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Abstract

This study provides a systematic and unified approach for constructing exact and static replications for exotic options, using the theory of integral equations. In particular, we focus on barrier-type options including standard, double and sequential barriers. Our primary approach to static options replication is the DEK method proposed by Derman et al. (1995). However, our solution approach is novel in the sense that we study its continuous-time version using integral equations. We prove the existence and uniqueness of hedge weights under certain conditions. Further, if the underlying dynamics is time-homogeneous, then hedge weights can be explicitly found via Laplace transforms. Based on our framework, we propose an improved version of the DEK method. This method is applicable under general Markovian diffusion with killing.

KEYWORDS: Static hedging, Integral equations, Markovian diffusion with killing, Barrier options, Exotic options

1 Introduction

The pricing principle via dynamic replication of Black and Scholes (1973) provides the rationale of dynamic hedging in addition to option pricing formulae. This dynamic hedging, however, has long been known to yield unsatisfactory outcomes especially for exotic options, which urged academics and practitioners to search for alternative static hedging methods. See Derman and Taleb (2005) for more details. For instance, the so called strike-spread approach of Carr et al. (1998) uses vanilla options with different strikes and the same maturity in order to replicate a target exotic option.

The underlying philosophy in this paper has been shared by many researchers and practitioners for the past two decades. Even textbooks such as Hull (2015) introduce the boundary matching approach (the DEK method hereafter) of Derman et al. (1995), who sparked a stream of literature. Later, Fink (2003) and Nalholm and Poulsen (2006) extended the DEK method for asset price

dynamics with random jumps and stochastic volatility. Chung et al. (2010) increased the performance of the DEK method by matching thetas of a target barrier option and a hedging portfolio. The scope of target options has also been enlarged. Chung and Shih (2009); Ruas et al. (2013) used calendar-spread approaches for American options whereas Chung et al. (2013a,b); Nunes et al. (2015) did for American barriers and Dias et al. (2015) for double barriers. Parisian options, which are classified as occupation time derivatives in Broadie and Detemple (2004), are added to the list (Kim and Lim, 2016). More recently, Kim and Lim (2019) proposed a recursive method for autocallable structured products based on the results developed in this paper.

Albeit the above achievements and a recent growing interest, there has been no detailed investigation of the theoretical validation of the DEK method such as its convergence or error analysis. One exception is Akahori et al. (2017) where the authors studied higher order semi-static hedges for American-style options and barrier options together with convergence properties with first and second order hedging errors. To answer these non-trivial issues, we propose a new systematic approach to constructing an *exact* static hedge for a wide class of financial products under a general Markovian diffusion with killing. This approach can be thought of as a continuous version of the DEK method, that provides a theoretical justification of the DEK method and enhances our understanding of hedging problems beyond a discrete-time model. The key feature of our approach is the use of integral equations whose rich theory provides an excellent vehicle for characterizing and quantifying static hedging portfolios. More specifically, we express the time-t value of a target option in terms of continuum of more basic options such as vanilla calls: for $0 \le t \le T$,

$$\Psi(t,T,S) = \int_0^{T-t} \mathbf{w}(u) C(t, T-u, S) \mathrm{d}u \tag{1}$$

where $\Psi(t,T,S)$ is the time-t value of a target option with maturity T and $C(t,T-u,S)$ is the time-t value of a hedging instrument with maturity $T-u$ and asset price S. The portfolio on the right hand side shall be constructed in a way that it matches the option value not only at 0 but also at any time t until maturity. This expression shows us how to construct an exact replicating portfolio, that is, we purchase $\mathbf{w}(u)\mathrm{d}u$ units of the hedging instrument with maturity $T-u$ for each u between 0 and T. The "weight" function $\mathbf{w} : [0,T] \to \mathbb{R}$ will be characterized via a certain integral equation which is based on the boundary information of the target option. Since vanilla options can be analytically computed under most popular underlying asset dynamics, our analytical representation reduces the complexity of hedging and pricing of exotic options down to that of vanilla options. Our main contributions in this paper are as follows:

- We establish (1) for a wide class of exotic options by imposing boundary matching conditions, which result in associated integral equations for $\mathbf{w}(\cdot)$. Main examples are standard single barrier options and barrier options of exotic type such as double barriers or sequential barriers.

- The existence and uniqueness of $\mathbf{w}(\cdot)$ is verified under certain conditions. To do this, we study the associated Volterra integral equation of the second kind and generalized Abel integral equations.

- Analytic expressions of Ψ and $\mathbf{w}$ are obtained by computing their Laplace transforms under the condition that the underlying asset price dynamics is time-homogeneous.

- Based on our framework, we propose a new variant of the DEK method that outperforms existing techniques. Furthermore, we devise an explicit method of evaluating hedging errors.

Before we proceed, it is worth noting that there has been another stream of literature on exact static hedges, called the strike-spread approach. In this case, combinations of basic options with the same maturity but different strikes can replicate some target exotic options as long as the asset price dynamics satisfies a certain symmetry condition. For instance, Carr and Chou (1997) and Carr et al. (1998) constructed an exact static hedging portfolio for single-barrier options under the Black-Scholes model and a symmetric local volatility model, respectively. Recently, Carr and Nadtochiy (2011) extended this idea to general time-homogeneous diffusion models for standard barrier options. In this case, the European payoff of a hedging instrument is not a vanilla type in general, leading to the approximations with vanilla options in practice. Funahashi and Kijima (2016) considered the problem of static hedging under the symmetrized volatility model, but the stringent assumptions on the volatility function in this paper or Carr et al. (1998) are inconsistent with market behaviors such as the leverage effect or the implied volatility skew. It is discussed later in the paper that the static hedge solution in Carr et al. (1998) can be represented as a solution to the integral equation based approach. In this sense, our proposal can be considered as a unified framework.

The paper is organized as follows. Section 2 presents a brief description of the asset price model and hedging instruments. Section 3 explains static options replication via integral equations. Also, we present some sufficient conditions for the existence and uniqueness of our static hedging portfolios. In the next section, such conditions are verified under mild assumptions on implied volatilities. In Section 5, we discuss applications of our framework for computing analytic solutions for the weight function, and designing a new variant of the DEK method. We give some concluding remarks in Section 6. In order to deliver our main results in a compact manner, we place all proofs regarding the JDCEV model and case studies in the appendix.

2 Preliminaries

2.1 The Model

Underlying assumptions are, first, the market is frictionless and there is no arbitrage and, second, equity holders do not receive any recovery in the event of default unless stated otherwise. The defaultable asset price is described by S_t for $t < \zeta$, and is sent to a cemetery state Δ, defined as zero, for $t \geq \zeta$ where ζ is a random time of default. Moreover, the pre-default asset price S_t is modeled as the following diffusion process under the risk-neutral measure $\mathbb{Q}$:

$$\frac{\mathrm{d}S_t}{S_t} = [r - q + \lambda(S_t, t)]\mathrm{d}t + \sigma(S_t, t)\mathrm{d}W_t \tag{2}$$

where $S_0 > 0$, the risk free interest rate $r \geq 0$, the continuous dividend yield $q \geq 0$, instantaneous volatility function $\sigma(S_t, t)$, default intensity function $\lambda(S_t, t)$ and W_t is a standard Wiener process defined under measure $\mathbb{Q}$ generating the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$. For notational convenience, we set $q = 0$ without loss of any generality. Default can occur either at the first hitting time of zero, $\tau_0 = \inf\{t \geq 0, S_t = 0\}$ or by a jump to default. This random time of jump to default $\widetilde{\zeta}$ is modeled by

$$\widetilde{\zeta} = \inf\left\{t \geq 0 : \int_0^t \lambda(S_u, u)du \geq \mathcal{E}\right\}.$$

where $\mathcal{E}$ is an exponential random variable with mean 1 and independent of $\{W_t, t \geq 0\}$. Therefore, the default time ζ is given by the smaller of the two, $\zeta = \tau_0 \wedge \widetilde{\zeta}$. Lastly, we introduce a default indicator process $\{D_t = 1_{t>\zeta}, t \geq 0\}$ generating the filtration $\mathbb{D} = \{\mathcal{D}_t, t \leq 0\}$ and an enlarged filtration $\mathbb{G} = \{\mathcal{G}_t, t \geq \leftrightarrows 0\}, \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$. We note that although it is one-dimensional, this setting encompasses important and practically useful specifications, such as local volatility models, that capture empirical features of financial markets.

2.2 Hedging Instruments

In our construction of hedging portfolios, we use European calls or puts. Binary options can also be used. Differently from the classical Black-Scholes model, a jump-to-default event needs to be separately handled particularly for put options. Following Carr and Linetsky (2006), we see that the payoff of put option $(K - S_T)^+$ with strike K can be decomposed into two parts, namely the put option part with zero recovery upon default and a recovery payment K at the option maturity if a jump-to-default event occurs.

Our notation is summarized in Table 1. For notational convenience, we suppress the dependence on r, σ, or λ when no confusion occurs. The time sensitivities of hedging instruments are defined

Table 1: Summary of notation. Put prices with zero recovery upon default are denoted by the subscript 0.

symbol	explanation
C^{E}, P^{E} (P_0^{E})	price of European call and put
C^{bin}, P^{bin} (P_0^{bin})	price of binary call and put
Θ^{C}, Θ^{P} (Θ_0^{P})	theta of European call and put
$\Theta^{\mathrm{C\cdot bin}}$, $\Theta^{\mathrm{P\cdot bin}}$ $(\Theta_0^{\mathrm{P\cdot bin}})$	theta of binary call and put
v_D	price of payment 1 at maturity upon default

as well. Then, we have the following relationships:

$$\begin{aligned} P^{\mathrm{E}}(t,T,S;K) &= P_0^{\mathrm{E}}(t,T,S;K) + Kv_D(t,T,S), \\ P^{\mathrm{bin}}(t,T,S;K) &= P_0^{\mathrm{bin}}(t,T,S;K) + v_D(t,T,S) \end{aligned}$$

where t is the current time, T is the option maturity, S is the stock price at t, and the option strike K under the assumption that default has not occurred by time t. The European put, binary put with no recovery $P_0^{\mathrm{E}}(t,T,S;K)$, $P_0^{\mathrm{bin}}(t,T,S;K)$ and one dollar recovery paid at the maturity upon default $v_D(t,T,S)$ are equal to

$$\begin{aligned} P_0^{\mathrm{E}}(t,T,S;K) &= \mathsf{E}\left[e^{-r(T-t)}(K-S_T)^+\mathbf{1}_{\{\zeta>T\}}\middle|\mathcal{G}_t\right] \\ P_0^{\mathrm{bin}}(t,T,S;K) &= \mathsf{E}\left[e^{-r(T-t)}\mathbf{1}_{\{S_T<K,\zeta>T\}}|\mathcal{G}_t\right] \\ v_D(t,T,S) &= \mathsf{E}\left[e^{-r(T-t)}\mathbf{1}_{\{\zeta\le T\}}\middle|\mathcal{G}_t\right]. \end{aligned}$$

3 Boundary Matching Approach

3.1 Integral Equations

The purpose of this paper is to find exact hedging portfolios for exotic options. Although our approach can be applied to more general types, at this stage, we restrict our presentation to up-and-in barrier options whose prices are denoted by $\Psi(t,T,S_t;\mathbb{U})$. Here T is the option maturity, S_t is the asset price at time t, and $\mathbb{U} := \{U_s\}_{t\le s\le T}$ is the barrier level where U_s is a continuous and deterministic function in s. Upon a knock-in event at $\tau := \inf\{s>0 : S_s = U_s\}$, the up-and-in barrier option has a value function $v(\tau,T,U_\tau)$ along the barrier, which is pre-specified by a contract:

$$\Psi(t,T,S_t;\mathbb{U}) = \mathsf{E}\left[e^{-r(\tau-t)}v(\tau,T,U_\tau)\mathbf{1}_{\{\tau\le T,\zeta>\tau\}}\middle|\mathcal{G}_t\right] \tag{3}$$

for $t \leq \tau \wedge \zeta$ and $S_t < U_t$. The equation (3) is useful in that it covers a variety of exotic options such as American options and exotic barrier options (e.g., general knock-in barrier options, knock-in knock-out options, and sequential barriers).

For example, a standard up-and-in barrier call is turned into a European call with time-to-maturity $T - \tau$, asset price U and a pre-specified strike K at τ. This makes $v(\tau, T, U)$ equal to $C^{\mathrm{E}}(\tau, T, U; K)$ and the time-t price of the barrier call is given by

$$\Psi(t, T, S_t; U) \quad = \quad \mathsf{E}\left[e^{-r(\tau - t)} C^{\mathrm{E}}(\tau, T, U; K) \mathbf{1}_{\{\tau \leq T, \zeta > \tau\}} \middle| \mathcal{G}_t\right].$$

Here, $\mathbb{U}$ is simplified to U for constant barriers. American put options are another example. Provided that the early exercise boundary $\mathbb{U}$ is given, $v(\tau, T, U_\tau)$ in (3) is replaced by the intrinsic value $K - U_\tau$. The boundary $\mathbb{U}$ can also be computed via the so called *smooth pasting condition*. We leave this extension as a separate topic to investigate in future in order to focus on a clear delivery of our idea.

The above up-and-in barrier call will be statically hedged by using European calls or binary calls. These hedging instruments have the same strike U and continuum of maturities from 0 to T. The function C in (1) is now written as $C(0, T - u, S_0; U)$. The central idea of boundary matching is to match values of the target option and the hedging portfolio along the barrier as well as at the option maturity.

Theorem 1 *Let $\Psi(t, T, S_t; U)$ be the time-t value of the up-and-in barrier option. Assume that the function $v(t, T, U)$ is continuous on $t \in [0, T]$ and that $v(T, T, U) = 0$. Then, the option price for $S_0 < U$ is given by*

$$\Psi(0, T, S_0; U) = \int_0^T \mathbf{w}(u) C(0, T - u, S_0; U) \mathrm{d}u$$

provided that there is a solution $\mathbf{w}(\cdot)$ to the following Volterra integral equation

$$\int_0^t \mathbf{w}(u) C(T - t, T - u, U; U) \mathrm{d}u = v(T - t, T, U), \quad 0 \leq t \leq T \tag{4}$$

where $C(t, s, S; K)$ is the time-t value of a European call or binary call with maturity s, asset price S at time t , and strike K.

Proof: Suppose that we have European calls or binary calls for all maturities in $(0, T]$. The hedging portfolio consists of $\mathbf{w}(u)\mathrm{d}u$ number of calls with maturity $T - u$, for $u \in [0, T)$. The price of each call is $C(0, T - u, S_0; U)$.

For each sample path, we have three possibilities. Firstly suppose $\zeta < \min\{T, \tau\}$ where ζ is the default time and $\tau = \inf\{t > 0 : S_t = U\}$. Then, both the target option and the replicating

portfolio expire worthless. Secondly suppose $T < \min\{\zeta, \tau\}$. Without knock-in, the target option expires worthless whereas the calls in the replicating portfolio never give positive payoffs because their strikes are U.

Lastly suppose $\tau \leq \min\{T, \zeta\}$. At this moment, the calls in the replicating portfolio have values $\mathbf{w}(u)C(\tau, T-u, U; U)\mathrm{d}u$ for $0 \leq u \leq T-\tau$. Other calls have expired worthless at τ. On the other hand, the target option has the value $v(\tau, T, U)$. Since $\mathbf{w}(\cdot)$ is assumed to satisfy (4), the replicating portfolio and the barrier option give the same payoff at τ. We note that the condition $v(T, T, U) = 0$ makes this equivalence valid even when $\tau = T$ or $t = 0$.

Consequently, the no-arbitrage principle implies that the time-0 value of the hedging portfolio must be equal to the barrier option price. ■

We note that the existence of a solution to (4) implies the continuity of $v(t, T, U_t)$ for $t \in [0, T]$. This result can be further extended to

- other types of barrier options such as down-barrier with some recovery value and knock-out cases; see Section 3.4,
- relaxation of $v(T, T, U) = 0$; see Section 3.4,
- non-constant barrier level (time-dependent boundaries),
- exotic barrier options; see Appendix D.

It is worth pointing out that the above representation shows the linkage between barrier options (with general payoff) and vanilla calls. The main equation in Theorem 1 can be interpreted as providing not only a static hedging portfolio, but also the "market consistent" price of the target barrier option in the sense that the prices of vanilla options are directly utilized.

In comparison with the existing literature on static hedge, the key difference in our approach is the use of integral equations. There is a rich theory of integral equations, and it can be shown that the boundary matching condition (4) is converted into a Volterra integral equation of the second kind or an Abel integral equation, depending on the choice of the hedging instrument. To handle these associated integral equations, we present some useful conditions for the existence of solutions.

Definition 1 *A function $\mathcal{K}(s, t)$ is said to be* weakly singular *if*

$$\mathcal{K}(s, t) = \frac{k(s, t)}{(t-s)^{\alpha}}$$

where $0 < \alpha < 1$ and $k(s, t)$ is continuous on $\{(s, t) | 0 \leq s \leq t \leq T\}$.

When binary options are utilized for constructing a static hedging portfolio in Theorem 1, (4) can be reduced to a Volterra integral equation of the second kind by differentiating with respect to time t. The following theorem provides sufficient conditions of the existence and uniqueness of a solution when the kernel is weakly singular.

Lemma 1 (Andras (2003)) *Consider the following Volterra integral equation of the second kind:*

$$f(t) = g(t) + \int_0^t f(s)\mathcal{K}(s,t)\mathrm{d}s \quad 0 \le t \le T. \tag{5}$$

If the kernel $\mathcal{K}(s,t)$ is weakly singular and $g(t)$ is continuous on $[0,T]$, then there exists a unique continuous solution $f(t)$ on $[0,T]$.

When European calls are used as hedging instrument in Theorem 1, the condition (4) becomes an Abel integral equation. However, existing results for Abel integral equations are not directly applicable to our problem. Thus, we modify existence conditions. Due to its technical nature, we defer the proof to Appendix A.

Lemma 2 *Consider the following generalized Abel integral equation:*

$$\int_0^t \frac{h_1(s,t)}{(t-s)^\alpha} f(s)\mathrm{d}s + \int_0^t h_2(s,t)f(s)\mathrm{d}s = g(t), \quad 0 \le t \le T \tag{6}$$

with $0 < \alpha < 1$. If

(i) $h_1(t,t) \neq 0$ for all t,

(ii) $h_i(s,t)$ are continuous for $0 \le s \le t \le T$, and $(\partial/\partial t)h_i(s,t)$ are weakly singular for $i = 1, 2$,

(iii) $g(t)$ is continuously differentiable on $[0,T]$,

then there exists a unique continuous solution $f(t)$ on $(0,T]$. If $g(0) = 0$, then $f(t)$ is continuous on $[0,T]$.

3.2 Remarks on Alternative Idea

There are alternative methods in constructing static hedging portfolios. The so called *strike-spread approach* requires standard options with continuum of strikes while the option maturities are equal to the maturity of the target option. This is in contrast with the static hedging portfolio constructed in Theorem 1 where we have continuous maturities but a constant strike.

In this subsection, we consider a standard down-and-in barrier call option in order to compare the boundary matching approach in the literature. The proof of the next result is similar to that of

Theorem 1. It can also be shown as a simple consequence of the main results of Carr and Nadtochiy (2011) from which we see an interesting relationship between the strike-spread approach and the theory of integral equations.

Proposition 1 *Let us consider a down-and-in barrier call option with maturity T, barrier level L. Assume that the payoff is the standard European call with strike $K > L$. Then, the option price for $S_0 > L$ is given by*

$$\int_0^L \mathbf{w}(u) P_0^{\mathrm{E}}(0, T, S_0; u) \mathrm{d}u$$

provided that there is a solution $\mathbf{w}(\cdot)$ to the following Fredholm integral equation of the first kind:

$$\int_0^L \mathbf{w}(u) P_0^{\mathrm{E}}(t, T, L; u) \mathrm{d}u = C^{\mathrm{E}}(t, T, L; K), \quad 0 \le t \le T.$$

Suppose that the underlying stock follows the Black-Scholes dynamics and that the risk-free rate is zero (Carr et al., 1998). Note that $P_0^{\mathrm{E}} = P^{\mathrm{E}}$ under the Black-Scholes model. If we allow generalized functions for solutions to the target equation, then one solution to the Fredholm equation above is

$$\mathbf{w}(u) = \frac{K}{L} \delta\left(\frac{L^2}{K}\right)$$

where $\delta(\cdot)$ is the Dirac delta function. Indeed,

$$\begin{aligned}
\int_0^L \mathbf{w}(u) P^{\mathrm{E}}(t, T, L; u) \mathrm{d}u &= \frac{K}{L} P^{\mathrm{E}}\left(t, T, L; \frac{L^2}{K}\right) \\
&= \frac{K}{L}\left[\frac{L^2}{K}\Phi\left(-d_2\left(L, \frac{L^2}{K}\right)\right) - L\Phi\left(-d_1\left(L, \frac{L^2}{K}\right)\right)\right] \\
&= L\Phi\left(d_1(L, K)\right) - K\Phi\left(d_2(L, K)\right) \\
&= C^{\mathrm{E}}(t, T, L; K).
\end{aligned}$$

Here, Φ is the cumulative distribution function of a standard normal distribution, and

$$d_1(x, k) = \frac{\log(x/k) + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(x, k) = d_1(x, k) - \sigma\sqrt{T-t}.$$

In the third equality, we use the relationships $d_1(x, k) = -d_2(x, x^2/k)$ and $d_2(x, k) = -d_1(x, x^2/k)$.

As a consequence, the barrier option price is given by

$$\int_0^L \mathbf{w}(u) P^{\mathrm{E}}(0, T, S_0; u) \mathrm{d}u = \frac{K}{L} P^{\mathrm{E}}\left(0, T, S_0; \frac{L^2}{K}\right).$$

If we consider a down-and-out barrier call option with barrier L and strike K, then its price is equal to the price of a European call minus the down-and-in barrier call price, which yields

$$C^{\mathrm{E}}(0, T, S_0; K) - \frac{K}{L} P^{\mathrm{E}}\left(0, T, S_0; \frac{L^2}{K}\right).$$

This coincides with the formula (7) in Carr et al. (1998).

It was already noted, e.g. Funahashi and Kijima (2016), that the approach of Carr et al. (1998) is not extendable even to the CEV model. On the other hand, Proposition 1 provides more flexibility when it comes to model selection. However, one caveat is that Fredholm integral equations are typically ill-posed. And this requires techniques different from what we do in this paper, for instance, see Carr and Nadtochiy (2011).

3.3 Existence and Uniqueness of Static Hedging Portfolio

In this subsection, we record conditions for the existence and uniqueness of a static hedging portfolio, as a solution to (4) when using European call or binary call as a hedging instrument. Since binary calls or puts have discontinuous payoffs when the asset price at the maturity is at strike, $S_T = K$, we define $C^{\text{bin}}(T,T,K;K) = \lim_{t\to T} C^{\text{bin}}(t,T,K;K) = 0.5$ as in Lemma 3 in the appendix. This technical assumption is used for the proofs throughout this paper.

Theorem 2 *Let $\Theta(t,T,U) = \frac{\partial v(t,T,U)}{\partial t}$ be the time sensitivity of the value function $v(t,T,U)$. Assume $v(T,T,U) = 0$.*

(i) Suppose that the hedging instrument is binary call and that $\Theta^{\text{C·bin}}(T-t,T-u,U;U)$ is weakly singular in (u,t). If $\Theta(T-t,T,U)$ is continuous on $t \in [0,T]$, then there exists a unique solution $\mathbf{w}$ to (4) that is continuous on $[0,T]$.

(ii) Suppose that the hedging instrument is European call and that $\Theta^{\text{C}}(T-t,T-u,U;U)$ is of the form $\frac{h_1(u,t)}{(t-u)^\alpha} + h_2(u,t)$ where α, h_1, h_2 satisfy the conditions of Lemma 2. If $\Theta(T-t,T,U)$ is continuously differentiable on $t \in [0,T]$, then there exists a unique solution $\mathbf{w}$ to (4) that is continuous on $(0,T]$.

In (ii), if $\Theta(T,T,U) = 0$, then $\mathbf{w}$ is continuous on $[0,T]$.

Proof: *Case (i)* Based on Theorem 1, it is enough to find $\mathbf{w}(u)$, a solution to (4)

$$\int_0^t \mathbf{w}(u) C^{\text{bin}}(T-t,T-u,U;U)\mathrm{d}u = v(T-t,T,U), \quad 0 \le t \le T.$$

Such a solution also satisfies the following Volterra equation of the second kind, which is obtained by differentiating the above equation with respect to t:

$$\frac{\mathbf{w}(t)}{2} - \int_0^t \mathbf{w}(u)\Theta^{\text{C·bin}}(T-t,T-u,U;U)\mathrm{d}u = -\Theta(T-t,T,U). \tag{7}$$

Since this integral equation has a weakly singular kernel and the right hand side is continuous on $[0,T]$ by assumption, Lemma 1 guarantees the existence and uniqueness of the solution $\mathbf{w}(u)$.

Case (ii) For vanilla call, (4) now reads

$$\int_0^t \mathbf{w}(u) C^{\mathrm{E}}(T-t, T-u, U; U)\mathrm{d}u = v(T-t, T, U), \quad 0 \le t \le T.$$

Differentiating this equation with respect to t, we get an Abel integral equation:

$$\int_0^t \mathbf{w}(u) \Theta^{\mathrm{C}}(T-t, T-u, U; U)\mathrm{d}u = \Theta(T-t, T, U). \tag{8}$$

The assumption on Θ^{C} allows us to apply Lemma 2, from which the desired conclusion easily follows. ■

Theorem 2 can be applied to standard up-and-in barrier put options with strike $U > K$. In this case, $v(t,T,U) = P^{\mathrm{E}}(t,T,U;K)$ and thus $v(T,T,U) = 0$. When $U < K$, the option is of reverse barrier type and $v(T,T,U) = K - U$ is nonzero. This requires a different treatment, which is the topic of the next subsection.

3.4 Reverse Barrier Options and Others

Reverse Barriers. The main results developed so far require that $v(T,T,U) = 0$. This condition rules out the possibility of applications for important exotic options such as reverse barrier options. When the barrier U is set in-the-money rather than out-of-the-money, we call the barrier option a reverse barrier option. In other words, the option is either knocked-in or knocked-out when it is in-the-money. For instance, standard up-and-in barrier put is of reverse type if the barrier is less than the strike. Likewise, standard down-and-in call is of reverse type if the barrier is greater than the strike.

It is well known that it is difficult to hedge reverse barrier options in dynamic hedging. We refer the reader to p.347 in Taleb (1997) for more information. In order to extend our boundary matching approach to reverse barrier options, we further utilize American binary options [1] as additional hedging instruments. In more detail, static hedging of reverse barrier option Ψ can be done similarly as in Theorem 1. The only difference is that we now utilize American binary calls C^{A}:

$$C^{\mathrm{A}}(t,T,S_t;U) = \mathsf{E}\left[e^{-r(\tau-t)}\mathbf{1}_{\{\tau \le T, \zeta > \tau\}}\middle|\mathcal{G}_t\right]$$

[1] The payoff of American binary option at the strike is fixed, but the time of the payoff is random. Carr and Picron (1999) and Akahori et al. (2017) showed that this timing risk can be statically hedged with European options.

for $t \leq \tau$ and $S_t < U$. The barrier option price then reads

$$\Psi(0,T,S_0;U) = \int_0^T \mathbf{w}(u) C(0,T-u,S_0;U)\mathrm{d}u + \Psi^* C^{\mathrm{A}}(0,T,S_0;U)$$

where C is the price of European call or binary call and $\Psi^* = v(T,T,U)$. And the weight function $\mathbf{w}(\cdot)$ is a solution of the following integral equation:

$$\int_0^t \mathbf{w}(u) C(T-t,T-u,U;U)\mathrm{d}u = v(T-t,T,U) - \Psi^*, \quad 0 \leq t \leq T. \tag{9}$$

Since the American binary call gives the option holder 1 as soon as the stock price hits the barrier level U, the above construction makes the values of the target option and the hedging portfolio match along the barrier U and at the maturity T.

Theorem 3 *Let $\Theta(t,T,U) = \frac{\partial v(t,T,U)}{\partial t}$ be the time sensitivity of the value function $v(t,T,U)$. Assume $\Psi^* = v(T,T,U)$ is nonzero. Then, the same conclusions in Theorem 2 hold for a solution $\mathbf{w}(u)$ to (9).*

Down-and-in Barriers. When an option has a down-and-in feature, we have a little more complications due to the possibility of (zero, partial, or full) recovery just like we have for European or binary put options. Also, hedging instruments in hedging portfolios are European or binary puts instead of calls. The price of a down and in barrier is written as for $t \leq \zeta \wedge T$ and $S_0 > L$,

$$\Psi(0,T,S_0;L) = \mathsf{E}\left[e^{-r\tau} v(\tau,T,L)\mathbf{1}_{\{\tau \leq T, \zeta > \tau\}} + e^{-rT} R \mathbf{1}_{\{\zeta \leq T\}}\right],$$

where L is a down barrier level and $\tau := \inf\{t > 0 : S_t = L\}$. The second term represents the recovery value since default activates the knock-in event.

Suppose that the target option Ψ has zero recovery upon default($R = 0$). The proof of Theorem 1 can be easily modified by using put prices with zero recovery P_0 instead of call prices C. Then, the option price Ψ can be written as

$$\Psi(0,T,S_0;L) = \int_0^T \mathbf{w}(u) P_0(0,T-u,S_0;L)\mathrm{d}u$$

and the weight function $\mathbf{w}(\cdot)$ is a solution to the following integral equation:

$$\int_0^t \mathbf{w}(u) P_0(T-t,T-u,L;L)\mathrm{d}u = v(T-t,T,L), \quad 0 \leq t \leq T \tag{10}$$

where L is the barrier level and $v(T,T,L) = 0$ is assumed.

If the hedging instrument is not P_0 but P (with full recovery), then the relationships $P^{\mathrm{E}} = P_0^{\mathrm{E}} + K v_D$ and $P^{\mathrm{bin}} = P_0^{\mathrm{bin}} + v_D$ can be used. Here, v_D is the value of payment 1 at maturity

upon the default of the reference entity (see Table 1). Recall that this v_D is one of three building block claims in Carr and Linetsky (2006). Hence, a static hedging portfolio consists of European or binary puts and credit derivatives in this case.

Similarly, if Ψ has a recovery component upon default, then the hedging portfolio must take into account such possibilities as well. For instance, a standard down-and-in put with strike K ($< L$) can be hedged by $\int_0^T \mathbf{w}(u)P_0(0, T-u, S_0; L)\mathrm{d}u + Kv_D(0, T, S_0)$ where $\mathbf{w}$ solves (10) in case that the target option pays K at maturity upon default. Lastly it should be noted that American binary puts can be incorporated for reverse barrier options with down-and-in features.

Knock-out Barriers. There are equally many barrier options with knock-out features instead of knock-in. The best way to deal with this case is to use the in-and-out parity. For instance, the price of a standard up-and-out put Ψ_{out} with zero recovery upon default is given by

$$\Psi_{\text{out}}(0, T, S_0; U) = P_0^{\text{E}}(0, T, S_0; K) - \Psi_{\text{in}}(0, T, S_0; U).$$

Here, U is the barrier level and K is the strike of the embedded European put.

4 Model Specification

In this section, we provide some concrete analysis in order to show that the conditions of Theorems 2 and 3 can indeed hold true. In particular, the JDCEV model will be the base model when we solve for the weight function $\mathbf{w}(\cdot)$ in Section 5.1 and when we investigate numerical techniques in Section 5.2. Additionally, extensions to other possible candidate models are discussed.

4.1 JDCEV Model

Let us briefly review the JDCEV model proposed by Carr and Linetsky (2006). To make the model consistent with market behaviors such as leverage effect, implied volatility skew and the positive relationship between credit default swap spreads and equity volatilities, $\sigma(S,t)$ and $\lambda(S,t)$ are specified by

$$\begin{aligned} \sigma(S,t) &= a_t S_t^{\beta}, \\ \lambda(S,t) &= b_t + c\sigma^2(S,t), \end{aligned}$$

where $\beta < 0$ is the volatility elasticity parameter, $a_t > 0$ is the time-dependent volatility scale parameter, $b_t \geq 0$ is a deterministic non-negative function of time and $c > 0$. Some additional parameters related to option prices from Dias et al. (2015) are introduced: $p = -(2|\beta|)^{-1}$, $\delta_+ =$

$2 + (2c+1)/|\beta|$, and

$$\begin{aligned}
\widetilde{x}(t,T,S) &= \frac{y^2(t,t,S)}{\theta(t,T)}, \\
\widetilde{y}(t,T,S) &= \frac{y^2(t,T,S)}{\theta(t,T)}, \\
y(t,T,S) &= \frac{1}{|\beta|} S^{|\beta|} e^{-|\beta| \int_t^T (r+b_s)\mathrm{d}s}, \\
\theta(t,T) &= \int_t^T a_u^2 e^{-2|\beta| \int_t^u (r+b_s)\mathrm{d}s} \mathrm{d}u.
\end{aligned}$$

Hereafter, we consider the time-homogeneous version of the JDCEV model, making a_t and b_t constant. In this case, the function $\theta(t,T)$ becomes simpler: with $\tau = T - t$,

$$\theta(\tau) = \begin{cases} a^2 \tau & \text{if } r + b = 0 \\ \dfrac{a^2}{2|\beta|(r+b)} \left(1 - e^{-2|\beta|(r+b)\tau}\right) & \text{if } r + b \neq 0. \end{cases}$$

Also, price formulas for European derivatives under the JDCEV model are fully available in Carr and Linetsky (2006) and Dias et al. (2015). In the JDCEV model in this paper, a jump to default almost surely precedes the first hitting time to zero for the diffusion process, $\tilde{\zeta} < \tau_0$ a.s., and $\zeta = \tilde{\zeta}$ a.s. We refer to Carr and Linetsky (2006) for detailed movement of the JDCEV process with respect to σ and λ.

Proposition 2 *Assume that the asset price S_t follows the JDCEV model. If $S = K$, then $\Theta^{\mathrm{C}}(t,T,K;K)$ and $\Theta^{\mathrm{C\cdot bin}}(t,T,K;K)$ satisfy conditions in Theorem 2. If $S \neq K$, then $\Theta^{\mathrm{C}}(t,T,S;K)$ and $\Theta^{\mathrm{C\cdot bin}}(t,T,S;K)$ are continuously differentiable on $[0,T]$.*

This result shows that the JDCEV model is sufficiently nice to guarantee the existence and uniqueness of the weight function $\mathbf{w}(\cdot)$. Its proof in the Appendix B relies on a careful study of asymptotic behaviors of basic option prices. The second statement of Proposition 2 is helpful when the target barrier option is converted into a European or binary option at the barrier so that the conditions on Θ in Theorem 2 are satisfied.

Remark 1 Recall the definitions of Θ^{P}, Θ_0^{P}, $\Theta^{\mathrm{P\cdot bin}}$ and $\Theta_0^{\mathrm{P\cdot bin}}$ in Table 1. The limiting behaviors of thetas can be understood by considering the following put-call parities:

$$C^{\mathrm{E}} - P^{\mathrm{E}} = S - Ke^{-r(T-t)}, \quad C^{\mathrm{bin}} + P^{\mathrm{bin}} = e^{-r(T-t)}$$

where S is the stock price at t, T is the maturity and K is the strike. Furthermore, $P^{\mathrm{bin}} = P_0^{\mathrm{bin}} + v_D$ allows us to compute the thetas of put options with zero recovery upon default. Indeed, the theta of v_D can be shown to converge to $-b - a^2 c / S^{2|\beta|}$ as t approaches T. ∎

4.2 General Case

For general diffusion models specified in Section 2, we state some sufficient conditions on the model implied volatility to ensure the existence and uniqueness of the weight function $\mathbf{w}(\cdot)$. Then, the much studied properties of implied volatilities convince us the usefulness of the boundary matching approach.

Proposition 3 *Suppose that $\sigma_{\rm imp}(t,T,S_t;K)$ is the implied volatility corresponding to the European call option price $C^{\rm E}(t,T,S_t;K)$ under the given asset dynamics. In other words, we have the relationship*

$$C^{\rm E}(t,T,S_t;K) = C^{\rm BS}(\tau, S_t, \sigma_{\rm imp}; K)$$

where the right hand side represents the Black-Scholes formula with time-to-maturity, $\tau = T-t$, and the volatility $\sigma_{\rm imp}$. If $\sigma_{\rm imp}(t,T,K;K)$ is continuously differentiable in t on $[0,T]$, then $\Theta^{\rm C}$ and $\Theta^{\rm C\cdot bin}$ satisfy the conditions in Theorem 2.

Proof: We suppress the parameters of $\sigma_{\rm imp}$ for notational convenience. Straightforward computations lead us to the following $\Theta^{\rm C} = (\partial/\partial t)C^{\rm E}$:

$$\Theta^{\rm C}(t,T,S_t;K) = S_t\phi(d_1)\left\{\sqrt{\tau}\frac{\partial\sigma_{\rm imp}}{\partial t} - \frac{\sigma_{\rm imp}}{2\sqrt{\tau}}\right\} - rKe^{-r\tau}\Phi(d_2).$$

Here ϕ and Φ stand for the density function and the distribution function of a standard normal random variable, respectively. The d_1, d_2 are the usual symbols for

$$d_1 = \frac{1}{\sigma_{\rm imp}\sqrt{\tau}}\left\{\log\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma_{\rm imp}^2\right)\tau\right\}, \quad d_2 = d_1 - \sigma_{\rm imp}\sqrt{\tau}.$$

Recall that Theorem 2 considers $\Theta^{\rm C}(T-t, T-u, U; U)$. Hence, it is clear that we need to set

$$\begin{aligned} h_1(u,t) &= -\frac{U}{2}\phi(d_1)\sigma_{\rm imp}(T-t,T-u,U;U), \\ h_2(u,t) &= U\phi(d_1)\sqrt{t-u}\frac{\partial\sigma_{\rm imp}}{\partial t}(T-t,T-u,U;U) - rKe^{-r(t-u)}\Phi(d_2) \end{aligned}$$

with suitable changes in d_1 and d_2. In order to check the conditions in Lemma 2, we see that first

$$h_1(t,t) = -\frac{U}{2\sqrt{2\pi}}\sigma^* \neq 0$$

as long as $\sigma^* = \lim_{u\uparrow t}\sigma_{\rm imp}(T-t,T-u,U;U)$ is a nonzero real number. Second, straightforward differentiation of h_1 and h_2 with respect to u shows that their partial derivatives are weakly singular given the assumption that $\sigma_{\rm imp}$ is smooth and finite near maturity.

The case of $\Theta^{\rm C\cdot bin}$ can be similarly treated, hence we omit its proof. ■

Implied volatilities are one key object in financial derivatives. Academics and practitioners have put enormous efforts in analyzing, modeling, and predicting implied volatilities. For instance, Gatheral (2006) discussed various models and asymptotic formulas for real and model implied volatilities. Empirical studies such as Dumas et al. (1998) assume smooth functions for the implied volatility function in time and strike. Details could vary, but the collective information in the literature seems to support the assumptions in Proposition 3. In order to illustrate this point, let us focus on one concrete case: local volatility models.

Since Dupire (1994) and Derman and Kani (1994), local volatility models (Eq. (2) with $\lambda \equiv 0$) have been used widely by practitioners. The central idea is to make the volatility coefficient at time 0 as a deterministic function of the asset price and time in such a way that the resulting diffusion process replicates all the given vanilla option prices. Indeed, it is well known that the volatility coefficient can be written explicitly using partial derivatives of $C^{\mathrm{E}}(0, T, S_0; K)$ with respect to T and K (Gatheral, 2006). Recently, Gatheral et al. (2012) found highly accurate approximations of the corresponding implied volatility function. More specifically, when the asset dynamics follows a time-inhomogeneous diffusion, the following approximate formula can be derived:

$$\sigma_{\mathrm{imp}}(t, T, S_t; K) = \alpha_1(t) + \alpha_2(t)\tau + \alpha_3(t)\tau^2 + O(\tau^3)$$

for suitable functions $\alpha_i(t)$'s and for time-to-maturity $\tau = T - t$.

Example 1 *As a simple example, we can consider the case of $\sigma(S_t, t) = \sigma(t)$ which we assume is positive and differentiable. Then, we obtain $\sigma_{\mathrm{imp}}(t, T, S_t; K) = \sqrt{\frac{1}{\tau}\int_t^T \sigma^2(u)\mathrm{d}u}$. It is not difficult to check that this function satisfies the conditions in Proposition 3.*

Remark 2 On the practical side, it is important that the information contained in implied volatility surfaces can be directly utilized in the integral equation based approach. We refer the reader to Cont and Da Fonseca (2002) for advantages of this practice; they are observables independent of models, quotations of vanilla options, and market risk indicators. Also, we can avoid numerical difficulties in the process of converting them into local volatilities. Typically, implied volatility surfaces are given in terms of moneyness and time-to-maturity. In (4), the kernel can be computed using at-the-money implied volatilities with time-to-maturity $t - u$. As long as the conditions of Proposition 3 are satisfied, we can find a solution $\mathbf{w}(\cdot)$ and obtain a market consistent price of the target exotic option. In practice, polynomial functions or Gaussian kernel are often used for modeling of implied volatility surfaces by many market participants. They indeed satisfy the conditions of the proposition (Dumas et al., 1998; Aït-Sahalia and Lo, 1998). ∎

Remark 3 When we consider asset price jumps or other types of randomness in S_t as in stochastic volatility models or time-changed processes, the boundary matching approach is still applicable. However, this comes with extra burden: multi-dimensional integral equations. For example, if S_t can have random jumps, then a knock-in event does not necessarily occur at the boundary. In order to handle such a possible overshoot, we need basic options in time as well as in strike. The resulting integral equation is of the mixed Volterra-Fredholm type. ∎

5 Applications

This section discusses applications of the proposed framework (1) in pricing and hedging. Section 5.1 introduces the method of Laplace transforms for an exact analytical solution for option prices and static hedges. Next, Section 5.2 proposes a variant of the DEK method that outperforms existing methods significantly.

5.1 Analytical Solution for Option Prices and Static Hedges

The most important component in the construction of (1) is the weight function $\mathbf{w}(\cdot)$. We find it in this subsection via Laplace transforms. The analytic solutions, in particular, are beneficial in computing the value of sequential barriers because the computational cost and error of numerical methods such as the DEK method can be substantial. See Appendix D for an example. In order to apply the method of Laplace transforms, the kernel function in the associated integral equations must be a difference kernel. This is indeed the case under time-homogeneous models. Otherwise, it is still possible to obtain $\mathbf{w}(\cdot)$ by a resolvent kernel; however, computations are much more involved in this case. We denote the Laplace transform of a given function $f(\cdot)$ by

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \mathrm{d}t.$$

The following theorem computes the Laplace transforms of $\mathbf{w}$ and the target option price Ψ. We do not exclude the possibility of nonzero $\Psi^* = v(T, T, U)$. Later in this section, we present Laplace transforms of hedging instruments.

Theorem 4 *Assume that the asset price process S_t is time-homogeneous. Let $\Psi(0, t, S_0; U)$ be the price of the up-and-in barrier option with maturity t. Then, $\widehat{\mathbf{w}}(\lambda)$ and $\widehat{\Psi}(\lambda, S_0; U)$ are given by*

$$\widehat{\mathbf{w}}(\lambda) = \frac{\lambda \widehat{v}(\lambda, U) - \Psi^*}{\lambda \widehat{C}(\lambda, U; U)},$$

and

$$\widehat{\Psi}(\lambda, S_0; U) = \widehat{\mathbf{w}}(\lambda) \widehat{C}(\lambda, S_0; U) + \Psi^* \widehat{C}^{\mathrm{A}}(\lambda, S_0; U)$$

provided that $\mathbf{w}(t)$ *and Laplace transforms of* $\mathbf{w}(t)$, $v(0,t,U)$, $C(0,t,S_0;U)$, $C^{\mathrm{A}}(0,t,S_0;U)$ *exist.*

Proof: For European or binary calls, the boundary matching condition (9) reads

$$v(T-t,T,U) = \int_0^t \mathbf{w}(u)C(T-t,T-u,U;U)\mathrm{d}u + \Psi^*, \quad 0 \le t \le T.$$

The time-homogeneity of the underlying model implies that

$$v(0,t,U) = \int_0^t \mathbf{w}(u)C(0,t-u,U;U)\mathrm{d}u + \Psi^*. \tag{11}$$

Now, we seek for a function $\mathbf{w}(\cdot)$ that solves (11) for every t and for a fixed Ψ^*. Such $\mathbf{w}(\cdot)$ can then be applied to up-and-in barrier options for any maturity with given U and Ψ^*.

We observe that

$$\begin{aligned}
\widehat{v}(\lambda,U) &= \int_0^\infty e^{-\lambda t} v(0,t,U)\mathrm{d}t \\
&= \int_0^\infty \int_0^t e^{-\lambda t}\mathbf{w}(u)C(0,t-u,U;U)\mathrm{d}u\mathrm{d}t + \Psi^* \int_0^\infty e^{-\lambda t}\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda u}\mathbf{w}(u) \int_u^\infty e^{-\lambda(t-u)} C(0,t-u,U;U)\mathrm{d}t\mathrm{d}u + \frac{1}{\lambda}\Psi^* \\
&= \widehat{\mathbf{w}}(\lambda)\widehat{C}(\lambda,U;U) + \frac{1}{\lambda}\Psi^*,
\end{aligned}$$

from which the first statement is immediate. Similarly we apply Laplace transforms to the hedging portfolios in Theorems 1 and 3:

$$\begin{aligned}
\widehat{\Psi}(\lambda,S_0;U) &= \int_0^\infty e^{-\lambda t}\Psi(0,t,S_0;U)\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t} \int_0^t \mathbf{w}(u)C(0,t-u,S_0;U)\mathrm{d}u\mathrm{d}t + \Psi^* \int_0^\infty e^{-\lambda t} C^{\mathrm{A}}(0,t,S_0;U)\mathrm{d}t \\
&= \widehat{\mathbf{w}}(\lambda)\widehat{C}(\lambda,S_0;U) + \Psi^*\widehat{C}^{\mathrm{A}}(\lambda,S_0;U).
\end{aligned}$$

■

In Appendix C, we provide some sufficient conditions for the existence of the Laplace transform of $\mathbf{w}(t)$. We further prove that such conditions are satisfied under the JDCEV specification. In order to implement the above results, we focus on the computations of $\widehat{C}^{\mathrm{E}}$, $\widehat{C}^{\mathrm{bin}}$, and $\widehat{C}^{\mathrm{A}}$ under the JDCEV model in the rest of this section. The function $\widehat{v}(\lambda,U)$ depends on contract details of the target option Ψ. But, our computations are applicable when the option is turned into European or binary options once knocked-in.

For this purpose, we introduce the following auxiliary functions, for $0 \le l < u \le \infty$,

$$\mathcal{I}_s(l,u;\alpha) = \int_l^u x^\alpha \psi_s(x)\mathfrak{m}(x)dx,$$

$$\mathcal{J}_s(l,u;\alpha) \;=\; \int_l^u x^\alpha \phi_s(x)\mathfrak{m}(x)dx$$

where $\mathfrak{m}(x)$ is the speed density of the JDCEV model and $\psi_s(x), \phi_s(x)$ are the increasing and decreasing fundamental solutions [2] to the ordinary differential equation:

$$\frac{1}{2}a^2x^{2\beta+2}f''(x) + \left(r+b+ca^2x^{2\beta}\right)xf'(x) - \left(s+b+ca^2x^{2\beta}\right)f(x) = 0.$$

The functions $\mathfrak{m}$, ψ_s, and ϕ_s can be found in Section 8.1 of Mendoza-Arriaga et al. (2010). Lemma 4 in the Appendix C records the explicit formulae for $\mathcal{I}_s$ and $\mathcal{J}_s$ under the assumption $\beta < 0$ and $r+b>0$. These are mild assumptions that are typically observed in financial markets.

Mendoza-Arriaga et al. (2010) proposed two approaches to the valuation of contingent claims under time-changed Markov processes, namely the Laplace transform-based approach and the spectral expansion approach. Particularly for the JDCEV model, European put and call prices based on spectral expansions are given in Theorem 8.4 of their paper. Propositions 4 and 5 below complement their results in that we provide Laplace transforms of European, binary, and American binary option prices as well as v_D the price of a credit derivative in Table 1.

Proposition 4 *Assume that the asset price S_t follows the JDCEV model and that $\beta < 0$ and $r+b>0$. The Laplace transforms of European option prices are given as follows:*

$$\begin{aligned}
\widehat{C}^{\mathrm{E}}(\lambda,S;K) &= \frac{\phi_{\lambda+r}(S)}{w_{\lambda+r}}\Big[\mathcal{I}_{\lambda+r}(K,K\vee S;1) - K\mathcal{I}_{\lambda+r}(K,K\vee S;0)\Big] \\
&\quad + \frac{\psi_{\lambda+r}(S)}{w_{\lambda+r}}\Big[\mathcal{J}_{\lambda+r}(K\vee S,\infty;1) - K\mathcal{J}_{\lambda+r}(K\vee S,\infty;0)\Big], \\
\widehat{P}_0^{\mathrm{E}}(\lambda,S;K) &= \frac{\phi_{\lambda+r}(S)}{w_{\lambda+r}}\Big[K\mathcal{I}_{\lambda+r}(0,K\wedge S;0) - \mathcal{I}_{\lambda+r}(0,K\wedge S;1)\Big] \\
&\quad + \frac{\psi_{\lambda+r}(S)}{w_{\lambda+r}}\Big[K\mathcal{J}_{\lambda+r}(K\wedge S,K;0) - \mathcal{J}_{\lambda+r}(K\wedge S,K;1)\Big], \\
\widehat{C}^{\mathrm{bin}}(\lambda,S;K) &= \frac{\phi_{\lambda+r}(S)}{w_{\lambda+r}}\mathcal{I}_{\lambda+r}(K,K\vee S;0) + \frac{\psi_{\lambda+r}(S)}{w_{\lambda+r}}\mathcal{J}_{\lambda+r}(K\vee S,\infty;0), \\
\widehat{P}_0^{\mathrm{bin}}(\lambda,S;K) &= \frac{\phi_{\lambda+r}(S)}{w_{\lambda+r}}\mathcal{I}_{\lambda+r}(0,K\wedge S;0) + \frac{\psi_{\lambda+r}(S)}{w_{\lambda+r}}\mathcal{J}_{\lambda+r}(K\wedge S,K;0),
\end{aligned}$$

where w_s is the Wronskian of the two fundamental solutions in the Appendix C.

Proposition 5 *Assume that the asset price S_t follows the JDCEV model and that $\beta < 0$ and $r+b>0$. The Laplace transforms of American binary option prices are given by*

$$\widehat{C}^{\mathrm{A}}(\lambda,S;K) = \frac{1}{\lambda}\frac{\psi_{\lambda+r}(S)}{\psi_{\lambda+r}(K)} \quad \textit{and} \quad \widehat{P}_0^{\mathrm{A}}(\lambda,S;K) = \frac{1}{\lambda}\frac{\phi_{\lambda+r}(S)}{\phi_{\lambda+r}(K)}.$$

[2] We refer to Carr and Linetsky (2006) and Borodin and Salminen (2002) for properties of two fundamental solutions at boundaries in detail.

Also, the Laplace transform of the price of v_D is given by

$$\widehat{v}_D(\lambda, S) = \frac{1}{\lambda + r} - \left[\frac{\phi_{\lambda+r}(S)}{w_{\lambda+r}} \mathcal{I}_{\lambda+r}(0, S; 0) + \frac{\psi_{\lambda+r}(S)}{w_{\lambda+r}} \mathcal{J}_{\lambda+r}(S, \infty; 0) \right].$$

5.2 A Variant of the DEK Method

The purpose of this subsection is to develop a variant of the DEK method by discretizing (4) on a fixed time grid. In the literature, the DEK method has been used as a tool for obtaining approximate prices and static hedges. We refer the reader to Chung and Shih (2009); Chung et al. (2010, 2013a,b), or Ruas et al. (2013); Dias et al. (2015); Nunes et al. (2015) for some recent references. This is quite valuable when the method of Laplace transforms is not applicable or when a practical static hedging portfolio with finitely many options is considered.

Within the proposed framework, we extends the study of the DEK method in three ways. First, it is possible to apply various efficient methods for obtaining the numerical solution of integral equations. Second, we resolve an unfavorable feature of the DEK method that the amount of hedging instruments for reverse barrier options tends to blow up as the time grid gets finer. Third, we can calculate the distribution of hedging errors explicitly that are typically evaluated based on (simulated) scenarios in the literature.

We briefly introduce the DEK method from the perspective of our approach. It turns out that the DEK method is identical to a simple but the most inefficient discretization method for obtaining the numerical solution of (4). In other words, the weights of the DEK method can be written as

$$\sum_{i=0}^{k-1} \mathbf{w}(t_i) C(T - t_k, T - t_i, U; U)(t_{i+1} - t_i) = v(T - t_k, T, U) \tag{12}$$

for a fixed time grid $\mathbb{T} = \{0 = t_0, t_1, \ldots, t_n = T\}$ and $k = 1, 2, \ldots, n$. The left hand side is simply an approximation to $\int_0^{t_k} \mathbf{w}(u) C(T - t_k, T - u, U; U) \mathrm{d}u$. It is easy to see that (12) admits unique $\mathbf{w}(t_i)$'s and they are found iteratively. Based on this, the hedging portfolio, say

$$\Psi_{\mathbb{T}}(0, T, S_0; U) = \sum_{i=0}^{n-1} \mathbf{w}(t_i) C(0, T - t_i, S_0; U)(t_{i+1} - t_i),$$

matches the prices of the target option Ψ on the event that the stock price hits the barrier U at some $T - t_k$. At the same time, $\Psi_{\mathbb{T}}$ is understood as an approximation to the price $\Psi(0, T, S_0; U)$ in Theorem 1.

On the other hand, it should be noted that (12) does not require price matching of $\Psi_{\mathbb{T}}$ and Ψ at T. This potential error is exacerbated if the DEK method is applied to reverse barrier options as detailed in Chung et al. (2010). To overcome this difficulty, the authors in this reference proposed

the idea of matching thetas as well as prices on the boundary. For instance, we can add binary calls with maturity T so that

$$\begin{aligned}\sum_{i=0}^{k-1}\mathbf{w}(t_i)C(T-t_k,T-t_i,U;U)(t_{i+1}-t_i)+\mathbf{w}_{\rm bin}C^{\rm bin}(T-t_k,T,U;U)&= \quad v(T-t_k,T,U),\\ \mathbf{w}(t_0)\frac{\partial C}{\partial t}(T-t_1,T,U;U)(t_1-t_0)+\mathbf{w}_{\rm bin}\Theta^{\rm C\cdot bin}(T-t_1,T,U;U)&= \quad \Theta(T-t_1,T,U).\end{aligned} \tag{13}$$

Note that the first equation holds for $k=1,2,\ldots,n$ and that the second equation matches the thetas of v and (a new) $\Psi_{\mathbb{T}}$ at time $T-t_1$. As in (12), $\mathbf{w}(t_i)$'s and $\mathbf{w}_{\rm bin}$ are uniquely determined by these $n+1$ linear equations.

From the view of our integral equation approach, this issue is related to the boundedness of the solution and it is easily solved by using American binary options to have a continuous weight function as in Theorem 3 and (9). If we apply the rectangular rule for a time gird $\mathbb{T}$, then we replace the right hand side of (12) with $v(T-t_k,T,U)-\Psi^*$. The resulting hedging portfolio is given by

$$\Psi_{\mathbb{T}}(0,T,S_0;U)=\sum_{i=0}^{n-1}\mathbf{w}(t_i)C(0,T-t_i,S_0;U)(t_{i+1}-t_i)+\Psi^*C^{\rm A}.$$

We call this the modified DEK method (mod_DEK in short).

If we further apply theta matching using European binary calls, then we use $v(T-t_k,T,U)-\mathbf{w}_{\rm A}$ for the right hand side in the first equation of (13). Here, $\mathbf{w}(t_i)$'s, $\mathbf{w}_{\rm bin}$, and $\mathbf{w}_{\rm A}$ are the solution to (13) plus $\Psi^*=0.5\mathbf{w}_{\rm bin}+\mathbf{w}_{\rm A}$. The resulting hedging portfolio is then

$$\Psi_{\mathbb{T}}(0,T,S_0;U)=\sum_{i=0}^{n-1}\mathbf{w}(t_i)C(0,T-t_i,S_0;U)(t_{i+1}-t_i)+\mathbf{w}_{\rm bin}C^{\rm bin}+\mathbf{w}_{\rm A}C^{\rm A}.$$

The binary calls in these portfolios have maturity T and strike U. We denote this version by mod_TM. The effectiveness of mod_DEK and mod_TM is presented in Figure 1. It is depicted that mod_DEK and mod_TM remarkably reduce the replication errors compared to the DEK and TM methods under the same time grid.

For analytically tractable models such as the JDCEV model, we can compute the distribution of hedging errors. A static hedging portfolio $\Psi_{\mathbb{T}}$ attempts to replicate Ψ in two ways:

(1) if the stock never hits U, then both expire worthless,

(2) if the option is knocked-in at $T-t_k$ for some $k=1,2,\ldots,n-1$, then $\Psi_{\mathbb{T}}=\Psi$.

If Ψ is a standard up-and-in call, then one can convert $\Psi_{\mathbb{T}}$ into a European call in the case of (2). In this sense, hedging operations end whenever the stock price hits the barrier level in the static

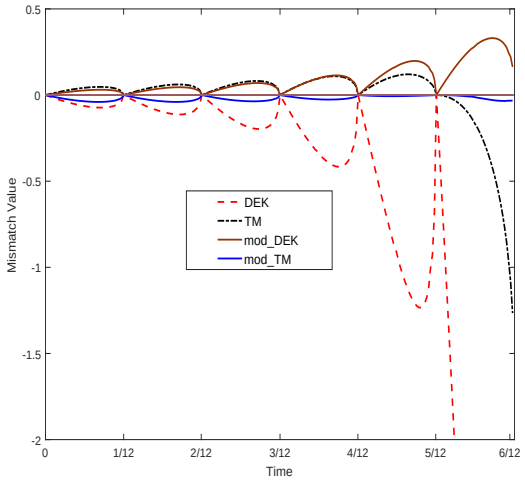

Figure 1: Differences between Ψ and $\Psi_{\mathbb{T}}$ on the boundary for a standard up-and-in call with barrier level 130, strike 115, maturity 0.5, and $\mathbb{T} = \{k/12 | k = 0, \ldots, 6\}$. Other parameters are $r = 5\%$, $a = 30$, $b = 0.05$, $c = 1$, $\beta = -1$.

hedging portfolio literature. Consequently, the discounted (relative) hedging error is given by

$$\boldsymbol{\varepsilon} \quad = \quad \mathbf{1}_{\{\tau \le T, \zeta > \tau\}} e(\tau)$$

where $\tau = \inf\{t > 0 : S_t = U\}$ and $e(t) = e^{-rt}|\Psi_{\mathbb{T}}(t, T, U; U) - v(t, T, U)|/\Psi(0, T, S_0; U)$.

Then, the hedging error distribution can be computed as follows: for $x \ge 0$,

$$\begin{aligned} \mathsf{P}(\boldsymbol{\varepsilon} \le x) &= \Big(1 - \mathsf{P}(\tau \le T, \zeta > \tau)\Big) + \mathsf{P}(\tau \le T, \zeta > \tau, e(\tau) \le x) \\ &= \bar{G}(T) + \int_0^T \mathbf{1}_{\{e(t) \le x\}} \mathrm{d}G(t). \end{aligned}$$

Here, $G(\cdot)$ is the distribution function of τ conditional on no default by τ and $\bar{G}(T) = 1 - \mathsf{P}(\tau \le T, \zeta > \tau)$. Its Laplace transform is given by

$$\mathsf{E}\left[e^{-\lambda\tau}\mathbf{1}_{\tau<\zeta}\right] = \frac{\psi_\lambda(S_0)}{\psi_\lambda(U)}$$

for $S_0 < U$. See the proof of Proposition 5. On the other hand, one should note that the distribution G and its Laplace transform are all given under the real world measure.

The remaining computational task is to find the region $\{t \in [0, T] | e(t) \le x\}$. As shown in Figure 1, this set appears to be a union of disjoint intervals for standard barrier options. If this is the case, say

$$e^{-1}\Big([0, x]\Big) = [t_0, t_1] \cup [t_2, t_3] \cup \cdots \cup [t_{n-1}, t_n], \quad t_0 < t_1 < \cdots < t_n,$$

then $\mathsf{P}(\varepsilon \le x) = \bar{G}(T) + \sum_{i=0}^{n}(-1)^{i+1}G(t_i)$. This procedure is computationally feasible as we can evaluate the function $e(t)$. However, some performance measures do not even require the knowledge of e^{-1}. For instance, the expected hedging error is easily found to be

$$\mathsf{E}[\varepsilon] = \mathsf{E}\Big[\mathbf{1}_{\{\tau \le T, \zeta > \tau\}} e(\tau)\Big] = \int_0^T e(t)\mathrm{d}G(t).$$

Other examples include maximum error $\|\varepsilon\|_\infty$.

We apply the above idea in order to compare hedging performances of different methods. Out of pure convenience, we continue to adopt the risk-neutral parameters in Figure 1. Table 2 reports mean, maximum error, value-at-risk (VaR), and expected shortfall (ES) of the original DEK method, TM method, and our mod_TM. It is noteworthy that mod_TM outperforms existing static hedging methods greatly. Particularly, there is a remarkable reduction in the tails of ε, which is also reflected in Figure 2.

Table 2: Risk measures for hedging errors of static hedging portfolios: DEK, TM and mod_TM. Parameters are given in Figure 1.

ε	mean	maximum	$\mathsf{VaR}_{0.1}$	$\mathsf{VaR}_{0.05}$	$\mathsf{ES}_{0.1}$	$\mathsf{ES}_{0.05}$
DEK	0.0195	2.6325	0.6907	1.3914	1.4710	1.9091
TM	0.0016	0.2571	0.0244	0.0727	0.0926	0.1416
mod_TM	0.0002	0.0081	0.0074	0.0079	0.0079	0.0080

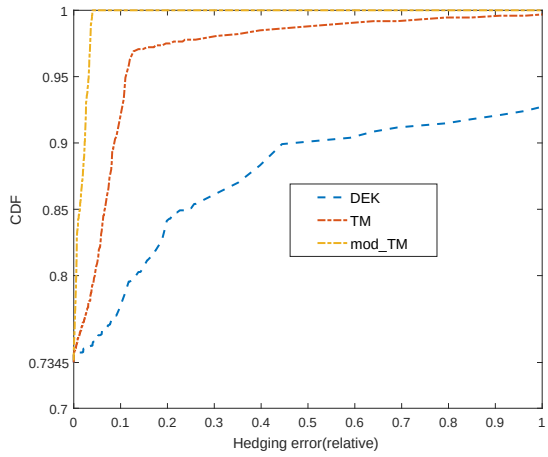

Figure 2: Cumulative hedging error distributions of static hedging portfolios: DEK, TM and mod_TM. Parameters are given in Figure 1.

The most interesting modification of the DEK method is to apply other numerical methods instead of the rectangular rule. When evaluating integrals via a finite number of function eval-

uations, it is known that a midpoint method achieves higher order convergence rates than the rectangular method. Linz (1969) showed that this assertion is valid for Volterra integral equations of the first kind. This fact motivates us to propose yet another discretization scheme, which we call mod_TM(mid). Particularly, we set $s_{i+1} = t_i + h/2$ for $i = 0, \ldots, n-1$ at which values of the target option and a portfolio are matched.

In mod_TM(mid), the linear equations for $\mathbf{w}(t_i)$'s, $\mathbf{w}_{\text{bin}}$, and $\mathbf{w}_{\text{A}}$ are now changed as follows:

$$\begin{aligned} \sum_{i=1}^{k-1} \mathbf{w}(t_i) C(T - s_k, T - t_i, U; U) h + \mathbf{w}(t_0) C(T - s_k, T - t_0, U; U) \frac{h}{2} \\ = \quad v(T - s_k, T, U) - \mathbf{w}_{\text{A}} - \mathbf{w}_{\text{bin}} C^{\text{bin}}(T - s_k, T, U; U), \end{aligned}$$

and the theta matching is given by

$$\mathbf{w}(t_0) \frac{\partial C}{\partial t}(T - s_1, T, U; U) \frac{h}{2} + \mathbf{w}_{\text{bin}} \Theta^{\text{C·bin}}(T - s_1, T, U; U) \quad = \quad \Theta(T - s_1, T, U).$$

Lastly $\Psi^* = 0.5\mathbf{w}_{\text{bin}} + \mathbf{w}_{\text{A}}$. We note that the above formulation contains the same set of hedging instruments in mod_TM.

Table 3 shows the numerical performance of mod_TM(mid). We consider standard up-and-in calls with constant barriers since the readily available solutions based on Laplace transform serve as benchmark prices. Total 8 different parameter settings are used.[3] For a fair comparison, the identical time grid is used to compute hedge weights of four different methods so that their computational costs are approximately the same. More details about a speed-accuracy tradeoff are available upon request. The mean absolute error shows that mod_TM and mod_TM(mid) significantly outperform the original DEK method and TM method.

The speeds of convergence are compared in Figure 3 by increasing the number of time steps. The averages of 8 *relative* errors for each of 9 different $\mathbb{T}$'s are depicted. The methods mod_TM, mod_TM(mid) achieve relative errors less than 0.1% even at $n < 5$.

[3]Parameter values for a, β are given to generate the same volatility level.

Table 3: Values of standard up-and-in calls with initial stock price 100, barrier level 130, maturity 0.5, and $\mathbb{T} = \{k/60 | k = 0, \ldots, 30\}$. Other parameters are $r = 5\%$, $b = 0.05$, $c = 1$. Mean absolute error is the average of 8 absolute price differences (average of each column).

K	β	a	exact solution	DEK	TM	mod_TM	mod_TM (mid)
115	-1	3.0E+01	4.9244	4.8461	4.9268	4.9245	4.9244
	-2	3.0E+03	3.8494	3.7683	3.8512	3.8493	3.8494
	-3	3.0E+05	2.8260	2.7476	2.8273	2.8258	2.8260
	-4	3.0E+07	1.8788	1.8100	1.8796	1.8786	1.8788
105	-1	3.0E+01	7.4114	7.2785	7.4139	7.4112	7.4114
	-2	3.0E+03	5.9647	5.8279	5.9668	5.9645	5.9647
	-3	3.0E+05	4.4614	4.3296	4.4631	4.4612	4.4615
	-4	3.0E+07	3.0017	2.8864	3.0030	3.0016	3.0018
mean absolute error				0.10293	0.00174	0.00016	0.00001

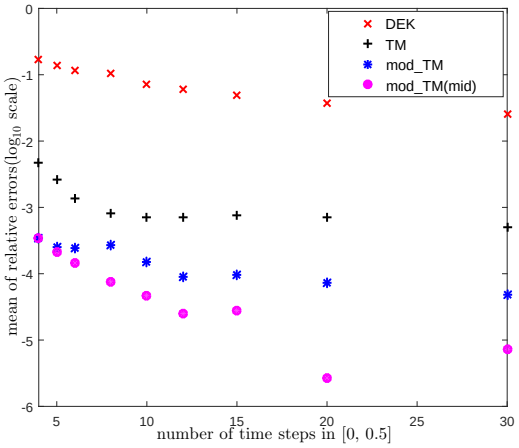

Figure 3: Mean relative errors for standard up-and-in calls versus the number of time steps n under 8 different parameter settings in Table 3. Relative errors are computed with respect to true prices based on Laplace transforms.

6 Conclusion

In this work, we presented a novel approach to static replication of exotic options under Markovian diffusions with random jump-to-default. Target options include a wide class of American options and barrier type options. Based on boundary matching conditions, we derived certain integral equations for hedge weights to satisfy. Those integral equations are Volterra integral equations of the second kind or generalized Abel integral equations, depending on a hedging instrument. One of main contributions is the derivation of existence and uniqueness conditions for hedge weights. Furthermore, target option prices as well as hedge weights can be explicitly computed by Laplace inversion if the underlying process is time-homogeneous. Their Laplace transforms are given in terms of Laplace transforms of more basic options such as vanilla or binary options. In this aspect, this paper enlarged the space of contingent claims whose (semi-explicit) pricing formulae are available.

We also paid a great deal of attention to more practical concerns regarding the construction of static hedges. Since there are finitely many basic options available in the market, we face two problems: how to determine hedge weights, and how to quantify hedging errors. The first question has been studied by many authors in the literature on calendar-spread approaches. Our new integral representations led us to another variant of the DEK method, and this new scheme performed better than existing schemes particularly for reverse barriers. For the second question, we were able to characterize the distribution function of hedging errors by which we compared hedging errors of three different calendar-spread approaches. Last but not least, such a static hedging portfolio on a discrete time grid can be useful as an approximate pricing method if exact replication is not possible; for instance when the barrier is curved. We can utilize existing numerical methods such as a midpoint rule for integral equations in order to enhance approximation qualities.

There have been many works on static replications on a discrete time grid for more than a decade. The central idea of this paper, however, lies in the representation of boundary matching conditions via integral equations, and subsequent analyses for the existence and the computations of solutions in the continuous-time setting. Based on this, we explicitly derived analytic expressions for the prices of certain exotic options for the first time, and demonstrated possibilities of better performing numerical methods for static hedges. Furthermore, quantification of hedging errors is a great advantage. Nevertheless, there are still many issues to be resolved so as to fully utilize our integral equations approach. For example, when asset price jumps or stochastic volatility are involved, boundary matching conditions need to be modified. This leads to mixed Volterra-Fredholm equations.

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Online Appendix

Static Replication of Barrier-type Options via Integral Equations

A Proof for the Abel Integral Equation

Proof of Lemma 2: First, we construct a second kind Volterra equation equivalent to (6). By multiplying both sides of (6) by the factor $(u-t)^{\alpha-1}\mathrm{d}t$ and integrating it with respect to t from 0 to u, we obtain

$$\int_0^u \int_0^t \frac{h_1(s,t)}{(u-t)^{1-\alpha}(t-s)^{\alpha}} f(s)\mathrm{d}s\mathrm{d}t + \int_0^u \int_0^t \frac{h_2(s,t)}{(u-t)^{1-\alpha}} f(s)\mathrm{d}s\mathrm{d}t = \int_0^u \frac{g(t)}{(u-t)^{1-\alpha}}\mathrm{d}t.$$

The exchange of the order of integrations gives us

$$\int_0^u \int_s^u \frac{h_1(s,t)}{(u-t)^{1-\alpha}(t-s)^{\alpha}} f(s)\mathrm{d}t\mathrm{d}s + \int_0^u \int_s^u \frac{h_2(s,t)}{(u-t)^{1-\alpha}} f(s)\mathrm{d}t\mathrm{d}s = \int_0^u \frac{g(t)}{(u-t)^{1-\alpha}}\mathrm{d}t. \tag{A.1}$$

This operation is validated once we identify a continuous solution f.

For $s < u$, define $L_1(s,u)$ and $L_2(s,u)$ as

$$\begin{aligned} L_1(s,u) &= \int_s^u \frac{h_1(s,t)}{(u-t)^{1-\alpha}(t-s)^{\alpha}}\mathrm{d}t = \int_0^1 \frac{h_1(s,s+(u-s)y)}{y^{\alpha}(1-y)^{1-\alpha}}\mathrm{d}y, \\ L_2(s,u) &= \int_s^u \frac{h_2(s,t)}{(u-t)^{1-\alpha}}\mathrm{d}t = (u-s)^{\alpha}\int_0^1 \frac{h_2(s,s+(u-s)y)}{(1-y)^{1-\alpha}}\mathrm{d}y. \end{aligned}$$

Then, after simple calculations, it is easy to see that

$$\begin{aligned} L_1(u,u) &:= \lim_{s\uparrow u} L_1(s,u) = h_1(u,u)\Gamma(1-\alpha)\Gamma(\alpha) \neq 0, \\ L_2(u,u) &:= \lim_{s\uparrow u} L_2(s,u) = 0. \end{aligned}$$

Furthermore, their derivatives are given by

$$\begin{aligned} \frac{\partial L_1(s,u)}{\partial u} &= \frac{1}{(u-s)^{\beta_1}}\int_0^1 \frac{y^{1-\alpha-\beta_1}}{(1-y)^{1-\alpha}}\widetilde{h}_1(s,s+(u-s)y)\mathrm{d}y, \\ \frac{\partial L_2(s,u)}{\partial u} &= \frac{\alpha}{(u-s)^{1-\alpha}}\int_0^1 \frac{h_2(s,s+(u-s)y)}{(1-y)^{\alpha}}\mathrm{d}y \\ &\quad + \frac{(u-s)^{\alpha}}{(u-s)^{\beta_2}}\int_0^1 \frac{y^{1-\beta_2}}{(1-y)^{1-\alpha}}\widetilde{h}_2(s,s+(u-s)y)\mathrm{d}y \end{aligned}$$

where we represent $(\partial/\partial t)h_i(s,t)$ as

$$\frac{\partial h_1(s,t)}{\partial t} = \frac{\widetilde{h}_1(s,t)}{(t-s)^{\beta_1}}, \quad \frac{\partial h_2(s,t)}{\partial t} = \frac{\widetilde{h}_2(s,t)}{(t-s)^{\beta_2}}$$

for some $0 < \beta_i < 1$ and continuous functions $\widetilde{h}_i(s,t)$ on $\{(s,t)|0 \le s \le t \le T\}$. Since $\widetilde{h}_i(s,s+(u-s)y)$ and $h_2(s,s+(u-s)y)$ are continuous functions in (s,u), all integrals in $(\partial/\partial u)L_i(s,u)$

are continuous as well. This implies $(\partial/\partial u)L(s,u)$ with $L(s,u) := L_1(s,u) + L_2(s,u)$ is weakly singular.

By differentiating Equation (A.1) with respect to u evaluated at t, we have

$$\begin{aligned} L_1(t,t)f(t) + \int_0^t \frac{\partial L(s,t)}{\partial t} f(s)\mathrm{d}s &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{g(s)}{(t-s)^{1-\alpha}}\mathrm{d}s \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\alpha}t^\alpha g(0) + \frac{1}{\alpha}\int_0^t (t-s)^\alpha g'(s)\mathrm{d}s\right]. \end{aligned}$$

Hence, the right hand side is continuous on $(0,T]$. This is a Volterra integral equation of the second kind. Now, we can apply Lemma 1 and conclude that there exists a unique continuous solution f on $(0,T]$. If $g(0) = 0$, then Lemma 1 can be applied to the interval $[0,T]$. ■

B Proofs for the Existence of the Weight Function under the JDCEV Model

Lemma 3 *The at-the-money prices of binary options under the JDCEV model are continuous in $t \in [0,T]$. In particular,*

$$\lim_{t\to T} C^{\mathrm{bin}}(t,T,K;K) = \lim_{t\to T} P_0^{\mathrm{bin}}(t,T,K;K) = 0.5.$$

Proof: In Dias et al. (2015), the binary call price is given by

$$C^{\mathrm{bin}}(t,T,K;K) = e^{-(r+b)(T-t)}\Big[\widetilde{x}(t,T,K)\Big]^{-p}\Phi_{+1}\Big(p, \widetilde{y}(t,T,K); \delta_+, \widetilde{x}(t,T,K)\Big)$$

where $\Phi_{+1}(p,y;\mu,x) = \mathsf{E}\left[X^p \mathbf{1}_{\{X>y\}}\right]$ is the truncated p-th moment of a noncentral chi-square random variable X with μ degrees of freedom and non-centrality parameter x.

Combining (D.1), (D.3), and (D.9) of Ruas et al. (2013) [1] and using the relation of Φ_{+1} and Φ_{-1} in (5.13) of Carr and Linetsky (2006), it is not difficult to see that

$$\begin{aligned} x^{-p}\Phi_{+1}(p,y;\mu,x) \sim{}& \left[\mathbf{1}_{\{\rho<1\}} + \frac{1}{2}\mathrm{sgn}(\rho-1)\mathrm{erfc}\left(\left|\sqrt{\frac{y}{2}} - \sqrt{\frac{x}{2}}\right|\right)\right]\sum_{m=0}^{\infty}\frac{C_m^\mu}{x^m} \\ &+ \frac{\exp\left[-\left(\sqrt{\frac{y}{2}} - \sqrt{\frac{x}{2}}\right)^2\right]}{\sqrt{2\pi x}}\sum_{m=0}^{\infty}\frac{D_m^\mu}{x^m} + (\rho-1-\gamma)\frac{\exp\left[-\left(\sqrt{\frac{y}{2}} - \sqrt{\frac{x}{2}}\right)^2\right]}{\sqrt{2\pi x}}\sum_{m=0}^{\infty}\frac{G_m^\mu}{x^m} \end{aligned} \quad \text{(B.1)}$$

as $x, y \to \infty$, where C_m^μ's, D_m^μ's and G_m^μ's are defined in (D.11) to (D.13) of Ruas et al. (2013). Lastly, $\gamma = (\rho-1)\mathbf{1}_{\{\rho\geq 1\}}$ with $\rho = \sqrt{y/x}$. The function erfc(s) is the complementary error function defined as $2/\sqrt{\pi}\int_s^\infty e^{-u^2}du$.

[1] We note that these formulas can be found in the supplementary material of Ruas et al. (2013)

For notational convenience, we simply denote $\widetilde{x}(t,T,K)$ by $\widetilde{x}$. Similarly $\widetilde{y}$ is used. By definition, we always have $\widetilde{y} < \widetilde{x}$ for $t < T$ so that $\rho = \sqrt{\widetilde{y}/\widetilde{x}} < 1$. On the other hand, $\lim_{t\to T} \widetilde{x} = \lim_{t\to T} \widetilde{y} = \infty$ and $\lim_{t\to T} \widetilde{x}/\widetilde{y} = 1$. Hence, as t approaches T, we can ignore all the terms except C_0^{μ} in (B.1) and obtain

$$\widetilde{x}^{-p}\Phi_{+1}(p,\widetilde{y};\delta_+,\widetilde{x}) \quad \sim \quad \left[1 - \frac{1}{2}\mathrm{erfc}\left(\left|\sqrt{\frac{\widetilde{y}}{2}} - \sqrt{\frac{\widetilde{x}}{2}}\right|\right)\right] C_0^{\delta_+}.$$

It is a simple matter to check $\lim_{t\to T}\left(\sqrt{\widetilde{x}} - \sqrt{\widetilde{y}}\right) = 0$. Hence, using $C_0^{\delta_+} = 1$, we obtain the desired result.

The value of a binary put option with zero recovery is given by (Dias et al., 2015))

$$P_0^{\mathrm{bin}}(t,T,K;K) = e^{-(r+b)(T-t)}\Big[\widetilde{x}(t,T,K)\Big]^{-p}\Phi_{-1}\Big(p,\widetilde{y}(t,T,K);\delta_+,\widetilde{x}(t,T,K)\Big)$$

where $\Phi_{-1}(p,y;\mu,x) = \mathsf{E}\left[X^p \mathbf{1}_{\{X\le y\}}\right]$ and X is as given above. Similar arguments confirm that the limit of P_0^{bin} is 0.5 as t approaches T. ■

Proof of Proposition 2: *Step 1* Let us first consider the case of binary calls. The proof is based on the explicit formula of theta given in Eq.(64) of Dias et al. (2015):

$$\begin{aligned}
\Theta^{\mathrm{C\cdot bin}}(t,T,S;K) \;=\;& e^{-(r+b)\tau}\Big(\widetilde{x}(t,T,S)\Big)^{-p}\Phi_{+1}\Big(p,\widetilde{y}(t,T,K);\delta_+,\widetilde{x}(t,T,S)\Big) \\
& \times\left[(r+b) - \frac{\theta'(\tau)}{\theta(\tau)}\left(p + \frac{\widetilde{x}(t,T,S)}{2}\right)\right] \\
& + e^{-(r+b)\tau}\Big(\widetilde{x}(t,T,S)\Big)^{-p}\frac{\theta'(\tau)}{\theta(\tau)}\widetilde{\Phi}_{+1}\Big(p,\widetilde{y}(t,T,K);\delta_+,\widetilde{x}(t,T,S)\Big) \\
& - e^{-(r+b)\tau}\Big(\widetilde{x}(t,T,S)\Big)^{-p}2^p e^{-\frac{1}{2}(\widetilde{y}(t,T,K)+\widetilde{x}(t,T,S))}\left(\frac{\widetilde{y}(t,T,K)}{2}\right)^{\frac{1}{2}\delta_+ + p} \\
& \times\left(2|\beta|(r+b) + \frac{\theta'(\tau)}{\theta(\tau)}\right) \times H(\widetilde{x}(t,T,S),\widetilde{y}(t,T,K),\delta_+) \qquad \text{(B.2)}
\end{aligned}$$

where $\tau = T - t$, $H(x,y,z)$ is in Eq.(61) of Dias et al. (2015), and $\widetilde{\Phi}_{+1}$ is in Eq.(35) of Ruas et al. (2013). Here θ' is the derivative of θ with respect to τ. We further simplify the formula by observing that

$$\begin{aligned}
H(x,y,z) &= \left(\frac{\sqrt{xy}}{2}\right)^{-\frac{z-2}{2}} I_{\frac{z-2}{2}}(\sqrt{xy}), \qquad \text{(B.3)} \\
\widetilde{\Phi}_{+1}(p,y;\mu,x) &= \frac{x}{2}\Phi_{+1}(p,y;\mu+2,x) \qquad \text{(B.4)}
\end{aligned}$$

where $I_\nu(\cdot)$ is the modified Bessel function of the first kind of order ν.

Next, straightforward calculations tell us that

$$\frac{1}{\theta} = \frac{1}{a^2\tau} + \frac{|\beta|(r+b)}{a^2} + O(\tau),$$

which in turn implies

$$\frac{\theta'}{\theta} = \frac{1}{\tau} - |\beta|(r+b) + O(\tau), \quad \frac{\theta'}{\theta^2} = \frac{1}{a^2\tau^2} + O(1).$$

The first two terms of $\Theta^{\text{C·bin}}$ then become

$$\begin{aligned} & e^{-(r+b)\tau}\widetilde{x}^{-p}\Phi_{+1}\left(p,\widetilde{y};\delta_{+},\widetilde{x}\right)\left(-\frac{p}{\tau} - \frac{1}{a^2\tau^2}\frac{1}{2|\beta|^2}S^{2|\beta|} + O(1)\right) \\ & + e^{-(r+b)\tau}\widetilde{x}^{-p}\Phi_{+1}\left(p,\widetilde{y};\delta_{++},\widetilde{x}\right)\left(\frac{1}{a^2\tau^2}\frac{1}{2|\beta|^2}S^{2|\beta|} + O(1)\right). \end{aligned} \tag{B.5}$$

Here the arguments of $\widetilde{x},\widetilde{y}$ are suppressed for simplicity and $\delta_{++} = \delta_{+} + 2$. We use the expansion (B.1) for $\widetilde{x}^{-p}\Phi_{+1}$.

Now set $S = K$. Then, it can be checked that $\rho = \sqrt{\widetilde{y}/\widetilde{x}} = e^{-c_1\tau/2} < 1$ with $c_1 = 2|\beta|(r+b)$. We also have

$$\rho - 1 = -\frac{c_1\tau}{2} + O(\tau^2).$$

In addition, it can be verified that

$$\frac{1}{\sqrt{2\pi\widetilde{x}}}\exp\left[-\left(\sqrt{\frac{\widetilde{y}}{2}} - \sqrt{\frac{\widetilde{x}}{2}}\right)^2\right] = \frac{1}{\sqrt{2\pi N_0}}\Big(\sqrt{\tau} + c_2\tau\sqrt{\tau} + O(\tau^2)\Big),$$

where $N_0 = K^{2|\beta|}/(a^2|\beta|^2)$ and c_2 is some constant. Then, (B.1) reads

$$\begin{aligned} \widetilde{x}^{-p}\Phi_{+1}(p,\widetilde{y};\delta_{+},\widetilde{x}) = & \left[1 - \frac{1}{2}\text{erfc}\left(\left|\sqrt{\frac{\widetilde{y}}{2}} - \sqrt{\frac{\widetilde{x}}{2}}\right|\right)\right]\left(C_0^{\delta_+} + \frac{C_1^{\delta_+}}{N_0}\tau + O(\tau^2)\right) \\ & + \frac{1}{\sqrt{2\pi N_0}}\left(\sqrt{\tau}D_0^{\delta_+} + c_3\tau\sqrt{\tau} + O(\tau^2)\right) + \frac{1}{\sqrt{2\pi N_0}}\left(-\frac{c_1 G_0^{\delta_+}}{2}\tau\sqrt{\tau} + O(\tau^2)\right) \end{aligned}$$

for some constant c_3. We have a similar expression when we have δ_{++} instead of δ_{+}. Using these expansions, (B.5) can be expressed in terms of τ^{-2}, $\tau^{-3/2}$, τ^{-1}, $\tau^{-1/2}$, and higher. Let us examine the coefficients of the followings as $\tau \to 0$:

$$\begin{aligned} \frac{1}{\tau^2}\text{term}: & \quad -\frac{N_0 C_0^{\delta_+}}{4} + \frac{N_0 C_0^{\delta_{++}}}{4} = 0, \\ \frac{1}{\tau^{3/2}}\text{term}: & \quad \frac{1}{\sqrt{2\pi N_0}}\left(-\frac{D_0^{\delta_+}N_0}{2} + \frac{D_0^{\delta_{++}}N_0}{2}\right) = \sqrt{\frac{N_0}{8\pi}}, \\ \frac{1}{\tau}\text{term}: & \quad -\frac{C_0^{\delta_+}p}{2} - \frac{C_1^{\delta_+}}{4} + \frac{C_1^{\delta_{++}}}{4} = 0, \end{aligned}$$

where we used the facts $C_0^{\mu} = 1$, $C_1^{\mu} = 0.5\tau_\mu(\tau_\mu - 1) - A_1(0.5\mu - 1)$, $D_0^{\mu} = \tau_\mu$ with $\tau_\mu = 2p + 0.5(\mu - 1)$, $A_1(x) = 0.5\Gamma(x+1.5)/\Gamma(x-0.5)$.

Let us turn our attention to the last term of $\Theta^{\text{C·bin}}$. Using (B.3) and the expansions of θ'/θ and $1/\theta$, the last term is seen to be

$$-e^{-(r+b)\tau}e^{-\frac{1}{2}(\widetilde{x}+\widetilde{y})}\frac{1}{2}\rho^{\frac{\delta_+}{2}+2p}\sqrt{\widetilde{x}\widetilde{y}}\left[\frac{1}{\tau} + \frac{c_1}{2} + O(\tau)\right]I_{\frac{\delta_+-2}{2}}(\sqrt{\widetilde{x}\widetilde{y}})$$

$$= \quad -e^{-(r+b+c_4)\tau} e^{-\frac{1}{2}(\widetilde{x}+\widetilde{y})} \frac{N_0}{2} \left[\frac{1}{\tau^2} + \frac{c_1}{\tau} + O(1) \right] I_{\frac{\delta_+ - 2}{2}}(\sqrt{\widetilde{x}\widetilde{y}})$$

for some constant c_4. Regarding the modified Bessel function, we apply Hankel's expansion

$$\begin{aligned} I_{\frac{\delta_+ - 2}{2}}(\sqrt{\widetilde{x}\widetilde{y}}) &= \frac{e^{\sqrt{\widetilde{x}\widetilde{y}}}}{\sqrt{2\pi\sqrt{\widetilde{x}\widetilde{y}}}} \left[1 - \frac{4\left(\frac{\delta_+ - 2}{2}\right)^2 - 1}{8\sqrt{\widetilde{x}\widetilde{y}}} + O\left(\frac{1}{\widetilde{x}\widetilde{y}}\right) \right] \\ &= \frac{e^{\sqrt{\widetilde{x}\widetilde{y}}}}{\sqrt{2\pi N_0}} \left[\sqrt{\tau} + O(\tau^2) \right] \left[1 + c_5\tau + O(\tau^2) \right] \\ &= \frac{e^{\sqrt{\widetilde{x}\widetilde{y}}}}{\sqrt{2\pi N_0}} \left[\sqrt{\tau} + c_5\tau^{3/2} + O(\tau^2) \right] \end{aligned}$$

for some constant c_5. Noting that $-(\widetilde{x}+\widetilde{y})/2 + \sqrt{\widetilde{x}\widetilde{y}} = O(\tau)$ after some calculations, we are led to the following expression for the third term of $\Theta^{\text{C·bin}}$:

$$-\sqrt{\frac{N_0}{8\pi}} \frac{1}{\tau^{3/2}} + O\left(\frac{1}{\sqrt{\tau}}\right).$$

Consequently, the coefficient of $\tau^{-3/2}$ term also disappears. Hence, $\Theta^{\text{C·bin}}$ consists of $\tau^{-1/2}$ or higher order terms and thus it is weakly singular. We have dealt with the case $r+b \neq 0$. It becomes easier and we arrive at the same conclusion if $r+b=0$.

Step 2 Now suppose $S < K$. As in the case of $S = K$, we can treat the first two terms and the third term of $\Theta^{\text{C·bin}}$ separately. For the former, we see that $\rho = \sqrt{\widetilde{y}/\widetilde{x}} > 1$ for all sufficiently small τ values. Then, we observe that

$$\frac{1}{\sqrt{2\pi\widetilde{x}}} \exp\left[-\left(\sqrt{\frac{\widetilde{y}}{2}} - \sqrt{\frac{\widetilde{x}}{2}} \right)^2 \right] = \frac{1}{\sqrt{2\pi\widetilde{x}}} \exp\left[-\frac{\widetilde{x}}{2}(\rho-1)^2 \right], \tag{B.6}$$

which converges to zero exponentially in $\widetilde{x}$. This is because $\widetilde{x} \to \infty$ but $\rho - 1$ converges to a nonzero constant as $\tau \to 0$. Furthermore, $\text{erfc}(s)$ expands as

$$\text{erfc}\left(\left| \sqrt{\frac{\widetilde{y}}{2}} - \sqrt{\frac{\widetilde{x}}{2}} \right| \right) = \text{erfc}\left(\left| \sqrt{\frac{\widetilde{x}}{2}}(\rho-1) \right| \right) = \frac{1}{\sqrt{\pi}} \left[\frac{\exp\left[-\frac{\widetilde{x}}{2}(\rho-1)^2 \right]}{\sqrt{\frac{\widetilde{x}}{2}}(\rho-1)} + \cdots \right] \tag{B.7}$$

and the convergence speed to zero is exponential in $\widetilde{x}$. As a result, all the terms in (B.5) converge to zero as $\tau \to 0$.

As for the third term of $\Theta^{\text{C·bin}}$, we can proceed as in Step 1 using Hankel's expansion. Then, careful counting reveals that its convergence is dominated by

$$\exp\left[-\frac{1}{2}(\widetilde{x}+\widetilde{y}) + \sqrt{\widetilde{x}\widetilde{y}} \right] = \exp\left[-\frac{1}{2}\left(\sqrt{\widetilde{x}} - \sqrt{\widetilde{y}} \right)^2 \right] = \exp\left[-\frac{\widetilde{x}}{2}(\rho-1)^2 \right], \tag{B.8}$$

which decreases exponentially fast in $\widetilde{x}$. Therefore, $\lim_{\tau\to 0}\Theta^{\text{C·bin}} = 0$.

When it comes to the differentiability of $\Theta^{\text{C·bin}}$, our only concern is at $\tau = 0$. But, the exponential rate of decrease of the theta in $\widetilde{x}$ implies the one-side derivative at $\tau = 0$ is zero. Actual derivatives of $\Theta^{\text{C·bin}}$ for $\tau > 0$ can also be computed by using the following recurrence relations:

$$\begin{aligned}
\frac{\partial\Phi_{+1}(p,y;\mu,x)}{\partial x} &= \frac{1}{2}\Big\{\Phi_{+}(p,y;\mu+2,x) - \Phi_{+}(p,y;\mu,x)\Big\}, && \text{(B.9)}\\
\frac{\partial\Phi_{+1}(p,y;\mu,x)}{\partial y} &= -2^{p-1}e^{-\frac{x+y}{2}}\left(\frac{y}{2}\right)^{\frac{\mu}{2}+p-1}H(x,y,\mu), && \text{(B.10)}\\
\frac{\partial H(x,y,\mu)}{\partial x} &= \frac{y}{4}H(x,y,\mu+2), && \text{(B.11)}\\
\frac{\partial H(x,y,\mu)}{\partial y} &= \frac{x}{4}H(x,y,\mu+2). && \text{(B.12)}
\end{aligned}$$

The derivation of these results are omitted as they are long but straightforward.

When $\rho > 1$, the above relations and (B.1) imply that the derivative of $\Theta^{\text{C·bin}}$ is a linear combination of terms such as

$$\operatorname{erfc}\left(\left|\sqrt{\frac{\widetilde{y}}{2}} - \sqrt{\frac{\widetilde{x}}{2}}\right|\right)\widetilde{x}^{n_1}\widetilde{y}^{n_2}, \quad \exp\left[-\left(\sqrt{\frac{\widetilde{y}}{2}} - \sqrt{\frac{\widetilde{x}}{2}}\right)^2\right]\widetilde{x}^{n_1}\widetilde{y}^{n_2}$$

for the first two terms of $\Theta^{\text{C·bin}}$, and

$$\exp\left[-\frac{1}{2}(\widetilde{x}+\widetilde{y}) + \sqrt{\widetilde{x}\widetilde{y}}\right]\widetilde{x}^{n_1}\widetilde{y}^{n_2},$$

for the third term of $\Theta^{\text{C·bin}}$ (using Hankel's expansion). Here n_1, n_2 are integers. Since we already showed that these terms shrink exponentially fast in $\widetilde{x}$, the derivative converges to zero. Hence, $\Theta^{\text{C·bin}}$ has a continuous derivative on $[0,T]$.

Step 3 Lastly for the binary call, suppose $S > K$. This makes $\rho = \sqrt{\widetilde{y}/\widetilde{x}} < 1$ for all τ. In this case as well, (B.6) to (B.8) decrease exponentially in $\widetilde{x}$ as ρ does not converge to zero as $\tau \to 0$. Hence, the only nontrivial terms in $\Theta^{\text{C·bin}}$ when we apply (B.1) and Hankel's expansion are

$$\begin{aligned}
\Theta^{\text{C·bin}} &= \left(C_0^{\delta_+} + \frac{C_1^{\delta_+}}{\widetilde{x}} + \frac{C_2^{\delta_+}}{\widetilde{x}^2} + O(\tau^3)\right)\left(r + b - \frac{\theta'}{\theta}\left(p + \frac{\widetilde{x}}{2}\right)\right)\\
&\quad + \left(C_0^{\delta_{++}} + \frac{C_1^{\delta_{++}}}{\widetilde{x}} + \frac{C_2^{\delta_{++}}}{\widetilde{x}^2} + O(\tau^3)\right)\frac{\theta'}{\theta}\frac{\widetilde{x}}{2} + o(\tau)\\
&= r + b - \frac{\theta'}{\theta}\left[\left(C_0^{\delta_+} + \frac{C_1^{\delta_+}}{\widetilde{x}} + \frac{C_2^{\delta_+}}{\widetilde{x}^2}\right)\left(p + \frac{\widetilde{x}}{2}\right) - \left(C_0^{\delta_{++}} + \frac{C_1^{\delta_{++}}}{\widetilde{x}} + \frac{C_2^{\delta_{++}}}{\widetilde{x}^2}\right)\frac{\widetilde{x}}{2}\right] + O(\tau)\\
&= r + b - \frac{\theta'}{\theta}\frac{2pC_1^{\delta_+} + C_2^{\delta_+} - C_2^{\delta_{++}}}{2\widetilde{x}} + O(\tau).
\end{aligned}$$

Here, calculations based on the Appendix D of Ruas et al. (2013) give us

$$C_2^{\mu} = \frac{\tau_\mu(\tau_\mu-1)(\tau_\mu-2)(\tau_\mu-3)}{8} - \frac{\tau_\mu(\tau_\mu-1)}{2}A_1\left(\frac{\mu}{2}-1\right) + A_2\left(\frac{\mu}{2}-1\right)$$

where τ_μ and $A_1(x)$ are given in Step 1, and $A_2(x) = 0.125\Gamma(x+2.5)/\Gamma(x-1.5)$. Then, the asymptotic expansion of θ'/θ plus long and tedious calculations result in the following limit:

$$\lim_{\tau\to 0}\Theta^{\text{C·bin}} = r + b + \frac{a^2 c}{S^{2|\beta|}}.$$

The differentiability of $\Theta^{\text{C·bin}}$ can be handled similarly as in Step 2, using the recurrence relations. Hence, we omit the details.

Step 4 We now look at the case of vanilla calls. For simplicity, we continue to use $\widetilde{x}$ and $\widetilde{y}$ instead of $\widetilde{x}(t,T,S)$ and $\widetilde{y}(t,T,K)$. When $S \neq K$, the theta for vanilla call is given in Eq.(60) of Dias et al. (2015):

$$\begin{aligned}\Theta^{\text{C}}(t,T,S;K) &= -K\Theta^{\text{C·bin}}(t,T,S;K) \\ &\quad -S\left[\widetilde{y}\,\mathsf{p}\left(\widetilde{y};\delta_+,\widetilde{x}\right)\left\{c_1 + \frac{\theta'(\tau)}{\theta(\tau)}\right\} - \widetilde{x}\,\mathsf{p}\left(\widetilde{y};\delta_{++},\widetilde{x}\right)\frac{\theta'(\tau)}{\theta(\tau)}\right] \qquad \text{(B.13)}\end{aligned}$$

where $c_1 = 2|\beta|(r+b)$ as in Step 1 and $\mathsf{p}(y;\mu,x)$ is the probability density function of a noncentral chi-square random variable with μ degrees of freedom and noncentrality parameter x. It is a known fact that $\mathsf{p}(y;\mu,x)$ is expressed as

$$\mathsf{p}(y;\mu,x) = \frac{1}{2}e^{-(x+y)/2}\left(\frac{y}{x}\right)^{\frac{\mu-2}{4}} I_{\frac{\mu-2}{2}}(\sqrt{xy}).$$

Then, the second and third terms of Θ^{C} are dominated by linear combinations of

$$\exp\left[-\frac{1}{2}(\widetilde{x}+\widetilde{y}) + \sqrt{\widetilde{x}\widetilde{y}}\right]\widetilde{x}^{n_1}\widetilde{y}^{n_2}$$

for some n_1 and n_2 using Hankel's expansion given in Step 1. As t approaches T, the blow-up behaviors of $\widetilde{x}$, $\widetilde{y}$ make such components decrease exponentially fast. Consequently, the asymptotic behavior of Θ^{C} is determined by that of binary theta.

Next, we turn our attention to the more complex case $S = K$. First, we will investigate the weak singularity of Θ^{C} with respect to $\tau = T - t$. For the second and third terms of the theta formula above, we apply Hankel's expansion for p and other simpler expansions for $\widetilde{x}$, $\widetilde{y}$, and θ'/θ as in Step 1. Then, it is not difficult to check that

$$\begin{aligned}&\widetilde{y}\,\mathsf{p}\left(\widetilde{y};\delta_+,\widetilde{x}\right)\left\{c_1 + \frac{\theta'(\tau)}{\theta(\tau)}\right\} - \widetilde{x}\,\mathsf{p}\left(\widetilde{y};\delta_{++},\widetilde{x}\right)\frac{\theta'(\tau)}{\theta(\tau)} \\ = \;& N_0\left[\frac{1}{\tau} - \frac{c_1}{2} + O(\tau)\right]\frac{1}{2\sqrt{2\pi N_0}}\left[\sqrt{\tau} + c_5\tau^{3/2} + O(\tau^2)\right]\times\left[\frac{1}{\tau} + \frac{c_1}{2} + O(\tau)\right]\end{aligned}$$

$$
\begin{aligned}
& -N_0\left[\frac{1}{\tau}+\frac{c_1}{2}+O(\tau)\right]\frac{1}{2\sqrt{2\pi N_0}}\Big[\sqrt{\tau}+c_6\tau^{3/2}+O(\tau^2)\Big]\times\left[\frac{1}{\tau}-\frac{c_1}{2}+O(\tau)\right] \\
= \quad & \frac{\sqrt{N_0}}{2\sqrt{2\pi}}\frac{c_5-c_6}{\sqrt{\tau}}+O(1)
\end{aligned}
$$

for the constant c_5 in Step 1 and for some new constant c_6. Combined with the fact that $\Theta^{\mathrm{C\cdot bin}}$ is weakly singular, we have the weak singularity of Θ^{C} with order $\sqrt{\tau}$.

This observation helps us re-write $\Theta^{\mathrm{C\cdot bin}}$ as $h_1/\sqrt{\tau}+h_2$ for some continuous h_1 and h_2. More precisely, we define

$$
\begin{aligned}
h_2(t,T;K) &= -K(r+b)C^{\mathrm{bin}}(t,T,K;K), \\
h_1(t,T;K) &= \sqrt{\tau}\Big[\Theta^{\mathrm{C}}(t,T,K;K)-h_2(t,T;K)\Big].
\end{aligned}
$$

It is clear that these functions are continuous on $[0,T]$ and that $(\partial/\partial T)h_2(t,T;K)$ is weakly singular, thanks to the weak singularity of $\Theta^{\mathrm{C\cdot bin}}$. It remains to show that $(\partial/\partial T)h_1(t,T;K)$ is weakly singular.

We are indeed able to prove that $\partial h_1/\partial\tau = O(\tau^{-1/2})$. Or equivalently,

$$
\tau\frac{\partial h_3}{\partial\tau}+\frac{h_3}{2}=O\left(1\right)
$$

where $h_3 := \Theta^{\mathrm{C}}-h_2$. Since the full derivation relies on long and tedious calculations, we record some important relations in order to compute $\partial h_3/\partial\tau$ and some important parameter values in (B.1) which are helpful in computing its asymptotics. For $\mathsf{p}(y;\mu,x)$, it is verifiable that

$$
\begin{aligned}
\widetilde{y}^{p+1}\mathsf{p}\left(\widetilde{y};\delta_+,\widetilde{x}\right) &= \widetilde{x}^{-p}2^p e^{-\frac{\widetilde{x}+\widetilde{y}}{2}}\left(\frac{\widetilde{y}}{2}\right)^{\frac{\delta_+}{2}+p}H\left(\widetilde{x},\widetilde{y},\delta_+\right), \qquad \text{(B.14)} \\
-\widetilde{y}^{p}\mathsf{p}\left(\widetilde{y};\delta_+,\widetilde{x}\right) &= \frac{\partial}{\partial\widetilde{y}}\Phi_{+1}\left(p,\widetilde{y};\delta_+,\widetilde{x}\right).
\end{aligned}
$$

Additionally, we utilize Equations (2), (3), and (9) of Cohen (1988) which describe recursive relations of p. For asymptotic expansion of the partial derivative of h_3, we also derive the following formulae: in (B.1),

$$
\begin{aligned}
D_1^{\mu} &= \frac{1}{3}\prod_{k=0}^{2}(\tau_\mu-k)-A_1\left(\frac{\mu}{2}-1\right)(\tau_\mu-1), \\
D_2^{\mu} &= \frac{1}{15}\prod_{k=0}^{4}(\tau_\mu-k)-A_1\left(\frac{\mu}{2}-1\right)\frac{1}{3}\prod_{k=0}^{2}(\tau_\mu-k)+A_2\left(\frac{\mu}{2}-1\right)(\tau_\mu-2), \\
G_0^{\mu} &= \frac{\tau_\mu(\tau_\mu-1)}{2}+(\rho-1)\frac{1}{6}\prod_{k=0}^{2}(\tau_\mu-k)+O\left((\rho-1)^2\right) \quad \text{as } \rho\to 1, \\
G_1^{\mu} &= \frac{1}{8}\prod_{k=0}^{3}(\tau_\mu-k)-A_1\left(\frac{\mu}{2}-1\right)\frac{\tau_\mu(\tau_\mu-1)}{2}+O\left((\rho-1)^2\right) \quad \text{as } \rho\to 1.
\end{aligned}
$$

Here, τ_μ, A_1 are given in Step 1 and A_2 in Step 3.

The reader may find that the definitions of h_1 and h_2 here have different parameterizations from those in Theorem 2. However, it does not cause any problem as they depend on the time-to-maturity only under the time-homogeneous JDCEV model. ■

C Proofs for the Laplace Transform under the JDCEV Model

In this appendix, we first give certain conditions under which the Laplace transform of $\mathbf{w}$ exists. Next, those conditions are shown to be valid under the JDCEV model.

Theorem 5 *Assume that the asset price process S_t is time-homogeneous, and that the conditions in Theorem 3 hold.*

(i) Suppose that the hedging instrument is binary call and that there exists a positive constant C independent of t such that $\left|\Theta^{\mathrm{C\cdot bin}}(0,t,U;U)\right| \leq \frac{C}{\sqrt{t}}$ for all $t>0$. If $\Theta(0,t,U;U)$ is continuous and $|\Theta(0,t,U;U)|$ is bounded above by an increasing exponential function for all $t \geq 0$, then $\widehat{\mathbf{w}}(\lambda)$ is valid for $\mathrm{Re}(\lambda) > D$ with some constant D.

(ii) Suppose that the hedging instrument is European call and that there exist positive constants C_1, C_2 and C_3 independent of t such that $|\widetilde{h}_1(t)| < C_1$, $|\widetilde{h}_2(t)| < C_2$ and $|h_2(t)| < C_3$ for all $t>0$. Here $\widetilde{h}_1$, $\widetilde{h}_2$ and h_2 are defined in the proof of Lemma 2. If $\Theta(0,t,U;U)$ is continuously differentiable and $|\mathrm{d}\Theta(0,t,U;U)/\mathrm{d}t|$ is bounded above by an increasing exponential function for all $t \geq 0$, then $\widehat{\mathbf{w}}(\lambda)$ is valid for $\mathrm{Re}(\lambda) > D$ with some constant D.

Proof: We have already proved the continuity of the weight function $\mathbf{w}(t)$ defined on $t \in [0,T]$ in Theorem 3. This fact can be naturally extended to the domain $[0,\infty)$. So, it is enough to show $\mathbf{w}(t)$ is exponentially bounded in order to ensure that $\widehat{\mathbf{w}}(\lambda)$ is well defined.

Case (i) Suppose $|\Theta(0,t,U;U)|$ is bounded by ae^{bt} for some positive constants a,b. We apply Theorem 2 in Medved (1997) to (7) for obtaining a weakly singular Gronwall inequality

$$\begin{aligned}
|\mathbf{w}(t)| &\leq 2|\Theta(0,t,U;U)| + 2\int_0^t |\mathbf{w}(u)||\Theta^{\mathrm{C\cdot bin}}(0,t-u,U;U)|\mathrm{d}u \\
&\leq 2ae^{bt} + 2\int_0^t \frac{C}{\sqrt{t-u}}|\mathbf{w}(u)|\mathrm{d}u \\
&\leq D_1 e^{D_2 t}
\end{aligned}$$

for all $t>0$ and some constants D_1 and D_2. We note that the conditions on $\Theta^{\mathrm{C\cdot bin}}$ and Θ are used to derive the second line.

Case (ii) For a fixed constant $T > 0$, we have the following Volterra integral equation of the second kind from Lemma 2 and (8):

$$L_1(0)\mathbf{w}(t) + \int_0^t \frac{\partial L(t-s)}{\partial t}\mathbf{w}(s)\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\Theta(0,s,U;U)}{\sqrt{t-s}}\mathrm{d}s$$

where $L_1(0) \neq 0$ and $L(u-s) := L_1(u-s) + L_2(u-s)$ is weakly singular. We note that each of functions is written in the difference form because we consider time homogeneous cases.

The derivatives $\frac{\partial L_1(t-s)}{\partial t}$ and $\frac{\partial L_2(t-s)}{\partial t}$ are expressed as

$$\begin{aligned}
\frac{\partial L_1(t-s)}{\partial t} &= \frac{1}{\sqrt{t-s}}\int_0^1 \frac{1}{\sqrt{1-y}}\widetilde{h}_1\left((t-s)y\right)\mathrm{d}y, \\
\frac{\partial L_2(t-s)}{\partial t} &= \frac{1}{2\sqrt{t-s}}\int_0^1 \frac{h_2\left((t-s)y\right)}{\sqrt{1-y}}\mathrm{d}y \\
&\quad + \int_0^1 \frac{\sqrt{y}}{\sqrt{1-y}}\widetilde{h}_2\left((t-s)y\right)\mathrm{d}y.
\end{aligned}$$

It is easy to show that $\frac{\partial L_1(t-s)}{\partial t}$, $\frac{\partial L_2(t-s)}{\partial t}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\Theta(0,s,U;U)}{\sqrt{t-s}}\mathrm{d}s$ are of class $L^1(0,T)$ by given conditions. Therefore, the above Volterra integral equation with the convolution kernel has a unique solution $\mathbf{w} \in L^1(0,T)$ by Lemma 1 of Miller and Feldstein (1997). This implies that the exponential boundedness of $\mathbf{w}(t)$ is required for t strictly bounded away from zero, for instance, $t \geq \varepsilon$ for some fixed $\varepsilon > 0$.

With this fact in mind, define $\widetilde{\mathbf{w}}(t) = \mathbf{w}(t+\varepsilon)$ for $t \geq 0$. Then, from the above integral equation, we obtain for $t \geq \varepsilon$,

$$\begin{aligned}
|\mathbf{w}(t)| &\leq \frac{1}{L_1(0)}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\Theta(0,s,U;U)}{\sqrt{t-s}}\mathrm{d}s\right| + \frac{1}{L_1(0)}\int_0^t \frac{C_4}{\sqrt{t-s}}\left|\mathbf{w}(s)\right|\mathrm{d}s \\
&= \frac{1}{L_1(0)}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\Theta(0,t-u,U;U)}{\sqrt{u}}\mathrm{d}u\right| + \frac{1}{L_1(0)}\int_0^\varepsilon \frac{C_4}{\sqrt{t-s}}\left|\mathbf{w}(s)\right|\mathrm{d}s \\
&\quad + \frac{1}{L_1(0)}\int_\varepsilon^t \frac{C_4}{\sqrt{t-s}}\left|\mathbf{w}(s)\right|\mathrm{d}s \\
&\leq \frac{C_5}{\sqrt{t}} + \frac{1}{L_1(0)}\int_0^t \frac{1}{\sqrt{u}}\left|\frac{\mathrm{d}}{\mathrm{d}t}\Theta(0,t-u,U;U)\right|\mathrm{d}u + C_6\sqrt{t} \\
&\quad + \frac{1}{L_1(0)}\int_\varepsilon^t \frac{C_4}{\sqrt{t-s}}\left|\mathbf{w}(s)\right|\mathrm{d}s.
\end{aligned}$$

Here we get bounding constants C_4 and C_5 thanks to the given assumptions. Also, utilizing the assumption on $|\mathrm{d}\Theta/\mathrm{d}t|$, it is not a difficult matter to see that the second term is bounded above by an exponential function. The term $C_6\sqrt{t}$ is obtained because $\mathbf{w}$ is L^1 in $(0,\varepsilon/2)$ and $1/\sqrt{t-s}$ is integrable over $(\varepsilon/2,\varepsilon)$.

Aggregating all these observations, the above inequality can be re-written in terms of $\widetilde{\mathbf{w}}$ as follows: for $t \geq 0$ and for some constants $a, b, C > 0$,

$$|\widetilde{\mathbf{w}}(t)| \leq ae^{bt} + \frac{1}{L_1(0)}\int_0^t \frac{C}{\sqrt{t-u}}\left|\widetilde{\mathbf{w}}(u)\right|\mathrm{d}u.$$

We are then able to apply Theorem 2 in Medved (1997) to conclude $|\widetilde{\mathbf{w}}(t)| \leq D_3 e^{D_4 t}$ for some $D_3, D_4 > 0$. ■

Proposition 6 *Assume that the asset price follows the JDCEV model with $r + b \neq 0$. Then,*

(i) $\lim_{t\to\infty} \sqrt{t}\Theta^{\mathrm{C\cdot bin}}(0,t,S;K) = 0$ *and* $\lim_{t\to\infty} \sqrt{t}\Theta^{\mathrm{C}}(0,t,S;K) = 0$ *for all $S > 0$ and $K > 0$.*

(ii) there exist positive constants C_1, C_2 and C_3, which are independent of t, such that $|\widetilde{h}_1(t)| < C_1$, $|\widetilde{h}_2(t)| < C_2$ and $|h_2(t)| < C_3$ for all $t > 0$.

Proof: (i) In the proof of Proposition 2, it is easy to check as $t \to \infty$

$$\begin{aligned}
&\theta(t) \to \frac{a^2}{2|\beta|(r+b)},\\
&\theta'(t) = a^2 e^{-2|\beta|(r+b)t} \to 0,\\
&\widetilde{x}(0,t,S) \to \frac{2(r+b)S^{2|\beta|}}{|\beta|a^2},\\
&\widetilde{y}(0,t,S) = \frac{2(r+b)S^{2|\beta|}e^{-2|\beta|(r+b)t}}{a^2|\beta|(1-e^{-2|\beta|(r+b)t})} \to 0.
\end{aligned}$$

By the monotone convergence theorem, we have

$$\begin{aligned}
\lim_{y\to 0} \Phi_{+1}(p,y;\mu,x) &= 2^p \sum_{n=0}^{\infty} e^{-x/2}\Big(\frac{x}{2}\Big)^n \frac{\Gamma(\nu+p+n+1,y/2)}{n!\Gamma(\nu+n+1)}\\
&= 2^p \sum_{n=0}^{\infty} e^{-x/2}\Big(\frac{x}{2}\Big)^n \frac{\Gamma(\nu+p+n+1)}{n!\Gamma(\nu+n+1)}.
\end{aligned}$$

Also, the right hand side is convergent and finite by the ratio test

$$\lim_{n\to\infty} \frac{x}{2(n+1)} \frac{\Gamma(\nu+n+1)\Gamma(\nu+p+n+2)}{\Gamma(\nu+n+2)\Gamma(\nu+p+n+1)} = \lim_{n\to\infty} \frac{x(\nu+p+n+1)}{2(n+1)(\nu+n+1)} = 0.$$

for all finite p, μ, x. Similarly, we can derive that $\lim_{y\to 0} H(x,y,z) = 0$ from the definition $H(x,y,z) = \sum_{n=0}^{\infty} \frac{(xy/4)^n}{n!\Gamma(z/2+n)}$, and that $\lim_{y\to 0} \widetilde{\Phi}_{+1}(p,y;\mu,x)$ is finite using (B.4). Combining these results to (B.2), we can see that $\sqrt{t}\Theta^{\mathrm{C\cdot bin}}(0,t,S;K)$ vanishes as $t \to \infty$. For the vanilla call case, it follows from (B.13) and (B.14).

(ii) From Step 4 in Proposition 2, h_1 and h_2 are determined by

$$\begin{aligned}
h_2(t;U) &= -U(r+b)C^{\mathrm{bin}}(0,t,U;U),\\
h_1(t;U) &= \sqrt{t}\Big[\Theta^{\mathrm{C}}(0,t,U;U) - h_2(t;U)\Big].
\end{aligned}$$

It is clear that $h_2(t)$ is uniformly bounded in t. Also, $\widetilde{h}_2(t) = \sqrt{t}\frac{\partial h_2(t)}{\partial t}$ is continuous and convergent to zero by the above results. Lastly, we need to compute $\frac{\partial \Theta^{\mathrm{C}}(0,t,U;U)}{\partial t}$ to check $\widetilde{h}_1(t)$. This computation is possible using the recursive relations (B.3), (B.14), (B.9), (B.10), (B.11) and (B.12). After tedious calculations, it is shown that $\widetilde{h}_1(t)$ converges to zero as $t \to \infty$. ■

In the remainder of this section, we present some integral formulae which are essential in computing Laplace transforms of basic contingent claims. We also give proofs for Propositions 4 and 5. Lastly, we comment on Laplace inversion.

To make our presentation self-contained, we record $\mathfrak{m}$, ψ_s, and ϕ_s from Mendoza-Arriaga et al. (2010):

$$\begin{aligned}\psi_s(x) &= x^{\frac{1}{2}+\beta-c}\exp\left(-\frac{1}{2}Ax^{-2\beta}\right)M_{\eta(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right),\\ \phi_s(x) &= x^{\frac{1}{2}+\beta-c}\exp\left(-\frac{1}{2}Ax^{-2\beta}\right)W_{\eta(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)\end{aligned}$$

where M, W are the first and the second Whittaker functions. Here, parameter values are given by $\nu = (1+2c)/(2|\beta|)$, $A = (r+b)/(a^2|\beta|)$, and

$$\eta(s) = \frac{\nu-1}{2} - \frac{s+\xi}{\omega}, \quad \omega = 2|\beta|(r+b), \quad \xi = 2c(r+b)+b.$$

Lastly, the speed density and the Wronskian of two fundamental solutions are

$$\begin{aligned}\mathfrak{m}(x) &= \frac{2}{a^2}x^{2c-2-2\beta}\exp\left(Ax^{-2\beta}\right),\\ w_s &= \frac{2(r+b)\Gamma(1+\nu)}{a^2\Gamma(\nu/2+1/2-\eta(s))}.\end{aligned}$$

The next lemma reports $\mathcal{I}(l,u;\alpha)$ and $\mathcal{J}(l,u;\alpha)$ for the JDCEV model, which are important ingredients of our Laplace transform based approach.

Lemma 4 *Suppose that $\beta<0$ and $r+b>0$. Then, if the real part of $\bar{p}(\alpha)+(\nu+1)/2$ is positive, then we have*

$$\mathcal{I}_s(0,K;\alpha) = \frac{A^{\frac{\nu+1}{2}}K^{|\beta|(2\bar{p}(\alpha)+\nu+1)}}{a^2|\beta|(\bar{p}(\alpha)+\frac{\nu+1}{2})}{}_2F_2\left[\bar{p}(\alpha)+\frac{\nu+1}{2},\frac{\nu+1}{2}-\eta(s);\bar{p}(\alpha)+\frac{\nu+3}{2},\nu+1;AK^{-2\beta}\right].$$

If the real part of $\bar{p}(\alpha)+\eta(s)$ is negative, then we have

$$\begin{aligned}\mathcal{J}_s(K,\infty;\alpha) = {}& \frac{A^{-\bar{p}(\alpha)}}{a^2|\beta|}\frac{\Gamma\left(\bar{p}(\alpha)+\frac{\nu+1}{2}\right)\Gamma\left(\bar{p}(\alpha)-\frac{\nu-1}{2}\right)\Gamma\left(-\bar{p}(\alpha)-\eta(s)\right)}{\Gamma\left(\frac{\nu+1}{2}-\eta(s)\right)\Gamma\left(\frac{1-\nu}{2}-\eta(s)\right)}\\ &-\frac{A^{\frac{\nu+1}{2}}K^{|\beta|(2\bar{p}(\alpha)+\nu+1)}}{a^2|\beta|(\bar{p}(\alpha)+\frac{\nu+1}{2})}\frac{\Gamma(-\nu)}{\Gamma\left(\frac{1-\nu}{2}-\eta(s)\right)}\\ &\quad\times{}_2F_2\left[\bar{p}(\alpha)+\frac{\nu+1}{2},\frac{\nu+1}{2}-\eta(s);\nu+1,\bar{p}(\alpha)+\frac{\nu+3}{2};AK^{-2\beta}\right]\\ &-\frac{A^{-\frac{\nu-1}{2}}K^{|\beta|(2\bar{p}(\alpha)-\nu+1)}}{a^2|\beta|(\bar{p}(\alpha)-\frac{\nu-1}{2})}\frac{\Gamma(\nu)}{\Gamma\left(\frac{\nu+1}{2}-\eta(s)\right)}\\ &\quad\times{}_2F_2\left[\bar{p}(\alpha)-\frac{\nu-1}{2},\frac{1-\nu}{2}-\eta(s);1-\nu,\bar{p}(\alpha)-\frac{\nu-3}{2};AK^{-2\beta}\right].\end{aligned}$$

Here, $\bar{p}(\alpha) = -(\alpha - \beta + c - 0.5)/(2\beta)$ *and* ${}_2F_2[a_1, a_2; b_1, b_2; z]$ *is the generalized hypergeometric function defined by*

$$ {}_2F_2[a_1, a_2; b_1, b_2; z] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!} $$

with Pochhammer symbols $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$. *In general, we have*

$$ \begin{aligned} \mathcal{I}_s(l, u; \alpha) &= \mathcal{I}_s(0, u; \alpha) - \mathcal{I}_s(0, l; \alpha), \\ \mathcal{J}_s(l, u; \alpha) &= \mathcal{J}_s(l, \infty; \alpha) - \mathcal{J}_s(u, \infty; \alpha). \end{aligned} $$

Proof: Computations are involved but straightforward by the change of variable $y = Ax^{-2\beta}$ and by utilizing some integrals involving Whittaker functions; specifically, see Equations 1.13.1.1 and 1.13.1.2 in Prundikov et al. (1990). ■

To invert Laplace transforms shown in this paper, we use the Talbot algorithm proposed by Abate and Valko (2004). The algorithm has one parameter M, the number of terms to be summed and we specify it as 32. In Lemma 4, the condition $\mathfrak{Re}(\bar{p}(\alpha) + \frac{\nu+1}{2}) > 0$ for $\mathcal{I}_s$ is always true for $\alpha \geq 0$. Also, the condition for $\mathcal{J}_s$ is

$$ \mathfrak{Re}(\bar{p}(\alpha) + \eta(s)) < 0 \Longleftrightarrow \mathfrak{Re}(s) > \alpha(r+b) - b. $$

However, in Proposition 4, we set $s = r + \lambda$ and α is either 0 or 1. Therefore, $\mathfrak{Re}(\lambda) > 0$ is enough to make the above conditions fulfilled. Thus, the formulae in Propositions 4 and 5 are valid as long as we use λ with positive real part. Consequently, we can successfully perform Laplace transform inversion.

Proof of Proposition 4: We consider the case of European call only. Other cases are almost identical and assume that default has not occurred by the current time 0. Let us first write $f(t, x) = \mathsf{E}_x \left[e^{-rt} (S_t - K)^+ \mathbf{1}_{\{\zeta > t\}} \right]$ with $S_0 = x$ and the expectation is defined with respect to the measure $\mathbb{Q}$. The Kolmogorov backward equation for f reads

$$ \frac{\partial f}{\partial t} = \mathfrak{G} f - rf $$

where the boundary condition is $f(0, x) = (x - K)^+$ and $\mathfrak{G}$ is the infinitesimal generator for the JDCEV model:

$$ \mathfrak{G} f = \frac{1}{2} a^2 x^{2\beta+2} \frac{\partial^2 f}{\partial x^2} + \left(r + b + ca^2 x^{2\beta} \right) x \frac{\partial f}{\partial x} - \left(b + ca^2 x^{2\beta} \right) f. $$

Then, the above partial differential equation is converted into the following equation for $\widehat{f}(\lambda, x)$ after Laplace transforms

$$ \frac{1}{2} a^2 x^{2\beta+2} \frac{\partial^2 \widehat{f}}{\partial x^2} + \left(r + b + ca^2 x^{2\beta} \right) x \frac{\partial \widehat{f}}{\partial x} - \left(\lambda + r + b + ca^2 x^{2\beta} \right) \widehat{f} + f(0, x) = 0. $$

with boundary conditions $\widehat{f}(\lambda,0)=0$ and $\widehat{f}(\lambda,\infty)=\lim_{x\to\infty}\frac{(x-K)}{\lambda}$. The latter condition shows the asymptotic rate of increase for $\widehat{f}(\lambda,x)$.

Recall that ψ_s and ϕ_s with $s=\lambda+r$ are the two linearly independent fundamental solutions of the above differential equation. See Borodin and Salminen (2002) for their boundary conditions and related explanations. The method of Green's functions then gives us the solution (see, e.g., Stakgold (1979)):

$$\widehat{f}(\lambda,x)=\int_0^\infty G_s(x,y)f(0,y)\mathrm{d}y+\left[\lim_{z\to\infty}\frac{(z-K)}{\lambda\psi_s(z)}\right]\psi_s(x)=\int_K^\infty G_s(x,y)(y-K)\mathrm{d}y \tag{C.1}$$

where Green's function G is defined as

$$G_s(x,y)=\frac{\mathfrak{m}(y)}{w_s}\begin{cases}\psi_s(x)\phi_s(y), & x\le y;\\ \psi_s(y)\phi_s(x), & x>y.\end{cases}$$

In (C.1), the second equality is obtained from the asymptotic properties of Whittaker functions (Linetsky, 2004):

$$\begin{aligned}\lim_{z\to\infty}\frac{(z-K)}{\lambda\psi_s(z)} &= \lim_{z\to\infty}\left[\lambda z^{-\frac{1}{2}+\beta-c}\exp\left(-\frac{1}{2}Az^{-2\beta}\right)M_{\eta(s),\frac{\nu}{2}}\left(Az^{-2\beta}\right)\right]^{-1}\\ &= \lim_{z\to\infty}\frac{C}{\lambda}z^{\frac{1}{2}-\beta+c-2\beta\eta(s)}\end{aligned}$$

for some constant C. The last limit becomes zero whenever the real part of λ is positive.

Then, we simply observe that, with $s=\lambda+r$,

$$\begin{aligned}\widehat{f}(\lambda,x) &= \int_K^{K\vee x}\frac{\mathfrak{m}(y)}{w_s}\psi_s(y)\phi_s(x)(y-K)\mathrm{d}y+\int_{K\vee x}^\infty\frac{\mathfrak{m}(y)}{w_s}\psi_s(x)\phi_s(y)(y-K)\mathrm{d}y\\ &= \frac{\phi_s(x)}{w_s}\left[\int_K^{K\vee x}y\psi_s(y)\mathfrak{m}(y)\mathrm{d}y-K\int_K^{K\vee y}\psi_s(y)\mathfrak{m}(y)\mathrm{d}y\right]\\ &\quad+\frac{\psi_s(x)}{w_s}\left[\int_{K\vee x}^\infty y\phi_s(y)\mathfrak{m}(y)\mathrm{d}y-K\int_{K\vee x}^\infty\phi_s(y)\mathfrak{m}(y)\mathrm{d}y\right]\\ &= \frac{\phi_s(x)}{w_s}\left[\mathcal{I}_s(K,K\vee x;1)-K\mathcal{I}(K,K\vee x;0)\right]\\ &\quad+\frac{\psi_s(x)}{w_s}\left[\mathcal{J}_s(K\vee x,\infty;1)-K\mathcal{J}_s(K\vee x,\infty;0)\right].\end{aligned}$$

This results in $\widehat{C}^{\mathrm{E}}$ in the statement. Repeat the same procedure for other basic claims. ■

Proof of Proposition 5: Let us first consider the case of American binary call with strike K, maturity t, and the initial stock price x. The hitting time of K is denoted by $\tau_K:=\inf\{u>0: S_u=K\}$, and the default time by ζ. Assume that default has not occurred by the current time 0. Then, the option price $f(t,x)$ is given by

$$f(t,x)=\mathsf{E}_x\left[e^{-r\tau_K}\mathbf{1}_{\{\tau_K\le t\}}\mathbf{1}_{\{\tau_K<\zeta\}}\right]$$

with $S_0 = x$ and the expectation is defined with respect to the measure $\mathbb{Q}$. Its Laplace transform is easily seen to be

$$\begin{aligned}
\widehat{f}(\lambda, x) &= \int_0^\infty e^{-\lambda t} f(t,x)\mathrm{d}t \\
&= \mathsf{E}_x\left[\int_0^\infty e^{-\lambda t - r\tau_K}\mathbf{1}_{\{\tau_K \le t\}}\mathbf{1}_{\{\tau_K<\zeta\}}\mathrm{d}t\right] \\
&= \frac{1}{\lambda}\mathsf{E}_x\left[e^{-(\lambda+r)\tau_K}\mathbf{1}_{\{\tau_K<\zeta\}}\right] \\
&= \frac{1}{\lambda}\mathsf{E}_x\left[e^{-(\lambda+r)\tau_K - \int_0^{\tau_K}\lambda(S_u)\mathrm{d}u}\right].
\end{aligned}$$

On the other hand, it is known to be

$$\mathsf{E}_x\left[e^{-(\lambda+r)\tau_K - \int_0^{\tau_K}\lambda(S_u)\mathrm{d}u}\right] = \begin{cases} \dfrac{\psi_s(x)}{\psi_s(K)}, & x \le K; \\ \dfrac{\phi_s(x)}{\phi_s(K)}, & x \ge K. \end{cases}$$

See p.18 of Borodin and Salminen (2002). Thus $\widehat{C}^{\mathrm{A}}$ is immediate. We can apply similar arguments for $\widehat{P}_0^{\mathrm{A}}$.

For the Laplace transform of v_D, let us denote the price of a defaultable zero-coupon bond with unit face value and zero recovery upon default by $B_0(0,t,x)$. Here, t is the bond maturity and x is the initial stock price. The very definition of v_D implies

$$v_D(0,t,x) + B_0(0,t,x) = e^{-rt},$$

from which we obtain

$$\begin{aligned}
\widehat{v}_D(\lambda, x) &= \int_0^\infty e^{-\lambda t}\left\{e^{-rt} - B_0(0,t,x)\right\}\mathrm{d}t \\
&= \frac{1}{\lambda + r} - \int_0^\infty e^{-\lambda t}\mathsf{E}_x\left[e^{-rt}\mathbf{1}_{\{t<\zeta\}}\right]\mathrm{d}t.
\end{aligned}$$

The proof of Proposition 4 indicates that this second term can be re-written using Green's function. In particular, any term involving $\psi_s(x)$ disappears because of the boundary condition of ψ_s at the natural boundary ∞. See Carr and Linetsky (2006) for boundary classification of the JDCEV model. Hence, we obtain

$$\begin{aligned}
\int_0^\infty G_s(x,y)\mathrm{d}y &= \int_0^x \frac{\mathfrak{m}(y)}{w_s}\psi_s(y)\phi_s(x)\mathrm{d}y + \int_x^\infty \frac{\mathfrak{m}(y)}{w_s}\psi_s(x)\phi_s(y)\mathrm{d}y \\
&= \frac{\phi_s(x)}{w_s}\mathcal{I}_s(0,x;0) + \frac{\psi_s(x)}{w_s}\mathcal{J}_s(x,\infty;0).
\end{aligned}$$

The proof is now complete. ■

D Case Studies

Our approach can be applied to a wide class of exotic options thanks to two reasons.

- The basic idea of the boundary matching approach applies to options with multiple barriers. Furthermore, each barrier can have independent features. Our first example is a general double knock-in option which has different knock-in payoffs, depending on which barrier is first hit. The second example is a KIKO option which has both knock-in and knock-out barriers.
- If we set the boundary $\mathbb{U} = \{U_s\}$ as the exercise boundary of an American put, then $\Psi(t, T, U_t; \mathbb{U}) = K - U_t$ and (1) solves the American option valuation problem. Or, it could be the value of another barrier option or American options, for example. This characteristic allows us to handle sequential barriers or double touch options.

D.1 General Double Barrier Knock-in

Our approach to up-and-in barrier options can be suitably modified to those options with flexible payoff structures at the knock-in or knock-out boundaries. A quite natural extension of standard barrier options is a general double barrier knock-in option, which becomes either a vanilla put or a vanilla call depending on which one of two barriers is hit first. We denote the time-0 price of this option by $\Psi(0, T, S_0; \{L, U\})$ with $L < S_0 < U$:

$$\Psi(0, T, S_0; \{L, U\}) = e^{-rT} \mathsf{E}\left[(K_1 - S_T)^+ \mathbf{1}_{\{\tau_U < \tau_L, \tau_U < T, \zeta > T\}} + (S_T - K_2)^+ \mathbf{1}_{\{\tau_L < \tau_U, \tau_L < T, \zeta > T\}}\right]$$

where $\tau_U = \{t > 0 : S_t = U\}$ and $\tau_L = \{t > 0 : S_t = L\}$. Pelsser (2000) computed double knock-out options under the Black-Scholes model by utilizing the Laplace transforms of relevant hitting times and their inversions. However, pricing general double knock-in options relies on numerical integration of those hitting times due to the lack of in-and-out parities.

Based on our boundary matching approach, we successfully derive the Laplace transform of the above double knock-in option Ψ. Furthermore, we have an exact static hedging portfolio. Since the option has up-and-in feature and down-and-in feature, we use European puts (or binary puts) with zero recovery and strike L as well as European calls (or binary calls) with strike U. The arguments used for up-and-in barrier can be applied to confirm that the following is our static hedging portfolio:

$$\begin{aligned}\Psi(0, T, S_0; \{L, U\}) = &\int_0^T \mathbf{w}_1(u) C(0, T-u, S_0; U) \mathrm{d}u + \int_0^T \mathbf{w}_2(u) P_0(0, T-u, S_0; L) \mathrm{d}u \\ &+ \Psi_1^* C^{\mathrm{A}}(0, T, S_0; U) + \Psi_2^* P_0^{\mathrm{A}}(0, T, S_0; L),\end{aligned} \tag{D.1}$$

Table 4: Prices of double barrier puts with $S_0 = 100$, $U = 120$, $L = 90$, $T = 0.5$, $b = 0.02$, $r = 10\%$ and $c = 0.5$; TM uses 1,000 time steps.

K	β	a	exact solution	Dias et al. (2015) TM	Dias et al. (2015) stopping time approach
95	-1	2.5E+01	4.5892	4.5892	4.5892
95	-2	2.5E+03	4.6725	4.6725	4.6725
95	-3	2.5E+05	4.8070	4.8070	4.8070
95	-4	2.5E+07	4.9944	4.9944	4.9944
100	-1	2.5E+01	5.9183	5.9183	5.9183
100	-2	2.5E+03	5.8789	5.8789	5.8789
100	-3	2.5E+05	5.8819	5.8819	5.8819
100	-4	2.5E+07	5.9263	5.9263	5.9263
105	-1	2.5E+01	7.5276	7.5276	7.5277
105	-2	2.5E+03	7.3542	7.3542	7.3542
105	-3	2.5E+05	7.2219	7.2219	7.2219
105	-4	2.5E+07	7.1247	7.1247	7.1248

where $\Psi_1^* = (K_1 - U)^+$ and $\Psi_2^* = (L - K_2)^+$ are introduced to handle reverse barriers. Furthermore, the American binary put with zero recovery P_0^{A} is considered. Here, it is implicitly assumed that the target option has zero recovery upon default. If knocked-in at time t, then Ψ becomes $P^{\mathrm{E}}(t, T, U; K_1)$ or $C^{\mathrm{E}}(t, T, L; K_2)$. These are matched to the values of the right hand side of (D.1), yielding 2-dimensional Volterra integral equations. Our previously developed theorems are naturally extended to this case, guaranteeing the existence and uniqueness of $\mathbf{w}_i$'s. We refer the reader to the Appendix E for integral equations and the Laplace transform of Ψ.

To test the effectiveness of our method, we use another double barrier option. Particularly, we price double barrier knock-in puts for which Dias et al. (2015) provided option values under 9 different parameter settings. Their pricing methods are the TM method (with 1000 time steps) and the stopping time approach proposed by Kuan and Webber (2003). For reader's convenience, we also record formulas for double barrier knock-in puts in the Appendix E. Table 4 reports valuation results which show almost identical option prices.

D.2 Double Barrier with Knock-in Knock-out

Another interesting variant of double barrier options is an option that has a knock-in feature for upper barrier and a knock-out feature for down barrier. This so called KIKO option used to be quite popular in the Korean foreign exchange market, with all the legal lawsuits that followed after the credit crisis. See Khil and Suh (2010) for more discussions of KIKO options.

More specifically, the option holder has a short position in up-and-in call and a long position in down-and-out put. To the authors' knowledge, our presentation is the first to give an analytic pricing formula for KIKO options. Its payoff structure at maturity is as follows:

$$\begin{cases} -\theta(S_T-K)^+ & \text{if the upper barrier } U \text{ is hit first before } T, \\ 0 & \text{if the lower barrier } L \text{ is hit first before } T, \\ (K-S_T)^+ & \text{otherwise.} \end{cases} \tag{D.2}$$

The time-0 price of KIKO is written by

$$\Psi(0,T,S_0;\{L,U\}) = e^{-rT}\mathsf{E}\left[(K-S_T)^+\mathbf{1}_{\{\tau_U>T,\tau_L>T,\zeta>T\}} - \theta(S_T-K)^+\mathbf{1}_{\{\tau_U<\tau_L,\tau_U\le T,\zeta>T\}}\right]$$

Here, θ is a leverage factor (usually two or three) and $L<K, S_0<U$. Our static hedging portfolio is then given by

$$\begin{aligned}\Psi(0,T,S_0;\{L,U\}) &= \int_0^T \mathbf{w}_1(u)C(0,T-u,S_0;U)\mathrm{d}u + \int_0^T \mathbf{w}_2(u)P_0(0,T-u,S_0;L)\mathrm{d}u \\ &\quad +\Psi_1^* C^{\mathrm{A}}(0,T,S_0;U) + \Psi_2^* P_0^{\mathrm{A}}(0,T,S_0;L) + P_0^{\mathrm{E}}(0,T,S_0;K).\end{aligned}$$

where we define $\Psi_1^* = -\theta(U-K)$ and $\Psi_2^* = -(K-L)$. It is again assumed that there is no recovery for Ψ. Similarly as in double knock-in options, we construct two dimensional Volterra integral equations to match boundary values of the target option and our hedging portfolio along two barriers U and L. See the Appendix E for the integral equations and the Laplace transform of KIKO options.

D.3 Sequential Barrier

A roll-down call is identical to a European call with strike K_0 if the asset price has not crossed the first lower barrier $L_1 < K_0$ before maturity. If L_1 is hit prior to maturity, the option strike is rolled down to a new strike K_1 between L_1 and K_0, but a knock-out barrier L_2 lower than L_1 newly appears:

$$\begin{aligned}&\Psi(0,T,S_0;\{K_0,K_1\},\{L_1,L_2\}) \\ &\quad = \quad e^{-rT}\left[(S_T-K_0)^+\mathbf{1}_{\{\zeta>T,\tau_{L_1}>T\}} + (S_T-K_1)^+\mathbf{1}_{\{\zeta>T,\tau_{L_1}\le T\,\tau_{L_2}>T\}}\right]\end{aligned}$$

where $\tau_{L_1} = \inf\{t > 0 : S_t = L_1\}$ and $\tau_{L_2} = \{t > 0 : S_t = L_2\}$ and assumption that there is no recovery value upon default. This double-barrier case of roll down options is naturally extendable to the case of arbitrary number of decreasing barriers and strikes. See Gastineau (1994) or Carr et al. (1998) for an introduction.

Actually Carr et al. (1998) described a static hedging method for roll down calls, by making the following observation:

$$\begin{aligned}
&\Psi(0, T, S_0; \{K_0, K_1\}, \{L_1, L_2\}) \\
&\quad = \quad \Psi_{\text{out}}(0, T, S_0; K_0, L_1) + \Psi_{\text{out}}(0, T, S_0; K_1, L_2) - \Psi_{\text{out}}(0, T, S_0; K_1, L_1). \qquad \text{(D.3)}
\end{aligned}$$

Here, the left side is the price of the target option, and $\Psi_{\text{out}}(t, T, S_t; K, L)$ is the price of a standard down-and-out call with barrier L and strike K. Carr et al. (1998) then applied their method of static replication of standard barrier options, under some assumption on the symmetry of the volatility function.

It is certainly possible to apply boundary matching to each of three down-and-out calls. (We would have three dimensional Volterra integral equations.) To demonstrate the flexibility of our approach, we derive analytic formulas without the aid of such a decomposition. Let us consider the last two terms of (D.3). Then, it is easy to see that this is a down-and-in option with barrier L_1 and the boundary value $\Psi_{\text{out}}(t, T, L_1; K_1, L_2)$ if L_1 is first hit at t. This down-and-in option, denoted by $\Psi_{\text{in}}(0, T, S_0; L_1)$, has the following representation:

$$\int_0^T \mathbf{w}_2(u) P_0(0, T-u, S_0; L_1) \mathrm{d}u.$$

Together with a static hedging portfolio for the first term in (D.3) as explained in Section 3.4, the price of the target roll down call is given by

$$\begin{aligned}
\Psi(0, T, S_0; \{K_0, K_1\}, \{L_1, L_2\}) \quad &= \quad C^{\text{E}}(0, T, S_0; K_0) - \int_0^T \mathbf{w}_1(u) P_0(0, T-u, S_0; L_1) \mathrm{d}u \\
&\quad + \int_0^T \mathbf{w}_2(u) P_0(0, T-u, S_0; L_1) \mathrm{d}u..
\end{aligned}$$

In order to find $\mathbf{w}_1$ and $\mathbf{w}_2$, we match boundary values of these standard down-and-out and exotic down-and-in options, which lead us to two-dimensional Volterra integral equations. The reader is referred to the Appendix E for the integral equations and Laplace transforms. Simply for an illustrative purpose, we provide Table 5 where some prices of KIKO options and roll down calls are given under different parameter settings.

Remark 4 To insure the existence of a static hedging portfolio, we need to show that the time derivative of $\Psi_{\text{out}}(t, T, L_1; K_1, L_2)$ is continuously differentiable. One simple way of seeing this is

Table 5: Option values for KIKO and roll down calls. Parameters are (i) KIKO: $S_0 = 100$, $U = 130$, $L = 80$, $T = 0.5$, $K = 120$, $\theta = 2$, $b = 0.02$, $r = 10\%$ and $c = 0.75$, (ii) roll-down call: $S_0 = 100$, $L_1 = 90$, $L_2 = 70$, $K_0 = 100$, $K_1 = 95$, $T = 1$, $b = 0.02$, $r = 10\%$ and $c = 0.5$.

KIKO			Roll-Down Call		
β	a	exact solution	β	a	exact solution
-1	2.5E+01	3.1875	-1	3.0E+01	13.5140
-2	2.5E+03	3.2853	-2	3.0E+03	13.4444
-3	2.5E+05	2.9026	-3	3.0E+05	13.3758
-4	2.5E+07	2.3081	-4	3.0E+07	13.2848

to consider its static hedging representation:

$$\Psi_{\text{out}}(t,T,L_1;K_1,L_2) = C^{\text{E}}(t,T,L_1;K_1) - \int_0^t \mathbf{w}(u) P_0^{\text{bin}}(T-t,T-u,L_1;L_2)\mathrm{d}u$$

for a suitable weight function $\mathbf{w}$. The basic options in this formula are known to have continuous derivatives. ∎

E Laplace Transforms for Case Studies

General Double Barrier Knock-in.

Integral equations: for $0 \le t \le T$,

$$\begin{aligned} P^{\text{E}}(T-t,T,U;K_1) &= \int_0^t \mathbf{w}_1(u) C(T-t,T-u,U;U)\mathrm{d}u + \int_0^t \mathbf{w}_2(u) P_0(T-t,T-u,U;L)\mathrm{d}u \\ &\quad + \Psi_1^* + \Psi_2^* P_0^{\text{A}}(T-t,T,U;L), \\ C^{\text{E}}(T-t,T,L;K_2) &= \int_0^t \mathbf{w}_1(u) C(T-t,T-u,L;U)\mathrm{d}u + \int_0^t \mathbf{w}_2(u) P_0(T-t,T-u,L;L)\mathrm{d}u \\ &\quad + \Psi_1^* C^{\text{A}}(T-t,T,L;U) + \Psi_2^* \end{aligned}$$

Laplace transform for the option price:

$$\widehat{\Psi}(\lambda,S;\{L,U\}) = \widehat{\mathbf{w}}_1(\lambda)\widehat{C}(\lambda,S;U) + \widehat{\mathbf{w}}_2(\lambda)\widehat{P}_0(\lambda,S;L) + \Psi_1^*\widehat{C}^{\text{A}}(\lambda,S;U) + \Psi_2^*\widehat{P}_0^{\text{A}}(\lambda,S;L)$$

Laplace transforms for weight functions:

$$\begin{pmatrix} \widehat{\mathbf{w}}_1(\lambda) \\ \widehat{\mathbf{w}}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \widehat{C}(\lambda,U;U) & \widehat{P}_0(\lambda,U;L) \\ \widehat{C}(\lambda,L;U) & \widehat{P}_0(\lambda,L;L) \end{pmatrix}^{-1} \begin{pmatrix} \widehat{P}^{\text{E}}(\lambda,U;K_1) - \frac{1}{\lambda}\Psi_1^* - \Psi_2^*\widehat{P}_0^{\text{A}}(\lambda,U;L) \\ \widehat{C}^{\text{E}}(\lambda,L;K_2) - \Psi_1^*\widehat{C}^{\text{A}}(\lambda,L;U) - \frac{1}{\lambda}\Psi_2^* \end{pmatrix}$$

Double Barrier Knock-in Put.

Static hedging portfolio: with $\Psi_1^* = (K-U)^+$ and $\Psi_2^* = (K-L)^+$,

$$\begin{aligned}\Psi(0,T,S;\{L,U\}) &= \int_0^T \mathbf{w}_1(u)C(0,T-u,S;U)\mathrm{d}u + \int_0^T \mathbf{w}_2(u)P_0(0,T-u,S;L)\mathrm{d}u \\ &\quad +\Psi_1^* C^{\mathrm{A}}(0,T,S;U) + \Psi_2^* P_0^{\mathrm{A}}(0,T,S;L) + Kv_D(0,T,S)\end{aligned}$$

Integral equations: for $0 \le t \le T$,

$$\begin{aligned}P^{\mathrm{E}}(T-t,T,U;K) &= \int_0^t \mathbf{w}_1(u)C(T-t,T-u,U;U)\mathrm{d}u + \int_0^t \mathbf{w}_2(u)P_0(T-t,T-u,U;L)\mathrm{d}u \\ &\quad +\Psi_1^* + \Psi_2^* P_0^{\mathrm{A}}(T-t,T,U;L) + Kv_D(0,T,U), \\ P^{\mathrm{E}}(T-t,T,L;K) &= \int_0^t \mathbf{w}_1(u)C(T-t,T-u,L;U)\mathrm{d}u + \int_0^t \mathbf{w}_2(u)P_0(T-t,T-u,L;L)\mathrm{d}u \\ &\quad +\Psi_1^* C^{\mathrm{A}}(T-t,T,L;U) + \Psi_2^* + Kv_D(0,T,L)\end{aligned}$$

Laplace transform for the option price:

$$\widehat{\Psi}(\lambda,S;\{L,U\}) = \widehat{\mathbf{w}}_1(\lambda)\widehat{C}(\lambda,S;U) + \widehat{\mathbf{w}}_2(\lambda)\widehat{P}_0(\lambda,S;L) + \Psi_1^*\widehat{C}^{\mathrm{A}}(\lambda,S;U) + \Psi_2^*\widehat{P}_0^{\mathrm{A}}(\lambda,S;L) + K\widehat{v}_D(\lambda,S)$$

Laplace transforms for weight functions:

$$\begin{pmatrix}\widehat{\mathbf{w}}_1(\lambda)\\ \widehat{\mathbf{w}}_2(\lambda)\end{pmatrix} = \begin{pmatrix}\widehat{C}(\lambda,U;U) & \widehat{P}_0(\lambda,U;L)\\ \widehat{C}(\lambda,L;U) & \widehat{P}_0(\lambda,L;L)\end{pmatrix}^{-1}\begin{pmatrix}\widehat{P}_0^{\mathrm{E}}(\lambda,U;K) - \frac{1}{\lambda}\Psi_1^* - \Psi_2^*\widehat{P}_0^{\mathrm{A}}(\lambda,U;L) - K\widehat{v}_D(\lambda,U)\\ \widehat{P}_0^{\mathrm{E}}(\lambda,L;K) - \Psi_1^*\widehat{C}^{\mathrm{A}}(\lambda,L;U) - \frac{1}{\lambda}\Psi_2^* - K\widehat{v}_D(\lambda,L)\end{pmatrix}$$

Knock-in Knock-out Option.

Integral equations: for $0 \le t \le T$,

$$\begin{aligned}-\theta C^{\mathrm{E}}(T-t,T,U;K) &= \int_0^t \mathbf{w}_1(u)C(T-t,T-u,U;U)\mathrm{d}u + \int_0^t \mathbf{w}_2(u)P_0(T-t,T-u,U;L)\mathrm{d}u \\ &\quad +\Psi_1^* + \Psi_2^* P_0^{\mathrm{A}}(T-t,T,U;L) + P_0^{\mathrm{E}}(T-t,T,U;K), \\ 0 &= \int_0^t \mathbf{w}_1(u)C(T-t,T-u,L;U)\mathrm{d}u + \int_0^t \mathbf{w}_2(u)P_0(T-t,T-u,L;L)\mathrm{d}u \\ &\quad +\Psi_1^* C^{\mathrm{A}}(T-t,T,L;U) + \Psi_2^* + P_0^{\mathrm{E}}(T-t,T,L;K).\end{aligned}$$

Laplace transform for the option price:

$$\begin{aligned}\widehat{\Psi}(\lambda,S;\{L,U\}) &= \widehat{\mathbf{w}}_1(\lambda)\widehat{C}(\lambda,S;U) + \widehat{\mathbf{w}}_2(\lambda)\widehat{P}_0(\lambda,S;L) \\ &\quad +\Psi_1^*\widehat{C}^{\mathrm{A}}(\lambda,S;U) + \Psi_2^*\widehat{P}_0^{\mathrm{A}}(\lambda,S;L) + \widehat{P}_0^{\mathrm{E}}(\lambda,S;K)\end{aligned}$$

Laplace transforms for weight functions:

$$\begin{pmatrix} \widehat{\mathbf{w}}_1(\lambda) \\ \widehat{\mathbf{w}}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \widehat{C}(\lambda,U;U) & \widehat{P}_0(\lambda,U;L) \\ \widehat{C}(\lambda,L;U) & \widehat{P}_0(\lambda,L;L) \end{pmatrix}^{-1} \times \begin{pmatrix} -\theta\widehat{C}_0^{\mathrm{E}}(\lambda,U;K) - \frac{1}{\lambda}\Psi_1^* - \Psi_2^*\widehat{P}_0^{\mathrm{A}}(\lambda,U;L) - \widehat{P}_0^{\mathrm{E}}(\lambda,U;K) \\ -\Psi_1^*\widehat{C}^{\mathrm{A}}(\lambda,L;U) - \frac{1}{\lambda}\Psi_2^* - \widehat{P}_0^{\mathrm{E}}(\lambda,L;K) \end{pmatrix}$$

Roll-down Call.

Static hedging portfolios:

$$\begin{aligned} \Psi_{\mathrm{out}}(\lambda,S;K_0,L_1) &= C^{\mathrm{E}}(0,T,S;K_0) - \int_0^T \mathbf{w}_1(u)P_0(0,T-u,S;L_1)\mathrm{d}u, \\ \Psi_{\mathrm{in}}(\lambda,S;L_1) &= \int_0^T \mathbf{w}_2(u)P_0(0,T-u,S;L_1)\mathrm{d}u \end{aligned}$$

Integral equations: for $0 \le t \le T$,

$$\begin{aligned} 0 &= C^{\mathrm{E}}(T-t,T,L_1;K_0) - \int_0^t \mathbf{w}_1(u)P_0(T-t,T-u,L_1;L_1)\mathrm{d}u \\ \Psi_{\mathrm{out}}(T-t,T,L_1;K_1,L_2) &= \int_0^t \mathbf{w}_2(u)P_0(T-t,T-u,L_1;L_1)\mathrm{d}u \end{aligned}$$

Laplace transform for the option price:

$$\widehat{\Psi}(\lambda,S;\{K_0,K_1\},\{L_1,L_2\}) = \widehat{\Psi}_{\mathrm{out}}(\lambda,S;K_0,L_1) + \widehat{\Psi}_{\mathrm{in}}(\lambda,S;L_1)$$

where

$$\begin{aligned} \widehat{\Psi}_{\mathrm{out}}(\lambda,S;K_0,L_1) &= \widehat{C}^{\mathrm{E}}(\lambda,S;K_0) - \widehat{C}^{\mathrm{E}}(\lambda,L_1;K_0)\frac{\widehat{P}_0(\lambda,S;L_1)}{\widehat{P}_0(\lambda,L_1;L_1)}, \\ \widehat{\Psi}_{\mathrm{in}}(\lambda,S;L_1) &= \widehat{\Psi}_{\mathrm{out}}(\lambda,L_1;K_1,L_2)\frac{\widehat{P}_0(\lambda,S;L_1)}{\widehat{P}_0(\lambda,L_1;L_1)} \\ &= \left(\widehat{C}^{\mathrm{E}}(\lambda,L_1;K_1) - \widehat{C}^{\mathrm{E}}(\lambda,L_2;K_1)\frac{\widehat{P}_0(\lambda,L_1;L_2)}{\widehat{P}_0(\lambda,L_2;L_2)}\right)\frac{\widehat{P}_0(\lambda,S;L_1)}{\widehat{P}_0(\lambda,L_1;L_1)} \\ &= \widehat{C}^{\mathrm{E}}(\lambda,L_1;K_1)\frac{\widehat{P}_0(\lambda,S;L_1)}{\widehat{P}_0(\lambda,L_1;L_1)} - \widehat{C}^{\mathrm{E}}(\lambda,L_2;K_1)\frac{\widehat{P}_0(\lambda,L_1;L_2)}{\widehat{P}_0(\lambda,L_2;L_2)}\frac{\widehat{P}_0(\lambda,S;L_1)}{\widehat{P}_0(\lambda,L_1;L_1)} \\ &= \widehat{C}^{\mathrm{E}}(\lambda,L_1;K_1)\frac{\widehat{P}_0(\lambda,S;L_1)}{\widehat{P}_0(\lambda,L_1;L_1)} - \widehat{C}^{\mathrm{E}}(\lambda,L_2;K_1)\frac{\widehat{P}_0(\lambda,S;L_2)}{\widehat{P}_0(\lambda,L_2;L_2)} \\ &= \widehat{\Psi}_{\mathrm{out}}(\lambda,S;K_1,L_2) - \widehat{\Psi}_{\mathrm{out}}(\lambda,S;K_1,L_1) \end{aligned}$$

Laplace transforms for weight functions:

$$\widehat{\mathbf{w}}_1(\lambda) = \frac{\widehat{C}^{\mathrm{E}}(\lambda,L_1;K_0)}{\widehat{P}_0(\lambda,L_1;L_1)},$$

$$\widehat{\mathbf{w}}_2(\lambda) \;=\; \left(\widehat{C}^{\mathrm{E}}(\lambda, L_1; K_1) - \widehat{C}^{\mathrm{E}}(\lambda, L_2; K_1)\frac{\widehat{P}_0(\lambda, L_1; L_2)}{\widehat{P}_0(\lambda, L_2; L_2)}\right)\frac{1}{\widehat{P}_0(\lambda, L_1; L_1)}$$

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