



# THE UNIVERSITY *of* EDINBURGH

## Edinburgh Research Explorer

### **(I1,I2)-RIP and Projected Back-Projection Reconstruction for Phase-Only Measurements**

**Citation for published version:**

Feuillen, T, Davies, ME, Vandendorpe, L & Jacques, L 2020, '(I1,I2)-RIP and Projected Back-Projection Reconstruction for Phase-Only Measurements', *IEEE Signal Processing Letters*, vol. 27. <https://doi.org/10.1109/LSP.2020.2973506>

**Digital Object Identifier (DOI):**

[10.1109/LSP.2020.2973506](https://doi.org/10.1109/LSP.2020.2973506)

**Link:**

[Link to publication record in Edinburgh Research Explorer](#)

**Document Version:**

Peer reviewed version

**Published In:**

IEEE Signal Processing Letters

**General rights**

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [openaccess@ed.ac.uk](mailto:openaccess@ed.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.



# $(\ell_1, \ell_2)$ -RIP and Projected Back-Projection Reconstruction for Phase-Only Measurements

Thomas Feuillen, Mike E. Davies, Luc Vandendorpe, and Laurent Jacques

**Abstract**—This letter analyzes the performances of a simple reconstruction method, namely the Projected Back-Projection (PBP), for estimating the direction of a sparse signal from its phase-only (or amplitude-less) complex Gaussian random measurements, *i.e.*, an extension of one-bit compressive sensing to the complex field. To study the performances of this algorithm, we show that complex Gaussian random matrices respect, with high probability, a variant of the Restricted Isometry Property (RIP) relating to the  $\ell_1$ -norm of the sparse signal measurements to their  $\ell_2$ -norm. This property allows us to upper-bound the reconstruction error of PBP in the presence of phase noise. Monte Carlo simulations are performed to highlight the performance of our approach in this phase-only acquisition model when compared to error achieved by PBP in classical compressive sensing.

## I. INTRODUCTION

One aspect of compressive sensing (CS) is to reduce the number of measurements needed to achieve (high) quality reconstruction of low-complexity signals (*e.g.*, sparse) [6], [8]. Recent research has also focused on reducing the accuracy of each measurement, *e.g.*, by lowering their resolution (or bit-depth) in specific quantization contexts [3], [11]–[13]. This paper investigates the consequences of removing the information about the amplitude of a complex signal, *i.e.*, using only the measurement phase for the reconstruction. While phase-only (PO) acquisition can serve as a stepping stone to study new quantizations schemes, *e.g.*, when quantizing the measurement phase [5], this sensing is tantamount to a complex form of one-bit quantization, *e.g.*, extensively studied in one-bit CS [3], [10], [13].

Oppenheim and co-authors [17], [18] proved in a few seminal contributions that real, bandlimited signals can be reconstructed, up to a lost amplitude, from the phase of their Fourier transform. More recently, for phase-only CS (PO-CS) with complex Gaussian random matrices, Boufounos determined that a specific distance between the measurement phases of two sparse signals encodes their angular distance up to an additive distortion [4]. While this distortion prevents us from proving perfect estimation of sparse signal direction, the author showed experimentally that this is achievable, thanks to a greedy algorithm enforcing the phase consistency between the signal estimate and the PO measurements.

TF, LV and LJ are with the ICTEAM institute, UCLouvain, Belgium (e-mail: {thomas.feuilleen, luc.vandendorpe, laurent.jacques}@uclouvain.be). MD is with the Institute for Digital Communications, University of Edinburgh, Edinburgh EH9 3JL, U.K. (e-mail: mike.davies@ed.ac.uk). MD would like to acknowledge partial support from the ERC project C-SENSE (ERC-ADG-2015-694888). LJ is funded by the F.R.S.-FNRS.

In this context, our contributions are as follows. While the question of perfect recovery of signal direction remains open, we here focus on a simple, non-iterative algorithm, the Projected Back-Projection (PBP, see Sec. II), and show that this method accurately estimates the direction of sparse signals in PO-CS (Sec. III). This is possible if the sensing matrix respects a variant of the RIP, the  $(\ell_1, \ell_2)$ -RIP in the complex field, which was previously introduced for (real) one-bit CS. Using tools from measure concentration [15], we then prove that complex Gaussian random matrices satisfy, with high probability (w.h.p.), the  $(\ell_1, \ell_2)$ -RIP if the number of measurements is large compared to the signal sparsity level (Sec. IV). Note that the  $\ell_1$ -norm of this RIP prevents a simple proof of this result by recasting the complex field to the real field. Finally, extensive Monte Carlo simulations confirm that the PBP estimation error for PO-CS compares favorably to the one of an unaltered, linear CS scheme (Sec. V).

**Notations and conventions:** We denote matrices and vectors with bold symbols, *e.g.*,  $\Phi \in \mathbb{C}^{m \times n}$ ,  $\mathbf{x} \in \mathbb{C}^n$ , and scalar values with light symbols. We will often use the following quantities:  $[d] := \{1, \dots, d\}$  with  $d \in \mathbb{N}$ ; the complex number  $i$  such that  $i^2 = -1$ ;  $\Re\{\lambda\}$  (or  $\lambda^{\Re}$ ) and  $\Im\{\lambda\}$  (or  $\lambda^{\Im}$ ) are the real and imaginary part of  $\lambda \in \mathbb{C}$ , respectively, and  $\lambda^*$  is its complex conjugate;  $\mathbf{A}^H$  is the conjugate transpose of  $\mathbf{A}$ ;  $\text{supp } \mathbf{x}$  is the support of  $\mathbf{x} \in \mathbb{C}^d$ ;  $|\mathcal{S}|$  is the cardinality of a finite set  $\mathcal{S}$ ;  $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^d x_i y_i^*$  is the scalar product between two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^d$ ; the  $\ell_p$ -norm of  $\mathbf{x}$  ( $p \geq 1$ ) is defined as  $\|\mathbf{x}\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ , with  $\|\mathbf{x}\|_\infty = \max_i |x_i|$  and  $\|\mathbf{u}\|_0 := |\text{supp}(\mathbf{u})|$ , and the  $\ell_{p,q}$ -norm of  $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_d)^T \in \mathbb{C}^{d \times d'}$  is  $\|\mathbf{A}\|_{p,q} = (\sum_{i=1}^d \|\mathbf{a}_i\|_p^q)^{1/q}$ ;  $\mathbb{B}^n := \{\mathbf{u} \in \mathbb{C}^n : \|\mathbf{u}\| \leq 1\}$ ; the Hadamard product is  $\odot$ ; and the angle operator (applied componentwise onto vectors) reads  $\angle(ce^{i\phi}) = \phi$  for  $c > 0$  and  $\phi \in [-\pi, \pi]$ . We denote by  $\mathcal{N}^{m \times n}(\mu, \sigma^2)$  and  $\mathbb{C}\mathcal{N}^{m \times n}(\mu, 2\sigma^2)$  (dropping the symbol  $n$  if  $n = 1$ ) the  $m \times n$  random matrices with entries independently and identically distributed (i.i.d.) as the normal distribution  $\mathcal{N}(\mu, \sigma^2)$  and the complex normal distribution  $\mathbb{C}\mathcal{N}(\mu, 2\sigma^2) \sim \mathcal{N}(\mu^{\Re}, \sigma^2) + i\mathcal{N}(\mu^{\Im}, \sigma^2)$ , respectively, for some mean  $\mu$  and variance  $\sigma^2$ . An  $s$ -sparse  $\mathbf{x}$  vector belongs to the set  $\bar{\Sigma}_s^n := \{\mathbf{u} \in \mathbb{C}^n, \|\mathbf{u}\|_0 \leq s\}$ . Given  $g, g' \sim \mathcal{N}(0, \sigma^2)$ , the random variable (*r.v.*)  $z := |g + ig'|$  is distributed as the Rayleigh distribution  $\mathcal{R}(\sigma)$  with parameter  $\sigma$  [19].

## II. PHASE-ONLY SENSING MODEL

Let us consider a complex  $s$ -sparse vector  $\mathbf{x}_0 \in \bar{\Sigma}_s^n$ . Given a complex matrix  $\Phi \in \mathbb{C}^{m \times n}$ , this work is concerned

with the following noisy non-linear sensing model [4], which generalizes one-bit CS [10], [22] to the complex field:

$$z = \text{sign}_{\mathbb{C}}(\Phi \mathbf{x}_0) \odot e^{i\xi}, \quad (1)$$

where  $\text{sign}_{\mathbb{C}}(\cdot)$  is the *complex signum* operator, applied component-wise onto vectors, *i.e.*,  $\text{sign}_{\mathbb{C}}(\lambda) = \lambda/|\lambda|$  for  $\lambda \in \mathbb{C} \setminus \{0\}$ , and  $\xi$  stands for a possible corruption of the measurement phase (with  $\xi_i \in [0, 2\pi)$ ,  $i \in [m]$ ). The matrix  $\Phi$  can be, *e.g.*, a complex Gaussian random matrix (see Sec. IV).

The sensing model (1) thus discards the amplitudes of the measurements  $\Phi \mathbf{x}_0$ ; estimating  $\mathbf{x}_0$  from  $z$  is possible only up to a global unknown normalization of  $\mathbf{x}_0$ , *i.e.*, only the direction  $\mathbf{x}_0/\|\mathbf{x}_0\|_2$  can be estimated.

We aim to show that the projected back projection (PBP) algorithm [10], [22] accurately estimates the direction of complex sparse signals provided the complex sensing matrix respects a variant of the RIP property (see Sec. III). Given  $s \in [n]$ , the sensing matrix  $\Phi$ , and the measurement vector  $z$ , this algorithm is simply defined as

$$\hat{\mathbf{x}} = H_s(\Phi^H z), \quad (\text{PBP})$$

where  $H_s(\mathbf{u})$  is the hard thresholding operator setting all of the components of the vector  $\mathbf{u}$  to zero but the  $s$  strongest in amplitude (which are unchanged). For CS, PBP is often used as the first iteration of more complex iterative methods such as iterative hard thresholding (IHT) [2], [22]. Despite its simplicity, analyzing PBP can thus lead to better iterative reconstruction algorithms for PO-CS.

### III. BOUND ON THE PBP RECONSTRUCTION ERROR

In CS theory, the error of most signal reconstruction algorithms is controlled by the restricted isometry property — or  $(\ell_2, \ell_2)$ -RIP — of the sensing matrix [9]. This amounts to asking that for some  $0 < \delta \leq 1$ ,

$$(1 - \delta)\|\mathbf{x}\|_2^2 \leq \|\Phi \mathbf{x}\|_2^2 \leq (1 + \delta)\|\mathbf{x}\|_2^2,$$

holds true for all sparse vectors  $\mathbf{x}$ . For instance, if the (real or complex) matrix  $\Phi$  respects the  $(\ell_2, \ell_2)$ -RIP over all  $2s$ -sparse vectors and one observes a  $s$ -sparse vector from the model  $\mathbf{y} = \Phi \mathbf{x}_0$ , the error of the estimate  $\hat{\mathbf{x}} = H_s(\Phi^H \mathbf{y})$  is bounded as  $\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2 = O(\delta)$  [9], [22].

As will be clear below, the capacity of PBP to estimate a sparse vector  $\mathbf{x}_0$  from its complex, phase-only observations  $z$  in (1) depends on the following RIP variant.

**Definition.** Given  $\delta > 0$ , the matrix  $\Phi \in \mathbb{C}^{m \times n}$  satisfies the  $(\ell_1, \ell_2)$ -RIP( $s, \delta$ ) if, for all  $\mathbf{x} \in \bar{\Sigma}_s^n$ ,

$$(1 - \delta)\|\mathbf{x}\|_2 \leq \|\Phi \mathbf{x}\|_1 \leq (1 + \delta)\|\mathbf{x}\|_2.$$

This property was introduced for real one-bit CS [10], [20]; with it, specific algorithms (including PBP) yield a good estimate of a real sparse signal from the sign of its random measurements. Moreover, provided that  $m$  is large compared to  $s$ , different types of real random matrix constructions, such as Gaussian random matrices [20, Lemma 2.1] [12] or randomly subsampled Gaussian circulant matrices [7], have been shown to respect the  $(\ell_1, \ell_2)$ -RIP( $s, \delta$ ) w.h.p.

To bound the reconstruction error of PBP, we first need the following lemma that is adapted from [10, Lemma 3].

**Lemma 1.** *If  $\Phi$  satisfies the  $(\ell_1, \ell_2)$ -RIP( $\delta, s$ ) for  $0 < \delta < 1$  and  $s \in [n]$ , then for any vector  $\mathbf{x} \in \mathbb{C}^n$  with unit  $\ell_2$ -norm such that  $\text{supp } \mathbf{x} \subset \mathcal{S} \subset [n]$  with  $|\mathcal{S}| = s$ ,*

$$\|H_{\mathcal{S}}(\Phi^H \text{sign}_{\mathbb{C}}(\Phi \mathbf{x})) - \mathbf{x}\|_2 \leq \sqrt{5\delta}.$$

We can now determine the main result of this section, which derives from an adaptation of [10, Thm 8] to the complex field.

**Theorem 2.** *If  $\Phi$  satisfies  $(\ell_1, \ell_2)$ -RIP( $2s, \delta$ ), then the PBP estimate  $\hat{\mathbf{x}}$  of any signal  $\mathbf{x}_0 \in \bar{\Sigma}_s^n$  with  $\|\mathbf{x}_0\|_2 = 1$  observed via (1) with  $\|\xi\|_{\infty} \leq \tau$  respects*

$$\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2 \leq 2\sqrt{5\delta} + 4\tau. \quad (2)$$

*Proof:* Let  $\mathcal{S}_0$  and  $\mathcal{T}$  be the  $s$ -sparse supports of  $\mathbf{x}_0$  and  $\hat{\mathbf{x}}$ , respectively. Writing  $\mathcal{S} := \mathcal{S}_0 \cup \mathcal{T}$  (with  $|\mathcal{S}| \leq 2s$ ) and  $\mathbf{a} = \Phi^H z$ , we first note that  $\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2 \leq \|\mathbf{x}_0 - H_{\mathcal{S}}(\mathbf{a})\|_2 + \|\hat{\mathbf{x}} - H_{\mathcal{S}}(\mathbf{a})\|_2$ , so that  $\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2 \leq 2\|\mathbf{x}_0 - H_{\mathcal{S}}(\mathbf{a})\|_2$  since  $\hat{\mathbf{x}}$  is the best  $s$ -term approximation of both  $\mathbf{a}$  and  $H_{\mathcal{S}}(\mathbf{a})$ . The triangle inequality and Lemma 1 then provide

$$\begin{aligned} \|\mathbf{x}_0 - H_{\mathcal{S}}(\mathbf{a})\|_2 &= \|\mathbf{x}_0 - H_{\mathcal{S}}(\Phi^H [\text{sign}_{\mathbb{C}}(\Phi \mathbf{x}_0) \odot \exp(i\xi)])\|_2 \\ &\leq \sqrt{5\delta} + \|H_{\mathcal{S}}(\Phi^H [\text{sign}_{\mathbb{C}}(\Phi \mathbf{x}_0) \odot (1 - e^{i\xi})])\|_2. \end{aligned}$$

Since  $\Phi$  respects the  $(\ell_1, \ell_2)$ -RIP( $2s, \delta$ ), we get, using Hölder's inequality and the fact the each measurement has an amplitude of 1

$$\begin{aligned} &\|H_{\mathcal{S}}(\Phi^H [\text{sign}_{\mathbb{C}}(\Phi \mathbf{x}_0) \odot (1 - e^{i\xi})])\|_2 \\ &= \sup_{\mathbf{u} \in \bar{\Sigma}_{2s}^n} \langle \Phi(H_{\mathcal{S}}(\mathbf{u})), \text{sign}_{\mathbb{C}}(\Phi \mathbf{x}_0) \odot (1 - e^{i\xi}) \rangle \\ &\leq \|1 - e^{i\xi}\|_{\infty} \sup_{\mathbf{u} \in \bar{\Sigma}_{2s}^n} \|\Phi(H_{\mathcal{S}}(\mathbf{u}))\|_1 \\ &\leq 2\|1 - e^{i\xi}\|_{\infty} \leq 2\|\xi\|_{\infty} \leq 2\tau. \end{aligned}$$

Gathering all bounds provides the result.  $\blacksquare$

Interestingly, (2) shows that one can still accurately estimate the direction of a complex sparse signal in PO-CS if  $\Phi$  is  $(\ell_1, \ell_2)$ -RIP( $2s, \delta$ ) with a small constant  $\delta$ .

Moreover, as clarified in Sec. IV, (2) allows us to understand how, for complex Gaussian sensing matrices, the error of PBP decays when  $m$  increases. Indeed, up to some missing log factors, we prove in Thm. 6 that complex Gaussian random matrices satisfy the  $(\ell_1, \ell_2)$ -RIP( $2s, \delta$ ) w.h.p. provided  $m \geq C\delta^{-2}s$  for some  $C > 0$ . By saturating this condition, we see that, for noiseless PO-CS, PBP achieves the error

$$\|\mathbf{x}_0 - \hat{\mathbf{x}}\|_2 = O\left(\sqrt[4]{s/m}\right) \quad (3)$$

when  $m$  increases, *i.e.*, which tends to zero for large  $m$ .

This evolution of the PBP error meets the one encountered for real one-bit CS [10] and non-linear CS [21]. However, this behavior is a bit pessimistic compared to the experimental decay in  $O(\sqrt{s/m})$  reached by simulations (see Sec. V). The exponent over  $\delta$  in (2) could thus be improved from  $\sqrt{\delta}$  to  $\delta$ . This would then match the performances of PBP in linear CS (see the beginning of this section) and dithered quantized CS [14], [22] where it reaches an error bounded by  $O(\delta)$  for  $(\ell_2, \ell_2)$ -RIP( $2s, \delta$ ) sensing matrices, *i.e.*, a decay in  $O(\sqrt{s/m})$  for Gaussian random sensing matrices.

#### IV. THE $(\ell_1, \ell_2)$ -RIP OF COMPLEX GAUSSIAN MATRICES

While one easily extends the  $(\ell_2, \ell_2)$ -RIP of certain random matrix constructions from the real to the complex fields — *e.g.*, by recasting the signal space  $\mathbb{C}^n$  and measurement domain  $\mathbb{C}^m$  to  $\mathbb{R}^{2n}$  and  $\mathbb{R}^{2m}$ , respectively [9] — such an extension for  $(\ell_1, \ell_2)$ -RIP matrices is not known.

Fortunately, using the tools of measure concentration [15], we prove below that complex Gaussian random matrices  $\Phi$  respects the  $(\ell_1, \ell_2)$ -RIP w.h.p. provided  $m$  is large compared to the signal sparsity. To show this, we first establish that, given  $\mathbf{x} \in \mathbb{C}^n$ ,  $\mathbb{E}\|\Phi\mathbf{x}\|_1$  is proportional to  $\|\mathbf{x}\|_2$  since each random variable  $|(\Phi\mathbf{x})_i|$  is Rayleigh distributed.

**Lemma 3.** *Given  $\mathbf{x} \in \mathbb{C}^n$  and a random matrix  $\Phi \sim \mathcal{CN}^{m \times n}(0, \sigma^2)$  with  $\sigma := \frac{1}{m} \frac{\sqrt{2}}{\sqrt{\pi}}$ , we have*

$$\mathbb{E}[\|\Phi\mathbf{x}\|_1] = \|\mathbf{x}\|_2.$$

*Proof:* By decomposing both the entries of  $\Phi$  and the components of  $\mathbf{x}$  into their real and imaginary parts, we get

$$\|\Phi\mathbf{x}\|_1 = \sum_{i=1}^m \left| \sum_{j=1}^n \Phi_{ij} x_j \right| = \sum_{i=1}^m \left| \sum_{j=1}^n g_{ij}^{\Re} + i g_{ij}^{\Im} \right|,$$

with  $g_{ij}^{\Re} := \Phi_{ij}^{\Re} x_j^{\Re} - \Phi_{ij}^{\Im} x_j^{\Im}$  and  $g_{ij}^{\Im} := \Phi_{ij}^{\Re} x_j^{\Im} + \Phi_{ij}^{\Im} x_j^{\Re}$ .

We note that, for all indices  $i, i' \in [m]$  and  $j, j' \in [n]$ ,  $\Phi_{ij}^{\Re}$  and  $\Phi_{ij}^{\Im}$  are Gaussian random variables with  $\mathbb{E}[\Phi_{ij}^{\Re}] = \mathbb{E}[\Phi_{ij}^{\Im}] = \mathbb{E}[\Phi_{ij}^{\Re} \Phi_{i'j'}^{\Re}] = 0$ . Therefore,  $g_{ij}^{\Re}, g_{ij}^{\Im} \sim_{\text{i.i.d.}} \mathcal{N}(0, \sigma^2 |x_j|^2)$  and a simple computation provides  $\mathbb{E}g_{ij}^{\Re} g_{i'j'}^{\Im} = 0$ . The r.v.s  $\Gamma_i^{\Re} := \sum_{j=1}^n g_{ij}^{\Re}$  and  $\Gamma_i^{\Im} := \sum_{j=1}^n g_{ij}^{\Im}$  are thus independent and distributed as  $\mathcal{N}(0, \sigma^2 \|\mathbf{x}\|_2^2)$  for all  $i \in [m]$ . Consequently,

$$\mathbb{E}[\|\Phi\mathbf{x}\|_1] = \sum_{i=1}^m \mathbb{E}[|\Gamma_i^{\Re} + i\Gamma_i^{\Im}|] = m\mathbb{E}[\Gamma_0],$$

where  $\Gamma_0$  follows a Rayleigh distribution  $\mathcal{R}(\sigma\|\mathbf{x}\|_2)$ . Since  $\mathbb{E}[\Gamma_0] = \sigma\sqrt{\frac{\pi}{2}}\|\mathbf{x}\|_2$  [19] and  $\sigma = \frac{1}{m}\sqrt{\frac{2}{\pi}}$ , we find  $\mathbb{E}[\|\Phi\mathbf{x}\|_1] = \sigma\|\mathbf{x}\|_2\sqrt{\frac{\pi}{2}}m = \|\mathbf{x}\|_2$ . ■

We also need this classical result from Ledoux and Talagrand [15, Eq. 1.6], see also [12, Lemma 5].

**Lemma 4.** *If the function  $F$  is Lipschitz with  $\lambda = \|F\|_{\text{Lip}}$ , then, for  $r > 0$  and  $\gamma \sim \mathcal{N}^m(0, 1)$ ,*

$$\mathbb{P}(|F(\gamma) - \mathbb{E}(F(\gamma))| > r) \leq 2 \exp(-\frac{1}{2}r^2\lambda^{-2}).$$

In our developments,  $F$  will be of the following kind.

**Lemma 5.** *The functions  $G : \mathbf{u} \in \mathbb{C}^m \mapsto \|\mathbf{u}\|_1 \in \mathbb{R}_+$  and of  $G' : (\mathbf{u}^{\Re}, \mathbf{u}^{\Im}) \in \mathbb{R}^{m \times 2} \mapsto \|(\mathbf{u}^{\Re}, \mathbf{u}^{\Im})\|_{2,1} \in \mathbb{R}_+$  have a Lipschitz constant equal to  $\sqrt{m}$ .*

*Proof:* For all  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ ,  $\|\mathbf{u}\|_1 - \|\mathbf{v}\|_1 \leq \|\mathbf{u} - \mathbf{v}\|_1 \leq \sqrt{m}\|\mathbf{u} - \mathbf{v}\|_2$ , which gives the Lipschitz constant of  $G$ . The one of  $G'$  follows from  $\|\mathbf{u}\|_1 = \|(\mathbf{u}^{\Re}, \mathbf{u}^{\Im})\|_{2,1}$ . ■

We are now ready to prove the main result of this section.

**Theorem 6.** *Let  $\delta \in (0, 1)$ ,  $\sigma = \frac{1}{m} \frac{\sqrt{2}}{\sqrt{\pi}}$ , and  $\Phi \sim \mathcal{CN}^{m \times n}(0, \sigma^2)$  be a complex Gaussian random matrix. If  $m \geq \frac{36}{\pi} \delta^{-2} [s \log(\frac{en}{s}(1 + \frac{6}{\delta})^2) + \log(\frac{2}{\eta})]$ , then, with probability exceeding  $1 - \eta$ , the matrix  $\Phi$  satisfies the  $(\ell_1, \ell_2)$ -RIP( $s, \delta$ ).*

*Proof:* The proof strategy<sup>1</sup> follows the one developed in [1] for proving that real Gaussian random matrices satisfy the  $(\ell_2, \ell_2)$ -RIP w.h.p.. By homogeneity of the  $(\ell_1, \ell_2)$ -RIP, it is enough to prove that complex Gaussian random matrices satisfy it w.h.p. for all vectors of  $\tilde{\Sigma}_s^n := \tilde{\Sigma}_s^n \cap \mathbb{B}^n$ .

We first show that for a fixed vector  $\mathbf{x} \in \mathbb{C}^n$ ,  $\|\Phi\mathbf{x}\|_1$  concentrates around  $\|\mathbf{x}\|_2$ . Using the r.v.s  $\Gamma_i^{\Re}, \Gamma_i^{\Im}$  defined in the proof of Lemma 3, we can write

$$\begin{aligned} p &:= \mathbb{P}(\|\Phi\mathbf{x}\|_1 - \|\mathbf{x}\|_2 > t\|\mathbf{x}\|_2) \\ &= \mathbb{P}(\left| \sum_{i=1}^m ((\Gamma_i^{\Re})^2 + (\Gamma_i^{\Im})^2)^{1/2} - \|\mathbf{x}\|_2 \right| > t\|\mathbf{x}\|_2) \\ &= \mathbb{P}(\left| \sum_{i=1}^m ((\gamma_i^{\Re})^2 + (\gamma_i^{\Im})^2)^{1/2} - m\sqrt{\frac{\pi}{2}} \right| > tm\sqrt{\frac{\pi}{2}}), \end{aligned}$$

where we defined the independent Gaussian random vectors  $\gamma^{\Re}, \gamma^{\Im} \sim_{\text{i.i.d.}} \mathcal{N}^m(0, 1)$ . Since  $\sum_{i=1}^m ((\gamma_i^{\Re})^2 + (\gamma_i^{\Im})^2)^{1/2} = \|(\gamma^{\Re}, \gamma^{\Im})\|_{2,1}$ , Lemma 4 provides

$$\begin{aligned} p &= \mathbb{P}(\left| \|(\gamma^{\Re}, \gamma^{\Im})\|_{2,1} - m\sqrt{\frac{\pi}{2}} \right| > tm\sqrt{\frac{\pi}{2}}) \\ &\leq 2 \exp\left(-\frac{\pi}{4}t^2m\right) \end{aligned} \quad (4)$$

by considering  $\gamma = (\gamma^{\Re}, \gamma^{\Im})$  as a  $2m$  Gaussian random vector, with the function  $F(\gamma) := \|(\gamma^{\Re}, \gamma^{\Im})\|_{2,1}$  whose Lipschitz constant is characterized in Lemma 5. Therefore, given  $\mathbf{x}$  and  $t > 0$ , we have

$$\|\Phi\mathbf{x}\|_1 - \|\mathbf{x}\|_2 \leq t\|\mathbf{x}\|_2$$

with probability exceeding  $1 - p \geq 1 - 2 \exp\left(-\frac{\pi}{4}t^2m\right)$ .

We now extend this result to all vectors of  $\tilde{\Sigma}_s^n$  by first determining when this concentration holds for all the vectors of a  $\rho$ -covering of this domain — that is a set such that all elements of  $\tilde{\Sigma}_s^n$  are no more than  $\rho > 0$  far apart from an element of this covering — and by finally extending this property to  $\tilde{\Sigma}_s^n$  by continuity.

Let us first build this covering. We note that  $\tilde{\Sigma}_s^n = \bigcup_{S \subset [n]: |S|=s} \tilde{\Sigma}^n(S)$ , with  $\tilde{\Sigma}^n(S) := \{\mathbf{u} \in \mathbb{B}^n : \text{supp } \mathbf{u} = S\}$ . Moreover,  $\tilde{\Sigma}^n(S)$  is isomorphic to  $\mathbb{B}^s$ , and thus to  $\mathbb{B}^{2s}$ . Since this last set, and thus  $\tilde{\Sigma}^n(S)$ , can be covered with no more than  $(1 + \frac{2}{\rho})^{2s}$  vectors [1], a covering  $\mathcal{J}_\rho$  of  $\tilde{\Sigma}_s^n$  can be reached by gathering all coverings —  $\binom{n}{s}$  in total — so that

$$|\mathcal{J}_\rho| \leq \binom{n}{s} (1 + \frac{2}{\rho})^{2s} \leq \left(\frac{en}{s}\right)^s (1 + \frac{2}{\rho})^{2s}.$$

Interestingly, by design, this covering is such that all  $\mathbf{x} \in \tilde{\Sigma}_s^n$  can be written as  $\mathbf{x} = \mathbf{u} + \mathbf{r}$  with  $\mathbf{u} \in \mathcal{J}_\rho \subset \tilde{\Sigma}_s^n$ ,  $\mathbf{r} \in \rho\mathbb{B}^n \cap \tilde{\Sigma}_s^n = \rho\tilde{\Sigma}_s^n$ , with  $\text{supp } \mathbf{x} = \text{supp } \mathbf{u} = \text{supp } \mathbf{r}$ .

Using (4), by union bound over all the vectors of  $\mathcal{J}_\rho$ , the event

$$\mathcal{E}_{\rho,t} : \|\Phi\mathbf{u}\|_1 - \|\mathbf{u}\|_2 \leq t, \quad \forall \mathbf{u} \in \mathcal{J}_\rho, \quad (5)$$

holds with failure probability  $p_{\rho,t} := \mathbb{P}(\mathcal{E}_{\rho,t}^c)$  at most

$$p_{\rho,t} \leq 2 \left(\frac{en}{s}\right)^s (1 + \frac{2}{\rho})^{2s} \exp\left(-\frac{\pi}{4}t^2m\right).$$

Let us assume  $\mathcal{E}_{\rho,t}$  holds and pick an arbitrary  $\mathbf{x} \in \tilde{\Sigma}_s^n$ . As explained above, we can write  $\mathbf{x} = \mathbf{u} + \mathbf{r}$  with  $\mathbf{u} \in \mathcal{J}_\rho$ ,  $\mathbf{r} \in \rho\tilde{\Sigma}_s^n$ , and  $\text{supp } \mathbf{x} = \text{supp } \mathbf{u} = \text{supp } \mathbf{r}$ .

<sup>1</sup>An alternative proof consists in using the majorizing measure theorem [16], which could possibly extend the  $(\ell_1, \ell_2)$ -RIP of complex Gaussian matrices to other low-complexity sets of  $\mathbb{R}^n$ .

Using (5), and the properties of the covering, we get

$$\begin{aligned} \|\Phi \mathbf{x}\|_1 - \|\mathbf{x}\|_2 &= \|\Phi(\mathbf{u} + \mathbf{r})\|_1 - \|(\mathbf{u} + \mathbf{r})\|_2 \\ &\leq \|\Phi \mathbf{u}\|_1 - \|\mathbf{u}\|_2 + \|\Phi(\mathbf{u} + \mathbf{r})\|_1 - \|\Phi \mathbf{u}\|_1 \\ &\quad + \|\mathbf{u} + \mathbf{r}\|_2 - \|\mathbf{u}\|_2 \leq t + \rho + \rho \|\Phi(\rho^{-1} \mathbf{r})\|_1, \end{aligned}$$

where we used multiple times the triangle inequality. However,  $\rho^{-1} \mathbf{r} \in \tilde{\Sigma}_s^n$  and we can recursively apply the same development to  $\|\Phi(\rho^{-1} \mathbf{r})\|_1$ , so that

$$\|\Phi \mathbf{x}\|_1 - \|\mathbf{x}\|_2 \leq (t + \rho) \sum_{k=0}^{+\infty} \rho^k = \frac{t + \rho}{1 - \rho}.$$

Setting  $t = \rho = \delta/3$  for some  $0 < \delta < 1$ , we get  $\frac{t + \rho}{1 - \rho} \leq \delta$ . From the analysis of  $\mathcal{E}_{\rho, t}$  above, we finally obtain that  $\|\Phi \mathbf{x}\|_1 - \|\mathbf{x}\|_2 \leq \delta$  holds true for all  $\mathbf{x} \in \tilde{\Sigma}_s^n$  — i.e., the  $(\ell_1, \ell_2)$ -RIP is verified — with failure probability at most

$$p_{\frac{\delta}{3}, \frac{\delta}{3}} \leq 2 \left(\frac{en}{s}\right)^s \left(1 + \frac{6}{\delta}\right)^{2s} \exp\left(-\frac{\pi}{36} \delta^2 m\right).$$

We conclude the proof by observing that  $p_{\frac{\delta}{3}, \frac{\delta}{3}} \leq \eta$  for  $0 < \eta < 1$  if  $m \geq \frac{36}{\pi} \delta^{-2} \left[s \log\left(\frac{en}{s}\right) \left(1 + \frac{6}{\delta}\right)^2 + \log\left(\frac{2}{\eta}\right)\right]$  ■

## V. SIMULATIONS

We now assess the tightness of our theoretical analysis through Monte Carlo simulations. We do not aim to demonstrate the superiority of (PBP) over other methods but to study the potentialities of such a simple algorithm in PO-CS.

As a first experiment, we have tested the estimation of complex sparse signals  $\mathbf{x}_0$  in  $\mathbb{C}^n$  with  $n = 256$  for different sparsity levels  $s \in [n]$  and measurement number  $m$ . Two acquisition strategies were compared: the phase-only acquisition fixed by the model (1), and classical compressive sensing where we directly acquire the measurement vector  $\mathbf{y} := \Phi \mathbf{x}_0$  without alteration. For each combination of  $s$  and  $m$ , the performances of both strategies have been tested over 100 000 generations of the sparse signal  $\mathbf{x}_0$  and the complex Gaussian random matrix  $\Phi \sim \mathcal{CN}(0, \sigma^2)$ , with  $\sigma^2$  set to  $2/(\pi m^2)$  and  $1/m$  for the phase-only and the CS scenario, respectively. Each sparse signal  $\mathbf{x}_0$  was created by picking a  $s$ -sparse support uniformly at random amongst the  $\binom{n}{s}$  possible supports, inserting in this support  $s$  i.i.d. complex values picked uniformly at random before normalizing. We analyzed the reconstruction error of the signal direction with the metric  $\mathcal{E}(\mathbf{x}_0, \hat{\mathbf{x}}) := \|\mathbf{x}_0 - \|\hat{\mathbf{x}}\|_2^{-1} \hat{\mathbf{x}}\|_2$ , where  $\hat{\mathbf{x}}$  is the (PBP) estimate. Comparing the two schemes in Fig. 1 for different sparsity levels, we observe that the reconstruction error achieved from phase-only measurements exhibits good performances given the absence of the amplitude information. The experimental convergence rate is also matching the one of the CS scheme; it scales as  $m^{-\frac{1}{2}}$  when  $m$  increases instead of the pessimistic rate in  $m^{-\frac{1}{4}}$  predicted by the theory in (3). The phase-only scheme seems to only suffer from a constant loss (in dB) when compared to the classic model.

In a second experiment, we have studied the performances of PBP in the presence of phase noise. In this new test, we kept the same parameters as above, restricting only the sparsity level and the number of measurements to  $s = 10$  and  $m = 64$ , respectively. The phase noise  $\xi$  in (1) was generated according to a uniform distribution between  $-\tau$  and  $\tau$ , with  $\tau \in [0, 4\pi]$ .

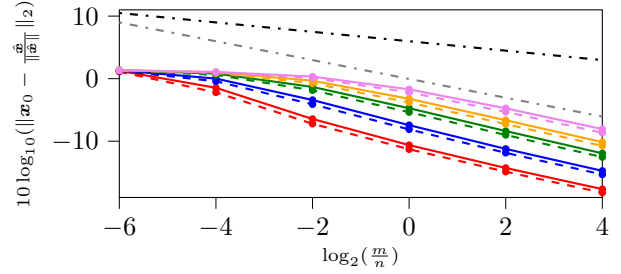


Fig. 1: (Best viewed in color) Reconstruction error of (PBP) for different measurement models. (dashed lines) compressive sensing; (solid lines) phase-only measurements. The colors represent the sparsity, namely  $s = 2$  in red,  $s = 4$  in blue,  $s = 10$  in green,  $s = 20$  in orange, and  $s = 50$  in pink. The dotted lines represent the rates of  $m^{-\frac{1}{2}}$  in gray and  $m^{-\frac{1}{4}}$  in black.

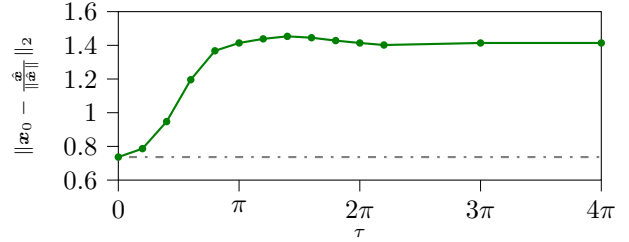


Fig. 2: Reconstruction error of (PBP) for noiseless (dashed lines) and noisy measurements (solid lines) for different  $\tau$  with  $s = 10$  and  $M = 64$ .

As established (2), the reconstruction error  $\mathcal{E}(\mathbf{x}_0, \hat{\mathbf{x}})$  increases almost linearly when  $\tau$  increases from 0 to  $\pi$ , before saturating at  $\sqrt{2}$  from  $\tau > \pi$ . In other words, from that noise level, phase-only measurements are too noisy and  $\langle \mathbf{x}_0, \hat{\mathbf{x}} \rangle \approx 0$ . Furthermore, the additive nature of the degradation in (2) is clearly visible when comparing the noiseless in dashed gray and noisy reconstruction in solid green.

## VI. CONCLUSION

In this paper, we have studied how to estimate the direction of complex sparse vectors from noisy phase-only measurements. We proved theoretically that the estimate yielded by the projected back projection of noisy phase-only measurement has bounded and stable reconstruction error provided that the sensing matrix satisfies an extension of the  $(\ell_1, \ell_2)$ -RIP in the complex field. Moreover, we showed that  $m \times n$  complex Gaussian random matrices respect w.h.p. this property with distortion  $\delta > 0$  provided that  $m$  is large compared to the signal sparsity level  $s$ , i.e.,  $m = O(\delta^{-2} s \log(\frac{n}{\delta s}))$ . The proof of this result leverages the tools of measure concentration since the  $\ell_1$ -norm prevents a simple recasting of the complex  $(\ell_1, \ell_2)$ -RIP to a real domain of larger dimension. We finally analyzed the tightness of our theoretical developments through Monte Carlo simulations. They confirmed that, despite the lack of amplitude information, we can reach arbitrary high accuracy on the estimation of sparse signal direction provided  $m/s$  is large, with an experimental error rate decaying as  $1/\sqrt{m}$  when  $m$  increases, thus faster than our theoretical error rate in  $1/m^{1/4}$ . The discrepancy between this two rates will be studied in future work, as well as the impact of phase quantization and additive noise on the phase-only sensing model.

## REFERENCES

- [1] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, "A Simple Proof of the Restricted Isometry Property for Random Matrices," *Constructive Approximation*, vol. 28, no. 3, pp. 253263, Jan. 2008.
- [2] T. Blumensath and M. E. Davies, "Iterative hard thresholding for compressed sensing," *Applied and Computational Harmonic Analysis*, vol. 27, no. 3, pp. 265274, Nov. 2009.
- [3] P. T. Boufounos and R. G. Baraniuk, "1-Bit compressive sensing," 2008 42nd Annual Conference on Information Sciences and Systems, Mar. 2008.
- [4] P. Boufounos, "Sparse signal reconstruction from phase-only measurements," *Proc. Int. Conf. Sampling Theory and Applications (SampTA)*, (July 1-5 2013). 2013.
- [5] P. T. Boufounos, "Angle-preserving quantized phase embeddings," *Wavelets and Sparsity XV*, Sep. 2013.
- [6] E. J. Candès, J. K. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Communications on Pure and Applied Mathematics*, vol. 59, no. 8, pp. 12071223, 2006.
- [7] S. Dirksen, H. C. Jung, and H. Rauhut, "One-bit compressed sensing with partial Gaussian circulant matrices," *Information and Inference: A Journal of the IMA*, iaz017, arXiv:1710.03287 (2017).
- [8] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 12891306, Apr. 2006.
- [9] S. Foucart and H. Rauhut, "A Mathematical Introduction to Compressive Sensing," *Applied and Numerical Harmonic Analysis*, 2013.
- [10] S. Foucart, "Flavors of Compressive Sensing," *Springer Proceedings in Mathematics & Statistics, Approximation Theory XV: San Antonio 2016*, pp. 61104, 2017.
- [11] C. S. Güntürk, M. Lammers, A. M. Powell, R. Saab, and Ö. Yılmaz, "Sobolev Duals for Random Frames and  $\Sigma\Delta$  Quantization of Compressed Sensing Measurements," *Foundations of Computational Mathematics*, vol. 13, no. 1, pp. 136, Oct. 2012.
- [12] L. Jacques, D. K. Hammond, and J. M. Fadili, "Dequantizing Compressed Sensing: When Oversampling and Non-Gaussian Constraints Combine," *IEEE Transactions on Information Theory*, vol. 57, no. 1, pp. 559571, Jan. 2011.
- [13] L. Jacques, J. N. Laska, P. T. Boufounos, and R. G. Baraniuk, "Robust 1-Bit Compressive Sensing via Binary Stable Embeddings of Sparse Vectors," *IEEE Transactions on Information Theory*, vol. 59, no. 4, pp. 20822102, Apr. 2013.
- [14] L. Jacques and V. Cambareri, "Time for dithering: fast and quantized random embeddings via the restricted isometry property," *Information and Inference: A Journal of the IMA*, vol. 6, no. 4, pp. 441476, Apr. 2017.
- [15] M. Ledoux and M. Talagrand, "Probability in Banach Spaces: Isoperimetry and Processes," Springer, 1991.
- [16] C. Liaw, A. Mehrabian, Y. Plan, and R. Vershynin, "A Simple Tool for Bounding the Deviation of Random Matrices on Geometric Sets," *Geometric Aspects of Functional Analysis*, pp. 277299, 2017.
- [17] A. V. Oppenheim and J. S. Lim, "The importance of phase in signals," *Proceedings of the IEEE*, vol. 69, no. 5, pp. 529541, 1981.
- [18] A. V. Oppenheim, M. H. Hayes, and J. S. Lim, "Iterative Procedures For Signal Reconstruction From Fourier Transform Phase," *Optical Engineering*, vol. 21, no. 1, p. 211122, Feb. 1982.
- [19] A. Papoulis, S. Pillai, "Probability, Random Variables and Stochastic Processes", Tata McGraw-Hill Education; 2002.
- [20] Y. Plan and R. Vershynin, "Dimension Reduction by Random Hyperplane Tessellations," *Discrete & Computational Geometry*, vol. 51, no. 2, pp. 438461, Dec. 2013.
- [21] Y. Plan and R. Vershynin, "The Generalized Lasso With Non-Linear Observations," *IEEE Transactions on Information Theory*, vol. 62, no. 3, pp. 15281537, Mar. 2016.
- [22] C. Xu and L. Jacques, "Quantized compressive sensing with rip matrices: The benefit of dithering," *Information and Inference: A Journal of the IMA*, 2019 (in press)