Oppositions in a point

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Abstract

Following a previous article (cf.[3]) in which logical oppositions are defined in a line segment, this article goes one step further and proposes a method defining them using a zero-dimensional object: a point.

1 Introduction

In the environment of a two-valued logic such as first-order logic, (logical) oppositions are regularly four: contradiction, contrariety, subcontrariety and subalternation. By the very beginning, these relations were displayed using the square of opposition and, later, Robert Blanché in [2] showed how the square induces another bidimensional object: the hexagon of opposition. Recently, some authors investigated generalizations of the square to hexagons and to three-dimensional solids and so on (cf. [1], [4] and [5]).

In [3], it is argued that a line segment of integers is enough to define logical oppositions in such a way that there is no need to use n-dimensional objects ($n \geq 2$) to get representations of standard logical oppositions. The procedure proposed works for the square of opposition and, with some adaptations, works as well for the hexagon of opposition. New connections with respect to the line segment have been established by Fabien Schang in [6].

Currently, a different direction is pursued but taking into account the results of [3]. Instead of a one-dimensional object such as a line segment of integers, this paper defines oppositions in an abstract dot, a point, a zero-dimensional object. Therefore, there is no need to use any dimension at all

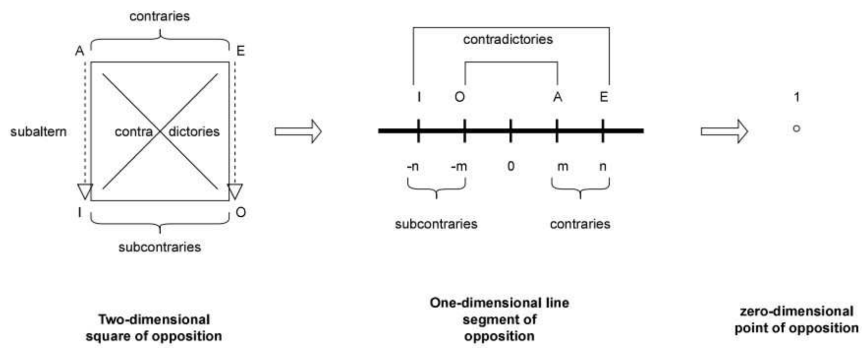

Figure 1: Reducing dimensions of diagrams

to define logical oppositions.[1]

Figure 1 reflects the scheme of reductions concerning dimensions of diagrams used to represent oppositions. The square is initially transformed into a line segment which, in its turn, is transformed into a mathematical point.

2 Oppositions without dimensions

In the construction suggested in [3], the set $\mathbb{Z}$ is taken into consideration and categorical statements are associated to positive and negative integers allowing oppositions to be defined in a line segment. This produces, therefore, a conversion of the square and the hexagon of opposition into a line segment of opposition (for precision and details cf. [3]).

In order to get started, consider classical two-valued first-order logic.[2]

[1] When I have developed the *basic construction* to convert the square and the hexagon into a line segment, a one-dimensional structure, Jean-Yves Béziau asked to me (in personal communication, 2019) whether it would be possible to define oppositions without dimensions. This article is a reply to his question. Thanks to Edelcio de Souza and Peter Arndt for comments and discussions.

[2] In an n-valued logic, with $n > 2$, other oppositions could appear.

The following construction defines all four oppositions by means of a mathematical dot, a point, a fragment of zero-dimensional space: it does not work, however, for all points in general. Differently, it works precisely for an unique and special point in the real line and the reasons for this fact will be clear in due time.

2.1 The square in a point

Let $exp(a)$ denotes the exponent of a (this notation is somewhat essential because the construction uses physical distinctions between exponents with the same base). Again, as in [3], let $\mathbf{C} = \{A, E, I, O\}$ be a set of categorical-like sentences containing statements of the form A, E, I and O (these are forms of sentences in a square of opposition), and for $m \neq n$ and $m, n \neq 0$, let $\mathbb{Z}' = \{-n, -m, m, n\} \subset \mathbb{Z}$ be a set of integers used to generate exponents of a given number a. Let $a \in \mathbb{Z}$ and let $\mathbb{Z}^\sharp$ be a set of the following type: $\mathbb{Z}^\sharp = \{a^{-n}, a^{-m}, a^m, a^n\}$. Now, consider a bijection $e : C \to \mathbb{Z}^\sharp$ such that $e(A) = a^m$, $e(E) = a^n$, $e(I) = a^{-n}$ and $e(O) = a^{-m}$. Thus, each member of $\mathbf{C}$ is assigned to a member of $\mathbb{Z}^\sharp$ in the following way: positive exponents are connected with universal statements and negative exponents are connected with existential statements. Let α, β be variables for elements of $\mathbf{C}$. So, $e(\alpha)$ and $e(\beta)$ are numbers of the form a^j, for $j \in \mathbb{Z}'$ while $exp(e(\alpha))$ and $exp(e(\beta))$ refer to the exponents of a.

Logical oppositions have to be redefined in this new setting according to the laws below:

1. α and β are *contradictories* if, and only if $exp(e(\alpha)) + exp(e(\beta)) = 0$;

2. α and β are *contraries* if, and only if $exp(e(\alpha))$ and $exp(e(\beta))$ are positive integers;

3. α and β are *subcontraries* if, and only if $exp(e(\alpha))$ and $exp(e(\beta))$ are negative integers;

4. β is subaltern of α if, and only if, $exp(e(\beta)) \neq -exp(e(\alpha))$ and $exp(e(\beta))$ is a negative integer.

It is easy to see that the above construction does not lead to a zero-dimensional object for all a such that $a \in \mathbb{Z}^*$.[3] Take, for instance, $a = 2$ and

[3]We always take $a \neq 0$ because exponentiation is defined with base a.

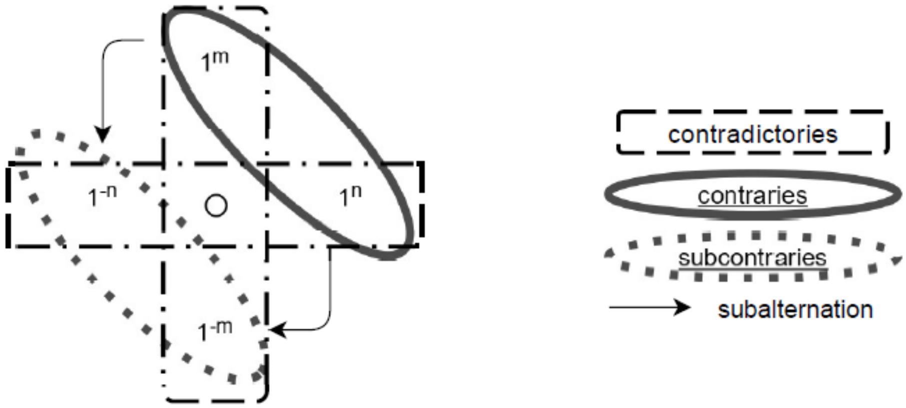

Figure 2: Oppositions in a zero-dimensional object

let $\mathbb{Z}' = \{-2, -1, 1, 2\}$. In this case, the function e generates: $e(A) = 2^1$, $e(E) = 2^2$, $e(I) = 2^{-2}$ and $e(O) = 2^{-1}$. Despite the fact that all oppositions are satisfied, considering $a = 2$, it induces a line segment in the interval $[1/4, 4]$. Indeed, more generally, everytime there is an $a \in \mathbb{Z}$ such that $a > 1$ and the greatest $exp(a)$ is k, then a line segment of the form $[1/a^k, a^k]$ is obtained. If $a < 1$, $a \neq 0$ and the greatest $exp(a)$ is k, then the line segment has the form $[a, a^k]$.

So, in order to get oppositions defined in a point, we need a special point a in $\mathbb{Z}$. This is the number 1, the *surprising point*: the unique point, in this method, in which all oppositions can be defined without leading to a line segment. Let's check why. Consider $a = 1$ and let $\mathbb{Z}' = \{-2, -1, 1, 2\}$. Therefore, the function e associates $e(A) = 1^1$, $e(E) = 1^2$, $e(I) = 1^{-2}$ and $e(O) = 1^{-1}$. So, in all cases, we have the number 1 and oppositions precisely defined in it. A and O (and E and I) are *contradictories* because the $exp(e(A)) + exp(e(O)) = 0$ ($exp(e(E)) + exp(e(I)) = 0$). We also have A and E are *contraries* because that $exp(e(A))$ and $exp(e(E))$ are positive integers. In the same way, I and O are *subcontraries* as $exp(e(I))$ and $exp(e(O))$ are negative integers; By the end, notice that I is subaltern of A given that $exp(e(I)) \neq -exp(e(A))$ and $exp(e(I))$ is a negative integer. The same holds for E and O. Figure 2 represents how oppositions can be defined in the point 1 (in the center) with its different manifestations using different exponents.

Mutatis mutandis, following the construction proposed in [3] combined with the technique above, the two-dimensional hexagon of opposition can also be converted to a point, and this is let as an exercise to the reader. So, both the square and the hexagon can be transformed step-by-step into a line segment and into a point.

3 Conclusion

The theory of logical oppositions, as it has been developed in the literature, uses n-dimensional diagrams ($n \geq 2$) to represent contradiction, contrariety, subcontrariety and subalternation. Nevertheless, it is possible to define oppositions using a line segment (one-dimensional space) and, as showed here, even no dimension at all is required, considering that oppositions can be defined in the surprising point 1.

A criticism to this approach has been raised by Peter Arndt from the University of Düsseldorf (in personal communication, 2020). He argues that, despite the fact that only the number 1 is used to define oppositions, the one-dimensional character of the construction appears in the exponents. This restores, therefore, one-dimensionality. It is somewhat reasonable to accept that there is, indeed, a parasitary connection between oppositions without dimensions defined in a point and those defined in a line segment. The worth and essential remark here is that all representations used of the number 1 refer to the same object, and logical oppositions make use of only a single mathematical dot to formulate all oppositions in a zero-dimensional diagram. The construction developed shows the zero-dimensional point of opposition, though it is dependent of a line segment in the exponents, what decides the question is the dimension of the diagram, and in the case of the point of opposition, it is its base which tell us in which dimension the diagram is, not the exponents. So, it is legitimate to speak about oppositions without dimensions. A next step would be to find a zero-dimensional point of opposition without any kind of dependence. Moreover, it is important to say that the constructions suggested in [3] and the method here are not unique ways to reduce dimensions of diagrams, maybe some other possible reductions could also be achieved.

Two-dimensional or three-dimensional diagrams to represent oppositions are very well-known, and they have been used widely in the literature (especially squares). The work of Alessio Moretti in [5] contains the foundations of what is called n-opposition theory with some exploitations concerning dimensions of simplexes, but there is no attempt there to define all oppositions simultaneously in a given simplex of dimension one or zero. Again, concerning the plurality of dimensions, a natural ontological point is to check whether Ockham's razor should be applied in the case of diagrams. Why should we proliferate dimensions using complicated diagrams if we can use a line or a point? Is there any special reason for this fact? The existence of simple diagrammatic abbreviations such as a *line segment of opposition* and a *point of opposition* challenges the limits of the researches on the square and its extensions.

References

[1] Beziau, J-Y. (2012). The power of the hexagon. *Logica Universalis*, 6(1), pp.1-43.

[2] Blanché, R. (1966). *Structures intellectuelles. Essai sur l'organisation systématique des concepts*. Paris: Vrin.

[3] Costa-Leite, A. (2018). Oppositions in a line segment. *South American Journal of Logic*, 4(1), pp.185-193.

[4] Moretti, A. (2004). Geometry of Modalities? Yes: Through n-Opposition Theory. In: *Aspects of Universal Logic*, J-Y.Beziau, A. Costa-Leite, A.Facchini (editors), *Travaux de Logique*, 17, pp. 102-145.

[5] Moretti, A. (2012). From the "Logical Square" to the "Logical Poly-Simplexes". In: *The Square of Opposition: a general framework for cognition*. J-Y. Beziau and G. Payette (editors), pp. 119-156. Bern: Peter Lang.

[6] Schang, F. (2018). End of the Square? *South American Journal of Logic*, 4(2), pp.485-505.