_{y1}^{INT} = \sigma/\epsilon \qquad 6.3.3.1$$

where EXT and INT refer to fields outside and inside the beam. This condition must be satisfied at the top and bottom of the beam:

$$E_{y1}^{EXT}(+a) - E_{y1}^{INT}(+a) = \sigma(a)/\epsilon_0$$

$$E_{y1}^{EXT}(-a) - E_{y1}^{INT}(-a) = \sigma(-a)/\epsilon_0$$

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Since the perturbed y velocity V gives a displacement of $V/(-i\omega+ikU_{ox})$ this leads to the following form for the density σ

$$\sigma = \frac{\pm \rho_o V(y)}{\left[-i\omega+ikU_{ox}\right]}$$ 6.3.3.2

which follows from the continuity equation, and the plus and minus signs refer to the top and bottom of the beam respectively.   We therefore have

$$E_{y1}^{EXT}(+a) - E_{y1}^{INT}(+a) = \frac{+\rho_o V(a)}{\left[-i\omega+ik\Delta a\right]}$$ 6.3.3.3

$$E_{y1}^{EXT}(-a) - E_{y1}^{INT}(-a) = \frac{+\rho_o V(-a)}{\left[-i\omega-ik\Delta a)\right]}$$

The external field is given by solving Laplace's equation in the gap between the beam and the electrodes and the internal field is given by solving for V as discussed in 6.3.2 and then solving for E_{y1}.   At this point we note that the boundary conditions depend only on the velocity V and E_{y1} which is readily derived from it.   This justifies the choice of V as the dependent variable in the discussion of beam dynamics in 6.2.2.

The external field will only be considered in the case where the conducting electrodes are plane conductors.   The case when the anode is a slow waveguide is more complex and a discussion of this case is given by Macfarlane and Hay (1950).

Since the space charge between the beam and electrode is zero equations 6.3.2.5 and 6.3.2.6 may be combined to give

$$\frac{\partial^2 EY}{\partial y^2} - k^2 EY = 0$$

which may be integrated to give

$$EY = Ae^{ky} + Be^{-ky}$$

and using 6.3.2.5 we also obtain

$$EX = -i(Ae^{ky} - Be^{-ky})$$

where the constants A and B are determined by the behaviour of the field at the electrodes and the beam edge.

It is common practice to combine both components of the electric field into a single parameter called the normalised E-Mode admittance by

$$Y = \frac{EY}{EX} \qquad\qquad 6.3.3.4$$

which in this case becomes

$$Y = -i \; \frac{Ae^{ky} + Be^{-ky}}{Ae^{-ky} - Be^{-ky}}$$

and is pure imaginary. The advantage of using this admittance is that it is easily shown that if the admittance is Q_1 at $y = y_1$ say then the admittance Q_2 at $y = y_1 + d$ across free space is given by:

$$Q_2 = \frac{Q_1 + \tanh(kd)}{1 + \tanh(kd)Q_1} \qquad\qquad 6.3.3.5$$

This powerful result concludes this discussion. If the admittance of the lower electrode is known then 6.3.3.5 immediately gives the admittance and hence the electric field at the lower beam edge. The same result is true between the upper electrode and upper beam edge. We may then use 6.3.3.3 and the results of 6.2.2 to relate these fields to the fields inside the beam and hence deduce the dispersion relation. We note two important results. Firstly the admittance of a conducting plate is infinite and hence the admittance of the upper and lower beam edges is given by $i\coth(kd)$ and $-i\coth(kd)$ if the beam is a distance d away from the electrodes. Secondly, as a direct result of the first result the admittance is effectively unity if kd is sufficiently large. This is valid if $kd \geqslant 4$, i.e. if the distance between the electrode and beam edge is greater than a quarter of the wavelength of the perturbation.

We now combine the results of 6.3.2 and 6.3.3 to derive the dispersion relation.

6.3.4 The dispersion relation

In this section we give some consideration to the general dispersion relation and consider different forms of this relation with approximations that give an analytic dispersion relation. We return to the differential equation describing the beam and summarise it here for convenience where we have dropped the prime on V' for the sake of clarity.

$$\frac{\Delta^2}{\omega_c^2} \frac{1}{V} \frac{\partial^2 V}{\partial s^2} = 1 + \frac{\Delta(3\Delta-\omega_c)}{\omega_c^2(1-s^2)} + \frac{3\Delta^2 s^2}{\omega_c^2(1-s^2)^2}$$

We now follow and correct Buneman, Levy and Leʷson (1966). The general solution of this equation is given by

$$V = AV_1 + BV_2$$

where V_1 and V_2 are the two linearly independent solutions. The boundary conditions 6.3.3 may now be written in terms of the variables V and s as follows. We write 6.3.3.3 in the form:

$$\left\{ EY + \frac{\rho_0 V(y)}{\epsilon_0(-i\omega+ikU_{xo})} \right\}_{T,B} = \left\{ \pm iEX \right\}_{T,B}$$

where T B again refer to the top and bottom of the beam. We now eliminate EY and EX in favour of the variable V using 6.3.2.2 and 6.3.2.3 to obtain:

$$\left\{ \frac{1}{ikV} \frac{\partial V}{\partial y} + \left[\frac{2\Delta(-i\omega+ikU_{xo})}{((-i\omega+ikU_{xo})^2+\omega_c^2)} \right] + \frac{\Delta}{(-i\omega+ikU_{xo})} \right\}_{T,B} = \pm 1$$

and substituting for V and s gives

$$\left\{ \frac{1}{V} \frac{dV}{ds} + \frac{1}{s(s^2-1)} \right\}_{T B} = \pm \frac{\omega_c}{\Delta} \qquad\qquad 6.3.4.1$$

We define a function $h_T(s)$ at the top of the beam which is given by

$$h_T(s) = \frac{s(1-s^2)\frac{\Delta}{\omega_c} \frac{dV_1}{ds} - \left[\frac{\Delta}{\omega_c} + s(1-s^2)\right]V_1}{s(1-s^2)\frac{\Delta}{\omega_c} \frac{dV_2}{ds} - \left[\frac{\Delta}{\omega_c} + s(1-s^2)\right]V_2}$$

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and a similar function $h_B(s)$ at the bottom of the beam which differs from $h_T(s)$ by a change of sign in the third term of numerator and denominator. These functions $h_T(s)$ and $h_B(s)$ are defined by substituting the expression for V into 6.3.4.1 and evaluating the ratio $-B/A$. Since both $h_T(s)$ and $h_B(s)$, evaluated at the top and bottom of the beam both equal $-B/A$ we may immediately write

$$h_T(s_T) = h_B(s_B) = -B/A \qquad\qquad 6.3.4.2$$

which determines s_T and s_B and hence the complex frequency ω through 6.3.2.11.

The two solutions V_1 and V_2 are related since the original differential equation is even in s. We therefore immediately write:

$$V_2(s) = V_1(-s)$$

and examination of the definitions of h_T and h_B shows the following form for 6.3.4.2:

$$h_T(s_T) \; h_B(-s_B) = 1 \qquad\qquad 6.3.4.3$$

We may now examine possible forms for the dispersion relation using this result. If s is small, a condition satisfied by either thin beams or perturbations of a long wavelength then both h_T and h_B vary as $1 + O(s^2)$. 6.3.4.3 then gives

$$s_T{}^2 + s_B{}^2 = 0$$

which combined with 6.3.2.11 shows

$$\omega_i = \pm ika\Delta \qquad\qquad 6.3.4.4$$

i.e. the longwavelength perturbations are unstable, with the growth rate being proportional to the beam width and the velocity slip. This particular instability is readily identified and will be discussed in more detail in 6.3.5. Buneman et al (1966) give an elegant discussion of the behaviour

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of 6.3.4.3 for increasing s and we merely summarise their conclusions here.
As s increases i.e. either the beam thickness increases or the wavelength
decreases, then the system remains unstable until a critical value of s is
reached, when the growth rate vanishes. This critical value is found to be
$s = 2\Delta^2/\omega_c^2$. Further increase in s gives a real frequency and hence stability
until the next critical value is reached. At this critical value a new
instability mechanism is found, which we later identify with the gyroresonance
and the critical value is given by $1 - 2ka\Delta^2/\omega_c^2$.

These results are summarised in the following figure:
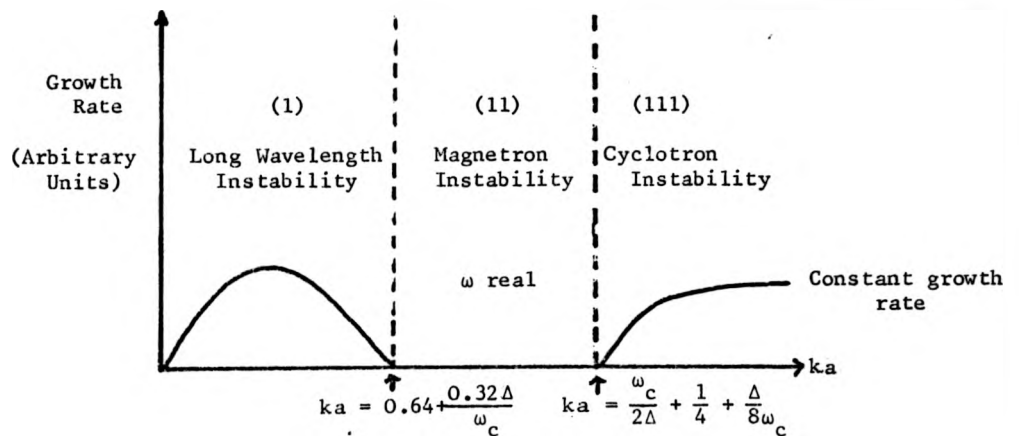
The system has been shown to exhibit instability in two distinct regions with
differing wavelengths and beam widths. The intermediate region with ω real
as shown above is not always stable. If the beam is confined by a plane
electrode and a slow wave structure than this region also exhibits instability.
We choose to call this instability the magnetron instability.

We now note the confusion that has arisen in the literature when insta-
bilities in plasmas in crossed fields are discussed. Knauer (1966), Gould
(1965), Gould (1957, Pierce (1956) derived the dispersion relation for region
1 and called the instability they found the diocotron or slipping stream
instability. (We shall see in 6.3.5 that this instability depends least on
the presence of a slipping stream). Macfarlane and Hay (1950), Buneman

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(1961a,b,1957), Chen (1959) and others derived the dispersion relation for region 3 and called it the slipping stream instability. Macfarlane and Hay (1950), Dombrowski (1957) and Gould (1957) considered the effect of a slow wave structure on the stability and derived the dispersion relation for region 2, again calling the effect the slipping stream instability.

We can therefore conclude that the crossed field instability or diocotron instability is a generic term describing a number of possible instabilities and in the next section give a discussion of the physical mechanisms leading to these instabilities.

6.3.5 Results and Discussion

We wish to discuss the long wavelength instability and the gyro instability in more detail but will not give any further consideration to the magnetron instability which has received considerable attention in the past.

To consider the long wavelength instability we initially ignore the velocity slip and let $\Delta = 0$ in equation 6.3.2.7. This gives

$$\left[(-i\omega+ikU_{xo})^2 + \omega_c^2\right]\left[\frac{\partial^2 V}{\partial y^2} - k^2 V\right] = 0$$

which has solutions

$$(\omega-kU_{xo})^2 = \omega_c^2 \qquad\qquad 6.3.5.1$$

and

$$\frac{\partial^2 V}{\partial y^2} - k^2 V = 0$$

It is more convenient to consider the electric field and, in view of 6.3.2.2 and 6.3.2.3 with $\Delta = 0$ we see that $V \propto EX$ and so we can write

$$\frac{\partial^2 EX}{\partial y^2} - k^2 EX = 0 \qquad\qquad 6.3.5.2$$

The dispersion relation 6.3.5.1 shows two stable waves in the bulk of the beam. The electrons move in circular orbits at the frequency ω_c. The other waves

described by 6.3.5.2 are waves associated with propagation in free space since there is no charge density and hence represent surface waves on the surface of an incompressible fluid.

It is a simple matter to derive the dispersion relation corresponding to the two surface waves 6.3.5.2 for a beam with a single surface. This is done by matching an exponentially decaying electric field outside the beam to a corresponding field below the surface. Using the equations 6.3.2.2 to 6.3.2.4 with the condition $\Delta = 0$ leads to the two waves with the dispersion relation

$$(\omega - kU_{ox}) = \frac{1}{2} \omega_c \pm \left(\frac{1}{4} \omega_c^2 \right)^{\frac{1}{2}} \qquad\qquad 6.3.5.3$$

The wave when the plus sign is taken is a fast wave and is a manifestation of the bulk wave 6.3.5.1 at the surface. When the minus sign is chosen, the wave is a slow wave which arises when the electric and magnetic forces oppose each other. (This is in contrast to the fast wave when the two forces act together). It is easily seen from 6.3.2.2 and 6.3.2.3 that if the slow surface wave is present that

$$V = EX/B_o \quad \text{and} \quad U = -EY/B_o$$

i.e. the motion of the electrons is pure guiding centre motion and the electrons move with their $\underline{E} \times \underline{B}$ drift velocities. This may appear to contradict the original assumption of zero velocity slip since the guiding centre motion exhibits a velocity gradient. Closer examination of 6.3.2.2 and 6.3.2.5 shows that the unperturbed velocity distribution has been retained as a slipping stream but that the derivative of the y directed velocity has been neglected. This is consistent with the conditions of incompressible flow in the beam and the presence of surface waves. The argument given above, under the assumption of zero velocity slip merely gives an indication of how an instability may arise and a more rigorous argument is given below.

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The argument above is found to be valid for a small velocity slip or a thin beam such that the gradient of V is small, i.e. $k\Delta a$ is small.

If we now suppose that the beam has two surfaces and that each surface supports such a slow wave then we find, under the assumption $k\Delta a \ll 1$, that the two surface waves interact across the beam and become unstable. This is the long wavelength instability for thick beams and small velocity gradients or a universal instability for thin beams such as considered by Pierce (1956) and Gould (1957). We can now derive the dispersion relation for this instability using the formalism developed in 6.3.3.

Equation 6.3.3.3 relates the fields across the beam edges. We substitute $V(y) = EX/B_o$ in the right-hand side and immediately see the simplification that arises, i.e. the relation is now entirely between electric fields. Since the electron beam and the space on either side are now effectively charge free we can immediately write:

$$EY^{EXT} \text{ (above beam)} = Be^{-ky}$$

$$EY^{EXT} \text{ (below beam)} = Ae^{ky}$$

Inside the beam we have

$$EY^{INT} = Ce^{ky} + De^{-ky}$$

Since the divergence of the electric field is zero we also have

$$EX^{EXT} \text{ (above beam)} = -iBe^{-ky}$$

$$EX^{EXT} \text{ (below beam)} = iAe^{+ky}$$

$$EX^{INT} = iCe^{ky} - iDe^{-ky}$$

The component of electric field parallel to the beam surface must be continuous and applying this condition at both beam surfaces gives

$$-Be^{-ka} = Ce^{ka} - De^{ka} \qquad\qquad 6.3.5.4$$

$$Ae^{-ka} = Ce^{-ka} - De^{ka} \qquad\qquad 6.3.5.5$$

Since we must eliminate the four constants A, B, C and D we require two further relations between them and these are given by 6.3.3.3 evaluated at both surfaces, i.e.

$$EY^{EXT}(a) - EY^{INT}(a) = \frac{\rho_o EX(a)}{-iB\epsilon_o(\omega-k\Delta a)}$$

and

$$EY^{EXT}(-a) - EY^{INT}(-a) = \frac{\rho_o EX^{EXT}(-a)}{-iB\epsilon_o(\omega+k\Delta a)}$$

Substituting for EY^{EXT} and EY^{INT} from above gives:

$$Ce^{ka} + De^{-ka} = Ae^{ka}\left\{1 + \frac{1}{\omega-k\Delta a}\right\} \qquad 6.3.5.6$$

$$Ce^{-ka} + De^{+ka} = -Be^{-ka}\left\{1 - \frac{1}{\omega+k\Delta a}\right\} \qquad 6.3.5.7$$

We now combine 6.3.5.4 to 6.3.5.7 and eliminate A, B, C and D to give the dispersion relation:

$$4(\omega/\Delta)^2 = (1 - 2ka)^2 - e^{-4ka} \qquad 6.3.5.8$$

This dispersion relation was derived by Gould (1955). Gould's dispersion relation was derived under the assumption of a thin beam whereas we have assumed a thick beam with a small velocity slip and long wavelength perturbations. His dispersion relation also differed from 6.3.5.8 in that instead of Δ, ω_p^2/ω_c appeared. Although the two results are equivalent the appearance of Δ in 6.3.5.8 shows that the instability is a result of the slipping stream and not a simple space charge effect as could be interpreted from Gould's equation.

For small ka the right-hand side of 6.3.5.8 vanishes at ka = 0.64. We therefore have a marginally stable state at ka = 0.64 with instability occurring if ka < 0.64. The growth rate is then given by

$$\omega = ika\Delta$$

which may be rewritten

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$$\omega_i = \frac{1}{2} k \times \text{velocity slip across beam.}$$

This result is identical to the result that would be obtained for the instability of two electron streams interacting electrostatically or of a classical incompressible fluid flow with a velocity profile. This point is discussed further in 6.4.

We can now show that the instability arises from the interaction of two surface waves. The total excess charge flow consistent with the discontinuity at the upper surface C can be written as:

$$-(-i\omega + ikV(a))\ \sigma - i\Delta\sigma e^{-2ka}$$

where σ is the excess charge. This result is proved by explicitly evaluating the coefficients C and D of equations 6.3.5.4 and 6.3.5.5 and substituting into 6.3.3.3. The dependence of this excess charge on the top surface on the second term arises from the dependence of the coefficient B on the coefficients C and D. The first term arises solely from the charge flow as a result of the surface wave on that surface. The second term arises from the electric field created by the excess charge on the other surface, i.e. if there is an excess of charge σ on this opposite surface this induces an x directed field $i\sigma e^{-2ka}$. This field produces an additional velocity perturbation $V' = -i\sigma e^{-2ka}/B_o$ which gives the excess charge flow $-i\Delta\sigma e^{-2ka}$.

In conclusion we note that these surface perturbations induce meander or sausage-like distortions in the original linear beam. If the instability is not limited then these meanders grow larger and larger, finally creating a turbulent beam.

The cyclotron resonance instability does not lend itself to a simple analysis as is possible for the long wavelength instability. We briefly discuss the behaviour of the beam equation near the cyclotron resonance defined by $s = \pm 1$ following Buneman et al (1966). By considering the indicial equation for 6.3.2.10 enables us to write series expansions for the two solutions V_1 and V_2 at $s = 1$. These series solutions are dominated

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by terms proportional to $\log_e(1-s)$. The equation 6.3.2.10 is then solved in the WKB approximation and these two sets of solutions are then matched at the points $s = \pm 1$. Expressions are then deduced for $h_T(s)$ and $h_B(s)$ which are complex functions of exponential integrals. A criterion is established for the onset of instability and an explicit expression is derived for the complex frequency:

$$\omega = k\Delta a - \frac{1}{2}\Delta + i(\pi/2e)e^{-2\Delta^2/\omega_c^2}$$

The imaginary part of the frequency is small and equal to 0.06 for Brillouin flow, a result first deduced by Macfarlane and Hay (1950).

The analysis leading to this result is complex and obscures the physical origins of the instability. At the cyclotron resonance condition $s = 1$ the wavelength of the instability is sufficiently small, so that the Doppler shifted frequency at the beam surface coincides with ω_c. The interaction takes place between the surface wave on the resonant edge of the beam and one of the compressive waves in the bulk of the beam. The surface waves then interact not through an incompressible fluid as in the long wavelength instability, but through a compressible fluid having a resonant layer. The analysis is therefore dominated by the beam dynamics, i.e. the differential equation 6.3.2.10 and not the surface waves.

6.4 Towards a nonlinear theory

We have demonstrated that the so-called crossed field instability is in fact a single name that describes three instabilities, i.e. the long wavelength instability, the magnetron instability and the cyclotron instability. The occurrence of these instabilities depends on the value of the product of the wavenumber and the beam width. For a true double-sided beam all three instabilities can occur but for a single-sided beam, i.e. with the lower edge in contact with the cathode only the last two instabilities can occur. In 6.3 we gave greater attention to the long wavelength instability than the other two since it can be discussed analytically. The other two instabil-

ities require a much more complex numerical analysis. The disadvantage of considering the long wavelength instability is that it is the least likely instability to occur in microwave devices. This is primarily because the wavenumbers encountered in crossed field tubes are beyond the value where the long wavelength instability ceases. In addition, we showed in 6.2. that the most likely beam configuration to be found in emitting cathode tubes is that with the lower edge of the beam in contact with the electrode. However, the long wavelength instability can be studied analytically and holds most promise for nonlinear analysis and so we will devote the remainder of this chapter to considering future work on a nonlinear analysis of one of the crossed field instabilities.

The growth rate of the instability was shown to be, equation 6.3.4.4 $ka\Delta$ where Δ is the velocity slip. This growth rate can also be written as $\frac{1}{2}$ x k x total velocity slip across the beam and as discussed in 6.3.5 is identical with the growth rate that would be found from a velocity induced instability between two interacting plasma streams. The instability is therefore hydrodynamic in nature and in the nonlinear regime wave-wave interactions would dominate to limit the growth (this point was discussed in the introduction to Chapter 5). The crossed field instability also exhibits a marginally stable point, a situation already encountered with the two-stream instability.

We therefore suggest that a nonlinear theory of the long wavelength instability can be formulated using the general techniques developed in Chapters 2 and 3. Indeed, the realisation that this instability is closely related to the two-stream instability stimulated the work which led to the general result of Chapter 2 and the more specific result in Chapter 5. Further the two-dimensional nature of the equations valid for this instability stimulated the general theory of nonlinear wave propagation in two dimensions as described in Chapter 3. Both these general results must now be combined and applied to the crossed field plasma equations. This analysis might

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be expected to lead to a single nonlinear equation for the wave amplitude, of nonlinear Schrödinger type, which will enable a definite statement to be made about the nonlinear stability or instability of a crossed field plasma in the long wavelength limit.   This analysis would not be complex due to the great simplification that arises in the equations describing the system under the guiding centre approximation.

We also suggest that the two-dimensional theory of Chapter 3 may be applied to the full equations of motion and so provide a theory of the nonlinear behaviour of the cyclotron instability.   This analysis would be much more complex since the solutions of the linear equations must be known to enable the analysis to be carried out.   However, analytic approximations to these solutions are in principle known and the necessary integrations could be performed analytically or numerically.

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