

Asymptotic freedom from the two loop term of the β -function in a cubic theory

J.A. Gracey,
Theoretical Physics Division,
Department of Mathematical Sciences,
University of Liverpool,
P.O. Box 147,
Liverpool,
L69 3BX,
United Kingdom.

Abstract. We renormalize a six dimensional cubic theory to four loops in the modified minimal subtraction ($\overline{\text{MS}}$) scheme where the scalar is in a bi-adjoint representation. The underlying model was originally derived in a problem relating to gravity being a double copy of Yang-Mills theory. As a field theory in its own right we find that it has a curious property in that while unexpectedly there is no one loop contribution to the β -function the two loop coefficient is negative. It therefore represents an example where asymptotic freedom is determined by the *two* loop term of the β -function. We also examine a multi-adjoint cubic theory in order to see whether this is a more universal property of these models.

1 Introduction.

Scalar ϕ^3 theory in six dimensions has proved to be a useful laboratory or tool to explore major ideas in quantum field theory. For instance, after the discovery of asymptotic freedom in Quantum Chromodynamics (QCD), [1, 2], it was used as a testing ground to study the implications of this property. This was because it was shown that six dimensional ϕ^3 theory was renormalizable and also asymptotically free, [3]. In other words the consequences of this characteristic of non-abelian gauge theories could be explored in a simple environment without the complications of the gauge structure. One reason why higher dimensional theories might be of interest for lower dimensional ones is that the ultraviolet behaviour of one theory could be related to the infrared dynamics of another, [4, 5]. That idea has yet to be realized concretely in the gauge theory context. However in the case of scalar ϕ^3 theory, where the calculations were easier to carry out, it has been used to explore certain infrared ideas associated with the strong interactions. For instance, it has proved useful as a toy model of Regge theory but in six dimensions where ladder diagrams were analysed in order to gain insight into the Regge slope. Several articles in this direction in ϕ^3 theory are [6, 7, 8], for example. Clearly there are limitations to such exercises. The obvious one is that of an unphysical spacetime dimension. A more serious one is the lack of a bounded Hamiltonian which means that true bound state analyses could not be fully credible. Despite this the theory played a valuable role as a sounding board for exploring new ideas in a gauge theory. Other applications of cubic scalar theory lie mainly in condensed matter physics and in particular critical phenomena. For instance various decorations of the scalar field with different symmetries allowed the critical exponents that relate to percolation and the Lee-Yang edge singularity problems, [9, 10, 11], to be determined very accurately in the ϵ expansion in integer dimensions below six.

More recently another perhaps surprising example of the connection a cubic scalar theory has with physics has emerged. In [12, 13] the idea that on-shell gravity could be interpreted as a double copy of Yang-Mills theory was initially put forward and generated a large amount of interest. For instance it was shown that there was a relation between the product of Yang-Mills n -point functions and the corresponding on-shell gravity amplitudes representing a connection with the KLT relations, [14]. This double copy of Yang-Mills appears to be widely accepted as an interesting interpretation and clearly is a direction to pursue in the quest for a theory of quantum gravity. One consequence of the double copy vision was the connection with a scalar field endowed with a bi-adjoint symmetry, [15, 16], although the gravity connection with a scalar cubic interaction was observed earlier in [17]. For instance in [15] it was shown that scattering amplitudes of the double copy theory could be related to the gluonic ones in pure Yang-Mills. Another direction that was followed in [18, 19] was to study classical solutions of a linearized version of Yang-Mills theory and their relation to double copies of scalar fields in the bi-adjoint cubic theory. These ideas were explored further in [20, 21] where new solutions were found further strengthening the double copy correspondence concept. While most of these studies were classical it is worth investigating the underlying quantum field theory in its own right to ascertain whether it has any other interesting properties. That is the purpose of this article. In particular we will renormalize the theory to four loops in six dimensions and deduce the renormalization group functions. Doing so will reveal a curious feature. It will transpire that for general Lie groups the one loop coefficient of the β -function vanishes despite there being a one loop contribution to the 3-point vertex function. This is a rather unusual and rare situation in a renormalizable theory although it is known that the one loop term of renormalization group functions, other than the β -function, can be zero in other models. What makes this bi-adjoint theory even more intriguing is that the non-zero two loop β -function coefficient is negative. Therefore this model appears to be one of the few cases where the property of asymptotic

freedom is determined purely from the *two* loop term of the β -function. We will explore some of the basic consequences of this property as well as finding the underlying reason why it emerges. As part of this investigation we will renormalize a generalization of the bi-adjoint model by allowing the field to take values in the adjoint representation of four (different) Lie groups which we will refer to as the quartic adjoint model.

The article is organized as follows. The background to the properties of the bi-adjoint cubic scalar theory such as the group theory connected with the computation are discussed in Section 2. That will also include the details of how we performed the computation the results of which are discussed in Section 3. The generalization to the quartic adjoint model is provided in the next section before the concluding remarks of Section 5. An appendix records full details of the renormalization group functions of the quartic adjoint model.

2 Background.

To begin with we define the six dimensional Lagrangian for the bi-adjoint cubic scalar theory that was derived from solutions of linearized Yang-Mills theory and related to the double copy of gravity. If we denote the basic scalar field by $\phi^{a_1 a_2}$ then the Lagrangian is, for example from [19, 20],

$$L = \frac{1}{2} (\partial_\mu \phi^{a_1 a_2})^2 + \frac{g}{6} f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} \phi^{a_1 a_2} \phi^{b_1 b_2} \phi^{c_1 c_2} . \quad (2.1)$$

Given that the fields take values in the group $G_1 \times G_2$ we use a different notation from [19, 20] as we will carry out a more general analysis subsequently. Therefore we note that the numerical label on the adjoint indices will correspond to the label on the respective subgroups in the overall symmetry group. The Roman letter that carries that label is the one that is summed over in any repetition. Moreover these indices will run over a set whose dimension is the dimension of the adjoint representation of the respective group and denoted by N_i . So for instance $1 \leq a_1 \leq N_1$ and $1 \leq b_2 \leq N_2$ or in more general terms $1 \leq a_i, b_i, c_i, d_i \leq N_i$ for each i . In other words

$$\delta^{a_i a_i} = N_i \quad (2.2)$$

where there is never a sum over the repeated label i which indicates the specific group. If $G_1 = SU(N_c)$ for example then $N_1 = N_c^2 - 1$ corresponding to the dimension of the adjoint representation. The respective structure functions of each group appearing in (2.1) are $f^{a_i b_i c_i}$ for $i = 1$ and 2. As we will be carrying out loop computations it is worth discussing related group theory quantities that will appear later. For example, we use a compact notation for the Casimirs of each with

$$f^{a_i c_i d_i} f^{b_i c_i d_i} = C_i \delta^{a_i b_i} \quad (2.3)$$

for each i . Ordinarily one uses C_A or $C_2(G)$ where A denotes the adjoint representation of the group G for what we now denote by C_i . It is not necessary however to include the representation designation since the adjoint will be used throughout the article unless stated otherwise. Beyond the first few loop orders higher rank group Casimirs will appear. This was noted when the four loop β -function of Quantum Chromodynamics (QCD) was determined in [22] and a comprehensive study was undertaken in [23] of general Lie group Casimirs in the context of perturbative computations. We briefly summarize the relevant aspects of that are needed here. For instance the fully symmetric rank 4 tensor defined by, [23],

$$d_{R_i}^{a_i b_i c_i d_i} = \frac{1}{6} \text{Tr} \left(T_{R_i}^{a_i} T_{R_i}^{(b_i} T_{R_i}^{c_i} T_{R_i}^{d_i)} \right) \quad (2.4)$$

will arise. Here we revert momentarily to representation R_i of the group G_i as (2.4) involves the group generators $T_{R_i}^{a_i}$. Within the computation the contracted product of these tensors will

produce additional group Casmirs independent of C_i . At four loops in Yang-Mills theory the only combination that appears turns out to be the simple product, [23],

$$d_{(i)44} = d_A^{a_i b_i c_i d_i} d_A^{a_i b_i c_i d_i} \quad (2.5)$$

in our notation where the bracketed label is used to avoid any potential ambiguity with the tensor rank when products of more groups are considered and A denotes the adjoint representation. In the four loop QCD β -function, [23], by contrast, products of $d_{R_i}^{a_i b_i c_i d_i}$ in the fundamental and adjoint representations also arise. As a reference point for results that appear later we recall that for $SU(N_c)$, [23],

$$d_{(i)44} = \frac{N_c^2 [N_c^2 + 36]}{24} N_i \quad (2.6)$$

and we have not substituted the explicit value for N_i as that quantity appears in the results for a general Lie group which is what we use throughout. Higher rank tensors beyond (2.4) have been discussed in [23]. A secondary motivation for studying the renormalization of (2.1) at large loop order is to ascertain whether such rank 4 tensor Casimirs first appear at the same loop order as that of QCD or not. In terms of other aspects of (2.1) for ease of comparison we retain the conventions that were used in [24]. In particular in [24] the sign of the coupling constant g was opposite to that used in earlier work by others such as [3, 25, 26, 27]. There ought not to be difficulty in translating where necessary.

Before renormalizing (2.1) we recall our notation. First if we denote bare entities with a subscript $_o$ then their relation to the renormalized counterparts are

$$\phi_o^{a_1 a_2} = \sqrt{Z_\phi} \phi^{a_1 a_2} \quad , \quad g_o = Z_g g \quad (2.7)$$

in six dimensions. However we will dimensionally regularize (2.1) in $d = 6 - 2\epsilon$ dimensions and determine the renormalization group functions in the $\overline{\text{MS}}$ scheme. To find Z_ϕ and Z_g to four loop order we have to compute the 2- and 3-point functions and we will follow the algorithm used in [24] where more details of the technicalities of this exercise can be found. However given the presence of the structure constants in the interaction we have had to adapt that method to determine Z_g in particular. For example there are 540 Feynman graphs to evaluate for the four loop vertex function which is a large number to handle. To circumvent this a shortcut was exploited which was to generate these and lower loop vertex graphs from the 2-point graphs by applying a simple mapping to each propagator. In other words setting

$$\frac{1}{k^2} \rightarrow \frac{1}{k^2} + \frac{\lambda}{(k^2)^2} \quad (2.8)$$

and retaining only terms with zero or one power of the parameter λ , means that the $O(\lambda)$ terms will formally correspond to 3-point function graphs where one external momentum is set to zero, [24]. In six dimensions this is infrared safe and evaluating all the graphs that are $O(\lambda)$ will produce the full 3-point function to that loop order. While this simple mapping is the essence of what one needs to do at the level of the graph generation it is not sufficient for (2.1) as the group theory factors need to be accommodated. To achieve this we adapt (2.8) by including the group structure of the propagator as well as the inserted vertex to produce the mapping for (2.1)

$$\frac{1}{k^2} \delta^{a_1 b_1} \delta^{a_2 b_2} \rightarrow \frac{1}{k^2} \delta^{a_1 b_1} \delta^{a_2 b_2} + \frac{\lambda g}{(k^2)^2} f^{a_1 b_1 c_1^e} f^{a_2 b_2 c_2^e} \quad (2.9)$$

where c_i^e are the external indices of the inserted leg of the generated 3-point function. One benefit of using this technique is that there are only 64 four loop graphs in the 2-point function and the insertion does not change the underlying graph topology. This is important and leads

to a more efficient computation since the same integration subroutine for that topology can be used to determine the divergences of the 2-point graph and its associated 3-point one where there is a nullified insertion on each propagator. While we have not included a mass term in (2.1) we can still compute the anomalous dimension of the mass by inserting the mass operator $\frac{1}{2}\phi^{a_1 a_2}\phi^{a_1 a_2}$ in a 2-point function. The renormalization constant associated with this operator is equivalent to that of the mass itself which we denote by Z_m . Therefore we can directly use (2.8) to determine Z_m as well as Z_g .

The computational strategy to evaluate the graphs of the 2- and 3-point functions is to use the Laporta integration by parts algorithm, [28]. This constructs relations between a set of Feynman integrals that can be algebraically solved in such a way that all the integrals are related to a relatively small set. These are termed the master integrals and have to be evaluated directly. In our case since all the 2-point four loop master integrals are available in four dimensions, [29], it was possible to connect these to the ones that emerge in our six dimensional computation, [24]. This is achieved by the Tarasov method, [30, 31], whereby integrals in d -dimensions can be related to the $(d+2)$ -dimensional integral with same topology and other topologies where one or more edges have been removed. Therefore the four loop six dimensional masters were deduced in [24]. To effect the Laporta algorithm we used the REDUZE implementation, [32, 33]. A useful feature of the package is that it allows the database that is generated to be written in the symbolic language FORM, [34, 35]. This is important since we have written an automatic programme in FORM to carry out the full computation. In particular the contributing Feynman graphs are generated with the QGRAF package, [36], and the topology mapping appended. This allows the automatic programme to proceed since the integration of each topology follows a separate path. The final stage is the summation of all the graphs and the implementation of the automatic renormalization to deduce Z_ϕ , Z_m and Z_g . To do this we follow [37] which means that the graphs are evaluated with bare parameters with the renormalization constants (2.7) being introduced at the end. For instance the 2-point function is multiplied overall by Z_ϕ which allows one to deduce the unknown counterterms. Equally for the extraction of the mass and coupling constant renormalization constants the parameter λ in each of (2.8) and (2.9) are multiplied by Z_m and $Z_g\sqrt{Z_\phi}$ respectively. One major tool that was used to carry out the manipulation of the large number of structure functions present at each vertex was the `color.h` package written in FORM and available from [34]. It encodes the group theory discussed in [23] in an efficient way particularly for the three and four loop graphs.

3 Results.

Having outlined our computational strategy we can now record the outcome of determining Z_ϕ , Z_m and Z_g that lead to the respective renormalization group functions $\gamma_\phi(g)$, $\gamma_m(g)$ and $\beta(g)$ in the $\overline{\text{MS}}$ scheme. First the anomalous dimension of the scalar field is

$$\begin{aligned}
\gamma_\phi(g) = & -\frac{g^2}{12}C_1C_2 - \frac{5g^4}{432}C_1^2C_2^2 - \frac{827g^6}{248832}C_1^3C_2^3 \\
& + \left[1032\zeta_3C_1^4C_2^4N_1N_2 + 108\zeta_4C_1^4C_2^4N_1N_2 - 960\zeta_5C_1^4C_2^4N_1N_2 - 943C_1^4C_2^4N_1N_2 \right. \\
& - 576\zeta_3C_1^4d_{(2)44}N_1 - 2592\zeta_4C_1^4d_{(2)44}N_1 - 11520\zeta_5C_1^4d_{(2)44}N_1 \\
& + 14976C_1^4d_{(2)44}N_1 - 576\zeta_3C_2^4d_{(1)44}N_2 - 2592\zeta_4C_2^4d_{(1)44}N_2 \\
& - 11520\zeta_5C_2^4d_{(1)44}N_2 + 14976C_2^4d_{(1)44}N_2 + 96768\zeta_3d_{(1)44}d_{(2)44} \\
& \left. + 62208\zeta_4d_{(1)44}d_{(2)44} - 138240\zeta_5d_{(1)44}d_{(2)44} \right] \frac{g^8}{497664N_1N_2} + O(g^{10}) \quad (3.1)
\end{aligned}$$

where ζ_z denotes the Riemann zeta function and we note that the rank 4 group tensors first appear at four loops. For the mass anomalous dimension we find

$$\begin{aligned}
\gamma_m(g) = & -\frac{g^2}{2}C_1C_2 + \frac{5g^4}{48}C_1^2C_2^2 + [432\zeta_3 - 2203] \frac{C_1^3C_2^3g^6}{13824} \\
& + \left[127764\zeta_3C_1^4C_2^4N_1N_2 + 1944\zeta_4C_1^4C_2^4N_1N_2 - 251640\zeta_5C_1^4C_2^4N_1N_2 \right. \\
& + 255517C_1^4C_2^4N_1N_2 + 88128\zeta_3C_1^4d_{(2)44}N_1 - 46656\zeta_4C_1^4d_{(2)44}N_1 \\
& - 492480\zeta_5C_1^4d_{(2)44}N_1 + 409536C_1^4d_{(2)44}N_1 + 88128\zeta_3C_2^4d_{(1)44}N_2 \\
& - 46656\zeta_4C_2^4d_{(1)44}N_2 - 492480\zeta_5C_2^4d_{(1)44}N_2 + 409536C_2^4d_{(1)44}N_2 \\
& + 3110400\zeta_3d_{(1)44}d_{(2)44} + 1119744\zeta_4d_{(1)44}d_{(2)44} \\
& \left. + 622080\zeta_5d_{(1)44}d_{(2)44} \right] \frac{g^8}{1492992N_1N_2} + O(g^{10}) \tag{3.2}
\end{aligned}$$

where the higher order Casimirs first appear at the same order.

To complete the set the four loop β -function is

$$\begin{aligned}
\beta(g) = & -\frac{5g^5}{1152}C_1^2C_2^2 \\
& + \left[108\zeta_3C_1^4C_2^4N_1N_2 - 161C_1^4C_2^4N_1N_2 - 2592\zeta_3C_1^4d_{(2)44}N_1 + 2592C_1^4d_{(2)44}N_1 \right. \\
& \left. - 2592\zeta_3C_2^4d_{(1)44}N_2 + 2592C_2^4d_{(1)44}N_2 + 62208\zeta_3d_{(1)44}d_{(2)44} \right] \frac{g^7}{41472C_1C_2N_1N_2} \\
& + \left[-368928\zeta_3C_1^4C_2^4N_1N_2 + 518400\zeta_5C_1^4C_2^4N_1N_2 - 101089C_1^4C_2^4N_1N_2 \right. \\
& - 3483648\zeta_3C_1^4d_{(2)44}N_1 + 6220800\zeta_5C_1^4d_{(2)44}N_1 - 2363904C_1^4d_{(2)44}N_1 \\
& - 3483648\zeta_3C_2^4d_{(1)44}N_2 + 6220800\zeta_5C_2^4d_{(1)44}N_2 - 2363904C_2^4d_{(1)44}N_2 \\
& \left. - 95551488\zeta_3d_{(1)44}d_{(2)44} + 119439360\zeta_5d_{(1)44}d_{(2)44} \right] \frac{g^9}{23887872N_1N_2} \\
& + O(g^{11}) . \tag{3.3}
\end{aligned}$$

This has the unusual feature in that the first non-zero term is at two loops rather than one loop. This is not the first or only case of the first term of a renormalization group function in a fully renormalizable field theory being absent. For instance, while the field anomalous dimension in four dimensional ϕ^4 theory is zero at one loop this is for the simple reason that the only graph contributing to the 2-point function is a snail. Therefore it is independent of the external momentum and its divergence contributes to the mass renormalization only. Here the situation is different in that the only one loop graph of the 3-point vertex is divergent but the residue of the simple pole is exactly cancelled by the contribution from the wave function renormalization. This is not the case for other symmetry decorations of the scalar field in scalar ϕ^3 theory in six dimensions, [3, 26, 27]. This curious property has an interesting consequence which is that since the coefficient of the now leading two loop term of $\beta(g)$ is negative then the theory is asymptotically free. Ordinarily when this is a feature of other field theories it is purely from the one loop term, [1, 2]. We note at this point that this sign of the two loop term would have emerged irrespective of the coupling constant sign convention alluded to earlier. One comment that deserves mention at this point concerns the scheme dependence of this particular β -function. Even though the one loop term is zero the three loop term of $\beta(g)$ still depends on the renormalization scheme. Unlike the other two renormalization group functions the rank 4 Casimirs first appear at three loop in the β -function rather than four. This is one order earlier than that of QCD, [22].

To gain more insight into the consequences of their being no one loop term of the β -function it is worth focussing on the case when both groups G_1 and G_2 are the same which we will denote by G . In this case (3.1), (3.2) and (3.3) become

$$\begin{aligned}
\gamma_\phi^{G \times G}(g) &= -\frac{C_1^2 g^2}{12} - \frac{5C_1^4 g^4}{432} - \frac{827C_1^6 g^6}{248832} \\
&+ \left[1032\zeta_3 C_1^8 N_1^2 - 1152\zeta_3 C_1^4 d_{(1)44} N_1 + 96768\zeta_3 d_{(1)44}^2 + 108\zeta_4 C_1^8 N_1^2 \right. \\
&\quad - 5184\zeta_4 C_1^4 d_{(1)44} N_1 + 62208\zeta_4 d_{(1)44}^2 - 960\zeta_5 C_1^8 N_1^2 - 23040\zeta_5 C_1^4 d_{(1)44} N_1 \\
&\quad \left. - 138240\zeta_5 d_{(1)44}^2 - 943C_1^8 N_1^2 + 29952C_1^4 d_{(1)44} N_1 \right] \frac{g^8}{497664N_1^2} \\
&+ O(g^{10}) \tag{3.4}
\end{aligned}$$

$$\begin{aligned}
\gamma_m^{G \times G}(g) &= -\frac{C_1^2 g^2}{2} + \frac{5C_1^4 g^4}{48} + C_1^6 [432\zeta_3 - 2203] \frac{g^6}{13824} \\
&+ \left[127764\zeta_3 C_1^8 N_1^2 + 176256\zeta_3 C_1^4 d_{(1)44} N_1 + 3110400\zeta_3 d_{(1)44}^2 + 1944\zeta_4 C_1^8 N_1^2 \right. \\
&\quad - 93312\zeta_4 C_1^4 d_{(1)44} N_1 + 1119744\zeta_4 d_{(1)44}^2 - 251640\zeta_5 C_1^8 N_1^2 \\
&\quad - 984960\zeta_5 C_1^4 d_{(1)44} N_1 + 622080\zeta_5 d_{(1)44}^2 + 255517C_1^8 N_1^2 \\
&\quad \left. + 819072C_1^4 d_{(1)44} N_1 \right] \frac{g^8}{1492992N_1^2} + O(g^{10}) \tag{3.5}
\end{aligned}$$

and

$$\begin{aligned}
\beta^{G \times G}(g) &= -\frac{5C_1^4 g^5}{1152} \\
&+ \left[108\zeta_3 C_1^8 N_1^2 - 5184\zeta_3 C_1^4 d_{(1)44} N_1 + 62208\zeta_3 d_{(1)44}^2 - 161C_1^8 N_1^2 \right. \\
&\quad \left. + 5184C_1^4 d_{(1)44} N_1 \right] \frac{g^7}{41472C_1^2 N_1^2} \\
&+ \left[-368928\zeta_3 C_1^8 N_1^2 - 6967296\zeta_3 C_1^4 d_{(1)44} N_1 - 95551488\zeta_3 d_{(1)44}^2 \right. \\
&\quad + 518400\zeta_5 C_1^8 N_1^2 + 12441600\zeta_5 C_1^4 d_{(1)44} N_1 + 119439360\zeta_5 d_{(1)44}^2 \\
&\quad \left. - 101089C_1^8 N_1^2 - 4727808C_1^4 d_{(1)44} N_1 \right] \frac{g^9}{23887872N_1^2} + O(g^{11}). \tag{3.6}
\end{aligned}$$

Specifying to the group $SU(3)$ we deduce

$$\begin{aligned}
\gamma_\phi^{SU(3) \times SU(3)}(g) &= -\frac{3}{42}g^2 - \frac{15}{16}g^4 - \frac{2481}{1024}g^6 \\
&\quad + 27 [4992\zeta_3 + 1728\zeta_4 - 11760\zeta_5 + 5297] \frac{g^8}{2048} + O(g^{10}) \\
\gamma_m^{SU(3) \times SU(3)}(g) &= -\frac{9}{2}g^2 + \frac{135}{16}g^4 + 27 [432\zeta_3 - 2203] \frac{g^6}{512} \\
&\quad + 9 [299484\zeta_3 + 31104\zeta_4 - 429840\zeta_5 + 426157] \frac{g^8}{2048} + O(g^{10}) \\
\beta^{SU(3) \times SU(3)}(g) &= -\frac{45}{128}g^5 + 9 [1728\zeta_3 + 919] \frac{g^7}{512} \\
&\quad + 9 [8294400\zeta_5 - 5967648\zeta_3 - 1086049] \frac{g^9}{32768} + O(g^{11}) \tag{3.7}
\end{aligned}$$

or

$$\begin{aligned}
\gamma_\phi^{SU(3)\times SU(3)}(g) &= -0.750000g^2 - 0.937500g^4 - 2.422852g^6 + 12.836235g^8 + O(g^{10}) \\
\gamma_m^{SU(3)\times SU(3)}(g) &= -4.500000g^2 + 8.437500g^4 - 88.789469g^6 + 1644.017718g^8 \\
&\quad + O(g^{10}) \\
\beta^{SU(3)\times SU(3)}(g) &= -0.351562g^5 + 52.666775g^7 + 93.711209g^9 + O(g^{11}) \tag{3.8}
\end{aligned}$$

numerically. From (3.8) it is clear that there is a Banks-Zaks fixed point, [38] stemming from the opposite signs of the first two terms of the β -function. In fact it is also the case that a similar fixed point is present for $SU(M) \times SU(N)$. Strictly the fixed point of [38] in QCD derives from the one and two loop scheme independent terms. We use it here in the sense of the first two non-zero terms although the second of these is scheme dependent unlike [38]. Moreover the value of the critical coupling for $N = 3$ only changes by around 1% when solving $\beta^{SU(3)\times SU(3)}(g) = 0$ at three and four loops. One of the reasons for providing this example is to show another interesting consequence of the absence of the one loop term. While the renormalization group functions are scheme dependent one can derive renormalization group invariants from them. These are critical exponents that are the evaluation of the functions at a non-trivial fixed point. One important such fixed point is the Wilson-Fisher one, [39, 40], where the critical coupling is defined by setting the d -dimensional β -function to zero and denoted by g_* . So in $d = 6 - 2\epsilon$ dimensions we have

$$\begin{aligned}
\eta^{SU(3)\times SU(3)} &= -\frac{2i\sqrt{5}}{5}\sqrt{\epsilon} + \frac{2}{25}[576\zeta_3 + 323]\epsilon \\
&\quad + \sqrt{5}i\left[1327104\zeta_3^2 + 779232\zeta_3 + 921600\zeta_5 + 271945\right]\frac{\epsilon^{\frac{3}{2}}}{500} \\
&\quad + \left[311040\zeta_4 - 143327232\zeta_3^2 - 96163200\zeta_3 - 85060800\zeta_5 - 31005017\right]\frac{\epsilon^2}{6750} \\
&\quad + O(\epsilon^{\frac{5}{2}}) \\
\eta_m^{SU(3)\times SU(3)} &= -\frac{12i\sqrt{5}}{5}\sqrt{\epsilon} + 16[432\zeta_3 + 211]\frac{\epsilon}{25} \\
&\quad + \frac{9i\sqrt{5}}{250}\left[442368\zeta_3^2 + 233664\zeta_3 + 307200\zeta_5 + 79175\right]\epsilon^{\frac{3}{2}} \\
&\quad + 2\left[110854656\zeta_3^2 + 55734372\zeta_3 + 155520\zeta_4 + 60058800\zeta_5 + 15279107\right]\frac{\epsilon^2}{1125} \\
&\quad + O(\epsilon^{\frac{5}{2}}) \\
\omega^{SU(3)\times SU(3)} &= 2\epsilon - \frac{2i\sqrt{5}}{5}[1728\zeta_3 + 919]\epsilon^{\frac{3}{2}} \\
&\quad + \left[35831808\zeta_3^2 + 8274528\zeta_3 + 41472000\zeta_5 + 4704487\right]\frac{\epsilon^2}{2250} + O(\epsilon^{\frac{5}{2}}) \tag{3.9}
\end{aligned}$$

where $\eta = \gamma_\phi(g_*)$, $\eta_m = \gamma_m(g_*)$ and $\omega = 2\beta'(g_*)$. The main key difference between these exponents and those from models where there is a non-zero one loop term is that the expansion is a function of $\sqrt{\epsilon}$ rather than ϵ . In addition the exponents are complex but this is due to having assumed ϵ is real and positive. If ϵ were real and negative then the exponents are real above six dimensions.

Finally we close this section by recalling that solving the β -function as a differential equation determines the functional dependence of the running coupling constant with the renormalization scale μ . Therefore we can compare the running coupling constants in the conventional case where

asymptotic freedom is determined by the one loop β -function with that for (2.1). For instance, if we formally define two β -functions by

$$\begin{aligned}\beta_1(g_1) &= -\bar{\beta}_1 g_1^3 + O(g_1^5) \\ \beta_2(g_2) &= -\bar{\beta}_2 g_2^5 + O(g_2^7)\end{aligned}\tag{3.10}$$

then we have

$$g_1^2(\mu) = -\frac{1}{\bar{\beta}_1 \ln(\mu^2/\Lambda_1^2)}\tag{3.11}$$

for the more conventional one loop β -function. By contrast solving the second case we find

$$g_2^2(\mu) = -\frac{1}{\sqrt{2\bar{\beta}_2} \ln(\mu^2/\Lambda_2^2)}\tag{3.12}$$

where Λ_1 and Λ_2 are the constants of integration. Clearly both running coupling constants have the same general behaviour in that they tend to zero as $\mu \rightarrow \infty$. However in the latter case where the one loop β -function term is absent, the coupling constant tends to zero at a much slower rate. So if this model, or one with the same property, was realized in Nature the constituent particles would only be effectively free at significantly high energies.

4 Quartic adjoint.

The absence of a one loop term in the β -function of (2.1) is an interesting property. In order to see whether this property is common to more general scalar ϕ^3 theories with adjoint decorations we have repeated the renormalization exercise for (2.1) for what we will term the quartic adjoint theory with Lagrangian

$$L = \frac{1}{2}(\partial_\mu \phi^{a_1 a_2 a_3 a_4})^2 + \frac{g}{6} f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} f^{a_3 b_3 c_3} f^{a_4 b_4 c_4} \phi^{a_1 a_2 a_3 a_4} \phi^{b_1 b_2 b_3 b_4} \phi^{c_1 c_2 c_3 c_4} .\tag{4.1}$$

Here we have a scalar field which takes values in the group $G_1 \times G_2 \times G_3 \times G_4$ and in particular the interaction involves the adjoint representation of the group generators. In (4.1) we use a similar notation to that introduced for (2.1) where there are now two additional labels due to the extra groups G_3 and G_4 . Equally the definition of the group Casimirs have an obvious natural extension of those given in (2.2), (2.3) and (2.5). We have followed the same process of renormalizing (4.1) as that for (2.1) together with similar consistency checks. Therefore we move to the discussion of the outcome. With the additional group structure it transpires that the four loop expressions for each of the renormalization group functions are more involved than those of (2.1). These have been recorded in the Appendix. Instead we illustrate the structure in the simpler case of the group $G \times G \times G \times G \equiv G^4$ and find

$$\begin{aligned}\gamma_\phi^{G^4}(g) &= -\frac{C_1^4 g^2}{12} - \frac{19C_1^8 g^4}{864} - \frac{40421C_1^{12} g^6}{3981312} \\ &+ \left[1910\zeta_3 C_1^{16} N_1^4 + 2112\zeta_3 C_1^{12} d_{(1)44} N_1^3 + 89856\zeta_3 C_1^8 d_{(1)44}^2 N_1^2 \right. \\ &\quad - 110592\zeta_3 C_1^4 d_{(1)44}^3 N_1 + 4644864\zeta_3 d_{(1)44}^4 + 9\zeta_4 C_1^{16} N_1^4 \\ &\quad - 864\zeta_4 C_1^{12} d_{(1)44} N_1^3 + 31104\zeta_4 C_1^8 d_{(1)44}^2 N_1^2 - 497664\zeta_4 C_1^4 d_{(1)44}^3 N_1 \\ &\quad + 2985984\zeta_4 d_{(1)44}^4 - 320\zeta_5 C_1^{16} N_1^4 - 15360\zeta_5 C_1^{12} d_{(1)44} N_1^3 \\ &\quad - 276480\zeta_5 C_1^8 d_{(1)44}^2 N_1^2 - 2211840\zeta_5 C_1^4 d_{(1)44}^3 N_1 - 6635520\zeta_5 d_{(1)44}^4 \\ &\quad - 150934C_1^{16} N_1^4 + 14976C_1^{12} d_{(1)44} N_1^3 + 179712C_1^8 d_{(1)44}^2 N_1^2 \\ &\quad \left. + 2875392C_1^4 d_{(1)44}^3 N_1 \right] \frac{g^8}{23887872N_1^4} + O(g^{10})\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}
\gamma_m^{G^4}(g) = & -\frac{C_1^4 g^2}{2} + \frac{C_1^8 g^2}{96} + [3024\zeta_3 - 43537] \frac{C_1^{12} g^6}{221184} \\
& + \left[-619077\zeta_3 C_1^{16} N_1^4 + 58176\zeta_3 C_1^{12} d_{(1)44} N_1^3 + 1389312\zeta_3 C_1^8 d_{(1)44}^2 N_1^2 \right. \\
& + 5640192\zeta_3 C_1^4 d_{(1)44}^3 N_1 + 49766400\zeta_3 d_{(1)44}^4 - 91800\zeta_4 C_1^{16} N_1^4 \\
& - 5184\zeta_4 C_1^{12} d_{(1)44} N_1^3 + 186624\zeta_4 C_1^8 d_{(1)44}^2 N_1^2 - 2985984\zeta_4 C_1^4 d_{(1)44}^3 N_1 \\
& + 17915904\zeta_4 d_{(1)44}^4 - 177630\zeta_5 C_1^{16} N_1^4 - 158400\zeta_5 C_1^{12} d_{(1)44} N_1^3 \\
& - 1762560\zeta_5 C_1^8 d_{(1)44}^2 N_1^2 - 31518720\zeta_5 C_1^4 d_{(1)44}^3 N_1 + 9953280\zeta_5 d_{(1)44}^4 \\
& + 2070250 C_1^{16} N_1^4 + 136512 C_1^{12} d_{(1)44} N_1^3 + 1638144 C_1^8 d_{(1)44}^2 N_1^2 \\
& \left. + 26210304 C_1^4 d_{(1)44}^3 N_1 \right] \frac{g^8}{23887872 N_1^4} + O(g^{10}) \tag{4.3}
\end{aligned}$$

for the field and mass anomalous dimensions. For the β -function we arrived at

$$\begin{aligned}
\beta^{G^4}(g) = & -\frac{3C_1^4 g^3}{32} - \frac{467C_1^8 g^5}{18432} \\
& + \left[48\zeta_3 C_1^{16} N_1^4 - 4608\zeta_3 C_1^{12} d_{(1)44} N_1^3 + 165888\zeta_3 C_1^8 d_{(1)44}^2 N_1^2 \right. \\
& - 2654208\zeta_3 C_1^4 d_{(1)44}^3 N_1 + 15925248\zeta_3 d_{(1)44}^4 - 125981 C_1^{16} N_1^4 \\
& + 13824 C_1^{12} d_{(1)44} N_1^3 + 165888 C_1^8 d_{(1)44}^2 N_1^2 \\
& \left. + 2654208 C_1^4 d_{(1)44}^3 N_1 \right] \frac{g^7}{10616832 C_1^4 N_1^4} \\
& + \left[394304\zeta_3 C_1^{16} N_1^4 - 21494784\zeta_3 C_1^{12} d_{(1)44} N_1^3 - 224169984\zeta_3 C_1^8 d_{(1)44}^2 N_1^2 \right. \\
& - 7582187520\zeta_3 C_1^4 d_{(1)44}^3 N_1 - 3301834752\zeta_3 d_{(1)44}^4 - 15552\zeta_4 C_1^{16} N_1^4 \\
& + 1492992\zeta_4 C_1^{12} d_{(1)44} N_1^3 - 53747712\zeta_4 C_1^8 d_{(1)44}^2 N_1^2 + 859963392\zeta_4 C_1^4 d_{(1)44}^3 N_1 \\
& - 5159780352\zeta_4 d_{(1)44}^4 + 325120\zeta_5 C_1^{16} N_1^4 + 18370560\zeta_5 C_1^{12} d_{(1)44} N_1^3 \\
& + 479969280\zeta_5 C_1^8 d_{(1)44}^2 N_1^2 + 4636016640\zeta_5 C_1^4 d_{(1)44}^3 N_1 + 24418713600\zeta_5 d_{(1)44}^4 \\
& - 134800515 C_1^{16} N_1^4 + 15123456 C_1^{12} d_{(1)44} N_1^3 + 189444096 C_1^8 d_{(1)44}^2 N_1^2 \\
& \left. + 2521497600 C_1^4 d_{(1)44}^3 N_1 \right] \frac{g^9}{18345885696 N_1^4} + O(g^{11}) \tag{4.4}
\end{aligned}$$

and there is no Banks-Zaks fixed point. By contrast to (3.3) and the parallel simplification of (3.6) we note that there is a non-zero one loop β -function coefficient unlike the bi-adjoint model. In the general group case this coefficient is a simple product of C_i for $i = 1$ to 4.

In order to compare with the bi-adjoint case we note that specifying to the group $SU(3)$ gives

$$\begin{aligned}
\gamma_\phi^{SU(3)^4}(g) &= -6.750000g^2 - 144.281250g^4 - 5395.552185g^6 + 2.4888387 \times 10^5 g^8 \\
&+ O(g^{10}) \\
\gamma_m^{SU(3)^4}(g) &= -40.500000g^2 + 68.343750g^4 - 95872.884627g^6 + 2.842688 \times 10^6 g^8 \\
&+ O(g^{10}) \\
\beta^{SU(3)^4}(g) &= -7.593750g^3 - 166.231934g^5 - 458.390411g^7 - 95378.353885g^9 \\
&+ O(g^{11}) . \tag{4.5}
\end{aligned}$$

With these we can illustrate the difference in the corresponding critical exponents at the Wilson-Fisher fixed point in $d = 6 - 2\epsilon$ dimensions by noting that

$$\begin{aligned}
\eta^{SU(3)^4} &= \frac{4}{9}\epsilon + \frac{11}{729}\epsilon^2 + 2[4608\zeta_3 + 12295]\frac{\epsilon^3}{59049} \\
&\quad + [88631400\zeta_3 + 4478976\zeta_4 - 115795200\zeta_5 + 33923953]\frac{\epsilon^4}{19131876} + O(\epsilon^5) \\
\eta_m^{SU(3)^4} &= \frac{8}{3}\epsilon + \frac{1006}{243}\epsilon^2 + [-22392\zeta_3 + 684799]\frac{\epsilon^3}{19683} \\
&\quad + [33249552\zeta_3 - 10882512\zeta_4 - 288886560\zeta_5 + 1436683139]\frac{\epsilon^4}{6377292} + O(\epsilon^5) \\
\omega^{SU(3)^4} &= \epsilon - \frac{467}{324}\epsilon^2 + [-18432\zeta_3 + 28807]\frac{\epsilon^3}{26244} \\
&\quad + [-467532144\zeta_3 - 17915904\zeta_4 + 508069440\zeta_5 - 192428981]\frac{\epsilon^4}{17006112} \\
&\quad + O(\epsilon^5) .
\end{aligned} \tag{4.6}$$

The non-zero one loop β -function coefficient produces the standard ϵ expansion in contrast to the bi-adjoint case where the exponents depend on $\sqrt{\epsilon}$.

Having considered a second scalar theory with a group theory structure similar to that of (2.1) which does not have a zero one loop β -function coefficient it is worth trying to understand how this arises for (2.1). There are two parts to determining the renormalization constant Z_g that leads to the β -function. These are the divergences from the 2- and 3-point functions. The former produces the value for Z_ϕ directly whereas the divergences of the latter do not immediately give Z_g . Instead it gives the combination $Z_g Z_\phi^{\frac{3}{2}}$. So for Z_g to have no simple pole at one loop means that the divergence from the 3-point function must exactly match that of Z_ϕ multiplied by $\frac{3}{2}$. From the explicit computation we find that the residue of the one loop simple pole of the 2-point function is $-\frac{1}{12}C_1C_2$ whereas that for the 3-point function is $\frac{1}{8}C_1C_2$. These are clearly in the required ratio. By contrast the respective numbers for (4.1) are $-\frac{1}{12}C_1C_2C_3C_4$ and $\frac{1}{32}C_1C_2C_3C_4$. Combining these to deduce Z_g at one loop gives the correct coefficient of $-\frac{3}{32}C_1C_2C_3C_4$ of the general β -function. Aside from the additional group theory factors the only discrepancy between both models is in the coefficient of the divergence from the 3-point function which is different by a factor of $\frac{1}{4}$. This is the origin of why (3.3) has no one loop term and rests in the group theory deriving from the one loop triangle graph which is the sole contribution at this loop order. Each subgroup G_i of the symmetry group produces the combination

$$f^{a_i p_i q_i} f^{b_i r_i q_i} f^{c_i p_i r_i} = \frac{1}{2} C_i f^{a_i b_i c_i} . \tag{4.7}$$

So for each subgroup this relation, derived from the Jacobi identity in the adjoint representation, gives a factor of $\frac{1}{2}$ to the residue of the simple pole of the 3-point function. As there is one factor of $\frac{1}{2}$ from the actual integration over the loop momentum then for the most general group $G_1 \times \dots \times G_n$ the 3-point function simple pole has a residue of $\frac{1}{2^{n+1}} \prod_{i=1}^n C_i$. Hence for this general group the one loop coefficient of the β -function, denoted by $\hat{\beta}_1(n)$, will be

$$\hat{\beta}_1(n) = \left[\frac{1}{2^{n+1}} - \frac{1}{8} \right] \prod_{i=1}^n C_i \tag{4.8}$$

which is a monotonically decreasing function and defined for all integers $n \neq 1$. The exception is because one has a free field theory for $n = 1$ since the interaction is $f^{a_1 b_1 c_1} \phi^{a_1} \phi^{b_1} \phi^{c_1}$ which vanishes due to the antisymmetry of the structure constants. Clearly $\hat{\beta}_1(2) = 0$ and so the

curiosity of (2.1) being asymptotically free as a consequence of the two loop term of the β -function is purely due to a group theory property. We note that the value of $\hat{\beta}_1(0)$ is consistent with the known low order β -function of the pure ϕ^3 theory, [3, 25]. Equally $\hat{\beta}_1(4)$ is in agreement with (4.4) and (A.3).

5 Discussion.

Scalar ϕ^3 theory has played an important role as a toy model in quantum field theory for many decades. For instance any Feynman graph generated from the basic cubic interaction can in turn generate the basic topologies that can occur in higher n -point interactions. This is achieved by formally deleting propagators in the graph theory sense and hence represents the initial point for combinatoric studies in quantum field theory. Where the theory has limitations in physics applications is that its critical dimension is six rather than four. However as noted earlier certain properties of scalar ϕ^3 theory are similar to the more involved field theories in four dimensions and hence the six dimensional model can be used to explore ideas. In this article we have studied a interesting modification whereby the scalar field is in a bi-adjoint representation of Lie groups. This is motivated by the double copy relation between Yang-Mills and on-shell gravity. While the studies of [18, 19, 20, 21] examined classical solutions to the scalar theory it has turned out that the six dimensional field theory has a peculiar property. It is unusual that asymptotic freedom is a consequence of the *two* loop term of the β -function rather than the first. However that is the case for (2.1). In studying the consequences it appears to be unique in the class of extensions that would be termed multi-adjoint as the analysis we carried out for the quartic adjoint demonstrates. It is not clear whether there is a parallel theory in four dimensions that is asymptotically free due to the two loop β -function for which the bi-adjoint six dimensional scalar field theory is the underlying laboratory. It was noted in [41] that a necessary condition for this is non-abelian gauge fields.

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A Full results for quartic adjoint.

In this appendix we record the full expressions for the renormalization group functions of the quartic adjoint scalar theory which uses similar notation to that used for the parallel expressions of the bi-adjoint case. First the field anomalous and mass anomalous dimensions are

$$\begin{aligned}
\gamma_\phi(g) = & - C_1 C_2 C_3 C_4 \frac{g^2}{12} - 19 C_1^2 C_2^2 C_3^2 C_4^2 \frac{g^4}{864} - 40421 C_1^3 C_2^3 C_3^3 C_4^3 \frac{g^6}{3981312} \\
& + \left[1910 \zeta_3 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 + 9 \zeta_4 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \right. \\
& \quad - 320 \zeta_5 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 - 150934 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \\
& \quad + 528 \zeta_3 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 - 216 \zeta_4 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 \\
& \quad - 3840 \zeta_5 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 + 3744 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 \\
& \quad + 528 \zeta_3 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 - 216 \zeta_4 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 \\
& \quad - 3840 \zeta_5 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 + 3744 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 \\
& \quad \left. + 14976 \zeta_3 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 + 5184 \zeta_4 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 \right]
\end{aligned}$$

$$\begin{aligned}
& - 46080\zeta_5 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 + 29952 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 \\
& + 528\zeta_3 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 - 216\zeta_4 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 \\
& - 3840\zeta_5 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 + 3744 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 \\
& + 14976\zeta_3 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 + 5184\zeta_4 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 \\
& - 46080\zeta_5 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 + 29952 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 \\
& + 14976\zeta_3 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 + 5184\zeta_4 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 \\
& - 46080\zeta_5 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 + 29952 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 \\
& - 27648\zeta_3 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 - 124416\zeta_4 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 \\
& - 552960\zeta_5 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 + 718848 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 \\
& + 528\zeta_3 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 - 216\zeta_4 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 \\
& - 3840\zeta_5 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 + 3744 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 \\
& + 14976\zeta_3 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 + 5184\zeta_4 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 \\
& - 46080\zeta_5 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 + 29952 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 \\
& + 14976\zeta_3 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 + 5184\zeta_4 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 \\
& - 46080\zeta_5 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 + 29952 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 \\
& - 27648\zeta_3 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 - 124416\zeta_4 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 \\
& - 552960\zeta_5 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 + 718848 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 \\
& + 14976\zeta_3 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 + 5184\zeta_4 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 \\
& - 46080\zeta_5 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 + 29952 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 \\
& - 27648\zeta_3 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 - 124416\zeta_4 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 \\
& - 552960\zeta_5 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 + 718848 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 \\
& - 27648\zeta_3 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 - 124416\zeta_4 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 \\
& - 552960\zeta_5 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 + 718848 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 \\
& + 4644864\zeta_3 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} + 2985984\zeta_4 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} \\
& - 6635520\zeta_5 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} \Big] \frac{g^8}{23887872 N_1 N_2 N_3 N_4} + O(g^{10}) \quad (\text{A.1})
\end{aligned}$$

and

$$\begin{aligned}
\gamma_m(g) = & - C_1 C_2 C_3 C_4 \frac{g^2}{2} + C_1^2 C_2^2 C_3^2 C_4^2 \frac{g^2}{96} + C_1^3 C_2^3 C_3^3 C_4^3 [3024\zeta_3 - 43537] \frac{g^6}{221184} \\
& + \left[-619077\zeta_3 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 - 91800\zeta_4 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \right. \\
& - 177630\zeta_5 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 + 2070250 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \\
& + 14544\zeta_3 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 - 1296\zeta_4 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 \\
& - 39600\zeta_5 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 + 34128 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 \\
& + 14544\zeta_3 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 - 1296\zeta_4 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 \\
& - 39600\zeta_5 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 + 34128 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 \\
& + 231552\zeta_3 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 + 31104\zeta_4 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 \\
& - 293760\zeta_5 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 + 273024 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 \\
& + 14544\zeta_3 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 - 1296\zeta_4 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 \\
& \left. - 39600\zeta_5 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 + 34128 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 \right]
\end{aligned}$$

$$\begin{aligned}
& + 231552\zeta_3 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 + 31104\zeta_4 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 \\
& - 293760\zeta_5 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 + 273024 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 \\
& + 231552\zeta_3 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 + 31104\zeta_4 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 \\
& - 293760\zeta_5 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 + 273024 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 \\
& + 1410048\zeta_3 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 - 746496\zeta_4 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 \\
& - 7879680\zeta_5 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 + 6552576 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 \\
& + 14544\zeta_3 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 - 1296\zeta_4 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 \\
& - 39600\zeta_5 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 + 34128 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 \\
& + 231552\zeta_3 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 + 31104\zeta_4 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 \\
& - 293760\zeta_5 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 + 273024 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 \\
& + 231552\zeta_3 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 + 31104\zeta_4 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 \\
& - 293760\zeta_5 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 + 273024 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 \\
& + 1410048\zeta_3 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 - 746496\zeta_4 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 \\
& - 7879680\zeta_5 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 + 6552576 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 \\
& + 231552\zeta_3 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 + 31104\zeta_4 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 \\
& - 293760\zeta_5 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 + 273024 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 \\
& + 1410048\zeta_3 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 - 746496\zeta_4 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 \\
& - 7879680\zeta_5 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 + 6552576 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 \\
& + 1410048\zeta_3 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 - 746496\zeta_4 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 \\
& - 7879680\zeta_5 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 + 6552576 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 \\
& + 49766400\zeta_3 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} + 17915904\zeta_4 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} \\
& + 9953280\zeta_5 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} \Big] \frac{g^8}{23887872 N_1 N_2 N_3 N_4} + O(g^{10}) . \quad (\text{A.2})
\end{aligned}$$

Finally, the β -function is

$$\begin{aligned}
\beta(g) = & - 3C_1 C_2 C_3 C_4 \frac{g^3}{32} - 467 C_1^2 C_2^2 C_3^2 C_4^2 \frac{g^5}{18432} \\
& + \left[48\zeta_3 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 - 125981 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \right. \\
& - 1152\zeta_3 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 + 3456 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 \\
& - 1152\zeta_3 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 + 3456 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 \\
& + 27648\zeta_3 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 + 27648 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 \\
& - 1152\zeta_3 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 + 3456 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 \\
& + 27648\zeta_3 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 + 27648 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 \\
& + 27648\zeta_3 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 + 27648 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 \\
& - 663552\zeta_3 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 + 663552 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 \\
& - 1152\zeta_3 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 + 3456 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 \\
& + 27648\zeta_3 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 + 27648 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 \\
& + 27648\zeta_3 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 + 27648 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 \\
& - 663552\zeta_3 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 + 663552 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 \\
& \left. + 27648\zeta_3 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 + 27648 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 \right]
\end{aligned}$$

$$\begin{aligned}
& - 663552\zeta_3 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 + 663552 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 \\
& - 663552\zeta_3 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 + 663552 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 \\
& + 15925248\zeta_3 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} \Big] \frac{g^7}{10616832 C_1 C_2 C_3 C_4 N_1 N_2 N_3 N_4} \\
+ & \Big[394304\zeta_3 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 - 15552 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \zeta_4 \\
& + 325120 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \zeta_5 - 134800515 C_1^4 C_2^4 C_3^4 C_4^4 N_1 N_2 N_3 N_4 \\
& - 5373696\zeta_3 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 + 373248\zeta_4 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 \\
& + 4592640\zeta_5 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 + 3780864 C_1^4 C_2^4 C_3^4 d_{(4)44} N_1 N_2 N_3 \\
& - 5373696\zeta_3 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 + 373248\zeta_4 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 \\
& + 4592640\zeta_5 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 + 3780864 C_1^4 C_2^4 C_4^4 d_{(3)44} N_1 N_2 N_4 \\
& - 37361664\zeta_3 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 - 8957952\zeta_4 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 \\
& + 79994880\zeta_5 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 + 31574016 C_1^4 C_2^4 d_{(3)44} d_{(4)44} N_1 N_2 \\
& - 5373696\zeta_3 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 + 373248\zeta_4 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 \\
& + 4592640\zeta_5 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 + 3780864 C_1^4 C_3^4 C_4^4 d_{(2)44} N_1 N_3 N_4 \\
& - 37361664\zeta_3 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 - 8957952\zeta_4 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 \\
& + 79994880\zeta_5 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 + 31574016 C_1^4 C_3^4 d_{(2)44} d_{(4)44} N_1 N_3 \\
& - 37361664\zeta_3 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 - 8957952\zeta_4 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 \\
& + 79994880\zeta_5 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 + 31574016 C_1^4 C_4^4 d_{(2)44} d_{(3)44} N_1 N_4 \\
& - 1895546880\zeta_3 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 + 214990848\zeta_4 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 \\
& + 1159004160\zeta_5 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 + 630374400 C_1^4 d_{(2)44} d_{(3)44} d_{(4)44} N_1 \\
& - 5373696\zeta_3 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 + 373248\zeta_4 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 \\
& + 4592640\zeta_5 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 + 3780864 C_2^4 C_3^4 C_4^4 d_{(1)44} N_2 N_3 N_4 \\
& - 37361664\zeta_3 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 - 8957952\zeta_4 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 \\
& + 79994880\zeta_5 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 + 31574016 C_2^4 C_3^4 d_{(1)44} d_{(4)44} N_2 N_3 \\
& - 37361664\zeta_3 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 - 8957952\zeta_4 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 \\
& + 79994880\zeta_5 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 + 31574016 C_2^4 C_4^4 d_{(1)44} d_{(3)44} N_2 N_4 \\
& - 1895546880\zeta_3 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 + 214990848\zeta_4 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 \\
& + 1159004160\zeta_5 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 + 630374400 C_2^4 d_{(1)44} d_{(3)44} d_{(4)44} N_2 \\
& - 37361664\zeta_3 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 - 8957952\zeta_4 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 \\
& + 79994880\zeta_5 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 + 31574016 C_3^4 C_4^4 d_{(1)44} d_{(2)44} N_3 N_4 \\
& - 1895546880\zeta_3 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 + 214990848\zeta_4 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 \\
& + 1159004160\zeta_5 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 + 630374400 C_3^4 d_{(1)44} d_{(2)44} d_{(4)44} N_3 \\
& - 1895546880\zeta_3 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 + 214990848\zeta_4 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 \\
& + 1159004160\zeta_5 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 + 630374400 C_4^4 d_{(1)44} d_{(2)44} d_{(3)44} N_4 \\
& - 3301834752\zeta_3 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} - 5159780352\zeta_4 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} \\
& + 24418713600\zeta_5 d_{(1)44} d_{(2)44} d_{(3)44} d_{(4)44} \Big] \frac{g^9}{18345885696 N_1 N_2 N_3 N_4} \cdot \\
& + O(g^{11})
\end{aligned} \tag{A.3}$$

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