On the untrustworthiness of axiomatic-founded science

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Abstract

The idea of science being the best – or the only – way to reach the truth about our cosmos has been a major belief of modern civilization. Yet, science has grown tall on fragile legs of clay. Every scientific theory uses axioms and assumptions that by definition cannot be proved. This poses a serious limitation to the use of science as a tool to find the truth. The only way to search for the latter is to redefine the former to its original glory. In the days well before Galileo and Newton, science and religion were not separated. They worked together to discover the truth and while the latter had God as its final destination, the former had God as its starting point. Science is based on the irrational (unproven) belief that the world is intelligible along many other assumptions. This poses a serious limitation to science that can only be overcome if we accept the irrationality of the cosmos. The motto "Credo quia absurdum" holds more truth than one can ever realize at first glance. There is nothing logical in logic, whereas there is deep wisdom in the irrational. For while the former tries to build castles on moving sand, the latter digs deep inside the depths of existence itself in order to build on the most concrete foundations that there can be: the cosmos itself. The only way forward is backwards. Backwards to a time when religion led the quest for knowledge by accepting what we cannot know, rather than trying to comprehend what we do not. Science was anyway based on that in the first place.

Tags: science; science philosophy; irrational; axioms, foundations

Περίληψις (Summary in Greek)

Η ιδέα ότι η επιστήμη είναι η καλύτερη ή η μόνη οδός για να φτάσουμε στην αλήθεια για τον κόσμο μας, είναι μια βασική πίστη του μοντέρνου πολιτισμού. Και όμως, η επιστήμη έχει μεγαλώσει πολύ στηριζόμενη όμως σε πήλινα εύθραυστα πόδια. Κάθε επιστημονική θεωρία χρησιμοποιεί αξιώματα και παραδοχές οι οποίες εξ' ορισμού δεν αποδεικνύονται. Αυτό θέτει σοβαρούς περιορισμούς στο τι μπορεί να κάνει η επιστήμη αναφορικά με την αναζήτηση της αλήθειας. Ο μόνος τρόπος να βρούμε την τελευταία είναι να αναθεωρήσουμε την πρώτη στην πρότερη της δόξα. Στις ημέρες πολύ πριν το Γαλιλαίο και τον Νεύτωνα, η επιστήμη και η θρησκεία δεν ήταν διαχωρισμένες. Εργαζόντουσαν μαζί για να ανακαλύψουν την αλήθεια και ενώ η δεύτερη είχε το Θεό ως τον τελικό προορισμό της, η πρώτη Τον είχε ως αφετηρία. Η επιστήμη βασίζεται στην (παράλογη) πίστη ότι ο κόσμος είναι κατανοήσιμος, μεταξύ άλλων παραδοχών. Αυτό θέτει ένα σοβαρό περιορισμό στην επιστήμη που μπορεί να ξεπεραστεί μόνο εάν αποδεχτούμε τον παράλογο του Κόσμου. Η φράση "Credo quia absurdum" κρατά περισσότερη αλήθεια από ό, τι μπορεί κανείς να συνειδητοποιήσει ποτέ με την πρώτη ματιά. Δεν υπάρχει τίποτα λογικό στη λογική, ενώ υπάρχει βαθιά σοφία στο παράλογο. Γιατί ενώ η πρώτη προσπαθεί να χτίσει κάστρα σε κινούμενη άμμο, το δεύτερο σκάβει βαθιά μέσα στα βάθη της ίδιας της ύπαρξης, προκειμένου να στηριχτεί στα πιο στέρεα θεμέλια που μπορεί να υπάρχουν: τον ίδιο τον Κόσμο. Ο μόνος δρόμος προς τα εμπρός είναι προς τα πίσω. Πίσω σε μια εποχή που η θρησκεία οδήγησε την αναζήτηση της γνώσης αποδεχόμενη αυτό που δεν μπορούμε να ξέρουμε, αντί να προσπαθούμε να κατανοήσουμε αυτό που δεν γνωρίζουμε. Η επιστήμη ούτως ή άλλως βασιζόταν σε αυτό εξ' αρχής.

Λέξεις κλειδιά: επιστήμη, φιλοσοφία της επιστήμης, αξιώματα, θεμέλια επιστήμης

1. INTRODUCTION

Science today is thought to be the cornerstone of human civilization. People trust science to find solutions to their problems, to search for explanations on how the cosmos is working, even to reach the ultimate goal that was the holy grail of human philosophy for thousands of years: Truth. However, science is inherently limited and cannot play such a central role in the search for answers. This paper will present arguments to justify this proposition and will also try to explain that the untrustworthiness of science can only be supplemented by religion in its most pure form. As Einstein once postulated, "Science without religion is lame, religion without science is blind". The pages that follow will try to extrapolate on that line of thought and delve into the depths of the foundations of science, which surprisingly contain more religiosity than one would expect.

2. PURPOSE OF THE STUDY

The purpose of this study is to show that the foundations of modern science are not capable to support a method of thinking which can positively answer any of the great metaphysical questions of humanity; questions related to the "why" and not the "how" which is usually the subject of modern natural sciences. In order for us to reach an understanding of the cosmos, we need something beyond an axiomatic-based science. This paper will show how and why such a way of thinking can only be found within the realm of religion.

3. RESEARCH METHODS

The topic under analysis was examined with the help of three tools: Scientific analysis, Philosophy and History. The latter was the tool that provided evidence for how science has progressed into what it is today. That analysis provided insights on what are the main limitations of science due to the axioms it is using. These axioms were further analyzed to show what Gödel has showed many years ago: That an axiomatic-based theory can never be proved consistent or full. The analysis was performed based on the basic premise that "If something needs an assumption to be held as true, then it cannot be considered as true per se" (this should not be considered an assumption but rather a tautology: what is not proved to be true cannot be considered true). Finally yet most importantly, the research tried to find the prerequisites for a method of thinking that could overcome those obstacles and regain the trustworthiness of humanity to science as a way to reach the truth. The path revealed is described at the last chapters of the current thesis.

4. FINDINGS

The present research analyzes the foundations of science and shows that they are unreliable markers for the quest for truth. This is done by examining not only one of the major exact sciences of today (physics) but also the major tool utilized by all exact sciences to elaborate on the workings of the cosmos (logic). The results of the exercise portray a picture that is widely different than the one promoted by mainstream science today: The holy grail of modern science (the "truth") cannot be found unless we stop searching for it…

In summary, the scientific method [1] tries to examine which assumptions best fit the observed phenomena[1] and then formulate the best scientific model that could explain those phenomena in a self-consistent way. The method (e.g. logical deduction) or the tools (e.g. statistical analysis) used to perform the modeling of the cosmos are not of importance here. What is important is that for the scientific models to exist, there need to exist some assumptions which scientists take for granted. Nothing can be built based on nothing (except 'Nothing' of course, but Parmenides would have objections on whether this exists).

What will be shown is that these foundations of science (assumptions, a.k.a. axioms or principles) do not have any inherent truthfulness or validity. And if the foundations are not to be trusted to lead us to the truth, then neither science as we know it today can be trusted as well for that task.

4.1 Axioms: From Euclid to quantum physics

The Elements of Euclid constituted the first time a sector of mathematics[2] was founded on axioms. Euclid made a great work in formulating a specific set of principles on which his theory was based upon and this way of thinking dominated the western scientific thought for millennia. Even today, every scientific field from mathematics to physics and chemistry is based upon specific axioms and scientists struggle to create (or discover – depending on the philosophical theoresis you adhere to) new theorems based on those axioms.

> Terminology note No. 1: Axioms are also known as 'principles' in other fields of science besides mathematics. It is important to note that the name really makes no difference. What is important is that we speak about unproven assumptions that we take for granted when developing a theory.

It is important to note that 'unproven' does not mean that there is no evidence whatsoever for the axioms or principles used. It simply means that these axioms are not proved with the certainty one would want to declare them 'true' in the philosophical sense of the word, meaning valid in every possible case. Nobody starts pulling axioms out of thin air because he

[1] The term "phenomena" is the right term and not "fact" which is widely used in science-related articles. We can never be certain of the truthfulness of what we can experience via our senses. This is an important question that is outside the scope of this paper.

[2] Geometry and mathematics were inherently connected up to very recently.

just thought about them; usually there is a long process of thought involving both observation and induction reasoning to derive a specific axiom. (Later on, we will see how indeed this could be futile anyway, but for now let's assume that this process is of great significance) So for example, physics contains the principle of conservation of energy, which may be proven in specific cases but which has certainly not been proven for every possible case. This is clearly self-evident, since only a specific set of observations has been conducted so far. Drawing a general universal principle based on such a limited set of information (limited in relation to what we will discover one million years from now and surely limited in relation to the extent of the cosmos) is inductive in nature and can in no case substantiate the principle as a 'fact' beyond any doubt.

In other words, the fact that axioms are based on scientific observations does not mean that they are proved, as a theorem is proved in geometry for example. It is true that most axioms or principles are evidence-based, but this means nothing to a scientist – proof of something entails a long and painful process that would make certain that something holds true in every possible scenario. Even though the conservation of energy is something that seems to be true based on what we have seen so far in the physical systems we have examined, we cannot possibly be absolutely (100%) certain that this principle is true in the whole universe or at every possible circumstance. Our "facts" are based – at best – on circumstantial evidence; examining all the possible physical systems in the whole universe is something we will never accomplish anyway. Hence, the conservation of energy is a 'principle' and not a 'fact'.

<u>Terminology note No. 2</u>: Principles of physics (and other areas of science) are many times called 'laws'. This can be really confusing to the uneducated reader, since the word 'law' implies something that has been proven to exist in the whole cosmos (universal laws). One should always keep in mind that regardless of the name used, we are still speaking about unproven assumptions that we take for granted when developing a theory.

Nevertheless, axioms or principles are a basic and fundamental element for science. They offer the foundations on which theories are built and evolved. Without these foundations observations are just set of meaningless data. Axioms offer the context in which a theory is built; the basic building blocks on which the whole structure of the theory is erected upon. Change those axioms and you end up with a different theory based on exactly the same observational data. It may be the case that some new evidence disprove one axiom/ principle, but in that case another axiom takes its place for the whole structure of theories to still be able to stand tall. Needless to say that this is neither a bad nor a good thing. It is just how science works. Any journey needs a starting point and these axioms offer that point of reference from which one can watch the cosmos and derive meaning from it.

This section will first examine the use of such axioms and principles in various cases of scientific endeavor. I will then show that such axioms cannot be reliable foundations for the philosophical quest for the truth.

Important Note: The term 'axioms' and 'principles' will be used interchangeably in this paper depending on the context. In essence, even though the former are used in mathematics and the latter in exact sciences, they denote the same thing: unproven assumptions that we use as a starting point to build a theory.

4.1.1 The case of the 5th axiom

The fifth axiom of Euclid (also known as the "parallels'' postulate") was accepted for centuries as self-evident [2]. Why shouldn't it be anyway? It is self-evident (this is a key phrase used for all axioms) that parallel lines never meet. And it is also self-evident that from one line only one parallel line can be drawn. It is quite interesting though that people – from the very beginning – felt quite uneasy with that axiom (and not the other four). Somehow, they felt (intuition?) that something was wrong with this axiom. That is why there were many attempts to deduce it from the other four as a theorem, with all the efforts failing. So the years passed and for centuries after centuries this axiom was part of the geometry everyone knew and used.

What is important in the case of the Euclidian geometry, is that geometry was not just 'geometry' back then. It was the tool used for doing math as well. Long before algebraic notation, mathematicians used geometry to calculate the squares and cubes of number simply by designing… squares and cubes. In that sense the Euclidian geometry and its validity was often correlated with the axiomatic validity of the known mathematics. This idealistic picture was soon to change with changes not only in mathematics, but in geometry as well.

Eventually, it was proved that the 5th axiom cannot be proved, i.e. that it was indeed an axiom and not a theorem based on the previous axioms. This opened the path towards other types of geometries in which we either have more than one parallel lines to a certain straight line, or no parallel line at all. Non-Euclidian geometries soon gained momentum and many practical applications to various sections of science, like in the Theory of Relativity in physics, were revealed. Again, the details of the other geometries are mute. What is of essence here is that the change of an axiom is not a matter of change in the observable phenomena, but purely a matter of choice. Can you imagine of a geometry where there are only three parallel lines to a straight line? There you go! Congratulations on your new geometry!

Axioms turning into dogmas.

What does the above example show us? In short, the obvious: The more one thinks of something as self-evident the harder it is for him to discard it. It took us literally thousand years to change the 5th axiom of Euclid and yet, the moment we did, the new way of thinking bred many new children. What was initially the basis of scientific thinking, ended up hindering scientific thinking. This is expected and totally human in nature. Our mind is very prone to prejudice. One needs to actively pursue being open-minded in order to be so.

How can we change axioms though? This is a question that could baffle people outside the domain of science and yet it is one of the easiest questions to answer. Quite simply, an axiom can change if you just change it. There is no other way to select a different starting point for your theories besides selecting a different starting point for your theories. It sounds tautological in nature and it is; tautology is anyway the only truth we can be certain about from a philosophical point of view. This way of changing an axiom was used when the 5th axiom of Euclid dead-end was reached: In an extreme simplification of the story, people simply decided to see what would happen if they opted for a different thesis regarding the existence of parallel lines and new geometries emerged based on these new options. The new axioms were as unproven as the old one. And yet, they too helped build a new theory which seemed consistent (waiting for Gödel to prove otherwise – see below) and had practical applications.

The open questions regarding what is valid and what is not valid were just made more intense with the new 'discoveries'. If it is so easy to change a basic axiom and produce equally valid theories, then what does that imply for the truthfulness of the axioms per se? Are they just ad hoc selected cornerstones for the theories we want to build and which can be changed with others at will? Are we limited in any way regarding their selection? If such a change can easily happen in one of the most important and fundamental areas of science what does this mean for other areas where the basic principles are still in open debate? One could argue that other areas of science – like physics – are based on evidence; but again wasn't the parallel lines axiom based on purely empirical evidence that two parallel lines never meet? And let us not forget the practical applications of the (now) three different geometries we have, indicating a strong connection of the theory to actual life which cannot be easily ignored.

Having in mind all of the above, it is important to examine the other basic cornerstone of science besides mathematics: Logic.

We must forget how to think in order to think!
~ Harmonia Philosophica

4.1.2 Logic as Organon

Aristotle founded Logic as an Organon (Όργανον, Greek for 'tool') in his work with the matching title ("The Organon") for the first time [3]. In this work he managed to document the basic types of logical syllogisms and formulate the basic rules with which these syllogisms could produce valid conclusions based on some initial premises. This work helped formulate the basis of mathematical logic for the next centuries, only to be amended with new elements rather recently. Even today when we think logically we instinctively use these rules, which are generally so much accepted and embedded in our thinking that are almost automatically seen as valid. However, as in the geometry axioms mentioned above, extra caution should be exercised whenever something is seen as self-evident.

Despite the extensive use of logic from the time of ancient Greece until today, the actual usefulness of logic eludes us. Even Aristotle himself did not know how this 'organon' would be used for. For some it would be seen as a linguistic instrument to drive conclusions without any relevance to the philosophical concept of the "truth" whatsoever; a merely game which allows us to create new conclusions based on the (potentially erroneous) input we give to its mechanism. Anyway, it is true that logic is as good as its premises. For others, logic is the basic tool to reach the truth and the high peak of human thought throughout our history on this planet. To understand what is the case we need to take a step back and see logic from a different perspective.

If we examine logic more carefully, we will discover that even logic has axioms on its own. For example, the Law of Excluded Middle (LEM) is one of the basic laws used not only by Aristotle but also today by modern mathematics: In essence this law (remember, people have the bad habit to refer to axioms/ principles as "laws") states that a logical proposition is either True or False (but never both or neither). This sounds like very logical and common sense, however in science nothing is (or should be) common sense. That is why this principle is used as an axiom in logic; it was also the basic element of Peano arithmetic, which is what we essentially use today in mathematics. Brouwer however chose not to accept this as self-evident in his intuitionistic mathematics and in this way attempted to build a different area of mathematics, even though historically this did not work out as intended due to mainly non-mathematical reasons.

4.1.3 Axioms in modern mathematics

As mentioned above, modern mathematics are based on axioms; with the more important ones being related to mathematical logic. Those axioms played (and still play) a major role in the attempts to formulate the foundations of mathematics in logic, an attempt the beginnings of which many trace back to Russel (Logicism). These efforts – regardless (or perhaps because) of their inconclusiveness – were very fruitful in seeding the scientific thought with perspective regarding what is or what could qualify as a founding principle for a specific field of knowledge.

This debate over the foundations included many other principles of mathematics. For example regarding set theory – initially proposed by Cantor and regarded as one of the building blocks of modern arithmetics – there were (and still are) many discussions regarding the overall validity of the set theory and its premises. Cantor started his quest for the theory by doing what every scientist should to: Question the obvious. And the obvious in this case stated that the whole bigger than the parts. This simple and 'self-evident' truth was at the end discarded; we now 'know' that the set of e.g. integer numbers is not larger than the set of positive numbers which are just a part of the first set. The counter-intuitive basis of Cantor's theory caused great dispute, with the most infamous one being the dispute between Kronecker and Cantor. The results of their clash resulted in the modern set theory being accepted across the mathematics society, something we today take for granted. It is interesting to note here that foundations forming with a bang usually end up being accepted in silence by the generations to come.

Questioning the obvious is hard in the context of this silence. After the foundations of algebra took shape, modern mathematics based on the set theory were also formulated. But the Set theory - as any other - has axioms. Axioms leading into paradoxes (see Russel's paradox). In set theory, Zermelo–Fraenkel set theory, named after mathematicians Ernst Zermelo and Abraham Fraenkel, is an axiomatic system that was proposed in the early twentieth century in order to formulate a theory of sets free of paradoxes such as Russell's paradox. Today, Zermelo–Fraenkel set theory, with the historically controversial axiom of choice (AC) included, is the standard form of axiomatic set theory and as such is the most common foundation of mathematics. Zermelo–Fraenkel set theory with the axiom of choice included is abbreviated ZFC, where C stands for "choice", and ZF refers to the axioms of Zermelo–Fraenkel set theory with the axiom of choice excluded [4].

There are many equivalent formulations of the ZFC axioms; for a discussion of this see Fraenkel, Bar-Hillel & Lévy 1973. For illustration purposes only, a set of ZFC axioms is documented below. The following particular axiom set is from Kunen (1980). All formulations of ZFC imply that at least one set exists. Kunen includes an axiom that directly asserts the existence of a set, in addition to the axioms given below (although he notes that he does so only “for emphasis”) [4].

ZFC axioms list [4]

- Axiom of extensionality
- Axiom of regularity (also called the axiom of foundation)
- Axiom schema of specification (also called the axiom schema of separation or of restricted comprehension)
- Axiom of pairing
- Axiom of union
- Axiom schema of replacement
- Axiom of infinity
- Axiom of power set
- Well-ordering theorem (the Axiom of Choice is equivalent to that)

One can easily find out the axioms of any other theory; they are indeed a fundamental element for every theory anyway. As mentioned already, their existence per se implies nothing regarding the validity of the theories built upon them. All theories which have their foundations on such unproven theses can and do have practical applications.

After someone realizes the plethora of axioms used by a theory, some ‘simple’ and ‘obvious’ questions come to mind: Can there exist a theory that is based on arbitrary axioms and not have such practical applications? Should we choose the axioms with a specific ‘common sense’ (or instinct) so as to have ‘valid’ theories? Is there a limitation on how we choose axioms or is it that ‘everything goes’, to paraphrase Feyerabend when he was talking about the anarchy which rules the scientific method?

By examining other realms of science and, especially, evidence-based ones will allow us to easier detect the methodological limitations (or even worse, the inexistence of such) of modern science.

4.1.4 Axioms in modern physics

For many, mathematics is simply a tool and not a science per se, at least not as physics is. The first can be based on arbitrarily selected axioms but the latter is based on observations and evidence. This picture is totally false. By examining how physics formulates and adheres to specific principles it will be made evident that physics also has its own axioms that can be changed much more easily than many people believe. To exhibit that, we first need to show the way physics selects these principles and show how potential changes in those principles can indeed lead to equally valid pictures for the cosmos, in a similar way as the change of axioms in geometry resulted in equally valid geometries.

To begin with, physics is – as geometry – based on several assumptions. These assumptions, when used widely are promoted to what we call 'principles', which play the similar role that axioms play in mathematics. Some of these principles are difficult to detect since most of them (especially the fundamental ones) are used in every piece of research without any explicit reference to it. These principles usually reflect specific philosophical dogmatic beliefs, which for many – especially those who adhere to them – seem self-evident.

As a side note, it must be noted that the principles on which modern science is based are often called 'presuppositions', like in the PEL (Presuppositions, Evidence, Logic) model [5]. The nomenclature though does not have any effect on the essence of the matter in hand. These presuppositions/ principles/ axioms are all some unproved statements which are taken for granted and on which we build the theories we build. From here on this paper will refer to the presuppositions as either axioms, assumptions or principles, recognizing that the word 'axiom' is mostly related to mathematics and geometry. The reader should be aware that what we are trying to do here is to examine the essence of the connection between science and the truth. In that quest, words are a hurdle that we must sooner rather than later overcome.

> Terminology note No. 3: Principles, axioms, assumptions or presuppositions are used interchangeably in this paper from hereon – all implying unproven assumptions that we take for granted when developing a scientific theory.

Assumptions that underlie any science – and physics in particular – may include:

(i) A world external to mind exists;
(ii) This external world is not chaotic, it is organized;
(iii) The external world is knowable (intelligible), including the world beyond appearances; and, last but not least,
(iv) (iv) This external organized world is only material. [6]

These principles need to be accepted in physics in order for any scientific endeavor to have meaning. One cannot start measuring things in a lab if he does not believe that all things are measurable. There is no reason for the scientist to start analyzing the cosmos if he does not believe that there is some kind of order in what he is analyzing. There is no meaning in doing physics (and science in general) if you do not believe that your limited view of the cosmos has any validity whatsoever. There is no point in analyzing matter to discover how the universe works, if you do not believe that the universe is created by matter and matter alone.

If these seem very similar to specific philosophical ideologies, you would be right.

Philosophy is in any case the mother of science.

And the latter, as any spoiled child, want's to ignore (or kill, if possible) its parents and behave as if it has done everything on its own. However it is this oblivion of the foundations of science which makes it so arrogant as to believe that it is founded on nothing more than 'facts'; a current worldview which gives birth to the monster of modern scientism. Taking the abovementioned principles one by one, one can see that the basis of science is faith. Not faith in a supreme being, but in everything that makes the existence of such a being necessary: The existence of eternal universal laws, a universe in order and the ability of tiny humans to understand and comprehend the above. Newton did believe that he could understand the cosmos and he blatantly admitted that what he did was nothing more than reading the mind of God himself.

Planck did mention that if religion aims at finding God, science starts from the belief that God exists. The principles selected are selected based on nothing more than philosophy and saying otherwise would just blatantly ignore the elephant in the room: There is no way to 'prove' that the universe is comprehensible (intelligible) and that our limited view (via our limited senses) is correct or not. There is no way to 'prove' or consider as 'self-evident' that us 'unimportant' (based on the modern materialistic view) humans can in any way confer any truth from what we see with our limited senses and limited brains. Science accepts that our senses work properly. Every aspect of science accepts that the cosmos is intelligible [7]. All scientific work is based on our belief in the intelligibility of the cosmos [8] and (this goes without saying) our ability to comprehend the cosmos [9]. If we didn't believe in that intelligibility, there would be no starting point at all to collect evidence on which science is based upon. If we did not believe that the cosmos is intelligible there would be no sense in trying to understand it anyway. And yet, for philosophers, the very notion of our ability to sense the cosmos is under scrutiny with no specific conclusion drawn yet. Such principles make as a whole the idea of scientific realism [5] which is currently the 'dark matter' of modern exact sciences.

The principle that all things we need to know are measurable is yet another axiom (principle) on which modern science (especially exact sciences) is almost explicitly based upon. But who can verify that what we need to know is indeed measurable? This idea is strongly related to the notion of materialism which transcends all modern science. Still, many – if not most – physicists today work in experiments while taking this principle as obvious. Another principle strongly related to the previous one is the mechanistic view of the cosmos. Even though it is centuries old and obviously not in any way proven, all scientists today look at the cosmos as if it was a well-designed machine the inner workings of which we can understand by analyzing its parts. And let's not even go to more fundamental principles for which philosophers still debate: For example are our senses a valid window to the cosmos? Do they work in a valid manner that could lead to our truthful understanding of the universe? Or are they a distraction from the truth of the cosmos, a distraction which makes us focus on phenomena rather than the essence of existence? The catalogue could go on forever.

And there are many more principles to add to the above. The fact that science today does not mention them explicitly makes it extremely hard to detect them, but they are there. Poisoning our thought to the point that we believe that it is not poisoned. Take for example the simple notion of 'Things change'. If Parmenides was alive he would argue that this is an assumption we should not take lightly[3]. But science does believe in the notion of 'change'. If it did not there would again be no meaning in doing science. (Religion on the other hand is very close to the eternal view of One made by Parmenides – this will be examined further in the latest chapter) The existence of time is another basic axiom (principle) used by modern physics. Even though time is one of the most elusive philosophical notions stirring wide debates among philosophers on its existence, physics takes it for granted and attempts of scientists to disprove its validity (e.g. Gödel who eliminated time in one of his proof for General Relativity in a revolving universe) are just considered as interesting and yet meaningless games.

The basic idea that all the above try to illustrate should be evident to the reader at this point. Non-proven philosophical principles are currently driving the quest for knowledge and we should be very careful as to how we use those principles.

Humans should try to think freely. But is this possible? Can we ever think without any axioms? Can we formulate science without principles? The thesis of this paper is that we can by using science. Although this science was not actually called that way a long time ago…

4.2 Questioning the assumptions

How axioms/ principles change in order to give birth to a new theory is a matter of debate. Some believe that the changes can only happen within a very limited frame set by our experiences and the observational data at hand. Extreme voices speak about our ability to arbitrary select axioms. The truth – despite to what one might expect – lies not in the middle.

4.2.1 Which assumptions? There are no assumptions!

As already mentioned, axioms are not provable principles which we use so, by definition, they cannot be not related to what we call truth. The notion of truth for philosophy has been referring to eternal certainties and one can never be certain if the axioms he uses are related to anything true. It might be the case that in a specific case an axiom is accidentally related to the truth, but this would only be accidental. There is no way to systematically and scientifically know whether a specific axiom/ principle holds any special connection to the notion of truth

[3] According to Parmenides, change is just an illusion. For the great thinker there is no way for A to change to B, for if it does then it wouldn't still be A. And there is no way for something which does not exist at all to come into existence or for something which exists to perish into nothingness.

whatsoever. And what is important is that the non-existence of this relationship does not affect at all the use of the theories built on those principles.

Some examples will better illustrate this.

4.2.2 Questioning the obvious

Science today is based – as mentioned already – in many assumptions. These assumptions (presuppositions) are so fundamental that rarely do people question them. Isn't it obvious that you see a table? Isn't it obvious that crossing the street without looking for incoming cars might get you killed? Do you need more evidence or proof to say that "this chair that you see with your own eyes" exists? Discussions about such things usually end up in arguments using the 'obvious' as evidence of the obvious. Science is based on some very few presuppositions which in any case are closely related to what we call 'common sense' [5]. Can or should we question common sense? Even children have it [5].

But the obvious can be a trap for free thinking.

Are we to trust a kid's opinion on the cosmos because it is pure? Fine. But let us also trust its opinion that unicorns exist (by the way, as Parmenides said, we wouldn't be able to even talk about them if they did not exist). Are we to trust common sense on the evidence we have and their interpretation? Should we trust this common sense when it cries out that someone must have created this cosmos? Are we to trust our sensory input? But what happens if someone else has a different sensory input? Are we just going to discard people with different common logic as 'crazy' so as to verify our own perception? Can we say that our senses reveal anything about the cosmos when they are so limited? Touch a table. Feel it. And yet, the table is consisted of atoms which are essentially empty. What does that say about the validity of the sensory input that we get? Touch that table again. Do you sense the bacteria crawling on it? If not, what does this mean about the validity of this sensory input? Are we to trust our belief into an objective cosmos and laugh at those who claim that this cosmos is just an illusion? If we have no solid proof, there is no room for laughter. What is common sense today is history tomorrow. Quantum mechanics has provided many counter-intuitive examples which totally break down what we would call 'common sense'. And no, we cannot use the argument 'if we question our senses then we do not have science' as an argument. We cannot have the result (science) as an argument to justify itself.

Let us continue the analysis by questioning some more obvious premises of science.

4.2.3 Are there axioms of higher validity?

As already mentioned, axioms do not hold any special connection to the notion of truth. Having said that, one would wander what could be the limitations in selecting those axioms be. And if he is unable to find such, he would be right: There are no limitations. Any set of whatever axioms or principles can help us build a new theory (by logically deriving theorems and theories from those axioms). However, can arbitrarily selected axioms lead to anything of use? Could randomly selected axioms lead to the formulation of a useful theory?

Even though axioms and principles are seen as non-provable propositions, nevertheless there has always been a common belief among scientists that they must have some connection to nature and reality. (by the way, the belief in the existence of an objective reality is another major assumption of modern science) For example, when recently three physicists discovered a way to connect eigenvectors and eigenvalues [10], thus contributing in a major way in the field of mathematics, this was hailed as another proof that mathematics must in any case be connected with nature. The same applies for physics, which is much more interrelated to nature than mathematics.

To examine the validity of the proposition "All axioms must have a connection with nature to produce useful theories" (where by nature we mean the connection with reality) we must find some counter-examples. If we do find cases of theories which do not have any connection whatsoever to nature and yet produce useful theories, then we would have shown that having a connection with nature is not a required component for a practically useful theory.

Such examples were already analyzed in the previous section and we can easily find more. For example the mathematics we know have multiple practical applications and they contain negative numbers, which we learn to handle from first grade. The mathematics we know have multiple practical applications and contain irrational numbers, which we learn to handle from fifth grade. And yet, there is nowhere in nature that one can find -1 oranges…

Besides the above, it is true that if we follow the modern scientific paradigm on the value of random processes in producing valid results (e.g. random mutations play a vital role in the creation of life, at least according to the main paradigm in biology, always through the rules of natural selection) or on the possibility of creating the whole cosmos from random fluctuations (see for example the theories which want the cosmos to be created from random quantum fluctuations of the void of space), we can easily think of random axioms being the seed to create something really powerful and magnificent. If random fluctuations can produce a cosmos with universal laws science can discover, why deny the possibility that randomly selected axioms can create a useful theory? (something which in any case is much less difficult to exist than a cosmos so finely tuned and with so universal laws as our own)

To this, we must add another important note: The notion of 'usefulness' is something that is subjective. Even the most useless of theories for me could be extremely useful for someone else and vice versa. Surely today's materialistic civilization finds modern science theories as useful, but again a more spiritual civilization could render all those theories as completely and utterly useless. Having accurate GPS signals does not solve human everyday problems anyway.

Last but not least, why in any case should we use the criterion of the 'usefulness' in the first place? (remember, questioning everything also entails questioning yourself) The most critical and important things in life are useless. Or at least we do not currently know their usefulness. Are our grandparents useful? Are we useful to anyone else in the cosmos? Is life per se useful? Why should an axiom be judged scientifically based on its usefulness? Would that be a scientific argument to begin with?

4.2.4 Research and intuition

For Husserl, it is clear that the ultimate principles must be epistemological principles that are a priori eidetic laws (Husserl 1984, 235–36). Husserl used a rather unambiguous terminology in order to point out that there is one principle that is more fundamental than all the others. This is his principle of all principles. As Husserl said, "No conceivable theory can make us err with respect to the principle of all principles: that every originary presentive intuition is a legitimizing source of cognition, that everything originarily (so to speak, in its "personal" actuality) offered to us in "intuition" is to be accepted simply as what it is presented as being, but also only within the limits in which it is presented there". In other words: Every originary presentive intuition is a source of immediate justification [11]. This is very close to what was briefly mentioned in the beginning regarding the epistemological anarchy proposed by Feyerabend. And was also the cornerstone of the vision of Brouwer regarding the foundations of mathematics.

Poincare was also a proponent of the use of intuition in science [12]. The pre-intuitionists (in which Poincare was included, others use the term semi-intuitionist [12] [13]), as defined by Luitzen Egbertus Jan Brouwer, differed from the formalist standpoint in several ways, particularly in regard to the introduction of natural numbers, or how the natural numbers are defined/ denoted. For Poincaré, the definition of a mathematical entity is the construction of the entity itself and not an expression of an underlying essence or existence. This is to say that no mathematical object exists without human construction of it, both in mind and language [14].

This sense of definition allowed Poincaré to argue with Bertrand Russell over Giuseppe Peano's axiomatic theory of natural numbers. For example, there had been discussions on whether the property of induction is valid or not.

The five axioms of Peano are [15]:

1. Zero is a natural number.
2. Every natural number has a successor in the natural numbers.
3. Zero is not the successor of any natural number.
4. If the successor of two natural numbers is the same, then the two original numbers are the same.
5. If a set contains zero and the successor of every number is in the set, then the set contains the natural numbers.

The fifth axiom is known as the principle of induction because it can be used to establish properties for an infinite number of cases without having to give an infinite number of proofs. In particular, Peano's fifth axiom states [15] [14]:

- Allow that; zero has a property P (= it belongs to natural numbers);
- If every natural number less than a number x has the property P then x also has the property P.
- Therefore every natural number has the property P.

This is the principle of complete induction, which establishes the property of induction as necessary to the system. Since Peano's axiom is as infinite as the natural numbers, it is difficult to prove that the property of P does belong to any x and also x + 1. What one can do is say that, if after some number n of trials that show a property P conserved in x and x + 1, then we may infer that it will still hold to be true after n + 1 trials. But this is itself induction. Hence, the argument is a vicious circle. From this Poincaré argues that if we fail to establish the consistency of Peano's axioms for natural numbers without falling into circularity, then the principle of complete induction is not provable by general logic. Thus arithmetic and mathematics in general is not analytic but synthetic. Logicism thus rebuked and Intuitionism is held up [14].

To cut the long story short, the issues debated by intuitionists and constructivists in relation to the validity of long held assumptions in mathematics are well-known [16]. The discussions surrounding this matter are extremely interesting for someone to follow; there have been for example cases where there is debate of whether using the Platonic system of numbers we use (instead of the intuitionistic one) results in our difficulty in describing the notion of time in physics [17]. The purpose of this paper is not to describe them all, but just to document the very existence of such debate even for many self-evident and widely used principles.

No principles can be accepted without faith in their validity (again, this is not an argument but a tautology actually). And this faith by itself is based on intuition. If we are to trust the intuition we have however, then we must come in terms with some very uncomfortable truths regarding science that could lead us – temporarily – away from it. Truths which can result from the decomposition of science as we know it, based on the various tools we currently know (like induction) and its re-composition based on a much 'cleaner' view of things as they are. In metaphysics, we anyway deal with things that exist 'as they are'. And the main sector of human thought which deals with metaphysics has always been religion [18].

It could be that the correct path is not to discard science, but to rediscover it.

The next lines of this paper will discuss how this can be performed.

4.3 Irrationality as the path to follow?

The question is simple. And, thus, as all simple questions, exceptionally difficult to answer.

How can mathematics and science progress and change paradigm in order to be able to reach the truth? The answer could be more irrational than we might have thought, but because of this, much more logical than it could ever be if we chose to follow logic. Because in any case logic will have rules, which you will have to accept on the irrational basis of belief in them. Whenever you start using assumptions then the whole system you have developed will always be in question. But the truth cannot be in question. If you use axioms and principles that cannot be proved, then your theorems will never be proved as well. But can the truth be left unproven?

Modern man understands that the above lead to a different path than the one we have taken thousands of years ago. And he is just afraid to follow that path because of the uncertainty it poses. (do not underestimate the more practical reasons which have a lot of times hindered scientific progress: the fact that all science and scientists today are based on these assumptions that we now question! It is hard to get funding to research on why all current research could be obsolete…)

Questioning things is a difficult path that few can or are willing to take. It is much easier to stand behind unproven assumptions and then move on with confidence as if you have proved everything. Modern civilization's arrogance along with deep hostility towards anything religious, have fed the monster of scientism to a level that it is no longer distinguishable from science per se.

And we must kill the beast if we are to go back to the right path…

4.3.1 Untrustworthiness of axiomatic-founded science

All justifications of science using assumptions like the ones mentioned above are essentially using the same reasoning: "These things are common sense. If we cannot trust them, what can we trust? Even kids can understand that these things are true".

But are we really ready to go to kids for the foundations of science? Children, with their raw wisdom, can be truly terrifying for modern scientists. Because kids can be irrational. And as they can "know" that 1 + 1 = 2 (something which took more than 300 pages to Whitehead and Russell to prove by the way, again based on unproved assumptions/ axioms) they can also easily know that unicorns exist.

People today do not believe in unicorns or miracles.

Because we are too logical to know life.

But we do believe in assumptions which lead to research funding…

Because we are too vulgar to know death.

Others could also argue: "We could not even have science if we questioned such fundamental things as the input of our senses". This especially is a very common and popular argument against any attempt to question the assumptions of science. This argument is based however on the dogma that science (as this term is referring mainly to the 'exact sciences' today) is and should be the only tool to search for the truth (a.k.a. 'scientism', which we must remember that it is a dogma and by no means a proven position).

The answer this argument is simple: So?

Why should we not accept the obvious because we could not then have science? Yes, the fact that we cannot prove the validity of our senses for sure makes us more anxious. But, so what? Should that make us not accept the elephant in the room?

It is crucial that we not lose sight of the questions science cannot ask, let alone the ones it cannot answer [19]. If we try to free our thought from axioms and dogma's, then perhaps we should also try to remove the biggest dogma of them all: that science, as we know it today, is the only method of thinking that can lead to answers.

Searching for the "new" science that could overcome the limitations of existing science is the next logical step in our quest.

4.3.2 The vision of Brouwer revisited

In mathematics, the battle between Platonists, formalists and constructivists/ intuitionists in mathematics still rages on. Even though the formalists have officially won (there is no school today questioning the law of excluded middle or teaching that formal mathematics may not lead someone to valid proofs), specific elements of other ways of thinking still persist. We are still taught that we should construct our proofs, we are still watching in awe when a mathematician finds out a new theory or develops a new idea (see for example the Riemann hypothesis [28]) by intuition alone and others struggle for centuries to prove his thesis (see for example the Riemann hypothesis). Despite the use of irrational numbers and negative numbers every day, we still discuss about philosophy and the idea of truth and reality, regardless of the practical usefulness of such constructs.

Platonists – Constructivists – Formalists definitions: Platonists believe that the 'truth' is somewhere 'out there' for us to discover. On the other hand, constructivists believe that only what we can practically construct is worth analyzing and talking about (that is why for them the notion of infinite has no meaning – the same goes for irrational numbers). Last but not least, the formalists care for neither of the above! They only care about the rules of the game. And as long as you set the rules, then you can play… [20] [24] [26]

In Brouwer's original intuitionism, the truth of a mathematical statement is a subjective claim [21] [25]. Accepting this should be our first step towards realization of the need for a new science. What is obvious is the source of all potential errors. The presuppositions one uses are really dangerous. You cannot easily question what you have been taught to take for granted.

However, the vision of Brouwer is limited. It reaches up to a point, but fails to reach the final destination. For us to reach our destination, we must let go of all the restraints put on us by thinking in general. As for Brouwer the Pre-Intuitionists failed to go as far as necessary in divesting mathematics from metaphysics, for they still used principium tertii exclusi (the "law of excluded middle") [14], we must say the same for Brouwer himself.

Even though he goes a long way in discarding basic assumptions, he still uses the most basic of all assumptions: That thinking itself is something that can lead to valid conclusions! Brower and his followers afterwards tried to formulate arithmetics without the axiom of the excluded middle [22]. In the same way, we must try to formulate science in general without the axiom that scientific thinking itself is what is needed to understand the cosmos!

It is a daring attempt, but true philosophy should not be deterred in the face of difficulty. I would say that the opposite is true: Whatever seems like the hard path, this should be the path we should try to follow.

4.3.3 Credo quia absurdum! (Shestov)

Shestov, one of the greatest thinkers of modern time, spoke eloquently about the irrational. His objections to logic were all too logical to deny. “If someone requires you to adhere to his logic in order to understand his arguments, then what value can those arguments have?” he wandered [23]. This is something so obvious that we tend to forget it as we speak about science today. Every theory we have is based on axioms and yet we always fail to truly understand what this simple fact means.

What is logical for you could be illogical for me and vice versa. And if this happens, this will not be due to a caprice, but due to fundamental differences in the assumptions we make when we start analyzing the problem at hand.

For Shestov the motto “Credo quia absurdum” holds more truth than one can ever realize at first glance. There is nothing logical in logic, whereas there is deep wisdom in the irrational. For while the former tries to build castles on moving sand, the latter digs deep inside the depths of existence itself in order to build on the most concrete foundations that there can be: the cosmos itself. And in that sense, one can only believe something not because it is not irrational (i.e. it is rational) but exactly because it is irrational!

What I mean with that will be more evident later on.

For the moment, just keep the idea lingering in your head…

To go back to the problem of the scientific foundations, it seems we are in a stalemate since there is no way to determine which assumptions are the best to use. Yet, this is one of the cases where the difficulty to answer a question denotes a third option we rarely consider in today’s knowledge-addicted civilization…

4.3.4 Religion as non-axiomatic Science

When a question – like the one above – is hard to answer, one should consider the obvious. Besides the various possible answers to the question, there might be another solution to the problem: Perhaps the question itself is wrong! Science is a way to interrogate nature [5]. And this can and should be done with no constraints whatsoever. Many people claim that science is great because it is based on so few assumptions [5]. Why not make it even greater and base a new science on zero assumptions?!

At the end, the meaning and the purpose of the cosmos are not to be analyzed with science. The truth is not to be seen with our eyes or sensed with our touch. As Jung wisely pointed out, our self is the only element of the cosmos that we experience without the mediation of anything. And this is the only truth we can ever lay our hands onto. Could it be that this is the only truth worth pursuing? Only time will tell. If time exists after all. At the end, what is certain is that we must question everything in order to reach the truth – if such a notion even exists. And this questioning should consider nothing as self-evident, not even our logic or senses. This seems like a heavy toll to pay. And it is. But the destination is of such importance that this toll is relative small in exchange.

Religion is a way to view the cosmos without assumptions or presuppositions. In that sense, it is much more scientific than science as we know it today. It just entails accepting the cosmos (weirdly enough in the same way science today accepts the intelligibility of the cosmos). Pure thought can only be possible without thought. As Shestov once explained, whenever we try to understand something we destroy it in our effort to make it fit into those little boxes we have built in our brains.

Religion is about accepting what we know. It is not about analyzing and understanding, but about being part of the cosmos as it is. Every scientific theory is based on unproven axioms, thus nothing. And this 'nothing' is nothing more than our acceptance of the obvious: The cosmos is here for us to know. And we can know it only because we are part of God himself! Religion has said that a long time ago. We have believed in our godly nature for thousands of years. Until we started denying our self. And resulted in a cosmos void of any meaning, except the comical belief in our science. There is no logic entailed in believing. Only the acceptance of the things your heart has already seen. We seek to understand the cosmos, but to paraphrase Pascal, we wouldn't be searching for understanding the universe if we hadn't already known it…

The eternal mystery of the world is its comprehensibility…
The fact that it is comprehensible is a miracle!
~ Albert Einstein [27]

5. CONCLUSION

I went for dinner with some friends from work once. At some point someone mentioned that a university did some empirical research to determine whether a tree falling in a forest void of humans will actually make any sound. I do not know whether that was a true story. I do know that everyone laughed at that though. And then, they kept on eating. But I wanted to cry. We are so much accustomed to our assumptions that even at the slightest hint of them being wrong we just… laugh!

Imagine two small worms living all their lives inside the earth. Discussing about philosophy. Being so positive that their senses are telling them all they need to know about the cosmos. Laughing at the possibility that this is not the case.

Questioning the validity of science or even our very senses seems like an exaggeration. But so is the quest for truth. If we want to play ball in the big league, then we should be prepared for some heavy loses. At the end, we might lose what we cherish the most (science), only to discover that all this time we missed something much more important.

Plato once told us about a cave. And we laughed at those inside the cave. But we never understood that we should also laugh at those outside the cave. If those inside are not getting valid inputs, what makes us so certain that those outside do? Why should the shadows inside the cave be invalid but the shadows outside it valid?

We have used to see science as the only way out of the cave.

But we have forgotten that it was originally called religion…

We have accustomed to think that thinking is the only way to reach the truth.

But we have forgotten that thinking is based on non-thinking…

Yes, the cosmos we live in is irrational.

Yes, the fact that it is intelligible seems like a miracle.

If we try to analyze it, we will end up denying it.

For there is nothing which can be said to prove it.

Accept it we must!

For we are part of it.

And believe it, exactly because of its irrationality!

We can understand the mind of God.

Only because we are part of Him!

Remember?

Credo quia absurdrum!

Science is (was) based on it! (or should we say Religion?)

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