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Langevin dynamic for the 2D Yang–Mills measure

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Abstract

We define a natural state space and Markov process associated to the stochastic Yang–Mills heat flow in two dimensions.

To accomplish this we first introduce a space of distributional connections for which holonomies along sufficiently regular curves (Wilson loop observables) and the action of an associated group of gauge transformations are both well-defined and satisfy good continuity properties. The desired state space is obtained as the corresponding space of orbits under this group action and is shown to be a Polish space when equipped with a natural Hausdorff metric.

To construct the Markov process we show that the stochastic Yang–Mills heat flow takes values in our space of connections and use the “DeTurck trick” of introducing a time dependent gauge transformation to show invariance, in law, of the solution under gauge transformations.

Our main tool for solving for the Yang–Mills heat flow is the theory of regularity structures and along the way we also develop a “basis-free” framework for applying the theory of regularity structures in the context of vector-valued noise – this provides a conceptual framework for interpreting several previous constructions and we expect this framework to be of independent interest.

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1 Introduction

The purpose of this paper and the companion article [CCHS] is to study the Langevin dynamic associated to the Euclidean Yang–Mills (YM) measure. Formally, the YM measure is written

$$d\mu_{\text{YM}}(A) = \mathcal{Z}^{-1} \exp[-S_{\text{YM}}(A)] dA, \quad (1.1)$$

where dA is a formal Lebesgue measure on the space of connections of a principal G -bundle $P \rightarrow M$, G is a compact Lie group, and \mathcal{Z} is a normalisation constant. The YM action is given by

$$S_{\text{YM}}(A) \stackrel{\text{def}}{=} \int_M |F_A(x)|^2 dx, \quad (1.2)$$

where F_A is the curvature 2-form of A , the norm $|F_A|$ is given by an Ad-invariant inner product on the Lie algebra \mathfrak{g} of G , and dx is a Riemannian volume measure

on the space-time manifold M . The YM measure plays a fundamental role in high energy physics, constituting one of the components of the Standard Model, and its rigorous construction largely remains open, see [JW06, Cha19] and the references therein. The action S_{YM} in addition plays a significant role in geometry, see e.g. [AB83, DK90].

For the rest of our discussion we will take $M = \mathbf{T}^d$, the d -dimensional torus equipped with a Euclidean inner product and normalised Haar measure, and the principal bundle P to be trivial. In particular, we will identify the space of connections on P with \mathfrak{g} -valued 1-forms on \mathbf{T}^d (implicitly fixing a global section). The results of this paper almost exclusively focus on the case $d = 2$, and the case $d = 3$ is studied in [CCHS].

A postulate of gauge theory is that all physically relevant quantities should be invariant under the action of the gauge group, which consists of the automorphisms of the principal bundle P . In our setting, the gauge group can be identified with maps $g \in \mathfrak{G}^\infty = \mathcal{C}^\infty(\mathbf{T}^d, G)$, and the corresponding action on connections is given by

$$A \mapsto A^g \stackrel{\text{def}}{=} gAg^{-1} - (dg)g^{-1} . \quad (1.3)$$

Equivalently, the 1-form A^g represents the same connection as A but in a new coordinate system (i.e. global section) determined by g . The physically relevant object is therefore not the connection A itself, but its orbit $[A]$ under the action (1.3).

In addition to the challenge of rigorously interpreting (1.1) due to the infinite-dimensionality of the space of connections, gauge invariance poses an additional difficulty that is not encountered in theories such as the Φ^p models. Indeed, since S_{YM} is invariant under the action of the infinite-dimensional gauge group \mathfrak{G}^∞ (as it should be to represent a physically relevant theory), the interpretation of (1.1) as a probability measure on the space of connections runs into the problem of the impossibility of constructing a measure that is “uniform” on each gauge orbit. Instead, one would like to quotient out the action of the gauge group and build the measure on the space of gauge orbits, but this introduces a new difficulty in that it is even less clear what the reference “Lebesgue measure” means in this case.

A natural approach to study the YM measure is to consider the Langevin dynamic associated with the action S_{YM} . Indeed, this dynamic is expected to be naturally gauge covariant and one can aim to use techniques from PDE theory to understand its behaviour. Denoting by d_A the covariant derivative associated with A and by d_A^* its adjoint, the equation governing the Langevin dynamic is formally given by

$$\partial_t A = -d_A^* F_A + \xi . \quad (1.4)$$

In coordinates this reads, for $i = 1, \dots, d$ and with summation over j implicit,¹

$$\partial_t A_i = \xi_i + \Delta A_i - \partial_{j^2}^2 A_j + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + [\partial_j A_j, A_i] , \quad (1.5)$$

¹Implicit summation over repeated indices will be in place throughout the paper with departures from this convention explicitly specified.

where ξ_1, \dots, ξ_d are independent \mathfrak{g} -valued space-time white noises on $\mathbf{R} \times \mathbf{T}^d$ with covariance induced by an ad-invariant scalar product on \mathfrak{g} . (We fix such a scalar product for the remainder of the discussion.) Equation (1.4) was the original motivation of Parisi–Wu [PW81] in their introduction of stochastic quantisation. This field has recently received renewed interest due to a development of tools able to study singular SPDEs [Hai14, GIP15], and has proven fruitful in the study and an alternative construction of the scalar Φ^4 quantum field theories [MW17b, MW17a, AK17, MW18, GH18] (see also [BG18] for a related construction).

A very basic issue with (1.4) is the lack of ellipticity of the term $d_A^* F_A$, which is a reflection of the invariance of the action S_{YM} under the gauge group. A well-known solution to this problem is to realise that if we take any sufficiently regular functional $A \mapsto H(A) \in C^\infty(\mathbf{T}^2, \mathfrak{g})$ and consider instead of (1.4) the equation

$$\partial_t A = -d_A^* F_A + d_A H(A) + \xi, \quad (1.6)$$

then, at least formally, solutions to (1.6) are gauge equivalent to those of (1.4) in the sense that there exists a time-dependent gauge transformation mapping one into the other one, at least in law. This is due to the fact that the tangent space of the gauge orbit at A (ignoring issues of regularity/topology for the moment) is given by terms of the form $d_A \omega$, where ω is an arbitrary \mathfrak{g} -valued 0-form.

A convenient choice of H is given by $H(A) = -d^* A$ which yields the so-called DeTurck–Zwanziger term [Zwa81, DeT83]

$$-d_A d^* A = dx_i (\partial_i + [A_i, \cdot]) \partial_j A_j.$$

This allows to cancel out the term $\partial_{ji}^2 A_j$ appearing in (1.5) and thus renders the equation parabolic, while still keeping the solution to the modified equation gauge-equivalent to the original one. We note that the idea to use this modified equation to study properties of the heat flow has proven a useful tool in geometric analysis [DeT83, Don85, CG13] and has appeared in works on stochastic quantisation in the physics literature [Zwa81, BHST87, DH87].

With this discussion in mind, the equation we focus on, also referred to in the sequel as the stochastic Yang–Mills (SYM) equation, is given in coordinates by

$$\partial_t A_i = \Delta A_i + \xi_i + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]]. \quad (1.7)$$

Our goal is to show the existence of a natural space of gauge orbits such that (appropriately renormalised) solutions to (1.7) define a canonical Markov process on this space. In addition, one desires a class of gauge invariant observables to be defined on this orbit space which is sufficiently rich to separate points; a popular class is that of Wilson loop observables (another being the lasso variables of Gross [Gro85]), which are defined in terms of holonomies of the connection and a variant of which is known to separate the gauge orbits in the smooth setting [Sen92].

One of the difficulties in carrying out this task is that any reasonable definition for the state space should be supported on gauge orbits of distributional connections,

and it is a priori not clear how to define holonomies (or other gauge-invariant observables) for such connections. In fact, it is not even clear how to carry out the construction to ensure that the orbits form a reasonable (e.g. Polish) space, given that the quotient of a Polish space by the action of a Polish group will typically yield a highly pathological object from a measure-theoretical perspective. (Think of even simple cases like the quotient of $L^2([0, 1])$ by the action of $H_0^1([0, 1])$ given by $(x, g) \mapsto x + \iota g$ with $\iota: H_0^1 \rightarrow L^2$ the canonical inclusion map or the quotient of the torus \mathbf{T}^2 by the action of $(\mathbf{R}, +)$ given by an irrational rotation.)

We now describe our main results on an informal level, postponing a precise formulation to Section 2, and mention connections with the existing literature and several open problems.

1.1 Outline of results

Our first contribution is to identify a natural space of distributional connections Ω_α^1 , which can be seen as a refined analogue of the classical Hölder–Besov spaces, along with an associated gauge group. An important feature of this space is that holonomies along all sufficiently regular curves (and thus Wilson loops and their variants) are canonically defined for each connection in Ω_α^1 and are continuous functions of the connection and curve. In addition, the associated space of gauge orbits is a Polish space and thus well-behaved from the viewpoint of probability theory. A byproduct of the construction of Ω_α^1 is a parametrisation-independent way of measuring the regularity of a curve which relates to α -Hölder regular curves with $\alpha > 1$ in a way that is strongly reminiscent of how p -variation relates to Hölder regularity for $\alpha \leq 1$.

In turn, we show that the SPDE (1.7) can naturally be solved in the space Ω_α^1 through mollifier approximations. More precisely, we show that for any mollifier χ^ε at scale $\varepsilon \in (0, 1]$ and $C \in L_G(\mathfrak{g}, \mathfrak{g})$ (where $L_G(\mathfrak{g}, \mathfrak{g})$ consists of all linear operators from \mathfrak{g} to itself which commute with Ad_g for any $g \in G$), the solutions to the renormalised SPDE

$$\partial_t A_i = \Delta A_i + \chi^\varepsilon * \xi_i + C A_i + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] \quad (1.8)$$

converge as Ω_α^1 -valued processes as $\varepsilon \rightarrow 0$ (with a possibility of finite-time blow-up). Observe that the addition of the mass term in (1.8) (as well as the choice of mollification with respect to a fixed coordinate system) breaks gauge-covariance for any $\varepsilon > 0$. Our final result is that gauge-covariance can be restored in the $\varepsilon \rightarrow 0$ limit. Namely, we show that for each non-anticipative mollifier χ , there exists a *unique* choice for C (depending on χ) such that in the limit $\varepsilon \rightarrow 0$, the law of the gauge orbit $[A(t)]$ is independent of χ and depends only on the gauge orbit $[A(0)]$ of the initial condition. This provides the construction of the aforementioned canonical Markov process associated to (1.7) on the space of gauge orbits.

We mention that a large part of the solution theory for (1.7) is now automatic and follows from the theory of regularity structures [Hai14, BHZ19, CH16, BCCH17]. In particular, these works guarantee that a suitable renormalisation procedure yields

convergence of the solutions inside some Hölder–Besov space. The points which are not automatic are that the limiting solution indeed takes values in the space Ω_α^1 , that it is gauge invariant, and that no diverging counterterms are required for the convergence of (1.8). One contribution of this article is to adapt the algebraic framework of regularity structures developed in [BHZ19, BCCH17] to address the latter point. Precisely, we give a natural renormalisation procedure for SPDEs with vector-valued noise and solution of the form

$$(\partial_t - \mathcal{L}_t)A_t = F_t(\mathbf{A}, \boldsymbol{\xi}), \quad t \in \mathcal{L}_+. \quad (1.9)$$

Here $(\mathcal{L}_t)_{t \in \mathcal{L}_+}$ are differential operators, \mathbf{A} and $\boldsymbol{\xi}$ represent the jet of $(A_t)_{t \in \mathcal{L}_+}$ and $(\xi_t)_{t \in \mathcal{L}_-}$ which take values in vector spaces $(W_t)_{t \in \mathcal{L}_+}$ and $(W_t)_{t \in \mathcal{L}_-}$ respectively, and the nonlinearities $(F_t)_{t \in \mathcal{L}_+}$ are smooth and local. We give a systematic way to build a regularity structure associated to (1.9) and to derive the renormalised equation without ever choosing a basis of the spaces W_t .

Example 1.1 *In addition to (1.7), an equation of interest which fits into this framework comes from the Langevin dynamic of the Yang–Mills–Higgs Lagrangian*

$$\int_{\mathbf{T}^d} \left(|F_A|^2 + |d_A \Phi|^2 - \frac{1}{2} m^2 |\Phi|^2 + \frac{1}{4} |\Phi|^4 \right) dx, \quad (1.10)$$

where A is a 1-form taking values in a Lie sub-algebra \mathfrak{g} of the anti-Hermitian operators on \mathbf{C}^N , and Φ is a \mathbf{C}^N -valued function. The associated SPDE (again with DeTurck term) reads

$$\begin{aligned} \partial_t A &= -d_A^* F_A - d_A d_A^* A + \mathbf{B}(\Phi \otimes d_A \Phi) + \xi^A, \\ \partial_t \Phi &= -d_A^* d_A \Phi + (d_A^* A) \Phi - \Phi |\Phi|^2 - m^2 \Phi + \xi^\Phi, \end{aligned} \quad (1.11)$$

where $\mathbf{B}: \mathbf{C}^N \otimes \mathbf{C}^N \rightarrow \mathfrak{g}$ is the \mathbf{R} -linear map that satisfies $\langle h, \mathbf{B}(x \otimes y) \rangle_{\mathfrak{g}} = 2\operatorname{Re}\langle hx, y \rangle_{\mathbf{C}^N}$ for all $h \in \mathfrak{g}$.

One of the consequences of our framework is that the renormalisation counterterms of (1.11) can all be constructed from iterated applications of \mathbf{B} , the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$, and the product $(A, \Phi) \mapsto A\Phi$.

1.2 Relation to previous work

There have been several earlier works on the construction of an orbit space. Mitter–Viallet [MV81] showed that the space of gauge orbits modelled on H^k for $k > \frac{d}{2} + 1$ is a smooth Hilbert manifold. More recently, Gross [Gro17] has made progress on the analogue in $H^{1/2}$ in dimension $d = 3$.

An alternative (but related) route to give meaning to the YM measure is to directly define a stochastic process indexed by a class of gauge invariant observables (e.g. Wilson loops). This approach was undertaken in earlier works on the 2D YM measure [Dri89, Sen97, Lévo3, Lévo6] which have successfully given explicit representations of the measure for general compact manifolds and principal

bundles. It is not clear, however, how to extract from these works a space of gauge orbits with a well-defined probability measure, which is somewhat closer to the physical interpretation of the measure. (This is a kind of non-linear analogue to Kolmogorov’s standard question of finding a probability measure on a space of “sufficiently regular” functions that matches a given consistent collection of n -point distributions.) In addition, this setting is ill-suited for the study of the Langevin dynamic since it is far from clear how to interpret a realisation of such a stochastic process as the initial condition for a PDE.

A partial answer was obtained in [Che19] where it was shown that a gauge-fixed version of the YM measure (for a simply-connected structure group G) can be constructed in a Banach space of distributional connections which could serve as the space of initial conditions of the PDE (1.7). Section 3 of this paper extends part of this earlier work by providing a strong generalisation of the spaces used therein (e.g. supporting holonomies along all sufficiently regular paths, while only axis-parallel paths are handled in [Che19]) and constructing an associated canonical space of gauge orbits.

Another closely related work was recently carried out in [She18]. It was shown there that the lattice gauge covariant Langevin dynamic of the scalar Higgs model (the Lagrangian of which is given by (1.10) without the $|\Phi|^4$ term and with an abelian Lie algebra) in $d = 2$ can be appropriately modified by a DeTurck–Zwanziger term and renormalised to yield local-in-time solutions in the continuum limit. The mass renormalisation term CA_i as in (1.8) is absent in [She18] due to the fact that the lattice gauge theory preserves the exact gauge symmetry, while a divergent mass renormalisation for the Higgs field Φ is still needed but preserves gauge invariance. In addition, convergence of a natural class of gauge-invariant observables was shown over short time intervals; but there was no description for the orbit space.

1.3 Open problems

It is natural to conjecture that the Markov process constructed in this paper possesses a unique invariant measure, for which the associated stochastic process indexed by Wilson loops agrees with the YM measure constructed in [Sen97, Lévo3, Lévo6]. Such a result would be one of the few known rigorous connections between the YM measure and the YM energy functional (1.2) (another connection is made in [LNo6] through a large deviations principle). A possible approach would be to show that the gauge-covariant lattice dynamic for the discrete YM measure converges to the solution to the SYM equation (1.7) identified in this paper. Combined with a gauge fixing procedure as in [Che19] and an argument of Bourgain [Bou94] along the lines of [HM18a], this convergence would prove the result (as well as strong regularity properties of the YM measure obtained from the description of the orbit space in this paper). The main difficulty to overcome is the lack of general stochastic estimates for the lattice which are available in the continuum thanks to [CH16].

Our results do not exclude finite-time blow-up of solutions to SYM (1.7), not even in the quotient space. (Since gauge orbits are unbounded, non-explosion of solutions to (1.7) is a stronger property than non-explosion of the Markov process on

gauge orbits constructed in this article). It would be of interest to determine whether the solution to SYM survives almost surely for all time for any initial condition. The weaker case of the Markov process would be handled by the above conjecture combined with the strong Feller property [HM18b] and irreducibility [HS19] which both hold in this case. The analogous result is known for the Φ_d^4 SPDE in $d = 2, 3$ [MW18]. Long-time existence of the deterministic YM heat flow in $d = 2, 3$ is also known [Rad92, CG13], but it is not clear how to adapt these methods to the stochastic setting.

It is also unclear how to extend the results of this paper to the 3D setting. In [CCHS] we analyse the SPDE (1.7) for $d = 3$ and show a form of gauge-covariance in law, which formally should give rise to a Markov process on the orbit space. However, it is unclear how to construct the orbit space, which is closely linked to the fact that Wilson loop observables become singular in $d = 3$. We give further details therein.

1.4 Outline of the paper

The paper is organised as follows. In Section 2, we give a precise formulation of our main results concerning the SPDE (1.7) and the associated Markov process on gauge orbits. In Section 3 we provide a detailed study of the space of distributional 1-forms Ω_α^1 used in the construction of the state space of the Markov process. In Section 4 we study the stochastic heat equation as an Ω_α^1 -valued process.

In Section 5 we give a canonical, basis-free framework for constructing regularity structures associated to SPDEs with vector-valued noise. Moreover, we generalise the main results of [BCCH17] on formulae for renormalisation counterterms in the scalar setting and obtain analogous vectorial formulae. We expect this framework to be useful in for a variety of systems of SPDE whose natural formulation involve vector-valued noise – in the context of (1.7) this framework allows us to directly obtain expressions for renormalisation counterterms in terms of Lie brackets and to use symmetry arguments coming from the Ad-invariance of the noises.

In Section 6 we prove local well-posedness of the SPDE (1.7), and in Section 7 we show that gauge covariance holds in law for a specific choice of renormalisation procedure which allows us to construct the canonical Markov process on gauge orbits.

1.5 Notation and conventions

We collect some notation and definitions used throughout the paper. We denote by \mathbf{R}_+ the interval $[0, \infty)$ and we identify the torus \mathbf{T}^2 with the set $[-\frac{1}{2}, \frac{1}{2})^2$. We equip \mathbf{T}^2 with the geodesic distance, which, by an abuse of notation, we denote $|x - y|$, and $\mathbf{R} \times \mathbf{T}^2$ with the parabolic distance $|(t, x) - (s, y)| = \sqrt{|t - s|} + |x - y|$.

A *mollifier* χ is a smooth function on space-time $\mathbf{R} \times \mathbf{R}^2$ (or just space \mathbf{R}^2) with support in the ball $\{z \mid |z| < \frac{1}{4}\}$ such that $\int \chi = 1$. We will assume that any space-

time mollifier χ we use satisfies $\chi(t, x_1, x_2) = \chi(t, -x_1, x_2) = \chi(t, x_1, -x_2)$.² A space-time mollifier is called *non-anticipative* if it has support in the set $\{(t, x) \mid t \geq 0\}$.

Consider a separable Banach space $(E, |\cdot|)$. For $\alpha \in [0, 1]$ and a metric space (F, d) , we denote by $\mathcal{C}^{\alpha\text{-Höl}}(F, E)$ the set of all functions $f: F \rightarrow E$ such that

$$|f|_{\alpha\text{-Höl}} \stackrel{\text{def}}{=} \sup_{x, y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty,$$

where the supremum is over all distinct $x, y \in F$. We further denote by $\mathcal{C}^\alpha(F, E)$ the space of all functions $f: F \rightarrow E$ such that

$$|f|_{\mathcal{C}^\alpha} \stackrel{\text{def}}{=} |f|_\infty + |f|_{\alpha\text{-Höl}} < \infty,$$

where $|f|_\infty \stackrel{\text{def}}{=} \sup_{x \in F} |f(x)|$. For $\alpha > 1$, we define $\mathcal{C}^\alpha(\mathbf{T}^2, E)$ (resp. $\mathcal{C}^\alpha(\mathbf{R} \times \mathbf{T}^2, E)$) to be the space of k -times differentiable functions (resp. functions that are k_0 -times differentiable in t and k_1 -times differentiable in x with $2k_0 + k_1 \leq k$), where $k \stackrel{\text{def}}{=} \lceil \alpha \rceil - 1$, with $(\alpha - k)$ -Hölder continuous k -th derivatives.

For $\alpha < 0$, let $r \stackrel{\text{def}}{=} -\lceil \alpha - 1 \rceil$ and \mathcal{B}^r denote the set of all smooth functions $\psi \in \mathcal{C}^\infty(\mathbf{T}^2)$ with $|\psi|_{\mathcal{C}^r} \leq 1$ and support in the ball $|z| < \frac{1}{4}$. Let $(\mathcal{C}^\alpha(\mathbf{T}^2, E), |\cdot|_{\mathcal{C}^\alpha})$ denote the space of distributions $\xi \in \mathcal{D}'(\mathbf{T}^2, E)$ for which

$$|\xi|_{\mathcal{C}^\alpha} \stackrel{\text{def}}{=} \sup_{\lambda \in (0, 1]} \sup_{\psi \in \mathcal{B}^r} \sup_{x \in \mathbf{T}^2} \frac{|\langle \xi, \psi_x^\lambda \rangle|}{\lambda^\alpha} < \infty,$$

where $\psi_x^\lambda(y) \stackrel{\text{def}}{=} \varepsilon^{-d} \psi(\varepsilon^{-1}(y - x))$. For $\alpha = 0$, we define \mathcal{C}^0 to simply be $L^\infty(\mathbf{T}^2, E)$, and use $\mathcal{C}(\mathbf{T}^2, E)$ to denote the space of continuous functions, both spaces being equipped with the L^∞ norm. For any $\alpha \in \mathbf{R}$, we denote by $\mathcal{C}^{0, \alpha}$ the closure of smooth functions in \mathcal{C}^α . We drop E from the notation and write simply $\mathcal{C}(\mathbf{T}^2)$, $\mathcal{C}^\alpha(\mathbf{T}^2)$, etc. whenever $E = \mathbf{R}$.

For a space \mathcal{B} of E -valued functions (or distributions) on \mathbf{T}^2 , we denote by $\Omega\mathcal{B}$ the space of E -valued 1-forms $A = \sum_{i=1}^2 A_i dx_i$ where $A_1, A_2 \in \mathcal{B}$. If \mathcal{B} is equipped with a (semi)norm $|\cdot|_{\mathcal{B}}$, we define

$$|A|_{\Omega\mathcal{B}} \stackrel{\text{def}}{=} \sum_{i=1}^2 |A_i|_{\mathcal{B}}.$$

When \mathcal{B} is of the form $\mathcal{C}(\mathbf{T}^2, E)$, $\mathcal{C}^\alpha(\mathbf{T}^2, E)$, etc., we write simply $\Omega\mathcal{C}$, $\Omega\mathcal{C}^\alpha$, etc. for $\Omega\mathcal{B}$.

Given two real vector spaces V and W we write $L(V, W)$ for the set of all linear operators from V to W . If V is equipped with a topology, we write V^* for its topological dual, and otherwise we write V^* for its algebraic dual. As mentioned in the introduction, we also write $L_G(\mathfrak{g}, \mathfrak{g}) = \{C \in L(\mathfrak{g}, \mathfrak{g}) : C \text{Ad}_g = \text{Ad}_g C \text{ for all } g \in G\}$.

²This assumption is for convenience, so that some constants in our renormalisation calculation vanish, but is not strictly necessary.

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2 Main results

In this section, we give a precise formulation of the main results described in the introduction.

2.1 State space and solution theory for SYM equation

Our first result concerns the state space of the Markov process. We collect the main features of this space in the following theorem along with precise references, and refer the reader to Section 3 for a detailed study.

Theorem 2.1 *For each $\alpha \in (\frac{2}{3}, 1)$, there exists a Banach space Ω_α^1 of distributional \mathfrak{g} -valued 1-forms on \mathbf{T}^2 with the following properties.*

- (i) *For each $A \in \Omega_\alpha^1$ and $\gamma \in \mathcal{C}^{1,\beta}([0, 1], \mathbf{T}^2)$ with $\beta \in (\frac{2}{\alpha} - 2, 1]$, the holonomy $\text{hol}(A, \gamma) \in G$ is well-defined and, on bounded balls of $\Omega_\alpha^1 \times \mathcal{C}^{1,\beta}([0, 1], \mathbf{T}^2)$, is a Hölder continuous function of (A, γ) with distances between γ 's measured in the supremum metric. In particular, Wilson loop observables are well-defined on Ω_α^1 . (See Theorem 3.18 and Proposition 3.21 combined with Young ODE theory [Ly094, FH14].)*
- (ii) *There are canonical embeddings with the classical Hölder–Besov spaces*

$$\Omega\mathcal{C}^{0,\alpha/2} \hookrightarrow \Omega_\alpha^1 \hookrightarrow \Omega\mathcal{C}^{0,\alpha-1}.$$

(See Section 3.3.)

- (iii) *Let $\mathfrak{G}^{0,\alpha}$ denote the closure of smooth functions in $C^{\alpha\text{-Hö}l}(\mathbf{T}^2, G)$. Then there is a continuous group action of $\mathfrak{G}^{0,\alpha}$ on Ω_α^1 such that $\mathfrak{D}_\alpha \stackrel{\text{def}}{=} \Omega_\alpha^1 / \mathfrak{G}^{0,\alpha}$ equipped with the quotient topology is a Polish space. (See Corollary 3.36 and Theorem 3.45.)*
- (iv) *Gauge orbits in \mathfrak{D}_α are uniquely determined by conjugacy classes of holonomies along loops. (See Proposition 3.35.)* ■

Remark 2.2 Analogous spaces could be defined on any manifold, but it is not clear whether higher dimensional versions are useful for the study of the stochastic YM equation.

Remark 2.3 Since $\text{hol}(A, \gamma)$ is independent of the parametrisation of γ , the “right” way of measuring its regularity should also be parametrisation-independent, which

is not the case of $\mathcal{C}^{1,\beta}$. This is done in Definition 3.16 which might be of independent interest.

We now turn to the results on the SPDE. Let us fix $\alpha \in (\frac{2}{3}, 1)$ and $\eta \in (-\frac{1}{2}, \alpha - 1]$. We denote $\Omega_{\alpha,T} \stackrel{\text{def}}{=} \mathcal{C}([0, T], \Omega_\alpha^1)$. Furthermore, let $\mathcal{Y} = \Omega\mathcal{C}^\eta \cup \{\mathfrak{Q}\}$ equipped with the topology whose basis sets are the balls of $\Omega\mathcal{C}^\eta$ and sets of the form $\{A \in \mathcal{Y} \mid |A|_{\mathcal{C}^\eta} > N\}$ for $N \geq 0$, where we use the convention $|\mathfrak{Q}|_{\mathcal{C}^\eta} \stackrel{\text{def}}{=} \infty$.

For $A \in \mathcal{C}(\mathbf{R}_+, \mathcal{Y})$ and $L \in (0, \infty]$, let

$$T^L(A) \stackrel{\text{def}}{=} \inf\{t \geq 0 \mid |A(t)|_{\mathcal{C}^\eta} \geq L\}.$$

We set

$$\Omega^{\text{sol}} \stackrel{\text{def}}{=} \left\{ A \in \mathcal{C}(\mathbf{R}_+, \mathcal{Y}) \mid A|_{[0, T^\infty(A)]} \in \Omega_{\alpha, T^\infty(A)}, A|_{[T^\infty(A), \infty)} \equiv \mathfrak{Q} \right\}.$$

We equip Ω^{sol} with the metric $d(\cdot, \cdot) \stackrel{\text{def}}{=} \sum_{L=1}^{\infty} 2^{-L} d_L(\cdot, \cdot)$, where

$$d_L(A, \bar{A}) \stackrel{\text{def}}{=} 1 \wedge \left\{ \sup_{t \in [0, L]} |\Theta_L(A)(t) - \Theta_L(\bar{A})(t)|_{\mathcal{C}^\eta} + |\Theta_L(A) - \Theta_L(\bar{A})|_{\Omega_{\alpha, L}} \right\}$$

and

$$\Theta_L(A) \stackrel{\text{def}}{=} A(t)\mathbf{1}_{t < T^L(A)} + A(T^L(A))\mathbf{1}_{t \geq T^L(A)}.$$

Note that $\Theta_L(A)$ is an element of $\Omega_{\alpha, T}$ for all $T > 0$.

Let us fix for the remainder of this section a space-time mollifier χ as defined in Section 1.5 and denote $\chi^\varepsilon(t, x) \stackrel{\text{def}}{=} \varepsilon^{-4} \chi(t\varepsilon^{-2}, \varepsilon^{-1}x)$. We also fix i.i.d. \mathfrak{g} -valued white noises $(\xi_i)_{i=1}^2$ on $\mathbf{R} \times \mathbf{T}^2$ and write $\xi_i^\varepsilon \stackrel{\text{def}}{=} \xi_i * \chi^\varepsilon$. Fix some $C \in L_G(\mathfrak{g}, \mathfrak{g})$ independent of ε , and for each $\varepsilon \in (0, 1]$ consider the system of PDEs on $\mathbf{R}_+ \times \mathbf{T}^2$, with $i \in \{1, 2\}$,

$$\begin{aligned} \partial_t A_i^\varepsilon &= \Delta A_i^\varepsilon + \xi_i^\varepsilon + C A_i^\varepsilon + [A_j^\varepsilon, 2\partial_j A_i^\varepsilon - \partial_i A_j^\varepsilon + [A_j^\varepsilon, A_i^\varepsilon]], \\ A^\varepsilon(0) &= a \in \Omega_\alpha^1. \end{aligned} \quad (2.1)$$

Theorem 2.4 (Local existence) *The solution A^ε converges in Ω^{sol} in probability as $\varepsilon \rightarrow 0$ to an Ω^{sol} -valued random variable A .*

Remark 2.5 Note that we could take the initial condition $a \in \Omega\mathcal{C}^\eta$, and the analogous statement would hold at the expense of changing the definition of $\Omega_{\alpha, T}$ above to $\mathcal{C}((0, T], \Omega_\alpha^1)$ (that is, we lose continuity at $t = 0$).

Remark 2.6 As one would expect, the roughest part of the solution A is already captured by the solutions Ψ to the stochastic heat equation. (In fact, one has $A = \Psi + B$ where B belongs to in $\mathcal{C}^{1-\kappa}$ for any $\kappa > 0$.) Hence, fine regularity properties of A can be inferred from those of Ψ . In particular, one could sharpen the above result to encode time regularity of the solution A at the expense of taking smaller values of α , cf. Theorem 4.13.

Remark 2.7 From our assumption that G is compact, it follows that \mathfrak{g} is reductive, namely it can be written as the direct sum of simple Lie algebras and an abelian Lie algebra. Note that if \mathfrak{h} is one of the simple components, then every $C \in L_G(\mathfrak{g}, \mathfrak{g})$ preserves \mathfrak{h} and its restriction to \mathfrak{h} is equal to $\lambda \text{id}_{\mathfrak{h}}$ for some $\lambda \in \mathbf{R}$; indeed, \mathfrak{h} is the Lie algebra of a compact Lie group (see the proof of [Kna02, Thm.4.29] for a similar statement) and thus its complexification is also simple, and the claim follows readily from Schur’s lemma. Furthermore, since these components are orthogonal under the Ad-invariant inner product on \mathfrak{g} introduced in (1.2), each white noise ξ_i also splits into independent noises, each valued in the abelian or a simple component. Eq. (2.1) then decouples into a system of equations each for a simple or abelian component, which means that it suffices to prove Theorem 2.4 in the case of a simple Lie algebra for which we can take $C \in \mathbf{R}$ (this is the approach we take in our analysis of this SPDE). In the abelian case, (2.1) is just a linear stochastic heat equation taking values in an abelian Lie algebra with C a linear map (commuting with Ad) from the abelian Lie algebra to itself, for which the solution theory is standard.

We give the proof of Theorem 2.4 in Section 6. In principle a large part of the proof is by now automatic and follows from the series of results [Hai14, CH16, BHZ19, BCCH17]. Key facts which don’t follow from general principles are that the solution takes values in the space Ω_{α}^1 (but this only requires one to show that the SHE takes values in it) and more importantly that no additional renormalisation is required. However, if one were to directly apply the framework of [Hai14, CH16, BHZ19, BCCH17], one would have to expand the system with respect to a basis of \mathfrak{g} into a system of equations driven by $d \times \dim(\mathfrak{g})$ independent \mathbf{R} -valued *scalar* space-time white noises. The renormalised equation computed using [BCCH17] would then have to be rewritten to be taken back to the setting of vector valued noises. In particular, verifying that the renormalisation counterterm takes the form prescribed above would be both laborious and not very illuminating. We instead choose to work with (2.1) intrinsically and, in Section 5, develop a framework for applying the theory of regularity structures and the formulae of [BCCH17] directly to equations with vector valued noise.

When working with scalar noises, a labelled decorated combinatorial tree τ , which represents some space-time process, corresponds to a one dimensional subspace of our regularity structure. On the other hand, if our noises take values in some vector space W , then it is natural³ for τ to index a subspace of our regularity structure isomorphic to a partially symmetrised tensor product of copies of W^* , where the particular symmetrisation is determined by the symmetries of τ .

One of our key constructions in Section 5 is a functor $\mathbf{F}(\bullet)$ which maps labelled decorated combinatorial trees, which we view as objects in a category of “symmetric sets”, to these partially symmetrised tensor product spaces in the category of vector of spaces. In other words, operations/morphisms between these trees analogous to the products and the coproducts of [BHZ19] are mapped, under this functor, to

³See Section 5.1 for more detail on why this is indeed natural.

corresponding linear maps between the vector spaces they index. This allows us to construct a regularity structure, with associated structure group and renormalisation group, without performing any basis expansions.

We also show that this functor behaves well under direct sum decompositions of the vector spaces W , which allows us to verify that our constructions in the vector noise setting are consistent with the regularity structure that would be obtained in the scalar setting if one performed a basis expansion. This last point allows us to transfer results from the setting of scalar noise to that of vector noise. One of our main results in that section is Proposition 5.65 which reformulates the renormalisation formulae of [BCCH17] in the vector noise setting.

2.2 Gauge covariance and the Markov process on orbits

The reader may wonder why we don't simply enforce $C = 0$ in (2.1) since this is allowed in our statement. One reason is that although the limit of A^ε exists for such a choice, it would depend in general on the choice of mollifier χ . More importantly, our next result shows that it is possible to counteract this by choosing C as a function of χ in such a way that not only the limit is independent of the choice of χ , but the canonical projection of A onto \mathfrak{D}_α is independent (in law!) of the choice of representative of the initial condition. This then allows us to use this SPDE to construct a ‘‘nice’’ Markov process on the gauge orbit space \mathfrak{D}_α , which would *not* be the case for any other choice of C .

We first discuss the (lack of) gauge invariance of the mollified equation (2.1) from a geometric perspective. Recall that the natural state space for A is the space \mathcal{A} of (for now smooth) connections on a principal G -bundle P (which we assume is trivial for the purpose of this article). The space of connections is an affine space modelled on the vector space $\Omega^1(\mathbf{T}^2, \text{ad}(P))$, the space of 1-forms on \mathbf{T}^2 with values in the adjoint bundle. In what follows, we drop the references to \mathbf{T}^2 and $\text{ad}(P)$.

Recall furthermore that the covariant derivative is a map $d_A: \Omega^k \rightarrow \Omega^{k+1}$ with adjoint $d_A^*: \Omega^{k+1} \rightarrow \Omega^k$. Hence, the correct geometric form of the DeTurck–Zwanziger term $d_A d_A^* A$ is really $d_A d_Z^*(A - Z) = d_A d_A^*(A - Z)$, where Z is the canonical flat connection associated with the global section of P which we implicitly chose at the very start (this choice for Z is only for convenience – any fixed ‘‘reference’’ connection Z will lead to a parabolic equation for A , e.g., the initial condition of A is used as Z in [DK90, Sec. 6.3]). The mollification operator $\chi^\varepsilon: \Omega^k \rightarrow \Omega^k$ also depends on our global section (or equivalently, on Z).

If we endow Ω^k with the distance coming from its natural L^2 Hilbert space structure then, for any $g \in \mathfrak{G}^\infty = \mathcal{C}^\infty(\mathbf{T}^2, G)$, the adjoint action $\text{Ad}_g: \Omega^k \rightarrow \Omega^k$ is an isometry with the covariance properties $\text{Ad}_g(A - Z) = A^g - Z^g$ and $\text{Ad}_g d_A \omega = d_{A^g} \text{Ad}_g \omega$. Finally, recall that F_A is a two-form in Ω^2 , and satisfies $\text{Ad}_g F_A = F_{A^g}$.

With these preliminaries in mind, for any $\xi^\varepsilon \in \mathcal{C}^\infty([0, T], \Omega^1)$, we rewrite the PDE (2.1) as

$$\partial_t A = -d_A^* F_A - d_A d_A^*(A - Z) + \xi^\varepsilon + C(A - Z), \quad A(0) = a \in \mathcal{A},$$

where $C \in \mathbf{R}$ is a constant. Note that the right and left-hand sides take values in Ω^1 . For a time-dependent gauge transformation $g \in C^\infty([0, T], \mathfrak{G}^\infty)$, we have that $B \stackrel{\text{def}}{=} A^g$ satisfies

$$\partial_t B = \text{Ad}_g \partial_t A - \text{d}_B [(\partial_t g)g^{-1}].$$

In particular, if g satisfies

$$(\partial_t g)g^{-1} = \text{d}_B^*(Z^g - Z), \quad (2.2)$$

then B solves

$$\partial_t B = -\text{d}_B^* F_B - \text{d}_B \text{d}_B^*(B - Z) + \text{Ad}_g \xi^\varepsilon + C(B - Z^g), \quad B(0) = a^{g(0)} \in \mathcal{A}. \quad (2.3)$$

The claimed gauge covariance of (2.1) is then a consequence of the non-trivial fact that one can choose the constant C in such a way that, as $\varepsilon \rightarrow 0$, B converges to the same limit in law as the SPDE (2.1) started from $a^{g(0)}$, i.e.

$$\partial_t \tilde{A} = -\text{d}_{\tilde{A}}^* F_{\tilde{A}} - \text{d}_{\tilde{A}} \text{d}_{\tilde{A}}^*(\tilde{A} - Z) + \xi^\varepsilon + C(\tilde{A} - Z), \quad \tilde{A}(0) = a^{g(0)} \in \mathcal{A}. \quad (2.4)$$

We now make this statement precise. Written in coordinates, the equations for the gauge transformed system are given by

$$\begin{aligned} \partial_t B_i &= \Delta B_i + g \xi_i^\varepsilon g^{-1} + C B_i + C(\partial_i g)g^{-1} \\ &\quad + [B_j, 2\partial_j B_i - \partial_i B_j + [B_j, B_i]], \quad B(0) = a^{g(0)} \in \Omega_\alpha^1, \\ (\partial_t g)g^{-1} &= \partial_j((\partial_j g)g^{-1}) + [B_j, (\partial_j g)g^{-1}], \quad g(0) \in \mathfrak{G}^{0,\alpha}. \end{aligned} \quad (2.5)$$

The desired gauge covariance is then stated as follows.

Theorem 2.8 (i) *For every space-time mollifier χ there exists a unique ε -independent $\bar{C} \in L_G(\mathfrak{g}, \mathfrak{g})$ with the following property. For every $C \in L_G(\mathfrak{g}, \mathfrak{g})$, $a \in \Omega_\alpha^1$, and $g(0) \in \mathfrak{G}^{0,\alpha}$, if (B, g) is the solution to (2.5) and (\bar{A}, \bar{g}) is the solution to*

$$\begin{aligned} \partial_t \bar{A}_i &= \Delta \bar{A}_i + \chi^\varepsilon * (\bar{g} \xi_i \bar{g}^{-1}) + C \bar{A}_i + (C - \bar{C})(\partial_i \bar{g})\bar{g}^{-1} \\ &\quad + [\bar{A}_j, 2\partial_j \bar{A}_i - \partial_i \bar{A}_j + [\bar{A}_j, \bar{A}_i]], \quad \bar{A}(0) = a^{g(0)}, \\ (\partial_t \bar{g})\bar{g}^{-1} &= \partial_j((\partial_j \bar{g})\bar{g}^{-1}) + [\bar{A}_j, (\partial_j \bar{g})\bar{g}^{-1}], \quad \bar{g}(0) = g(0), \end{aligned} \quad (2.6)$$

then \bar{A} and B converge in probability to the same limit in Ω^{sol} as $\varepsilon \rightarrow 0$.

(ii) *If χ is a non-anticipative mollifier and \bar{C} is as stated in item (i), then the solution A to (2.1) with $C = \bar{C}$ is independent of χ .*

As discussed above, $B = A^g$ (i.e. B is pathwise gauge equivalent to A) for any choice of C . On the other hand, if χ is non-anticipative, then $\chi^\varepsilon * (\bar{g} \xi_i \bar{g}^{-1})$ is equal in law to ξ_i^ε by Itô isometry since \bar{g} is adapted, so that when $C = \bar{C}$, the law of \bar{A} does not depend on \bar{g} anymore and \bar{A} is equal in law to the process \tilde{A} defined in (2.4), obtained by starting the dynamics for A from $a^{g(0)}$. The theorem therefore proves the desired form of gauge covariance for the choice $C = \bar{C}$.

Remark 2.9 Again, as in Remark 2.7 it suffices to prove Theorem 2.8 in the case of a simple Lie algebra for which one has $\bar{C} \in \mathbf{R}$. In this case, for non-anticipative χ , $\bar{C} = -\lambda \lim_{\varepsilon \downarrow 0} \int dz \chi^\varepsilon(z)(K * K^\varepsilon)(z)$, where K is the heat kernel, $K^\varepsilon = \chi^\varepsilon * K$, and $\lambda < 0$ is such that $\lambda \text{id}_{\mathfrak{g}}$ is the quadratic Casimir in the adjoint representation.

We finally turn to the associated Markov process on gauge orbits. To state the way in which our Markov process is canonical we introduce a particular class of Ω_α^1 -valued processes which essentially captures the “nice” ways to run the SPDE (2.1) and restart it from different representatives of gauge orbits. For an interval $I \subset \mathbf{R}$ and a metric space X , let $D(I, X)$ denote the Skorokhod space of càdlàg functions $A: I \rightarrow X$. For the remainder of this section, by a “white noise” we mean a pair of i.i.d. \mathfrak{g} -valued white noises $\xi = (\xi_1, \xi_2)$ on $\mathbf{R} \times \mathbf{T}^2$.

Definition 2.10 Setting $\hat{\Omega}_\alpha^1 \stackrel{\text{def}}{=} \Omega_\alpha^1 \cup \{\bar{\omega}\}$, a probability measure μ on $D(\mathbf{R}_+, \hat{\Omega}_\alpha^1)$ is called generative if there exists a filtered probability space $(\mathcal{O}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ supporting a white noise ξ for which the filtration $(\mathcal{F}_t)_{t \geq 0}$ is admissible (i.e. ξ is adapted to $(\mathcal{F}_t)_{t \geq 0}$ and $\xi \upharpoonright [t, \infty)$ is independent of \mathcal{F}_t for all $t \geq 0$), and a random variable $A: \mathcal{O} \rightarrow D(\mathbf{R}_+, \hat{\Omega}_\alpha^1)$ with the following properties.

1. The law of A is μ and $A(0)$ is \mathcal{F}_0 -measurable.
2. For any $0 \leq s \leq t$, let $\Phi_{s,t}: \hat{\Omega}_\alpha^1 \rightarrow \hat{\Omega}_\alpha^1$ denote the (random) solution map in the $\varepsilon \rightarrow 0$ limit of (2.1) with a non-anticipative mollifier χ and constant $C = \bar{C}$ from part (i) of Theorem 2.8.⁴ There exists a sequence of stopping times $(\sigma_j)_{j=0}^\infty$, such that $\sigma_0 = 0$ almost surely and, for all $j \geq 0$,
 - (a) if $\sigma_j = \infty$, then $\sigma_{j+1} = \infty$,
 - (b) if $\sigma_j < \infty$ and $A(\sigma_j) = \bar{\omega}$, then $\sigma_{j+1} = \sigma_j$; if $A(\sigma_j) \neq \bar{\omega}$ then $\sigma_j < \sigma_{j+1}$ and $A(t) = \Phi_{\sigma_j, t}(A(\sigma_j))$ for all $t \in [\sigma_j, \sigma_{j+1})$,
 - (c) if $\sigma_{j+1} < \infty$, then there exists an $\mathcal{F}_{\sigma_{j+1}}$ -measurable random variable $g_j: \mathcal{O} \rightarrow \mathfrak{G}^{0, \alpha}$ such that $A(\sigma_{j+1})^{g_j} = \Phi_{\sigma_j, \sigma_{j+1}}(A(\sigma_j))$. (We use the convention $\bar{\omega}^g = \bar{\omega}$ for all $g \in \mathfrak{G}^{0, \alpha}$.)
3. Let $T^* \stackrel{\text{def}}{=} \inf\{t \geq 0 \mid A(t) = \bar{\omega}\}$. If $T^* < \infty$, then $A \equiv \bar{\omega}$ on $[T^*, \infty)$ and for any non-decreasing sequence of stopping times $\tau_n \nearrow T^*$

$$\lim_{n \rightarrow \infty} \inf_{g \in \mathfrak{G}^{0, \alpha}} |A(\tau_n)^g|_\alpha = \infty.$$

If there exists $a \in \hat{\Omega}_\alpha^1$ such that $A(0) = a$ almost surely, then we call a the initial condition of μ .

Remark 2.11 In the setting of Definition 2.10, if $B: \mathcal{O} \rightarrow \hat{\Omega}_\alpha^1$ is \mathcal{F}_s -measurable, then $t \mapsto \Phi_{s,t}(B)$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$. In particular, the conditions on the process A imply that A is adapted to $(\mathcal{F}_t)_{t \geq 0}$.

Denote $\hat{\mathfrak{D}}_\alpha \stackrel{\text{def}}{=} \mathfrak{D}_\alpha \cup \{\bar{\omega}\}$ and let $\pi: \hat{\Omega}_\alpha^1 \rightarrow \hat{\mathfrak{D}}_\alpha$ denote the projection map. Note that if μ is generative, then the pushforward $\pi_*\mu$ is a probability measure

⁴Note that Φ exists and is independent of the choice of χ by part (ii) of Theorem 2.8.

on $\mathcal{C}(\mathbf{R}_+, \hat{\mathcal{D}}_\alpha)$ (rather than just on $D(\mathbf{R}_+, \hat{\mathcal{D}}_\alpha)$) thanks to items 2b, 2c, and 3 of Definition 2.10. With these notations, the Markov process on the space of gauge orbits announced in the introduction is given by the following.

Theorem 2.12 (i) *For every $a \in \hat{\Omega}_\alpha^1$, there exists a generative probability measure μ with initial condition a . Moreover, one can take in Definition 2.10 $(\mathcal{F}_t)_{t \geq 0}$ to be the filtration generated by any white noise and the process A itself to be Markov.*

(ii) *There exists a unique $\hat{\mathcal{D}}_\alpha$ -valued Markov process X such that, for every $x \in \hat{\mathcal{D}}_\alpha$, $a \in x$, and generative probability measure μ with initial condition a , the pushforward $\pi_*\mu$ is the law of X^x .*

3 Construction of the state space

The aim of this section is to find a space of distributional 1-forms and a corresponding group of gauge transformations which can be used to construct the state space for our Markov process. We would like our space to be sufficiently large to contain 1-forms with components that “look like” a free field, but sufficiently small that there is a meaningful notion of integration along smooth enough curves. Our space of 1-forms is a strengthened version of that constructed in [Che19], the main difference being that we do not restrict our notion of integration to axis-parallel paths.

3.1 Additive functions on line segments

Let \mathcal{X} denote the set of oriented line segments in \mathbf{T}^2 of length at most $\frac{1}{4}$. Specifically, denoting $B_r \stackrel{\text{def}}{=} \{v \in \mathbf{R}^2 \mid |v| \leq r\}$, we define $\mathcal{X} \stackrel{\text{def}}{=} \mathbf{T}^2 \times B_{1/4}$ (first coordinate is the initial point, second coordinate is the direction). We fix for the remainder of this section a Banach space E .

Definition 3.1 We say that $\ell = (x, v), \bar{\ell} = (\bar{x}, \bar{v}) \in \mathcal{X}$ are joinable if $\bar{x} = x + v$ and there exist $w \in \mathbf{R}^2$ and $c, \bar{c} \in [-\frac{1}{4}, \frac{1}{4}]$ such that $|w| = 1, v = cw, \bar{v} = \bar{c}w$, and $|c + \bar{c}| \leq \frac{1}{4}$. In this case, we denote $\ell \sqcup \bar{\ell} \stackrel{\text{def}}{=} (x, (c + \bar{c})w) \in \mathcal{X}$. We say that a function $A: \mathcal{X} \rightarrow E$ is additive if $A(\ell \sqcup \bar{\ell}) = A(\ell) + A(\bar{\ell})$ for all joinable $\ell, \bar{\ell} \in \mathcal{X}$. Let $\Omega = \Omega(\mathbf{T}^2, E)$ denote the space of all measurable E -valued additive functions on \mathcal{X} .

Note that additivity implies that $A(x, 0) = 0$ for all $x \in \mathbf{T}^2$ and $A \in \Omega$.

Remark 3.2 For $A \in \Omega$, one should think of $A(\ell)$ as the line integral along ℓ of a homogeneous function on the tangent bundle of \mathbf{T}^2 . To wit, any measurable function $B: \mathbf{T}^2 \times \mathbf{R}^2 \rightarrow E$ which is bounded on $\mathbf{T}^2 \times B_1$ and homogeneous in the sense that $B(x, cv) = cB(x, v)$ for all $(x, v) \in \mathbf{T}^2 \times \mathbf{R}^2$ and $c \in \mathbf{R}$, defines an element $A \in \Omega$ by

$$A(x, v) \stackrel{\text{def}}{=} \int_0^1 B(x + tv, v) dt .$$

We will primarily be interested in the case that B is a 1-form, i.e., $B(x, v)$ is linear in v , and we discuss this situation in Section 3.3. However, many definitions and estimates turn out to be more natural in the general setting of Ω .

For $\ell = (x, v) \in \mathcal{X}$, let us denote by $\ell_i \stackrel{\text{def}}{=} x$ and $\ell_f \stackrel{\text{def}}{=} x + v$ the initial and final point of ℓ respectively. We define a metric d on \mathcal{X} by

$$d(\ell, \bar{\ell}) \stackrel{\text{def}}{=} |\ell_i - \bar{\ell}_i| \vee |\ell_f - \bar{\ell}_f| .$$

For $\ell = (x, v) \in \mathcal{X}$, let $|\ell| \stackrel{\text{def}}{=} |v|$ denote its length.

Definition 3.3 We say that $\ell, \bar{\ell} \in \mathcal{X}$ are *far* if $d(\ell, \bar{\ell}) > \frac{1}{4}(|\ell| \wedge |\bar{\ell}|)$. Define the function $\varrho: \mathcal{X}^2 \rightarrow [0, \infty)$ by

$$\varrho(\ell, \bar{\ell}) \stackrel{\text{def}}{=} \begin{cases} |\ell| + |\bar{\ell}| & \text{if } \ell, \bar{\ell} \text{ are far,} \\ |\ell_i - \bar{\ell}_i| + |\ell_f - \bar{\ell}_f| + \text{Area}(\ell, \bar{\ell})^{1/2} & \text{otherwise,} \end{cases}$$

where $\text{Area}(\ell, \bar{\ell})$ is the area of the convex hull of the points $(\ell_i, \ell_f, \bar{\ell}_f, \bar{\ell}_i)$ (which is well-defined whenever $\ell, \bar{\ell}$ are not far).

Remark 3.4 If $\ell, \bar{\ell} \in \mathcal{X}$ are not far, then their lengths are of the same order, and $\text{Area}(\ell, \bar{\ell})$ is of the same order as $|\ell|[d(\bar{\ell}_i, \ell) + d(\bar{\ell}_f, \ell)]$, where, denoting $\ell = (x, v)$, we have set $d(y, \ell) \stackrel{\text{def}}{=} \inf_{t \in [-1, 2]} |x + tv - y|$ (note the set $[-1, 2]$ in the infimum instead of $[0, 1]$). In particular, it readily follows that although ϱ isn't a metric in general, it is a semimetric admitting a constant $C \geq 1$ such that for all $a, b, c \in \mathcal{X}$

$$\varrho(a, b) \leq C(\varrho(b, c) + \varrho(b, c)) . \quad (3.1)$$

For $\alpha \in [0, 1]$, we define the (extended) norm on Ω

$$|A|_\alpha \stackrel{\text{def}}{=} \sup_{\varrho(\ell, \bar{\ell}) > 0} \frac{|A(\ell) - A(\bar{\ell})|}{\varrho(\ell, \bar{\ell})^\alpha} . \quad (3.2)$$

We also write Ω_α for the Banach space $\{A \in \Omega \mid |A|_\alpha < \infty\}$ equipped with the norm $|\cdot|_\alpha$.

Remark 3.5 By additivity, any element of Ω extends uniquely to an additive function on all line segments, not just those of length less than $1/4$ and we will use this extension in the sequel without further mention. However, the supremum in (3.2) is restricted to these ‘‘short’’ line segments.

Remark 3.6 Since we know that $A \in \Omega$ vanishes on line segments of zero length, it follows that $|A(\ell)| \leq |A|_\alpha |\ell|^\alpha$, so that despite superficial appearances (3.2) is a norm on Ω_α and not just a seminorm.

We now introduce several other (semi)norms which will be used in the sequel.

Definition 3.7 Define the (extended) norm on Ω

$$|A|_{\alpha\text{-gr}} \stackrel{\text{def}}{=} \sup_{|\ell|>0} \frac{|A(\ell)|}{|\ell|^\alpha}.$$

Let $\Omega_{\alpha\text{-gr}}$ denote the Banach space $\{A \in \Omega \mid |A|_{\alpha\text{-gr}} < \infty\}$ equipped with the norm $|\cdot|_{\alpha\text{-gr}}$.

Definition 3.8 We say that a pair $\ell, \bar{\ell} \in \mathcal{X}$ form a *vee* if they are not far, have the same length $|\ell| = |\bar{\ell}|$, and have the same initial point $\ell_i = \bar{\ell}_i$. Define the (extended) seminorm on Ω

$$|A|_{\alpha\text{-vee}} \stackrel{\text{def}}{=} \sup_{\ell \neq \bar{\ell}} \frac{|A(\ell) - A(\bar{\ell})|}{\text{Area}(\ell, \bar{\ell})^{\alpha/2}},$$

where the supremum is taken over all distinct $\ell, \bar{\ell} \in \mathcal{X}$ forming a vee.

Definition 3.9 For a line segment $\ell = (x, v) \in \mathcal{X}$, let us denote the associated subset of \mathbf{T}^2 by

$$\iota(\ell) \stackrel{\text{def}}{=} \iota(x, v) \stackrel{\text{def}}{=} \{x + cv : c \in [0, 1)\}.$$

For an integer $n \geq 3$, an n -gon is a tuple $P = (\ell^1, \dots, \ell^n) \in \mathcal{X}^n$ such that

- $\ell_i^1 = \ell_f^n$, and $\ell_i^j = \ell_f^{j-1}$ for all $j = 2, \dots, n$,
- $\iota(\ell^j) \cap \iota(\ell^k) = \emptyset$ for all distinct $j, k \in \{1, \dots, n\}$, and
- $\iota(\ell^1) \cup \dots \cup \iota(\ell^n)$ has diameter at most $\frac{1}{4}$.

A 3-gon is called a *triangle*.

Note that an n -gon P splits \mathbf{T}^2 into two connected components, one of which is simply connected and we denote by \mathring{P} . We further note that this split allows us to define when two n -gons have the same orientation. For measurable subsets $X, Y \in \mathbf{T}^2$, let $X \Delta Y$ denote their symmetric difference, and $|X|$ denote the Lebesgue measure of X . For n -gons P_1, P_2 , let us denote $|P_1| \stackrel{\text{def}}{=} |\mathring{P}_1|$ and

$$|P_1; P_2| \stackrel{\text{def}}{=} \begin{cases} |\mathring{P}_1 \Delta \mathring{P}_2| & \text{if } P_1, P_2 \text{ have the same orientation} \\ |\mathring{P}_1| + |\mathring{P}_2| & \text{otherwise,} \end{cases}$$

which we observe defines a metric on the set of n -gons.

Definition 3.10 Let $P = (\ell^1, \dots, \ell^n)$ be an n -gon. For $A \in \Omega$, we denote

$$A(\partial P) \stackrel{\text{def}}{=} \sum_{j=1}^n A(\ell^j).$$

For $\alpha \in [0, 1]$ we define the quantities

$$|A|_{\alpha\text{-tr}} \stackrel{\text{def}}{=} \sup_{|P|>0} \frac{|A(\partial P)|}{|P|^{\alpha/2}}, \quad (3.3)$$

where the supremum is taken over all triangles P with $|P| > 0$, and

$$|A|_{\alpha\text{-sym}} \stackrel{\text{def}}{=} \sup_{|P; \bar{P}| > 0} \frac{|A(\partial P) - A(\partial \bar{P})|}{|P; \bar{P}|^{\alpha/2}},$$

where the supremum is taken over all triangles P, \bar{P} with $|P; \bar{P}| > 0$.

The motivation behind each norm is the following.

- The norm $|\cdot|_{\alpha\text{-vee}}$ facilitates the analysis of gauge transformations (Section 3.4).
- The norm $|\cdot|_{\alpha\text{-sym}}$ is helpful in extending the domain of definition of $A \in \Omega_\alpha$ to a wider class of curves (Section 3.2).
- The norm $|\cdot|_{\alpha\text{-tr}}$ is simpler but equivalent to $|\cdot|_{\alpha\text{-sym}}$. Furthermore, the values $A(\partial P)$ can be evaluated using Stokes' theorem (e.g., as in Lemma 4.8).

We show now that each of these norms, when combined with $|\cdot|_{\alpha\text{-gr}}$, is equivalent to $|\cdot|_\alpha$.

Theorem 3.11 *There exists $C \geq 1$ such that for all $\alpha \in [0, 1]$ and $A \in \Omega$*

$$C^{-1}|A|_\alpha \leq |A|_{\alpha\text{-gr}} + |A|_\bullet \leq C|A|_\alpha,$$

where \bullet is any one of $\alpha\text{-vee}$, $\alpha\text{-tr}$, or $\alpha\text{-sym}$.

For the proof, we require the following lemmas.

Lemma 3.12 *For $\alpha \in [0, 1]$ and $n \geq 3$, it holds that*

$$\sup_{|P| > 0} \frac{|A(\partial P)|}{|P|^{\alpha/2}} \leq C_n |A|_{\alpha\text{-tr}},$$

where the supremum is taken over all n -gons P with $|P| > 0$, and where $C_3 \stackrel{\text{def}}{=} 1$ and for $n \geq 4$

$$C_n \stackrel{\text{def}}{=} \frac{C_{n-1} + (C_{n-1}^{-2/(2-\alpha)})^{\alpha/2}}{(1 + C_{n-1}^{-2/(2-\alpha)})^{\alpha/2}}.$$

Proof. This readily follows by induction, using the two ears theorem and the fact that C_n is the optimal constant such that $x^{\alpha/2} + C_{n-1}y^{\alpha/2} \leq C_n(x+y)^{\alpha/2}$ for all $x, y \geq 0$. \square

Lemma 3.13 *There exists $C \geq 1$ such that for all $\alpha \in [0, 1]$ and $A \in \Omega$*

$$|A|_{\alpha\text{-tr}} \leq |A|_{\alpha\text{-sym}} \leq C|A|_{\alpha\text{-tr}}.$$

Proof. The first inequality is obvious by taking \bar{P} in the definition of $|A|_{\alpha\text{-sym}}$ as any degenerate triangle. For the second, let P_1, P_2 be two triangles. We need only consider the case that P_1, P_2 are oriented in the same direction. Observe that there exist $k \leq 6$ and Q_1, \dots, Q_k , where each Q_i is an n -gon with $n \leq 7$, such that $|\dot{P}_1 \triangle \dot{P}_2| = \sum_{i=1}^k |\dot{Q}_i|$, and such that $A(\partial P_1) - A(\partial P_2) = \sum_{i=1}^k A(\partial Q_i)$. It then follows from Lemma 3.12 that

$$|A(\partial P_1) - A(\partial P_2)| \lesssim |A|_{\alpha\text{-tr}} \sum_{i=1}^k |\dot{Q}_i|^{\alpha/2} \lesssim |A|_{\alpha\text{-tr}} |\dot{P}_1 \triangle \dot{P}_2|^{\alpha/2},$$

as required. \square

Proof of Theorem 3.11. We show first

$$C^{-1}|A|_{\alpha} \leq |A|_{\alpha\text{-gr}} + |A|_{\alpha\text{-vee}} \leq C|A|_{\alpha}. \quad (3.4)$$

The second inequality in (3.4) is clear (without even assuming that A is additive) since $|\ell| = \varrho(\ell, \bar{\ell})$ whenever $|\bar{\ell}| = 0$, and for any $\ell, \bar{\ell} \in \mathcal{X}$ forming a vee, we have $\varrho(\ell, \bar{\ell}) \lesssim \text{Area}(\ell, \bar{\ell})^{1/2}$.

It remains to show the first inequality in (3.4). If $\ell, \bar{\ell}$ are far, then clearly $|A(\ell) - A(\bar{\ell})| \lesssim \varrho(\ell, \bar{\ell})^{\alpha} |A|_{\alpha\text{-gr}}$. Supposing now that $\ell, \bar{\ell}$ are not far, we want to show that

$$|A(\ell) - A(\bar{\ell})| \lesssim \varrho(\ell, \bar{\ell})^{\alpha} (|A|_{\alpha\text{-gr}} + |A|_{\alpha\text{-vee}}). \quad (3.5)$$

Consider the line segment a with initial point ℓ_i and endpoint $\bar{\ell}_f$, and the line segment $\bar{a} \in \mathcal{X}$ such that $\bar{a} = (\ell_i, c(\bar{\ell}_f - \ell_i))$ for some $c > 0$ and $|\bar{a}| = |\ell|$. Note that it is possible that $|a| > \frac{1}{4}$, and thus $a \notin \mathcal{X}$, however $A(a)$ still makes sense by additivity of A . Observe that $|a_f - \bar{a}_f| \lesssim |\ell_f - \bar{\ell}_f|$ and $\text{Area}(\ell, \bar{a}) \lesssim \text{Area}(\ell, \bar{\ell})$ (for the latter, note that a is contained inside the convex hull of $\ell, \bar{\ell}$, and that \bar{a} is at most twice the length of a).

Suppose first that \bar{a} and ℓ form a vee. Then breaking up $A(a)$ into $A(\bar{a})$ and a remainder, we see by additivity of A that

$$|A(\ell) - A(a)| \lesssim |A|_{\alpha\text{-vee}} \text{Area}(\ell, \bar{\ell})^{\alpha/2} + |A|_{\alpha\text{-gr}} |\ell_f - \bar{\ell}_f|^{\alpha}.$$

Suppose now that \bar{a} and ℓ do not form a vee. Then we have $|\ell|^2 \lesssim |\text{Area}(\bar{a}, \ell)|$, and thus

$$|A(\ell) - A(a)| \lesssim |A|_{\alpha\text{-gr}} (|\ell|^{\alpha} + |\ell_f - \bar{\ell}_f|^{\alpha}) \lesssim |A|_{\alpha\text{-gr}} (\text{Area}(\ell, \bar{\ell})^{\alpha/2} + |\ell_f - \bar{\ell}_f|^{\alpha}).$$

By symmetry, one obtains

$$|A(\bar{\ell}) - A(a)| \lesssim (|A|_{\alpha\text{-vee}} + |A|_{\alpha\text{-gr}}) (\text{Area}(\ell, \bar{\ell})^{\alpha/2} + |\ell_i - \bar{\ell}_i|^{\alpha}),$$

which proves (3.5).

For the remaining inequalities, one can readily see that

$$|A|_{\alpha\text{-vee}} \lesssim |A|_{\alpha\text{-gr}} + |A|_{\alpha\text{-tr}},$$

so that the claim follows if we can show that

$$|A|_{\alpha\text{-tr}} \lesssim |A|_{\alpha} . \quad (3.6)$$

For this, consider a triangle $P = (\ell^1, \ell^2, \ell^3)$ and assume without loss of generality that $|\ell^1| \geq |\ell^2| \geq |\ell^3|$. Suppose first that P is right-angled. If ℓ^1, ℓ^2 are far, then $\sum_{j=1}^3 |\ell^j| \lesssim |P|^{1/2}$, while if ℓ^1, ℓ^2 are not far, then $\varrho(\ell^1, \ell^2) \lesssim |P|^{1/2}$. In either case, $|A(\partial P)| \leq |A|_{\alpha} |P|^{1/2}$. For general P , we can split P into two right-angled triangles P_1, P_2 with $|P_1| + |P_2| = |P|$ and $A(\partial P) = A(\partial P_1) + A(\partial P_2)$ and apply the previous case, which proves (3.6). The conclusion follows from Lemma 3.13. \square

For $A \in \Omega$ and $\ell = (x, v) \in \mathcal{X}$, define the function $\ell_A : [0, 1] \rightarrow E$ by

$$\ell_A(t) \stackrel{\text{def}}{=} A(x, tv) .$$

Lemma 3.14 *There exists a constant $C > 0$ such that for all $\alpha \in [0, 1]$, $A \in \Omega$, and $\ell, \bar{\ell} \in \mathcal{X}$ forming a vee, one has*

$$|\ell_A|_{\alpha\text{-H\"ol}} \leq |\ell|_{\alpha} |A|_{\alpha\text{-gr}} , \quad |\ell_A - \bar{\ell}_A|_{\frac{\alpha}{2}\text{-H\"ol}} \leq C \text{Area}(\ell, \bar{\ell})^{\alpha/2} |A|_{\alpha} . \quad (3.7)$$

Proof. The first inequality is obvious by additivity of A . For the second, let $0 \leq s < t \leq 1$ and denote by $\ell^{s,t}$ the sub-segment of $\ell = (x, v)$ with initial point $x + sv$ and final point $x + tv$. We claim that

$$\varrho(\ell^{s,t}, \bar{\ell}^{s,t}) \lesssim |t - s|^{1/2} \text{Area}(\ell, \bar{\ell})^{1/2} . \quad (3.8)$$

Indeed, observe that $\text{Area}(\ell, \bar{\ell}) \asymp |\ell| |\ell_f - \bar{\ell}_f|$ as a consequence of the fact that $|\ell| = |\bar{\ell}|$ and $\ell_i = \bar{\ell}_i$ by the definition of ‘‘forming a vee’’. One furthermore has the identities $|\ell^{s,t}| = |\bar{\ell}^{s,t}| = |t - s| |\ell|$, and $|\ell_f^{s,t} - \bar{\ell}_f^{s,t}| = t |\ell_f - \bar{\ell}_f|$. Hence, if $\ell^{s,t}, \bar{\ell}^{s,t}$ are far, then we must have

$$|t - s| |\ell| \lesssim t |\ell_f - \bar{\ell}_f| \leq |\ell_f - \bar{\ell}_f| ,$$

and thus

$$\varrho(\ell^{s,t}, \bar{\ell}^{s,t}) = 2|t - s| |\ell| \lesssim |t - s|^{1/2} \text{Area}(\ell, \bar{\ell})^{1/2} .$$

On the other hand, if $\ell^{s,t}, \bar{\ell}^{s,t}$ are not far, then we must have

$$t |\ell_f - \bar{\ell}_f| \lesssim |t - s| |\ell| ,$$

and thus

$$t^{1/2} |\ell_f - \bar{\ell}_f| \lesssim |t - s|^{1/2} |\ell|^{1/2} |\ell_f - \bar{\ell}_f|^{1/2} \asymp |t - s|^{1/2} \text{Area}(\ell, \bar{\ell})^{1/2} .$$

It follows that

$$\varrho(\ell^{s,t}, \bar{\ell}^{s,t}) \leq 2t |\ell_f - \bar{\ell}_f| + |t - s|^{1/2} \text{Area}(\ell, \bar{\ell})^{1/2} \lesssim |t - s|^{1/2} \text{Area}(\ell, \bar{\ell})^{1/2} ,$$

which proves (3.8). It follows that

$$|\ell_A(t) - \ell_A(s) - \bar{\ell}_A(t) + \bar{\ell}_A(s)| = |A(\ell^{s,t}) - A(\bar{\ell}^{s,t})| \lesssim |t - s|^{\alpha/2} \text{Area}(\ell, \bar{\ell})^{\alpha/2} |A|_{\alpha} ,$$

concluding the proof. \square

3.2 Extension to regular curves

In this subsection we show that any element $A \in \Omega_\alpha$ extends to a well-defined functional on sufficiently regular curves $\gamma: [0, 1] \rightarrow \mathbf{T}^2$. Given that $A(\gamma)$ should be invariant under reparametrisation of γ , we first provide a way to measure the regularity of γ in a parametrisation invariant way, and later provide relations to more familiar spaces of paths (namely paths in $\mathcal{C}^{1,\beta}$).

For a function $\gamma: [s, t] \rightarrow \mathbf{T}^2$, we denote by $\text{diam}(\gamma) \stackrel{\text{def}}{=} \sup_{u,v \in [s,t]} |\gamma(u) - \gamma(v)|$ the diameter of γ . We assume throughout this subsection that all functions $\gamma: [s, t] \rightarrow \mathbf{T}^2$ under consideration have diameter at most $\frac{1}{4}$.

We call a *partition* of an interval $[s, t]$ a finite collection of subintervals $D = \{[t_i, t_{i+1}] \mid i \in \{0, \dots, n-1\}\}$, with $t_0 = s < t_1 < \dots < t_{n-1} < t_n = t$, and we write $\mathcal{D}([s, t])$ for the set of all partitions. For a function $\gamma: [s, t] \rightarrow \mathbf{T}^2$ and $D \in \mathcal{D}([s, t])$, let γ^D be the piecewise affine interpolation of γ along D . Note that if γ is piecewise affine, then there exists $D \in \mathcal{D}([s, t])$ and elements $\ell_i = (x_i, v_i) \in \mathcal{X}$ such that, for $u \in [t_i, t_{i+1}]$, one has $\gamma(u) = x_i + v_i(u - t_i)/(t_{i+1} - t_i)$. $A(\gamma)$ is then canonically defined by $A(\gamma) = \sum_i A(\ell_i)$. (This is independent of the choice of t_i and ℓ_i parametrising γ .)

Definition 3.15 Let $A \in \Omega$ and $\gamma: [0, 1] \rightarrow \mathbf{T}^d$. We say that A *extends* to γ if the limit

$$\gamma_A(t) \stackrel{\text{def}}{=} \lim_{|D| \rightarrow 0} A(\gamma^D \upharpoonright_{[0,t]}) \quad (3.9)$$

exists for all $t \in [0, 1]$, where $D \in \mathcal{D}([0, 1])$ and $|D| \stackrel{\text{def}}{=} \max_{[a,b] \in D} |b - a|$.

The following definition provides a convenient, parametrisation invariant way to determine if a given $A \in \Omega$ extends to γ .

Definition 3.16 Let $\gamma: [0, 1] \rightarrow \mathbf{T}^2$ be a function. The *triangle process* associated to γ is defined to be the function P defined on $[0, 1]^3$, taking values in the set of triangles, such that P_{sut} is the triangle formed by $(\gamma(s), \gamma(u), \gamma(t))$.

For two functions $\gamma, \bar{\gamma}: [0, 1] \rightarrow \mathbf{T}^2$, and a subinterval $[s, t] \subset [0, 1]$, define

$$|\gamma; \bar{\gamma}|_{[s,t]} \stackrel{\text{def}}{=} \sup_{u \in [s,t]} |P_{sut}; \bar{P}_{sut}|^{1/2},$$

where P, \bar{P} are the triangle processes associated to $\gamma, \bar{\gamma}$ respectively. For $\alpha \in [0, 1]$, define further

$$|\gamma; \bar{\gamma}|_{\alpha; [s,t]} \stackrel{\text{def}}{=} \sup_{D \in \mathcal{D}([s,t])} \sum_{[a,b] \in D} |\gamma; \bar{\gamma}|_{[a,b]}^\alpha.$$

We denote $|\gamma|_{\alpha; [s,t]} \stackrel{\text{def}}{=} |\gamma; \bar{\gamma}|_{\alpha; [s,t]}$ where $\bar{\gamma}$ is any constant path. We drop the reference to the interval $[s, t]$ whenever $[s, t] = [0, 1]$.

We note the following basic properties of $|\cdot; \cdot|_\alpha$:

- $|\cdot; \cdot|_\alpha$ is symmetric and satisfies the triangle inequality but defines only a pseudometric rather than a metric since any two affine paths are at distance 0 from each other.
- The map $\alpha \mapsto |\gamma; \bar{\gamma}|_\alpha$ is decreasing in α for any $\gamma, \bar{\gamma}: [0, 1] \rightarrow \mathbf{T}^2$.
- For a typical smooth curve, $|P_{sut}|$ is of order $|t - s|^3$ (cf. (3.13) below). It follows that $|\gamma|_\alpha < \infty$ for all smooth $\gamma: [0, 1] \rightarrow \mathbf{T}^2$ if and only if $\alpha \geq \frac{2}{3}$.

Recall (see, e.g., [FV10, Def.1.6]) that a *control* is a continuous, super-additive function $\omega: \{(s, t) \mid 0 \leq s \leq t \leq 1\} \rightarrow \mathbf{R}_+$ such that $\omega(t, t) = 0$. Here super-additivity means that $\omega(s, t) + \omega(t, u) \leq \omega(s, u)$ for any $s \leq t \leq u$.

Lemma 3.17 *Let $\gamma, \bar{\gamma} \in \mathcal{C}([0, 1], \mathbf{T}^2)$ such that $|\gamma; \bar{\gamma}|_\alpha < \infty$. Then $\omega: (s, t) \mapsto |\gamma; \bar{\gamma}|_{\alpha; [s, t]}$ is a control.*

The proof of Lemma 3.17 follows in the same way as the more classical statement that $(s, t) \mapsto |\gamma|_{p\text{-var}; [s, t]}^p$ is a control, see e.g. the proof of [FV10, Prop. 5.8] (note that continuity is the only subtle part).

Theorem 3.18 *Let $0 \leq \alpha < \bar{\alpha} \leq 1$ and denote $\theta \stackrel{\text{def}}{=} \bar{\alpha}/\alpha$. Let $A \in \Omega$ with $|A|_{\bar{\alpha}\text{-sym}} < \infty$ and $\gamma \in \mathcal{C}([0, 1], \mathbf{T}^2)$ such that $|\gamma|_\alpha < \infty$. Then A extends to γ and for any partition D of $[0, 1]$*

$$|A(\gamma^D) - A(\gamma)| \leq 2^\theta \zeta(\theta) |A|_{\bar{\alpha}\text{-sym}} \sum_{[s, t] \in D} |\gamma|_{\alpha; [s, t]}^\theta, \quad (3.10)$$

where ζ is the classical Riemann zeta function. Let $\bar{\gamma} \in \mathcal{C}([0, 1], \mathbf{T}^2)$ be another path such that $|\bar{\gamma}|_\alpha < \infty$. Then

$$|A(\gamma) - A(\bar{\gamma})| \leq |A(\ell) - A(\bar{\ell})| + 2^\theta \zeta(\theta) |A|_{\bar{\alpha}\text{-sym}} |\gamma; \bar{\gamma}|_\alpha^\theta, \quad (3.11)$$

where $\ell, \bar{\ell} \in \mathcal{X}$ are the line segments connecting $\gamma(0), \gamma(1)$ and $\bar{\gamma}(0), \bar{\gamma}(1)$ respectively.

Proof. Define $\omega(s, t) \stackrel{\text{def}}{=} |\gamma; \bar{\gamma}|_{\alpha; [s, t]}$, which we note is a control by Lemma 3.17. Let D be a partition of $[0, 1]$. We will apply Young's partition coarsening argument to show that

$$|A(\gamma^D) - A(\bar{\gamma}^D)| \leq |A(\ell) - A(\bar{\ell})| + |\gamma; \bar{\gamma}|_\alpha^\theta 2^\theta \zeta(\theta) |A|_{\bar{\alpha}\text{-sym}}. \quad (3.12)$$

Let n denote the number of points in D . If $n = 2$, then the claim is obvious. If $n \geq 3$, then by superadditivity of ω there exist two adjacent subintervals $[s, u], [u, t] \in D$ such that $\omega(s, t) \leq 2\omega(0, 1)/(n-1)$. Let P, \bar{P} denote the triangle process associated with $\gamma, \bar{\gamma}$ respectively. Observe that

$$\begin{aligned} |A(\partial P_{sut}) - A(\partial \bar{P}_{sut})| &\leq |\gamma; \bar{\gamma}|_{\alpha; [s, t]}^\theta |A|_{\bar{\alpha}\text{-sym}} \leq \omega(s, t)^\theta |A|_{\bar{\alpha}\text{-sym}} \\ &\leq (2\omega(0, 1)/(n-1))^\theta |A|_{\bar{\alpha}\text{-sym}}. \end{aligned}$$

Merging the intervals $[s, u], [u, t] \in D$ into $[s, t]$ yields a coarser partition D' and we see that

$$\begin{aligned} |A(\gamma^D) - A(\bar{\gamma}^D) - (A(\gamma^{D'}) - A(\bar{\gamma}^{D'}))| &= |A(\partial P_{sut}) - A(\partial \bar{P}_{sut})| \\ &\leq (2\omega(0, 1)/(n-1))^\theta |A|_{\bar{\alpha}\text{-sym}}. \end{aligned}$$

Proceeding inductively, we obtain (3.12). It remains only to show that (3.9) exists for $t = 1$ and satisfies (3.10). By Lemma 3.17, we have

$$\lim_{\varepsilon \rightarrow 0} \sup_{|D| < \varepsilon} \sum_{[s,t] \in D} |\gamma|_{\alpha; [s,t]}^\theta = 0.$$

Observe that if D' is a refinement of D , then we can apply the uniform bound (3.12) to every $[s, t] \in D$ to obtain

$$|A(\gamma^D) - A(\gamma^{D'})| \leq 2^\theta \zeta(\theta) |A|_{\bar{\alpha}\text{-sym}} \sum_{[s,t] \in D} |\gamma|_{\alpha; [s,t]}^\theta,$$

from which the existence of (3.9) and the bound (3.10) follow. \square

For a metric space (X, d) and $p \in [1, \infty)$, recall that the p -variation $|x|_{p\text{-var}}$ of a path $x: [0, 1] \rightarrow X$ is given by

$$|x|_{p\text{-var}}^p \stackrel{\text{def}}{=} \sup_{D \in \mathcal{D}([0,1])} \sum_{[s,t] \in D} d(x(s), x(t))^p.$$

Our interest in p -variation stems from the Young integral [You36, FH14] which ensures that ODEs driven by finite p -variation paths are well-defined.

Corollary 3.19 *Let $0 \leq \alpha < \bar{\alpha} \leq 1$, $\eta \in (0, 1]$, and $p \geq \frac{1}{\eta}$. Consider $\gamma \in \mathcal{C}([0, 1], \mathbf{T}^2)$ with $|\gamma|_\alpha < \infty$ and $A \in \Omega$ with $|A|_{\bar{\alpha}\text{-sym}} + |A|_{\eta\text{-gr}} < \infty$. Then*

$$|\gamma_A|_{p\text{-var}} \leq |A|_{\eta\text{-gr}} |\gamma|_{p\eta\text{-var}}^\eta + 2^{\bar{\alpha}/\alpha} \zeta(\bar{\alpha}/\alpha) |A|_{\bar{\alpha}\text{-sym}} |\gamma|_\alpha^{\bar{\alpha}/\alpha}.$$

Proof. For any $[s, t] \subset [0, 1]$, (3.11) implies that

$$|\gamma_A(t) - \gamma_A(s)| \leq |A|_{\eta\text{-gr}} |\gamma(t) - \gamma(s)|^\eta + 2^{\bar{\alpha}/\alpha} \zeta(\bar{\alpha}/\alpha) |A|_{\bar{\alpha}\text{-sym}} |\gamma|_{\alpha; [s,t]}^{\bar{\alpha}/\alpha},$$

from which the conclusion follows by Minkowski's inequality. \square

The following result provides a convenient (now parametrisation dependent) way to control the quantity $|\gamma; \bar{\gamma}|_\alpha$. For $\beta \in [0, 1]$, let $\mathcal{C}^{1,\beta}([0, 1], \mathbf{T}^2)$ denote the space of differentiable functions $\gamma: [0, 1] \rightarrow \mathbf{T}^2$ with $\dot{\gamma} \in \mathcal{C}^\beta$. Recall that $|\cdot|_\infty$ denotes the supremum norm.

Proposition 3.20 *There exists $C > 0$ such that for all $\alpha \in [\frac{2}{3}, 1]$, $\beta \in [\frac{2}{\alpha} - 2, 1]$, and $\kappa \in [\frac{2-\alpha}{\alpha(1+\beta)}, 1]$, and $\gamma, \bar{\gamma} \in \mathcal{C}^{1,\beta}([0, 1], \mathbf{T}^2)$, it holds that*

$$|\gamma; \bar{\gamma}|_\alpha \leq C \left[(|\dot{\gamma}|_\infty + |\dot{\bar{\gamma}}|_\infty) (|\dot{\gamma}|_{\beta\text{-Hö}} + |\dot{\bar{\gamma}}|_{\beta\text{-Hö}})^\kappa |\gamma - \bar{\gamma}|_\infty^{1-\kappa} \right]^{\alpha/2}.$$

Proof. Let P, \bar{P} denote the triangle process associated to $\gamma, \bar{\gamma}$ respectively. For $0 \leq s < u < t \leq 1$, observe that

$$\begin{aligned} |P_{sut}| &\leq |\gamma(t) - \gamma(s)| \left| \gamma(u) - \gamma(s) - \frac{u-s}{t-s} (\gamma(t) - \gamma(s)) \right| \\ &\leq |t-s| |\dot{\gamma}|_\infty \left| \frac{1}{t-s} \int_s^u \int_s^t |\dot{\gamma}(r) - \dot{\gamma}(q)| \, dq \, dr \right| \\ &\leq |t-s|^{2+\beta} |\dot{\gamma}|_\infty |\dot{\gamma}|_{\beta\text{-Hö}}. \end{aligned} \quad (3.13)$$

Furthermore,

$$|P_{sut}; \bar{P}_{sut}| \lesssim \sum_{q \neq r} (|\gamma(q) - \gamma(r)| + |\bar{\gamma}(q) - \bar{\gamma}(r)|) (|\gamma(q) - \bar{\gamma}(q)| + |\gamma(r) - \bar{\gamma}(r)|),$$

where the sum is over all 2-subsets $\{q, r\}$ of $\{s, u, t\}$, whence

$$|P_{sut}; \bar{P}_{sut}| \lesssim |t-s| (|\dot{\gamma}|_\infty + |\dot{\bar{\gamma}}|_\infty) |\gamma - \bar{\gamma}|_\infty. \quad (3.14)$$

Interpolating between (3.13) and (3.14), we have for any $\kappa \in [0, 1]$

$$|P_{sut}; \bar{P}_{sut}| \lesssim (|\dot{\gamma}|_\infty + |\dot{\bar{\gamma}}|_\infty) (|\dot{\gamma}|_{\beta\text{-Hö}} + |\dot{\bar{\gamma}}|_{\beta\text{-Hö}})^\kappa |t-s|^{1+\kappa+\beta\kappa} |\gamma - \bar{\gamma}|_\infty^{1-\kappa}.$$

The conclusion following taking $\kappa \geq \frac{2-\alpha}{\alpha(1+\beta)}$ so that $\alpha(1+\kappa+\beta\kappa)/2 \geq 1$. \square

We end this subsection with a result on the continuity in p -variation of γ_A jointly in $(A, \gamma) \in \Omega_\alpha \times \mathcal{C}^{1,\beta}([0, 1], \mathbf{T}^2)$. For $\beta \in [0, 1]$, a ball in $\mathcal{C}^{1,\beta}$ is any set of the form

$$\{\gamma \in \mathcal{C}^{1,\beta}([0, 1], \mathbf{T}^2) \mid |\dot{\gamma}|_\infty + |\dot{\gamma}|_{\beta\text{-Hö}} \leq R\}$$

for some $R \geq 0$.

Proposition 3.21 *Let $\alpha \in (\frac{2}{3}, 1]$, $p > \frac{1}{\alpha}$, and $\beta \in (\frac{2}{\alpha} - 2, 1]$. There exists $\delta > 0$ such that for all $A, \bar{A} \in \Omega_\alpha$*

$$|\gamma_A - \bar{\gamma}_{\bar{A}}|_{p\text{-var}} \lesssim |A - \bar{A}|_\alpha + |A|_\alpha |\gamma - \bar{\gamma}|_\infty^\delta$$

uniformly over $\gamma, \bar{\gamma}$ in balls of $\mathcal{C}^{1,\beta}$.

Proof. Note that $|\gamma|_{1\text{-var}}$ is trivially bounded by $|\dot{\gamma}|_\infty$. Furthermore, for $\bar{\alpha} \in [\frac{2}{3}, 1]$, it follows from Proposition 3.20 that $|\gamma|_{\bar{\alpha}}$ is uniformly bounded on balls in $\mathcal{C}^{1,\bar{\beta}}$ with $\bar{\beta} = \frac{2}{\bar{\alpha}} - 2$. As a consequence (using that $\beta > \frac{2}{\alpha} - 2$), it follows from Corollary 3.19 that $|\gamma_A|_{\frac{1}{\alpha}\text{-var}} \lesssim |A|_\alpha$ uniformly on balls in $\mathcal{C}^{1,\beta}$.

On the other hand, by (3.11) and Proposition 3.20 (using again that $\beta > \frac{2}{\alpha} - 2$), there exists $\varepsilon > 0$ such that $|\gamma_A - \bar{\gamma}_{\bar{A}}|_\infty \lesssim |A|_\alpha |\gamma - \bar{\gamma}|_\infty^\varepsilon$ uniformly over balls in $\mathcal{C}^{1,\beta}$. Applying the interpolation estimate for $p > \frac{1}{\alpha}$ and $x: [0, 1] \rightarrow E$

$$|x|_{p\text{-var}} \leq (|x|_{\frac{1}{\alpha}\text{-var}})^{\frac{1}{\alpha p}} (2|x|_\infty)^{1-\frac{1}{\alpha p}},$$

it follows that for some $\delta > 0$

$$|\gamma_A - \bar{\gamma}_A|_{p\text{-var}} \lesssim |A|_\alpha |\gamma - \bar{\gamma}|_\infty^\delta$$

uniformly on balls in $\mathcal{C}^{1,\beta}$.

Finally, it follows again from Corollary 3.19 and Proposition 3.20 that

$$|\gamma_A - \gamma_{\bar{A}}|_{\frac{1}{\alpha}\text{-var}} = |\gamma_{A-\bar{A}}|_{\frac{1}{\alpha}\text{-var}} \lesssim |A - \bar{A}|_\alpha$$

uniformly over balls in $\mathcal{C}^{1,\beta}$, from which the conclusion follows. \square

3.3 Closure of smooth 1-forms

Recall that $\Omega\mathcal{C} \stackrel{\text{def}}{=} \Omega\mathcal{C}(\mathbf{T}^2, E)$ denotes the Banach space of continuous E -valued 1-forms. Following Remark 3.2, there exists a canonical map

$$\iota: \Omega\mathcal{C} \rightarrow \Omega_{1\text{-gr}} \tag{3.15}$$

defined by

$$\iota A(x, v) \stackrel{\text{def}}{=} \int_0^1 \sum_{i=1}^2 A_i(x + tv) v_i dt,$$

which is injective and satisfies $|\iota A|_{1\text{-gr}} \leq |A|_\infty$.

Definition 3.22 For $\alpha \in [0, 1]$, let Ω_α^1 and $\Omega_{\alpha\text{-gr}}^1$ denote the closure of $\iota(\Omega\mathcal{C}^\infty)$ in Ω_α and $\Omega_{\alpha\text{-gr}}$ respectively.

Remark 3.23 Recalling notation from Section 1.5, it is easy to see that ι embeds $\Omega\mathcal{C}^{\alpha/2}$ and $\Omega\mathcal{C}^{0,\alpha/2}$ continuously into Ω_α and Ω_α^1 respectively (and that the exponent $\alpha/2$ is sharp in the sense that $\Omega\mathcal{C}^\beta$ and $\Omega\mathcal{C}^{0,\beta}$ do not embed into Ω_α and Ω_α^1 for any $\beta < \alpha/2$).

Remark 3.24 Note that since any element of $\Omega\mathcal{C}^\infty$ can be approximated by a trigonometric polynomial with rational coefficients, Ω_α^1 is a separable Banach space whenever E is separable.

We now construct a continuous, linear map $\pi: \Omega_{\alpha\text{-gr}} \rightarrow \Omega\mathcal{C}^{\alpha-1}$ which is a left inverse to ι and which we will use to classify the space $\Omega_{\alpha\text{-gr}}^1$. Consider $\alpha \in (0, 1]$ and $A \in \Omega_{\alpha\text{-gr}}$. For $\ell = (z, w) \in \mathcal{X}$, $v \in B_{1/4}$, and $s \in [0, 1]$, define $X_v^{s,\ell} \in \mathcal{C}^{\alpha\text{-H\"ol}}([0, 1], E)$ by $X_v^{s,\ell}(t) \stackrel{\text{def}}{=} A(z + sw, tv)$ (note that $|X_v^{s,\ell}|_{\alpha\text{-H\"ol}} \leq |v|^\alpha |A|_{\alpha\text{-gr}}$ by Lemma 3.14). In a similar way, for $\psi \in \mathcal{C}^1(\mathbf{T}^2)$, consider $Y_v^{s,\ell} \in \mathcal{C}^1([0, 1], \mathbf{R})$ given by $Y_v^{s,\ell}(t) \stackrel{\text{def}}{=} \psi(z + sw + tv)$. We define the E -valued distribution $\pi_{\ell,v} A \in \mathcal{D}'(\mathbf{T}^2, E)$ by

$$\langle \pi_{\ell,v} A, \psi \rangle \stackrel{\text{def}}{=} |v_1 w_2 - w_1 v_2| \int_0^1 ds \int_0^1 Y_v^{s,\ell}(t) dX_v^{s,\ell}(t),$$

where the inner integral is in the Young sense.

To motivate this definition, consider the parallelogram

$$P(\ell, v) \stackrel{\text{def}}{=} \{z + sw + tv \mid s, t \in [0, 1]\} \subset \mathbf{T}^2.$$

Note that the factor $|v_1 w_2 - w_1 v_2|$ is the area of $P(\ell, v)$. By additivity of A , one has the following basic properties:

1. if ψ has support inside $P(\ell, v) \cap P(\bar{\ell}, v)$, then $\langle \pi_{\ell, v} A, \psi \rangle = \langle \pi_{\bar{\ell}, v} A, \psi \rangle$,
2. if ℓ and $\bar{\ell}$ are joinable (in which case $P(\ell \sqcup \bar{\ell}, v) = P(\ell, v) \cup P(\bar{\ell}, v)$ and $P(\ell, v), P(\bar{\ell}, v)$ intersect only on their boundaries) then $\pi_{\ell \sqcup \bar{\ell}, v} A = \pi_{\ell, v} A + \pi_{\bar{\ell}, v} A$,
3. similarly, if $cv, \bar{c}v, (c + \bar{c})v \in B_{1/4}$ for some $v \in \mathbf{R}^2$ and $c, \bar{c} \in \mathbf{R}$, then $\pi_{\ell, (c+\bar{c})v} A = \pi_{\ell, cv} A + \pi_{\ell, \bar{c}v} A$.

With these considerations, one should interpret $\langle \pi_{\ell, v} A, \psi \rangle$ as the integral of A against ψ inside $P(\ell, v)$ in the direction v . In fact, for $A \in \Omega\mathcal{C}$, one has the identity

$$\langle \pi_{\ell, v} \iota A, \psi \rangle = \sum_{i=1}^2 v_i \int_{P(\ell, v)} A_i(x) \psi(x) dx. \quad (3.16)$$

Finally, we define $\pi: \Omega_{\alpha\text{-gr}} \rightarrow \Omega\mathcal{C}^{\alpha-1}$ by setting

$$(\pi A)_i \stackrel{\text{def}}{=} \sum_{n=1}^{16} \pi_{\ell_n, \frac{1}{4}e_i} A,$$

where $\ell_n \in \mathcal{X}$ for $n = 1, \dots, 16$ are such that $P(\ell_n, \frac{1}{4}e_i)$ partition $\mathbf{T}^2 \simeq [-\frac{1}{2}, \frac{1}{2}]^2$ into 16 squares (note that this definition does not depend on the particular choice). One can show that $|\pi A|_{\Omega\mathcal{C}^{\alpha-1}} \lesssim |A|_{\Omega_{\alpha\text{-gr}}}$ (see [Che19, Prop. 3.21]) and that π is a left inverse of ι in the sense that, for all $A \in \Omega\mathcal{C}$, $\pi(\iota A) = A$ as distributions.

Observe that for $A \in \iota\Omega\mathcal{C}$, $\ell \in \mathcal{X}$, and $v \in B_{1/4}$, (3.16) implies the linearity property

$$\langle \pi_{\ell, v} A, \psi \rangle = \sum_{i=1}^2 v_i \langle (\pi A)_i, \psi \rangle \quad (3.17)$$

for all $\psi \in \mathcal{C}^1(\mathbf{T}^2)$ with support in $P(\ell, v)$. The following result shows that this property essentially characterises the space $\Omega_{\alpha\text{-gr}}^1$. Furthermore, we see that π is injective on $\Omega_{\alpha\text{-gr}}^1$ and one has a direct way to recover $A \in \Omega_{\alpha\text{-gr}}^1$ from πA through mollifier approximations. In particular, we can identify $\Omega_{\alpha\text{-gr}}^1$ (and a fortiori Ω_{α}^1) with a subspace of $\Omega\mathcal{C}^{\alpha-1}$.

Recall the definition of a mollifier from Section 1.5. For $\varepsilon \in (0, 1]$, a mollifier χ on \mathbf{R}^2 , and $A \in \Omega_{\alpha\text{-gr}}$, define $A^{\chi, \varepsilon} \in \Omega\mathcal{C}^\infty$ by

$$A_i^{\chi, \varepsilon}(z) \stackrel{\text{def}}{=} \langle (\pi A)_i, \chi^\varepsilon(\cdot - z) \rangle.$$

We then have the following characterisation of these spaces.

Proposition 3.25 *Let $\alpha \in (0, 1)$ and $A \in \Omega_{\alpha\text{-gr}}$. Then the following are equivalent:*

- (i) $A \in \Omega_{\alpha\text{-gr}}^1$
- (ii) A is a continuous function on \mathcal{X} ,

$$\lim_{\varepsilon \rightarrow 0} \sup_{|\ell| < \varepsilon} |\ell|^{-\alpha} |A(\ell)| = 0, \quad (3.18)$$

and (3.17) holds for every $\ell \in \mathcal{X}$, $v \in B_{1/4}$, and $\psi \in \mathcal{C}^1(\mathbf{T}^2)$ with support in $P(\ell, v)$.

- (iii) for every $\delta > 0$ there exists $\varepsilon > 0$ such that $|A - \iota A^{\chi, \varepsilon}|_{\alpha\text{-gr}} \leq \delta$ for all mollifiers χ on \mathbf{R}^2 .

Proof. The implication (iii) \Rightarrow (i) is obvious. We now show (i) \Rightarrow (ii). Let $A \in \Omega_{\alpha\text{-gr}}^1$ and $\delta > 0$. Consider a sequence $(A^n)_{n \geq 1}$ in $\Omega\mathcal{C}^\infty$ such that $\lim_{n \rightarrow \infty} |\iota A^n - A|_{\alpha\text{-gr}} = 0$. Since (3.17) holds for every ιA^n , we see by continuity that (3.17) holds for A . Define functions $B^n: \mathcal{X} \rightarrow E$ by

$$B^n(\ell) \stackrel{\text{def}}{=} \begin{cases} |\ell|^{-\alpha} \iota A^n(\ell) & \text{if } |\ell| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We define $B: \mathcal{X} \rightarrow E$ in the same way with ιA^n replaced by A . Observe that, since $A^n \in \Omega\mathcal{C}^\infty$, B^n is a continuous function on \mathcal{X} . Furthermore, $\lim_{n \rightarrow \infty} |\iota A^n - A|_{\alpha\text{-gr}} = 0$ is equivalent to

$$\lim_{n \rightarrow \infty} \sup_{\ell \in \mathcal{X}} |B^n(\ell) - B(\ell)| = 0.$$

Hence B is continuous on \mathcal{X} , from which continuity of A and (3.18) follow. This completes the proof of (ii).

It remains to show (ii) \Rightarrow (iii). Suppose that (ii) holds and let $\delta > 0$. Define $B: \mathcal{X} \rightarrow E$ as above, which is uniformly continuous by (ii) and compactness of \mathcal{X} . In particular, there exists $\varepsilon > 0$ such that $|B(y, v) - B(x, v)| \leq \delta$ for all $(x, v) \in \mathcal{X}$ and $y \in \mathbf{T}^2$ such that $|x - y| \leq \varepsilon$. Hence, for any $\ell = (x, v) \in \mathcal{X}$ and mollifier χ on \mathbf{T}^2 ,

$$\left| A(\ell) - \int_{\mathbf{T}^2} \chi^\varepsilon(h) A(x + h, v) dh \right| \leq \delta |\ell|^\alpha.$$

For $\ell = (x, v) \in \mathcal{X}$ and a mollifier χ , define $\chi_\ell^\varepsilon \in \mathcal{C}^\infty(\mathbf{T}^2)$ by

$$\chi_\ell^\varepsilon(z) \stackrel{\text{def}}{=} \int_0^1 \chi^\varepsilon(z - x - tv) dt.$$

Note that the support of χ_ℓ^ε shrinks to $\iota(\ell)$ as $\varepsilon \rightarrow 0$. Thus, taking $|\ell| < \frac{1}{8}$ and ε sufficiently small, we can find $\bar{\ell} \in \mathcal{X}$ such that $P(\bar{\ell}, v)$ contains the support of χ_ℓ^ε . Hence

$$\int_{\mathbf{T}^2} \chi^\varepsilon(h) A(x + h, v) dh = \langle \pi_{\bar{\ell}, v} A, \chi_\ell^\varepsilon \rangle = \sum_{i=1}^2 v_i \langle (\pi A)_i, \chi_\ell^\varepsilon \rangle = \iota A^{\chi, \varepsilon}(\ell),$$

where we used (3.17) in the second equality, and the first and third equalities follow readily from definitions and additivity of A . We conclude that, for any $\ell \in \mathcal{X}$ and mollifier χ , $|A(\ell) - \iota A^{\chi, \varepsilon}(\ell)| \leq \delta |\ell|^\alpha$, from which (iii) follows. \square

3.4 Gauge transformations

For the remainder of the section, we fix a compact Lie group G with Lie algebra \mathfrak{g} . We equip \mathfrak{g} with an arbitrary norm and henceforth take $E = \mathfrak{g}$ as our Banach space. Since G is compact, we can assume without loss of generality that G (resp. \mathfrak{g}) is a Lie subgroup of unitary matrices (resp. Lie subalgebra of anti-Hermitian matrices), so that both G and \mathfrak{g} are embedded in some normed linear space F of matrices. For $g \in G$, we denote by $\text{Ad}_g: \mathfrak{g} \rightarrow \mathfrak{g}$ the adjoint action $\text{Ad}_g(X) = gXg^{-1}$.

For $\alpha \in [0, 1]$ and a function $g: \mathbf{T}^2 \rightarrow F$, recall the definition of the seminorm $|g|_{\alpha\text{-Höl}}$ and norm $|g|_\infty$. We denote by \mathfrak{G}^α the subset $\mathcal{C}^\alpha(\mathbf{T}^2, G)$, which we note is a topological group.

Definition 3.26 Let $\alpha \in (0, 1]$, $A \in \Omega_{\alpha\text{-gr}}$, $\beta \in (\frac{1}{2}, 1]$ with $\alpha + \beta > 1$, and $g \in \mathfrak{G}^\beta$. Define $A^g \in \Omega$ by

$$A^g(\ell) \stackrel{\text{def}}{=} \int_0^1 (\text{Ad}_{g(x+tv)} d\ell_A(t) - [dg(x+tv)]g^{-1}(x+tv)),$$

where $\ell = (x, v) \in \mathcal{X}$, and where both terms make sense as \mathfrak{g} -valued Young integrals since $\alpha + \beta > 1$ and $\beta > \frac{1}{2}$. In the case $\alpha > \frac{1}{2}$, for $A, \bar{A} \in \Omega_{\alpha\text{-gr}}$ we write $A \sim \bar{A}$ if there exists $g \in \mathfrak{G}^\alpha$ such that $A^g = \bar{A}$.

Note that, in the case that A is a continuous 1-form and g is \mathcal{C}^1 , we have $d\ell_A(t) = A(x+tv)(v) dt$, hence

$$A^g(x) = \text{Ad}_{g(x)}A(x) - [dg(x)]g^{-1}(x),$$

as one expects from interpreting A as a connection. However, in the interpretation of A as a 1-form, the more natural map is $A \mapsto A^g - 0^g$, which is linear and makes sense for any $\beta \in (0, 1]$ such that $\alpha + \beta > 1$ (here 0 is an element of $\Omega_{\alpha\text{-gr}}$ and, despite the notation, 0^g is in general non-zero).

The main result of this subsection is the following.

Theorem 3.27 Let $\beta \in (\frac{2}{3}, 1]$ and $\alpha \in (0, 1]$ such that $\alpha + \frac{\beta}{2} > 1$ and $\frac{\alpha}{2} + \beta > 1$. Then the map $(A, g) \mapsto A^g$ is a continuous map from $\Omega_\alpha \times \mathfrak{G}^\beta$ (resp. $\Omega_{\alpha\text{-gr}} \times \mathfrak{G}^\beta$) into $\Omega_{\alpha \wedge \beta}$ (resp. $\Omega_{\alpha \wedge \beta\text{-gr}}$). If $\alpha \leq \beta$, then this map defines a left-group action, i.e., $(A^h)^g = A^{gh}$.

We give the proof of Theorem 3.27 at the end of this subsection. We begin by analysing the case $A = 0$.

Proposition 3.28 Let $\alpha \in (\frac{2}{3}, 1]$ and $g \in \mathfrak{G}^\alpha$. Then $|0^g|_\alpha \lesssim |g|_{\alpha\text{-Höl}} \vee |g|_{\alpha\text{-Höl}}^2$, where the proportionality constant depends only on α .

For the proof of Proposition 3.28, we require several lemmas.

Lemma 3.29 *Let $\alpha \in [0, 1]$, $g \in \mathfrak{G}^\alpha$, and $\ell = (x, v)$, $\bar{\ell} = (\bar{x}, \bar{v}) \in \mathcal{X}$ forming a vee. Consider the path $\ell_g: [0, 1] \rightarrow G$ given by $\ell_g(t) = g(x + tv)$, and similarly for $\bar{\ell}_g$. Then*

$$|\ell_g|_{\alpha\text{-HöI}} \lesssim |\ell|^\alpha |g|_{\alpha\text{-HöI}} \quad (3.19)$$

and

$$|\ell_g - \bar{\ell}_g|_{\alpha/2\text{-HöI}} \lesssim |g|_{\alpha\text{-HöI}} \text{Area}(\ell, \bar{\ell})^{\alpha/2} \quad (3.20)$$

for universal proportionality constants.

Proof. We have $|\ell_g(t) - \ell_g(s)| \leq |g|_{\alpha\text{-HöI}} |t - s|^\alpha |\ell|^\alpha$, which proves (3.19). For (3.20) we have

$$\begin{aligned} |\ell_g(t) - \ell_g(s) - \bar{\ell}_g(t) + \bar{\ell}_g(s)| &\leq |g|_{\alpha\text{-HöI}} [(2t^\alpha |\ell_f - \bar{\ell}_f|^\alpha) \wedge (|t - s|^\alpha |\ell|^\alpha)] \\ &\lesssim 2|g|_{\alpha\text{-HöI}} |t - s|^{\alpha/2} \text{Area}(\ell, \bar{\ell})^{\alpha/2}, \end{aligned}$$

where in the second inequality we used interpolation and the fact that $\text{Area}(\ell, \bar{\ell}) \asymp |\ell_f - \bar{\ell}_f| |\ell|$. \square

Lemma 3.30 *Let $\alpha \in (\frac{1}{2}, 1]$ and $g \in \mathfrak{G}^\alpha$. Then $|0^g|_{\alpha\text{-gr}} \lesssim |g|_{\alpha\text{-HöI}} \vee |g|_{\alpha\text{-HöI}}^2$, where the proportionality constant depends only on α .*

Proof. Let $\ell = (x, v) \in \mathcal{X}$. Then by (3.19) and Young's estimate

$$|0^g(\ell)| = \left| \int_0^1 dg(x + tv) g^{-1}(x + tv) \right| \lesssim (1 + |g|_{\alpha\text{-HöI}} |\ell|^\alpha) |\ell|^\alpha |g|_{\alpha\text{-HöI}},$$

which implies the claim. \square

Lemma 3.31 *Let $\alpha \in (\frac{2}{3}, 1]$ and $g \in \mathfrak{G}^\alpha$. Then $|0^g|_{\alpha\text{-vee}} \lesssim |g|_{\alpha\text{-HöI}} \vee |g|_{\alpha\text{-HöI}}^2$, where the proportionality constant depends only on α .*

Proof. Let $\ell = (x, v)$, $\bar{\ell} = (\bar{x}, \bar{v}) \in \mathcal{X}$ form a vee. Then, denoting $Y_t \stackrel{\text{def}}{=} g^{-1}(x + tv)$, $\bar{Y}_t \stackrel{\text{def}}{=} g^{-1}(x + t\bar{v})$, and $X_t \stackrel{\text{def}}{=} g(x + tv)$, $\bar{X}_t \stackrel{\text{def}}{=} g(x + t\bar{v})$, we have

$$\begin{aligned} |0^g(\ell) - 0^g(\bar{\ell})| &= \left| \int_0^1 Y_t dX_t - \int_0^1 \bar{Y}_t d\bar{X}_t \right| \\ &\leq \left| \int_0^1 (Y_t - \bar{Y}_t) dX_t \right| + \left| \int_0^1 \bar{Y}_t d(X_t - \bar{X}_t) \right|. \end{aligned}$$

Using (3.20), (3.19), Young's estimate, and the fact that $Y_0 = \bar{Y}_0$, we have

$$\begin{aligned} \left| \int_0^1 (Y_t - \bar{Y}_t) dX_t \right| &\lesssim |Y - \bar{Y}|_{\alpha/2\text{-HöI}} |X|_{\alpha\text{-HöI}} \\ &\lesssim |g|_{\alpha\text{-HöI}}^2 \text{Area}(\ell, \bar{\ell})^{\alpha/2} |\ell|^\alpha \end{aligned}$$

and

$$\begin{aligned} \left| \int_0^1 \bar{Y}_t d(X_t - \bar{X}_t) \right| &\lesssim (1 + |\bar{Y}|_{\alpha\text{-H\"{o}l}}) |X - \bar{X}|_{\alpha/2\text{-H\"{o}l}} \\ &\lesssim (1 + |\ell|^\alpha |g|_{\alpha\text{-H\"{o}l}}) |g|_{\alpha\text{-H\"{o}l}} \text{Area}(\ell, \bar{\ell})^{\alpha/2}, \end{aligned}$$

thus concluding the proof. \square

Proof of Proposition 3.28. Combining the equivalence of norms $|\cdot|_\alpha \asymp |\cdot|_{\alpha\text{-gr}} + |\cdot|_{\alpha\text{-vee}}$ from Theorem 3.11 with Lemmas 3.30 and 3.31 yields the proof. \square

For the lemmas which follow, recall that the quantity $A^g - 0^g$ makes sense for all $A \in \Omega_{\alpha\text{-gr}}$ and $g \in \mathfrak{G}^\beta$ provided that $\alpha, \beta \in (0, 1]$ with $\alpha + \beta > 1$.

Lemma 3.32 *Let $\alpha, \beta \in (0, 1]$ such that $\alpha + \beta > 1$, $A \in \Omega_{\alpha\text{-gr}}$, and $g \in \mathfrak{G}^\beta$. Then*

$$|A^g - 0^g - A|_{\alpha\text{-gr}} \lesssim (|g - 1|_\infty + |g|_{\beta\text{-H\"{o}l}}) |A|_{\alpha\text{-gr}}, \quad (3.21)$$

where the proportionality constant depends only on α and β .

Proof. Let $\ell = (x, v) \in \mathcal{X}$, $A \in \Omega$, and $g: \mathbf{T}^2 \rightarrow G$. Using notation from Lemma 3.29, note that

$$(A^g - 0^g - A)(\ell) = \int_0^1 (\text{Ad}_{\ell_g(t)} - 1) d\ell_A(t). \quad (3.22)$$

Using (3.7), (3.19), and Young's estimate, we obtain

$$|A^g(\ell) - 0^g(\ell) - A(\ell)| \lesssim (|g - 1|_\infty + |\ell|^\beta |g|_{\beta\text{-H\"{o}l}}) |\ell|^\alpha |A|_{\alpha\text{-gr}},$$

which proves (3.21). \square

Lemma 3.33 *Let $\alpha, \beta \in (0, 1]$ such that $\frac{\beta}{2} + \alpha > 1$ and $\frac{\alpha}{2} + \beta > 1$, $A \in \Omega_\alpha$, and $g \in \mathfrak{G}^\beta$. Then*

$$|A^g - 0^g - A|_{\alpha \wedge \beta\text{-vee}} \lesssim (|g - 1|_\infty + |g|_{\beta\text{-H\"{o}l}}) |A|_\alpha, \quad (3.23)$$

where the proportionality constant depends only on α and β .

Proof. Let $\ell, \bar{\ell} \in \mathcal{X}$ form a vee. Recall the identity (3.22). By (3.7), (3.20), and Young's estimate (since $\frac{\beta}{2} + \alpha > 1$), we have

$$\left| \int_0^1 (\text{Ad}_{\ell_g(t)} - \text{Ad}_{\bar{\ell}_g(t)}) d\ell_A(t) \right| \lesssim |g|_{\beta\text{-H\"{o}l}} \text{Area}(\ell, \bar{\ell})^{\beta/2} |\ell|^\alpha |A|_{\alpha\text{-gr}}.$$

Similarly, using (3.7), (3.19), and Young's estimate (since $\beta + \frac{\alpha}{2} > 1$), we have

$$\left| \int_0^1 (\text{Ad}_{\bar{\ell}_g(t)} - 1) d(\ell_A(t) - \bar{\ell}_A(t)) \right| \lesssim (|g - 1|_\infty + |g|_{\beta\text{-H\"{o}l}} |\ell|^\beta) |A|_\alpha \text{Area}(\ell, \bar{\ell})^{\alpha/2}.$$

Note that the integrals on the left-hand sides of the previous two bounds add to $(A^g - 0^g - A)(\ell) - (A^g - 0^g - A)(\bar{\ell})$, from which (3.23) follows. \square

Proof of Theorem 3.27. The fact that the action of \mathfrak{G}^β maps $\Omega_{\alpha\text{-gr}}$ into $\Omega_{\alpha\wedge\beta\text{-gr}}$ follows from Proposition 3.28 and Lemma 3.32. The fact that the action of \mathfrak{G}^β maps Ω_α into $\Omega_{\alpha\wedge\beta}$ follows by combining the equivalence of norms $|\cdot|_\alpha \asymp |\cdot|_{\alpha\text{-gr}} + |\cdot|_{\alpha\text{-vee}}$ from Theorem 3.11 with Proposition 3.28 and Lemmas 3.32 and 3.33. The fact that $(A, g) \mapsto A^g$ is continuous in both cases follows from writing

$$A^g - B^h = ((A - B)^h - 0^h - (A - B)) - ((A^g)^{hg^{-1}} - 0^{hg^{-1}} - A^g) - 0^{hg^{-1}} + (A - B)$$

and noting that all four terms vanish in $\Omega_{\alpha\text{-gr}}$ (resp. Ω_α) as $(A, g) \rightarrow (B, h)$ in $\Omega_{\alpha\text{-gr}} \times \mathfrak{G}^\beta$ (resp. $\Omega_\alpha \times \mathfrak{G}^\beta$) again by Proposition 3.28 and Lemmas 3.32 and 3.33. Finally, if $\alpha \leq \beta$, the fact that $A^{gh} = (A^h)^g$ follows from the identity

$$d(gh)(gh)^{-1} = (dg)g^{-1} + \text{Ad}_g[(dh)h^{-1}].$$

Combining all of these claims completes the proof. \square

3.5 Holonomies and recovering gauge transformations

The main result of this subsection, Proposition 3.35, provides a way to recover the gauge transformation that transforms between gauge equivalent elements of $\Omega_{\alpha\text{-gr}}$. This result can be seen as a version of [Seng2, Prop. 2.1.2] for the non-smooth case (see also [LNo6, Lem. 3]).

Let us fix $\alpha \in (\frac{1}{2}, 1]$ throughout this subsection. For $\ell \in \mathcal{X}$ and $A \in \Omega_{\alpha\text{-gr}}$, the ODE

$$dy(t) = y(t) d\ell_A(t), \quad y(0) = 1,$$

admits a unique solution $y: [0, 1] \rightarrow G$ as a Young integral (thanks to Lemma 3.14). Furthermore, the map $\ell_A \mapsto y$ is locally Lipschitz when both sides are equipped with $|\cdot|_{\alpha\text{-Höl}}$. We define the holonomy of A along ℓ as $\text{hol}(A, \ell) \stackrel{\text{def}}{=} y(1)$. As usual, we extend the definition $\text{hol}(A, \gamma)$ to any piecewise affine path $\gamma: [0, 1] \rightarrow \mathbf{T}^2$ by taking the ordered product of the holonomies along individual line segments.

Remark 3.34 Recall from item (i) of Theorem 2.1 that, provided $\alpha \in (\frac{2}{3}, 1]$ and $\beta \in (\frac{2}{\alpha} - 2, 1]$, the holonomy $\text{hol}(A, \gamma)$ is well-defined for all paths γ piecewise in $\mathcal{C}^{1,\beta}$ (rather than only piecewise affine) and all $A \in \Omega_\alpha^1$.

For any $g \in \mathfrak{G}^\alpha$ and any piecewise affine path γ , note the familiar identity

$$\text{hol}(A^g, \gamma) = g(\gamma(0)) \text{hol}(A, \gamma) g(\gamma(1))^{-1}. \quad (3.24)$$

For $x, y \in \mathbf{T}^2$, let \mathcal{L}_{xy} denote the set of piecewise affine paths $\gamma: [0, 1] \rightarrow \mathbf{T}^2$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Proposition 3.35 *Let $\alpha \in (\frac{1}{2}, 1]$ and $A, \bar{A} \in \Omega_{\alpha\text{-gr}}$. Then the following are equivalent:*

- (i) $A \sim \bar{A}$.

- (ii) there exists $x \in \mathbf{T}^2$ and $g_0 \in G$ such that $\text{hol}(\bar{A}, \gamma) = g_0 \text{hol}(A, \gamma) g_0^{-1}$ for all $\gamma \in \mathcal{L}_{xx}$.
- (iii) for every $x \in \mathbf{T}^2$ there exists $g_x \in G$ such that $\text{hol}(\bar{A}, \gamma) = g_x \text{hol}(A, \gamma) g_x^{-1}$ for all $\gamma \in \mathcal{L}_{xx}$.

Furthermore, if (ii) holds, then there exists a unique $g \in \mathfrak{G}^\alpha$ such that $g(x) = g_0$ and $A^g = \bar{A}$. The element g is determined by

$$g(y) = \text{hol}(\bar{A}, \gamma_{xy})^{-1} g_0 \text{hol}(A, \gamma_{xy}), \quad (3.25)$$

where γ_{xy} is any element of \mathcal{L}_{xy} , and satisfies

$$|g|_{\alpha\text{-Höl}} \lesssim |A|_{\alpha\text{-gr}} + |\bar{A}|_{\alpha\text{-gr}}. \quad (3.26)$$

Proof. The implication (i) \Rightarrow (iii) is clear from (3.24) and the implication (iii) \Rightarrow (ii) is trivial. Hence suppose (ii) holds. Let us define $g(y)$ using (3.25), which we note does not depend on the choice of path $\gamma_{xy} \in \mathcal{L}_{xy}$. Then one can readily verify the bound (3.26) and that $A^g = \bar{A}$, which proves (i). The fact that g is the unique element in \mathfrak{G}^α such that $g(x) = g_0$ and $A^g = \bar{A}$ follows again from (3.24). \square

3.6 The orbit space

We define and study in this subsection the space of gauge orbits of the Banach space Ω_α^1 . Let $\mathfrak{G}^{0,\alpha}$ denote the closure of $\mathcal{C}^\infty(\mathbf{T}^2, G)$ in \mathfrak{G}^α . The following is a simple corollary of Theorem 3.27.

Corollary 3.36 *Let $\alpha \in (\frac{2}{3}, 1]$. Then $(A, g) \mapsto A^g$ is a continuous left group action of $\mathfrak{G}^{0,\alpha}$ on Ω_α^1 and on $\Omega_{\alpha\text{-gr}}^1$.*

Proof. It holds that $A^g \in \mathfrak{v}\Omega\mathcal{C}^\infty$ whenever $A \in \mathfrak{v}\Omega\mathcal{C}^\infty$ and $g \in \mathcal{C}^\infty(\mathbf{T}^2, G)$. The conclusion follows from Theorem 3.27 by continuity of $(A, g) \mapsto A^g$. \square

We are now ready to define our desired space of orbits.

Definition 3.37 For $\alpha \in (\frac{2}{3}, 1]$, let \mathfrak{D}_α denote the space of orbits $\Omega_\alpha^1 / \mathfrak{G}^{0,\alpha}$ equipped with the quotient topology. For every $A \in \Omega_\alpha^1$, let $\mathfrak{D}_\alpha \ni [A] \stackrel{\text{def}}{=} \{A^g : g \in \mathfrak{G}^{0,\alpha}\} \subset \Omega_\alpha^1$ denote the corresponding gauge orbit.

We next show that the restriction to the subgroup $\mathfrak{G}^{0,\alpha}$ is natural in the sense that $\mathfrak{G}^{0,\alpha}$ is precisely the stabiliser of Ω_α^1 . For this, we use the following version of the standard fact that the closure of smooth functions yield the ‘‘little Hölder’’ spaces.

Lemma 3.38 *For $\alpha \in (0, 1)$, one has $g \in \mathfrak{G}^{0,\alpha}$ if and only if $\lim_{\varepsilon \rightarrow 0} \sup_{|x-y| < \varepsilon} |x-y|^{-\alpha} |g(x) - g(y)| = 0$.* \square

We then have the following general statement.

Proposition 3.39 *Let $\alpha \in (\frac{1}{2}, 1)$ and $A \in \Omega_{\alpha\text{-gr}}^1$. Suppose that $A^g \in \Omega_{\alpha\text{-gr}}^1$ for some $g \in \mathfrak{G}^\alpha$. Then $g \in \mathfrak{G}^{0,\alpha}$.*

Proof. By part (ii) of Proposition 3.25,

$$\lim_{\varepsilon \rightarrow 0} \sup_{|\ell| < \varepsilon} \frac{|A(\ell)| + |A^g(\ell)|}{|\ell|^\alpha} = 0. \quad (3.27)$$

Combining (3.27) with the expression for g in (3.25), we conclude that $g \in \mathfrak{G}^{0,\alpha}$ by Lemma 3.38. \square

In general, the quotient of a Polish space by the continuous action of a Polish group has no nice properties. In the remainder of this subsection, we show that the space \mathfrak{D}_α for $\alpha \in (\frac{2}{3}, 1)$ is itself a Polish space and we exhibit a metric D_α for its topology. We first show that these orbits are very well-behaved in the following sense.

Lemma 3.40 *Let $\alpha \in (\frac{2}{3}, 1)$. For every $A \in \Omega_\alpha^1$, the gauge orbit $[A]$ is closed.*

Proof. Since Ω_α^1 is a separable Banach space, it suffices to show that, for every $B \in \Omega_\alpha^1$ and any sequence $A_n \in [B]$ such that $A_n \rightarrow A$ in Ω_α^1 , one has $A \in [B]$. Since the A_n are uniformly bounded, the corresponding gauge transformations g_n such that $A_n = B^{g_n}$ are uniformly bounded in \mathfrak{G}^α by (3.26). Since $\mathfrak{G}^\alpha \subset \mathfrak{G}^\beta$ compactly for $\beta < \alpha$, we can assume modulo passing to a subsequence that $g_n \rightarrow g$ in \mathfrak{G}^β , which implies that $A = B^g$ by Theorem 3.27. Since however we know that $A \in \Omega_\alpha^1$, we conclude that $g \in \mathfrak{G}^{0,\alpha}$ by Proposition 3.39, so that $A \in [B]$ as required. \square

In the next step, we introduce a complete metric k_α on Ω_α^1 which generates the same topology as $|\cdot|_\alpha$, but shrinks distances at infinity so that, for large r , points on the sphere with radius r are close to each other but such that the spheres with radii r and $2r$ are still far apart. We then define the metric D_α on \mathfrak{D}_α as the Hausdorff distance associated with k_α .

Definition 3.41 Let $\alpha \in (0, 1]$. For $A, B \in \Omega_\alpha^1$, set

$$K_\alpha(A, B) \stackrel{\text{def}}{=} \frac{||A|_\alpha - |B|_\alpha| + 1}{(|A|_\alpha \wedge |B|_\alpha) + 1} (|A - B|_\alpha \wedge 1),$$

and define the metric

$$k_\alpha(A, B) \stackrel{\text{def}}{=} \inf_{Z_0, \dots, Z_n} \sum_{i=1}^n K_\alpha(Z_{i-1}, Z_i), \quad (3.28)$$

where the inf is over all finite sequences $Z_0, \dots, Z_n \in \Omega_\alpha^1$ with $Z_0 = A$ and $Z_n = B$.

Note that

$$k_\alpha(A, B) \leq K_\alpha(A, B) \leq \frac{1}{r+1} \quad (3.29)$$

for all A, B in the sphere

$$S_\alpha^r \stackrel{\text{def}}{=} \{C \in \Omega_\alpha^1 : |C|_\alpha = r\}.$$

On the other hand, for $r_1, r_2 > 0$, if $A \in S_\alpha^{r_1}$ and $B \in S_\alpha^{r_2}$, then

$$K_\alpha(A, B) \geq \frac{|r_1 - r_2|}{r_1 \wedge r_2 + 1}, \quad (3.30)$$

and if $r > 0$ and A, B are in the ball

$$B_\alpha^r \stackrel{\text{def}}{=} \{C \in \Omega_\alpha^1 : |C|_\alpha \leq r\},$$

then

$$K_\alpha(A, B) \geq \frac{|A - B|_\alpha \wedge 1}{r+1}. \quad (3.31)$$

Lemma 3.42 *Let $\alpha \in (0, 1]$. If $A \in S_\alpha^r$ and $B \in S_\alpha^{r+h}$ for some $r, h > 0$, then*

$$k_\alpha(A, B) \geq \frac{h}{r+h+1}.$$

Proof. Consider a sequence $Z_0 = A, Z_1, \dots, Z_n = B$. Let $r_i \stackrel{\text{def}}{=} |Z_i|_\alpha$ and $R \stackrel{\text{def}}{=} \max_{i=0, \dots, n} r_i$. Then

$$\sum_{i=1}^n K_\alpha(Z_{i-1}, Z_i) \geq \frac{R-r}{R+1} \geq \frac{h}{r+h+1},$$

where in the first bound we used (3.30) and $R-r \leq \sum_{i=1}^n |r_{i-1} - r_i|$, and in the second bound we used $h \leq R-r$ and that $0 \leq \lambda \mapsto \frac{\lambda}{r+\lambda+1}$ is increasing. \square

Proposition 3.43 *Let $\alpha \in (0, 1]$. The metric space $(\Omega_\alpha^1, k_\alpha)$ is complete and k_α metrises the original topology of Ω_α^1 .*

Proof. It is obvious that k_α is weaker than the metric induced by $|\cdot|_\alpha$. On the other hand, if A_n is a k_α -Cauchy sequence, then $\sup |A_n|_\alpha < \infty$ by Lemma 3.42. It readily follows from (3.31) that $k_\alpha(A_n, A) \rightarrow 0$ and $|A - A_n|_\alpha \rightarrow 0$ for some $A \in \Omega_\alpha^1$. \square

Definition 3.44 Let $\alpha \in (\frac{2}{3}, 1]$. We denote by D_α the Hausdorff distance on \mathfrak{D}_α associated with k_α .

Theorem 3.45 *The metric space $(\mathfrak{D}_\alpha, D_\alpha)$ is complete and D_α metrises the quotient topology on \mathfrak{D}_α . In particular, \mathfrak{D}_α is a Polish space.*

For the proof of Theorem 3.45 we require several lemmas.

Lemma 3.46 *Let $\alpha \in (\frac{2}{3}, 1]$ and $x \in \mathfrak{D}_\alpha$. Then for all $r > \inf_{A \in x} |A|_\alpha$ there exists $A \in x$ with $|A|_\alpha = r$.*

Proof. For any $A \in \Omega_\alpha^1$, from the identity (3.24), we can readily construct a continuous function $g: [0, \infty) \rightarrow \mathfrak{G}^{0, \alpha}$ such that $g(0) \equiv 1$ and $\lim_{t \rightarrow \infty} |A^{g(t)}|_\alpha = \infty$. The conclusion follows by continuity of $g \mapsto |A^g|_\alpha$ (Corollary 3.36). \square

Lemma 3.47 *Suppose $\alpha \in (\frac{2}{3}, 1]$ and that $k_\alpha(A_n, A) \rightarrow 0$. Then $D_\alpha([A], [A_n]) \rightarrow 0$.* ■

Proof. Consider $\varepsilon > 0$. Observe that (3.29), the fact that $\sup_n |A_n|_\alpha < \infty$ by Lemma 3.42, and Lemma 3.46 together imply that there exists $r > 0$ sufficiently large such that the Hausdorff distance for k_α between $[A_n] \cap (\Omega_\alpha^1 \setminus B_\alpha^r)$ and $[A] \cap (\Omega_\alpha^1 \setminus B_\alpha^r)$ is at most ε for all n sufficiently large. On the other hand, for any $r > 0$, $g \in \mathfrak{G}^{0, \alpha}$, and $X, Y \in \Omega_\alpha^1$ such that $X, X^g \in B_\alpha^r$, it follows from Lemmas 3.32 and 3.33 and the identity

$$X^g - Y^g = ((X - Y)^g - 0^g - (X - Y)) + (X - Y)$$

that

$$|X^g - Y^g|_\alpha \lesssim (1 + |g|_{\alpha\text{-H\"{o}l}})|X - Y|_\alpha$$

where $|g|_{\alpha\text{-H\"{o}l}} \lesssim r$ due to (3.26). It follows that

$$\sup_{X \in [A_n] \cap B_\alpha^r} \inf_{Y \in [A]} k_\alpha(X, Y) + \sup_{X \in [A] \cap B_\alpha^r} \inf_{Y \in [A_n]} k_\alpha(X, Y) \rightarrow 0,$$

which concludes the proof. \square

Lemma 3.48 *Let $\alpha \in (\frac{2}{3}, 1]$ and suppose that $[A_n]$ is a D_α -Cauchy sequence. Then there exist $B \in \Omega_\alpha^1$ and representatives $B_n \in [A_n]$ such that $k_\alpha(B_n, B) \rightarrow 0$.*

Proof. We can assume that A_n are ‘‘almost minimal’’ representatives of $[A_n]$ in the sense that $|A_n|_\alpha \leq 1 + \inf_{g \in \mathfrak{G}^{0, \alpha}} |A_n^g|_\alpha$. By Lemma 3.42 and the definition of the Hausdorff distance, we see that $\sup_{n \geq 1} |A_n|_\alpha < \infty$, from which it is easy to extract a k_α -Cauchy sequence $B_n \in [A_n]$. \square

Proof of Theorem 3.45. Since every $x \in \mathfrak{D}_\alpha$ is closed by Lemma 3.40, $D_\alpha(x, y) = 0$ if and only if $x = y$. The facts that D_α is a complete metric and that it metrises the quotient topology both follow from Lemmas 3.47 and 3.48. \square

4 Stochastic heat equation

We investigate in this section the regularity of the stochastic heat equation (which is the ‘‘rough part’’ of the SYM) with respect to the spaces introduced in Section 3. For the remainder of the article, we will focus on the space of ‘‘1-forms’’ Ω_α^1 .

4.1 Regularising operators

The main result of this subsection, Proposition 4.1, provides a convenient way to extend regularising properties of an operator K to the spaces Ω_α^1 . This will be particularly helpful in deriving Schauder estimates and controlling the effect of mollifiers (Corollaries 4.2 and 4.4).

Let E be a Banach space throughout this subsection, and consider a linear map $K : \mathcal{C}^\infty(\mathbf{T}^2, E) \rightarrow \mathcal{C}(\mathbf{T}^2, E)$. We denote also by K the linear map $K : \Omega\mathcal{C}^\infty \rightarrow \Omega\mathcal{C}$ obtained by componentwise extension. We denote by $\hat{K} : \iota\Omega\mathcal{C}^\infty \rightarrow \iota\Omega\mathcal{C}$ the natural ‘‘lift’’ of K given by $\hat{K}(\iota A) \stackrel{\text{def}}{=} \iota(KA)$. We say that K is translation invariant if K commutes with all translation operators $T_v : f \mapsto f(\cdot + v)$. For $\theta \geq 0$, we denote

$$|K|_{\mathcal{C}^\theta \rightarrow L^\infty} \stackrel{\text{def}}{=} \sup\{|K(f)|_\infty \mid f \in \mathcal{C}^\infty(\mathbf{T}^2, E), |f|_{\mathcal{C}^\theta} = 1\}.$$

In general, for normed spaces X, Y and a linear map $K : D(K) \rightarrow Y$, where $D(K) \subset X$, we denote

$$|K|_{X \rightarrow Y} \stackrel{\text{def}}{=} \sup\{|K(x)|_Y \mid x \in D(K), |x|_X = 1\}.$$

If $D(K)$ is dense in X , then K does of course extend uniquely to all of X if $|K|_{X \rightarrow Y} < \infty$. The reason for this setting is that it will be convenient to consider $D(K)$ as fixed and to allow X to vary.

Proposition 4.1 *Let $0 < \bar{\alpha} \leq \alpha \leq 1$. Let $K : \mathcal{C}^\infty(\mathbf{T}^2, E) \rightarrow \mathcal{C}(\mathbf{T}^2, E)$ be a translation invariant linear map. Then*

$$|\hat{K}|_{\Omega_\alpha^1 \rightarrow \Omega_{\bar{\alpha}}^1} \lesssim |K|_{\mathcal{C}^{(\alpha-\bar{\alpha})/2} \rightarrow L^\infty}. \quad (4.1)$$

Furthermore, if $\bar{\alpha} \in [\frac{\alpha}{2}, \alpha]$, then for all $A \in \iota\Omega\mathcal{C}^\infty$

$$|\hat{K}A|_{\bar{\alpha}\text{-gr}} \lesssim |K|_{\mathcal{C}^{\alpha-\bar{\alpha}} \rightarrow L^\infty} |A|_\alpha^{2(\alpha-\bar{\alpha})/\alpha} |A|_{\alpha\text{-gr}}^{(2\bar{\alpha}-\alpha)/\alpha}. \quad (4.2)$$

The proportionality constants in both inequalities are universal.

Proof. We suppose that $|K|_{\mathcal{C}^{\alpha-\bar{\alpha}} \rightarrow L^\infty} < \infty$, as otherwise there is nothing to prove. Let $A \in \Omega\mathcal{C}^\infty$ and observe that, for $(x, v) \in \mathcal{X}$,

$$\begin{aligned} \int_0^1 (KA_i)(x + tv) dt &= \int_0^1 T_{tv}(KA_i)(x) dt = \int_0^1 K[T_{tv}A_i](x) dt \\ &= K\left[\int_0^1 T_{tv}A_i dt\right](x) = K\left[\int_0^1 A_i(\cdot + tv) dt\right](x), \end{aligned}$$

where we used translation invariance of K in the second equality, and the boundedness of K in the third equality. In particular, it follows from the definition of $\iota : \Omega\mathcal{C} \rightarrow \Omega$ that

$$\hat{K}(\iota A)(x, v) = K[\iota A(\cdot, v)](x). \quad (4.3)$$

We will first prove (4.2). We claim that for any $\theta \in [0, 1]$

$$|\iota A(\cdot, v)|_{\mathcal{C}(\theta\alpha/2)} \lesssim |\iota A|_{\alpha}^{\theta} |\iota A|_{\alpha\text{-gr}}^{1-\theta} |v|^{\alpha(1-\theta/2)} \quad (4.4)$$

for a universal proportionality constant. Indeed, note that $|\iota A(x, v)|_{\infty} \leq |\iota A|_{\alpha\text{-gr}} |v|^{\alpha}$ which is bounded above by the right-hand side of (4.4) for any $\theta \in [0, 1]$. Furthermore, we have for all $x, y \in \mathbf{T}^2$

$$|\iota A(x, v) - \iota A(y, v)| \lesssim [|\iota A|_{\alpha} |v|^{\alpha/2} |x - y|^{\alpha/2}] \wedge [|\iota A|_{\alpha\text{-gr}} |v|^{\alpha}], \quad (4.5)$$

for a universal proportionality constant, from which (4.4) follows by interpolation. If $\bar{\alpha} \in [\frac{\alpha}{2}, \alpha]$, then we can take $\theta = 2(\alpha - \bar{\alpha})/\alpha$ in (4.4) and combine with (4.3) to obtain (4.2).

We now prove (4.1). Consider $\ell = (x, v), \bar{\ell} = (\bar{x}, \bar{v}) \in \mathcal{X}$. If $\ell, \bar{\ell}$ are far, then the necessary estimate follows from (4.2). Hence, suppose $\ell, \bar{\ell}$ are not far. Consider the function $\Psi \in \mathcal{C}^{\infty}(\mathbf{T}^2, E)$ given by $\Psi(y) \stackrel{\text{def}}{=} \iota A(y, v) - \iota A(y + \bar{x} - x, \bar{v})$. Note that (4.3) implies

$$(K\Psi)(x) = \hat{K}(\iota A)(\ell) - \hat{K}(\iota A)(\bar{\ell}). \quad (4.6)$$

We claim that for any $\theta \in [0, 1]$

$$|\Psi|_{\mathcal{C}^{\alpha\theta/2}} \lesssim |A|_{\alpha} |v|^{\alpha\theta/2} \varrho(\ell, \bar{\ell})^{\alpha(1-\theta)}. \quad (4.7)$$

Indeed, note that $|\Psi|_{\infty} \leq |A|_{\alpha} \varrho(\ell, \bar{\ell})^{\alpha}$, which is bounded above (up to a universal constant) by the right-hand side of (4.7) for any $\theta \in [0, 1]$. Furthermore, since $(y, v), (y + \bar{x} - x, \bar{v})$ are also not far for every $y \in \mathbf{T}^2$, and since $|v| \asymp |\bar{v}|$, we have

$$\begin{aligned} |\Psi(y) - \Psi(z)| &= |A(y, v) - A(y + \bar{x} - x, \bar{v}) - A(z, v) + A(z + \bar{x} - x, \bar{v})| \\ &\lesssim |A|_{\alpha} [\varrho(\ell, \bar{\ell})^{\alpha} \wedge (|v|^{\alpha/2} |y - z|^{\alpha/2})] \end{aligned} \quad (4.8)$$

for a universal proportionality constant, from which (4.7) follows by interpolation. Taking $\theta = \frac{\alpha - \bar{\alpha}}{\alpha} \Leftrightarrow \bar{\alpha} = \alpha(1 - \theta)$ in (4.7) and combining with (4.6) proves (4.1). \square

As a consequence of Proposition 4.1, any linear map $K : \mathcal{C}^{\infty}(\mathbf{T}^2, E) \rightarrow \mathcal{C}(\mathbf{T}^2, E)$ with $|K|_{\mathcal{C}(\alpha - \bar{\alpha})/2 \rightarrow L^{\infty}} < \infty$ uniquely determines a bounded linear map $\hat{K} : \Omega_{\alpha}^1 \rightarrow \Omega_{\bar{\alpha}}^1$ which intertwines with K through the embedding $\iota : \Omega\mathcal{C} \rightarrow \Omega$. The same applies to $\Omega_{\alpha\text{-gr}}^1 \rightarrow \Omega_{\bar{\alpha}\text{-gr}}^1$ if $|K|_{L^{\infty} \rightarrow L^{\infty}} < \infty$. In the sequel, we will denote \hat{K} by the same symbol K without further notice.

We give two useful corollaries of Proposition 4.1. For $t \geq 0$, let $e^{t\Delta}$ denote the heat semigroup acting on $\mathcal{C}^{\infty}(\mathbf{T}^2, E)$.

Corollary 4.2 *Let $0 < \bar{\alpha} \leq \alpha \leq 1$. Then for all $A \in \Omega_{\alpha}^1$, it holds that*

$$|(e^{t\Delta} - 1)A|_{\bar{\alpha}} \lesssim t^{(\alpha - \bar{\alpha})/4} |A|_{\alpha}, \quad (4.9)$$

where the proportionality constant depends only on $\alpha - \bar{\alpha}$.

Proof. Recall the classical estimate for $\kappa \in [0, 1]$

$$|(e^{t\Delta} - 1)|_{\mathcal{C}^\kappa \rightarrow L^\infty} \leq |(e^{t\Delta} - 1)|_{\mathcal{C}^{\kappa-\text{Hö}} \rightarrow L^\infty} \lesssim t^{\kappa/2}.$$

The claim then follows from (4.1) by taking $\kappa = (\alpha - \bar{\alpha})/2$. \square

Remark 4.3 The appearance of $t^{(\alpha-\bar{\alpha})/4}$ in (4.9) may seem unusual since one instead has $t^{(\alpha-\bar{\alpha})/2}$ in the classical Schauder estimates for the Hölder norm $|\cdot|_{\mathcal{C}^\alpha}$. The exponent $(\alpha - \bar{\alpha})/4$ is however sharp (which can be seen by looking at the Fourier basis), and is consistent with the embedding of $\mathcal{C}^{\alpha/2}$ into Ω_α (Remark 3.23).

Corollary 4.4 *Let $0 < \bar{\alpha} \leq \alpha \leq 1$ and $\kappa \in [0, 1]$. Let χ be a mollifier on $\mathbf{R} \times \mathbf{R}^2$ and consider a function $A: \mathbf{R} \rightarrow \Omega_\alpha^1$. Then for any interval $I \subset \mathbf{R}$*

$$\sup_{t \in I} |(\chi^\varepsilon * A)(t) - A(t)|_{\bar{\alpha}} \lesssim |\chi|_{L^1} \left(\varepsilon^{(\alpha-\bar{\alpha})/2} \sup_{t \in I_\varepsilon} |A(t)|_\alpha + \varepsilon^{2\kappa} |A|_{\mathcal{C}^{\kappa-\text{Hö}}(I_\varepsilon, \Omega_{\bar{\alpha}})} \right),$$

where I_ε is the ε^2 fattening of I , and the proportionality constant is universal.

Proof. For $t \in \mathbf{R}$ define $m(t) \stackrel{\text{def}}{=} \int_{\mathbf{T}^2} \chi^\varepsilon(t, x) dx$ and denote by $\chi^\varepsilon(t)$ the convolution operator $[\chi^\varepsilon(t)f](x) \stackrel{\text{def}}{=} \langle \chi^\varepsilon(t, x - \cdot), f(\cdot) \rangle$ for $f \in \mathcal{D}'(\mathbf{T}^2)$. Observe that for any $\theta \in [0, 1]$

$$|m(t)f - \chi^\varepsilon(t)f|_{L^\infty} \leq \varepsilon^\theta |\chi^\varepsilon(t, \cdot)|_{L^1(\mathbf{T}^2)} |f|_{\mathcal{C}^{\theta-\text{Hö}}(\mathbf{T}^2)}.$$

In particular, $|m(t) - \chi^\varepsilon(t)|_{\mathcal{C}^{(\alpha-\bar{\alpha})/2-\text{Hö}} \rightarrow L^\infty} \leq \varepsilon^{(\alpha-\bar{\alpha})/2} |\chi^\varepsilon(t, \cdot)|_{L^1(\mathbf{T}^2)}$. Hence, for any $t \in I$,

$$\begin{aligned} |(\chi^\varepsilon * A)(t) - A(t)|_{\bar{\alpha}} &\leq \int_{\mathbf{R}} |(\chi^\varepsilon(s) - m(s))A(t-s)|_{\bar{\alpha}} ds \\ &\quad + \int_{\mathbf{R}} |m(s)(A(t-s) - A(t))|_{\bar{\alpha}} ds \\ &\lesssim \int_{\mathbf{R}} |\chi^\varepsilon(s, \cdot)|_{L^1} \varepsilon^{(\alpha-\bar{\alpha})/2} |A(t-s)|_\alpha ds \\ &\quad + \int_{\mathbf{R}} |m(s)| |s|^\kappa |A|_{\mathcal{C}^{\kappa-\text{Hö}}(I_\varepsilon, \Omega_{\bar{\alpha}})} ds \\ &\leq |\chi|_{L^1} \varepsilon^{(\alpha-\bar{\alpha})/2} \sup_{t \in I_\varepsilon} |A(t)|_\alpha + \varepsilon^{2\kappa} |\chi|_{L^1} |A|_{\mathcal{C}^{\kappa-\text{Hö}}(I_\varepsilon, \Omega_{\bar{\alpha}})}, \end{aligned}$$

where we used (4.1) in the second inequality. \square

Another useful property is that the heat semigroup is strongly continuous on Ω_α^1 . To show this, we need the following lemma. For a function $\omega: \mathbf{R}_+ \rightarrow \mathbf{R}_+$, let $\mathcal{C}^\omega(\mathbf{T}^2, E)$ denote the space of continuous functions $f: \mathbf{T}^2 \rightarrow E$ with

$$|f|_{\mathcal{C}^\omega} \stackrel{\text{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)} < \infty.$$

Lemma 4.5 *Let $\alpha \in (0, 1]$ and $K: \mathcal{C}^\infty(\mathbf{T}^2, E) \rightarrow \mathcal{C}(\mathbf{T}^2, E)$ be a translation invariant linear map with $|K|_{L^\infty \rightarrow L^\infty} < \infty$. Let $A \in \Omega_\alpha^1$ and $\omega: \mathbf{R}_+ \rightarrow \mathbf{R}_+$, and suppose that for all $x, y \in \mathbf{T}^2$, $v, \bar{v} \in B_{1/4}$, and $h \in \mathbf{R}^2$,*

$$|A(x, v) - A(y, v)| \leq |A|_{\alpha\text{-gr}} |v|^\alpha \omega(|x - y|) \quad (4.10)$$

and

$$|A(x, v) - A(x + h, \bar{v}) - A(y, v) + A(y + h, \bar{v})| \leq |A|_\alpha \varrho(\ell, \bar{\ell})^\alpha \omega(|x - y|), \quad (4.11)$$

where $\ell = (x, v)$ and $\bar{\ell} = (x + h, \bar{v})$. Then

$$|KA|_{\alpha\text{-gr}} \leq |K|_{\mathcal{C}^\omega \rightarrow L^\infty} |A|_{\alpha\text{-gr}} \quad \text{and} \quad |KA|_\alpha \leq |K|_{\mathcal{C}^\omega \rightarrow L^\infty} |A|_\alpha.$$

Proof. The proof is essentially the same as that of Proposition 4.1; one simply replaces (4.5) by (4.10) and (4.8) by (4.11). \square

Proposition 4.6 *Let $\alpha \in (0, 1]$. The heat semigroup $e^{t\Delta}$ is strongly continuous on $\Omega_{\alpha\text{-gr}}^1$ and Ω_α^1 .*

Proof. Observe that for every $A \in \iota\Omega\mathcal{C}^\infty$ there exists a bounded modulus of continuity $\omega: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that (4.10) and (4.11) hold. On the other hand, recall that for every bounded modulus of continuity $\omega: \mathbf{R}_+ \rightarrow \mathbf{R}_+$

$$\lim_{t \rightarrow 0} |e^{t\Delta} - 1|_{\mathcal{C}^\omega \rightarrow L^\infty} = 0.$$

It follows from Lemma 4.5 that $\lim_{t \rightarrow 0} |e^{t\Delta} A - A|_{\alpha\text{-gr}} = 0$ for every $A \in \Omega\mathcal{C}^\infty$, and the same for the norm $|\cdot|_\alpha$, from which the conclusion follows by density of $\iota\Omega\mathcal{C}^\infty$ in $\Omega_{\alpha\text{-gr}}^1$ and Ω_α^1 . \square

4.2 Kolmogorov bound

In this subsection, let ξ be a \mathfrak{g} -valued Gaussian random distribution on $\mathbf{R} \times \mathbf{T}^2$. We assume that there exists $C_\xi > 0$ such that

$$\mathbf{E}[|\langle \xi, \varphi \rangle|^2] \leq C_\xi |\varphi|_{L^2(\mathbf{R} \times \mathbf{T}^2)}^2 \quad (4.12)$$

for all smooth compactly supported $\varphi: \mathbf{R} \times \mathbf{T}^2 \rightarrow \mathbf{R}$. Let ξ_1, ξ_2 be two i.i.d. copies of ξ , and let $\Psi = \sum_{i=1}^2 \Psi_i dx_i$ solve the stochastic heat equation $(\partial_t - \Delta)\Psi = \xi$ on $\mathbf{R}_+ \times \mathbf{T}^2$ with initial condition $\iota\Psi(0) \in \Omega_\alpha^1$.

Lemma 4.7 *Let P be a triangle with inradius h . Let $\kappa \in (0, 1)$ and $p \in [1, \kappa^{-1})$, and let $W^{\kappa, p}$ denote the Sobolev–Slobodeckij space on \mathbf{T}^2 . Then $|\mathbf{1}_P|_{W^{\kappa, p}}^p \lesssim |P| h^{-\kappa p}$, where the proportionality constant depends only on κp .*

Proof. Using the definition of Sobolev–Slobodeckij spaces, we have

$$\begin{aligned}
|\mathbf{1}_{\dot{P}}|_{W^{\kappa,p}}^p &= \int_{\mathbf{T}^2} \int_{\mathbf{T}^2} \frac{|\mathbf{1}_{\dot{P}}(x) - \mathbf{1}_{\dot{P}}(y)|^p}{|x - y|^{\kappa p + 2}} dx dy \\
&= 2 \int_{\dot{P}} \int_{\mathbf{T}^2 \setminus \dot{P}} |x - y|^{-\kappa p - 2} dx dy \\
&\leq 2 \int_{\dot{P}} dx \int_{|y-x| > d(x, \partial P)} dy |x - y|^{-\kappa p - 2} \\
&\lesssim \int_{\dot{P}} d(x, \partial P)^{-\kappa p} dx \\
&= \int_0^1 |\{x \in \dot{P} \mid d(x, \partial P) < \delta\}| \delta^{-1 - \kappa p} d\delta.
\end{aligned}$$

Note that the integrand is non-zero only if $\delta < h$, in which case $|\{x \in \dot{P} \mid d(x, \partial P) < \delta\}| \lesssim |\partial P| \delta$, where $|\partial P|$ denotes the length of the perimeter. Hence, since $\kappa p < 1$,

$$|\mathbf{1}_{\dot{P}}|_{W^{\kappa,p}}^p \lesssim \int_0^h |\partial P| \delta^{-\kappa p} d\delta \lesssim |\partial P| h^{-\kappa p + 1} \asymp |P| h^{-\kappa p},$$

as claimed. \square

Lemma 4.8 *Let $\kappa \in (0, 1)$ and suppose $\Psi(0) = 0$. Then for any triangle P with inradius h*

$$\mathbf{E}[|\Psi(t)(\partial P)|^2] \lesssim C_\xi t^\kappa |P| h^{-2\kappa} \leq C_\xi t^\kappa |P|^{1-\kappa},$$

where the proportionality constant depends only on κ .

Proof. By Stokes' theorem, we have

$$|\Psi(t)(\partial P)| = |\langle \partial_1 \Psi_2(t) - \partial_2 \Psi_1(t), \mathbf{1}_{\dot{P}} \rangle|.$$

Observe that

$$\langle \partial_1 \Psi_2(t), \mathbf{1}_{\dot{P}} \rangle = \int_{\mathbf{R} \times \mathbf{T}^2} \xi_2(s, y) \mathbf{1}_{s \in [0, t]} [e^{(t-s)\Delta} \partial_1 \mathbf{1}_{\dot{P}}](y) ds dy.$$

Hence, by (4.12),

$$\mathbf{E}[|\langle \partial_1 \Psi_2(t), \mathbf{1}_{\dot{P}} \rangle|^2] \leq C_\xi \int_0^t |e^{s\Delta} \partial_1 \mathbf{1}_{\dot{P}}|_{L^2}^2 ds.$$

By the estimate $|e^{s\Delta} f|_{L^2} \lesssim s^{(-1+\kappa)/2} |f|_{H^{-1+\kappa}}$, we have

$$|e^{s\Delta} \partial_1 \mathbf{1}_{\dot{P}}|_{L^2}^2 \lesssim s^{-1+\kappa} |\partial_1 \mathbf{1}_{\dot{P}}|_{H^{-1+\kappa}}^2.$$

Since $|\partial_1 f|_{H^{-1+\kappa}} \lesssim |f|_{H^\kappa}$, we have by Lemma 4.7

$$\mathbf{E}[|\langle \partial_1 \Psi_2(t), \mathbf{1}_{\dot{P}} \rangle|^2] \lesssim C_\xi \int_0^1 s^{-1+\kappa} |P| h^{-2\kappa} ds \lesssim C_\xi t^\kappa |P| h^{-2\kappa}.$$

Likewise for the term $\langle \partial_2 \Psi_1(t), \mathbf{1}_{\dot{P}} \rangle$, and the conclusion follows from the inequality $\pi h^2 \leq |P|$. \square

Lemma 4.9 *Let $\ell = (x, v) \in \mathcal{X}$ and consider the distribution $\langle \delta_\ell, \psi \rangle \stackrel{\text{def}}{=} \int_0^1 |v| \psi(x+tv) dt$. Then, for any $\kappa \in (\frac{1}{2}, 1)$,*

$$|\delta_\ell|_{H^{-\kappa}} \lesssim |\ell|^\kappa,$$

where the proportionality constant depends only on κ .

Proof. By rotation and translation invariance, we may assume $\ell = (0, |\ell|e_1)$. For $k = (k_1, k_2) \in \mathbf{Z}^2$, we have $\langle \delta_\ell, e^{2\pi i \langle k, \cdot \rangle} \rangle = (e^{2\pi i k_1 |\ell|} - 1)/(2\pi i k_1)$. Hence

$$\begin{aligned} |\delta_\ell|_{H^{-\kappa}}^2 &= \sum_{k \in \mathbf{Z}^2} |\langle \delta_\ell, e^{2\pi i \langle k, \cdot \rangle} \rangle|^2 (1 + k_1^2 + k_2^2)^{-\kappa} \\ &\lesssim \sum_{k \in \mathbf{Z}^2} (|\ell|^2 \wedge k_1^{-2}) (1 + k_1^2 + k_2^2)^{-\kappa} \\ &\lesssim \sum_{k \in \mathbf{Z}} (|\ell|^2 \wedge k^{-2}) (1 + k)^{1-2\kappa}. \end{aligned}$$

Splitting the final sum into $|k| \leq |\ell|^{-1}$ and $|k| > |\ell|^{-1}$ yields the desired result. \square

Lemma 4.10 *Let $\kappa \in (0, \frac{1}{2})$ and suppose $\Psi(0) = 0$. Then for any $\ell \in \mathcal{X}$*

$$\mathbf{E}[|\Psi(t)(\ell)|^2] \lesssim C_\xi t^\kappa |\ell|^{2-2\kappa},$$

where the proportionality constant depends only on κ .

Proof. Observe that $\Psi(t)(\ell) = \sum_{i=1}^2 |v|^{-1} v_i \langle \Psi_i(t), \delta_\ell \rangle$, where we used the notation of Lemma 4.9. Furthermore,

$$\langle \Psi_i(t), \delta_\ell \rangle = \int_{\mathbf{R} \times \mathbf{T}^2} \xi_i(s, y) \mathbf{1}_{s \in [0, t]} [e^{(t-s)\Delta} \delta_\ell](y) ds dy.$$

Hence, by (4.12),

$$\mathbf{E}[\langle \Psi_i(t), \delta_\ell \rangle^2] \leq C_\xi \int_0^t |e^{s\Delta} \delta_\ell|_{L^2}^2 ds.$$

The estimate $|e^{s\Delta} f|_{L^2} \lesssim s^{(\kappa-1)/2} |f|_{H^{\kappa-1}}$ implies $|e^{s\Delta} \delta_\ell|_{L^2}^2 \lesssim s^{\kappa-1} |\delta_\ell|_{H^{\kappa-1}}^2$. Hence, by Lemma 4.9,

$$\mathbf{E}[\langle \Psi_i(t), \delta_\ell \rangle^2] \lesssim C_\xi \int_0^1 s^{-1+\kappa} |\ell|^{2-2\kappa} ds \lesssim C_\xi t^\kappa |\ell|^{2-2\kappa},$$

and the claim follows from the bound $||v|^{-1} v_i| \lesssim 1$. \square

Since our ‘‘index space’’ \mathcal{X} and ‘‘distance’’ function ϱ are not entirely standard, we spell out the following Kolmogorov-type criterion.

Lemma 4.11 *Let A be a \mathfrak{g} -valued stochastic process indexed by \mathcal{X} such that, for all joinable $\ell, \bar{\ell} \in \mathcal{X}$, $A(\ell \sqcup \bar{\ell}) = A(\ell) + A(\bar{\ell})$ almost surely. Suppose that there exist $p \geq 1$, $M > 0$, and $\alpha \in (0, 1]$ such that for all $\ell \in \mathcal{X}$*

$$\mathbf{E}[|A(\ell)|^p] \leq M|\ell|^{p\alpha},$$

and for all triangles P

$$\mathbf{E}[|A(\partial P)|^p] \leq M|P|^{p\alpha/2}.$$

Then there exists a modification of A (which we denote by the same letter) which is a.s. a continuous function on \mathcal{X} . Furthermore, for every $\bar{\alpha} \in (0, \alpha - \frac{16}{p})$, there exists $\lambda > 0$, depending only on $p, \alpha, \bar{\alpha}$, such that

$$\mathbf{E}[|A|_{\bar{\alpha}}^p] \leq \lambda M.$$

Proof. Observe that for any $\ell, \bar{\ell} \in \mathcal{X}$, we can write $A(\ell) - A(\bar{\ell}) = A(\partial P_1) + A(\partial P_2) + A(a) - A(b)$, where $|P_1| + |P_2| \leq \varrho(\ell, \bar{\ell})^2$ and $|a| + |b| \leq \varrho(\ell, \bar{\ell})$ (if $\ell, \bar{\ell}$ are far, then $a = \ell, b = \bar{\ell}$, and P_1, P_2 are empty). It follows that for all $\ell, \bar{\ell} \in \mathcal{X}$

$$\mathbf{E}[|A(\ell) - A(\bar{\ell})|^p] \lesssim M\varrho(\ell, \bar{\ell})^{p\alpha}, \quad (4.13)$$

where the proportionality constant depends only on p, α . For $N \geq 1$ let D_N denote the set of line segments in \mathcal{X} whose start and end points have dyadic coordinates of scale 2^{-N} , and let $D = \cup_{N \geq 1} D_N$. For $r > 0, \ell \in \mathcal{X}$, define $B_\varrho(r, \ell) \stackrel{\text{def}}{=} \{\bar{\ell} \in \mathcal{X} \mid \varrho(\bar{\ell}, \ell) \leq r\}$. From Definition 3.3 and Remark 3.4, we see that for some $K > 0$, the family $\{B_\varrho(K2^{-N}, \ell)\}_{\ell \in D_{2N}}$ covers \mathcal{X} (quite wastefully) for every $N \geq 1$. It readily follows, using (3.1), that for any $\bar{\alpha} \in (0, 1]$

$$\sup_{\ell, \bar{\ell} \in D} \frac{|A(\ell) - A(\bar{\ell})|^p}{\varrho(\ell, \bar{\ell})^{p\bar{\alpha}}} \lesssim \sum_{N \geq 1} \sum_{\substack{a, b \in D_{2N} \\ \varrho(a, b) \leq K2^{-N}}} 2^{N\bar{\alpha}p} |A(a) - A(b)|^p. \quad (4.14)$$

Observe that $|D_{2N}| \leq 2^{8N}$, and thus the second sum has at most 2^{16N} terms. Hence, for $\bar{\alpha} \in (0, \alpha - \frac{16}{p}) \Leftrightarrow 16 + p(\bar{\alpha} - \alpha) < 0$, we see from (4.14) that the expectation of the right-hand side of (4.14) is bounded by $\lambda(p, \alpha, \bar{\alpha})M$. The conclusion readily follows as in the classical Kolmogorov continuity theorem. \square

By equivalence of moments for Gaussian random variables, Lemmas 4.8, 4.10, and 4.11 yield the following lemma.

Lemma 4.12 *Suppose $\Psi(0) = 0$. Then for any $p > 16$, $\alpha \in (0, 1)$, and $\bar{\alpha} \in (0, \alpha - \frac{16}{p})$, there exists $C > 0$, depending only on $p, \bar{\alpha}, \alpha$, such that for all $t \geq 0$*

$$\mathbf{E}[|\Psi(t)|_{\bar{\alpha}}^p] \leq CC_\xi^{p/2} t^{p(1-\alpha)/2}.$$

We are now ready to prove the following continuity theorem.

Theorem 4.13 *Let $0 < \bar{\alpha} < \alpha < 1$, $\kappa \in (0, \frac{\alpha - \bar{\alpha}}{4})$, and suppose $\Psi(0) \in \Omega_\alpha$. Then for all $p \geq 1$ and any $T > 0$*

$$\mathbf{E} \left[\sup_{0 \leq s < t \leq T} \frac{|\Psi(t) - \Psi(s)|_\alpha^p}{|t - s|^{p\kappa}} \right]^{1/p} \lesssim |\Psi(0)|_\alpha + C_\xi^{1/2}$$

where the proportionality constant depends only on $p, \alpha, \bar{\alpha}, T$.

Proof. Let $0 \leq s \leq t \leq T$ and observe that

$$\Psi(t) - \Psi(s) = (e^{(t-s)\Delta} - 1)e^{s\Delta}\Psi(0) + (e^{(t-s)\Delta} - 1)\tilde{\Psi}(s) + \hat{\Psi}(t),$$

where $\tilde{\Psi}: [0, s] \rightarrow \Omega_{\bar{\alpha}}$ and $\hat{\Psi}: [s, t] \rightarrow \Omega_{\bar{\alpha}}$ driven by ξ with zero initial conditions. By Corollary 4.2,

$$|(e^{(t-s)\Delta} - 1)e^{s\Delta}\Psi(0)|_{\bar{\alpha}} \lesssim |t - s|^{(\alpha - \bar{\alpha})/4} |\Psi(0)|_\alpha.$$

Likewise, by Corollary 4.2 and Lemma 4.12, for any $\beta > \alpha + \frac{16}{p}$

$$\mathbf{E}[|(e^{(t-s)\Delta} - 1)\tilde{\Psi}(s)|_{\bar{\alpha}}^p] \lesssim C_\xi^{p/2} |t - s|^{p(\alpha - \bar{\alpha})/4} s^{p(1-\beta)/2}.$$

Finally, by Lemma 4.12, for any $\beta > \bar{\alpha} + \frac{16}{p}$

$$\mathbf{E}[|\hat{\Psi}(t)|_{\bar{\alpha}}^p] \lesssim C_\xi^{p/2} |t - s|^{p(1-\beta)/2}.$$

In conclusion, for all p sufficiently large,

$$\mathbf{E}[|\Psi(t) - \Psi(s)|_{\bar{\alpha}}^p] \lesssim |t - s|^{p(\alpha - \bar{\alpha})/4} (|\Psi(0)|_\alpha^p + C_\xi^{p/2}).$$

The conclusion follows by the classical Kolmogorov continuity criterion. \square

Corollary 4.14 *Let χ be a mollifier on $\mathbf{R} \times \mathbf{T}^2$. Suppose that ξ is a \mathfrak{g} -valued white noise and denote $\xi^\varepsilon \stackrel{\text{def}}{=} \chi^\varepsilon * \xi$. Suppose that $\Psi(0) = 0$ and let Ψ^ε solve $(\partial_t - \Delta)\Psi^\varepsilon = \xi^\varepsilon$ on $\mathbf{R}_+ \times \mathbf{T}^2$ with zero initial condition $\Psi^\varepsilon(0) = 0$. Let $\alpha \in (0, 1)$, $T > 0$, $\kappa \in (0, \frac{1-\alpha}{4})$, and $p \geq 1$. Then*

$$\mathbf{E} \left[\sup_{t \in [0, T]} |\Psi^\varepsilon(t) - \Psi(t)|_\alpha^p \right]^{1/p} \lesssim \varepsilon^{2\kappa} |\chi|_{L^1},$$

where the proportionality constant depends only on α, κ, T, p .

Proof. Observe that, by Theorem 4.13,

$$\mathbf{E} \left[\sup_{t \in [0, \varepsilon^2]} |\Psi(t)|_\alpha^p \right]^{1/p} \lesssim \varepsilon^{2\kappa}. \quad (4.15)$$

Furthermore, for any $\varphi \in L^2(\mathbf{R} \times \mathbf{T}^2)$, by Young's inequality,

$$\mathbf{E}[(\xi^\varepsilon, \varphi)^2] = |\chi^\varepsilon * \varphi|_{L^2}^2 \leq |\varphi|_{L^2}^2 |\chi^\varepsilon|_{L^1}^2.$$

Hence ξ^ε satisfies (4.12) with $C_{\xi^\varepsilon} \stackrel{\text{def}}{=} |\chi|_{L^1}^2$. It follows again by Theorem 4.13 that

$$\mathbf{E} \left[\sup_{t \in [0, \varepsilon^2]} |\Psi^\varepsilon(t)|_\alpha^p \right]^{1/p} \lesssim \varepsilon^{2\kappa} |\chi|_{L^1}. \quad (4.16)$$

It remains to estimate $\mathbf{E}[\sup_{t \in [\varepsilon^2, T]} |\Psi(t) - \Psi^\varepsilon(t)|_\alpha^p]$.

Denoting $I \stackrel{\text{def}}{=} [\varepsilon^2, T]$, observe that by Corollary 4.4, for any $\bar{\alpha} \in [\alpha, 1]$

$$\begin{aligned} & \mathbf{E} \left[\sup_{t \in I} |\Psi(t) - \chi^\varepsilon * \Psi(t)|_\alpha^p \right]^{1/p} \\ & \lesssim |\chi|_{L^1} \left\{ \varepsilon^{(\bar{\alpha}-\alpha)/2} \mathbf{E} \left[\sup_{t \in I_\varepsilon} |\Psi(t)|_{\bar{\alpha}}^p \right]^{1/p} + \varepsilon^{2\kappa} \mathbf{E} \left[|\Psi|_{C^{\kappa-\text{H\"{o}l}(I_\varepsilon, \Omega_\alpha)} }^p \right]^{1/p} \right\}. \end{aligned}$$

Both expectations are finite provided $\bar{\alpha} < 1$, and thus the right-hand side is bounded above by a multiple of $\varepsilon^{2\kappa} |\chi|_{L^1}$.

We now estimate $\mathbf{E}[\sup_{t \in I} |\chi^\varepsilon * \Psi - \Psi^\varepsilon|_\alpha^p]$. Let us denote by $\mathbf{1}_+$ the indicator on the set $\{(t, x) \in \mathbf{R} \times \mathbf{T}^2 \mid t \geq 0\}$. Observe that $\mathbf{1}_+(\chi^\varepsilon * \xi)(t, x)$ and $\chi^\varepsilon * (\mathbf{1}_+ \xi)(t, x)$ both vanish if $t < -\varepsilon^2$ and agree if $t > \varepsilon^2$. In particular, $\chi^\varepsilon * \Psi$ and Ψ^ε both solve the (inhomogeneous) heat equation on $[\varepsilon^2, \infty) \times \mathbf{T}^2$ with the same source term but with possibly different initial conditions. To estimate these initial conditions, for $s \in [-\varepsilon^2, \varepsilon^2]$, let us denote by $\chi^\varepsilon(s)$ the convolution operator $[\chi^\varepsilon(s)f](x) \stackrel{\text{def}}{=} \langle \chi^\varepsilon(s, x - \cdot), f(\cdot) \rangle$ for $f \in \mathcal{D}'(\mathbf{T}^2)$. Observe that

$$|\chi^\varepsilon(s)|_{L^\infty \rightarrow L^\infty} \leq \mu(s) \stackrel{\text{def}}{=} \int_{\mathbf{T}^2} |\chi^\varepsilon(s, x)| \, dx,$$

and thus $|\chi^\varepsilon(s)A|_\alpha \lesssim \mu(s)|A|_\alpha$ for any $A \in \Omega_\alpha$ by (4.1). Hence

$$\begin{aligned} \mathbf{E}[|\chi^\varepsilon * \Psi(\varepsilon^2)|_\alpha^p]^{1/p} &= \mathbf{E} \left[\left| \int_{\mathbf{R}} \chi^\varepsilon(s) \Psi(\varepsilon^2 - s) \, ds \right|_\alpha^p \right]^{1/p} \\ &\lesssim |\chi|_{L^1} \mathbf{E} \left[\sup_{t \in [0, 2\varepsilon^2]} |\Psi|_\alpha^p \right]^{1/p} \lesssim |\chi|_{L^1} \varepsilon^{2\kappa}. \end{aligned}$$

As a result, by Theorem 4.13 and recalling that ξ^ε satisfies (4.12) with $C_{\xi^\varepsilon} = |\chi|_{L^1}^2$, we obtain

$$\mathbf{E}[|\chi^\varepsilon * \Psi(\varepsilon^2)|_\alpha^p]^{1/p} + \mathbf{E}[|\Psi^\varepsilon(\varepsilon^2)|_\alpha^p]^{1/p} \lesssim \varepsilon^{2\kappa} |\chi|_{L^1}.$$

Finally,

$$\mathbf{E} \left[\sup_{t \geq \varepsilon^2} |\chi^\varepsilon * \Psi(t) - \Psi^\varepsilon(t)|_\alpha^p \right]^{1/p} \lesssim \mathbf{E}[|\chi^\varepsilon * \Psi(\varepsilon^2) - \Psi^\varepsilon(\varepsilon^2)|_\alpha^p]^{1/p} \lesssim \varepsilon^{2\kappa} |\chi|_{L^1}$$

where we used Corollary 4.2 in the first inequality. \square

5 Regularity structures for vector-valued noises

5.1 Motivation

As already mentioned in the introduction, the aim of this section is to provide a solution/renormalisation theory for SPDEs of the form

$$(\partial_t - \mathcal{L}_t)A_t = F_t(\mathbf{A}, \boldsymbol{\xi}), \quad t \in \mathcal{L}_+, \quad (5.1)$$

where the nonlinearities $(F_t)_{t \in \mathcal{L}_+}$, linear operators $(\mathcal{L}_t)_{t \in \mathcal{L}_+}$, and noises $(\xi_t)_{t \in \mathcal{L}_-}$, satisfy the assumptions required for the general theory of [Hai14, CH16, BCCH17, BHZ19] to apply. The problem is that this theory assumes that the different components of the solutions A_t and of its driving noises ξ_t are scalar-valued. While this is not a restriction in principle (simply expand solutions and noises according to some arbitrary basis of the corresponding spaces), it makes it rather unwieldy to obtain an expression for the precise form of the counterterms generated by the renormalisation procedure described in [BCCH17].

Instead, one would much prefer a formalism in which the vector-valued natures of both the solutions and the driving noises are preserved. To motivate our construction, consider the example of a \mathfrak{g} -valued noise ξ , where \mathfrak{g} is some finite-dimensional vector space. One way of describing it in the context of [Hai14] would be to choose a basis $\{e_1, \dots, e_n\}$ of \mathfrak{g} and to consider a regularity structure \mathcal{T} with basis vectors Ξ_i endowed with a model Π such that $\Pi \Xi_i = \xi_i$ with ξ_i such that $\xi = \sum_{i=1}^n \xi_i e_i$. We could then also consider the element $\Xi \in \mathcal{T} \otimes \mathfrak{g}$ obtained by setting $\Xi = \sum_{i=1}^n \Xi_i \otimes e_i$. When applying the model to Ξ (or rather its first factor), we then obtain $\Pi \Xi = \sum_{i=1}^n \xi_i e_i = \xi$ as expected. A cleaner coordinate-independent way of achieving the same result is to view the subspace $\mathcal{T}[\Xi] \subset \mathcal{T}$ spanned by the Ξ_i as a copy of \mathfrak{g}^* , with Π given by $\Pi \Xi_g = g(\xi)$ for any element Ξ_g in this copy of \mathfrak{g}^* . In this way, $\Xi \in \mathcal{T}[\Xi] \otimes \mathfrak{g} \simeq \mathfrak{g}^* \otimes \mathfrak{g}$ is simply given by $\Xi = \text{id}_{\mathfrak{g}}$, where $\text{id}_{\mathfrak{g}}$ denotes the identity map $\mathfrak{g} \rightarrow \mathfrak{g}$, modulo the canonical correspondence $L(X, Y) \simeq X^* \otimes Y$. (Here and below we will use the notation $X \simeq Y$ to denote the existence of a *canonical* isomorphism between objects X and Y .)

Remark 5.1 This viewpoint is consistent with the natural correspondence between a \mathfrak{g} -valued rough path \mathbf{X} and a model Π . Indeed, while \mathbf{X} takes values in \mathcal{H}^* , the tensor series/Grossman–Larson algebra over \mathfrak{g} , one evaluates the model Π against elements of its predual \mathcal{H} , the tensor/Connes–Kreimer algebra over \mathfrak{g}^* , see [Hai14, Sec. 4.4] and [BCFP19, Sec. 6.2].

Imagine now a situation in which we are given \mathfrak{g}_1 and \mathfrak{g}_2 -valued noises $\xi_1 = \bullet$ and $\xi_2 = \bullet$, as well as an integration kernel K which we draw as a plain line, and consider the symbol \blacktriangledown . It seems natural in view of the above discussion to associate it with a subspace of \mathcal{T} isomorphic to $\mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$ (let's borrow the notation from [GH19] and denote this subspace as $\blacktriangledown \otimes \mathfrak{g}_1^* \otimes \mathfrak{g}_2^*$) and to have the canonical model act on it as

$$\Pi(\blacktriangledown \otimes g_1 \otimes g_2) = (K \star g_1(\xi_1)) (K \star g_2(\xi_2)).$$

It would appear that such a construction necessarily breaks the commutativity of the product since in the same vein one would like to associate to $\blacktriangleright \blacktriangleleft$ a copy of $\mathfrak{g}_2^* \otimes \mathfrak{g}_1^*$, but this can naturally be restored by simply postulating that in \mathcal{T} one has the identity

$$(\blacktriangleright \otimes g_1) \cdot (\blacktriangleleft \otimes g_2) = \blacktriangleright \blacktriangleleft \otimes g_1 \otimes g_2 = \blacktriangleleft \blacktriangleright \otimes g_2 \otimes g_1 = (\blacktriangleleft \otimes g_2) \cdot (\blacktriangleright \otimes g_1). \quad (5.2)$$

This then forces us to associate to $\blacktriangleright \blacktriangleleft$ a copy of the symmetric tensor product $\mathfrak{g}_1^* \otimes_s \mathfrak{g}_1^*$. The goal of this section is to provide a functorial description of such considerations which allows us to transfer algebraic identities for regularity structures of trees of the type considered in [BHZ19] to the present setting where each noise (or edge) type \mathfrak{t} is associated to a vector space $\mathfrak{g}_{\mathfrak{t}}^*$. This systematises previous constructions like [GH19, Sec. 3.1] or [Sch18, Sec. 3.1] where similar considerations were made in a rather ad hoc manner. Our construction bears a resemblance to that of [CW16] who introduced a similar formalism in the context of rough paths, but our formalism is more functorial and better suited for our purposes.

5.2 Symmetric sets and symmetric tensor products

Fix a collection \mathcal{L} of types and recall that a ‘‘typed set’’ T consists of a finite set (which we denote again by T) together with a map $\mathfrak{t}: T \rightarrow \mathcal{L}$. For any two typed sets T and \bar{T} , write $\text{Iso}(T, \bar{T})$ for the set of all type-preserving bijections from $T \rightarrow \bar{T}$.

Definition 5.2 A *symmetric set* \mathfrak{s} consists of a non-empty index set $A_{\mathfrak{s}}$, as well as a triple $\mathfrak{s} = (\{T_{\mathfrak{s}}^a\}_{a \in A_{\mathfrak{s}}}, \{\mathfrak{t}_{\mathfrak{s}}^a\}_{a \in A_{\mathfrak{s}}}, \{\Gamma_{\mathfrak{s}}^{a,b}\}_{a,b \in A_{\mathfrak{s}}})$ where $(T_{\mathfrak{s}}^a, \mathfrak{t}_{\mathfrak{s}}^a)$ is a typed set and $\Gamma_{\mathfrak{s}}^{a,b} \subset \text{Iso}(T_{\mathfrak{s}}^b, T_{\mathfrak{s}}^a)$ are non-empty sets such that, for any $a, b, c \in A_{\mathfrak{s}}$,

$$\begin{aligned} \gamma \in \Gamma_{\mathfrak{s}}^{a,b} &\Rightarrow \gamma^{-1} \in \Gamma_{\mathfrak{s}}^{b,a}, \\ \gamma \in \Gamma_{\mathfrak{s}}^{a,b}, \bar{\gamma} \in \Gamma_{\mathfrak{s}}^{b,c} &\Rightarrow \gamma \circ \bar{\gamma} \in \Gamma_{\mathfrak{s}}^{a,c}. \end{aligned}$$

In other words, a symmetric set is a connected groupoid inside $\text{Set}_{\mathcal{L}}$, the category of typed sets endowed with type-preserving maps.

Remark 5.3 Each of the sets $\Gamma_{\mathfrak{s}}^{a,a}$ forms a group and, by connectedness, these are all (not necessarily canonically) isomorphic. We will call this isomorphism class the ‘‘local symmetry group’’ of \mathfrak{s} .

Given a typed set T and a symmetric set \mathfrak{s} , we define

$$\overline{\text{Hom}}(T, \mathfrak{s}) = \left(\bigcup_{a \in A_{\mathfrak{s}}} \text{Iso}(T, T_{\mathfrak{s}}^a) \right) / \Gamma_{\mathfrak{s}},$$

i.e. we postulate that $\text{Iso}(T, T_{\mathfrak{s}}^a) \ni \varphi \sim \tilde{\varphi} \in \text{Iso}(T, T_{\mathfrak{s}}^b)$ if and only if there exists $\gamma \in \Gamma_{\mathfrak{s}}^{a,b}$ such that $\varphi = \gamma \circ \tilde{\varphi}$. Note that, by connectedness of $\Gamma_{\mathfrak{s}}$, any equivalence class in $\overline{\text{Hom}}(T, \mathfrak{s})$ has, for every $a \in A_{\mathfrak{s}}$, at least one representative $\varphi^a \in \text{Iso}(T, T_{\mathfrak{s}}^a)$.

Given two symmetric sets \mathfrak{s} and $\bar{\mathfrak{s}}$, we also define the set $\overline{\text{SHom}}(\mathfrak{s}, \bar{\mathfrak{s}})$ of “sections” by

$$\overline{\text{SHom}}(\mathfrak{s}, \bar{\mathfrak{s}}) = \{ \Phi = (\Phi_a)_{a \in A_{\mathfrak{s}}} : \Phi_a \in \text{Vec}(\overline{\text{Hom}}(T_{\mathfrak{s}}^a, \bar{\mathfrak{s}})) \},$$

where $\text{Vec}(X)$ denotes the real vector space spanned by a set X . We then have the following definition.

Definition 5.4 A *morphism* between two symmetric sets \mathfrak{s} and $\bar{\mathfrak{s}}$ is a $\Gamma_{\mathfrak{s}}$ -invariant section; namely, an element of

$$\text{Hom}(\mathfrak{s}, \bar{\mathfrak{s}}) = \{ \Phi \in \overline{\text{SHom}}(\mathfrak{s}, \bar{\mathfrak{s}}) : \Phi_b = \Phi_a \circ \gamma_{a,b} \quad \forall a, b \in A_{\mathfrak{s}}, \forall \gamma_{a,b} \in \Gamma_{\mathfrak{s}}^{a,b} \}.$$

Here, we note that right composition with $\gamma_{a,b}$ gives a well-defined map from $\overline{\text{Hom}}(T_{\mathfrak{s}}^b, \bar{\mathfrak{s}})$ to $\overline{\text{Hom}}(T_{\mathfrak{s}}^a, \bar{\mathfrak{s}})$ and we extend this to $\text{Vec}(\overline{\text{Hom}}(T_{\mathfrak{s}}^b, \bar{\mathfrak{s}}))$ by linearity. Composition of morphisms is defined in the natural way by

$$(\bar{\Phi} \circ \Phi)_a = \bar{\Phi}_{\bar{a}} \circ \Phi_a^{(\bar{a})},$$

where $\Phi_a^{(\bar{a})}$ denotes an arbitrary representative of Φ_a in $\text{Vec}(\text{Iso}(T_{\mathfrak{s}}^a, T_{\mathfrak{s}}^{\bar{a}}))$ and composition is extended bilinearly. It is straightforward to verify that this is independent of the choice of \bar{a} and of representative $\Phi_a^{(\bar{a})}$ thanks to the invariance property $\bar{\Phi}_{\bar{a}} \circ \gamma_{\bar{a}, \bar{b}} = \bar{\Phi}_{\bar{b}}$, as well as the postulation of the equivalence relation in the definition of $\overline{\text{Hom}}(T_{\mathfrak{s}}^a, \bar{\mathfrak{s}})$.

Remark 5.5 A natural generalisation of this construction is obtained by replacing \mathfrak{L} by an arbitrary finite category. In this case, typed sets are defined as before, with each element having as type an object of \mathfrak{L} . Morphisms between typed sets A and \bar{A} are then given by maps $\varphi: A \rightarrow \bar{A} \times \text{Hom}_{\mathfrak{L}}$ such that, writing $\varphi = (\varphi_0, \vec{\varphi})$, one has $\vec{\varphi}(a) \in \text{Hom}_{\mathfrak{L}}(\mathfrak{t}(a), \mathfrak{t}(\varphi_0(a)))$ for every $a \in A$. Composition is defined in the obvious way by “following the arrows”, namely

$$(\psi \circ \varphi)_0 = \psi_0 \circ \varphi_0, \quad (\psi \circ \varphi)(a) = \vec{\psi}(\varphi_0(a)) \circ \vec{\varphi}(a),$$

where the composition on the right takes place in $\text{Hom}_{\mathfrak{L}}$. The set $\text{Iso}(A, \bar{A})$ is then defined as those morphisms φ such that φ_0 is a bijection, but we do *not* impose that $\vec{\varphi}(a)$ is an isomorphism in \mathfrak{L} for $a \in A$.

Remark 5.6 Note that, for any symmetric set \mathfrak{s} , there is a natural identity element $\text{id}_{\mathfrak{s}} \in \text{Hom}(\mathfrak{s}, \mathfrak{s})$ given by $a \mapsto [\text{id}_{T_{\mathfrak{s}}^a}]$, with $[\text{id}_{T_{\mathfrak{s}}^a}]$ denoting the equivalence class of $\text{id}_{T_{\mathfrak{s}}^a}$ in $\overline{\text{Hom}}(T_{\mathfrak{s}}^a, \mathfrak{s})$. In particular, symmetric sets form a category, which we denote by SSet (or $\text{SSet}_{\mathfrak{L}}$).

Remark 5.7 We choose to consider formal linear combinations in our definition of $\overline{\text{SHom}}$ since otherwise the resulting definition of $\text{Hom}(\mathfrak{s}, \bar{\mathfrak{s}})$ would be too small for our purpose.

Remark 5.8 An important special case is given by the case when $A_\mathfrak{s}$ and $A_{\bar{\mathfrak{s}}}$ are singletons. In this case, $\text{Hom}(\mathfrak{s}, \bar{\mathfrak{s}})$ can be viewed as a subspace of $\text{Vec}(\overline{\text{Hom}}(T_\mathfrak{s}, \bar{\mathfrak{s}}))$, $\overline{\text{Hom}}(T_\mathfrak{s}, \bar{\mathfrak{s}}) = \text{Iso}(T_\mathfrak{s}, T_{\bar{\mathfrak{s}}})/\Gamma_{\bar{\mathfrak{s}}}$, and $\Gamma_{\bar{\mathfrak{s}}}$ is a subgroup of $\text{Iso}(T_{\bar{\mathfrak{s}}}, T_{\bar{\mathfrak{s}}})$.

Remark 5.9 An alternative, more symmetric, way of viewing morphisms of SSet is as two-parameter maps

$$A_\mathfrak{s} \times A_{\bar{\mathfrak{s}}} \ni (a, \bar{a}) \mapsto \Phi_{\bar{a}, a} \in \text{Vec}(\text{Iso}(T_\mathfrak{s}^a, T_{\bar{\mathfrak{s}}}^{\bar{a}})) ,$$

which are invariant in the sense that, for any $\gamma_{a,b} \in \Gamma_\mathfrak{s}^{a,b}$ and $\bar{\gamma}_{\bar{a}, \bar{b}} \in \Gamma_{\bar{\mathfrak{s}}}^{\bar{a}, \bar{b}}$, one has the identity

$$\Phi_{\bar{a}, a} \circ \gamma_{a,b} = \bar{\gamma}_{\bar{a}, \bar{b}} \circ \Phi_{\bar{b}, b} . \quad (5.3)$$

Composition is then given by

$$(\bar{\Phi} \circ \Phi)_{\bar{a}, a} = \bar{\Phi}_{\bar{a}, \bar{a}} \circ \Phi_{\bar{a}, a} ,$$

for any *fixed* choice of \bar{a} (no summation). Indeed, it is easy to see that for any choice of \bar{a} , $\bar{\Phi} \circ \Phi$ satisfies (5.3). To see that our definition does not depend on the choice of \bar{a} , note that, for any $\bar{b} \in A_{\bar{\mathfrak{s}}}$, we can take an element $\gamma_{\bar{a}, \bar{b}} \in \Gamma_{\bar{\mathfrak{s}}}^{\bar{a}, \bar{b}}$ (which is non-empty set by definition) and use (5.3) to write

$$\text{id}_{T_\mathfrak{s}^{\bar{a}}, T_{\bar{\mathfrak{s}}}^{\bar{a}}} \circ \bar{\Phi}_{\bar{a}, \bar{b}} \circ \Phi_{\bar{b}, a} = \bar{\Phi}_{\bar{a}, \bar{a}} \circ \bar{\gamma}_{\bar{a}, \bar{b}} \circ \Phi_{\bar{b}, a} = \bar{\Phi}_{\bar{a}, \bar{a}} \circ \Phi_{\bar{a}, a} \circ \text{id}_{T_\mathfrak{s}^a, T_{\bar{\mathfrak{s}}}^a} .$$

We write $\text{Hom}_2(\mathfrak{s}, \bar{\mathfrak{s}})$ of the set of morphisms, as described above, between \mathfrak{s} and $\bar{\mathfrak{s}}$. To see that this notion of morphism gives an equivalent category note that the map(s) $\iota_{\mathfrak{s}, \bar{\mathfrak{s}}}: \text{Hom}(\mathfrak{s}, \bar{\mathfrak{s}}) \rightarrow \text{Hom}_2(\mathfrak{s}, \bar{\mathfrak{s}})$, given by mapping $\Gamma_\mathfrak{s}$ equivalence classes to their symmetrised sums, is a bijection and maps compositions in Hom to the corresponding compositions in Hom_2 .

Remark 5.10 The category SSet of symmetric sets just described is an \mathbf{R} -linear symmetric monoidal category, with tensor product $\mathfrak{s} \otimes \bar{\mathfrak{s}}$ given by



$$A = A_\mathfrak{s} \times A_{\bar{\mathfrak{s}}} , \quad T^{(a, \bar{a})} = T_\mathfrak{s}^a \sqcup T_{\bar{\mathfrak{s}}}^{\bar{a}} , \quad \mathfrak{t}^{(a, \bar{a})} = \mathfrak{t}_\mathfrak{s}^a \sqcup \mathfrak{t}_{\bar{\mathfrak{s}}}^{\bar{a}} , \\ \Gamma^{(a, \bar{a}), (b, \bar{b})} = \{ \gamma \sqcup \bar{\gamma} : \gamma \in \Gamma_\mathfrak{s}^{a,b} , \bar{\gamma} \in \Gamma_{\bar{\mathfrak{s}}}^{\bar{a}, \bar{b}} \} .$$


and unit object $\mathbb{1}$ given by $A_{\mathbb{1}} = \{ \bullet \}$ a singleton and $T_{\mathbb{1}}^\bullet = \emptyset$.

Remark 5.11 We will sometimes encounter the situation where a pair $(\mathfrak{s}, \bar{\mathfrak{s}})$ of symmetric sets naturally comes with elements $\Phi_a \in \overline{\text{Hom}}(T_\mathfrak{s}^a, \bar{\mathfrak{s}})$ such that $\Phi \in \text{Hom}(\mathfrak{s}, \bar{\mathfrak{s}})$. In this case, Φ is necessarily an isomorphism which we call the ‘‘canonical isomorphism’’ between \mathfrak{s} and $\bar{\mathfrak{s}}$. Note that this notion of ‘‘canonical’’ is not intrinsic to SSet but relies on additional structure in general.

More precisely, consider a category \mathcal{C} that is concrete over typed sets (i.e. such that objects of \mathcal{C} can be viewed as typed sets and morphisms as type-preserving maps between them). Then, any collection $(T^a)_{a \in A}$ of isomorphic objects of \mathcal{C} yields a

symmetric set \mathfrak{s} by taking for Γ the groupoid of all \mathcal{C} -isomorphisms between them. Two symmetric sets obtained in this way such that the corresponding collections $(T^a)_{a \in A}$ and $(\bar{T}^b)_{b \in \bar{A}}$ consist of objects that are \mathcal{C} -isomorphic are then canonically isomorphic (in SSet) by taking for Φ_a the set of all \mathcal{C} -isomorphisms from T^a to any of the \bar{T}^b . Note that this does *not* in general mean that there isn't another isomorphism between these objects in SSet !

Example 5.12 *An example of a symmetric set is obtained via “a tree with \mathfrak{L} -typed leaves”. A concrete tree τ is defined by fixing a vertex set V , which we can take without loss of generality as a finite subset of \mathbf{N} (which we choose to play the role of the set of all possible vertices), together with an (oriented) edge set $E \subset V \times V$ so that the resulting graph is a rooted tree with all edges oriented towards the root, as well as a labelling $\mathfrak{t}: L \rightarrow \mathfrak{L}$, with $L \subset V$ the set of leaves. However, when we draw⁵ a “tree with \mathfrak{L} -typed leaves” such as , we are actually specifying a isomorphism class of trees since we are not specifying V , E , and \mathfrak{t} as concrete sets. Thus  corresponds to an infinite isomorphism class of trees τ with each $\tau \in \tau$ being a concrete representative of τ .*

For any two concrete representatives $\tau_1 = (V_1, E_1, \mathfrak{t}_1)$ and $\tau_2 = (V_2, E_2, \mathfrak{t}_2)$ in the isomorphism class , we have two distinct tree isomorphisms $\gamma: V_1 \sqcup E_1 \rightarrow V_2 \sqcup E_2$ which preserve the typed tree structure, since it doesn't matter how the two vertices of type \bullet get mapped onto each other. In this way, we have an unambiguous way of viewing $\tau = \text{img alt="A tree diagram with a root node (red triangle) and three leaf nodes (blue circle, purple circle, green circle) connected by edges." data-bbox="548 488 585 505} as an object in $\langle \tau \rangle \in \text{SSet}$, with typed set (L, \mathfrak{t}) and local symmetry group isomorphic to \mathbf{Z}_2 .$

Example 5.13 *We now give an example where we compute $\text{Hom}(\cdot, \cdot)$ and $\overline{\text{Hom}}(\cdot, \cdot)$. Consider $\tau = \text{img alt="A tree diagram with a root node (red triangle) and three leaf nodes (blue circle, purple circle, green circle) connected by edges." data-bbox="548 546 585 563}$, fix some representative $\tau \in \tau$, and write $T_\tau = \{x, y, z\} \subset V_\tau \subset \mathbf{N}$, with $\mathfrak{t}_\tau(x) = \mathfrak{t}_\tau(y) = \bullet$ and $\mathfrak{t}_\tau(z) = \blacklozenge$ —the local symmetry group is then isomorphic to \mathbf{Z}_2 , acting on T_τ by permuting $\{x, y\}$. We also introduce a second isomorphism class $\bar{\tau} = \text{img alt="A tree diagram with a root node (red triangle) and three leaf nodes (green circle, blue circle, purple circle) connected by edges." data-bbox="548 596 585 613} which has trivial local symmetry group and fix a representative $\bar{\tau}$ of $\bar{\tau}$ which coincides, as a typed set, with τ .$*

It is easy to see that $\overline{\text{Hom}}(T_\tau, \langle \tau \rangle)$ consists of only one equivalence class, while $\overline{\text{Hom}}(T_\tau, \langle \bar{\tau} \rangle)$ consists of two equivalence classes, which we call φ and $\tilde{\varphi}$. $\text{Hom}(\langle \tau \rangle, \langle \bar{\tau} \rangle)$ then consists of the linear span of a “section” Φ such that, restricting to the representative τ , $\Phi_\tau = \varphi + \tilde{\varphi}$, since the action of \mathbf{Z}_2 on τ swaps φ and $\tilde{\varphi}$.

5.2.1 Symmetric tensor products

A *space assignment* V for \mathfrak{L} is a tuple of vector spaces $V = (V_t)_{t \in \mathfrak{L}}$. We say a space assignment V is finite-dimensional if $\dim(V_t) < \infty$ for every $t \in \mathfrak{L}$. For the rest of this subsection we fix an arbitrary (not necessarily finite-dimensional) space assignment $(V_t)_{t \in \mathfrak{L}}$.

For any (finite) typed set T , we write $V^{\otimes T}$ for the tensor product defined as the linear span of elementary tensors of the form $v = \bigotimes_{x \in T} v_x$ with $v_x \in V_{\mathfrak{t}(x)}$,

⁵By convention, we always draw the root at the bottom of the tree.

subject to the usual identifications suggested by the notation. Given $\psi \in \text{Iso}(T, \bar{T})$ for two typed sets, we can then interpret it as a linear map $V^{\otimes T} \rightarrow V^{\otimes \bar{T}}$ by

$$v = \bigotimes_{x \in T} v_x \mapsto \psi \cdot v = \bigotimes_{y \in \bar{T}} v_{\psi^{-1}(y)}. \quad (5.4)$$

In particular, given a symmetric set \mathfrak{s} , elements $a, b \in A_{\mathfrak{s}}$, and $\gamma \in \Gamma_{\mathfrak{s}}^{a,b}$, we view γ as a map from $V^{\otimes T_{\mathfrak{s}}^b}$ to $V^{\otimes T_{\mathfrak{s}}^a}$. We then define the vector space $V^{\otimes \mathfrak{s}} \subset \prod_{a \in A_{\mathfrak{s}}} V^{\otimes T_{\mathfrak{s}}^a}$ by

$$V^{\otimes \mathfrak{s}} = \left\{ (v^{(a)})_{a \in A_{\mathfrak{s}}} : v^{(a)} = \gamma_{a,b} \cdot v^{(b)} \quad \forall a, b \in A_{\mathfrak{s}}, \forall \gamma_{a,b} \in \Gamma_{\mathfrak{s}}^{a,b} \right\}. \quad (5.5)$$

Note that for every $a \in A_{\mathfrak{s}}$, we have a natural symmetrisation map $\pi_{\mathfrak{s},a}: V^{\otimes T_{\mathfrak{s}}^a} \rightarrow V^{\otimes \mathfrak{s}}$ given by

$$(\pi_{\mathfrak{s},a}v)^{(b)} = \frac{1}{|\Gamma_{\mathfrak{s}}^{b,a}|} \sum_{\gamma \in \Gamma_{\mathfrak{s}}^{b,a}} \gamma \cdot v, \quad (5.6)$$

an important property of which is that

$$\pi_{\mathfrak{s},a} \circ \gamma_{a,b} = \pi_{\mathfrak{s},b}, \quad \forall a, b \in A_{\mathfrak{s}}, \quad \forall \gamma_{a,b} \in \Gamma_{\mathfrak{s}}^{a,b}. \quad (5.7)$$

Furthermore, these maps are left inverses to the natural inclusions $\iota_{\mathfrak{s},a}: V^{\otimes \mathfrak{s}} \rightarrow V^{\otimes T_{\mathfrak{s}}^a}$ given by $(v^{(b)})_{b \in A_{\mathfrak{s}}} \mapsto v^{(a)}$.

Remark 5.14 Suppose that we are given a symmetric set \mathfrak{s} . For each $a \in A_{\mathfrak{s}}$, if we view \mathfrak{s}_a as a symmetric set in its own right with $A_{\mathfrak{s}_a} = \{a\}$, then $V^{\otimes \mathfrak{s}_a}$ is a partially symmetrised tensor product. In particular $V^{\otimes \mathfrak{s}_a}$ is again characterised by a universal property, namely it allows one to uniquely factorise multilinear maps on $V^{\otimes T_{\mathfrak{s}}^a}$ that are, for every $\gamma \in \Gamma_{\mathfrak{s}}^{a,a}$, γ -invariant in the sense that they are invariant under a permutation of their arguments like (5.4) with $\psi = \gamma$.

This construction (where $|A_{\mathfrak{s}}| = 1$) is already enough to build the vector spaces that we would want to associate to combinatorial trees as described in Section 5.1. A concrete combinatorial tree, that is a tree with a fixed vertex set and edge set along with an associated type map, will allow us to construct a symmetric set with $|A_{\mathfrak{s}}| = 1$.

We now turn to another feature of our construction, namely that we allow $|A_{\mathfrak{s}}| > 1$. The main motivation is that when we work with combinatorial trees, what we really want is to work with *isomorphism classes of such trees*, and so we want our construction to capture that we can allow for many different ways for the same concrete combinatorial tree to be realised. In particular we will use $A_{\mathfrak{s}}$ to index a variety of different ways to realise the same combinatorial trees as a concrete set of vertices and edges with type map. The sets $\Gamma_{\mathfrak{s}}^{a,b}$ then encode a particular set of chosen isomorphisms linking different combinatorial trees in the same isomorphism class. Once they are fixed, the maps (5.6) allow us to move between the different vector spaces that correspond to different concrete realisations

of our combinatorial trees. In particular, once \mathfrak{s} has been fixed, for every $a, b \in A_{\mathfrak{s}}$ one has fixed canonical isomorphisms

$$V^{\otimes \mathfrak{s}a} \simeq V^{\otimes \mathfrak{s}b} \simeq V^{\otimes \mathfrak{s}} \quad (5.8)$$

which can be written explicitly using the maps $\pi_{\mathfrak{s}, \bullet}$ of (5.6).

In addition to meaningfully resolving⁶ the ambiguity between working with isomorphism classes of objects like trees and concrete instances in those isomorphism classes, this flexibility is crucial for the formulation and proof of Proposition 5.28.

Remark 5.15 Given a space assignment V there is a natural notion of a dual space assignment given by $V^* = (V_t^*)_{t \in \mathcal{L}}$. There is then a canonical inclusion

$$(V^*)^{\otimes \mathfrak{s}} \hookrightarrow (V^{\otimes \mathfrak{s}})^* . \quad (5.9)$$

Thanks to (5.8) it suffices to prove (5.9) when $A_{\mathfrak{s}} = \{a\}$.

Let ι be the canonical inclusion from $(V^*)^{\otimes T_{\mathfrak{s}}^a}$ into $(V^{\otimes T_{\mathfrak{s}}^a})^*$ and let r be the canonical surjection from $(V^{\otimes T_{\mathfrak{s}}^a})^*$ to $(V^{\otimes \mathfrak{s}})^*$. The desired inclusion in (5.9) is then given by the restriction of $r \circ \iota$ to $(V^*)^{\otimes \mathfrak{s}}$. To see the claimed injectivity of this map, suppose that for some $w \in (V^*)^{\otimes \mathfrak{s}}$ one has $\iota(w)(v) = 0$ for all $v \in V^{\otimes \mathfrak{s}}$. Then, we claim that $\iota(w) = 0$ since for arbitrary $v' \in V^{\otimes T_{\mathfrak{s}}^a}$ we have

$$\iota(w)(v') = \iota\left(|\Gamma_{\mathfrak{s}}^{a,a}|^{-1} \sum_{\gamma \in \Gamma_{\mathfrak{s}}^{a,a}} \gamma \cdot w\right)(v') = \iota(w)\left(|\Gamma_{\mathfrak{s}}^{a,a}|^{-1} \sum_{\gamma \in \Gamma_{\mathfrak{s}}^{a,a}} \gamma^{-1} \cdot v'\right) = 0$$

where in the first equality we used that $w \in (V^*)^{\otimes \mathfrak{s}}$ while in the last equality we used that the sum in the expression before is in $V^{\otimes \mathfrak{s}}$.

If the space assignment V is finite-dimensional then our argument above shows that we have a canonical isomorphism

$$(V^*)^{\otimes \mathfrak{s}} \simeq (V^{\otimes \mathfrak{s}})^* . \quad (5.10)$$

Example 5.16 Continuing with Example 5.13 and denoting $\mathfrak{s} = \langle \tau \rangle$, given vector spaces V_{\bullet} , V_{\circ} , an element in $V^{\otimes \mathfrak{s}}$ can be identified with a formal sum over all representations of vectors of the form

$$v_{x_1}^{\bullet} \otimes v_{y_1}^{\bullet} \otimes v_{z_1}^{\circ} + v_{y_1}^{\bullet} \otimes v_{x_1}^{\bullet} \otimes v_{z_1}^{\circ} ,$$

namely, it is partially symmetrised such that for another representation, the two choices of γ both satisfy the requirement in (5.5). The projection $\pi_{\mathfrak{s}, a}$ then plays the role of symmetrisation.

We now fix two symmetric sets \mathfrak{s} and $\bar{\mathfrak{s}}$. Given $\Phi \in \overline{\text{Hom}}(T_{\mathfrak{s}}^a, \bar{\mathfrak{s}})$, it naturally defines a linear map $F_{\Phi}^a: V^{\otimes T_{\mathfrak{s}}^a} \rightarrow V^{\otimes \bar{\mathfrak{s}}}$ by

$$F_{\Phi}^a v = \pi_{\bar{\mathfrak{s}}, \bar{a}}(\Phi^{(\bar{a})} \cdot v) , \quad (5.11)$$

⁶See for instance Remarks 5.29 and 5.30.

where $\bar{a} \in A_{\bar{s}}$ and $\Phi^{(\bar{a})}$ denotes any representative of Φ in $\text{Iso}(T_{\bar{s}}^a, T_{\bar{s}}^{\bar{a}})$. Since any other such choice \bar{b} and $\Phi^{(\bar{b})}$ is related to the previous one by composition to the left with an element of $\Gamma_{\bar{s}}^{\bar{a}, \bar{b}}$, it follows from (5.7) that (5.11) is independent of these choices. We extend (5.11) to $\text{Vec}(\overline{\text{Hom}}(T_{\bar{s}}^a, \bar{s}))$ by linearity.

Since $(\varphi \circ \psi) \cdot v = \varphi \cdot (\psi \cdot v)$ by the definition (5.4), we conclude that, for $v = (v^{(a)})_{a \in A_{\bar{s}}} \in V^{\otimes \bar{s}}$, $\Phi = (\Phi_a)_{a \in A_{\bar{s}}} \in \text{Hom}(\bar{s}, \bar{s})$, as well as $\gamma_{a,b} \in \Gamma_{\bar{s}}^{a,b}$, we have the identity

$$F_{\Phi_b}^b v^{(b)} = F_{\Phi_a \circ \gamma_{a,b}}^b v^{(b)} = F_{\Phi_a}^a (\gamma_{a,b} \cdot v^{(b)}) = F_{\Phi_a}^a v^{(a)},$$

so that F_{Φ} is well-defined as a linear map from $V^{\otimes \bar{s}}$ to $V^{\otimes \bar{s}}$ by

$$F_{\Phi} v = F_{\Phi_a}^a \iota_{\bar{s}, a} v, \quad (5.12)$$

which we have just seen is independent of the choice of a .

The following lemma shows that this construction defines a monoidal functor \mathbf{F}_V mapping \bar{s} to $V^{\otimes \bar{s}}$ and Φ to F_{Φ} between the category SSet of symmetric sets and the category Vec of vector spaces.

Lemma 5.17 *Consider symmetric sets $\bar{s}, \bar{\bar{s}}, \bar{\bar{\bar{s}}}$, and morphisms $\Phi \in \text{Hom}(\bar{s}, \bar{\bar{s}})$ and $\bar{\Phi} \in \text{Hom}(\bar{\bar{s}}, \bar{\bar{\bar{s}}})$. Then $F_{\bar{\Phi} \circ \Phi} = F_{\bar{\Phi}} \circ F_{\Phi}$.*

Proof. Since we have by definition

$$F_{\bar{\Phi}} F_{\Phi} v = F_{\bar{\Phi}} \pi_{\bar{\bar{s}}, \bar{a}} (\Phi_a^{(\bar{a})} \cdot \iota_{\bar{s}, a} v) = \pi_{\bar{\bar{s}}, \bar{a}} (\bar{\Phi}_{\bar{b}}^{(\bar{a})} \cdot \iota_{\bar{\bar{s}}, \bar{b}} \pi_{\bar{\bar{s}}, \bar{a}} (\Phi_a^{(\bar{a})} \cdot \iota_{\bar{s}, a} v)),$$

for any arbitrary choices of $a \in A_{\bar{s}}$, $\bar{a}, \bar{b} \in A_{\bar{\bar{s}}}$, $\bar{\bar{a}} \in A_{\bar{\bar{\bar{s}}}}$, it suffices to note that

$$\bar{\Phi}_{\bar{b}}^{(\bar{a})} \cdot \iota_{\bar{\bar{s}}, \bar{b}} \pi_{\bar{\bar{s}}, \bar{a}} w = \frac{1}{|\Gamma_{\bar{\bar{s}}}^{\bar{b}, \bar{a}}|} \sum_{\gamma \in \Gamma_{\bar{\bar{s}}}^{\bar{b}, \bar{a}}} (\bar{\Phi}_{\bar{b}}^{(\bar{a})} \circ \gamma) \cdot w = \bar{\Phi}_{\bar{a}}^{(\bar{a})} \cdot w, \quad \forall w \in V^{\otimes T_{\bar{\bar{s}}}^{\bar{a}}},$$

as an immediate consequence of the definition of $\text{Hom}(\bar{\bar{s}}, \bar{\bar{\bar{s}}})$. \square

Remark 5.18 A useful property is the following. Given two space assignments V and W and a collection of linear maps $U_t: V_t \rightarrow W_t$, this induces a natural transformation $\mathbf{F}_V \rightarrow \mathbf{F}_W$. Indeed, for any typed set \bar{s} , it yields a collection of linear maps $U_{\bar{s}}^a: V^{\otimes T_{\bar{s}}^a} \rightarrow W^{\otimes T_{\bar{s}}^a}$ by

$$U_{\bar{s}}^a \bigotimes_{x \in T_{\bar{s}}^a} v_x = \bigotimes_{x \in T_{\bar{s}}^a} U_{t_{\bar{s}}^a(x)} v_x.$$

This in turn defines a linear map $U_{\bar{s}}: V^{\otimes \bar{s}} \rightarrow W^{\otimes \bar{s}}$ in the natural way. It is then immediate that, for any $\Phi \in \text{Hom}(\bar{s}, \bar{s})$, one has the identity

$$U_{\bar{s}} \circ \mathbf{F}_V(\Phi) = \mathbf{F}_W(\Phi) \circ U_{\bar{s}},$$

so that this is indeed a natural transformation.

Remark 5.19 In the more general context of Remark 5.5, this construction proceeds similarly. The only difference is that now a space assignment V is a functor $\mathcal{L} \rightarrow \text{Vec}$ mapping objects \mathfrak{t} to spaces $V_{\mathfrak{t}}$ and morphisms $\varphi \in \text{Hom}_{\mathcal{L}}(\mathfrak{t}, \bar{\mathfrak{t}})$ to linear maps $V_{\varphi} \in L(V_{\mathfrak{t}}, V_{\bar{\mathfrak{t}}})$. In this case, an element $\varphi \in \text{Iso}(T, \bar{T})$ naturally yields a linear map $V^{\otimes T} \rightarrow V^{\otimes \bar{T}}$ by

$$v = \bigotimes_{x \in T} v_x \mapsto \varphi \cdot v = \bigotimes_{y \in \bar{T}} V_{\varphi(\varphi_0^{-1}(y))} v_{\varphi_0^{-1}(y)}.$$

The remainder of the construction is then essentially the same.

Remark 5.20 One may want to restrict oneself to a smaller category than Vec by enforcing additional “nice” properties on the spaces $V_{\mathfrak{t}}$. For example, it will be convenient below to replace it by some category of topological vector spaces.

5.2.2 Typed structures

It will be convenient to consider the larger category TStruc of *typed structures*.

Definition 5.21 We define TStruc to be the category obtained by freely adjoining countable products to SSet . We write $\text{TStruc}_{\mathcal{L}}$ for TStruc when we want to emphasize the dependence of this category on the underlying label set \mathcal{L} .

Remark 5.22 An object \mathcal{S} in the category TStruc can be viewed as a countable (possibly finite) index set \mathcal{A} and, for every $\alpha \in \mathcal{A}$, a symmetric set $\mathfrak{s}_{\alpha} \in \text{Ob}(\text{SSet})$. This typed structure is then equal to $\prod_{\alpha \in \mathcal{A}} \mathfrak{s}_{\alpha}$, where \prod denotes the categorical product. (When \mathcal{A} is finite it coincides with the coproduct and we will then also write $\bigoplus_{\alpha \in \mathcal{A}} \mathfrak{s}_{\alpha}$ and call it the “direct sum” in the sequel.) Morphisms between \mathcal{S} and $\bar{\mathcal{S}}$ can be viewed as “infinite matrices” $M_{\bar{\alpha}, \alpha}$ with $\alpha \in \mathcal{A}$, $\bar{\alpha} \in \bar{\mathcal{A}}$, $M_{\bar{\alpha}, \alpha} \in \text{Hom}(\mathfrak{s}_{\alpha}, \mathfrak{s}_{\bar{\alpha}})$ and the property that, for every $\bar{\alpha} \in \bar{\mathcal{A}}$, one has $M_{\bar{\alpha}, \alpha} = 0$ for all but finitely many values of α . Composition of morphisms is performed in the natural way, analogous to matrix multiplication.

We remark that the index set \mathcal{A} here has nothing to do with the index set $A_{\mathfrak{s}}$ in Definition 5.2.

Note that TStruc is still symmetric monoidal with the tensor product behaving distributively over the direct sum if we enforce $(\prod_{\alpha \in \mathcal{A}} \mathfrak{s}_{\alpha}) \otimes (\prod_{\beta \in \mathcal{B}} \mathfrak{s}_{\beta}) = \prod_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} (\mathfrak{s}_{\alpha} \otimes \mathfrak{s}_{\beta})$, with $\mathfrak{s}_{\alpha} \otimes \mathfrak{s}_{\beta}$ as in Remark 5.10, and define the tensor product of morphisms in the natural way.

Remark 5.23 The choice of adjoining countable products (rather than coproducts) is that we will use this construction in Section 5.8 to describe the general solution to the algebraic fixed point problem associated to (5.1) as an infinite formal series.

If the space assignments V_t are finite-dimensional, the functor \mathbf{F}_V then naturally extends to an additive monoidal functor from TStruc to the category of topological vector spaces (see for example [ST12, Sec. 4.5]). Note that in particular one has $\mathbf{F}_V(\mathcal{S}) = \prod_{\alpha \in \mathcal{A}} \mathbf{F}_V(\mathfrak{s}_\alpha)$.

5.3 Direct sum decompositions of symmetric sets

In Section 5.2 we showed how, given a set of labels \mathcal{L} and space assignment $(V_t)_{t \in \mathcal{L}}$, we can “extend” this space assignment so that we get an appropriately symmetrised vector space $V^{\otimes \mathfrak{s}}$ for any symmetric set \mathfrak{s} (or, more generally, for any typed structure) typed by \mathcal{L} . In this subsection we will investigate how this construction behaves under a direct sum decomposition for the space assignment $(V_t)_{t \in \mathcal{L}}$ that is encoded via a corresponding “decomposition” on the set \mathcal{L} .

Definition 5.24 Let $\mathcal{P}(A)$ denote the powerset of a set A . Given two distinct finite sets of labels \mathcal{L} and $\bar{\mathcal{L}}$ as well as a map $\mathbf{p}: \mathcal{L} \rightarrow \mathcal{P}(\bar{\mathcal{L}}) \setminus \{\emptyset\}$, such that $\{\mathbf{p}(t) : t \in \mathcal{L}\}$ is a partition of $\bar{\mathcal{L}}$, we call $\bar{\mathcal{L}}$ a type decomposition of \mathcal{L} under \mathbf{p} . If we are also given space assignments $(V_t)_{t \in \mathcal{L}}$ for \mathcal{L} and $(\bar{V}_l)_{l \in \bar{\mathcal{L}}}$ for $\bar{\mathcal{L}}$ with the property that

$$V_t = \bigoplus_{l \in \mathbf{p}(t)} \bar{V}_l \quad \text{for every } t \in \mathcal{L}, \quad (5.13)$$

then we say that $(\bar{V}_l)_{l \in \bar{\mathcal{L}}}$ is a decomposition of $(V_t)_{t \in \mathcal{L}}$. For $l \in \mathbf{p}(t)$, we write $\mathbf{P}_l: V_t \rightarrow \bar{V}_l$ for the projection induced by (5.13).

For the remainder of this subsection we fix a set of labels \mathcal{L} , a space assignment $(V_t)_{t \in \mathcal{L}}$, along with a type decomposition $\bar{\mathcal{L}}$ of \mathcal{L} under \mathbf{p} and a space assignment $(\bar{V}_l)_{l \in \bar{\mathcal{L}}}$ that is a decomposition of $(V_t)_{t \in \mathcal{L}}$. To shorten notations, for functions $t: B \rightarrow \mathcal{L}$ and $l: B \rightarrow \bar{\mathcal{L}}$ with any set B , we write $l \rightsquigarrow t$ as a shorthand for the relation $l(p) \in \mathbf{p}(t(p))$ for every $p \in B$. Given any symmetric set \mathfrak{s} with label set \mathcal{L} and any $a \in A_\mathfrak{s}$, we write $\hat{L}_\mathfrak{s}^a = \{l: T_\mathfrak{s}^a \rightarrow \bar{\mathcal{L}} : l \rightsquigarrow t_\mathfrak{s}^a\}$, and we consider on $\hat{L}_\mathfrak{s} = \bigcup_{a \in A_\mathfrak{s}} \hat{L}_\mathfrak{s}^a$ the equivalence relation \sim given by

$$\hat{L}_\mathfrak{s}^a \ni l \sim \bar{l} \in \hat{L}_\mathfrak{s}^b \iff \exists \gamma_{b,a} \in \Gamma_\mathfrak{s}^{b,a} : l = \bar{l} \circ \gamma_{b,a}. \quad (5.14)$$

We denote by $L_\mathfrak{s} \stackrel{\text{def}}{=} \hat{L}_\mathfrak{s} / \sim$ the set of such equivalence classes, which we note is finite.

Example 5.25 *In this example we describe the vector space associated to \heartsuit . For our space assignment we start with a labelling set $\mathcal{L} = \{\star\}$, where \star represents the noise, and the noise takes values in a vector space V_\star . If we wanted to expand our noise into two components we can encode this via a direct sum decomposition $V_\star = V_{(\star,1)} \oplus V_{(\star,2)}$, where we introduce a new set of labels $\bar{\mathcal{L}} = \mathcal{L} \times \{1, 2\}$ and we define $\mathbf{p}(\star) = \{(\star, 1), (\star, 2)\}$.*

Recall our convention described in Example 5.12: \heartsuit represents an isomorphism class of trees and any concrete tree τ in that class is realised by a vertex

set which is a subset of 3 elements of \mathbf{N} and in which 2 of those elements are the leaves labelled by \star . Let δ be the symmetric set associated to \circlearrowleft and $\tau \in A_\delta$ be a concrete tree with $T_\tau = \{x, y\}$, $\mathfrak{t}_\tau(x) = \mathfrak{t}_\tau(y) = \star$. Then the local symmetry group is isomorphic to \mathbf{Z}_2 .

The set \hat{L}_δ^τ consists of 4 elements which we denote by

$$\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}, \text{ and } \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}. \quad (5.15)$$

In the symbols above, the left leaf corresponds to x and the right one to y . We thus obtain four labellings on T_τ by $\bar{\mathfrak{L}}$ where $(\star, 1)$ is associated to \bullet and $(\star, 2)$ is associated to \bullet .

However, if we were to interpret the symbols of (5.15) as isomorphism classes of trees labelled by $\bar{\mathfrak{L}}$ then $\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}$ and $\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}$ are the same isomorphism class and this is reflected by the fact that $|L_\delta| = 3$. The isomorphism class of \circlearrowleft is associated to a vector space isomorphic to $V_\star \otimes_s V_\star$. Our construction will decompose (see Proposition 5.28) the vector space for \circlearrowleft into a direct sum of three vector spaces corresponding to the isomorphism classes $\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}$, $\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}$, and $\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}$ which are, respectively, isomorphic to $V_{(\star,1)} \otimes_s V_{(\star,1)}$, $V_{(\star,2)} \otimes_s V_{(\star,2)}$, and $V_{(\star,1)} \otimes_s V_{(\star,2)}$.

Given an equivalence class $Y \in L_\delta$ and $a \in A_\delta$, we define $Y_a = Y \cap \hat{L}_\delta^a$ (which we note is non-empty due to the connectedness of Γ_δ). We then define a symmetric set δ_Y by

$$A_{\delta_Y} = \{(a, \mathfrak{l}) : a \in A_\delta, \mathfrak{l} \in Y_a\}, \quad T_{\delta_Y}^{(a, \mathfrak{l})} = T_\delta^a, \quad \mathfrak{t}_{\delta_Y}^{(a, \mathfrak{l})} = \mathfrak{l}, \\ \Gamma_{\delta_Y}^{(a, \mathfrak{l}), (b, \bar{\mathfrak{l}})} = \{\gamma \in \Gamma_\delta^{a, b} : \bar{\mathfrak{l}} = \mathfrak{l} \circ \gamma\}.$$

Remark 5.26 The definition (5.14) of our equivalence relation guarantees that Γ_{δ_Y} is connected. The definition of Γ_{δ_Y} furthermore yields a morphism of groupoids $\Gamma_{\delta_Y} \rightarrow \Gamma_\delta$ which is easily seen to be surjective.

With these notations at hand, we can define a functor \mathbf{p}^* from $\text{SSet}_\mathfrak{L}$ to $\text{TStruc}_\mathfrak{L}$ as follows. Given any $\delta \in \text{Ob}(\text{SSet}_\mathfrak{L})$, we define

$$\mathbf{p}^*\delta = \bigoplus_{Y \in L_\delta} \delta_Y \in \text{Ob}(\text{TStruc}_\mathfrak{L}). \quad (5.16)$$

To describe how \mathbf{p}^* acts on morphisms, let us fix two symmetric sets $\delta, \bar{\delta} \in \text{Ob}(\text{SSet}_\mathfrak{L})$, a choice of $Y \in L_\delta$, $(a, \mathfrak{l}) \in A_{\delta_Y}$, as well as an element $\varphi \in \overline{\text{Hom}}(T_\delta^a, \bar{\delta})$. We then let $\varphi \cdot \mathfrak{l} \subset \hat{L}_{\bar{\delta}}$ be given by

$$\varphi \cdot \mathfrak{l} = \mathfrak{l} \circ \varphi^{-1} \stackrel{\text{def}}{=} \{\bar{\mathfrak{l}} : \exists \psi \in \varphi \text{ with } \bar{\mathfrak{l}} = \mathfrak{l} \circ \psi^{-1}\},$$

where we recall that $\varphi \subset \bigcup_{\bar{a} \in A_{\bar{\delta}}} \text{Iso}(T_\delta^a, T_{\bar{\delta}}^{\bar{a}})$ is a $\Gamma_{\bar{\delta}}$ -equivalence class of bijections. It follows from the definitions of the equivalence relation on $\hat{L}_{\bar{\delta}}$ and of $\overline{\text{Hom}}(T_\delta^a, \bar{\delta})$

that one actually has $\varphi \cdot \mathfrak{l} \in L_{\bar{\mathfrak{y}}}$. We then define

$$\mathbf{p}_{(a,\mathfrak{l})}^* \varphi \in \bigoplus_{\bar{Y} \in L_{\bar{\mathfrak{y}}}} \text{Vec}(\overline{\text{Hom}}(T_{\bar{\mathfrak{y}}_Y}^{(a,\mathfrak{l})}, \bar{\mathfrak{y}}_{\bar{Y}})) ,$$

by simply setting

$$\mathbf{p}_{(a,\mathfrak{l})}^* \varphi = \varphi \in \overline{\text{Hom}}(T_{\bar{\mathfrak{y}}_Y}^{(a,\mathfrak{l})}, \bar{\mathfrak{y}}_{\varphi \cdot \mathfrak{l}}) \subset \bigoplus_{\bar{Y} \in L_{\bar{\mathfrak{y}}}} \text{Vec}(\overline{\text{Hom}}(T_{\bar{\mathfrak{y}}_Y}^{(a,\mathfrak{l})}, \bar{\mathfrak{y}}_{\bar{Y}})) , \quad (5.17)$$

which makes sense since $T_{\bar{\mathfrak{y}}_Y}^{(a,\mathfrak{l})} = T_{\bar{\mathfrak{y}}}^a$, $T_{\bar{\mathfrak{y}}_Y}^{(\bar{a},\hat{\mathfrak{l}})} = T_{\bar{\mathfrak{y}}}^{\bar{a}}$, and since $\Gamma_{\bar{\mathfrak{y}}_{\varphi \cdot \mathfrak{l}}} \rightarrow \Gamma_{\bar{\mathfrak{y}}}$ is surjective.

We take a moment to record an important property of this construction. Given any $(a, \mathfrak{l}), (b, \hat{\mathfrak{l}}) \in A_{\bar{\mathfrak{y}}_Y}$, $\gamma \in \Gamma_{\bar{\mathfrak{y}}_Y}^{(a,\mathfrak{l}), (b,\hat{\mathfrak{l}})}$, and $\varphi \in \overline{\text{Hom}}(T_{\bar{\mathfrak{y}}}^a, \bar{\mathfrak{y}})$, it follows from our definitions that

$$(\mathbf{p}_{(a,\mathfrak{l})}^* \varphi) \circ \gamma = \mathbf{p}_{(b,\hat{\mathfrak{l}})}^* (\varphi \circ \gamma) \quad (5.18)$$

where on the right-hand side of (5.18) we are viewing γ as an element of $\Gamma_{\bar{\mathfrak{y}}}^{a,b}$, which indeed maps $\overline{\text{Hom}}(T_{\bar{\mathfrak{y}}}^a, \bar{\mathfrak{y}})$ into $\overline{\text{Hom}}(T_{\bar{\mathfrak{y}}}^b, \bar{\mathfrak{y}})$ by right composition.

Extending (5.17) by linearity, we obtain a map

$$\mathbf{p}_{(a,\mathfrak{l})}^* : \text{Vec}(\overline{\text{Hom}}(T_{\bar{\mathfrak{y}}}^a, \bar{\mathfrak{y}})) \rightarrow \bigoplus_{\bar{Y} \in L_{\bar{\mathfrak{y}}}} \text{Vec}(\overline{\text{Hom}}(T_{\bar{\mathfrak{y}}_Y}^{(a,\mathfrak{l})}, \bar{\mathfrak{y}}_{\bar{Y}})) .$$

We then use this to construct a map $\mathbf{p}_Y^* : \text{Hom}(\mathfrak{y}, \bar{\mathfrak{y}}) \rightarrow \text{Hom}(\mathfrak{y}_Y, \mathbf{p}^* \bar{\mathfrak{y}})$ as follows. For any $\Phi = (\Phi_a)_{a \in A_{\bar{\mathfrak{y}}}} \in \text{Hom}(\mathfrak{y}, \bar{\mathfrak{y}})$, we set

$$(\mathbf{p}_Y^* \Phi)_{(a,\mathfrak{l})} = \mathbf{p}_{(a,\mathfrak{l})}^* \Phi_a , \quad \forall (a, \mathfrak{l}) \in A_{\bar{\mathfrak{y}}_Y} .$$

To show that this indeed belongs to $\text{Hom}(\mathfrak{y}_Y, \mathbf{p}^* \bar{\mathfrak{y}})$, note that, for any $(a, \mathfrak{l}), (b, \hat{\mathfrak{l}}) \in A_{\bar{\mathfrak{y}}_Y}$ and $\gamma \in \Gamma_{\bar{\mathfrak{y}}_Y}^{(a,\mathfrak{l}), (b,\hat{\mathfrak{l}})} \subset \Gamma_{\bar{\mathfrak{y}}}^{a,b}$, we have

$$(\mathbf{p}_Y^* \Phi)_{(a,\mathfrak{l})} \circ \gamma = (\mathbf{p}_{(a,\mathfrak{l})}^* \Phi_a) \circ \gamma = (\mathbf{p}_{(b,\hat{\mathfrak{l}})}^* (\Phi_a \circ \gamma)) = (\mathbf{p}_{(b,\hat{\mathfrak{l}})}^* \Phi_b) = (\mathbf{p}_Y^* \Phi)_{(b,\hat{\mathfrak{l}})} .$$

In the second equality we used the property (5.18) and in the third equality we used that $\Phi_a \circ \gamma = \Phi_b$ which follows from our assumption that $\Phi \in \text{Hom}(\mathfrak{y}, \bar{\mathfrak{y}})$ – recall that here we are viewing γ as an element of $\Gamma_{\bar{\mathfrak{y}}}^{a,b}$.

Finally, we then obtain the desired map $\mathbf{p}^* : \text{Hom}(\mathfrak{y}, \bar{\mathfrak{y}}) \rightarrow \text{Hom}(\mathbf{p}^* \mathfrak{y}, \mathbf{p}^* \bar{\mathfrak{y}})$ by setting, for $\Phi \in \text{Hom}(\mathfrak{y}, \bar{\mathfrak{y}})$,

$$\mathbf{p}^* \Phi = \bigoplus_{Y \in L_{\bar{\mathfrak{y}}}} \mathbf{p}_Y^* \Phi .$$

The fact that \mathbf{p}^* is a functor (i.e. preserves composition of morphisms) is an almost immediate consequence of (5.17). Indeed, given $\varphi \in \overline{\text{Hom}}(T_{\bar{\mathfrak{y}}}^a, \bar{\mathfrak{y}})$ and $\bar{\Phi} \in \text{Hom}(\bar{\mathfrak{y}}, \bar{\bar{\mathfrak{y}}})$, it follows immediately from (5.17) that

$$\mathbf{p}_{\varphi \cdot \mathfrak{l}}^* \bar{\Phi} \circ \mathbf{p}_{(a,\mathfrak{l})}^* \varphi = \mathbf{p}_{(a,\mathfrak{l})}^* (\bar{\Phi} \circ \varphi) ,$$

where we view $\bar{\Phi} \circ \varphi$ as an element of $\text{Vec}(\overline{\text{Hom}}(T_3^a, \bar{\delta}))$. It then suffices to note that $\mathbf{p}_Y^* \bar{\Phi} \circ \mathbf{p}_{(a,l)}^* \varphi = 0$ for $\bar{Y} \neq \varphi \cdot l$, which then implies that

$$\mathbf{p}^* \bar{\Phi} \circ \mathbf{p}_{(a,l)}^* \varphi = \mathbf{p}_{(a,l)}^* (\bar{\Phi} \circ \varphi) ,$$

and the claim follows. Note also that \mathbf{p}^* is monoidal in the sense that $\mathbf{p}^*(\delta \otimes \bar{\delta}) = \mathbf{p}^*(\delta) \otimes \mathbf{p}^*(\bar{\delta})$ and similarly for morphisms, modulo natural transformations.

Remark 5.27 One property that can be verified in a rather straightforward way is that if we define $(\bar{\mathbf{p}} \circ \mathbf{p})(t) = \bigcup_{l \in \mathbf{p}(t)} \bar{\mathbf{p}}(l)$, then

$$(\bar{\mathbf{p}} \circ \mathbf{p})^* = \bar{\mathbf{p}}^* \circ \mathbf{p}^* ,$$

again modulo natural transformations. This is because triples (a, l, \bar{l}) with $\bar{l} \rightsquigarrow l \rightsquigarrow t_a$ are in natural bijection with pairs (a, \bar{l}) . Note that this identity crucially uses that the sets $\mathbf{p}(t)$ are all disjoint.

Our main interest in the functor \mathbf{p}^* is that it will perform the corresponding direct sum decompositions at the level of partially symmetric tensor products of the spaces V_l . This claim is formulated as the following proposition.

Proposition 5.28 *One has $\mathbf{F}_{\bar{V}} \circ \mathbf{p}^* = \mathbf{F}_V$, modulo natural transformation.*

Proof. Fix $\delta \in \text{SSet}_{\mathcal{L}}$. Given any $a \in A_\delta$ and elementary tensor $v^{(a)} \in V^{\otimes T_3^a}$ of the form $v^{(a)} = \bigotimes_{x \in T_3^a} v_x^{(a)}$, we first note that we have the identity

$$v^{(a)} = \bigotimes_{x \in T_3^a} \sum_{l \in \mathbf{p}(t_3(x))} \mathbf{P}_l v_x^{(a)} = \sum_{l \rightsquigarrow t_3} \bigotimes_{x \in T_3^a} \mathbf{P}_{l(x)} v_x^{(a)} , \quad (5.19)$$

where \mathbf{P}_l is defined below (5.13). This suggests the following definition for a map

$$t_\delta : \prod_{a \in A_\delta} V^{\otimes T_3^a} \rightarrow \prod_{(a,l) \in \hat{L}_\delta} \bar{V}^{\otimes T_{\delta[a,l]}^{(a,l)}} ,$$

where $[a, l] \in L_\delta$ is the equivalence class that (a, l) belongs to. Given $v = (v^{(a)})_{a \in A_\delta}$ with $v^{(a)} = \bigotimes_{x \in T_3^a} v_x^{(a)}$, we set

$$(t_\delta v)_{(a,l)} \stackrel{\text{def}}{=} \bigotimes_{x \in T_3^a} \mathbf{P}_{l(x)} v_x^{(a)} ,$$

which is clearly invertible with inverse given by $(t_\delta^{-1} w)_a = \sum_{l \rightsquigarrow t_a} \bigotimes_{x \in T_3^a} w_x^{(a,l)}$. Note now that

$$\begin{aligned} \mathbf{F}_V(\delta) &\subset \prod_{a \in A_\delta} V^{\otimes T_3^a} , \\ \mathbf{F}_{\bar{V}}(\mathbf{p}^* \delta) &= \bigoplus_{Y \in L_\delta} \mathbf{F}_{\bar{V}}(\delta_Y) \subset \bigoplus_{Y \in L_\delta} \prod_{(a,l) \in A_{\delta_Y}} \bar{V}^{\otimes T_{\delta_Y}^{(a,l)}} \simeq \prod_{(a,l) \in \hat{L}_\delta} \bar{V}^{\otimes T_{\delta[a,l]}^{(a,l)}} , \end{aligned}$$

where we used that L_δ is finite in the final line. Furthermore, both ι_δ and ι_δ^{-1} preserve these subspaces, and we can thus view ι_δ as an isomorphism of vector spaces between $\mathbf{F}_V(\delta)$ and $\mathbf{F}_{\bar{V}}(\mathbf{p}^*\delta)$. The fact that, for $\Phi \in \text{Hom}(\delta, \bar{\delta})$, one has

$$\iota_{\bar{\delta}} \circ \mathbf{F}_V(\Phi) = \mathbf{F}_{\bar{V}}(\mathbf{p}^*\Phi) \circ \iota_\delta,$$

is then straightforward to verify. \square

5.4 Symmetric sets from trees and forests

Most of the symmetric sets entering our constructions will be generated from *finite labelled rooted trees* (sometimes just called “trees” for simplicity) and their associated automorphisms. A finite labelled rooted tree $\tau = (T, \varrho, \mathfrak{t}, \mathfrak{n})$ consists of a tree $T = (V, E)$ with finite vertex set V , edge set $E \subset V \times V$ and root $\varrho \in V$, endowed with a type $\mathfrak{t}: E \rightarrow \mathfrak{L}$ and label $\mathfrak{n}: V \cup E \rightarrow \mathbf{N}^{d+1}$. We also write $\mathfrak{e}: E \rightarrow \mathfrak{L} \times \mathbf{N}^{d+1}$ for the map $\mathfrak{e} = (\mathfrak{t}, \mathfrak{n} \upharpoonright E)$. Note that the “smallest” possible tree, usually denoted by $\mathbf{1}$, is given by $V = \{\varrho\}$, $E = \emptyset$ and $\mathfrak{n}(\varrho) = 0$; we denote by \mathbf{X}^k with $k \in \mathbf{N}^{d+1}$ the same tree but with $\mathfrak{n}(\varrho) = k$. For convenience, we consider edges as directed towards the root in the sense that we always have $e = (e_-, e_+)$ with e_+ nearer to the root. Note that one can naturally extend the map \mathfrak{t} to $V \setminus \{\varrho\}$ by setting $\mathfrak{t}(v) = \mathfrak{t}(e)$ for the unique edge e such that $e_- = v$.

An *isomorphism* between two labelled rooted trees is a bijection between their edge and vertex sets that preserves their connective structure, their roots, and their labels \mathfrak{t} and \mathfrak{n} . We then denote by \mathfrak{T} the set of isomorphism classes of rooted labelled trees with vertex sets that are subsets of \mathbf{N} .⁷

Given $\tau \in \mathfrak{T}$, we assign to it a symmetric set $\delta = \langle \tau \rangle$. In particular, we fix $A_\delta = \tau$ and, for every $\tau \in \tau$, we set $T_\delta^\tau = E_\tau$ (the set of edges of τ), \mathfrak{t}_δ^τ the type map of τ , and, for $\tau_1, \tau_2 \in \tau$, we let $\Gamma_\delta^{\tau_1, \tau_2}$ be the set of all elements of $\text{Iso}(T_\delta^{\tau_2}, T_\delta^{\tau_1})$ obtained from taking a tree isomorphism from τ_2 to τ_1 and then restricting this map to the set of edges E_{τ_2} . We also define the object $\langle \mathfrak{T} \rangle$ in TStruc given by $\langle \mathfrak{T} \rangle = \prod_{\tau \in \mathfrak{T}} \langle \tau \rangle$.

Given an arbitrary labelled rooted tree τ , we also write $\langle \tau \rangle$ for the symmetric set with $A_{\langle \tau \rangle}$ a singleton, $T_{\langle \tau \rangle}$ and $\mathfrak{t}_{\langle \tau \rangle}$ as above, and Γ the set of all automorphisms of τ . The following remark is crucial for our subsequent use of notations.

Remark 5.29 By definition, given any labelled rooted tree τ , there exists exactly one $\tau \in \mathfrak{T}$ such that its elements are tree isomorphic to τ and exactly *one* element of $\overline{\text{Hom}}(T_{\langle \tau \rangle}, \langle \tau \rangle)$ whose representatives are tree isomorphisms, so we are in the setting of Remark 5.11. As a consequence, we can, for all intents and purposes, identify $\langle \tau \rangle$ with $\langle \tau \rangle$. As an example, in Section 5.4.1 this observation allows us to define various morphisms on $\langle \mathfrak{T} \rangle$ and $\langle \mathfrak{T} \rangle \otimes \langle \mathfrak{T} \rangle$ by defining operations at the level of trees $\tau \in \tau \in \mathfrak{T}$ with fully specified vertex and edge sets rather than working with the isomorphism class τ .

⁷The choice of \mathbf{N} here is of course irrelevant; any set of infinite cardinality would do. The only reason for this restriction is to make sure that elements of \mathfrak{T} are sets.

Remark 5.30 In the example in Section 5.1, the two labelled rooted trees $\tau = \bullet \blacktriangledown \bullet$ and $\bar{\tau} = \bullet \blacktriangledown \bullet$ (both with trivial labels $n = 0$ say) are isomorphic. The canonical isomorphism $\Phi \in \text{Hom}(\langle \tau \rangle, \langle \bar{\tau} \rangle)$ is then simply the map matching the same types. The map $\mathbf{F}_V(\Phi): V^{\otimes \langle \tau \rangle} \rightarrow V^{\otimes \langle \bar{\tau} \rangle}$ is then a canonical isomorphism – this is where the middle identity in the motivation (5.2) is encoded.

A labelled rooted forest $f = (F, \mathfrak{R}, \mathfrak{t}, \mathfrak{n})$ is defined as consisting of a finite forest⁸ $F = (V, E)$, where again V is the set of vertices, E the set of edges, each connected component T of F has a unique distinguished root ϱ with $\mathfrak{R} \subset V$ the set of all these roots, and \mathfrak{t} and \mathfrak{n} are both as before. Note that we allow for the empty forest, that is the case where $V = E = \emptyset$, and that any finite labelled rooted tree $(T, \varrho, \mathfrak{t}, \mathfrak{n})$ is also a forest (where $\mathfrak{R} = \{\varrho\}$).

Two labelled rooted forests are considered isomorphic if there is a bijection between their edge and vertex sets that preserves their connective structure, their roots, and their labels \mathfrak{t} and \mathfrak{n} – note that we allow automorphisms of labelled rooted forests to swap connected components of the forest. Given a labelled rooted forest f , we then write $\langle f \rangle$ for the corresponding symmetric set constructed similarly to above, now with tree automorphisms replaced by forest automorphisms.

We denote by \mathfrak{F} the set of isomorphism classes of rooted labelled forests with vertex sets in \mathbf{N} , which can naturally be viewed as the unital commutative monoid generated by \mathfrak{T} with unit given by the empty forest. In the same way as above, we assign to an element $\mathbf{f} \in \mathfrak{F}$ a symmetric set $\langle \mathbf{f} \rangle$ and we write $\langle \mathfrak{F} \rangle = \prod_{\mathbf{f} \in \mathfrak{F}} \langle \mathbf{f} \rangle \in \text{Ob}(\text{TStruc})$.

Before continuing our discussion we take a moment to describe where we are going. In many previous works on regularity structures, in particular in [BHZ19], the vector space underlying a regularity structure is given by $\text{Vec}(\mathfrak{T}(R))$ for a subset $\mathfrak{T}(R) \subset \mathfrak{T}$ determined by some rule R . The construction and action of the structure and renormalisation groups was then described by using combinatorial operations on elements of \mathfrak{T} and \mathfrak{F} .

Here our point of view is different. Our concrete regularity structure will be obtained by applying the functor \mathbf{F}_V to $\langle \mathfrak{T}(R) \rangle$, an object in TStruc . We will refer to $\langle \mathfrak{T}(R) \rangle$ as an “abstract” regularity structure. In particular, trees $\tau \in \mathfrak{T}(R)$ will not be interpreted as basis vectors for our regularity structure anymore, but instead serve as an indexing set for subspaces canonically isomorphic to $\mathbf{F}_V(\langle \tau \rangle)$. In the case when $V_{\mathfrak{t}} \simeq \mathbf{R}$ for all $\mathfrak{t} \in \mathfrak{L}$, this is of course equivalent, but in general it is not. Operations like integration, tree products, forest products, and co-products on the regularity structures defined in [BHZ19] were previously given in terms of operations on \mathfrak{T} and/or \mathfrak{F} . In order to push these operations to our concrete regularity structure, we will in the next section describe how to interpret them as morphisms between the corresponding typed structures, which then allows us to push them through to “concrete” regularity structures using \mathbf{F}_V .

⁸Recall that a forest is a graph without cycles, so that every connected component is a tree.

Remark 5.31 Although the definition of $\mathfrak{T}_{\mathcal{L}}$ depends on the choice of \mathcal{L} , this definition is compatible with \mathbf{p}^* in the following sense. Given $\tau \in \mathfrak{T}_{\mathcal{L}}$, if $L_{\langle \tau \rangle}$ is defined as in the definition immediately below (5.14), then $L_{\langle \tau \rangle}$ can be identified with a subset of $\mathfrak{T}_{\bar{\mathcal{L}}}$. In particular, we overload notation and define a map $\mathbf{p}: \mathfrak{T}_{\mathcal{L}} \rightarrow \mathcal{P}(\mathfrak{T}_{\bar{\mathcal{L}}}) \setminus \{\emptyset\}$ by setting $\mathbf{p}(\tau) = L_{\langle \tau \rangle}$ so $\mathbf{p}^*\langle \tau \rangle \simeq \bigoplus_{\bar{\tau} \in \mathbf{p}(\tau)} \langle \bar{\tau} \rangle$. It is also easy to see that $\{\mathbf{p}(\tau) : \tau \in \mathfrak{T}_{\mathcal{L}}\}$ is a partition of $\mathfrak{T}_{\bar{\mathcal{L}}}$ so that

$$\mathbf{p}^*\langle \mathfrak{T}_{\mathcal{L}} \rangle \simeq \langle \mathfrak{T}_{\bar{\mathcal{L}}} \rangle.$$

Analogous statements hold for the sets of forests $\mathfrak{F}_{\mathcal{L}}$ and $\mathfrak{F}_{\bar{\mathcal{L}}}$.

5.4.1 Integration and products

We start by recalling the tree product. Given two rooted labelled trees $\tau = (T, \varrho, \mathfrak{t}, \mathfrak{n})$ and $\bar{\tau} = (\bar{T}, \bar{\varrho}, \bar{\mathfrak{t}}, \bar{\mathfrak{n}})$ the tree product of τ and $\bar{\tau}$, which we denote $\tau\bar{\tau}$, is a rooted labelled tree defined as follows. Writing $\tau\bar{\tau} = (\hat{T}, \hat{\varrho}, \hat{\mathfrak{t}}, \hat{\mathfrak{n}})$, one sets $\hat{T} \stackrel{\text{def}}{=} (T \sqcup \bar{T})/\{\varrho, \bar{\varrho}\}$, namely \hat{T} is the rooted tree obtained by taking the rooted trees T and \bar{T} and identifying the roots ϱ and $\bar{\varrho}$ into a new root $\hat{\varrho}$. Writing $T = (V, E)$, $\bar{T} = (\bar{V}, \bar{E})$, and $\hat{T} = (\hat{V}, \hat{E})$, we have a canonical identification of \hat{E} with $E \sqcup \bar{E}$ and $\hat{V} \setminus \{\hat{\varrho}\}$ with $(V \sqcup \bar{V}) \setminus \{\varrho, \bar{\varrho}\}$. With these identifications in mind, $\hat{\mathfrak{t}}$ is obtained from the concatenation of \mathfrak{t} and $\bar{\mathfrak{t}}$. We also set

$$\hat{\mathfrak{n}}(a) \stackrel{\text{def}}{=} \begin{cases} \mathfrak{n}(a) & \text{if } a \in E \sqcup (V \setminus \{\varrho\}), \\ \bar{\mathfrak{n}}(a) & \text{if } a \in \bar{E} \sqcup (\bar{V} \setminus \{\bar{\varrho}\}), \\ \mathfrak{n}(\varrho) + \bar{\mathfrak{n}}(\bar{\varrho}) & \text{if } a = \hat{\varrho}. \end{cases}$$

We remark that the tree product is well-defined and commutative at the level of isomorphism classes.

In order to push this tree product through our functor, we want to encode it as a morphism $\mathcal{M} \in \text{Hom}(\langle \mathfrak{T} \rangle \otimes \langle \mathfrak{T} \rangle, \langle \mathfrak{T} \rangle)$. It is of course sufficient for this to define elements $\mathcal{M} \in \text{Hom}(\langle \tau \rangle \otimes \langle \bar{\tau} \rangle, \langle \tau\bar{\tau} \rangle)$ for any $\tau, \bar{\tau} \in \mathfrak{T}$, which in turn is given by

$$\langle \tau \rangle \otimes \langle \bar{\tau} \rangle \simeq \langle \tau \rangle \otimes \langle \bar{\tau} \rangle \rightarrow \langle \tau\bar{\tau} \rangle \simeq \langle \tau\bar{\tau} \rangle, \quad (5.20)$$

where the two canonical isomorphisms are the ones given by Remark 5.11 and the morphism in $\text{Hom}(\langle \tau \rangle \otimes \langle \bar{\tau} \rangle, \langle \tau\bar{\tau} \rangle)$ is obtained as follows. Note that the same typed set $(\hat{E}, \hat{\mathfrak{t}})$ underlies both the symmetric sets $\langle \tau \rangle \otimes \langle \bar{\tau} \rangle$ and $\langle \tau\bar{\tau} \rangle$ and that $\Gamma_{\langle \tau \rangle \otimes \langle \bar{\tau} \rangle}$ is a subgroup (possibly proper) of $\Gamma_{\langle \tau\bar{\tau} \rangle}$. Therefore, the only natural element of $\text{Hom}(\langle \tau \rangle \otimes \langle \bar{\tau} \rangle, \langle \tau\bar{\tau} \rangle)$ is the equivalence class of the identity in $\overline{\text{Hom}}(E \sqcup \bar{E}, \langle \tau\bar{\tau} \rangle)$ (in the notation of Remark 5.8). It is straightforward to verify that \mathcal{M} constructed in this way is independent of the choices $\tau \in \tau$ and $\bar{\tau} \in \bar{\tau}$.

The ‘‘neutral element’’ $\eta \in \text{Hom}(I, \langle \mathfrak{T} \rangle)$ for \mathcal{M} , where I denotes the unit object in TStruc (corresponding to the empty symmetric set), is given by the canonical isomorphism $I \rightarrow \langle \mathbf{1} \rangle$ with $\mathbf{1}$ denoting the tree with a unique vertex and $\mathfrak{n} = 0$ as before, composed with the canonical inclusion $\langle \mathbf{1} \rangle \rightarrow \langle \mathfrak{T} \rangle$. One does indeed

have $\mathcal{M} \circ (\eta \otimes \text{id}) = \mathcal{M} \circ (\text{id} \otimes \eta) = \text{id}$, with equalities holding modulo the identifications $\langle \mathfrak{T} \rangle \simeq \langle \mathfrak{T} \rangle \otimes I \simeq I \otimes \langle \mathfrak{T} \rangle$. Associativity holds in a similar way, namely $\mathcal{M} \circ (\text{id} \otimes \mathcal{M}) = \mathcal{M} \circ (\mathcal{M} \otimes \text{id})$ as elements of $\text{Hom}(\langle \mathfrak{T} \rangle \otimes \langle \mathfrak{T} \rangle \otimes \langle \mathfrak{T} \rangle, \langle \mathfrak{T} \rangle)$.

Remark 5.32 Another important remark is that the construction of the product \mathcal{M} respects the functors \mathbf{p}^* in the same way as the construction of $\langle \mathfrak{T} \rangle$ does.

As mentioned above, \mathfrak{F} is viewed as the free unital commutative monoid generated by \mathfrak{T} with unit given by the empty forest (which we denote by \emptyset). We can interpret this product in the following way. Given two rooted labelled forests $f = (F, \mathfrak{P}, \mathfrak{t}, \mathfrak{n})$ and $\bar{f} = (\bar{F}, \bar{\mathfrak{P}}, \bar{\mathfrak{t}}, \bar{\mathfrak{n}})$ we define the forest product $f \cdot \bar{f} = (\hat{F}, \hat{\mathfrak{P}}, \hat{\mathfrak{t}}, \hat{\mathfrak{n}})$ by $\hat{F} = F \sqcup \bar{F}$, $\hat{\mathfrak{P}} = \mathfrak{P} \sqcup \bar{\mathfrak{P}}$, $\hat{\mathfrak{t}} = \mathfrak{t} \sqcup \bar{\mathfrak{t}}$, and $\hat{\mathfrak{n}} = \mathfrak{n} \sqcup \bar{\mathfrak{n}}$. Again, it is easy to see that this product is well-defined and commutative at the level of isomorphism classes. As before, writing $\hat{F} = (\hat{V}, \hat{E})$ and noting that the same typed set $(\hat{E}, \hat{\mathfrak{t}}) = (E \sqcup \bar{E}, \mathfrak{t} \sqcup \bar{\mathfrak{t}})$ underlies both symmetric sets $\langle f \rangle \otimes \langle \bar{f} \rangle$ and $\langle f \cdot \bar{f} \rangle$ and that the symmetry group of the former is a subgroup of that of the latter, there is a natural morphism $\text{Hom}(\langle f \rangle \otimes \langle \bar{f} \rangle, \langle f \cdot \bar{f} \rangle)$ given by the equivalence class of the identity. As before, this yields a product morphism in $\text{Hom}(\langle \mathfrak{F} \rangle \otimes \langle \mathfrak{F} \rangle, \langle \mathfrak{F} \rangle)$, this time with the canonical isomorphism between I and $\langle \emptyset \rangle$ (with \emptyset the empty forest) playing the role of the neutral element.

We now turn to integration. Given any $\mathfrak{l} \in \mathfrak{L}$ and $\tau = (T, \varrho, \mathfrak{t}, \mathfrak{n}) \in \mathfrak{T}$ we define a new rooted labelled tree $\mathcal{J}_{(\mathfrak{l}, 0)}(\tau) = (\bar{T}, \bar{\varrho}, \bar{\mathfrak{t}}, \bar{\mathfrak{n}}) \in \mathfrak{T}$ as follows. The tree $\bar{T} = (\bar{V}, \bar{E})$ is obtained from $T = (V, E)$ by setting $\bar{V} \stackrel{\text{def}}{=} V \sqcup \{\bar{\varrho}\}$ and $\bar{E} = E \sqcup \{\bar{e}\}$ where $\bar{e} = (\varrho, \bar{\varrho})$, that is one adds a new root vertex to the tree T and connects it to the old root with an edge. We define $\bar{\mathfrak{t}}$ to be the extension of \mathfrak{t} to \bar{E} obtained by setting $\mathfrak{t}(\bar{e}) = \mathfrak{l}$ and $\bar{\mathfrak{n}}$ to be the extension of \mathfrak{n} obtained by setting $\bar{\mathfrak{n}}(\bar{e}) = \bar{\mathfrak{n}}(\varrho) = 0$. We encode this into a morphism $\mathcal{J}_{(\mathfrak{l}, 0)} \in \text{Hom}(\langle \mathfrak{T} \rangle \otimes \langle \mathfrak{l} \rangle, \langle \mathfrak{T} \rangle)$ where $\langle \mathfrak{l} \rangle$ denotes the symmetric set with a single element \bullet of type \mathfrak{l} . For this, it suffices to exhibit natural morphisms

$$\text{Hom}(\langle \tau \rangle \otimes \langle \mathfrak{l} \rangle, \langle \mathcal{J}_{(\mathfrak{l}, 0)}(\tau) \rangle), \quad (5.21)$$

which are given by the equivalence class of $\iota: E \sqcup \{\bullet\} \rightarrow \bar{E}$ in $\overline{\text{Hom}}(E \sqcup \{\bullet\}, \langle \mathcal{J}_{(\mathfrak{l}, 0)}(\tau) \rangle)$, where ι is the identity on E and $\iota(\bullet) = \bar{e}$. It is immediate that this respects the automorphisms of τ and therefore defines indeed an element of $\text{Hom}(\langle \tau \rangle \otimes \langle \mathfrak{l} \rangle, \langle \mathcal{J}_{(\mathfrak{l}, 0)}(\tau) \rangle)$. The construction above also gives us corresponding morphisms $\mathcal{J}_{(\mathfrak{l}, p)} \in \text{Hom}(\langle \mathfrak{T} \rangle \otimes \langle \mathfrak{l} \rangle, \langle \mathfrak{T} \rangle)$, for any $p \in \mathbf{N}^{d+1}$, if we exploit the canonical isomorphism $\langle \mathcal{J}_{(\mathfrak{l}, 0)}(\tau) \rangle \simeq \langle \mathcal{J}_{(\mathfrak{l}, p)}(\tau) \rangle$ where $\mathcal{J}_{(\mathfrak{l}, p)}(\tau)$ is constructed just as $\mathcal{J}_{(\mathfrak{l}, 0)}(\tau)$, the only difference being that one sets $\bar{\mathfrak{n}}(\bar{e}) = p$.

5.4.2 Coproducts

In order to build a regularity structure, we will also need analogues of the maps Δ^+ and Δ^- as defined in [BHZ19]. The following construction will be very useful: given $\tau \in \mathfrak{T}$ and $f \in \mathfrak{F}$, we write $f \hookrightarrow \tau$ for the specification of an injective map $\iota: T_f \rightarrow T_\tau$ which preserves connectivity, orientation, and type

(but roots of f may be mapped to arbitrary vertices of τ). We also impose that $\mathbf{n}_f(e) = \mathbf{n}_\tau(\iota e)$ for every edge $e \in E_f$ and that polynomial vertex labels are increased by ι in the sense that $\mathbf{n}_f(x) \leq \mathbf{n}_\tau(\iota x)$ for all $x \in V_f$. Given $f \hookrightarrow \tau$, we also write $\partial E_f \subset E_\tau \setminus \iota(E_f)$ for the set of edges e “incident to f ” in the sense that $e_+ \in \iota(V_f)$. We consider inclusions $\iota: f \hookrightarrow \tau$ and $\bar{\iota}: \bar{f} \hookrightarrow \tau$ to be “the same” if there exists a forest isomorphism $\varphi: f \rightarrow \bar{f}$ such that $\iota = \bar{\iota} \circ \varphi$. (We do however consider them as distinct if they differ by a tree isomorphism of the target τ !)

Given a label $\mathfrak{e}: \partial E_f \rightarrow \mathbf{N}^{d+1}$, we write $\pi \mathfrak{e}: V_f \rightarrow \mathbf{N}^{d+1}$ for the map given by $\pi \mathfrak{e}(x) = \sum_{e_+ = \iota x} \mathfrak{e}(e)$ and we write $f_\mathfrak{e}$ for the forest f , but with \mathbf{n}_f replaced by $\mathbf{n}_f + \pi \mathfrak{e}$. We then write $\tau/f_\mathfrak{e} \in \mathfrak{T}$ for the tree constructed as follows. Its vertex set is given by V_τ / \sim_f , where \sim_f is the equivalence relation given by $x \sim_f y$ if and only if $x, y \in \iota(V_f)$ and $\iota^{-1}x$ and $\iota^{-1}y$ belong to the same connected component of f . The edge set of $\tau/f_\mathfrak{e}$ is given by $E_\tau \setminus \iota(E_f)$, and types and the root are inherited from τ . Its edge label is given by $e \mapsto \mathbf{n}_\tau(e) + \mathfrak{e}(e)$. Noting that vertices of $\tau/f_\mathfrak{e}$ are subsets of V_τ , its vertex label is given by $x \mapsto \sum_{y \in x} (\mathbf{n}_\tau(y) - \mathbf{n}_f(\iota^{-1}y))$ with the convention that \mathbf{n}_f is extended additively to subsets. (This is positive by our definition of “inclusion”.)

This construction then naturally defines an ‘extraction/contraction’ operation $(f \hookrightarrow \tau)_\mathfrak{e} \in \overline{\text{Hom}}(E_\tau, \langle f_\mathfrak{e} \rangle \otimes \langle \tau/f_\mathfrak{e} \rangle)$ similarly to above. (Using ι , the edge set of τ is canonically identified with the disjoint union of the edge set of $f_\mathfrak{e}$ with that of $\tau/f_\mathfrak{e}$.) Note that this is well-defined in the sense that two identical (in the sense specified above) inclusions yield identical (in the sense of canonically isomorphic) elements of $\overline{\text{Hom}}(E_\tau, \langle f_\mathfrak{e} \rangle \otimes \langle \tau/f_\mathfrak{e} \rangle)$. We also define $\bar{f}/f_\mathfrak{e}$ and $(\bar{f} \hookrightarrow \tau)_\mathfrak{e}$ for a forest \bar{f} in the analogous way.

We also define a “cutting” operation in a very similar way. Given two trees τ and $\bar{\tau}$, we write $\bar{\tau} \xrightarrow{\iota} \tau$ if $\bar{\tau} \hookrightarrow \tau$ (viewing $\bar{\tau}$ as a forest with a single tree) and the injection ι furthermore maps the root of $\bar{\tau}$ onto that of τ . With this definition at hand, we define “extraction” and “cutting” operators

$$\begin{aligned} \Delta^{\text{ex}}[\tau] &\in \overline{\text{Hom}}(E_\tau, \langle \mathfrak{F} \rangle \otimes \langle \mathfrak{T} \rangle), & \Delta^{\text{ex}}[\tau] &= \sum_{f \hookrightarrow \tau} \sum_{\mathfrak{e}} \frac{1}{\mathfrak{e}!} \binom{\tau}{f} (f \hookrightarrow \tau)_\mathfrak{e}, \\ \Delta^{\text{cut}}[\tau] &\in \overline{\text{Hom}}(E_\tau, \langle \mathfrak{T} \rangle \otimes \langle \mathfrak{T} \rangle), & \Delta^{\text{cut}}[\tau] &= \sum_{\bar{\tau} \xrightarrow{\iota} \tau} \sum_{\mathfrak{e}} \frac{1}{\mathfrak{e}!} \binom{\tau}{\bar{\tau}} (\bar{\tau} \xrightarrow{\iota} \tau)_\mathfrak{e}. \end{aligned}$$

Here, the inner sum runs over $\mathfrak{e}: \partial E_f \rightarrow \mathbf{N}^d$ ($\mathfrak{e}: \partial E_{\bar{\tau}} \rightarrow \mathbf{N}^d$ in the second case) and the binomial coefficient $\binom{\tau}{f}$ is defined as

$$\binom{\tau}{f} = \prod_{x \in V_f} \binom{\mathbf{n}_\tau(\iota x)}{\mathbf{n}_f(x)}.$$

We also view $\overline{\text{Hom}}(E_\tau, \langle f_\mathfrak{e} \rangle \otimes \langle \tau/f_\mathfrak{e} \rangle)$ as a subset of $\overline{\text{Hom}}(E_\tau, \langle \mathfrak{F} \rangle \otimes \langle \mathfrak{T} \rangle)$ via the canonical maps $\langle \tau \rangle \simeq \langle \tau \rangle \hookrightarrow \langle \mathfrak{T} \rangle$ and similarly for $\langle \mathfrak{F} \rangle$.

Lemma 5.33 *One has $\Delta^{\text{ex}}[\tau] \in \text{Hom}(\langle \tau \rangle, \langle \mathfrak{F} \rangle \otimes \langle \mathfrak{T} \rangle)$ as well as $\Delta^{\text{cut}}[\tau] \in \text{Hom}(\langle \tau \rangle, \langle \mathfrak{T} \rangle \otimes \langle \mathfrak{F} \rangle)$. \blacksquare*

Proof. Given any isomorphism φ of τ , it suffices to note that, in $\text{Hom}(\langle\tau\rangle, \langle\mathfrak{F}\rangle \otimes \langle\mathfrak{T}\rangle)$, we have the identity

$$(f \hookrightarrow \tau)_\epsilon \circ \varphi = (f\varphi \hookrightarrow \tau)_{\epsilon\varphi},$$

where, if $f \hookrightarrow \tau$ is represented by ι , then $f\varphi \hookrightarrow \tau$ is represented by $\varphi^{-1} \circ \iota$ and $\epsilon\varphi = \epsilon \circ \varphi$. It follows that $\Delta^{\text{ex}}[\tau] \circ \varphi = \Delta^{\text{ex}}[\tau]$ as required. The argument for Δ^{cut} is virtually identical. \square

It also follows from our construction that, given $\tau \in \mathfrak{T}$, $\Delta^{\text{ex}}[\tau]$ and $\Delta^{\text{cut}}[\tau]$ are independent of $\tau \in \tau$, modulo canonical isomorphism as in Remark 5.11 (see also (5.20) above), so that we can define $\Delta^{\text{ex}}[\tau] \in \text{Hom}(\langle\tau\rangle, \langle\mathfrak{F}\rangle \otimes \langle\mathfrak{T}\rangle)$ and similarly for Δ^{cut} .

5.5 Regularity structures generated by rules

We now show how regularity structures generated by rules as in [BHZ19] can be recast in this framework. This then allows us to easily formalise constructions of the type ‘‘attach a copy of V to every noise / kernel’’ as was done in a somewhat ad hoc fashion in [GH19, Sec. 3.1].

We will restrict ourselves to the setting of *reduced* abstract regularity structures (as in the language of [BHZ19, Section 6.4]). The extended label (as in [BHZ19, Section 6.4]) will not play an explicit role here but appears behind the scenes when we use the black box of [BHZ19] to build a corresponding scalar reduced regularity structure, which is then identified, via the natural transformation of Proposition 5.28, with the concrete regularity structure obtained by applying \mathbf{F}_V to our abstract regularity structure.

We start by fixing a degree map $\text{deg}: \mathfrak{L} \rightarrow \mathbf{R}$ (where \mathfrak{L} was our previously fixed set of labels), a ‘‘space’’ dimension $d \in \mathbf{N}$, and a scaling $\mathfrak{s} \in [1, \infty)^{d+1}$. Multiindices $k \in \mathbf{N}^{d+1}$ are given a scaled degree $|k|_{\mathfrak{s}} = \sum_{i=0}^d k_i \mathfrak{s}_i$. (We use the convention that the 0-component denotes the time direction.)

We then define, as in [BHZ19, Eq. (5.5)], the sets \mathcal{E} of edge labels and \mathcal{N} of node types by

$$\mathcal{E} = \mathfrak{L} \times \mathbf{N}^{d+1}, \quad \mathcal{N} = \hat{\mathcal{P}}(\mathcal{E}),$$

where $\hat{\mathcal{P}}(A)$ denotes the set of all multisets with elements from A . With this notation, we fix a ‘‘rule’’ $R: \mathfrak{L} \rightarrow \mathcal{P}(\mathcal{N}) \setminus \{\emptyset\}$. We will only consider rules that are subcritical and complete in the sense of [BHZ19, Def. 5.22].

Given $\tau \in \tau \in \mathfrak{T}$ with underlying tree $T = (V, E)$, every vertex $v \in V$ is naturally associated with a node type $\mathcal{N}(v) \stackrel{\text{def}}{=} (o(e) : e_+ = v) \in \mathcal{N}$, where we set $o(e) = (t(e), n(e)) \in \mathcal{E}$. The degree of τ is given by

$$\text{deg } \tau = \sum_{v \in V} |\mathcal{N}(v)|_{\mathfrak{s}} + \sum_{e \in E} (\text{deg } t(e) - |\mathcal{N}(e)|_{\mathfrak{s}}).$$

We say that τ *strongly conforms* to R if

- (i) for every $v \in V \setminus \{\varrho\}$, one has $\mathcal{N}(v) \in R(t(v))$, and
- (ii) there exists $t \in \mathcal{L}$ such that $\mathcal{N}(\varrho) \in R(t)$.

We say that τ is *planted* if $\sharp\mathcal{N}(\varrho) = 1$ and $\mathfrak{n}(\varrho) = 0$ and *unplanted* otherwise.

The map \deg , the above properties, and the label $\mathfrak{n}(\varrho) \in \mathbf{N}^{d+1}$ depend only on the isomorphism class $\tau \ni \tau$, and we shall use the same terminology for τ . We write $\mathfrak{T}(R) \subset \mathfrak{T}$ for the set of τ that strongly conform to R . We further write $\mathfrak{T}_*(R) \subset \mathfrak{T}$ for the set of planted trees satisfying condition (i).

We now introduce the algebras of trees / forests that are used for negative and positive renormalisation. We write $\mathfrak{F}(R) \subset \mathfrak{F}$ for the unital monoid generated (for the forest product) by $\mathfrak{T}(R)$, $\mathfrak{F}_-(R) \subset \mathfrak{F}(R)$ for the unital monoid generated by

$$\mathfrak{T}_-(R) \stackrel{\text{def}}{=} \{\tau \in \mathfrak{T}(R) : \deg \tau < 0, \mathfrak{n}(\varrho) = 0, \tau \text{ unplanted}\}, \quad (5.22)$$

and $\mathfrak{T}_+(R) \subset \mathfrak{T}$ for the unital monoid generated (for the tree product) by

$$\{\mathbf{X}^k : k \in \mathbf{N}^d\} \cup \{\tau \in \mathfrak{T}_*(R) : \deg \tau > 0\}.$$

We are now ready to construct our abstract regularity structure in the category TStruc that was defined in Definition 5.21. We define $\mathcal{T}, \mathcal{T}_+, \mathcal{T}_-, \mathcal{F}, \mathcal{F}_- \in \text{Ob}(\text{TStruc})$ by

$$\begin{aligned} \mathcal{T} &\stackrel{\text{def}}{=} \langle \mathfrak{T}(R) \rangle, & \mathcal{T}_+ &\stackrel{\text{def}}{=} \langle \mathfrak{T}_+(R) \rangle, & \mathcal{T}_- &\stackrel{\text{def}}{=} \langle \mathfrak{T}_-(R) \rangle, \\ \mathcal{F} &\stackrel{\text{def}}{=} \langle \mathfrak{F}(R) \rangle, & \mathcal{F}_- &\stackrel{\text{def}}{=} \langle \mathfrak{F}_-(R) \rangle. \end{aligned}$$

As mentioned earlier, we think of \mathcal{T} as an ‘‘abstract’’ regularity structure with ‘‘characters on \mathcal{T}_+ ’’ forming its structure group and ‘‘characters on \mathcal{F}_- ’’ forming its renormalisation group. Write

$$\pi_+ \in \text{Hom}(\mathcal{T}, \mathcal{T}_+), \quad \pi_- \in \text{Hom}(\mathcal{F}, \mathcal{F}_-),$$

for the natural projections. These allow us to define

$$\begin{aligned} \text{Hom}(\mathcal{T}, \mathcal{T} \otimes \mathcal{T}_+) \ni \Delta^+ &\stackrel{\text{def}}{=} \sum_{\tau \in \mathfrak{T}(R)} (\text{id} \otimes \pi_+) \Delta^{\text{cut}}[\tau], \\ \text{Hom}(\mathcal{T}, \mathcal{F}_- \otimes \mathcal{T}) \ni \Delta^- &\stackrel{\text{def}}{=} \sum_{\tau \in \mathfrak{T}(R)} (\pi_- \otimes \text{id}) \Delta^{\text{ex}}[\tau]. \end{aligned}$$

Remark 5.34 Here and below, it is not difficult to see that these expressions do indeed define morphisms of TStruc. Regarding Δ^+ for example, it suffices to note that, given any $\tau \in \mathfrak{T}(R)$ and $\tau_+ \in \mathfrak{T}_+(R)$, there exist only finitely many pairs of trees $\tau^{(2)} \xrightarrow{\epsilon} \tau^{(1)}$ in $\mathfrak{T}(R)$ and edge labels ϵ in Section 5.4.2, such that $\tau_+ = \tau^{(1)}/\tau_\epsilon^{(2)}$ and $\tau = \tau_\epsilon^{(2)}$.

In an analogous way, we also define

$$\Delta_s^+ \in \text{Hom}(\mathcal{T}_+, \mathcal{T}_+ \otimes \mathcal{T}_+), \quad \Delta_s^- \in \text{Hom}(\mathcal{F}_-, \mathcal{F}_- \otimes \mathcal{F}_-),$$

by

$$\Delta_s^+ \stackrel{\text{def}}{=} \sum_{\tau \in \mathfrak{T}_+(R)} (\pi_+ \otimes \pi_+) \Delta^{\text{cut}}[\tau], \quad \Delta_s^- \stackrel{\text{def}}{=} \sum_{f \in \mathfrak{F}_-(R)} (\pi_- \otimes \pi_-) \Delta^{\text{ex}}[f].$$

As in [BHZ19], one has the identities

$$(\Delta_s^- \otimes \text{id}) \circ \Delta^- = (\text{id} \otimes \Delta^-) \circ \Delta^-, \quad (\Delta_s^- \otimes \text{id}) \circ \Delta^- = (\text{id} \otimes \Delta^-) \circ \Delta^-,$$

as well as the coassociativity property for Δ_s^- , multiplicativity of Δ^+ and Δ_s^+ with respect to the tree product, and multiplicativity of Δ^- and Δ_s^- with respect to the forest product.

5.6 Concrete regularity structure

Recall that the label set \mathfrak{L} splits as $\mathfrak{L} = \mathfrak{L}_+ \cup \mathfrak{L}_-$, where \mathfrak{L}_- indexes the set of “noises” while \mathfrak{L}_+ indexes the set of kernels, which in the setting of [BCCH17] equivalently indexes the components of the class of SPDEs under consideration.

It is then natural to introduce another space assignment called a target space assignment $(W_t)_{t \in \mathfrak{L}}$ where, for each $t \in \mathfrak{L}$, the vector space W_t is the target space for the corresponding noise or component of the solution. As we already saw in the discussion at the start of Section 5.1, given a noise taking values in some space W_t for some $t \in \mathfrak{L}_-$, it is natural to assign to it a subspace of the regularity structure that is isomorphic to the (algebraic) dual space W_t^* . Then, for fixing the space assignment $(V_t)_{t \in \mathfrak{L}}$ used in the category theoretic constructions earlier in this section, this motivates space assignments of the form

$$V_t \stackrel{\text{def}}{=} \begin{cases} W_t^* & \text{for } t \in \mathfrak{L}_-, \\ \mathbf{R} & \text{for } t \in \mathfrak{L}_+. \end{cases} \quad (5.23)$$

Given a space assignment V of the form (5.23), we then use the functor \mathbf{F}_V defined in Section 5.2 to define the vector spaces $\mathcal{T}, \mathcal{T}_+, \mathcal{T}_-, \mathcal{F}, \mathcal{F}_-$ by

$$\mathcal{U} \stackrel{\text{def}}{=} \mathbf{F}_V(\mathcal{U}) = \prod_{\tau \in \mathfrak{U}(R)} \mathcal{U}[\tau], \quad \mathcal{U}[\tau] \stackrel{\text{def}}{=} \mathbf{F}_V(\langle \tau \rangle) = V^{\otimes \langle \tau \rangle} \quad (5.24)$$

where, respectively,

- \mathcal{U} is one of $\mathcal{T}, \mathcal{T}_+, \mathcal{T}_-, \mathcal{F}, \mathcal{F}_-$,
- \mathcal{U} is one of $\mathcal{T}, \mathcal{T}_+, \mathcal{T}_-, \mathcal{F}, \mathcal{F}_-$, and
- \mathfrak{U} is one of $\mathfrak{T}, \mathfrak{T}_+, \mathfrak{T}_-, \mathfrak{F}, \mathfrak{F}_-$ (τ in (5.24) can denote either a tree or a forest).

We also adopt a similar notation for linear maps $h: \mathcal{U} \rightarrow X$ (for any vector space X) by writing $h[\tau]$ for the restriction of h to $\mathcal{U}[\tau]$ for any $\tau \in \mathfrak{U}(R)$. Note that the forest product turns \mathcal{F} and \mathcal{F}_- into algebras, but that \mathcal{T} (or $\mathcal{T}_+, \mathcal{T}_-$) are not algebras in general since they may not be preserved by the tree product. We call \mathcal{T} the concrete regularity structure built from \mathcal{T} . The notion of sectors of \mathcal{T} is defined as before in [Hai14, Def. 2.5].

Remark 5.35 Given a label decomposition $\bar{\mathcal{L}}$ of \mathcal{L} under \mathbf{p} , we will naturally “extend” \mathbf{p} to a map $\mathbf{p}: \mathcal{E} \rightarrow \mathcal{P}(\bar{\mathcal{E}})$, where $\bar{\mathcal{E}} = \bar{\mathcal{L}} \times \mathbf{N}^{d+1}$, by setting, for $o = (\mathfrak{t}, p) \in \mathcal{E}$, $\mathbf{p}(o) = \mathbf{p}(\mathfrak{t}) \times \{p\}$.

If we have a splitting of labels $\mathcal{L} = \mathcal{L}_+ \sqcup \mathcal{L}_-$ then we implicitly work with a corresponding splitting $\bar{\mathcal{L}} = \bar{\mathcal{L}}_+ \sqcup \bar{\mathcal{L}}_-$ given by $\bar{\mathcal{L}}_{\pm} = \bigsqcup_{\mathfrak{t} \in \mathcal{L}_{\pm}} \mathbf{p}(\mathfrak{t})$.

Additionally, given a rule R with respect to the labelling set \mathcal{L} , we obtain a corresponding rule \bar{R} with respect to $\bar{\mathcal{L}}$ by setting, for each $\bar{\mathfrak{t}} \in \bar{\mathcal{L}}$,

$$\bar{R}(\bar{\mathfrak{t}}) = \left\{ \bar{N} \in \hat{\mathcal{P}}(\bar{\mathcal{E}}) : \exists N \in R(\mathfrak{t}) \quad \text{with} \quad \bar{N} \rightsquigarrow N \right\},$$

where \mathfrak{t} is the unique element of \mathcal{L} with $\bar{\mathfrak{t}} \in \mathbf{p}(\mathfrak{t})$ and we say that $\bar{N} \rightsquigarrow N$ if there is a bijection from \bar{N} to N respecting⁹ \mathbf{p} . If we are also given a notion of degree $\text{deg}: \mathcal{L} \rightarrow \mathbf{R}$ then we also have an induced degree $\text{deg}: \bar{\mathcal{L}} \rightarrow \mathbf{R}$ by setting, for $\bar{\mathfrak{t}}$ and \mathfrak{t} as above, $\text{deg}(\bar{\mathfrak{t}}) = \text{deg}(\mathfrak{t})$. It then follows that subcriticality or completeness hold for \bar{R} if and only if they hold for R .

Linking back to Remark 5.31, we also mention that $\{\mathbf{p}(\tau) : \tau \in \mathfrak{T}_{\mathcal{L}}(R)\}$ is a partition of $\mathfrak{T}_{\bar{\mathcal{L}}}(\bar{R})$.

We say that a decomposition \mathbf{p} acts trivially on $\mathfrak{l} \in \mathcal{L}$ if $|\mathbf{p}(\mathfrak{l})| = 1$ and in this case we will just write $\mathbf{p}(\mathfrak{l}) = \{\mathfrak{l}\}$.

Remark 5.36 We call a decomposition $\bar{\mathcal{L}}$ of \mathcal{L} under \mathbf{p} that acts trivially on \mathcal{L}_+ a *noise decomposition*. The regularity structure \mathcal{T} defined in (5.24) is then fixed, up to natural transformation, under noise decompositions of \mathcal{L} . In this case, if $(\bar{W}_{\mathfrak{l}})_{\mathfrak{l} \in \bar{\mathcal{L}}}$ is a decomposition¹⁰ of the original target space assignment $(W_{\mathfrak{t}})_{\mathfrak{t} \in \mathcal{L}}$ then $\bar{V} = (\bar{V}_{\mathfrak{l}})_{\mathfrak{l} \in \bar{\mathcal{L}}}$ built from the target space assignment $(\bar{W}_{\mathfrak{l}})_{\mathfrak{l} \in \bar{\mathcal{L}}}$ using (5.23) is a decomposition of $(V_{\mathfrak{t}})_{\mathfrak{t} \in \mathcal{L}}$. Note that this is not the case if the decomposition acts non-trivially on elements of \mathcal{L}_+ . In what follows, we will only directly apply the framework of this subsection to handle noise decompositions.

Remark 5.37 We call a target space assignment $(W_{\mathfrak{t}})_{\mathfrak{t} \in \mathcal{L}}$ with $\dim(W_{\mathfrak{t}}) = 1$ for every $\mathfrak{t} \in \mathcal{L}_-$ a *scalar noise target space assignment*, and we will say that we are working with scalar noises. Our construction of regularity structures and renormalisation groups in this section will match the constructions in [Hai14, BHZ19] when we have a scalar noise target space assignment and so we will have all the machinery developed in [Hai14, CH16, BHZ19, BCCH17] available.

Given a set of labels \mathcal{L} and target space assignment $(W_{\mathfrak{t}})_{\mathfrak{t} \in \mathcal{L}}$, we say $\bar{\mathcal{L}}$ and $(\bar{W}_{\mathfrak{l}})_{\mathfrak{l} \in \bar{\mathcal{L}}}$ are a *scalar noise decomposition* of \mathcal{L} and $(W_{\mathfrak{t}})_{\mathfrak{t} \in \mathcal{L}}$ if $\bar{\mathcal{L}}$ is a noise decomposition and if $(\bar{W}_{\mathfrak{l}})_{\mathfrak{l} \in \bar{\mathcal{L}}}$ is scalar noise target space assignment. In this situation natural transformations given by Proposition 5.28 allow us to identify the regularity structure and renormalisation group built from \mathcal{L} and $(W_{\mathfrak{t}})_{\mathfrak{t} \in \mathcal{L}}$ with those built from $\bar{\mathcal{L}}$ and $(\bar{W}_{\mathfrak{l}})_{\mathfrak{l} \in \bar{\mathcal{L}}}$. Thanks to this we can leverage the machinery of

⁹Respecting \mathbf{p} means that if $\bar{N} \ni \bar{o} \mapsto o \in N$ then $\bar{o} \in \mathbf{p}(o)$ where $\mathbf{p}: \mathcal{E} \rightarrow \mathcal{P}(\bar{\mathcal{E}})$ as above.

¹⁰cf. (5.13) (note that a target space assignment is also a space assignment)

[Hai14, CH16, BHZ19, BCCH17] for the regularity structure and renormalisation group built from \mathcal{L} and $(W_t)_{t \in \mathcal{L}}$.

Remark 5.38 One remaining difference between the setting of a scalar noise target space assignment and the setting of [Hai14, BHZ19] is that in [Hai14, BHZ19] one also enforces $\dim(W_t) = 1$ for $t \in \mathcal{L}_+$ – this constraint enforces solutions to also be scalar-valued. However, while a scalar noise assignment allows $\dim(W_t) > 1$ for $t \in \mathcal{L}_+$, our decision to enforce $V_t = \mathbf{R}$ in (5.23) means that we require that the “integration” encoded by edges of type t acts diagonally on W_t , i.e., it doesn’t mix components. In particular, with this constraint the difference between working with vector-valued solutions versus the corresponding system of equations with scalar solutions is completely cosmetic – the underlying regularity structures are the same and the only difference is how one organises the space of modelled distributions.

Remark 5.39 While the convention (5.23) is natural in our setting, an example where it must be discarded is the setting of [GH19]. In [GH19] combinatorial trees also index subspaces of the regularity structure which generically are not one-dimensional. To start translating [GH19] into our framework one would want to take $\mathcal{L}_+ = \{t_+\}$ and set $V_{t_+} = \mathcal{B}$ for \mathcal{B} an appropriate space of distributions. However, since \mathcal{B} is infinite-dimensional in this case, the machinery we develop in the remainder of this section does not immediately extend to this context. See however [GHM20] for a trick allowing to circumvent this in some cases.

At this point we make the following assumption.

Assumption 5.40 *Our target space assignments W are always finite-dimensional space assignments (which means the corresponding V given by (5.23) are finite-dimensional).*

Remark 5.41 There are several ways in which we use Assumption 5.40 in the rest of this section. One key fact is that for vector spaces X and Y one has $L(X, Y) \simeq X^* \otimes Y$ provided that either X or Y is finite-dimensional – this is especially important in the context of Remark 5.48. Another convenience of working with finite-dimensional space assignments is that we are then allowed to assume the existence of a scalar noise decomposition which lets us leverage the machinery of [Hai14, CH16, BHZ19, BCCH17].

5.7 The renormalisation group

Our construction also provides us with a “renormalisation group” that remains fixed under noise decompositions. Recalling the set of forests $\mathfrak{F}_-(R)$ and the associated algebra \mathcal{F}_- introduced in Section 5.6 (in particular Eq. (5.24)), we note that the map $\Delta_{\mathfrak{s}}$ introduced in Section 5.5 – or rather its image under the functor \mathbf{F}_V – turns \mathcal{F}_- into a bialgebra. Moreover, we can use the number of edges of each element in $\mathfrak{F}_-(R)$ to grade \mathcal{F}_- . Since \mathcal{F}_- is connected by (5.22) (i.e. its subspace of degree 0 is generated by the unit), it admits an antipode \mathcal{A}_- turning it into a

commutative Hopf algebra and we denote by \mathcal{G}_- the associated group of characters. It is immediate that, up to natural isomorphisms, the Hopf algebra \mathcal{F}_- and character group \mathcal{G}_- remain fixed under noise decompositions. Given $\ell \in \mathcal{G}_-$, we define a corresponding renormalisation operator M_ℓ , which is a linear operator

$$M_\ell: \mathcal{T} \rightarrow \mathcal{T}, \quad M_\ell \stackrel{\text{def}}{=} (\ell \otimes \text{id}_{\mathcal{T}})\Delta^- . \quad (5.25)$$

Remark 5.42 Note that the action of M_ℓ would not in general be well defined on the direct product $\mathbf{F}_V(\mathfrak{T})$ but it is well-defined on \mathcal{T} thanks to the assumption of subcriticality.

Also note that there is a canonical isomorphism

$$\mathcal{G}_- \simeq \bigoplus_{\tau \in \mathfrak{T}_-(R)} \mathcal{T}[\tau]^* . \quad (5.26)$$

In particular, given $\ell \in \mathcal{G}_-$ and $\tau \in \mathfrak{T}_-(R)$, we write $\ell[\tau]$ for the component of ℓ in $\mathcal{T}[\tau]^*$ above.

5.7.1 Canonical lifts

For the remainder of this section we impose the following assumption.

Assumption 5.43 *The rule R satisfies $R(\mathfrak{l}) = \{\emptyset\}$ for every $\mathfrak{l} \in \mathfrak{L}_-$.*

A kernel assignment is a collection of kernels $K = (K_{\mathfrak{t}} : \mathfrak{t} \in \mathfrak{L}_+)$ where each $K_{\mathfrak{t}}$ is a smooth compactly supported scalar function on $\mathbf{R}^{d+1} \setminus \{0\}$. A smooth noise assignment is a tuple $\zeta = (\zeta_{\mathfrak{t}} : \mathfrak{t} \in \mathfrak{L}_-)$ where each $\zeta_{\mathfrak{t}}$ is a smooth function from \mathbf{R}^{d+1} to $W_{\mathfrak{t}}$.

Note that the set of kernel (or smooth noise) assignments for \mathfrak{L} and W can be identified with the set of kernel (or smooth noise) assignments for any $\bar{\mathfrak{L}}$ and \bar{W} obtained via noise decomposition of the label set \mathfrak{L} and W . For smooth noise assignments this identification is given by the correspondence

$$(\zeta_{\mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_-) \leftrightarrow (\zeta_{\bar{\mathfrak{l}}} = \mathbf{P}_{\bar{\mathfrak{l}}}\zeta_{\mathfrak{l}} : \mathfrak{l} \in \mathfrak{L}_-, \bar{\mathfrak{l}} \in \mathbf{p}(\mathfrak{l})) .$$

If we are working with scalar noises then, upon fixing kernel and smooth noise assignments K and ζ , [Hai14] introduces a map $\mathbf{\Pi}_{\text{can}}$ which takes trees $\tau \in \mathfrak{T}(R)$ into $\mathcal{C}^\infty(\mathbf{R}^{d+1})$. This map gives a correspondence between combinatorial trees and the space-time functions/distributions they represent (without incorporating any negative or positive renormalisation), and $\mathbf{\Pi}_{\text{can}}$ is extended linearly to \mathcal{T} . In the general case with vector valued noise we can appeal to any scalar noise decomposition $\bar{\mathfrak{L}}$ of \mathfrak{L} and W to again obtain a linear map $\mathbf{\Pi}_{\text{can}} : \mathcal{T}_{\bar{\mathfrak{L}}} \rightarrow \mathcal{C}^\infty(\mathbf{R}^{d+1})$ —this map is of course independent of the particular scalar noise decomposition we appealed to for its definition.

In order to make combinatorial arguments which use the structure of the trees of our abstract regularity structure, it is convenient to have an explicit vectorial formula for $\mathbf{\Pi}_{\text{can}}$.

Given $\tau \in \mathfrak{T}(R)$, we write $\langle L(\tau) \rangle$ for the symmetric set obtained by restricting the tree symmetries of every $\tau \in \tau$ to the set

$$L(\tau) \stackrel{\text{def}}{=} \{e \in E_\tau : \mathfrak{t}(e) \in \mathfrak{L}_-\} .$$

Recalling the choice (5.23), we have

$$W^{\otimes \langle L(\tau) \rangle} \simeq (V^*)^{\otimes \langle \tau \rangle} \simeq \mathcal{T}[\tau]^* , \quad (5.27)$$

where we used Assumption 5.40 and (5.10) for the second canonical isomorphism. Writing, for any $\tau \in \mathfrak{T}(R)$, $\mathbf{\Pi}_{\text{can}}[\tau]$ for the restriction of $\mathbf{\Pi}_{\text{can}}$ to $\mathcal{T}[\tau]$, we will realise $\mathbf{\Pi}_{\text{can}}[\tau]$ as an element

$$\mathbf{\Pi}_{\text{can}}[\tau] \in \mathcal{C}^\infty(\mathbf{R}^{d+1}, W^{\otimes \langle L(\tau) \rangle}) ,$$

where we remind the reader that (5.27) gives us

$$\mathcal{C}^\infty(\mathbf{R}^{d+1}, W^{\otimes \langle L(\tau) \rangle}) \simeq L(\mathcal{T}[\tau], \mathcal{C}^\infty(\mathbf{R}^{d+1})) .$$

The explicit vectorial formula for $\mathbf{\Pi}_{\text{can}}$ mentioned above is then given by

$$\begin{aligned} \mathbf{\Pi}_{\text{can}}[\tau](z) = & \int_{(\mathbf{R}^{d+1})^{N(\tau)}} dx_{N(\tau)} \delta(x_\varrho - z) \left(\prod_{v \in N(\tau)} x_v^{n(v)} \right) \quad (5.28) \\ & \left(\prod_{e \in K(\tau)} D^{n(e)} K_{\mathfrak{t}(e)}(x_{e_+} - x_{e_-}) \right) \left(\bigotimes_{e \in L(\tau)} D^{n(e)} \zeta_{\mathfrak{t}(e)}(x_{e_+}) \right) \end{aligned}$$

where we have taken an arbitrary $\tau \in \tau$ and set

$$K(\tau) \stackrel{\text{def}}{=} E_\tau \setminus L(\tau) \text{ and } N(\tau) \stackrel{\text{def}}{=} \{v \in V_\tau : v \neq e_- \text{ for any } e \in L(\tau)\} . \quad (5.29)$$

Thanks to Assumption 5.43, all the integration variables $x_v \in \mathbf{R}^{d+1}$ appearing on the right-hand side of (5.28) satisfy $v \in N(\tau)$. Moreover, while the right-hand side of (5.28) is written as an element of $\bigotimes_{e \in L(\tau)} W_{\mathfrak{t}(e)}$, due to the symmetry of the integrand it can canonically be identified with an element of $W^{\otimes \langle L(\tau) \rangle}$.

5.7.2 The BPHZ character

In the scalar noise setting, upon fixing a kernel assignment K and a *random*¹¹ smooth noise assignment ζ , [BHZ19, Sec. 6.3] introduces a multiplicative linear functional $\bar{\mathbf{\Pi}}_{\text{can}}$ (denoted by g^- ($\mathbf{\Pi}_{\text{can}}$) therein) on \mathcal{F}_- obtained by setting, for $\tau \in \mathfrak{T}(R)$, $\bar{\mathbf{\Pi}}_{\text{can}}[\tau] = \mathbf{E}(\mathbf{\Pi}_{\text{can}}[\tau](0))$ and then extending multiplicatively and linearly to

¹¹With appropriate finite moment conditions.

the algebra \mathcal{F}_- . [BHZ19] also introduces a corresponding BPHZ renormalisation character $\ell_{\text{BPHZ}} \in \mathcal{G}_-$ given by

$$\ell_{\text{BPHZ}} = \bar{\Pi}_{\text{can}} \circ \tilde{\mathcal{A}}_-, \quad (5.30)$$

where $\tilde{\mathcal{A}}_-$ is the *negative twisted antipode*, an algebra homomorphism from \mathcal{F}_- to \mathcal{F} determined by enforcing the condition that

$$\mathcal{M}(\tilde{\mathcal{A}}_- \otimes \text{id})\Delta^- = 0 \quad (5.31)$$

on the subspace of \mathcal{T} generated by $\mathfrak{T}_-(R)$. Here, \mathcal{M} denotes the (forest) multiplication map from $\mathcal{F} \otimes \mathcal{T}$ into \mathcal{F} . We remind the reader that condition (5.31), combined with multiplicativity of $\tilde{\mathcal{A}}_-$, gives a recursive method for computing $\tilde{\mathcal{A}}_- \tau$ where the recursion is in $|E_\tau|$.

In the general vector-valued case we note that, analogously to (5.31), we can consider for a morphism $\tilde{\mathcal{A}}_- \in \text{Hom}(\mathcal{F}_-, \mathcal{F})$ the identity

$$\mathcal{M}(\tilde{\mathcal{A}}_- \otimes \text{id})\Delta^- = 0 \quad (5.32)$$

as an identity in $\text{Hom}(\mathcal{T}_-, \mathcal{F})$. If we furthermore impose that $\tilde{\mathcal{A}}_-$ is multiplicative in the sense that $\tilde{\mathcal{A}}_- \circ \mathcal{M} = \mathcal{M} \circ (\tilde{\mathcal{A}}_- \otimes \tilde{\mathcal{A}}_-)$ as morphisms in $\text{Hom}(\mathcal{F}_- \otimes \mathcal{F}_-, \mathcal{F})$ and $\tilde{\mathcal{A}}_-[\emptyset] = \text{id}_{\mathcal{F}[\emptyset]}$,¹² then we can proceed again by induction on $|E_\tau|$ to uniquely determine $\tilde{\mathcal{A}}_- \in \text{Hom}(\mathcal{F}_-, \mathcal{F})$.

Analogously to (5.27), one has $\mathcal{T}[\mathbf{f}]^* \simeq W^{\otimes \langle L(\mathbf{f}) \rangle}$ where $\langle L(\mathbf{f}) \rangle$ is the symmetric set obtained by restricting the forest symmetries of every $f \in \mathbf{f}$ to the set of leaves $L(f) = \bigcup_{\tau \in f} L(\tau)$, where the union runs over all the trees τ in f . This shows that, if we set again

$$\bar{\Pi}_{\text{can}}[\tau] = \mathbf{E}(\Pi_{\text{can}}[\tau](0)),$$

with Π_{can} given in (5.28), we can view $\bar{\Pi}_{\text{can}}[\tau]$ as an element of $\mathcal{T}[\tau]^*$, and, extending its definition multiplicatively, as an element of $\mathcal{T}[\mathbf{f}]^*$. Hence (5.30) yields again an element of \mathcal{G}_- , provided that we set $\tilde{\mathcal{A}}_- = \mathbf{F}_V(\tilde{\mathcal{A}}_-)$.

Remark 5.44 This construction is consistent with [BHZ19] in the sense that if we consider ℓ_{BPHZ} as in [BHZ19] for any scalar noise decomposition \mathbf{p} of \mathcal{L} , then this agrees with the construction we just described, provided that the corresponding spaces are identified via the functor \mathbf{p}^* .

5.8 Non-linearities, coherence, and the map Υ

In this subsection we will use type decompositions to show that the formula for the Υ map which appears in the description of the renormalisation of systems of scalar equations in [BCCH17] has an analogue in our setting of vector-valued regularity structures. We again fix a finite label set $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}_+ \sqcup \mathcal{L}_-$ and a target space assignment $(W_t)_{t \in \mathcal{L}}$, which determines the space assignment $(V_t)_{t \in \mathcal{L}}$ by (5.23).

¹²Recall that \emptyset denotes the empty forest and is the unit of the algebras \mathcal{F} and \mathcal{F}_- , while $\mathbf{1}$ is the tree with a single vertex and 0 is just the zero of a vector space, so these three notations are completely different.

Remark 5.45 Up to now we were consistently working with isomorphism classes $\tau \in \mathfrak{T}$. For brevity, we will henceforth work with concrete trees $\tau \in \mathcal{T}$, all considerations for which will depend only on the symmetric set $\langle \tau \rangle$, which, by Remark 5.29, we canonically identify with $\langle \tau \rangle$. We will correspondingly abuse notation and write $\tau \in \mathfrak{T}$.

Remark 5.46 In what follows we will often identify \mathfrak{L} with a subset of $\mathcal{E} = \mathfrak{L} \times \mathbf{N}^{d+1}$ by associating $\mathfrak{t} \mapsto (\mathfrak{t}, 0)$.

We define $\mathfrak{A} \stackrel{\text{def}}{=} \prod_{o \in \mathcal{E}} W_o$ where the $(W_{(\mathfrak{t}, k)} : k \in \mathbf{N}^{d+1})$ are distinct copies of the space $W_{\mathfrak{t}}$. One should think of $\mathbf{A} \in \mathfrak{A}$ as describing the jet of both the noise and the solution to a system of PDEs of the form (5.1). We equip \mathfrak{A} with the product topology.

Given any two topological vector spaces U and B , we write $\mathcal{C}^\infty(U, B)$ for the space of all maps $F : U \rightarrow B$ with the property that, for every element $\ell \in B^*$, there exists a continuous linear map $\bar{\ell} : U \rightarrow \mathbf{R}^n$ and a smooth function $F_{\ell, \bar{\ell}} : \mathbf{R}^n \rightarrow \mathbf{R}$ such that, for every $u \in U$,

$$\langle \ell, F(u) \rangle = F_{\ell, \bar{\ell}}(\bar{\ell}(u)) . \quad (5.33)$$

When our domain is $U = \mathfrak{A}$ we often just write $\mathcal{C}^\infty(B)$ instead of $\mathcal{C}^\infty(\mathfrak{A}, B)$. Note that when B is finite-dimensional then for each $F \in \mathcal{C}^\infty(B)$, $F(\mathbf{A})$ is a smooth function of $(A_o : o \in \mathcal{E}_F)$ for some *finite* subset $\mathcal{E}_F \subset \mathcal{E}$.

Remark 5.47 One difference in the point of view of the present article versus that of [BCCH17] is that here we will treat the solution and noise on a more equal footing. As an example, in [BCCH17] the domain of our smooth functions would be a direct product indexed by $\mathfrak{L}_+ \times \mathbf{N}^{d+1}$ rather than one indexed by \mathcal{E} . The fact that, in the case of the stochastic Yang–Mills equations considered here, the dependence on the noise variables has to be affine is enforced when assuming that the nonlinearity obeys our rule R , see Definition 5.51 below.

In particular, when defining the Υ map in [BCCH17] through an induction on trees τ , the symbols associated to the noises (and derivatives and products thereof)¹³ were treated as “generators” – the base case of the Υ induction. In our setting, however, the sole such generator will be the symbol $\mathbf{1}$ and noises will be treated as branches/edges $\mathcal{J}_{\mathfrak{t}}(\mathbf{1})$ for $\mathfrak{t} \in \mathfrak{L}_-$. See also Remark 5.56 below.

A specification of the right-hand side of our equation determines an element in

$$\mathring{\mathcal{Q}} \stackrel{\text{def}}{=} \mathcal{C}^\infty\left(\bigoplus_{\mathfrak{t} \in \mathfrak{L}} (V_{\mathfrak{t}} \otimes W_{\mathfrak{t}})\right) \simeq \bigoplus_{\mathfrak{t} \in \mathfrak{L}} \mathcal{C}^\infty(V_{\mathfrak{t}} \otimes W_{\mathfrak{t}}) . \quad (5.34)$$

Recalling that $V_{\mathfrak{t}} = \mathbf{R}$ for $\mathfrak{t} \in \mathfrak{L}_+$ (see (5.23)) and writing an element $F \in \mathring{\mathcal{Q}}$ as $F = \bigoplus_{\mathfrak{t} \in \mathfrak{L}} F_{\mathfrak{t}}$ with $F_{\mathfrak{t}} \in \mathcal{C}^\infty(V_{\mathfrak{t}} \otimes W_{\mathfrak{t}})$, note that $F_{\mathfrak{t}}$ for $\mathfrak{t} \in \mathfrak{L}_+$ plays the role of

¹³Here we are referring to the “drivers” of [BCCH17].

the function appearing on the right-hand side of (5.1), namely a smooth function in the variable¹⁴ $\mathbf{A} \in \mathcal{A}$ taking values in $V_t \otimes W_t \simeq W_t$. In our case, the space $V_t = \mathbf{R}$ for $t \in \mathcal{L}_+$ plays not much of a role, but in general it can be used to encode additional information about the integration kernel as in [GH19].

Remark 5.48 While the definition of the vector space \mathcal{A} depends on the label set \mathcal{L} and target space assignment $(W_t)_{t \in \mathcal{L}}$, it is natural to treat \mathcal{A} as remaining fixed under decompositions of \mathcal{L} and $(W_t)_{t \in \mathcal{L}}$ since the direct product that defines \mathcal{A} would just re-sum our decomposition.

Since we chose to simply set $V_t = \mathbf{R}$ for $t \in \mathcal{L}_+$, the spaces $\mathcal{C}^\infty(V_t \otimes W_t)$ with $t \in \mathcal{L}_+$ are also invariant (modulo canonical isomorphisms) under decompositions of \mathcal{L} . This is *not* the case for these spaces with $t \in \mathcal{L}_-$. Indeed, given a finite-dimensional vector space B , the identity id_B is the unique (up to multiplication by a scalar) element of $L(B, B) \simeq B^* \otimes B$ such that, for every decomposition $B = \bigoplus_i B_i$ one has $\text{id}_B \in \bigoplus_i L(B_i, B_i) \simeq \bigoplus_i (B_i^* \otimes B_i)$. This suggests that if we want to have nonlinearities that are invariant under decomposition (and constant for $l \in \mathcal{L}_-$ as enforced by Assumption 5.43), we should set $F_l = \text{id}_{W_l}$ for $l \in \mathcal{L}_-$. This is indeed the case and will be enforced in Definition 5.52 below.

5.8.1 Derivatives

Just as in [BCCH17] we introduce two families of differentiation operators, the first $\{D_o\}_{o \in \mathcal{E}}$ corresponding to derivatives with respect to the components of the jet \mathcal{A} and the second $\{\partial_j\}_{j=0}^d$ corresponding to derivatives in the underlying space-time.

Consider locally convex topological vector spaces U and B . Suppose that $B = \prod_{i \in I} B_i$, where each B_i is finite-dimensional, equipped with the product topology. Let $F \in \mathcal{C}^\infty(U, B)$ and $\ell \in B^*$, $\bar{\ell}$, and $F_{\ell, \bar{\ell}}$ as in (5.33). For $m \geq 0$ and $u \in U$, consider the symmetric m -linear map

$$U^m \ni (v_1, \dots, v_m) \mapsto D^m F_{\ell, \bar{\ell}}(\bar{\ell}(u))(\bar{\ell}(v_1), \dots, \bar{\ell}(v_m)) \in \mathbf{R}. \quad (5.35)$$

For fixed u, v_1, \dots, v_m , the right-hand side of (5.35) defines a linear function of ℓ which one can verify is independent of the choice of $\bar{\ell}$. Since $B^* = \bigoplus_{i \in I} B_i^*$, the algebraic dual of which is again B , there exists an element $D^m F(u)(v_1, \dots, v_m) \in B$ such that $\langle \ell, D^m F(u)(v_1, \dots, v_m) \rangle$ agrees with the right-hand side of (5.35). It is immediate that $U^m \ni (v_1, \dots, v_m) \mapsto D^m F(u)(v_1, \dots, v_m) \in B$ is symmetric and m -linear for every $u \in U$.

Turning to the case $U = \mathcal{A}$, for $o_1, \dots, o_m \in \mathcal{E}$ and $\mathbf{A} \in \mathcal{A}$, we define

$$D_{o_1} \cdots D_{o_m} F(\mathbf{A}) = D^m F(\mathbf{A}) \upharpoonright_{W_{o_1} \times \dots \times W_{o_m}} \in L(W_{o_1}, \dots, W_{o_m}; B).$$

Due to the finite-dimensionality of W_{o_i} by Assumption 5.40, the map

$$D_{o_1} \cdots D_{o_m} F: \mathbf{A} \mapsto D_{o_1} \cdots D_{o_m} F(\mathbf{A})$$

¹⁴The (\mathbf{A}, ξ) written in (5.1) corresponds to the \mathbf{A} here, see Remark 5.47.

is an element of $\mathcal{C}^\infty(L(W_{o_1}, \dots, W_{o_m}; B))$. The operators $\{D_o\}_{o \in \mathcal{E}}$ naturally commute, modulo reordering the corresponding factors.

Remark 5.49 Given a decomposition of our labelling set \mathcal{L} and target space assignment $(W_t)_{t \in \mathcal{L}}$ into $\bar{\mathcal{L}}$ and $(\bar{W}_t)_{t \in \bar{\mathcal{L}}}$, our definitions give us another set of derivative operators $\{D_{\bar{o}}\}_{\bar{o} \in \bar{\mathcal{E}}}$. Via the identifications $W_o = \bigoplus_{\bar{o} \in \mathbf{p}(o)} W_{\bar{o}}$ for any $o \in \mathcal{E}$, we have for any $F \in \mathcal{C}^\infty(B)$

$$D_o F \simeq \bigoplus_{\bar{o} \in \mathbf{p}(o)} D_{\bar{o}} F, \quad (5.36)$$

where $\mathbf{p}(o)$ is understood as in Remark 5.35. Analogous identities involving iterated direct sums hold for iterated derivatives.

For $j \in \{0, 1, \dots, d\}$ and $o = (t, p) \in \mathcal{E}$, we first define $\partial_j o \in \mathcal{E}$ by $\partial_j o = (t, p + e_j)$. We then define operators ∂_j on $\mathcal{C}^\infty(B)$ by setting, for $\mathbf{A} = (\mathbf{A}_o)_{o \in \mathcal{E}} \in \mathfrak{A}$ and $F \in \mathcal{C}^\infty(B)$,

$$(\partial_j F)(\mathbf{A}) \stackrel{\text{def}}{=} \sum_{o \in \mathcal{E}} (D_o F)(\mathbf{A}) \mathbf{A}_{\partial_j o}. \quad (5.37)$$

Note that the operators $\{\partial_j\}_{j=0}^d$ commute amongst themselves and so ∂^p is well-defined for any $p \in \mathbf{N}^{d+1}$.

Remark 5.50 Combining (5.36) with (5.37), we see that the definition of the derivatives $\{\partial_j\}_{j=0}^d$ remains unchanged under decompositions of \mathcal{L} and $(W_t)_{t \in \mathcal{L}}$, thus justifying our notation.

Just as in [BCCH17], we want to restrict ourselves to $F \in \mathring{\mathcal{Q}}$ that *obey*¹⁵ the rule R we use to construct our regularity structure.

Definition 5.51 We say $F \in \mathring{\mathcal{Q}}$ obeys a rule R if, for each $t \in \mathcal{L}$ and $o_1, \dots, o_n \in \mathcal{E}$,

$$(o_1, \dots, o_n) \notin R(t) \quad \Rightarrow \quad D_{o_1} \cdots D_{o_n} F_t = 0. \quad (5.38)$$

Note that, for any type decomposition $\bar{\mathcal{L}}$ of \mathcal{L} under \mathbf{p} , F obeys a rule R if and only if it obeys \bar{R} as defined in Remark 5.35.

Definition 5.52 Given a subcritical and complete rule R , define $\mathcal{Q} \subset \mathring{\mathcal{Q}}$ to be the set of F obeying R such that furthermore $F_l(\mathbf{A}) = \text{id}_{W_l}$ for all $l \in \mathcal{L}_-$.

Recall that, by Assumption 5.43, for any F obeying the rule R , $F_l(\mathbf{A})$ is independent of \mathbf{A} for $l \in \mathcal{L}_-$. The reason for imposing the specific choice $F_l(\mathbf{A}) = \text{id}_{W_l}$ is further discussed in Remarks 5.48 and 5.60 below.

¹⁵The notion of obey (and the set \mathcal{Q}) we choose here is analogous to item (ii) of [BCCH17, Prop. 3.13] rather than [BCCH17, Def. 3.10]. In particular, our definition is not based on expanding a nonlinearity in terms of a polynomial in the rough components of \mathbf{A} and a smooth function in the regular components of \mathbf{A} . This will mean we don't need to impose [BCCH17, Assump. 3.12] to use the main results of [BCCH17].

5.8.2 Coherence and the definition of Υ

In this subsection we formulate the notion of coherence from [BCCH17, Sec. 3] in the setting of vector regularity structures. In particular, in Theorem 5.59, we show that the coherence constraint is preserved under noise decompositions.

We first introduce some useful notation. For $o = (t, p) \in \mathcal{E}$ we set

$$\mathcal{B} \stackrel{\text{def}}{=} \mathbf{F}_V(\langle \mathcal{T} \rangle) = \prod_{\tau \in \mathcal{T}} V^{\otimes \langle \tau \rangle}, \quad \mathcal{B}_o \stackrel{\text{def}}{=} \mathbf{F}_V(\langle \mathcal{J}_o \mathcal{T} \rangle) \subset \mathcal{B},$$

and equip \mathcal{B} with the product topology. As usual, we use the notation $\mathcal{B}_t = \mathcal{B}_{(t,0)}$. Note that \mathcal{B} is an algebra when equipped with the tree product, or rather its image under \mathbf{F}_V . The following remark, where we should have in mind the case $\mathcal{B}_+ = \mathbf{F}_V(\langle \mathcal{T} \setminus \{\mathbf{1}\} \rangle)$, is crucial for the formulation of our construction.

Remark 5.53 Let \mathcal{B}_+ be an algebra such that $\mathcal{B}_+ = \varprojlim_n \mathcal{B}_+^{(n)}$ with each $\mathcal{B}_+^{(n)}$ nilpotent, let U and B be locally convex spaces where B is of the same form as in Section 5.8.1, and let $F \in \mathcal{C}^\infty(U, B)$. Write $\mathcal{B} = \mathbf{R} \oplus \mathcal{B}_+$, which is then a unital algebra. Then F can be extended to a map $\mathcal{B} \otimes U \rightarrow \mathcal{B} \otimes B$ as follows: for $u \in U$ and $\tilde{u} \in \mathcal{B}_+ \otimes U$, we set

$$F(\mathbf{1} \otimes u + \tilde{u}) \stackrel{\text{def}}{=} \sum_{m \in \mathbf{N}} \frac{D^m F(u)}{m!}(\tilde{u}, \dots, \tilde{u}), \quad (5.39)$$

where $D^m F(u) \in L(U, \dots, U; B)$ for each $u \in U$ naturally extends to a m -linear map $(\mathcal{B} \otimes U)^m \rightarrow \mathcal{B} \otimes B$ by imposing that

$$D^m F(u)(b_1 \otimes v_1, \dots, b_m \otimes v_m) = (b_1 \cdots b_m) \otimes D^m F(u)(v_1, \dots, v_m). \quad (5.40)$$

Note that the first term of this series (5.39) belongs to $\mathbf{R} \otimes B$ while all other term belong to $\mathcal{B}_+ \otimes B$. Since $\tilde{u} \in \mathcal{B}_+ \otimes U$, the projection of this series onto any of the spaces $\mathcal{B}_+^{(n)} \otimes B$ contains only finitely many non-zero terms by nilpotency, so that it is guaranteed to converge. If B and all the $\mathcal{B}_+^{(n)}$ are finite-dimensional, then the extension of F defined in (5.39) actually belongs to $\mathcal{C}^\infty(\mathcal{B} \otimes U, \mathcal{B} \otimes B)$, where \mathcal{B}_+ is equipped with the projective limit topology, under which it is nuclear, and $\mathcal{B} \otimes U$, is equipped with the projective tensor product. Furthermore, in this case, every element of $\mathcal{C}^\infty(\mathcal{B} \otimes U, \mathcal{B} \otimes B)$ extends to an element of $\mathcal{C}^\infty(\mathcal{B} \hat{\otimes} U, \mathcal{B} \otimes B)$, where $\hat{\otimes}$ denotes the (completion of the) projective tensor product.

We introduce a space¹⁶ of expansions $\mathcal{H} = \bigoplus_{t \in \mathcal{E}} \mathcal{H}_t$ with

$$\mathcal{H}_t \stackrel{\text{def}}{=} (\mathcal{B}_t \oplus \bar{\mathcal{T}}) \otimes W_t \subset \mathcal{B} \otimes W_t, \quad \bar{\mathcal{T}} \stackrel{\text{def}}{=} \prod_{k \in \mathbf{N}^{d+1}} \mathcal{T}[\mathbf{X}^k].$$

¹⁶The space \mathcal{H} introduced here plays the role of the space \mathcal{H}^{ex} in [BCCH17, Sec. 3.7].

Given $\mathcal{A} \in \mathcal{H}$, we write $\mathcal{A} = \sum_{t \in \mathcal{L}} \mathcal{A}_t$ with

$$\mathcal{A}_t = \mathcal{A}_t^R + \left(\sum_{k \in \mathbb{N}^{d+1}} \frac{\mathbf{X}^k}{k!} \otimes \mathbf{A}_{(t,k)} \right) \in \mathcal{H}_t, \quad \mathcal{A}_t^R \in \mathcal{B}_t \otimes W_t, \quad (5.41)$$

and write $\mathbf{A}^{\mathcal{A}} = (\mathbf{A}_o)_{o \in \mathcal{E}} \in \mathcal{A}$, where the coefficients $\mathbf{A}_{(t,k)}$ are as in (5.41).¹⁷ Note that (5.41) gives a natural inclusion $\mathcal{A} \subset \mathcal{H}$.

Remark 5.54 In (5.41) and several places to follow, we write \sum to denote an element of a direct product. This will simplify several expressions below, e.g. (5.53).

For $o = (t, p) \in \mathcal{E}$ we also define

$$\mathcal{A}_o \stackrel{\text{def}}{=} \mathcal{A}_o^R + \left(\sum_{k \in \mathbb{N}^{d+1}} \frac{\mathbf{X}^k}{k!} \otimes \mathbf{A}_{(t,p+k)} \right) \in \mathcal{B} \otimes W_o, \quad (5.42)$$

where $\mathcal{A}_o^R \in \mathcal{B}_o \otimes W_o$ is given by the image of \mathcal{A}_t^R under the canonical isomorphism $\mathcal{B}_o \otimes W_o \simeq \mathcal{B}_t \otimes W_t$ (but note that \mathcal{B}_o and \mathcal{B}_t are different subspaces of \mathcal{B} when $p \neq 0$). Collecting the \mathcal{A}_o into one element $\hat{\mathcal{A}} \stackrel{\text{def}}{=} (\mathcal{A}_o : o \in \mathcal{E})$, we see that $\hat{\mathcal{A}}$ is naturally viewed as an element of $\mathcal{B} \hat{\otimes} \mathcal{A}$.

Note also that, for any $o = (t, p) \in \mathcal{E}$, our construction of the morphism \mathcal{F}_o on symmetric sets built from trees in Section 5.4.1 gives us, via the functor \mathbf{F}_V , an isomorphism $\mathcal{F}_o: \mathcal{B} \otimes V_t \rightarrow \mathcal{B}_o$.

We fix for the rest of this subsection a choice of $F \in \mathbb{Q}$. Then the statement that $\mathcal{A} \in \mathcal{H}$ algebraically solves (5.1) corresponds to

$$\mathcal{A}_t^R = (\mathcal{F}_t \otimes \text{id}_{W_t})(F_t(\hat{\mathcal{A}})). \quad (5.43)$$

Here, we used Remark 5.53 to view $F_t: \mathcal{A} \rightarrow V_t \otimes W_t$ as a map from $\mathcal{B} \hat{\otimes} \mathcal{A}$ into $\mathcal{B} \otimes V_t \otimes W_t$, which \mathcal{F}_t then maps into $\mathcal{B}_t \otimes W_t$.

The coherence condition then encodes the constraint (5.43) as a functional dependence of $\mathcal{A}^R = \bigoplus_{t \in \mathcal{L}} \mathcal{A}_t^R$ on $\mathbf{A}^{\mathcal{A}}$. This functional dependence will be formulated by defining a pair of (essentially equivalent) maps Υ and $\tilde{\Upsilon}$ where

$$\tilde{\Upsilon} \in \prod_{t \in \mathcal{L}} \mathcal{C}^\infty(\mathcal{B} \otimes V_t \otimes W_t) \quad \text{and} \quad \Upsilon \in \prod_{o \in \mathcal{E}} \mathcal{C}^\infty(\mathcal{B}_o \otimes W_o). \quad (5.44)$$

The coherence condition on $\mathcal{A} \in \mathcal{H}$ will be formulated as $\mathcal{A}^R = \Upsilon(\mathbf{A}^{\mathcal{A}})$.

To define $\tilde{\Upsilon}$ and Υ , we first define corresponding maps $\tilde{\Upsilon}$ and Υ (belonging to the same respective spaces) from which $\tilde{\Upsilon}$ and Υ will be obtained by including some combinatorial factors (see (5.52)). We will write, for $t \in \mathcal{L}$, $o \in \mathcal{E}$, and $\tau \in \mathcal{T}$, $\Upsilon_o[\tau]$ for the component of Υ in $\mathcal{C}^\infty(\mathcal{B}[\mathcal{F}_o\tau] \otimes W_o)$ and $\tilde{\Upsilon}_t[\tau]$ for its component

¹⁷As a component of $\mathbf{A}^{\mathcal{A}} \in \mathcal{A}$, $\mathbf{A}_o \in W_o$, while as a term of (5.41), $\mathbf{A}_o \in W_t$. This is of course not a problem since $W_o \simeq W_t$.

in $\mathcal{C}^\infty(\mathcal{B}[\tau] \otimes V_t \otimes W_t)$.¹⁸ We will define Υ and $\bar{\Upsilon}$ by specifying the components $\Upsilon_o[\tau]$ and $\bar{\Upsilon}_t[\tau]$ through an induction in τ . Before describing this induction, we make another remark about notation.

Remark 5.55 For $t \in \mathcal{L}$, $G \in \mathcal{C}^\infty(V_t \otimes W_t)$, $(o_1, \tau_1), \dots, (o_m, \tau_m) \in \mathcal{E} \times \mathcal{T}$, and $k \in \mathbf{N}^{d+1}$, consider the map

$$\partial^k D_{o_1} \cdots D_{o_m} G \in \mathcal{C}^\infty(L(W_{o_1}, \dots, W_{o_m}; V_t \otimes W_t)).$$

It follows from Remark 5.53 that if we are given elements $\Theta_i \in \mathcal{C}^\infty(\mathcal{B} \otimes W_{o_i})$ we have a canonical interpretation for

$$(\partial^k D_{o_1} \cdots D_{o_m} G(\mathbf{A}))(\Theta_1(\mathbf{A}), \dots, \Theta_m(\mathbf{A})) \in \mathcal{B} \otimes V_t \otimes W_t, \quad (5.45)$$

which, as a function of \mathbf{A} , is an element of $\mathcal{C}^\infty(\mathcal{B} \otimes V_t \otimes W_t)$ which we denote by $(\partial^k D_{o_1} \cdots D_{o_m} G)(\Theta_1, \dots, \Theta_m)$.

We further note that if $\Theta_i \in \mathcal{C}^\infty(\mathcal{B}_i \otimes W_{o_i})$ for some subspaces $\mathcal{B}_i \subset \mathcal{B}$, then (5.45) belongs to $\hat{\mathcal{B}} \otimes V_t \otimes W_t$, where $\hat{\mathcal{B}} \subset \mathcal{B}$ is the smallest closed linear space containing all products of the form $b_1 \cdots b_m$ with $b_i \in \mathcal{B}_i$.

Now consider an isomorphism class of trees $\tau \in \mathcal{T}$. Then τ can be written as

$$\mathbf{X}^k \prod_{i=1}^m \mathcal{J}_{o_i}(\tau_i), \quad (5.46)$$

where $k = \mathbf{n}(\varrho)$, $m \geq 0$, $\tau_i \in \mathcal{T}$, and $o_i \in \mathcal{E}$.

Remark 5.56 Following up on Remark 5.47, in the analogous expression [BCCH17] Eq. (2.11) a tree τ may also contain a factor Ξ representing a noise. However, in (5.46) a noise (or a derivative of a noise) is represented by $\mathcal{J}_{(l,p)}(\mathbf{1})$ with $l \in \mathcal{L}_-$.

Given $t \in \mathcal{L}$ and τ of the form (5.46), $\bar{\Upsilon}_t[\tau]$ and $\Upsilon_{(t,p)}[\tau]$ are inductively defined by first setting

$$\bar{\Upsilon}_t[\mathbf{1}] \stackrel{\text{def}}{=} \mathbf{1} \otimes F_t, \quad (5.47)$$

which belongs to $\mathcal{B}[\mathbf{1}] \otimes \mathcal{C}^\infty(V_t \otimes W_t) \simeq \mathcal{C}^\infty(\mathcal{B}[\mathbf{1}] \otimes V_t \otimes W_t) \subset \mathcal{C}^\infty(\mathcal{B} \otimes V_t \otimes W_t)$ (the first isomorphism follows from the fact that $V_t \otimes W_t$ is finite dimensional by assumption) so this is indeed of the desired type. We then set

$$\begin{aligned} \bar{\Upsilon}_t[\tau] &\stackrel{\text{def}}{=} \mathbf{X}^k \left[\partial^k D_{o_1} \cdots D_{o_m} \bar{\Upsilon}_t[\mathbf{1}] \right] (\Upsilon_{o_1}[\tau_1], \dots, \Upsilon_{o_m}[\tau_m]), \\ \Upsilon_{(t,p)}[\tau] &\stackrel{\text{def}}{=} (\mathcal{J}_{(t,p)} \otimes \text{id}_{W_t})(\bar{\Upsilon}_t[\tau]). \end{aligned} \quad (5.48)$$

¹⁸This means that our notation for $\Upsilon[\tau]$ breaks the notational convention we've used so far for other elements of spaces of this type (direct products of $\mathbf{F}_V(\langle \tau \rangle)$, possibly tensorised with some fixed space). The reason we do this is to be compatible with the notations of [BCCH17], and also to keep notations in Sections 6.2 and 7.3 cleaner.

We explain some of the notation and conventions used in (5.48). By Remark 5.55, the term following \mathbf{X}^k in the right-hand side for $\tilde{\Upsilon}_t[\tau]$ is an element of

$$\mathcal{C}^\infty\left(\mathcal{B}\left[\prod_{j=1}^m \mathcal{J}_{o_j}(\tau_j)\right] \otimes V_t \otimes W_t\right).$$

We then interpret $\mathbf{X}^{k\bullet}$ as the canonical isomorphism

$$\mathcal{B}\left[\prod_{j=1}^m \mathcal{J}_{o_j}(\tau_j)\right] \simeq \mathcal{B}\left[\mathbf{X}^k \prod_{j=1}^m \mathcal{J}_{o_j}(\tau_j)\right] = \mathcal{B}[\tau], \quad (5.49)$$

acting on the first factor of the tensor product, hence the right-hand side of the definition of $\tilde{\Upsilon}_t[\tau]$ belongs to $\mathcal{C}^\infty(\mathcal{B}[\tau] \otimes V_t \otimes W_t)$, which is mapped to $\mathcal{C}^\infty(\mathcal{B}[\mathcal{J}_{(t,p)}\tau] \otimes W_t)$ by $\mathcal{J}_{(t,p)} \otimes \text{id}_{W_t}$ as desired.

Remark 5.57 We have two important consequences of (5.48) and (5.47):

- (i) Since F obeys R , we have, for any $o = (t, p) \in \mathcal{E}$, $\Upsilon_o[\tau] = 0$ and $\tilde{\Upsilon}_t[\tau] = 0$ unless $\mathcal{J}_t(\tau) \in \mathfrak{T}(R)$.
- (ii) For any $t \in \mathfrak{L}_-$, $p \in \mathbf{N}^{d+1}$ and $\tau \in \mathfrak{T} \setminus \{\mathbf{1}\}$, one has $\Upsilon_{(t,p)}[\tau] = 0$ and $\tilde{\Upsilon}_t[\tau] = 0$, due to annihilation by the operators ∂ and D .

In particular, our assumption that the rule R is subcritical guarantees that for any given degree γ , only finitely many of the components $\tilde{\Upsilon}_t[\tau]$ with $\deg \tau < \gamma$ are non-vanishing.

Remark 5.58 Although it plays exactly the same role, the map Υ introduced in [BCCH17] is of a slightly different type than the maps Υ and $\tilde{\Upsilon}$ introduced here. More precisely, in [BCCH17], $\Upsilon_o[\tau](\mathbf{A}) \in \mathbf{R}$ played the role of a coefficient¹⁹ of a basis vector in the regularity structure. In the present article on the other hand, $\Upsilon_o[\tau](\mathbf{A}) \in \mathcal{B}[\mathcal{J}_o\tau] \otimes W_o$. In the setting of [BCCH17], these spaces are canonically isomorphic to \mathbf{R} and our definitions are consistent modulo this isomorphism.

The $\tilde{\Upsilon}$ and Υ defined in (5.48) are missing the combinatorial symmetry factors $S(\tau)$ associated to a tree $\tau \in \mathfrak{T}$, which we define in the same way as in [BCCH17]. For this we represent τ more explicitly than (5.46) by writing

$$\tau = \mathbf{X}^k \prod_{j=1}^{\ell} \mathcal{J}_{o_j}(\tau_j)^{\beta_j}, \quad (5.50)$$

with $\ell \geq 0$, $\beta_j > 0$, and *distinct* $(o_1, \tau_1), \dots, (o_\ell, \tau_\ell) \in \mathcal{E} \times \mathfrak{T}$, and define

$$S(\tau) \stackrel{\text{def}}{=} k! \left(\prod_{j=1}^{\ell} S(\tau_j)^{\beta_j} \beta_j! \right). \quad (5.51)$$

¹⁹The coefficient of $\mathcal{J}_o(\tau)$ in a coherent jet and the coefficient of τ in the expansion of the non-linearities evaluated on a coherent jet.

We then set, for $\mathfrak{t} \in \mathfrak{L}$, $o \in \mathcal{E}$, and $\tau \in \mathfrak{T}$,

$$\begin{aligned} \Upsilon &= \sum_{o \in \mathcal{E}, \tau \in \mathfrak{T}} \Upsilon_o[\tau], & \Upsilon_o[\tau] &\stackrel{\text{def}}{=} \Upsilon_o[\tau]/S(\tau), \\ \bar{\Upsilon} &= \sum_{\mathfrak{t} \in \mathfrak{L}, \tau \in \mathfrak{T}} \bar{\Upsilon}_{\mathfrak{t}}[\tau], & \bar{\Upsilon}_{\mathfrak{t}}[\tau] &\stackrel{\text{def}}{=} \bar{\Upsilon}_{\mathfrak{t}}[\tau]/S(\tau). \end{aligned} \quad (5.52)$$

We can now state the main theorem of this section – here we specialise to noise decompositions (i.e. those acting trivially on \mathfrak{L}_+) described in Remark 5.36.

Theorem 5.59 *$\bar{\Upsilon}$ and Υ as defined in (5.52) are left unchanged under noise decompositions of \mathfrak{L} and $(W_{\mathfrak{t}})_{\mathfrak{t} \in \mathfrak{L}}$. Precisely, given a noise decomposition $\bar{\mathfrak{L}}$ of \mathfrak{L} under \mathbf{p} with associated target space decomposition $(\bar{W}_{\mathfrak{t}})_{\mathfrak{t} \in \bar{\mathfrak{L}}}$, one has, for any $\mathfrak{t} \in \mathfrak{L}$, $\tau \in \mathfrak{T}$ and $\mathbf{A} \in \mathcal{A}$,*

$$\Upsilon_{\mathfrak{t}}[\tau](\mathbf{A}) = \sum_{\mathfrak{t} \in \mathbf{p}(\mathfrak{t}), \bar{\tau} \in \mathbf{p}(\tau)} \Upsilon_{\mathfrak{t}}[\bar{\tau}](\mathbf{A}), \quad (5.53)$$

where $(\Upsilon_{\mathfrak{t}})_{\mathfrak{t} \in \bar{\mathfrak{L}}}$ on the right-hand side is defined as above but with the decomposed labelling set and target space assignment used in its construction, and $\mathbf{p}(\tau)$ is defined as in Remark 5.31. The equality (5.53) also holds when Υ is replaced by $\bar{\Upsilon}$.

Proof. We prove (5.53) inductively in the number of edges of τ . Writing τ in the form (5.46), our base case corresponds to $m = 0$, i.e. $\tau = \mathbf{X}^k$ for $k \in \mathbf{N}^{d+1}$, so that $\mathbf{p}(\mathbf{X}^k) = \{\mathbf{X}^k\}$ and $S(\mathbf{X}^k) = k!$. This case is covered by Remark 5.48. For our inductive step, we may assume that $m \geq 1$ in (5.46). By item (ii) of Remark 5.57 there is nothing to check if $\mathfrak{t} \in \mathfrak{L}_-$ so we turn to the situation where $\mathfrak{t} \in \mathfrak{L}_+$.

Since \mathbf{p} acts trivially on \mathfrak{L}_+ we can write $\mathbf{p}(\mathfrak{t}) = \{\mathfrak{t}\}$. Then, inserting our inductive hypothesis in (5.48) and also applying Remark 5.49 we see that (5.53) follows if we can show that

$$\begin{aligned} \sum_{(l, \sigma) \in d(\tau)} (\mathcal{F}_{\mathfrak{t}} \otimes \text{id}_{W_{\mathfrak{t}}}) \left[\mathbf{X}^k \left(\partial^k (D_{l_1} \cdots D_{l_m} \bar{\Upsilon}_{\mathfrak{t}}[\mathbf{1}])(\Upsilon_{l_1}[\sigma_1], \dots, \Upsilon_{l_m}[\sigma_m]) \right) \right] \\ = \sum_{\bar{\tau} \in \mathbf{p}(\tau)} \frac{S(\tau)}{S(\bar{\tau})} \Upsilon_{\mathfrak{t}}[\bar{\tau}], \end{aligned} \quad (5.54)$$

where $d(\tau)$ consists of all pairs of tuples (l, σ) with $l = (l_i)_{i=1}^m$, $\sigma = (\sigma_i)_{i=1}^m$, $l_i \in \mathbf{p}(o_i)$ and $\sigma_i \in \mathbf{p}(\tau_i)$. Given $(l, \sigma) \in d(\tau)$ we write $\tau(l, \sigma) \stackrel{\text{def}}{=} \mathbf{X}^k \prod_{i=1}^m \mathcal{J}_{l_i}(\sigma_i) \in \mathfrak{T}_{\bar{\mathfrak{L}}}$. Clearly one has $\tau(l, \sigma) \in \mathbf{p}(\tau)$ and for fixed (l, σ) the corresponding summand on the left-hand side of (5.54) is simply $\Upsilon_{\mathfrak{t}}[\tau(l, \sigma)]$. Finally, for any $\bar{\tau} \in \mathbf{p}(\tau)$, it is straightforward to prove, using a simple induction and manipulations of multinomial coefficients, that

$$\frac{S(\tau)}{S(\bar{\tau})} = |\{(l, \sigma) \in d(\tau) : \tau(l, \sigma) = \bar{\tau}\}|,$$

which shows (5.54). Using natural isomorphisms between the spaces where $\tilde{\Upsilon}$ and Υ live, it follows that (5.53) also holds for $\tilde{\Upsilon}$. \square

Remark 5.60 We used Remark 5.48 in a crucial way to start the induction, which shows that since we consider rules with $R(\mathfrak{l}) = \{\emptyset\}$ for $\mathfrak{l} \in \mathfrak{L}_-$, the choice $F_{\mathfrak{l}} = \text{id}_{W_{\mathfrak{l}}}$ is the only one that complies with (5.38) and also guarantees invariance under noise decompositions. See however [GHM20] for an example where $R(\mathfrak{l}) \neq \{\emptyset\}$ and it is natural to make a different choice for $F_{\mathfrak{l}}$.

We now precisely define coherence in our setting. For $L \in \mathbf{N} \cup \{\infty\}$, we denote by $p_{\leq L}$ the projection map on $\bigoplus_{\mathfrak{t} \in \mathfrak{L}} \mathcal{B} \otimes W_{\mathfrak{t}}$ which vanishes on any subspace of the form $\mathcal{T}[\tau] \otimes W_{\mathfrak{t}}$ if

$$|E_{\tau}| + \sum_{v \in V_{\tau}} |\mathfrak{n}(v)| > L$$

and is the identity otherwise. Above we write E_{τ} for the set of edges of τ , V_{τ} for the set of nodes of τ , and \mathfrak{n} for the label on τ . Note that $p_{\leq \infty}$ is just the identity operator.

Definition 5.61 We say $\mathcal{A} \in \mathcal{H}$ is coherent to order $L \in \mathbf{N} \cup \{\infty\}$ with F if

$$p_{\leq L} \mathcal{A}^R = p_{\leq L} \Upsilon(\mathbf{A}^{\mathcal{A}}), \quad (5.55)$$

where $\mathcal{A}^R = \bigoplus_{\mathfrak{t} \in \mathfrak{L}} \mathcal{A}_{\mathfrak{t}}^R$ with $\mathcal{A}_{\mathfrak{t}}^R$ determined from \mathcal{A} as in (5.41).

Note that, by Theorem 5.59, coherence to any order L is preserved under noise decompositions. Thanks to Theorem 5.59 we can reformulate [BCCH17, Lem. 3.21] to show that our definition of Υ encodes the condition (5.43); we state this as a lemma.

Lemma 5.62 $\mathcal{A} \in \mathcal{H}$ is coherent to order $L \in \mathbf{N} \cup \{\infty\}$ with F if and only if, for each $\mathfrak{t} \in \mathfrak{L}$,

$$p_{\leq L} \mathcal{A}_{\mathfrak{t}}^R = p_{\leq L} (\mathcal{J}_{\mathfrak{t}} \otimes \text{id}_{W_{\mathfrak{t}}}) F_{\mathfrak{t}}(\mathcal{A}). \quad (5.56)$$

Remark 5.63 Combining Lemma 5.62 with Definition 5.61 shows that Υ does indeed have the advertised property, namely it yields a formula for the “non-standard part” of the expansion of any solution to the algebraic counterpart (5.43) of the mild formulation of the original problem (5.1).

Conversely, this provides us with an alternative method for computing $\Upsilon(\mathbf{A})[\tau]$ for any $\tau \in \mathfrak{T}$. Given $\mathbf{A} \in \mathcal{A}$, set $\mathcal{A}^{(0)} = \mathbf{A} \in \mathcal{H}$ (recall (5.41) for the identification of \mathcal{A} as a subspace of \mathcal{H}) and then proceed iteratively by setting

$$\mathcal{A}_{\mathfrak{t}}^{(n+1)} = \mathbf{A}_{\mathfrak{t}} + (\mathcal{J}_{\mathfrak{t}} \otimes \text{id}_{W_{\mathfrak{t}}}) F_{\mathfrak{t}}(\mathcal{A}^{(n)}).$$

Subcriticality then guarantees that any of the projections $\mathcal{A}^{(n)}[\tau]$ stabilises after a finite number of steps, and one has $\Upsilon_{\mathfrak{t}}[\tau](\mathbf{A}) = \mathcal{A}^{(\infty)}[\tau]$.

Remark 5.64 The material discussed in Section 5.8 up to this point has been devoted to treating (5.1) as an *algebraic* fixed point problem in the space \mathcal{H} . We also want to solve an *analytic* fixed point problem in a space of modelled distributions, namely in a space of \mathcal{H} -valued functions over some space-time domain.

Posing the analytic fixed point problem requires us to start with more input than we needed for the algebraic one. After fixing $F \in \mathbb{Q}$ one also needs to fix suitable²⁰ regularity exponents $(\gamma_t : t \in \mathcal{L})$ for the modelled distribution spaces involved and initial data $(u_t : t \in \mathcal{L}_+)$ for the problem. Moreover, one prescribes a modelled distribution expansion for each noise, namely for every $l \in \mathcal{L}_-$, we fix a modelled distribution \mathcal{O}_l of regularity \mathcal{D}^{γ_l} of the form

$$\mathcal{O}_l(z) = \sum_{k \in \mathbf{N}^{d+1}} O_{(l,k)}(z) \mathbf{X}^k + \mathcal{J}_l(\mathbf{1}). \quad (5.57)$$

The corresponding analytic fixed point problem [BCCH17, Eq. (5.6)] is then posed on a space of modelled distributions $\mathcal{U} = (\mathcal{U}_t : t \in \mathcal{L}_+)$ such that $\mathcal{U}_t \in \mathcal{D}^{\gamma_t}$ (at least locally). On some space-time domain D (typically of the form $[0, T] \times \mathbf{R}^d$), the fixed point problem is of the form

$$\mathcal{U}_t = \mathcal{P}_t \mathbf{1}_{t>0} F_t(\mathcal{U} \sqcup \mathcal{O})(\bullet) + G_t u_t.$$

In this identity, \mathcal{P}_t is an operator of the form

$$(\mathcal{P}_t \mathcal{F})(z) = \mathfrak{p}_{\leq \gamma_t} \mathcal{J}_t \mathcal{F}(z) + (\dots),$$

where (\dots) takes values in $\bar{\mathcal{T}}_t \stackrel{\text{def}}{=} \bigoplus_{k \in \mathbf{N}^{d+1}} \mathcal{T}[\mathbf{X}^k] \otimes W_t$, and G_t is the “harmonic extension map” as in [Hai14, (7.13)] associated to $(\partial_t - \mathcal{L}_t)^{-1}$ (possibly with suitable boundary conditions). Here, $\mathfrak{p}_{\leq \gamma_t}$ is the projection onto components of degree less than γ_t . Since G_t also takes values in $\bar{\mathcal{T}}_t$, it follows that for any solution \mathcal{U} to such a fixed point problem and any space-time point $z \in D$, $\mathcal{U}(z) \sqcup \mathcal{O}(z)$ is coherent with F to some order L which depends on the exponents $(\gamma_t : t \in \mathcal{L})$; see [BCCH17, Thm. 5.7] for a precise statement.

Note that, depending on the degrees of our noises, there can be some freedom in our choice of (5.57) depending on how we choose to have our model act on symbols $\mathcal{J}_l(\mathbf{1})$ for $l \in \mathcal{L}_-$ – the key fact is that \mathcal{O}_l represents the corresponding driving noise in our problem, not necessarily $\mathcal{J}_l(\mathbf{1})$. However, when $\deg(l) < 0$, a natural choice for the input (5.57) is to simply set $O_{(l,k)}(z) = 0$ for all $k \in \mathbf{N}^{d+1}$ and this is the convention we use in Sections 6 and 7.

5.8.3 Renormalised equations

We now describe the action of the renormalisation group \mathcal{G}_- on nonlinearities, which is how it produces counterterms in equations. We no longer treat $F \in \mathbb{Q}$ as

²⁰Here, “suitable” means sufficiently large so that the fixed point problem is well-posed. Subcriticality guarantees that setting $\gamma_t = \gamma$ for all t is a suitable choice provided that $\gamma \in \mathbf{R}_+$ is sufficiently large.

fixed and when we want to make the dependence of $\tilde{\Upsilon}$ on $F \in \mathbb{Q}$ explicit we write $\tilde{\Upsilon}^F$. We re-formulate the main algebraic results of [BCCH17] in the following proposition; the proof is obtained by using Theorem 5.59 to restate [BCCH17, Lem. 3.22, Lem. 3.23, and Prop. 3.24].

Proposition 5.65 *Fix $\ell \in \mathcal{G}_-$. There is a map $F \mapsto M_\ell F$, taking \mathbb{Q} to itself, defined by, for $\mathfrak{t} \in \mathcal{L}$ and $\mathbf{A} \in \mathcal{A}$,*

$$\begin{aligned} (M_\ell F)_\mathfrak{t}(\mathbf{A}) &\stackrel{\text{def}}{=} (p_{\mathbf{1},\mathfrak{t}} M_\ell \otimes \text{id}_{V_\mathfrak{t} \otimes W_\mathfrak{t}}) \tilde{\Upsilon}_\mathfrak{t}^F(\mathbf{A}) \\ &= F_\mathfrak{t}(\mathbf{A}) + \sum_{\tau \in \mathfrak{T}_-(R)} (\ell \otimes \text{id}_{V_\mathfrak{t} \otimes W_\mathfrak{t}}) \tilde{\Upsilon}_\mathfrak{t}^F[\tau](\mathbf{A}), \end{aligned} \quad (5.58)$$

where $p_{\mathbf{1},\mathfrak{t}}$ denotes the projection onto $\mathfrak{T}[\mathbf{1}]$ and the operator M_ℓ on the right-hand side is given by (5.25).

Moreover, for any $L \in \mathbf{N} \cup \{\infty\}$,

$$p_{\leq L} \left(\bigoplus_{\mathfrak{t} \in \mathcal{L}} M_\ell \otimes \text{id}_{V_\mathfrak{t} \otimes W_\mathfrak{t}} \right) \tilde{\Upsilon}^F = p_{\leq L} \tilde{\Upsilon}^{M_\ell F}, \quad (5.59)$$

and there exists $\bar{L} \in \mathbf{N} \cup \{\infty\}$ (which can be taken finite if L is finite) such that if $\mathcal{A} \in \mathcal{H}$ is coherent to order \bar{L} with $F \in \mathbb{Q}$ then $(M_\ell \otimes \text{id})\mathcal{A}$ is coherent to order L with $M_\ell F$.

Remark 5.66 Note that for $\mathfrak{t} \in \mathcal{L}$, $\tilde{\Upsilon}_\mathfrak{t}^F[\tau](\mathbf{A}) \in \mathfrak{T}[\tau] \otimes V_\mathfrak{t} \otimes W_\mathfrak{t}$, so every term on the right-hand side of (5.58) is an element of $V_\mathfrak{t} \otimes W_\mathfrak{t}$.

6 Solution theory of the SYM equation

In this section we make rigorous the solution theory for (1.7) and provide the proof of Theorem 2.4. In particular, we explicitly identify the counterterms appearing in the renormalised equation as this will be needed for the proof of gauge covariance in Section 7. Recalling Remark 2.7 we make the following assumption.

Assumption 6.1 *The Lie algebra \mathfrak{g} is simple.*

The current section is split into two parts. In Section 6.1 we recast (1.7) into the framework of regularity structures with vector-valued noise using Section 5. In Section 6.2 we invoke the black box theory of [Hai14, CH16, BHZ19, BCCH17] to prove convergence of our mollified/renormalised solutions and then explicitly compute our renormalised equation (using Proposition 5.65) in order to show that, when $d = 2$, the one counterterm appearing converges to a finite value as $\varepsilon \downarrow 0$.

6.1 Regularity structure and models for the SYM equation

We set up our regularity structure for formulating (1.7) in $d = 2$ or $d = 3$ space dimensions. However, when, computing counterterms, we will again fix $d = 2$. We write $[d] \stackrel{\text{def}}{=} \{1, \dots, d\}$.

Our space-time scaling $\mathfrak{s} \in [1, \infty)^{d+1}$ is given by setting $\mathfrak{s}_0 = 2$ and $\mathfrak{s}_i = 1$ for $i \in [d]$. We define $\mathfrak{L}_+ \stackrel{\text{def}}{=} \{\mathfrak{a}_i\}_{i=1}^d$ and $\mathfrak{L}_- \stackrel{\text{def}}{=} \{\mathfrak{l}_i\}_{i=1}^d$. We define a degree $\text{deg}: \mathfrak{L} \rightarrow \mathbf{R}$ on our label set by setting

$$\text{deg}(\mathfrak{t}) \stackrel{\text{def}}{=} \begin{cases} 2 & \mathfrak{t} \in \mathfrak{L}_+, \\ -d/2 - 1 - \kappa & \mathfrak{t} \in \mathfrak{L}_- \end{cases} \quad (6.1)$$

where we fix $\kappa \in (0, 1/4)$.

Looking at equation (1.8) leads us to consider the rule \mathring{R} given by setting, for each $i \in [d]$, $\mathring{R}(\mathfrak{l}_i) \stackrel{\text{def}}{=} \{\emptyset\}$ and

$$\mathring{R}(\mathfrak{a}_i) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \{(\mathfrak{l}_i, 0)\}, \{(\mathfrak{a}_i, 0), (\mathfrak{a}_j, 0), (\mathfrak{a}_j, 0)\} \\ \{(\mathfrak{a}_j, 0), (\mathfrak{a}_j, e_i)\}, \{(\mathfrak{a}_j, 0), (\mathfrak{a}_i, e_j)\} \end{array} : j \in [d] \right\}. \quad (6.2)$$

It is straightforward to verify that \mathring{R} is subcritical. The rule \mathring{R} has a smallest normal [BHZ19, Definition 5.22] extension and this extension admits a completion R as constructed in [BHZ19, Proposition 5.21] – this rule R is also subcritical will be the rule that will be used to define our regularity structure.

We fix our target space assignment $(W_{\mathfrak{t}})_{\mathfrak{t} \in \mathfrak{L}}$ by setting

$$W_{\mathfrak{t}} \stackrel{\text{def}}{=} \mathfrak{g} \quad \forall \mathfrak{t} \in \mathfrak{L}. \quad (6.3)$$

The space assignment $(V_{\mathfrak{t}})_{\mathfrak{t} \in \mathfrak{L}}$ used in the construction of our concrete regularity structure via the functor \mathbf{F}_V is then given by (5.23).

Remark 6.2 While the notation $\mathbf{A} = (\mathbf{A}_{(\mathfrak{t}, p)} : (\mathfrak{t}, p) \in \mathfrak{E}) \in \mathcal{A}$ was convenient for the formulation and proof of the statements of Section 5.8, it would make the computations of this section and Section 7 harder to follow. We thus go back to using the symbol A for the components $\mathbf{A}_{(\mathfrak{t}, p)}$ of \mathbf{A} with $\mathfrak{t} \in \mathfrak{L}_+$ and the symbol ξ for the components $\mathbf{A}_{(\mathfrak{l}, 0)}$ with $\mathfrak{l} \in \mathfrak{L}_-$. To streamline notations, we also write the subscript p as a derivative, namely we write

$$\xi_i = \mathbf{A}_{(\mathfrak{l}_i, 0)}, \quad A_i = \mathbf{A}_{(\mathfrak{a}_i, 0)} \quad \text{and} \quad \partial_j A_i = \mathbf{A}_{(\mathfrak{a}_i, e_j)}. \quad (6.4)$$

Regarding the specification of the right-hand side $F = \bigoplus_{\mathfrak{t} \in \mathfrak{L}} F_{\mathfrak{t}} \in \mathbb{Q}$, we set, and for each $i \in [d]$ and $\mathbf{A} \in \mathcal{A}$, $F_{\mathfrak{l}_i}(\mathbf{A}) = \text{id}_{\mathfrak{g}}$ and

$$\begin{aligned} F_{\mathfrak{a}_i}(\mathbf{A}) &= \mathbf{A}_{(\mathfrak{l}_i, 0)} + \sum_{j=1}^d [\mathbf{A}_{(\mathfrak{a}_j, 0)}, 2\mathbf{A}_{(\mathfrak{a}_i, e_j)} - \mathbf{A}_{(\mathfrak{a}_j, e_i)} + [\mathbf{A}_{(\mathfrak{a}_j, 0)}, \mathbf{A}_{(\mathfrak{a}_i, 0)}]] \\ &= \xi_i + \sum_{j=1}^d [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]], \end{aligned} \quad (6.5)$$

where the identification of the two lines uses the notations of Remark 6.2.

The right-hand side of (6.5) is clearly a polynomial in \mathbf{A} taking values in $W_{\mathfrak{a}_i} \simeq \mathfrak{g}$ as described in Section 5.8, so it is indeed the case that $F \in \mathring{\mathbb{Q}}$. The derivatives $D_{o_1} \cdots D_{o_m} F_{\mathfrak{a}_i}(\mathbf{A})$, for $o_1, \dots, o_m \in \mathcal{E}$, are not difficult to compute. For instance, for fixed $\mathbf{A} \in \mathcal{A}$, $D_{(\mathfrak{a}_j, 0)} F_{\mathfrak{a}_i}(\mathbf{A}) \in L(W_{(\mathfrak{a}_j, 0)}, W_{(\mathfrak{a}_i, 0)}) \simeq L(\mathfrak{g}, \mathfrak{g})$ is given by

$$(D_{(\mathfrak{a}_j, 0)} F_{\mathfrak{a}_i}(\mathbf{A}))(\bullet) = [\bullet, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + [A_j, [\bullet, A_i]] + \delta_{i,j} \sum_{k=1}^d [A_k, [A_k, \bullet]] .$$

It is then straightforward to see that $F \in \mathring{\mathbb{Q}}$, namely, it obeys the rule R in the sense of Definition 5.51.

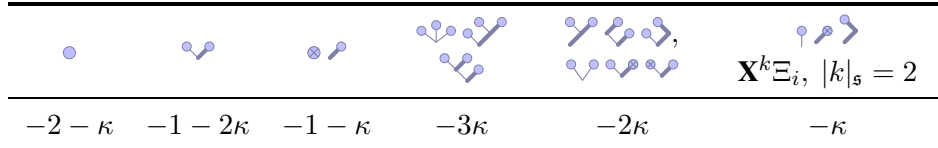
6.2 The BPHZ model/counterterms for the SYM equation in $d = 2$

We remind the reader that we now restrict to the case $d = 2$ and $|\mathfrak{l}|_{\mathfrak{s}} = -2 - \kappa$ for every $\mathfrak{l} \in \mathfrak{L}_-$. We also enforce that $\kappa < 1/2$.

As mentioned in Remark 5.56, we will use the symbol $\Xi_i \stackrel{\text{def}}{=} \mathcal{F}_{(\mathfrak{l}_i, 0)}(\mathbf{1})$ for the noise. Similarly to Remark 6.2, we also use the notations \mathcal{F}_i and $\mathcal{F}_{i,j}$ as shorthands for $\mathcal{F}_{(\mathfrak{a}_i, 0)}$ and $\mathcal{F}_{(\mathfrak{a}_i, e_j)}$ respectively.

Below we introduce a graphical notation to describe forms of relevant trees. The noises Ξ_i are circles \bullet , noises with polynomials $\mathbf{X}^{e_j} \Xi$ with $j \in [d]$ are crossed circles \otimes , and edges \mathcal{F}_i and $\mathcal{F}_{i,j}$ are thin and thick grey lines respectively. It is always assumed that the indices i and j appearing on occurrences of Ξ_i , \mathcal{F}_i , and $\mathcal{F}_{i,j}$ throughout the tree are constrained by the requirement that our trees conform to the rule R .

We now give a complete list of the forms of trees in $\mathfrak{T}(R)$ with negative degree, the form is listed on the top and the degree below it.



Note that each symbol above actually corresponds to a family of trees, determined by assigning indices in a way that conforms to the rule R . For instance, when we say that τ is of the form , then τ could be any tree of the type

$$\mathcal{F}_{i_1}(\mathcal{F}_{i_2}(\Xi_{i_2})\mathcal{F}_{i_3, j_3}(\Xi_{i_3}))\mathcal{F}_{i_4, j_4}(\Xi_{i_4})$$

for any $i_1, i_2, i_3, i_4, j_3, j_4 \in [d]$ satisfying both of the following two constraints: first, one must have either $i_1 = i_4$ or $j_4 = i_1$, and, second, one must have either $i_2 = i_3$ and $j_3 = i_1$ or $i_2 = j_3$ and $i_3 = i_1$.

Note that a circle \bullet or a crossed circle \otimes actually represents an *edge* when we think of any of the corresponding typed combinatorial trees; for instance, in the sense of Section 5.4, has *four* edges and not two. In Section 6.2.4, we will further colour our graphical symbols to encode constraints on indices.

6.2.1 Kernel and noise assignments for (1.7)

We fix a kernel assignment by setting, for every $t \in \mathcal{L}_+$, $K_t = K$ where we fix K to be a truncation of the Green's function $G(z)$ of the heat operator which satisfies the following properties:

1. $K(z)$ is smooth on $\mathbf{R}^{d+1} \setminus \{0\}$.
2. $K(z) = G(z)$ for $0 \leq |z|_s \leq 1/2$.
3. $K(z) = 0$ for $|z|_s > 1$.
4. Writing $z = (t, x)$ with $x \in \mathbf{T}^2$, $K(0, x) = 0$ for $x \neq 0$, and $K(t, x) = 0$ for $t < 0$.
5. Writing $z = (t, x_1, x_2)$ with x_1, x_2 the spatial components, $K(t, x_1, x_2) = K(t, -x_1, x_2) = K(t, x_1, -x_2)$ and $K(t, x_1, x_2) = K(t, x_2, x_1)$.

We will also use the shorthand $K^\varepsilon = K * \chi^\varepsilon$.

Remark 6.3 Property 4 is not strictly necessary for the proof of Theorem 2.1 but will be convenient for proving item (ii) of Theorem 2.8 in Section 7 so we include it here for convenience. Property 5 is also not strictly necessary, but convenient if we want certain BPHZ renormalisation constants to vanish rather than just being finite.

Note that we do not assume a moment vanishing condition here as in [Hai14, Assumption 5.4] – the only real change from the framework of [Hai14] that dropping this assumption entails is that, for $p, k \in \mathbf{N}^{d+1}$, we can have presence of expressions such as $\mathcal{F}_{(m,p)}(\mathbf{X}^k)$ when we write out trees in $\mathfrak{T}(R)$. Works such as [CH16], [BHZ19], [BCC17] already assume trees containing such expressions are allowed to be present.

Next, we overload notation and introduce a random smooth noise assignment $\zeta^\varepsilon = (\zeta_t)_{t \in \mathcal{L}_-}$ by setting, for $i \in [d]$, $\zeta_{t(i)} = \xi_i^\varepsilon$ where we recall that $\xi_i^\varepsilon = \xi_i * \chi^\varepsilon$ and $(\xi_i)_{i=1}^d$ are the i.i.d. \mathfrak{g} -valued space-time white noises introduced as the beginning of the paper. With this fixed choice of kernel assignment and random smooth noise assignment ζ^ε for $\varepsilon > 0$ we have a corresponding BPHZ renormalised model $Z_{\text{BPHZ}}^\varepsilon$. We also write $\ell_{\text{BPHZ}}^\varepsilon \in \mathcal{G}_-$ for the corresponding BPHZ character.

6.2.2 Convergence of models for (1.7)

We now apply [CH16, Theorem 2.15] to prove the following.



Lemma 6.4 *The random models $Z_{\text{BPHZ}}^\varepsilon$ converge in probability, as $\varepsilon \downarrow 0$, to a limiting random model Z_{BPHZ} .*

Proof. We note that [CH16, Theorem 2.15] is stated for the scalar noise setting so to be precise one must verify its conditions after applying some choice of scalar noise decomposition. However, it is not hard to see that the conditions of the theorem are completely insensitive to the choice of scalar noise decomposition. Let $\zeta = (\zeta_t)_{t \in \mathcal{L}_-}$ be the unmollified random noise assignment, that is, $\zeta_{t(i)} = \xi_i$.

For any scalar noise decomposition, it is straightforward to verify the condition that the random smooth noise assignments ζ^ε are a uniformly compatible family of Gaussian noises that converge to the Gaussian noise ζ (again, seen as a rough, random, noise assignment for scalar noise decomposition in the natural way).

The first three listed conditions of [CH16, Theorem 2.15] refer to power-counting considerations written in terms of the degrees of the combinatorial trees spanning the scalar regularity structure and the degrees of the noises. Since this power-counting is not affected by decompositions, they can be checked directly on the trees of $\mathfrak{T}(R)$. We note that

$$\min\{\deg(\tau) : \tau \in \mathfrak{T}(R), |N(\tau)| > 1\} = -1 - 2\kappa > -2$$

is achieved for τ of the form —here $N(\tau)$ is as defined in (5.29). This is greater than $-|\mathfrak{s}|/2$, so the third criterion is satisfied. Combining this with the fact that $\deg(\mathfrak{l}) = -|\mathfrak{s}|/2 - \kappa$ for every $\mathfrak{l} \in \mathfrak{L}_-$ guarantees that the second criterion is satisfied. Finally, the worst case scenario for the first condition is for τ of the form  and $A = \{a\}$ with type $\mathfrak{t}(a) = \mathfrak{l}$ for any element $\mathfrak{l} \in \mathfrak{L}_-$ and for which we have

$$\deg(\tau) + \deg(\bar{\mathfrak{l}}) + |\mathfrak{s}| = 2 - 4\kappa > 0,$$

as required. □

6.2.3 The BPHZ renormalisation constants














The set of trees $\mathfrak{T}_-(R)$ is given by all trees of the form

$$\text{, , , , , , , , or .$$

Our remaining objective for this section is to compute the counterterms

$$\sum_{\tau \in \mathfrak{T}_-(R)} (\ell_{\text{BPHZ}}^\varepsilon[\tau] \otimes \text{id}_{W_t}) \tilde{\mathbf{Y}}_t^F[\tau](\mathbf{A}) \quad (6.6)$$

for each $\mathfrak{t} \in \mathfrak{L}_+$. In what follows, we perform separate computations for the character $\ell_{\text{BPHZ}}^\varepsilon[\tau]$ and for $\tilde{\mathbf{Y}}^F$, before combining them to compute (6.6). The following lemma identifies some cases where $\ell_{\text{BPHZ}}^\varepsilon[\tau] = 0$.

- Lemma 6.5** (i) $\ell_{\text{BPHZ}}^\varepsilon[\tau] = 0$ for each τ consisting of an odd number of noises, that is any τ of the form , , and .
- (ii) On every subspace $\mathcal{T}[\tau]$ of the regularity structure with τ of the form , , , or , one has $\tilde{\mathcal{A}}_- = -\text{id}$, so that $\ell_{\text{BPHZ}}^\varepsilon[\tau] = -\tilde{\Pi}_{\text{can}}[\tau]$.
- (iii) $\ell_{\text{BPHZ}}^\varepsilon[\tau] = 0$ for every τ of the form .
- (iv) For τ of the form , , or , one has $\ell_{\text{BPHZ}}^\varepsilon[\tau] = 0$ unless the two noises Ξ_{i_1} and Ξ_{i_2} appearing in τ carry the same index, that is $i_1 = i_2$.
- (v) For τ of the form , or , one has $\ell_{\text{BPHZ}}^\varepsilon[\tau] = 0$ unless the two spatial derivatives appearing on the two thick edges in τ carry the same index.

Proof. Item (i) is true for every Gaussian noise. For item (ii), the statement about the abstract regularity structure is a direct consequence of the definition (5.32) of the twisted antipode (see also [BHZ19, Prop. 6.6]) and the statement about $\ell_{\text{BPHZ}}^\varepsilon[\tau]$ then follows from (5.30).

For item (iii) if we write $\tau = \mathcal{F}_i(\Xi_i)\mathcal{F}_{j,l}(\Xi_j)$ then

$$\bar{\Pi}_{\text{can}}[\tau] = \int du dv K(-u)\partial_l K(-v)\mathbf{E}[\xi_i^\varepsilon(u) \otimes \xi_j^\varepsilon(v)] .$$

Performing a change of variable by flipping the sign of the l -component of v , followed by exploiting the equality in law of ξ^ε and the change in sign of $\partial_l K$ under such a reflection, shows that the integral above vanishes.

For item (iv), the fact that $\mathbf{E}[\xi_i^\varepsilon(u) \otimes \xi_j^\varepsilon(v)] = 0$ if $i \neq j$ enforces the desired constraint.

For item (v), the argument is similar to that of item (iii) - namely the presence of precisely one spatial derivative in a given direction allows one to argue that $\bar{\Pi}_{\text{can}}[\tau]$ vanishes by performing a reflection in the appropriate integration variable in that direction. \square

Remark 6.6 We now start to use splotches of colour such as $\color{red}{\bullet}$ or $\color{blue}{\bullet}$ to represent indices in $[d]$, since they will allow us to work with expressions that would become unwieldy when using Greek or Roman letters. We also use Kronecker notation to enforce the equality of indices represented by colours, for instance writing $\delta_{\color{red}{\bullet},\color{red}{\bullet}}$.

We can use colours to include indices in our graphical notation for trees in an unobtrusive way, for instance writing $\color{red}{\uparrow} = \mathcal{F}_{\color{red}{\bullet}}(\bullet) = \mathcal{F}_{\color{red}{\bullet}}(\Xi_{\color{red}{\bullet}})$. Note that the splotch of $\color{red}{\bullet}$ in the symbol $\color{red}{\uparrow}$ fixes the two indices in $\mathcal{F}_{\color{red}{\bullet}}(\Xi_{\color{red}{\bullet}})$ which have to be equal for any tree conforming to our rule R . The edges corresponding to integration can be decorated by derivatives which introduce a new index, so we introduce notation such as $\color{red}{\uparrow} = \mathcal{F}_{\color{red}{\bullet},\color{red}{\bullet}}(\Xi_{\color{red}{\bullet}})$, where the colour of a thick edge determines the index of its derivative.

For drawing a tree like $\mathcal{F}_{\color{red}{\bullet}}(\Xi_{\color{red}{\bullet}})\mathcal{F}_{\color{red}{\bullet},\color{red}{\bullet}}(\mathcal{F}_{\color{red}{\bullet},\color{red}{\bullet}}(\Xi_{\color{red}{\bullet}}))$, our earlier way of drawing $\color{red}{\uparrow}$ didn't give us a node to colour $\color{red}{\bullet}$, so we add small triangular nodes to our drawings to allow us to display the colour determining the type of the edge incident to that node from below, for example $\color{red}{\uparrow} = \mathcal{F}_{\color{red}{\bullet}}(\Xi_{\color{red}{\bullet}})\mathcal{F}_{\color{red}{\bullet},\color{red}{\bullet}}(\mathcal{F}_{\color{red}{\bullet},\color{red}{\bullet}}(\Xi_{\color{red}{\bullet}}))$.

We will see by Lemma 6.13 that $\Upsilon_i[\tau] = 0$ for any τ of the form $\color{red}{\uparrow}$ or $\color{red}{\uparrow}$. Therefore, (6.6) will only have contributions from trees of the form

$$\color{red}{\uparrow}, \color{red}{\uparrow}, \text{ or } \color{red}{\downarrow} . \tag{6.7}$$

Thanks to the invariance of our driving noises under the action of the Lie group we will see in Lemma 6.9 below that $\ell_{\text{BPHZ}}^\varepsilon[\tau]$ has to be a scalar multiple of the Casimir element (in particular, it belongs to the subspace $\mathfrak{g} \otimes_s \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$). This is an immediate consequence of using noise that is white with respect to our inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . In particular, note that this inner product on \mathfrak{g} induces an inner

product $\langle \cdot, \cdot \rangle_2$ on $\mathfrak{g} \otimes \mathfrak{g}$ and that there is a unique element $\text{Cas} \in \mathfrak{g} \otimes_s \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ with the property that, for any $h_1, h_2 \in \mathfrak{g}$,

$$\langle \text{Cas}, h_1 \otimes h_2 \rangle_2 = \langle h_1, h_2 \rangle . \quad (6.8)$$

One can write Cas explicitly as $\text{Cas} = \sum_i e_i \otimes e_i$ for any orthonormal basis of \mathfrak{g} but we will refrain from doing so since we want to perform computations without fixing a basis. Cas should be thought of as the covariance of \mathfrak{g} -valued white noise, in particular for $i, j \in \{1, 2\}$ we have

$$\mathbf{E}[\xi_i(t, x) \otimes \xi_j(s, y)] = \delta_{i,j} \delta(t-s) \delta(x-y) \text{Cas} . \quad (6.9)$$

Thanks to (6.8), Cas is invariant under the action of the Lie group G in the sense that

$$(\text{Ad}_g \otimes \text{Ad}_g) \text{Cas} = \text{Cas} , \quad \forall g \in G . \quad (6.10)$$

The identity (6.10) is of course just a statement about the rotation invariance of our noise. Alternatively, we can interpret Cas as an element of $U(\mathfrak{g})$, the universal enveloping algebra of \mathfrak{g} . The following standard fact will be crucial in the sequel.

Lemma 6.7 *Cas belongs to the centre of $U(\mathfrak{g})$.*

Proof. Let $h \in \mathfrak{g}$ and let θ be a random element of \mathfrak{g} with $\mathbf{E}(\theta \otimes \theta) = \text{Cas}$. Differentiating $\mathbf{E}[\text{Ad}_g \theta \otimes \text{Ad}_g \theta]$ at $g = e$ in the direction of h yields

$$\mathbf{E}([h, \theta] \otimes \theta) = -\mathbf{E}(\theta \otimes [h, \theta]) . \quad (6.11)$$

We conclude that

$$[h, \text{Cas}] = [h, \mathbf{E}(\theta \otimes \theta)] = \mathbf{E}(h \otimes \theta \otimes \theta - \theta \otimes \theta \otimes h) = \mathbf{E}([h, \theta] \otimes \theta + \theta \otimes [h, \theta]) = 0 ,$$

as claimed, where we used (6.11) in the last step. \square

Remark 6.8 We note that Cas is of course just the quadratic Casimir. Moreover, recall that every element $h \in U(\mathfrak{g})$ yields a linear operator $\text{ad}_h : \mathfrak{g} \rightarrow \mathfrak{g}$ by setting

$$\text{ad}_{h_1 \otimes \dots \otimes h_k} X = [h_1, \dots [h_k, X] \dots] .$$

With this notation, Lemma 6.7 implies that ad_{Cas} commutes with every other operator of the form ad_h for $h \in \mathfrak{g}$ (and therefore also $h \in U(\mathfrak{g})$). If \mathfrak{g} is simple, then this implies that $\text{ad}_{\text{Cas}} = \lambda \text{id}_{\mathfrak{g}}$.

We now describe $\ell_{\text{BPHZ}}^\varepsilon[\tau]$ for τ of the form (6.7). We define

$$\bar{C}^\varepsilon \stackrel{\text{def}}{=} \int dz K^\varepsilon(z)^2, \quad \hat{C}^\varepsilon \stackrel{\text{def}}{=} \int dz \partial_j K^\varepsilon(z) (\partial_j K * K^\varepsilon)(z), \quad C_{\text{SYM}}^\varepsilon \stackrel{\text{def}}{=} \bar{C}^\varepsilon - 4\hat{C}^\varepsilon \quad (6.12)$$

where, on the right-hand side of the second equation one can choose either $j = 1$ or 2 – they both give the same value.

Lemma 6.9 For \hat{C}^ε and \bar{C}^ε as in (6.12), one has

$$\ell_{\text{BPHZ}}^\varepsilon[\text{↗↘}] = -\ell_{\text{BPHZ}}^\varepsilon[\text{↘↗}] = \hat{C}^\varepsilon \text{Cas}, \quad \ell_{\text{BPHZ}}^\varepsilon[\text{↖↗}] = \bar{C}^\varepsilon \text{Cas}. \quad (6.13)$$

Furthermore, $C_{\text{SYM}}^\varepsilon$ as defined in (6.12) converges to a finite value C_{SYM} as $\varepsilon \rightarrow 0$.

Remark 6.10 The last statement is specific to two space dimensions. In three dimensions, the mass renormalisations do not add up to a finite constant.

Remark 6.11 The first identity of (6.13) makes sense since, even though there are two natural isomorphisms $\mathcal{T}[\text{↗↘}]^* \approx \mathfrak{g} \otimes \mathfrak{g}$ and $\mathcal{T}[\text{↘↗}]^* \approx \mathfrak{g} \otimes \mathfrak{g}$ (corresponding to the two ways of matching the two noises), Cas is invariant under that transposition. For the second identity, note that $\ell_{\text{BPHZ}}^\varepsilon[\text{↖↗}] \simeq \mathfrak{g} \otimes_s \mathfrak{g}$.

Proof. The identities (6.13) readily follow from direct computation once one uses that in all cases $\ell_{\text{BPHZ}}^\varepsilon[\tau] = -\bar{\Pi}_{\text{can}}[\tau]$ (this is item (ii) of Lemma 6.5), writes down the corresponding expectation/integral, moves the mollification from the noises to the kernels, and uses (6.9).

Regarding the last claim of the lemma, since K is a truncation of the heat kernel, observe that

$$(\partial_t - \Delta)K = \delta_0 + Q, \quad (6.14)$$

where Q is smooth and supported away from the origin. Using the shorthand $\text{Int}[F] = \int dz F(z)$ it follows that

$$\text{Int}[(\Delta K * K^\varepsilon)K^\varepsilon] = -\text{Int}[(K^\varepsilon)^2] + \text{Int}[(\partial_t K * K^\varepsilon)K^\varepsilon] - \text{Int}[(Q * K^\varepsilon)K^\varepsilon]. \quad (6.15)$$

On the other hand, we also have

$$\begin{aligned} \text{Int}[(\Delta K * K^\varepsilon)K^\varepsilon] &= \text{Int}[(K * K^\varepsilon)\Delta K^\varepsilon] \\ &= -\text{Int}[(K * K^\varepsilon)\chi^\varepsilon] - \text{Int}[(\partial_t K * K^\varepsilon)K^\varepsilon] - \text{Int}[(K * K^\varepsilon)(Q * \chi^\varepsilon)]. \end{aligned} \quad (6.16)$$

Observe that $\text{Int}[(K * K^\varepsilon)\chi^\varepsilon]$, $\text{Int}[(Q * K^\varepsilon)K^\varepsilon]$, and $\text{Int}[(K * K^\varepsilon)(Q * \chi^\varepsilon)]$ all converge²¹ as $\varepsilon \rightarrow 0$. Hence, adding (6.15) and (6.16), we obtain that

$$\begin{aligned} 2\text{Int}[(\Delta K * K^\varepsilon)K^\varepsilon] + \text{Int}[(K^\varepsilon)^2] \\ = -\text{Int}[(K * K^\varepsilon)\chi^\varepsilon] - \text{Int}[(K * K^\varepsilon)(Q * \chi^\varepsilon)] - \text{Int}[(Q * K^\varepsilon)K^\varepsilon] \end{aligned} \quad (6.17)$$

converges as $\varepsilon \rightarrow 0$. We now note that the quantity above equals $C_{\text{SYM}}^\varepsilon$ since $\bar{C}^\varepsilon = \text{Int}[(K^\varepsilon)^2]$ and, by integration by parts,

$$\hat{C}^\varepsilon = -\frac{1}{2}\text{Int}[(\Delta K * K^\varepsilon)K^\varepsilon],$$

which completes the proof. \square

²¹These facts follow easily from the fact that $K * K$ is well-defined and bounded and also continuous away from the origin. To see this note that the semigroup property gives $(G * G)(t, x) = tG(t, x)$ and $G - K$ is smooth and supported away from the origin.

6.2.4 Computation of $\tilde{\Upsilon}^F$

Before continuing, we introduce some notational conventions that will be convenient when we calculate $\tilde{\Upsilon}^F$.

Recall that, for $n \in [d]$, the symbol Ξ_n is a tree that indexes a subspace $\mathcal{T}[\Xi_n]$ of our concrete regularity structure \mathcal{T} . We introduce a corresponding notation $\Xi_n \in \mathcal{T}[\Xi_n] \otimes \mathfrak{g}$ which, under the isomorphism $\mathcal{T}[\Xi_n] \otimes \mathfrak{g} \simeq \mathfrak{g}^* \otimes \mathfrak{g} \simeq L(\mathfrak{g}, \mathfrak{g})$, is given by $\Xi_n = \text{id}_{\mathfrak{g}}$. The expression Ξ_n really represents the corresponding noise in the sense that $(\Pi_{\text{can}} \Xi_n)(\cdot) = \xi_n^\varepsilon(\cdot)$, where we are abusing notation by having Π_{can} only act on the left factor of the tensor product.

Continuing to develop this notation, we also define

$$\Psi_n = \mathcal{J}_n \Xi_n \in \mathcal{T}[\bullet] \otimes \mathfrak{g} .$$

where we continue the same notation abuse, namely \mathcal{J}_n acts only on the left factors appearing in Ξ_n . In particular, we have $\Pi_{\text{can}} \Psi_n(\cdot) = (K * \xi_n^\varepsilon)(\cdot)$. We also have a corresponding notation

$$\Psi_{n,n} = \mathcal{J}_{n,n} \Xi_n \in \mathcal{T}[\dagger] \otimes \mathfrak{g} .$$

We now show how this notation is used for products / non-linear expressions. Given some $h \in \mathfrak{g}$, we may write an expression such as

$$[\Psi_n, [h, \Psi_{n,n}]] \in \mathcal{T}[\bullet \blacktriangleright \dagger] \otimes \mathfrak{g} . \quad (6.18)$$

In an expression like this, we apply the multiplication $\mathcal{T}[\bullet] \otimes \mathcal{T}[\dagger] \rightarrow \mathcal{T}[\bullet \blacktriangleright \dagger]$ to combine the left factors of Ψ_n and $\Psi_{n,n}$. The right \mathfrak{g} -factors of Ψ_n and $\Psi_{n,n}$ are used as the actual arguments of the brackets above, yielding the new \mathfrak{g} -factor on the right.

Remark 6.12 For what follows, given $i, j \in [d]$, we write Υ_i and $\Upsilon_{i,j}$ for Υ_{a_i} and $\Upsilon_{(a_i, e_j)}$ respectively. In particular, we will use notation such as Υ_n and $\Upsilon_{n,n}$. We extend this convention, also writing Υ_n and Υ_n along with W_n and $W_{n,n}$.

With these conventions in place the following computations follow quite easily from our definitions:

$$\tilde{\Upsilon}_n^F[\bullet] = \delta_{n,n} \Xi_n , \quad \Upsilon_n^F[\bullet] = \delta_{n,n} \Psi_n , \quad \Upsilon_{n,n}^F[\bullet] = \delta_{n,n} \Psi_{n,n} . \quad (6.19)$$

(Since $S(\tau) = 1$ for these trees, the corresponding Υ are identical.) Moreover, for $k \in \mathbb{N}^{d+1}$ with $k \neq 0$,

$$\Upsilon_n^F[\mathbf{X}^k \Xi_n] = \Upsilon_{n,n}^F[\mathbf{X}^k \Xi_n] = 0 . \quad (6.20)$$

Note that the left-hand sides of (6.19) are in principle allowed to depend on an argument $\mathbf{A} \in \mathcal{A}$, but here they are constant in \mathbf{A} , so we are using a canonical identification of constants with constant functions here. ²²

We now compute $\tilde{\Upsilon}^F$ for all the trees appearing in (6.7).

²²In (6.19) we are also exploiting canonical isomorphisms between W_n and $W_{n,n}$ and \mathfrak{g} . For instance, one also has $\tilde{\Upsilon}_{n,n}^F[\bullet] = \delta_{n,n} \Xi_n$ but here the last \mathfrak{g} factor on the right-hand side should be interpreted via the isomorphism with $W_{n,n}$ rather than W_n as in the first equality of (6.19).

Lemma 6.13

$$\tilde{\Upsilon}_n^F[\bullet \blacktriangledown \bullet](\mathbf{A}) = \mathbf{1}_{n \neq n}[\Psi_n, [\Psi_n, A_n]] \quad (6.21)$$

$$\tilde{\Upsilon}_n^F[\blacktriangleleft \blacktriangleright](\mathbf{A}) = (2\delta_{n,n}\delta_{n,n} - \delta_{n,n}\delta_{n,n})[[2\delta_{n,n}A_n - \delta_{n,n}A_n, \mathcal{F}_n(\Psi_{n,n})], \Psi_{n,n}]$$

$$\tilde{\Upsilon}_n^F[\bullet \blacktriangleright \bullet](\mathbf{A}) = (2\delta_{n,n}\delta_{n,n} - \delta_{n,n}\delta_{n,n})[\Psi_n, [2\delta_{n,n}A_n - \delta_{n,n}A_n, \mathcal{F}_n(\Psi_{n,n})]]$$

Moreover, for any τ of the form $\bullet \blacktriangleright \bullet$ or $\bullet \blacktriangledown \bullet$, $\tilde{\Upsilon}_t^F[\tau] = 0$ for every $t \in \mathfrak{L}_+$.

Proof. Let $\tau = \mathcal{F}_n(\tau_1)\mathcal{F}_n(\tau_2)$ for some choice of trees τ_1 and τ_2 , one has, by (5.48),

$$\begin{aligned} \tilde{\Upsilon}_n^F[\tau](\mathbf{A}) &= \mathbf{1}_{n \neq n \neq n} \left([\Upsilon_n^F[\tau_1](\mathbf{A}), [\Upsilon_n^F[\tau_2](\mathbf{A}), A_n]] \right. \\ &\quad \left. + [\Upsilon_n^F[\tau_2](\mathbf{A}), [\Upsilon_n^F[\tau_1](\mathbf{A}), A_n]] \right) \\ &\quad + \mathbf{1}_{n \neq n \neq n} \left([\Upsilon_n^F[\tau_2](\mathbf{A}), [A_n, \Upsilon_n^F[\tau_1](\mathbf{A})]] \right. \\ &\quad \left. + [A_n, [\Upsilon_n^F[\tau_2](\mathbf{A}), \Upsilon_n^F[\tau_1](\mathbf{A})]] \right). \end{aligned}$$

Specifying to $\tau = \bullet \blacktriangledown \bullet$, using (6.19) in the above identity gives

$$\begin{aligned} \tilde{\Upsilon}_n^F[\bullet \blacktriangledown \bullet](\mathbf{A}) &= \mathbf{1}_{n \neq n \neq n} \left([\Psi_n, [\Psi_n, A_n]] + [\Psi_n, [\Psi_n, A_n]] \right) \\ &\quad + \mathbf{1}_{n \neq n \neq n} \left([\Psi_n, [A_n, \Psi_n]] + [A_n, [\Psi_n, \Psi_n]] \right). \end{aligned} \quad (6.22)$$

Therefore,²³

$$\tilde{\Upsilon}_n^F[\bullet \blacktriangledown \bullet](\mathbf{A}) = 2\mathbf{1}_{n \neq n}[\Psi_n, [\Psi_n, A_n]]. \quad (6.23)$$

By (5.51) we have $S(\bullet \blacktriangledown \bullet) = (2!)S(\bullet)^2 = 2$ and so the first identity of (6.21) follows.

Before moving onto the second identity we recall that, again using (5.48),

$$\begin{aligned} \tilde{\Upsilon}_n^F[\blacktriangleright](\mathbf{A}) &= [2\delta_{n,n}A_n - \delta_{n,n}A_n, \Upsilon_{n,n}^F[\bullet](\mathbf{A})] \\ &= [2\delta_{n,n}A_n - \delta_{n,n}A_n, \Psi_{n,n}]. \end{aligned} \quad (6.24)$$

It follows that

$$\begin{aligned} \tilde{\Upsilon}_n^F[\blacktriangleleft \blacktriangleright](\mathbf{A}) &= (2\delta_{n,n}\delta_{n,n} - \delta_{n,n}\delta_{n,n})[\Upsilon_n^F[\blacktriangleright](\mathbf{A}), \Upsilon_{n,n}^F[\bullet](\mathbf{A})] \\ &= (2\delta_{n,n}\delta_{n,n} - \delta_{n,n}\delta_{n,n})[[2\delta_{n,n}A_n - \delta_{n,n}A_n, \mathcal{F}_n(\Psi_{n,n})], \Psi_{n,n}] \end{aligned}$$

where again, in $\mathcal{F}_n(\Psi_{n,n}) \in \mathcal{T}[\blacktriangleright] \otimes \mathfrak{g}$, the operator \mathcal{F}_n is acting only on the left factor of $\Psi_{n,n} \in \mathcal{T}[\blacktriangleright] \otimes \mathfrak{g}$. We then obtain the second identity of (6.21) since $S(\blacktriangleleft \blacktriangleright) = 1$.

For the third identity we recall that, by (5.48) and (6.24),

$$\Upsilon_{n,n}^F[\blacktriangleright] = \mathcal{F}_{n,n}(\tilde{\Upsilon}_n^F[\blacktriangleright]) = [2\delta_{n,n}A_n - \delta_{n,n}A_n, \mathcal{F}_{n,n}(\Psi_{n,n})],$$

²³Note that $\mathbf{1}_{n \neq n}$ here is necessary, because different colours only means “not necessarily identical”!

so that

$$\begin{aligned}\tilde{\Upsilon}_{\#}^F[\bullet \rightarrow \bullet](\mathbf{A}) &= (2\delta_{\#, \#} \delta_{\#, \#} - \delta_{\#, \#} \delta_{\#, \#})[\Upsilon_{\#}^F[\bullet](\mathbf{A}), \Upsilon_{\#, \#}^F[\uparrow](\mathbf{A})] \\ &= (2\delta_{\#, \#} \delta_{\#, \#} - \delta_{\#, \#} \delta_{\#, \#})[\Psi_{\#}, [2\delta_{\#, \#} A_{\#} - \delta_{\#, \#} A_{\#}, \mathcal{F}_{\#, \#}(\Psi_{\#, \#})]] .\end{aligned}$$

Since $S(\bullet \rightarrow \bullet) = 1$ we obtain the desired result.

The final claim of the lemma regarding trees of the form $\bullet \rightarrow \bullet$ or $\bullet \rightarrow \bullet$ follows immediately from the induction (5.48) combined with (6.20). \square

6.2.5 Putting things together and proving Theorem 2.4

Before proceeding, we give more detail on how to use our notation for computations. We note that, given any $w \in \mathfrak{g} \otimes_s \mathfrak{g}$, one can use the isomorphism²⁴ $\mathcal{T}[\bullet \rightarrow \bullet]^* \simeq \mathfrak{g} \otimes \mathfrak{g}$ to view w as acting on the expression (6.18) via an adjoint action, namely,

$$(w \otimes \text{id}_{\mathfrak{g}})[\Psi_{\#}, [h, \Psi_{\#, \#}]] = -(w \otimes \text{id}_{\mathfrak{g}})[\Psi_{\#}, [\Psi_{\#, \#}, h]] = -\text{ad}_w h . \quad (6.25)$$

Lemma 6.14

$$\sum_{\tau \in \mathfrak{S}_-(R)} (\ell_{\text{BPHZ}}^{\varepsilon}[\tau] \otimes \text{id}_{\mathfrak{g}}) \tilde{\Upsilon}_{\#}^F[\tau](\mathbf{A}) = \lambda C_{\text{SYM}}^{\varepsilon} A_{\#} , \quad (6.26)$$

where λ is the constant given in Remark 6.8 and $C_{\text{SYM}}^{\varepsilon}$ is as in (6.12).

Proof. By (6.21) and Lemma 6.9,

$$\begin{aligned}& \sum_{\#, \#, \#} (\ell_{\text{BPHZ}}^{\varepsilon}[\bullet \rightarrow \bullet] \otimes \text{id}_{\mathfrak{g}}) \tilde{\Upsilon}_{\#}^F[\bullet \rightarrow \bullet](\mathbf{A}) \\ &= (\hat{C}^{\varepsilon} \text{Cas} \otimes \text{id}_{\mathfrak{g}}) \left(4[[A_{\#}, \mathcal{F}_{\#} \Psi_{\#, \#}], \Psi_{\#, \#}] - \sum_{\#} 2[[A_{\#}, \mathcal{F}_{\#} \Psi_{\#, \#}], \Psi_{\#, \#}] \right. \\ & \quad \left. - \sum_{\#} 2[[A_{\#}, \mathcal{F}_{\#} \Psi_{\#, \#}], \Psi_{\#, \#}] + [[A_{\#}, \mathcal{F}_{\#} \Psi_{\#, \#}], \Psi_{\#, \#}] \right) \\ &= -3\hat{C}^{\varepsilon} \text{ad}_{\text{Cas}} A_{\#} ,\end{aligned}$$

where we used $d = 2$ to sum over the free indices $\#$ and $\#$, as well as (6.25) in the last step.

Also, by (6.21) and Lemma 6.9

$$\begin{aligned}& \sum_{\#, \#, \#} (\ell_{\text{BPHZ}}^{\varepsilon}[\bullet \rightarrow \bullet] \otimes \text{id}_{\mathfrak{g}}) \tilde{\Upsilon}_{\#}^F[\bullet \rightarrow \bullet](\mathbf{A}) \\ &= (-\hat{C}^{\varepsilon} \text{Cas} \otimes \text{id}_{\mathfrak{g}}) \left(4[\Psi_{\#}, [A_{\#}, \mathcal{F}_{\#, \#} \Psi_{\#, \#}]] - 2[\Psi_{\#}, [A_{\#}, \mathcal{F}_{\#, \#} \Psi_{\#, \#}]] \right)\end{aligned}$$

²⁴Again, this is only canonical up to permutation of the factors, but doesn't matter since w is symmetric.

$$\begin{aligned}
& - \sum_{\#} 2[\Psi_{\#}, [A_{\#}, \mathcal{F}_{\#, \#} \Psi_{\#, \#}]] + [\Psi_{\#}, [A_{\#}, \mathcal{F}_{\#, \#} \Psi_{\#, \#}]] \\
& = -\hat{C}^{\varepsilon} \text{ad}_{\text{Cas}} A_{\#},
\end{aligned}$$

where $d = 2$ and (6.25) are used again. Finally by (6.21) and Lemma 6.9,

$$\sum_{\#} (\ell_{\text{BPHZ}}^{\varepsilon}[\bullet] \otimes \text{id}_{\mathfrak{g}}) \bar{\Upsilon}_{\#}^F[\bullet](\mathbf{A}) = \bar{C}^{\varepsilon} \text{ad}_{\text{Cas}} A_{\#}.$$

Adding these three terms and recalling Remark 6.8 gives (6.26). \square

With these calculation, we are ready for the proof of Theorem 2.4.

Remark 6.15 It would be desirable to apply the black box convergence theorem [BCCH17, Thm. 2.21] directly. However, we are slightly outside its scope since we are working with non-standard spaces Ω_{α}^1 and are required to show continuity at time $t = 0$ for the solution $A^{\varepsilon} : [0, T] \rightarrow \Omega_{\alpha}^1$. Nonetheless, we can instead use several more general results from [BCCH17, BHZ19, CH16, Hai14].

Proof of Theorem 2.4. Consider the lifted equation associated to (2.1) in the bundle of modelled distributions $\mathcal{D}_{-\kappa}^{\gamma, \eta} \times \mathcal{M}$ for $\gamma > 1 + \kappa$ (and η as before). Note that $\gamma > 1 + \kappa$ and $\eta > -\frac{1}{2}$ ensure that, by [Hai14, Thm. 7.8], the lifted equation admits a unique fixed point $\mathcal{A} \in \mathcal{D}_{-\kappa}^{\gamma, \eta}$ and is locally Lipschitz in $(a, Z) \in \Omega\mathcal{C}^{\eta} \times \mathcal{M}$, where \mathcal{M} is the space of models on the associated regularity structure. Specialising to $Z = Z_{\text{BPHZ}}^{\varepsilon}$, the computation of Lemma 6.14 along with [BCCH17, Thm. 5.7] (and its partial reformulation in the vector case via Proposition 5.65) show that the reconstruction $A^{\varepsilon} \stackrel{\text{def}}{=} \mathcal{R}\mathcal{A}(a, Z_{\text{BPHZ}}^{\varepsilon})$ is the maximal solution in $\Omega\mathcal{C}^{\eta}$ to the PDE (2.1) starting from a with C replaced by C^{ε} given by

$$C^{\varepsilon} = \lambda C_{\text{SYM}}^{\varepsilon}, \tag{6.27}$$

where $C_{\text{SYM}}^{\varepsilon}$ is as in Lemma 6.9. We now show that A^{ε} converges in the space Ω^{sol} . To this end, let us decompose $A^{\varepsilon} = \Psi^{\varepsilon} + B^{\varepsilon}$, where Ψ^{ε} solves $\partial_t \Psi^{\varepsilon} = \Delta \Psi^{\varepsilon} + \xi^{\varepsilon}$ on $\mathbf{R}_+ \times \mathbf{T}^2$ with initial condition $a \in \Omega_{\alpha}^1$. Write also Ψ for the solution to $\partial_t \Psi = \Delta \Psi + \xi$ on $\mathbf{R}_+ \times \mathbf{T}^2$ with initial condition $a \in \Omega_{\alpha}^1$. Combining Theorem 4.13 and Proposition 4.6, we see that $\Psi \in \mathcal{C}(\mathbf{R}_+, \Omega_{\alpha}^1)$, and, by Corollary 4.14, $\Psi^{\varepsilon} \rightarrow \Psi$ in $\mathcal{C}(\mathbf{R}_+, \Omega_{\alpha}^1)$. Moreover, observe that $B^{\varepsilon} = \mathcal{R}(\mathcal{P}\mathbf{1}_+ F(\mathcal{A}))$, where $F(\mathcal{A}) \in \mathcal{D}_{-1-2\kappa}^{\gamma-1-\kappa, 2\eta-1}$. From the embedding $\Omega\mathcal{C}^{\alpha/2} \hookrightarrow \Omega_{\alpha}$ (Remark 3.23), the convergence of models given by Lemma 6.4, the continuity of the reconstruction map, and [Hai14, Thm. 7.1], we see that B^{ε} converges in the space Ω^{sol} as $\varepsilon \rightarrow 0$.

Finally, observe that we can perturb the constants C^{ε} in (2.1) by any bounded quantity while retaining convergence of maximal solutions to (2.1) and so, thanks to the convergence of (6.27) promised by Lemma 6.9, we obtain the desired convergence for *any* family $(C^{\varepsilon})_{\varepsilon \in (0,1]}$ such that $\lim_{\varepsilon \rightarrow 0} C^{\varepsilon}$ exists and is finite. \square

Remark 6.16 The proof of Theorem 2.4 allows us to make the following important observation about dependence of the solution on the mollifier. Namely, given any fixed constant $\delta C \in \mathbf{R}$, the limiting maximal solution to (2.1) obtained as one takes $\varepsilon \downarrow 0$ with $C = \delta C + \lambda C_{\text{SYM}}$ is independent of the choice of mollifier, namely all dependence of the solution on the mollifier is cancelled by C_{SYM} 's dependence on the mollifier.²⁵

Moreover, recall that C_{SYM} is the $\varepsilon \downarrow 0$ limit of the right-hand side of (6.17) and

$$\lim_{\varepsilon \downarrow 0} \int dz (K * K^\varepsilon)(z)(Q * \chi^\varepsilon)(z) \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \int dz (Q * K^\varepsilon)(z)K^\varepsilon(z)$$

are both independent of χ , where Q is as in (6.14). In particular, if one chooses

$$C = -\lambda \lim_{\varepsilon \downarrow 0} \int dz \chi^\varepsilon(z)(K * K^\varepsilon)(z)$$

then the limiting solution to (2.1) is independent of the mollifier χ .

7 Gauge covariance

The aim of this section is to show that the projected process $[A_t]$ on the orbit space is again a Markov process, which is a very strong form of gauge invariance. The first three subsections will be devoted to proving Theorem 2.8 – most of our work will be devoted to part (i) and we will obtain part (ii) afterwards by a short computation with renormalisation constants. We close with Section 7.4 where we construct the desired Markov process.

7.1 The full gauge transformed system of equations

One obstruction we encounter when trying to directly treat the systems (2.5) and (2.6) using currently available tools is that the evolution for g takes place in the non-linear space G . Fortunately, it is possible to rewrite the equations in such a way that the role of g is played by objects living in linear spaces. For this, given a smooth function $g: \mathbf{T}^2 \rightarrow G$, we define the functions $h: \mathbf{T}^2 \rightarrow \mathfrak{g}^2$ and $U: \mathbf{T}^2 \rightarrow L(\mathfrak{g}, \mathfrak{g})$ by

$$h \stackrel{\text{def}}{=} (dg)g^{-1} \quad \text{and} \quad U \stackrel{\text{def}}{=} \text{Ad}_g. \quad (7.1)$$

Straightforward algebraic manipulations yield the following lemma.

Lemma 7.1 *Given solutions B and g to (2.5) and defining h and U by (7.1), one has*

$$\begin{aligned} \partial_t h_i &= \Delta h_i - [h_j, \partial_j h_i] + [[B_j, h_j], h_i] + \partial_i [B_j, h_j], \\ \partial_t U &= \Delta U - [h_j, [h_j, \cdot]] \circ U + [[B_j, h_j], \cdot] \circ U, \\ \partial_t B_i &= \Delta B_i + [B_j, 2\partial_j B_i - \partial_i B_j + [B_j, B_i]] \\ &\quad + U J^\varepsilon(\xi_i) + C B_i + C h_i. \end{aligned} \quad (7.2)$$

²⁵This is because λC_{SYM} is the renormalisation arising from limiting BPHZ model Z_{BPHZ} which is independent of the mollifier.

Similarly, given solutions \bar{A} and \bar{g} to (2.6) and defining \bar{h} and \bar{U} by (7.1) in terms of \bar{g} , one has

$$\begin{aligned} \partial_t \bar{A}_i &= \Delta \bar{A}_i + [\bar{A}_j, 2\partial_j \bar{A}_i - \partial_i \bar{A}_j + [\bar{A}_j, \bar{A}_i]] \\ &\quad + J^\varepsilon(\bar{U}\xi_i) + C\bar{A}_i + (C - \bar{C})\bar{h}_i, \end{aligned} \quad (7.3)$$

and the equations for \bar{h} and \bar{U} are as in (7.2) with B replaced by \bar{A} .

Note that in the above equations we have omitted the ε -dependence in the notation $B, \bar{A}, U, \bar{U}, h, \bar{h}$, while ξ is the white noise which is really independent of ε . The term $\bar{U}\xi_i$ appearing in (7.3) is well-defined since \bar{A}_i is smooth for any given $\varepsilon > 0$.

Proof. By definition (7.1) of h and the equation for g in (2.5), one has the following identities

$$\begin{aligned} (\partial_t g)g^{-1} &= \operatorname{div} h + [B_j, h_j], \\ \partial_j h_i - \partial_i h_j &= [h_j, h_i], \\ \Delta h_i - \partial_i \operatorname{div} h &= \partial_j [h_j, h_i], \end{aligned} \quad (7.4)$$

where the last identity follows from the second. One then obtains

$$\begin{aligned} \partial_t h_i &= [(\partial_t g)g^{-1}, h_i] + \partial_i((\partial_t g)g^{-1}) \\ &= [\operatorname{div} h + [B_j, h_j], h_i] + \partial_i \operatorname{div} h + \partial_i [B_j, h_j] \\ &= \Delta h_i - [h_j, \partial_j h_i] + [[B_j, h_j], h_i] + \partial_i [B_j, h_j]. \end{aligned}$$

For the U equation, we start by noting that

$$\partial_i U = [h_i, \cdot] \circ U \quad (7.5)$$

and therefore

$$\Delta U = [\operatorname{div} h, \cdot] \circ U + [h_i, [h_i, \cdot]] \circ U. \quad (7.6)$$

By the first identity in (7.4)

$$\partial_t U = [(\partial_t g)g^{-1}, \cdot] \circ U = [\operatorname{div} h + [B_j, h_j], \cdot] \circ U, \quad (7.7)$$

and the claim follows from (7.6). The equations for $\bar{A}, \bar{h}, \bar{U}$ are derived in the same way. \square

Remark 7.2 Note that the knowledge of U and h is sufficient to describe the action of the corresponding g on connections since $A^g = UA - h$. This means that, for the purpose of our argument, we never need to go back and recover the evolution of g from that of U and h . The same can of course be said for \bar{g}, \bar{U} , and \bar{h} .

7.2 Regularity structure for the gauge transformed system

To recast (7.2) in the language of regularity structures, we use the label sets

$$\mathfrak{L}_+ \stackrel{\text{def}}{=} \{\mathfrak{a}_i, \mathfrak{h}_i, \mathfrak{m}_i\}_{i=1}^d \cup \{\mathfrak{u}\} \quad \text{and} \quad \mathfrak{L}_- \stackrel{\text{def}}{=} \{\bar{\mathfrak{l}}_i, \mathfrak{l}_i\}_{i=1}^d .$$

Our approach is to work with *one single* regularity structure to study both systems (7.2) and (7.3), allowing us to compare their solutions at the abstract level of modelled distributions.

Our particular choice of label sets and abstract non-linearities also involves some pre-processing to allow us to use the machinery of Section 5.8.3 to obtain the form of our renormalised equation. The label \mathfrak{h}_i indexes the solutions h_i or \bar{h}_i , \mathfrak{u} indexes the solutions U or \bar{U} , and \mathfrak{l}_i indexes the noise ξ_i (while $\bar{\mathfrak{l}}_i$ indexes a noise mollified at scale ε , see below).

The other labels are used to describe the B_i equation within system (7.2) and the \bar{A}_i equation within system (7.3). To explain our strategy, we first note that (ignoring for the moment the contribution coming from the initial condition) the equation for \bar{A} can be written as the integral fixed point equation

$$\bar{A}_i = G * ([\bar{A}_j, 2\partial_j \bar{A}_i - \partial_i \bar{A}_j + [\bar{A}_j, \bar{A}_i]] + C\bar{A}_i + (C - \bar{C})\bar{h}_i) + G_\varepsilon * (\bar{U}\xi_i) .$$

where $G_\varepsilon = \chi^\varepsilon * G$. While this can be cast as an abstract fixed point problem at the level of jets/modelled distributions, it does not quite fit into the framework of Section 5.8.3 since it involves multiple kernels on the right-hand side. We can deal with this problem by introducing a component \mathfrak{m}_i to index a new component of our solution that is only used to represent the term $G_\varepsilon * (\bar{U}\xi_i)$. The label \mathfrak{a}_i then represents the first term on the right-hand side above.

Turning to the equation for B , the corresponding fixed point problem is

$$B_i = G * ([B_j, 2\partial_j B_i - \partial_i B_j + [B_j, B_i]] + CB_i + Ch_i + UJ^\varepsilon(\xi_i))$$

Note that we cannot combine the mollification operator J^ε with a heat kernel, so we instead we use the label $\bar{\mathfrak{l}}_i$ to represent $J^\varepsilon(\xi_i)$ which we treat, at a purely algebraic level, as a completely separate noise from ξ_i .

Turning to our space assignment $(W_t)_{t \in \mathfrak{L}}$ we set

$$W_t \stackrel{\text{def}}{=} \begin{cases} \mathfrak{g} & t = \mathfrak{a}_i, \mathfrak{h}_i, \mathfrak{l}_i, \bar{\mathfrak{l}}_i, \text{ or } \mathfrak{m}_i , \\ L(\mathfrak{g}, \mathfrak{g}) & t = \mathfrak{u} , \end{cases} \quad (7.8)$$

and the space assignment $(V_t)_{t \in \mathfrak{L}}$ is given by (5.23) as before. We also define $\text{deg} : \mathfrak{L} \rightarrow \mathbf{R}$ by setting

$$\text{deg}(t) \stackrel{\text{def}}{=} \begin{cases} 2 - \kappa & t = \mathfrak{a}_i \text{ or } \mathfrak{m}_i , \\ 2 & t = \mathfrak{h}_i \text{ or } \mathfrak{u} , \\ -d/2 - 1 - \kappa & t = \mathfrak{l}_i \text{ or } \bar{\mathfrak{l}}_i , \end{cases}$$

where $\kappa \in (0, 1/12)$.

The systems of equations (7.2) and (7.3) and earlier discussion about the roles of our labels lead us to the rule \mathring{R} given by setting²⁶

$$\begin{aligned} \mathring{R}(\iota_i) &= \mathring{R}(\bar{\iota}_i) = \{\mathbf{1}\}, \quad \mathring{R}(\mathfrak{m}_i) = \{\mathfrak{ul}_i\}, \\ \mathring{R}(\mathfrak{u}) &= \{\mathfrak{uh}_j^2, \mathfrak{uq}_j\mathfrak{h}_j : \mathfrak{q} \in \{\mathfrak{a}, \mathfrak{m}\}, j \in [d]\}, \\ \mathring{R}(\mathfrak{h}_i) &= \{\mathfrak{h}_j\partial_j\mathfrak{h}_i, \mathfrak{q}_j\mathfrak{h}_j\mathfrak{h}_i, \mathfrak{h}_j\partial_i\mathfrak{q}_j, \mathfrak{q}_j\partial_i\mathfrak{h}_j : \mathfrak{q} \in \{\mathfrak{a}, \mathfrak{m}\}, j \in [d]\}, \\ \mathring{R}(\mathfrak{a}_i) &= \{\mathfrak{q}_i, \mathfrak{h}_i, \mathfrak{q}_i\hat{\mathfrak{q}}_j\tilde{\mathfrak{q}}_j, \mathfrak{q}_j\partial_i\hat{\mathfrak{q}}_j, \mathfrak{q}_j\partial_j\hat{\mathfrak{q}}_i, \mathfrak{ul}_i, \mathfrak{ul}_i : \mathfrak{q}, \hat{\mathfrak{q}}, \tilde{\mathfrak{q}} \in \{\mathfrak{a}, \mathfrak{m}\}, j \in [d]\}. \end{aligned} \quad (7.9)$$

Here we are using monomial notation for node types: a type $\mathfrak{t} \in \mathcal{L}$ should be associated with $(\mathfrak{t}, 0)$ and the symbol $\partial_j\mathfrak{t}$ represents (\mathfrak{t}, e_j) . We write products to represent multisets, for instance $\mathfrak{a}_j^2\partial_k\mathfrak{a}_i = \{(\mathfrak{a}_i, e_k), (\mathfrak{a}_j, 0), (\mathfrak{a}_j, 0)\}$. We write \mathfrak{q} , $\hat{\mathfrak{q}}$, and $\tilde{\mathfrak{q}}$ as dummy symbols since any occurrence of B_i or \bar{A}_i can correspond to an occurrence of \mathfrak{a}_i or \mathfrak{m}_i .

It is straightforward to check that \mathring{R} is subcritical and as in Section 6.1 the rule \mathring{R} has a smallest normal extension which admits a completion R which is also subcritical. This is the rule that is used to define the set of trees $\mathfrak{T}(R)$ which is used to build our regularity structure.

We adopt conventions analogous to those of Remark 6.2 and (6.4), writing (using our monomial notation)

$$\begin{aligned} \bar{A}_i &= \mathbf{A}_{\mathfrak{a}_i} + \mathbf{A}_{\mathfrak{m}_i}, \quad \partial_j\bar{A}_i = \mathbf{A}_{\partial_j\mathfrak{a}_i} + \mathbf{A}_{\partial_j\mathfrak{m}_i}, \\ B_i &= \mathbf{A}_{\mathfrak{a}_i}, \quad \partial_j B_i = \mathbf{A}_{\partial_j\mathfrak{a}_i}, \quad U = \bar{U} = \mathbf{A}_{\mathfrak{u}}, \quad \partial_i U = \partial_i \bar{U} = \mathbf{A}_{\partial_i\mathfrak{u}}, \\ J^\varepsilon(\xi_i) &= \xi_{\bar{\iota}_i}, \quad \xi_i = \xi_{\iota_i}, \quad h_i = \bar{h}_i = \mathbf{A}_{\mathfrak{h}_i}, \quad \partial_j h_i = \partial_j \bar{h}_i = \mathbf{A}_{\partial_j\mathfrak{h}_i}. \end{aligned} \quad (7.10)$$

Here, we choose to typeset components of \mathbf{A} in purple in order to be able to identify them at a glance as a solution²⁷-dependent element. This will be convenient later on when we manipulate expressions belonging to $\mathfrak{T} \otimes W$ for some vector space W (typically $W = \mathfrak{g}$ or $W = L(\mathfrak{g}, \mathfrak{g})$), in which case purple variables are always elements of the second factor W . Note that, when referring to components of $\mathbf{A} = (\mathbf{A}_o)_{o \in \mathcal{E}}$ the symbols \bar{U} , \bar{h} , and $\partial_j \bar{h}_i$ are identical to their unbarred versions but we still use both notations depending on which system of equations we are working with.

We now fix two non-linearities $F = \bigoplus_{\mathfrak{t} \in \mathcal{L}} F_{\mathfrak{t}}$, $\bar{F} = \bigoplus_{\mathfrak{t} \in \mathcal{L}} \bar{F}_{\mathfrak{t}} \in \mathbb{Q}$, which encode our systems (7.2) and (7.3), respectively. For some constants \mathring{C}_1 , and \mathring{C}_2 to

²⁶The choice to include $\{\mathfrak{ul}_i\} \in \mathring{R}(\mathfrak{a}_i)$ isn't directly motivated by (7.2) and (7.3) but is needed when we want to write an expression like (7.17) where H is the modelled distribution representing \bar{U} .

²⁷Components of \mathbf{A} corresponding to the noise such as ξ_{ι_i} and $\xi_{\bar{\iota}_i}$ will be left in black since their values are not solution-dependent.

be fixed later²⁸ we set F_t and \bar{F}_t to be $\text{id}_{\mathfrak{g}}$ for $t \in \mathfrak{L}_-$ and

$$F_t(\mathbf{A}) \stackrel{\text{def}}{=} \begin{cases} [B_j, 2\partial_j B_i - \partial_i B_j + [B_j, B_i]] + \dot{C}_1 B_i + \dot{C}_2 h_i + U J^\varepsilon(\xi_i) & \text{if } t = \mathfrak{a}_i, \\ -[h_j, \partial_j h_i] + [[B_j, h_j], h_i] + \partial_i [B_j, h_j] & \text{if } t = \mathfrak{h}_i, \\ -[h_j, [h_j, \cdot]] \circ U + [[B_j, h_j], \cdot] \circ U & \text{if } t = \mathfrak{u}, \\ 0 & \text{if } t = \mathfrak{m}_i. \end{cases}$$

(The term $\partial_i [B_j, h_j]$ should be interpreted by formally applying the Leibniz rule.)

For $\bar{F}_t(\mathbf{A})$ we set

$$\bar{F}_t(\mathbf{A}) \stackrel{\text{def}}{=} \begin{cases} [\bar{A}_j, 2\partial_j \bar{A}_i - \partial_i \bar{A}_j + [\bar{A}_j, \bar{A}_i]] + \dot{C}_1 \bar{A}_i + \dot{C}_2 \bar{h}_i & \text{if } t = \mathfrak{a}_i, \\ -[\bar{h}_j, \partial_j \bar{h}_i] + [[\bar{A}_j, \bar{h}_j], \bar{h}_i] + \partial_i [\bar{A}_j, \bar{h}_j] & \text{if } t = \mathfrak{h}_i, \\ -[\bar{h}_j, [\bar{h}_j, \cdot]] \circ \bar{U} + [[\bar{A}_j, \bar{h}_j], \cdot] \circ \bar{U} & \text{if } t = \mathfrak{u}, \\ \bar{U} \xi_i & \text{if } t = \mathfrak{m}_i. \end{cases}$$

For $j \in \{0\} \sqcup [d]$ we also introduce the shorthands

$$\begin{aligned} \Xi_i &= \mathcal{F}_{(\mathfrak{a}_i, 0)}(\mathbf{1}), \quad \bar{\Xi}_i = \mathcal{F}_{(\bar{\mathfrak{a}}_i, 0)}(\mathbf{1}), \quad \mathcal{F}_{i,j}(\cdot) = \mathcal{F}_{(\mathfrak{a}_i, j)}(\cdot), \quad \bar{\mathcal{F}}_{i,j}(\cdot) = \mathcal{F}_{(\mathfrak{m}_i, j)}(\cdot), \\ \mathcal{F}_{i,j}^h(\cdot) &= \mathcal{F}_{(\mathfrak{h}_i, j)}(\cdot), \quad \mathcal{F}^u(\cdot) = \mathcal{F}_{(\mathfrak{u}, 0)}(\cdot). \end{aligned}$$

When $j = 0$ in the above notation we sometimes suppress this index, for instance writing $\mathcal{F}_i(\cdot)$ instead of $\mathcal{F}_{i,0}(\cdot)$.

7.2.1 Kernel/noise assignments and BPHZ models

We write $K^{(\varepsilon)} = (K_t^{(\varepsilon)} : t \in \mathfrak{L}_+)$ for the kernel assignment given by setting

$$K_t^{(\varepsilon)} = \begin{cases} K & \text{for } t = \mathfrak{a}_i, \mathfrak{h}_i, \text{ or } \mathfrak{u}, \\ K^\varepsilon = K * \chi^\varepsilon & \text{for } t = \mathfrak{m}_i. \end{cases} \quad (7.11)$$

We also write \mathcal{M} for the space of all models and, for $\varepsilon \in [0, 1]$, we write $\mathcal{M}_\varepsilon \subset \mathcal{M}$ for the family of $K^{(\varepsilon)}$ -admissible models.

Note that in our choice of degrees we enforced $\deg(\mathfrak{a}_i) = \deg(\mathfrak{m}_i) = -2 - \kappa$ rather than -2 . The reason is that this allows us to extract a factor ε^κ from any occurrence of $K - K^\varepsilon$, which is crucial for Lemma 7.8, as well as the proof of (7.23) and (7.25) in Lemma 7.9.

We make this more precise now. Recall first the notion of a β -regularising kernel from [Hai14, Assumption 5.1]. We introduce some terminology so that we can use that notion in a slightly more quantitative sense. For $\beta, R > 0, r \geq 0$ we say that a kernel J is (r, R, β) -regularising, if one can find a decomposition of the form [Hai14, (5.3)] such that the estimates [Hai14, (5.4), (5.5)] hold with the same choice of $C = R$ for all multi-indices k, l with $|k|_s, |l|_s < r$. We use the norms

²⁸One will see that these constants are shifts of the constant C by some finite constants that depend only on our truncation of the heat kernel K .

$\|\cdot\|_{\alpha,m}$ on functions with prescribed singularities at the origin that were defined in [Hai14, Definition 10.12]. If J is a smooth function (except for possibly the origin), satisfies [Hai14, Assumption 5.4] with for some $r \geq 0$, and is supported on the ball $|x| \leq 1$, then it is straightforward to show that J is $(r, 2\|J\|_{\beta-|\mathfrak{g}|,r}, \beta)$ -regularising. We then have the following key estimate.

Lemma 7.3 *For any $m \in \mathbf{N}$ one has $\|K\|_{2,m} < \infty$ and there exists R such that $K - K^\varepsilon$ is $(m, \varepsilon^\kappa R, 2 - \kappa)$ -regularising for all $\varepsilon \in [0, 1]$.*

Proof. The first statement is standard. The second statement follows from combining the first statement, our conditions on the kernel K , [Hai14, Lemma 10.17], and the observations made above. \square

We now turn to our random noise assignments. In (7.2) and (7.3) both a mollified noise $J^\varepsilon(\xi_i) = \xi_i^\varepsilon$ and an un-mollified noise ξ appear. In order to start our analysis with smooth models, we replace the un-mollified noise with one mollified at scale δ . In particular, given $\varepsilon, \delta \in (0, 1]$ we define a random noise assignment $\zeta^{\delta,\varepsilon} = (\zeta_l : l \in \mathfrak{L}_-)$ by setting

$$\zeta_l = \begin{cases} \chi^\delta * \xi_i = \xi_i^\delta & \text{for } l = \underline{l}_i \\ \chi^\varepsilon * \xi_i = \xi_i^\varepsilon & \text{for } l = \bar{l}_i. \end{cases}$$

We also define $Z_{\text{BPHZ}}^{\delta,\varepsilon} = (\Pi^{\delta,\varepsilon}, \Gamma^{\delta,\varepsilon}) \in \mathcal{M}_\varepsilon$ to be the BPHZ lift associated to the kernel assignment $K^{(\varepsilon)}$ and random noise assignment $\zeta^{\delta,\varepsilon}$. In our analysis we will first take $\delta \downarrow 0$ followed by $\varepsilon \downarrow 0$ – the first limit is a minor technical point while the second limit is the limit referenced in part (i) of Theorem 2.8.

Note that we have “doubled” our noises in our noise assignment by having two sets of noise labels $\{\underline{l}_i\}_{i=1}^d$ and $\{\bar{l}_i\}_{i=1}^d$ – we will want to use the fact that these two sets of noises take values in the same space \mathfrak{g} (and in practice, differ only by mollification). This is formalised by noting that there are canonical isomorphisms $\mathcal{T}[\Xi_i] \simeq \mathfrak{g}^* \simeq \mathcal{T}[\bar{\Xi}_i]$ for each $i \in [d]$, which we combine into an isomorphism

$$\sigma : \bigoplus_{i=1}^d \mathcal{T}[\Xi_i] \rightarrow \bigoplus_{i=1}^d \mathcal{T}[\bar{\Xi}_i]. \quad (7.12)$$

7.2.2 ε -dependent regularity structures

In the framework of regularity structures, analytic statements regarding models and modelled distributions reference norms $\|\cdot\|_\ell$ on the vector space T_ℓ of all elements of degree $\ell \in \text{deg}(R) = \{\text{deg}(\tau) : \tau \in \mathfrak{T}(R)\}$ – in our setting this is given by

$$T_\ell = \bigoplus_{\tau \in \mathfrak{T}(\ell,R)} \mathcal{T}[\tau],$$

where $\mathfrak{T}(\ell, R) = \{\tau \in \mathfrak{T}(R) : \text{deg}(\tau) = \ell\}$. In many applications the spaces T_ℓ are finite-dimensional and there is no need to specify the norm $\|\cdot\|_\ell$ on T_ℓ (since they are all equivalent).

While the spaces T_ℓ are also finite-dimensional in our setting, we want to encode the fact that K^ε is converging to K and ξ^ε is converging to ξ as $\varepsilon \downarrow 0$ in a way that allows us to treat discrepancies between these quantities as small at the level of our abstract formulation of the fixed point problem. We achieve this by defining, for each $\ell \in \deg(R)$, a family of norms $\{\|\cdot\|_{\ell,\varepsilon} : \varepsilon \in (0, 1]\}$ on T_ℓ . Our definition will depend on a small parameter $\theta \in (0, \kappa)$ which we treat as fixed in what follows.

Heuristically and pretending for a moment that we are in the scalar noise setting, we define these $\|\cdot\|_{\ell,\varepsilon}$ norms by performing a “change of basis” and writing out trees in terms of the noises $\Xi_i, \bar{\Xi}_i - \Xi_i$, operators $\mathcal{F}_{i,p}, \bar{\mathcal{F}}_{i,p} - \mathcal{F}_{i,p}, \mathcal{F}_{i,p}^h$ and \mathcal{F}^u instead of $\Xi_i, \bar{\Xi}_i, \mathcal{F}_{i,p}, \bar{\mathcal{F}}_{i,p}, \mathcal{F}_{i,p}^h$ and \mathcal{F}^u , respectively. For instance, we rewrite

$$\bar{\mathcal{F}}_i(\bar{\Xi}_i) = (\bar{\mathcal{F}}_i - \mathcal{F}_i)(\bar{\Xi}_i - \Xi_i) + (\bar{\mathcal{F}}_i - \mathcal{F}_i)\Xi_i + \mathcal{F}_i(\bar{\Xi}_i - \Xi_i) + \mathcal{F}_i\Xi_i .$$

We then define, for any $\ell \in \deg(R)$ and $v = \sum_{\tau \in \mathfrak{T}(\ell, R)} v_\tau \tau \in T_\ell$,

$$\|v\|_{\ell,\varepsilon} = \max\{\varepsilon^{m(\tau)\theta} |v_\tau| : \tau \in \mathfrak{T}(\ell, R)\} ,$$

where $m(\tau)$ counts the number of occurrence of $\bar{\mathcal{F}}_{i,p} - \mathcal{F}_{i,p}$ and $\bar{\Xi}_i - \Xi_i$ in τ .

We now make this idea more precise and formulate it our setting of vector-valued noise. Recall that in our new setting the trees serve as indices for subspaces of our regularity structure, instead of basis vectors, so we do not really “change basis”. We note that there is a (unique) isomorphism $\Theta : \mathcal{T} \rightarrow \mathcal{T}$ with the properties that

- Θ preserves the domain of $\mathcal{F}_{i,p}, \mathcal{F}_{i,p}^h$ and \mathcal{F}^u and commutes with these operators on their domain.
- For any $\tau, \bar{\mathcal{F}}_{i,j}(\tau) \in \mathfrak{T}(R)$ one has $\Theta \circ \bar{\mathcal{F}}_{i,j} = (\bar{\mathcal{F}}_{i,j} + \mathcal{F}_{i,j}) \circ \Theta$.
- For any $u, v \in \mathcal{T}$ with $uv \in \mathcal{T}$ one has $\Theta(u)\Theta(v) = \Theta(uv)$ —here we are referencing the partially defined product on \mathcal{T} induced by the partially defined tree product on $\mathfrak{T}(R)$.
- The restriction of Θ to $\mathcal{T}[\bar{\Xi}_i]$ is given by $\text{id} + \sigma^{-1}$ where σ^{-1} is the inverse of the map σ given in (7.12).
- Θ restricts to the corresponding identity map on both $\mathcal{T}[\mathbf{X}^k]$ and $\mathcal{T}[\Xi_i]$.

It is immediate that Θ furthermore preserves T_ℓ for every $\ell \in \deg(R)$.

We now fix, for every $\tau \in \mathfrak{T}(R)$, some norm $\|\cdot\|_\tau$ on $\mathcal{T}[\tau]$. Since each $\mathcal{T}[\tau]$ is isomorphic to a subspace of $(\mathfrak{g}^*)^{\otimes n}$ and the isomorphism is furthermore canonical up to permutation of the factors, this can be done by choosing a norm on \mathfrak{g}^* as well as a choice of uniform crossnorm (for example the projective crossnorm).

We then define a norm $\|\cdot\|_{\ell,\varepsilon}$ on T_ℓ by setting, for any $v \in T_\ell$,

$$\|v\|_{\ell,\varepsilon} = \max\{\varepsilon^{m(\tau)\theta} \|P_\tau v\|_\tau : \tau \in \mathfrak{T}(R, \ell)\} ,$$

where P_τ is the projection from T_ℓ to $\mathcal{T}[\tau]$ and now $m(\tau)$ counts the number of occurrences of the labels $\{\bar{l}_i, \mathbf{m}_i\}_{i=1}^d$ appearing in τ . Finally, the norm $\|\cdot\|_{\ell,\varepsilon}$ is given by setting $\|v\|_{\ell,\varepsilon} = \|\Theta v\|_{\ell,\varepsilon}$.

The following lemma, which is straightforward to prove, states that these norms have the desired qualities.

Lemma 7.4

- Let $\ell \in \deg(R)$ and $v \in T_\ell$ with v is in the domain of the operator $\bar{\mathcal{F}}_{i,p} - \mathcal{F}_{i,p}$. Then one has, uniform in ε ,

$$\|(\bar{\mathcal{F}}_{i,p} - \mathcal{F}_{i,p})(v)\|_{\ell+2-\kappa,\varepsilon} \lesssim \varepsilon^\theta \|v\|_{\ell,\varepsilon} . \quad (7.13)$$

- For any $u \in \mathcal{T}[\Xi_i]$ one has, uniformly in ε ,

$$\|(\sigma(u) - u)\|_{-d/2-1-\kappa,\varepsilon} \lesssim \varepsilon^\theta \|u\|_{-d/2-1-\kappa,\varepsilon} . \quad (7.14)$$

Once we fix these ε -dependent norms on our regularity structure we also obtain corresponding

- ε -dependent seminorms and metrics on models which we denote by $\|\cdot\|_\varepsilon$ and $d_\varepsilon(\cdot, \cdot)$ respectively; and
- ε -dependent norms on $\mathcal{D}^{\gamma,\eta} \times \mathcal{M}_\varepsilon$ which we denote by $|\cdot|_{\gamma,\eta,\varepsilon}$.

Remark 7.5 Recall that modelled distributions in the scalar setting take values in the regularity structure \mathcal{T} and therefore in the definition of a norm on modelled distributions we reference norms $\|\cdot\|_\ell$ on the spaces T_ℓ . When we allow our noises / solutions to live in finite-dimensional vector spaces our modelled distributions will take values in $\mathcal{T} \otimes W$ for some finite-dimensional vector space W and so when specifying a norm on such modelled distributions we will need to reference norms on $T_\ell \otimes W$

We assume that we have already fixed, for any such space W appearing, an ε -independent norm $\|\cdot\|_W$. Then we view our norm on $T_\ell \otimes W$ as induced by the norm on T_ℓ by taking some choice of crossnorm (the particular choice does not matter).

Remark 7.6 Clearly, all of our ε -dependent seminorms/metrics on models are equivalent for different values of $\varepsilon \in (0, 1]$, but not uniformly so as $\varepsilon \downarrow 0$. The distances for controlling models (resp. modelled distributions) become stronger (resp. weaker) as one takes ε smaller.

Remark 7.7 In general, one would not expect the estimates of the extension theorem [Hai14, Theorem 5.14] to hold uniformly as we take $\varepsilon \downarrow 0$. However, it is straightforward to see from the proof of [Hai14, Theorem 5.14] that they do hold uniformly in ε for models in \mathcal{M}_ε (and, more trivially, \mathcal{M}_0) thanks to Lemma 7.3 and the fact that $\theta \leq \kappa$.

7.2.3 Comparing fixed point problems

For sufficiently small $\tilde{\theta} > 0$, one has a classical Schauder estimate

$$|G * f - G_\varepsilon * f|_{\mathcal{C}^{2+\ell}} \lesssim \varepsilon^{\tilde{\theta}} |f|_{\mathcal{C}^{\ell+\tilde{\theta}}} ,$$

which holds for all distributions f and non-integer regularity exponents. Our conditions on our ε -dependent norms let us prove an analogous estimate at the level of modelled distributions. In what follows we write $\mathcal{K}_i, \bar{\mathcal{K}}_i$ for the abstract integration operators on modelled distributions associated to \mathfrak{a}_i , and \mathfrak{m}_i , respectively.

Lemma 7.8 *Fix $i \in [d]$ and let \mathcal{V} be a sector of regularity α in our regularity structure which is in the domain of both \mathcal{F}_i and $\bar{\mathcal{F}}_i$. Fix $\gamma \in (0, 2]$ and $\eta < \gamma$ such that $\gamma + 2 - \kappa \notin \mathbf{N}$, $\eta + 2 - \kappa \notin \mathbf{N}$, and $\eta \wedge \alpha > -2$.*

Then, for fixed $M > 0$, one has

$$|\mathcal{K}_i H - \bar{\mathcal{K}}_i H|_{\gamma+2-\kappa, \bar{\eta}, \varepsilon} \lesssim \varepsilon^\theta |H|_{\gamma, \eta, \varepsilon}$$

uniformly in $\varepsilon \in (0, 1]$, $Z \in \mathcal{M}_\varepsilon$ with $\|Z\|_\varepsilon \leq M$, and $H \in \mathcal{D}^{\gamma, \eta}(\mathcal{V}) \times Z$, where $\theta \in (0, \kappa)$ is the fixed small parameter as above and $\bar{\eta} = (\eta \wedge \alpha) + 2 - \kappa$.

Proof. This result follows from the proof of [Hai14, Theorem 5.12 and Proposition 6.16]. Indeed, in the context of this reference, and working with some fixed norm on the given regularity structure, if the abstract integrator $\mathcal{F}(\cdot)$ of order β in question has an operator norm (as an operator on the regularity structure) bounded by \tilde{M} , and the kernel \mathcal{F} realises is $(\gamma + \beta, \tilde{M}, \beta)$ -regularising, then as long as $\gamma + \beta \notin \mathbf{N}$ and $\eta + \beta \notin \mathbf{N}$, one has

$$|\mathcal{K}H|_{\gamma+\beta, (\eta \wedge \alpha)+\beta} \lesssim \tilde{M} |H|_{\gamma, \eta}.$$

Here, \mathcal{K} is the corresponding integration on modelled distributions and the proportionality constant only depends on the size of the model in the model norm (which corresponds to the fixed norm on the regularity structure).

Our result then follows by combining this observation with the fact that we can view $\mathcal{F}_i - \bar{\mathcal{F}}_i$ as an abstract integrator of order $2 - \kappa$ on our regularity structure realising the kernel $K - K^\varepsilon$ which is $(m, \varepsilon^\kappa R, 2 - \kappa)$ -regularising by Lemma 7.3, and the fact that $\mathcal{F}_i - \bar{\mathcal{F}}_i$ has norm bounded by ε^θ by (7.13). \square

We specialise the above lemma to the particular estimate that we will need in our comparison of abstract fixed point problems. We first define, as in Section 6, $\Xi_i \in \mathcal{T}[\Xi_i] \otimes \mathfrak{g}$ and $\bar{\Xi}_i \in \mathcal{T}[\bar{\Xi}_i] \otimes \mathfrak{g}$ to be given by “ $\text{id}_{\mathfrak{g}}$ ” via the canonical isomorphisms $\mathcal{T}[\bar{\Xi}_i] \otimes \mathfrak{g} \simeq \mathcal{T}[\Xi_i] \otimes \mathfrak{g} \simeq \mathfrak{g}^* \otimes \mathfrak{g} \simeq L(\mathfrak{g}, \mathfrak{g})$. Note that we have $\sigma \Xi_i = \bar{\Xi}_i$ where we continue our abuse of notation with σ acting only on the left factor. We also remark that, as \mathfrak{g} -valued modelled distributions, $\Xi_i, \bar{\Xi}_i \in \mathcal{D}_{-d/2-1-\kappa}^{\infty, \infty}$. Then, thanks to (7.14) and Remark 7.5 one has, uniform in $\varepsilon \in (0, 1]$,

$$\|\Xi_i - \bar{\Xi}_i\|_{\infty, \infty, \varepsilon} \lesssim \varepsilon^\theta. \quad (7.15)$$

Lemma 7.9 Fix $i \in [d]$. Let γ be such that $\gamma - \frac{d}{2} - 1 - \kappa \in (0, 2)$ and $\eta > 0$. Let $\bar{\gamma} = \gamma - \frac{d}{2} + 1 - 2\kappa \notin \mathbf{N}$ and $\bar{\eta} = \eta - \frac{d}{2} + 1 - 2\kappa \notin \mathbf{N}$. Then, for any $M > 0$, one has

$$|\mathcal{K}_i(H\bar{\Xi}_i) - \bar{\mathcal{K}}_i(H\Xi_i)|_{\bar{\gamma}, \bar{\eta}, \varepsilon} \lesssim \varepsilon^\theta |H|_{\gamma, \eta, \varepsilon}, \quad (7.16)$$

uniformly over all $\varepsilon \in (0, 1]$, $Z \in \mathcal{M}_\varepsilon$ with $\|Z\|_\varepsilon \leq M$, and $L(\mathfrak{g}, \mathfrak{g})$ -valued modelled distributions $H \in \mathcal{D}^{\gamma, \eta}(\mathcal{V}) \times Z$. Here \mathcal{V} is a sector of regularity 0 which admits multiplication with any element of $\mathcal{T}[\Xi_i]$ or $\mathcal{T}[\bar{\Xi}_i]$ and subsequent integration with \mathcal{I}_i or $\bar{\mathcal{I}}_i$.

Proof. We write

$$\mathcal{K}_i(H\bar{\Xi}_i) - \bar{\mathcal{K}}_i(H\Xi_i) = \mathcal{K}_i(H(\bar{\Xi}_i - \Xi_i)) + (\mathcal{K}_i - \bar{\mathcal{K}}_i)(H\Xi_i). \quad (7.17)$$

Note that

$$\begin{aligned} |\mathcal{K}_i(H(\bar{\Xi}_i - \Xi_i))|_{\bar{\gamma}, \bar{\eta}, \varepsilon} &\lesssim |H(\bar{\Xi}_i - \Xi_i)|_{\gamma - \frac{d}{2} - 1 - \kappa, \eta - \frac{d}{2} - 1 - \kappa, \varepsilon} \\ &\lesssim |H|_{\gamma, \eta, \varepsilon} \|\bar{\Xi}_i - \Xi_i\|_{\infty, \infty, \varepsilon} \lesssim \varepsilon^\theta |H|_{\gamma, \eta, \varepsilon}, \end{aligned}$$

where we used the standard multi-level Schauder estimate [Hai14, Proposition 6.16] in the first inequality, the standard multiplication bounds [Hai14, Proposition 6.12] in the second inequality, and (7.15).

To finish the proof we observe that

$$|(\mathcal{K}_i - \bar{\mathcal{K}}_i)(H\Xi_i)|_{\bar{\gamma}, \bar{\eta}, \varepsilon} \lesssim \varepsilon^\theta |H\Xi_i|_{\gamma - \frac{d}{2} - 1 - \kappa, \eta - \frac{d}{2} - 1 - \kappa, \varepsilon} \lesssim \varepsilon^\theta |H|_{\gamma, \eta, \varepsilon},$$

where we used Lemma 7.8 in the first inequality and the standard multiplication bound in the second inequality. \square

We now write out the analytic fixed point problems for (7.2) and (7.3). We introduced the labels \mathfrak{m}_i just to assist with deriving the renormalised equation and so when we pose our analytic fixed point problem we stray from the formulation given in Remark 5.64 and instead eliminate the components \mathfrak{m}_i appearing in (7.3) by performing a substitution.

In what follows, we write \mathcal{R} for the reconstruction operator. Recall that $\mathcal{K}_i, \bar{\mathcal{K}}_i$ are the abstract integration operators associated to \mathfrak{a}_i and \mathfrak{m}_i ; we also write $\mathcal{K}_{\mathfrak{h}_i}$ and $\mathcal{K}_{\mathfrak{u}}$ for the abstract integration operators on modelled distributions corresponding to $\mathcal{F}_i^{\mathfrak{h}}$ and $\mathcal{F}_{\mathfrak{u}}$, and R the operator realising convolution with $G - K$ as a map from appropriate HÅulder–Besov functions into modelled distributions as in [Hai14, (7.7)].

Given initial data

$$(B^{(0)}, U^{(0)}, h^{(0)}) \in \Omega\mathcal{C}^\eta \times \mathcal{C}^\alpha(\mathbf{T}^2, L(\mathfrak{g}, \mathfrak{g})) \times \mathcal{C}^{\alpha-1}(\mathbf{T}^2, \mathfrak{g}), \quad (7.18)$$

the fixed point problem associated with (7.2) for the \mathfrak{g} -valued modelled distributions $(\mathcal{B}_i)_{i=1}^d, (\mathcal{H}_i)_{i=1}^d$ and $L(\mathfrak{g}, \mathfrak{g})$ -valued modelled distribution \mathcal{U} is

$$\mathcal{B}_i = \mathcal{G}_i \mathbf{1}_+ \left([\mathcal{B}_j, 2\partial_j \mathcal{B}_i - \partial_i \mathcal{B}_j + [\mathcal{B}_j, \mathcal{B}_i]] \right)$$

$$\begin{aligned}
& + \mathring{C}_1 \mathcal{B}_i + \mathring{C}_2 \mathcal{H}_i + \mathcal{U} \bar{\Xi}_i \Big) + G B_i^{(0)} \quad (7.19) \\
\mathcal{H}_i &= \mathcal{G}_{\mathfrak{h}_i} \mathbf{1}_+ \left([\mathcal{H}_j, \partial_j \mathcal{H}_i] + [[\mathcal{B}_j, \mathcal{H}_j], \mathcal{H}_i] + \partial_i [\mathcal{B}_j, \mathcal{H}_j] \right) + G \mathfrak{h}_i^{(0)} \\
\mathcal{U} &= \mathcal{G}_{\mathfrak{u}} \mathbf{1}_+ \left(- [\mathcal{H}_j, [\mathcal{H}_j, \cdot]] \circ \mathcal{U} + [[\mathcal{B}_j, \mathcal{H}_j], \cdot] \circ \mathcal{U} \right) + G U^{(0)},
\end{aligned}$$

where $\mathbf{1}_+$ is the map that restricts modelled distributions to non-negative times, $\mathcal{G}_t \stackrel{\text{def}}{=} \mathcal{K}_t + R\mathcal{R}$ and finally $G\cdot$ refers to the ‘‘harmonic extension’’ map of [Hai14, (7.13)].

The modelled distribution fixed point problem for the $(\bar{A}, \bar{U}, \bar{h})$ system (7.3) is the same as (7.19) except that the first equation is replaced by

$$\begin{aligned}
\mathcal{B}_i &= \mathcal{G}_i \mathbf{1}_+ \left([\mathcal{B}_j, 2\partial_j \mathcal{B}_i - \partial_i \mathcal{B}_j + [\mathcal{B}_j, \mathcal{B}_i]] + \mathring{C}_1 \mathcal{B}_i + \mathring{C}_2 \mathcal{H}_i \right) \quad (7.20) \\
&+ \bar{\mathcal{G}}_i \mathbf{1}_+ (\mathcal{U} \bar{\Xi}_i) + G B_i^{(0)},
\end{aligned}$$

where $\bar{\mathcal{G}}_i \stackrel{\text{def}}{=} \bar{\mathcal{K}}_i + \bar{R}\mathcal{R}$ with \bar{R} defined just like R but with $G - K$ replaced by $G^\varepsilon - K^\varepsilon$. In (7.20) we have written \mathcal{B}_i instead of something like \bar{A}_i to make it clearer that we are comparing two fixed point problems which have ‘‘almost’’ the same form – only the terms $\mathcal{G}_i \mathbf{1}_+ (\mathcal{U} \bar{\Xi}_i)$ and $\bar{\mathcal{G}}_i \mathbf{1}_+ (\mathcal{U} \bar{\Xi}_i)$ are different. We can now make precise what we mean by the two problems being ‘‘close’’.

Lemma 7.10 *For initial data $(B^{(0)}, U^{(0)}, h^{(0)})$ as in (7.18), the fixed point problems (7.19) and (7.20) are well-posed on the bundle of modelled distributions*

$$\left(\bigoplus_{\mathfrak{t}=\mathfrak{a}_i, \mathfrak{h}_i, \mathfrak{u}} \mathcal{D}_{\alpha_{\mathfrak{t}}}^{\gamma_{\mathfrak{t}}, \eta_{\mathfrak{t}}} \right) \times \mathcal{M} \text{ where } (\gamma_{\mathfrak{t}}, \alpha_{\mathfrak{t}}, \eta_{\mathfrak{t}}) = \begin{cases} (1 + 2\kappa, -\kappa, \eta) & \text{if } \mathfrak{t} = \mathfrak{a}_i, \\ (2 + 2\kappa, 0, \eta + 1) & \text{if } \mathfrak{t} = \mathfrak{u}, \\ (1 + 2\kappa, 0, \eta) & \text{if } \mathfrak{t} = \mathfrak{h}_i \end{cases} \quad (7.21)$$

For any $T \in (0, 1]$ and $L > 0$, let $\mathcal{S}^{L,T}(\cdot)$ and $\bar{\mathcal{S}}^{L,T}(\cdot)$ be the corresponding solution maps with cut-off size L and cut-off time T for systems (7.19) and (7.20).

Then, for any $\bar{R} > 0$, and uniform in $\varepsilon \in (0, 1]$ with $Z \in \mathcal{M}_\varepsilon$ with $\|Z\|_\varepsilon \leq \bar{R}$ we have the estimate

$$|\mathcal{S}^{L,T}(Z) - \bar{\mathcal{S}}^{L,T}(Z)|_{\bar{\gamma}, \bar{\eta}, \varepsilon} \lesssim \varepsilon^\theta \quad \text{and} \quad \|\mathcal{R}^Z \mathcal{S}^L(Z) - \mathcal{R}^Z \bar{\mathcal{S}}^L(Z)\|_{\bar{\alpha}} \lesssim \varepsilon^\theta, \quad (7.22)$$

where $|\cdot|_{\bar{\gamma}, \bar{\eta}, \varepsilon}$ is a corresponding multi-component modelled distribution norm for (7.21), \mathcal{R}^Z is the reconstruction operator associated to Z , and $\|\cdot\|_{\bar{\alpha}}$ is the norm on

$$\bigoplus_{\mathfrak{t}=\mathfrak{a}_i, \mathfrak{h}_i, \mathfrak{u}} C^{\alpha_{\mathfrak{t}}}([0, T] \times \mathbf{T}^d, W_{\mathfrak{t}}).$$

Proof. We first note that the two fixed point problems differ in the \mathcal{B}_i components, namely by the two terms

$$\mathcal{K}_i \mathbf{1}_+ (\mathcal{U} \bar{\Xi}_i) - \bar{\mathcal{K}}_i \mathbf{1}_+ (\mathcal{U} \bar{\Xi}_i) \quad \text{and} \quad R\mathcal{R} \mathbf{1}_+ (\mathcal{U} \bar{\Xi}_i) - \bar{R}\mathcal{R} \mathbf{1}_+ (\mathcal{U} \bar{\Xi}_i).$$

By Lemma 7.9, the first term gives a contribution of order ε^θ . The second term follows by an analogous argument as the one used in the proof of Lemma 7.9 but easier since we are dealing with the smooth part of the integration map. Namely we first write the second term as $R\mathcal{R}\mathbf{1}_+\mathcal{U}(\Xi_i - \bar{\Xi}_i) + (R - \bar{R})\mathcal{R}\mathbf{1}_+\mathcal{U}\Xi_i$. The first piece can be estimated in the same way as before but using the estimate of [Hai14, Lemma 7.3] instead of the Schauder estimate for \mathcal{H}_i . It is straightforward, by referring to the definition [Hai14, (7.7)] of $R - \bar{R}$, to argue that the second piece is of order ε since, for any $\alpha > 0$, one has $\|(G - K) - (G^\varepsilon - K^\varepsilon)\|_{\mathcal{C}^\alpha} \lesssim \varepsilon$.

With these bounds in hand, the first estimate of (7.22) then follows by the stability of the fixed point established in the proof of [Hai14, Theorem 7.8] and the second estimate is a consequence of the reconstruction bound (note that the reconstruction bound is uniform in ε even though we are using ε -dependent norms on both models and modelled distributions). \square

7.2.4 Control over the BPHZ models

Our key input in our argument regarding stochastic control of our models is given by the following lemma.

Lemma 7.11 *One has, for any $p \geq 1$,*

$$\sup_{\varepsilon \in (0,1]} \sup_{\delta \in (0,\varepsilon)} \mathbf{E}[\|Z_{\text{BPHZ}}^{\delta,\varepsilon}\|_\varepsilon^p] < \infty. \quad (7.23)$$

Moreover, there exist models $Z_{\text{BPHZ}}^{0,\varepsilon} \in \mathcal{M}_\varepsilon$ for $\varepsilon \in (0, 1]$ such that, for any such ε ,

$$\lim_{\delta \downarrow 0} Z_{\text{BPHZ}}^{\delta,\varepsilon} = Z_{\text{BPHZ}}^{0,\varepsilon} \quad (7.24)$$

in probability with respect to the topology of $d_\varepsilon(\cdot, \cdot)$.

Finally, there exists a model $Z_{\text{BPHZ}}^{0,0} \in \mathcal{M}_0$ such that

$$\lim_{\varepsilon \downarrow 0} Z_{\text{BPHZ}}^{0,\varepsilon} = Z_{\text{BPHZ}}^{0,0} \quad (7.25)$$

in probability with respect to the topology of $d_1(\cdot, \cdot)$.

Proof. As in Lemma 6.4 we proceed by using the results of [CH16]. We start by proving (7.24) and here we appeal to [CH16, Theorem 2.15]. We first note that for any scalar noise decomposition, it is straightforward to verify that the random smooth noise assignments $\zeta^{\delta,\varepsilon}$ are a uniformly compatible family of Gaussian noises that converge to the Gaussian noise $\zeta^{0,\varepsilon}$. The verification of the first three listed power-counting conditions of [CH16, Theorem 2.15] is analogous to how they were checked for Lemma 6.4. This gives the existence of the limiting models $Z_{\text{BPHZ}}^{0,\varepsilon}$ and the desired convergence statement (note that for fixed $\varepsilon > 0$, the metric $d_\varepsilon(\cdot, \cdot)$ is equivalent to $d_1(\cdot, \cdot)$).

To prove (7.25) we will show

$$\lim_{\varepsilon \downarrow 0} \sup_{\delta, \tilde{\varepsilon} \in (0, \varepsilon)} \mathbf{E}[d_1(Z_{\text{BPHZ}}^{\delta, \varepsilon}, Z_{\text{BPHZ}}^{\delta, \tilde{\varepsilon}})^2] = 0. \quad (7.26)$$

By using Fatou to take the limit $\delta \downarrow 0$, this gives us that $Z^{0, \varepsilon}$ is Cauchy in L^2 as $\varepsilon \downarrow 0$ and so we obtain the desired limiting model $Z_{\text{BPHZ}}^{0, 0}$ and the desired convergence statement.

To prove (7.26) we will use the more quantitative [CH16, Theorem 2.31]. Here we take \mathcal{L}_{cum} to be the set of all pairings of \mathcal{L}_- and so the three power-counting conditions we verified for [CH16, Theorem 2.15] also imply the super-regularity assumption of [CH16, Theorem 2.31]. Since we only work with pairings, the cumulant homogeneity \mathfrak{c} is determined by our degree assignment on our noises. After rewriting the difference of the action of models as a telescoping sum which allows one to factor the corresponding difference in the kernel assignment $K - K^\varepsilon$ or noise assignment $\xi_i^{\tilde{\varepsilon}} - \xi_i^\varepsilon$, one is guaranteed at least one factor of order ε^κ the right-hand side of the bound [CH16, (2.15)] – coming from $\|K - K^\varepsilon\|_{2-\kappa, k}$ in the first case or the contraction $\xi_i^{\tilde{\varepsilon}} - \xi_i^\varepsilon$ with another noise measured in the $\|\cdot\|_{-4-2\kappa, k}$ kernel norm in the second case. This gives us the estimate (7.26).

The above argument for obtaining (7.26) can also be applied to obtain (7.23), namely, with the constraint that $\delta \in (0, \varepsilon)$, occurrence of $\mathcal{F}_{t,p} - \bar{\mathcal{F}}_{t,p}$ gives a factor of ε^κ through the difference $K - K^\varepsilon$ and any occurrence of $\Xi_{i_1} - \bar{\Xi}_{i_1}$ gives a factor ε^κ through the difference $\xi_i^\delta - \xi_i^\varepsilon$ and since $\theta \in (0, \kappa)$ this gives the suitable uniform in ε bounds on the moments of the model norm $\|\cdot\|_\varepsilon$. \square

7.3 Renormalisation for the gauge transformed system

















In this section we derive the renormalised equations for the B system and the \bar{A} system and prove that they converge to the same limit, i.e. Proposition 7.22.




7.3.1 Identification of the renormalised equation

Given $\delta \in [0, 1]$ and $\varepsilon \in (0, 1]$, we write $\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\cdot]$ for the BPHZ renormalisation group character that goes between the canonical lift and $Z_{\text{BPHZ}}^{\delta, \varepsilon}$. The rule given below (7.9) determines the set $\mathfrak{T}_-(R)$ of trees as in (5.22) and we only list the trees in $\mathfrak{T}_-(R)$ that are relevant to deriving the renormalised equations in the following two tables (for the F system and the \bar{F} system respectively). The reason that we will only need to be concerned with these trees will be clear by Lemma 7.12 below, which follows easily from the definition of $\Upsilon_i[\cdot]$ and the parity constraints on the noises and spatial derivatives that are necessary for $\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\cdot]$ not to vanish.

Here the graphic notation is similarly as in Section 6: (thick) lines denote (derivatives of) \mathcal{F} , colors denote spatial indices, and the color of a tiny triangle labels the spatial index for the kernel immediately below it. Moreover, we draw a circle (resp. crossed circle) for Ξ (resp. $\mathbf{X}\Xi$), with a convention that the line immediately below it understood as \mathcal{F} , and a square (resp. crossed square) for Ξ (resp. $\mathbf{X}\Xi$), with a convention that the line immediately below it understood as $\bar{\mathcal{F}}$.

We also draw a zigzag line $\text{---}\text{---}\text{---}$ for \mathcal{F}^u and a wavy line $\text{~}\text{~}\text{~}$ for \mathcal{F}^h . Their thick versions and tiny triangles above them are understood as before.

Table 1		Table 2	
	$\mathcal{F}_n(\bar{\Xi}_n)^2$		$\bar{\mathcal{F}}_n(\Xi_n)^2$
	$\mathcal{F}_n(\bar{\Xi}_n)\mathcal{F}_{n,n}(\mathcal{F}_{n,n}(\bar{\Xi}_n))$		$\bar{\mathcal{F}}_n(\Xi_n)\mathcal{F}_{n,n}(\bar{\mathcal{F}}_{n,n}(\Xi_n))$
	$\mathcal{F}_{n,n}(\bar{\Xi}_n)\mathcal{F}_n(\mathcal{F}_{n,n}(\bar{\Xi}_n))$		$\bar{\mathcal{F}}_{n,n}(\Xi_n)\mathcal{F}_n(\bar{\mathcal{F}}_{n,n}(\Xi_n))$
	$\mathcal{F}_n(\bar{\Xi}_n)\mathcal{F}_{n,n}(\mathbf{X}_n\bar{\Xi}_n)$		$\bar{\mathcal{F}}_n(\Xi_n)\bar{\mathcal{F}}_{n,n}(\mathbf{X}_n\Xi_n)$
	$\mathcal{F}_n(\mathbf{X}_n\bar{\Xi}_n)\mathcal{F}_{n,n}(\bar{\Xi}_n)$		$\bar{\mathcal{F}}_n(\mathbf{X}_n\Xi_n)\bar{\mathcal{F}}_{n,n}(\Xi_n)$
	$\mathcal{F}^u(\mathcal{F}_n(\bar{\Xi}_n))\bar{\Xi}_n$		$\mathcal{F}^u(\bar{\mathcal{F}}_n(\Xi_n))\Xi_n$
	$\mathcal{F}_{n,n}(\bar{\Xi})\mathcal{F}_n^h(\mathcal{F}_{n,n}(\bar{\Xi}))$		$\bar{\mathcal{F}}_{n,n}(\Xi)\mathcal{F}_n^h(\bar{\mathcal{F}}_{n,n}(\Xi))$
	$\mathcal{F}_n(\bar{\Xi})\mathcal{F}_{n,n}^h(\mathcal{F}_{n,n}(\bar{\Xi}))$		$\bar{\mathcal{F}}_n(\Xi)\mathcal{F}_{n,n}^h(\bar{\mathcal{F}}_{n,n}(\Xi))$

The first five trees in each of the two tables have the same structure as the ones that appeared in Section 6, except that now the noises are understood as Ξ or $\bar{\Xi}$, and edges understood as \mathcal{F} or $\bar{\mathcal{F}}$. An important difference from Section 6 is that the trees of the type  and  had vanishing Υ in Section 6 and therefore no effect on the renormalised equation, but this is not the case now, as we will see below, due to the term $UJ^\varepsilon\xi$ (or $J^\varepsilon(\bar{U}\xi)$) in our equation. Moreover, the tables also show trees in $\mathfrak{T}_-(R)$ such as those of the form  which do not have any counterpart in Section 6.

Lemma 7.12 *If $\tau \in \mathfrak{T}_-(R)$ is not of any of the forms listed in Table 1 (resp. Table 2) then either $\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\tau] = 0$ or $\Upsilon_t^F[\tau] = 0$ (resp. either $\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\tau] = 0$ or $\Upsilon_t^{\bar{F}}[\tau] = 0$) for every $t \in \mathfrak{L}_+$.*

Proof. The proof of this lemma follows similar lines as Lemma 6.5, so we do not repeat the details. We only remark that for trees with a “polynomial” \mathbf{X} , namely

$$\mathcal{F}_{n,n}(\bar{\Xi}_n)\mathcal{F}_n(\mathbf{X}_n\bar{\Xi}_n), \quad \mathcal{F}_n(\bar{\Xi}_n)\mathcal{F}_{n,n}(\mathbf{X}_n\bar{\Xi}_n), \quad \bar{\mathcal{F}}_{n,n}(\Xi_n)\bar{\mathcal{F}}_n(\mathbf{X}_n\Xi_n), \quad \bar{\mathcal{F}}_n(\Xi_n)\bar{\mathcal{F}}_{n,n}(\mathbf{X}_n\Xi_n),$$

the polynomial can be dealt with in the same way as for the derivative in Lemma 6.5; for instance for the first tree, if $n \neq n$, then flipping the sign of the n -component (or, n -component) of the appropriate integration variable shows that $\bar{\Pi}_{\text{can}}[\tau] = 0$. \square

We now state a sequence of lemmas with identities for $\tilde{\Upsilon}^F$ and $\tilde{\Upsilon}^{\bar{F}}$, but we will not give the detailed calculations within the proof of each lemma, since these are straightforward (for instance they follow similarly as in Section 6). We first show that in both F and \bar{F} systems we don’t see any renormalisation of the u or h_i equations.

Lemma 7.13 *For any of the τ of the form listed in Table 1 (resp. Table 2) one has $\bar{\Upsilon}_u^F[\tau] = 0$ (resp. $\bar{\Upsilon}_u^{\bar{F}}[\tau] = 0$). Moreover, we have*

$$\sum_{\tau \in \mathfrak{T}_-(R)} (\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\tau] \otimes \text{id}) \bar{\Upsilon}_{h, \bullet}^F[\tau](\mathbf{A}) = \sum_{\tau \in \mathfrak{T}_-(R)} (\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\tau] \otimes \text{id}) \bar{\Upsilon}_{h, \bullet}^{\bar{F}}[\tau](\mathbf{A}) = 0 .$$

Proof. The fact that $\bar{\Upsilon}_u^F[\tau] = \bar{\Upsilon}_u^{\bar{F}}[\tau] = 0$ for τ appearing in the tables follows from direct computation. One has $\bar{\Upsilon}_{h, \bullet}^F[\tau] = 0$ (resp. $\bar{\Upsilon}_{h, \bullet}^{\bar{F}}[\tau] = 0$) for any τ in Table 1 (resp. Table 2) of the first six shapes. For the other trees one has, by integration by parts,

$$\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\text{tree}] = -\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\text{tree}], \quad \ell_{\text{BPHZ}}^{\delta, \varepsilon}[\text{tree}] = -\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\text{tree}]. \quad (7.27)$$

Additionally, one has

$$\Upsilon_{h, \bullet}[\text{tree}] = \Upsilon_{h, \bullet}[\text{tree}], \quad \Upsilon_{h, \bullet}[\text{tree}] = \Upsilon_{h, \bullet}[\text{tree}]. \quad (7.28)$$

Above we are exploiting the canonical isomorphisms between the spaces where the objects above live – namely for any two trees $\tau, \bar{\tau}$ of any of the four forms appearing above, one has a canonical isomorphism $\mathcal{T}[\tau] \simeq \mathcal{T}[\bar{\tau}]$ by using Remark 5.11 and the canonical isomorphisms between these trees obtained by only keeping their tree structure. Combining (7.27) with (7.28) then yields the last claim. \square

We define a subset $\bar{\mathcal{A}} \subset \mathcal{A}$ that encodes additional constraints on the jet of our solutions which come from (7.1) and (7.5). These constraints will also help us simplify the counterterms for the \mathbf{a}_i and \mathbf{m}_i equations.

Definition 7.14 We define $\bar{\mathcal{A}}$ to be the collection of all $\mathbf{A} = (\mathbf{A}_o)_{o \in \mathcal{E}} \in \mathcal{A}$ such that

- \mathbf{A}_u is unitary.
- For all $a, b \in \mathfrak{g}$, $\mathbf{A}_u[a, b] = [\mathbf{A}_u a, \mathbf{A}_u b]$.
- $\mathbf{A}_{\partial_j u} = [\mathbf{A}_{h_j}, \cdot] \circ \mathbf{A}_u$.

Remark 7.15 Lemma 7.13 guarantees that the renormalised reconstruction obtained via the models $(Z_{\text{BPHZ}}^{0, \varepsilon} : \varepsilon \in [0, 1])$, of our equations for U and h (resp. \bar{U} and \bar{h}) will be the same as what appears for these components in (7.2).

This observation means that, for initial data for U and h satisfying (7.1) for some fixed initial $g(0)$, the abstract solution for the F system obtained via the models $(Z_{\text{BPHZ}}^{0, \varepsilon} : \varepsilon \in [0, 1])$, together with their derivatives, will take values in $\bar{\mathcal{A}}$ pointwise. The analogous statement holds true for \bar{U} and \bar{h} equations and the \bar{F} system.

To argue this we first note that, since the constraint imposed by $\bar{\mathcal{A}}$ defines a closed set and the abstract solution map is continuous with respect to the model, it suffices to prove the claim when $\varepsilon > 0$. In this case, irrespective of the form of the renormalised equation for B , we can repeat the computations of Lemma 7.1 to show that if we start with initial $U(0)$ and $h(0)$ as above then, for positive existence

times $t > 0$, $U(t)$ and $h(t)$ satisfy (7.1) with respect to the gauge transformation $g(t)$ given by the evolution of $g(0)$ by the g equation given in (2.5) for some smooth process B .

This means that, for the sake of proving Theorem 2.8, we can assume the relations given in Definition 7.14 hold when computing the renormalised equations for B and \bar{A} for the models $(Z_{\text{BPHZ}}^{0,\varepsilon} : \varepsilon \in [0, 1])$.

We now turn to explicitly identifying the renormalisation counterterms for the \mathfrak{a}_i and \mathfrak{m}_i equations in the \bar{F} system.

We start by collecting formulae for the the renormalisation constants. Write $K^{\delta,\varepsilon} = K^\varepsilon * \chi^\delta = K * \chi^\varepsilon * \chi^\delta$ and recall the constants \hat{C}^ε and \bar{C}^ε defined in (6.12). We then define the variants

$$\bar{C}^{\delta,\varepsilon} \stackrel{\text{def}}{=} \int dz K^{\delta,\varepsilon}(z)^2, \quad \hat{C}^{\delta,\varepsilon} \stackrel{\text{def}}{=} \int dz \partial_j K^{\delta,\varepsilon}(z) (\partial_j K * K^{\delta,\varepsilon})(z), \quad (7.29)$$

where one can choose any $j \in \{1, 2\}$ as in (6.12). We then have the following lemma.

Lemma 7.16 *For \hat{C}^ε and \bar{C}^ε as in (6.12), one has*

$$\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = -\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = \hat{C}^\varepsilon \text{Cas}, \quad \ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = \bar{C}^\varepsilon \text{Cas}. \quad (7.30)$$

For $\hat{C}^{\delta,\varepsilon}$ and $\bar{C}^{\delta,\varepsilon}$ defined as in (7.29) one has

$$\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = -\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = \hat{C}^{\delta,\varepsilon} \text{Cas}, \quad \ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = \bar{C}^{\delta,\varepsilon} \text{Cas}. \quad (7.31)$$

Finally, for any $\varepsilon > 0$, one has $\lim_{\delta \downarrow 0} \hat{C}^{\delta,\varepsilon} = \hat{C}^\varepsilon$ and $\lim_{\delta \downarrow 0} \bar{C}^{\delta,\varepsilon} = \bar{C}^\varepsilon$.

Proof. The statements (7.30) and (7.31) follow in the same way as Lemma 6.9. The final statement about convergence as $\delta \downarrow 0$ of the renormalisation constants is obvious. \square

We introduce additional renormalisation constants

$$\tilde{C}^\varepsilon \stackrel{\text{def}}{=} \int dz \chi^\varepsilon(z) (K * K^\varepsilon)(z), \quad \tilde{C}^{\delta,\varepsilon} \stackrel{\text{def}}{=} \int dz \chi^\delta(z) (K * K^{\delta,\varepsilon})(z).$$

The following lemma is straightforward to prove.

Lemma 7.17 *One has*

$$\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = \tilde{C}^\varepsilon \text{Cas}, \quad \ell_{\text{BPHZ}}^{\delta,\varepsilon}[\text{diagram}] = \tilde{C}^{\delta,\varepsilon} \text{Cas},$$

and furthermore $\lim_{\delta \downarrow 0} \tilde{C}^{\delta,\varepsilon} = (K * K^\varepsilon)(0) \stackrel{\text{def}}{=} \tilde{C}^{0,\varepsilon}$. Additionally, there are finite constants C_{GSYM} and \bar{C}_{GSYM} such that

$$\lim_{\varepsilon \downarrow 0} \tilde{C}^\varepsilon = C_{\text{GSYM}} \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \tilde{C}^{0,\varepsilon} = \bar{C}_{\text{GSYM}}.$$

Finally, we have that $\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\blacksquare\blacktriangleright]$, $\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\bullet\blacktriangleright]$, $\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\bullet\blacktriangleright]$, and $\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\blacksquare\blacktriangleright]$ are each given by a multiple of Cas where the prefactor only depends on δ, ε and the form²⁹ of the tree.

The rest of our computation of the renormalised equation is summarised in the following lemmas. In what follows we refer to the constant λ fixed by Remark 6.8. We also introduce the shorthand³⁰

$$\Psi_{\#} = \mathcal{J}_{\#} \bar{\Xi}_{\#}, \quad \Psi_{\#,\#} = \mathcal{J}_{\#,\#} \bar{\Xi}_{\#}, \quad \bar{\Psi}_{\#} = \bar{\mathcal{J}}_{\#} \bar{\Xi}_{\#}, \quad \bar{\Psi}_{\#,\#} = \bar{\mathcal{J}}_{\#,\#} \bar{\Xi}_{\#}.$$

We now walk through the computation of renormalisation counterterms for the system of equations given by \bar{F} . We will directly give the expressions for $\bar{\Upsilon}^{\bar{F}}$ such as (7.32) and (7.35) below, which follow by straightforward calculations from the definitions.

Recall the convention (7.10) for writing components of \mathbf{A} as B, \bar{A}, U , etc. The following lemma gives the renormalisation for the m_i equation in this system.

Lemma 7.18 $\bar{\Upsilon}_{m,\#}^{\bar{F}}[\tau] = 0$ for all τ of the form in Table 2 except for $\tau = \blacksquare$ where

$$\bar{\Upsilon}_{m,\#}^{\bar{F}}[\blacksquare](\mathbf{A}) = \delta_{\#,\#} [[\bar{U} \mathcal{J}^u \bar{\Psi}_{\#}, \bar{h}_{\#}], \bar{U} \bar{\Xi}_{\#}]. \quad (7.32)$$

In particular, for $\mathbf{A} \in \bar{\mathcal{A}}$,

$$\sum_{\tau \in \mathfrak{S}_-(R)} (\ell_{\text{BPHZ}}^{\delta,\varepsilon}[\tau] \otimes \text{id}) \bar{\Upsilon}_{m,\#}^{\bar{F}}[\tau](\mathbf{A}) = -\lambda \tilde{C}^{\delta,\varepsilon} \bar{h}_{\#}. \quad (7.33)$$

Proof. Using the assumption that $\mathbf{A} \in \bar{\mathcal{A}}$ we have

$$[[\bar{U} \mathcal{J}^u(\bar{\Psi}_{\#}), \bar{h}_{\#}], \bar{U} \bar{\Xi}_{\#}] = -\bar{U} [\bar{\Xi}_{\#}, [\mathcal{J}^u(\bar{\Psi}_{\#}), \bar{U}^{-1} \bar{h}_{\#}]].$$

Inserting this into the left-hand side of (7.33) and combining it with Lemma 7.17, we see that it is equal to

$$-\tilde{C}^{\delta,\varepsilon} (\text{Cas} \otimes \text{id}) \bar{U} [\bar{\Xi}_{\#}, [\mathcal{J}^u(\bar{\Psi}_{\#}), \bar{U}^{-1} \bar{h}_{\#}]] = -\tilde{C}^{\delta,\varepsilon} \bar{U} \text{ad}_{\text{Cas}} \bar{U}^{-1} \bar{h}_{\#} = -\lambda \tilde{C}^{\delta,\varepsilon} \bar{h}_{\#}$$

since $\text{ad}_{\text{Cas}} = \lambda \text{id}_{\mathfrak{g}}$. \square

For the a_i components we have the following lemma.

Lemma 7.19 $\bar{\Upsilon}_{a,\#}^{\bar{F}}[\blacksquare] = 0$, and

$$\bar{\Upsilon}_{a,\#}^{\bar{F}}[\blacksquare\blacktriangleright](\mathbf{A}) = \mathbf{1}_{\# \neq \#} [\bar{U} \bar{\Psi}_{\#}, [\bar{U} \bar{\Psi}_{\#}, \bar{A}_{\#}]], \quad (7.34)$$

²⁹That is, they do not depend on the specific colors/spatial indices appearing in the tree as long as they obey the constraints given in Tables 1 and 2.







³⁰Note that our use of the notations $\Psi_{\#}$ and $\Psi_{\#,\#}$ differs slightly from Section 6.

$$\begin{aligned}\bar{\Upsilon}_{\alpha, \#}^{\bar{F}}[\text{red triangle}](\mathbf{A}) &= (2\delta_{\#, \#}\delta_{\#, \#} - \delta_{\#, \#}\delta_{\#, \#}) [[2\delta_{\#, \#}\bar{A}_{\#} - \delta_{\#, \#}\bar{A}_{\#}, \bar{U}\mathcal{J}_{\#}(\bar{\Psi}_{\#, \#})], \bar{U}\bar{\Psi}_{\#, \#}], \\ \bar{\Upsilon}_{\alpha, \#}^{\bar{F}}[\text{red triangle with arrow}](\mathbf{A}) &= (2\delta_{\#, \#}\delta_{\#, \#} - \delta_{\#, \#}\delta_{\#, \#}) [\bar{U}\bar{\Psi}_{\#}, [2\delta_{\#, \#}\bar{A}_{\#} - \delta_{\#, \#}\bar{A}_{\#}, \bar{U}\mathcal{J}_{\#}(\bar{\Psi}_{\#, \#})]], \\ \bar{\Upsilon}_{\alpha, \#}^{\bar{F}}[\text{red square}](\mathbf{A}) &= \delta_{\#, \#}(2\delta_{\#, \#} - 1) [\bar{U}\bar{\Psi}_{\#}, \partial_{\#}\bar{U}\bar{\mathcal{J}}_{\#, \#}(\mathbf{X}_{\#}\Xi_{\#})], \\ \bar{\Upsilon}_{\alpha, \#}^{\bar{F}}[\text{red square with arrow}](\mathbf{A}) &= \delta_{\#, \#}(2\delta_{\#, \#} - 1) [\partial_{\#}\bar{U}\bar{\mathcal{J}}_{\#, \#}(\mathbf{X}_{\#}\Xi_{\#}), \bar{U}\bar{\Psi}_{\#, \#}].\end{aligned}\tag{7.35}$$


$$\bar{\Upsilon}_{\alpha, \#}^{\bar{F}}[\text{red square with arrow}](\mathbf{A}) = \delta_{\#, \#}(2\delta_{\#, \#} - 1) [\partial_{\#}\bar{U}\bar{\mathcal{J}}_{\#, \#}(\mathbf{X}_{\#}\Xi_{\#}), \bar{U}\bar{\Psi}_{\#, \#}].\tag{7.36}$$

In particular, for $\mathbf{A} \in \bar{\mathcal{A}}$,

$$\sum_{\tau \in \bar{\Sigma}_{-}(R)} (\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\tau] \otimes \text{id}) \bar{\Upsilon}_{\alpha, \#}^{\bar{F}}[\tau](\mathbf{A}) = (\bar{C}^{\delta, \varepsilon} - 4\hat{C}^{\delta, \varepsilon}) \lambda \bar{A}_{\#}.\tag{7.37}$$

Proof. The right-hand side of (7.37) comes from the contribution of trees of the form , , and , which can be shown as in Lemma 6.14, combined with the condition that $\mathbf{A} \in \bar{\mathcal{A}}$ (namely, the second relation of Definition 7.14) to cancel the factors of \bar{U} . The total contributions from the trees of the form  and those of the form  each vanish. For the case of trees of form  this total contribution is given by

$$\check{C} \sum_{\# = 1, 2} (2\delta_{\#, \#} - 1) (\text{Cas} \otimes \text{id}) [\bar{U}\bar{\Psi}_{\#}, \partial_{\#}\bar{U}\bar{\mathcal{J}}_{\#, \#}(\mathbf{X}_{\#}\Xi_{\#})],$$

for some constant \check{C} . Other than the factor $(2\delta_{\#, \#} - 1)$, the summand above does not depend on $\#$ and since $\sum_{\# = 1, 2} (2\delta_{\#, \#} - 1) = 0$ it follows that the sum above vanishes as claimed. A similar argument takes care of the case of . \square

The computation of the renormalisation of the α_i components in the F system of equations mirrors the computations we have just done for the \bar{F} system with the one difference that the term $U\bar{\xi}_i$, which is the analogue of the term $U\xi_i$ that was part of $\bar{F}_{m, i}$, is included in $F_{\alpha, i}$. In particular, $\bar{\Upsilon}_{\alpha}^F[\tau]$ for $\tau \in \{\text{red triangle}, \text{red triangle with arrow}, \text{red square}, \text{red square with arrow}, \text{red square with arrow pointing left}, \text{red square with arrow pointing right}\}$ are given by formulas as in (7.34) and (7.35) with the following replacement

$$\bar{A} \mapsto B, \quad \bar{U} \mapsto U, \quad \bar{\Psi} \mapsto \Psi, \quad \bar{\mathcal{J}}(\mathbf{X}\Xi) \mapsto \mathcal{J}(\mathbf{X}\Xi)$$

$$\text{and } \bar{\Upsilon}_{\alpha, \#}^F[\text{red square with arrow pointing left}](\mathbf{A}) = \delta_{\#, \#} [[U\mathcal{J}^u\Psi_{\#}, h_{\#}], U\Xi_{\#}].$$

By using the renormalisation constants given in Lemma 7.17 and performing again computations of the type found in Lemmas 7.18 and 7.19, one obtains the following lemma.

Lemma 7.20 For $\mathbf{A} \in \bar{\mathcal{A}}$,

$$\sum_{\tau \in \bar{\Sigma}_{-}(R)} (\ell_{\text{BPHZ}}^{\delta, \varepsilon}[\tau] \otimes \text{id}) \bar{\Upsilon}_{\alpha, \#}^F[\tau](\mathbf{A}) = \lambda C_{\text{SYM}}^{\varepsilon} B_{\#} - \lambda \check{C}^{\varepsilon} h_{\#},$$

where $C_{\text{SYM}}^{\varepsilon}$ is as in (6.12).

Remark 7.21 Lemmas 7.18, 7.19, and 7.20, still hold if one replaces the first condition of Definition 7.14 by only requiring the invertibility of \mathbf{A}_u .

The main result of this section is the following proposition.

Proposition 7.22 *Fix any constants \mathring{C}_1 and \mathring{C}_2 and initial data $\bar{a} \in \Omega_\alpha^1$ and $g(0) \in \mathfrak{G}^\alpha$. Consider the system of equations*

$$\begin{aligned} \partial_t \bar{A}_i &= \Delta \bar{A}_i + \chi^\varepsilon * (\bar{g} \xi_i \bar{g}^{-1}) + \bar{C}_1^\varepsilon \bar{A}_i + \bar{C}_2^\varepsilon (\partial_i \bar{g}) \bar{g}^{-1} \\ &+ [\bar{A}_j, 2\partial_j \bar{A}_i - \partial_i \bar{A}_j + [\bar{A}_j, \bar{A}_i]], \quad \bar{A}(0) = \bar{a}, \end{aligned} \quad (7.38)$$

and

$$\begin{aligned} \partial_t B_i &= \Delta B_i + g \xi_i^\varepsilon g^{-1} + C_1^\varepsilon B_i + C_2^\varepsilon (\partial_i g) g^{-1} \\ &+ [B_j, 2\partial_j B_i - \partial_i B_j + [B_j, B_i]], \quad B(0) = \bar{a}, \end{aligned} \quad (7.39)$$

where \bar{g} and g are given by running the corresponding equations in (2.6) and (2.5) started with the same initial data $\bar{g}(0) = g(0)$ and we have defined the constants

$$\begin{aligned} \bar{C}_1^\varepsilon &= \mathring{C}_1 + \lambda C_{\text{SYM}}^\varepsilon, & \bar{C}_2^\varepsilon &= \mathring{C}_2 - \lambda \tilde{C}^{0,\varepsilon}, \\ C_1^\varepsilon &= \mathring{C}_1 + \lambda C_{\text{SYM}}^\varepsilon, & C_2^\varepsilon &= \mathring{C}_2 - \lambda \tilde{C}^\varepsilon. \end{aligned} \quad (7.40)$$

Then, \bar{A} and B converge in probability in Ω^{sol} to the same limit as $\varepsilon \downarrow 0$.

Proof. We claim that (7.38) is just the renormalised equation obtained via the reconstruction (with respect to $Z_{\text{BPHZ}}^{0,\varepsilon}$) of the fixed point problem (7.20). Since $Z_{\text{BPHZ}}^{0,\varepsilon}$ is not a smooth model, the justification of this claim goes via obtaining the corresponding renormalised equation for the model $Z_{\text{BPHZ}}^{\delta,\varepsilon}$ and then taking the limit $\delta \downarrow 0$ (which is justified by the convergence (7.24)).

We deploy [BCCH17, Thm. 5.7] and Proposition 5.65 to get the renormalised reconstruction of the equation (7.20). In terms of the indeterminates $\mathbf{A} = (\mathbf{A}_o)_{o \in \mathcal{E}}$ and nonlinearity \bar{F} , this amounts to *summing* the renormalised and reconstructed integral fixed point equations for the indeterminates $\mathbf{A}_{\mathbf{a}_i}$ and $\mathbf{A}_{\mathbf{m}_i}$ with nonlinearity \bar{F} , and recalling (7.11).

The claim then follows by using Lemma 7.13 and Remark 7.15 to allow us to go between $\text{Ad}_{\bar{g}}$ and $(\partial_i \bar{g}) \bar{g}^{-1}$ and \bar{U} and \bar{h}_i , then using the explicit computations of counter-terms in Lemmas 7.18 and 7.19, and then taking the limit $\delta \downarrow 0$ of renormalisation constants as given in Lemmas 7.16 and 7.17.

A similar argument shows that (7.39) is the renormalised equation obtained via the reconstruction (with respect to $Z_{\text{BPHZ}}^{0,\varepsilon}$) of the fixed point problem (7.19) with the minor differences that one is aiming for the renormalised and reconstructed integral fixed point equations for just the indeterminates $\mathbf{A}_{\mathbf{a}_i}$ with non-linearity F so the computations of Lemma 7.18 and 7.19 are replaced by that of Lemma 7.20.

We now turn to proving the statements concerning convergence in probability as $\varepsilon \downarrow 0$. We first show that the statement holds if, in the definition of Ω^{sol} , one replaced $\Omega_{\alpha,T}$ with $\mathcal{C}([0, T], C^{\alpha-1})$. The convergence of \bar{A} and B individually follow from the convergence of the models $Z_{\text{BPHZ}}^{0,\varepsilon}$ given in Lemma 7.11 and standard arguments using the continuity of the machinery of regularity structures as given in

[Hai14]. The statement that $d(\bar{A}, B) \rightarrow 0$ in probability as $\varepsilon \downarrow 0$ follows from the second estimate of (7.22) from the statement of Lemma 7.9. To obtain the control over models needed to apply this lemma it suffices to point out that by combining statements (7.23) and (7.24) of Lemma 7.11 we have, for any $p \geq 1$,

$$\sup_{\varepsilon \in (0,1]} \mathbf{E}[\|Z_{\text{BPHZ}}^{0,\varepsilon}\|_\varepsilon^p] < \infty. \quad (7.41)$$

To prove the desired statement for Ω^{sol} we first note that by using the same argument used in the proof of Theorem 2.4 at the end of Section 6.2.5 (namely, splitting into a linear part with more regular remainder) one can show \bar{A}, B both individually converge in probability in Ω^{sol} as $\varepsilon \downarrow 0$.

To show that $d(\bar{A}, B) \rightarrow 0$ in probability as $\varepsilon \downarrow 0$ we first fix $\alpha' \in (0, \alpha)$ and note that every ball in Ω_α is compact in $\Omega_{\alpha'}$, and thus also in $\mathcal{C}^{\alpha'-1}$. Since any two comparable Hausdorff topologies on a set which render it compact coincide, convergence in $\mathcal{C}^{\alpha'-1}$ with uniform bounds in Ω_α implies convergence in $\Omega_{\alpha'}$. Hence, since $|\bar{A}(t) - B(t)|_{\mathcal{C}^{\alpha'-1}} \rightarrow 0$ and $B(t)$ and $\bar{A}(t)$ a.s. stay bounded in a ball in Ω_α for each $t \in [0, T^*]$ as $\varepsilon \rightarrow 0$, we obtain $|\bar{A}(t) - B(t)|_{\alpha'} \rightarrow 0$. \square

Proof of Theorem 2.8. We first prove statement (i). Given $C \in \mathbf{R}$, which is assumed to be a real constant by Remark 2.7 and Assumption 6.1, we fix $\check{C}_1 = C - \lambda C_{\text{SYM}}$ and $\check{C}_2 = C + \lambda C_{\text{GSYM}}$. We then take the \bar{C} as claimed in the theorem as

$$\bar{C} \stackrel{\text{def}}{=} \lambda(\bar{C}_{\text{GSYM}} - C_{\text{GSYM}}).$$

With these choices and the definitions of (7.40), together with

$$\tilde{C}^\varepsilon - C_{\text{GSYM}} = o(1), \quad C_{\text{SYM}} - C_{\text{SYM}}^\varepsilon = o(1),$$

it follows that, as $\varepsilon \downarrow 0$,

$$C = C_1^\varepsilon + o(1) = C_2^\varepsilon + o(1) = \bar{C}_1^\varepsilon + o(1), \quad C - \bar{C} = \bar{C}_2^\varepsilon + o(1).$$

The desired statement then follows from Proposition 7.22.

We now prove (ii). Note that if χ is non-anticipative, then $\tilde{C}^{0,\varepsilon} = 0$ for every $\varepsilon > 0$ and so $\bar{C}_{\text{GSYM}} = 0$. It follows that $\bar{C} = -\lambda C_{\text{GSYM}} = -\lambda \lim_{\varepsilon \downarrow 0} \int dz \chi^\varepsilon(z)(K * K^\varepsilon)(z)$ and so the desired statement follows from Remark 6.16. \square

7.4 Construction of the Markov process

In this subsection, we prove Theorem 2.12. We begin with several lemmas.

Lemma 7.23 *Let $\alpha \in (\frac{2}{3}, 1]$ and $A, B \in \Omega_\alpha^1$. Then*

$$\left| \inf_{g \in \mathfrak{G}^{0,\alpha}} |B^g|_\alpha - \inf_{g \in \mathfrak{G}^{0,\alpha}} |A^g|_\alpha \right| \lesssim (1 + |A|_\alpha + |B|_\alpha) |A - B|_\alpha, \quad (7.42)$$

where the proportionality constant depends only on α .

Proof. As in the proof of Theorem 3.27, for $g \in \mathfrak{G}^{0,\alpha}$ we can write

$$A^g - B^g = ((A - B)^g - 0^g - (A - B)) + (A - B),$$

from which it follows by Lemmas 3.32 and 3.33 that

$$|A^g - B^g|_\alpha \lesssim (1 + |g|_{\alpha\text{-H\"{o}l}})|A - B|_\alpha, \quad (7.43)$$

where the proportionality constant depends only on α . Consider a minimising sequence $g_n \in \mathfrak{G}^{0,\alpha}$ for A . Then, by Proposition 3.35, $\limsup_{n \rightarrow \infty} |g_n|_{\alpha\text{-H\"{o}l}} \lesssim |A|_\alpha$, and thus by (7.43)

$$\inf_{g \in \mathfrak{G}^{0,\alpha}} |B^g|_\alpha - \inf_{g \in \mathfrak{G}^{0,\alpha}} |A^g|_\alpha \leq \limsup_{n \rightarrow \infty} |B^{g_n}|_\alpha - |A^{g_n}|_\alpha \lesssim (1 + |A|_\alpha)|A - B|_\alpha.$$

Swapping A and B and applying the same argument, we obtain (7.42). \square

Lemma 7.24 *Let $\lambda > 1$. Then there exists a measurable (Borel) selection $S : \mathfrak{D}_\alpha \rightarrow \Omega_\alpha^1$ such that $|S(x)|_\alpha \leq \lambda \inf_{A \in x} |A|_\alpha$ for all $x \in \mathfrak{D}_\alpha$.* \blacksquare

Proof. Consider the subset $Y \stackrel{\text{def}}{=} \{A \in \Omega_\alpha^1 \mid |A|_\alpha \leq \lambda \inf_{g \in \mathfrak{G}^{0,\alpha}} |A^g|_\alpha\}$, which is closed due to Lemma 7.23. In particular, Y is Polish and, by Lemma 3.40, the gauge equivalence classes in Y are closed. Finally, since $\pi^{-1}(\pi(U)) = \cup_{g \in \mathfrak{G}^{0,\alpha}} U^g$ is open for every open subset $U \subset \Omega_\alpha^1$, the conclusion follows by the Rokhlin–Kuratowski–Ryll–Nardzewski selection theorem [Bog07, Thm. 6.9.3]. \square

For the rest of the section, let us fix a non-anticipative mollifier χ and set $C = \bar{C}$, the constant from part (i) of Theorem 2.8. By a “white noise” we again mean a pair of i.i.d. \mathfrak{g} -valued white noises $\xi = (\xi_1, \xi_2)$ on $\mathbf{R} \times \mathbf{T}^2$.

Proof of Theorem 2.12. (i) By Lemma 7.24, there exists a measurable selection $S : \hat{\mathfrak{D}}_\alpha \rightarrow \hat{\Omega}_\alpha^1$ such that for all $x \in \hat{\mathfrak{D}}_\alpha$

$$|S(x)|_\alpha \leq 2 \inf_{a \in x} |a|_\alpha \quad (7.44)$$

and $S(\varpi) = \varpi$. Let ξ be white noise and let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by ξ .

Consider any $a \in \hat{\Omega}_\alpha^1$. We define a càdlàg Markov process $A : \mathbf{R}_+ \rightarrow \hat{\Omega}_\alpha^1$ and a sequence of stopping times $(\sigma_j)_{j=0}^\infty$ as follows. For $j = 0$, set $\sigma_0 = 0$ and $A(0) = a$. Consider now $j \geq 0$. If $\sigma_j = \infty$, then we set $\sigma_{j+1} = \infty$. Otherwise, if $\sigma_j < \infty$, suppose that A is defined on $[0, \sigma_j]$. If $A(\sigma_j) = \varpi$, then define $\sigma_j = \sigma_{j+1}$. Otherwise, define $\Theta \in \mathcal{C}([\sigma_j, \infty), \hat{\Omega}_\alpha^1)$ by $\Theta(t) = \Phi_{\sigma_j, t}(A(\sigma_j))$, where we used the notation $\Phi_{s,t}$ as in Definition 2.10, and set

$$\sigma_{j+1} = \inf\{t > \sigma_j \mid |\Theta(t)|_\alpha > 1 + 2 \inf_{g \in \mathfrak{G}^{0,\alpha}} |\Theta(t)^g|_\alpha\}.$$

We then define $A(t) = \Theta(t)$ for all $t \in (\sigma_j, \sigma_{j+1})$ and $A(\sigma_{j+1}) = S([\Theta(\sigma_{j+1})])$. Observe that (σ_j, σ_{j+1}) is a.s. non-empty due to Lemma 7.23, the condition (7.44),

and the continuity of Θ at σ_j . In fact, defining $M(t) = \inf_{g \in \mathfrak{G}^{0,\alpha}} |A(t)^g|_\alpha$, then by decomposing Θ into the SHE with initial condition $A(\sigma_j)$ and a remainder as in the proof of Theorem 2.4, we see that the law of $\sigma_{j+1} - \sigma_j$ depends only on $A(\sigma_j)$ and can be stochastically bounded from below by a strictly positive random variable depending only on $M(\sigma_j)$. In particular, if the quantity $T^* \stackrel{\text{def}}{=} \lim_{j \rightarrow \infty} \sigma_j$ is finite, then a.s. $\lim_{t \nearrow T^*} M(t) = \infty$. In this case, we have defined A on $[0, T^*)$, and then set $A \equiv \mathfrak{Q}$ on $[T^*, \infty)$. If $T^* = \infty$, then we have defined A on \mathbf{R}_+ and the construction is complete. Note that, in either case, a.s. $T^* = \inf\{t \geq 0 \mid A(t) = \mathfrak{Q}\}$. To complete the proof of (i), we need only remark that items 2 and 3 of Definition 2.10 are satisfied by the construction of $(\sigma_j)_{j=0}^\infty$ and the above discussion.

(ii) The idea of the proof is to couple any generative probability measure $\bar{\mu}$ to the law of the process A constructed in part (i). Consider a white noise $\bar{\xi}$ with an admissible filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$, a $\bar{\mathcal{F}}$ -stopping time σ , a solution $\bar{A} \in \mathcal{C}([s, \sigma], \Omega_\alpha^1)$ to the SYM driven by $\bar{\xi}$, and a gauge equivalent initial condition $A(s) = \bar{A}(s)^{g(s)}$. Remark that, by part (i) of Theorem 2.8, we can construct on the same probability space a stopping time τ , a time-dependent gauge transformation $g \in \mathcal{C}([s, \tau], \mathfrak{G}^{0,\alpha})$ (namely $g^{-1} = \bar{g}$, the solution to that component of (2.6) driven by \bar{A} started with initial data $\bar{g}(s) = g^{-1}(s)$) and a solution $A \in \mathcal{C}([s, \tau], \Omega_\alpha^1)$ to the SYM driven by the white noise $\xi \stackrel{\text{def}}{=} \text{Ad}_g \bar{\xi}$ such that $\bar{A}^g = A$ on $[s, \tau)$. Moreover, by the bound (3.26) in Proposition 3.35, $|g|_{\alpha\text{-Höl}}$ cannot blow-up before $|\bar{A}|_{\alpha\text{-gr}} + |A|_{\alpha\text{-gr}}$ does. Since $\Omega_{\alpha\text{-gr}}^1 \hookrightarrow \Omega\mathcal{C}^{0,\alpha-1}$ (see Section 3.3), and since by Theorem 2.4 we can start the SYM from any initial condition in $\Omega\mathcal{C}^\eta$, $\eta \in (-\frac{1}{2}, 0)$, it follows that we can take $\tau = \sigma \wedge T^*$ where T^* is the blow-up time of $|A|_\alpha$. Note also that g and ξ are adapted to the filtration generated by $\bar{\xi}$, and A is adapted to the filtration generated by ξ .

Consider $a, \bar{a} \in \hat{\Omega}_\alpha^1$ with $[a] = [\bar{a}]$ and a generative probability measure $\bar{\mu}$ on $D(\mathbf{R}_+, \hat{\Omega}_\alpha^1)$ with initial condition \bar{a} . Let $\bar{A} \in D(\mathbf{R}_+, \hat{\Omega}_\alpha^1)$ denote the corresponding process with filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$, white noise $\bar{\xi}$, and blow-up time \bar{T}^* as in Definition 2.10. It readily follows from the above remark and the conditions in Definition 2.10 that there exist, on the same probability space,

- a process $g: \mathbf{R}_+ \rightarrow \mathfrak{G}^{0,\alpha}$ adapted to $(\bar{\mathcal{F}}_t)_{t \geq 0}$, which is càdlàg on the interval $[0, \bar{T}^*)$ and remains constant $g \equiv 1$ on $[\bar{T}^*, \infty)$, and
- a Markov process $A \in D(\mathbf{R}_+, \hat{\Omega}_\alpha^1)$ constructed as in part (i) using the white noise $\xi \stackrel{\text{def}}{=} \text{Ad}_g \bar{\xi}$ such that $A = \bar{A}^g$ and $A(0) = a$.

(Specifically, the process g is constructed to have jumps in $[0, \bar{T}^*)$ only at the jump times of \bar{A} and A , and $\bar{g} = g^{-1}$ solves (2.6) driven by \bar{A} on its intervals of continuity.) In particular, the pushforwards $\pi_* \bar{\mu}$ and $\pi_* \mu$ coincide, where μ is the law of A .

To complete the proof, it remains only to show that for the process A from part (i) with any initial condition $a \in \hat{\Omega}_\alpha^1$, the projected process $\pi A \in \mathcal{C}(\mathbf{R}_+, \hat{\mathcal{D}}_\alpha)$ is Markov. However, this follows from the Markov property of A and from taking

$\bar{\mu}$ in the above argument as the law of A with initial condition $\bar{a} \sim a$. \square

Appendix A Symbolic index

We collect in this appendix commonly used symbols of the article, together with their meaning and, if relevant, the page where they first occur.

Symbol	Meaning	Page
$ \cdot _\alpha$	Extended norm on Ω	17
$\ \cdot\ _{\ell,\varepsilon}$	ε -dependent norms on regularity structure of degree ℓ	100
$\ \cdot\ _\varepsilon, d_\varepsilon$	ε -dependent seminorms and metrics on models	101
$ \cdot _{\gamma,\eta,\varepsilon}$	ε -dependent norms on modelled distributions	101
\mathcal{A}	Target space of the jet of the noise and the solution	72
$\tilde{\mathcal{A}}_-, \tilde{\mathcal{A}}_-$	Negative twisted antipode and its abstract version	71
\mathbf{A}^A	An element of \mathcal{A} describing the polynomial part of $\mathcal{A} \in \mathcal{H}$	76
Cas	Covariance of \mathfrak{g} valued white noise = quadratic Casimir	88
$\bar{C}^\varepsilon, \hat{C}^\varepsilon$	Renormalisation constants for stochastic YM equation	88
$C_{\text{SYM}}^\varepsilon, C_{\text{SYM}}$	Combination of renormalisation constants and its limit	88
E	A generic Banach space	16
\mathfrak{F}	Isomorphism classes of labelled forests	60
\mathbf{F}_V	The monoidal functor between SSet and Vec	53
G	Compact Lie group	29
\mathcal{G}_-	Renormalisation group	69
\mathfrak{g}	Lie algebra of G	29
\mathfrak{G}^α	α -Hölder continuous gauge transformations	29
$\mathfrak{G}^{0,\alpha}$	Closure of smooth functions in \mathfrak{G}^α	33
\mathcal{H}	Set of expansions with polynomial part and tree part	76
$\text{Hom}(\mathfrak{s}, \bar{\mathfrak{s}})$	Morphisms between two symmetric sets \mathfrak{s} and $\bar{\mathfrak{s}}$	48
$K^{(\varepsilon)}$	Kernel assignment for gauge transformed system	98
ℓ_{BPHZ}	BPHZ renormalisation character	71
\mathcal{M}_ε	The family of $K^{(\varepsilon)}$ -admissible models	98
Ω	Space of additive E -valued functions on \mathcal{X}	16
Ω_α	Banach space $\{A \in \Omega \mid A _\alpha < \infty\}$	17
$\Omega\mathcal{B}$	E -valued 1-forms with components in a function space \mathcal{B}	9
Ω_α^1	Closure of smooth E -valued 1-forms in Ω_α	26
\mathfrak{D}_α	Space of orbits $\Omega_\alpha^1 / \mathfrak{G}^{0,\alpha}$	33
$\mathcal{P}(A)$	Powerset of a set A	55
\mathbf{p}^*	Functor from SSet $_{\mathfrak{L}}$ to TStruc $_{\bar{\mathfrak{L}}}$	56
$\mathcal{C}^\infty(B)$	Space of smooth functions from \mathcal{A} to B	72

Symbol	Meaning	Page
$\mathring{\mathbb{Q}}$ (resp. \mathbb{Q})	The set of choices of RHS of SPDE (resp. obeying R)	72
ϱ	Distance function on \mathcal{X}	17
R	Subcritical, complete rule	64
\mathfrak{s}	A generic symmetric set	47
$\mathbb{S}\text{Set}_{\mathfrak{L}}$	The category of symmetric sets with types \mathfrak{L}	48
$\mathbb{T}\text{Struc}$	Category of typed structures, with objects of form $\prod_{\alpha \in \mathcal{A}} \mathfrak{s}_{\alpha}$	54
$\langle \tau \rangle$	The symmetric set for a labelled rooted tree τ	59
\mathfrak{T}	Isomorphism classes of labelled trees	59
$\mathfrak{T}(R)$	Trees strongly conforming to R	65
$\mathfrak{T}_{-}(R)$	Negative degree unplanted trees in $\mathfrak{T}(R)$ with $n(\varrho) = 0$	65
$\mathfrak{T}, \mathfrak{F}$	Our abstract regularity structures	65
\mathcal{T}, \mathcal{F}	Vector spaces for concrete regularity structure	66
$V^{\otimes \mathfrak{s}}$	Symmetric tensor product determined by symmetric set \mathfrak{s}	51
\mathcal{X}	Set of line segments	16
Ξ_i	Symbol for noise, defined as $\mathcal{J}_{(i,0)}(\mathbf{1})$ for $i \in \mathfrak{L}_{-}$	84
$\Upsilon, \bar{\Upsilon}$	Maps describing coherence of expansions	79

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