Spring 2020

The Game of Cops and Robbers on Planar Graphs

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THE GAME OF COPS AND ROBBERS ON PLANAR GRAPHS

A Master's Thesis

Presented to

The Graduate College of

Missouri State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

In graph theory, the game of cops and robbers is played on a finite, connected graph. The players take turns moving along edges as the cops try to capture the robber and the robber tries to evade capture forever. This game has received quite a bit of recent attention including several conjectures that have yet to be proven. In this paper, we restricted our attention to planar graphs in order to try to prove the conjecture that the dodecahedron graph is the smallest planar graph, in terms of vertices, that has cop number three. Along the way we discuss several other graphs with interesting properties connected with the cop number including a proof that the Tutte graph has cop number two.

KEYWORDS: cop number, graph, retraction, planar, cop win, dodecahedron, tutte graph

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ACKNOWLEDGEMENTS

I would like to thank many people who have helped me greatly throughout this process. Especially my wife Emily and daughter Harmony who keep me on track and off respectively. As well as my family, friends, and Luthien my rabbit (who claims the whole paper was his idea). Of course, none of my work would be possible without the great tutelage from professors like Dr. Richard Belshoff, Dr. Matthew Wright, Dr. Steven Senger, and Dr. Les Reid. Who each brought ideas, suggestions, and critiques.

I'd also like to recognize the following for allowing me the use of their images: David Benbennick for K_7, R. A. Nonenmacher for the Dodecahedron, Leshabirukov for the Peterson Graph, and Koko90 for the Tutte Graph and the Tutte Fragment.

Most importantly I would like to thank God for His grace and mercy, and ordering this universe in a way that we can study it.

TABLE OF CONTENTS

List of Figures

1. INTRODUCTION

The game of Cops and Robbers played on graphs was originally defined by Quilliot and then independently by Nowakowski and Winkler in 1978 and 1983 respectively [2]. How the game is played is the cop or cops first choose their places and then the robber chooses theirs. Next they take turns moving along the edges of the graph starting with the cop. The cops win if they can 'catch' the robber by moving to the same vertex on their turn. If the robber has a strategy to always move to a safe vertex, then the robber wins.

1.1. Notation and Definitions

A graph, $G(V, E)$, is comprised of a set of vertices, V, and edges, E, which are two element subsets of V. For instance let $V = \{1, 2, 3\}$ and $E = \{(12), (23), (31)\}$, then $G(V, E)$ is a graph with three vertices and three edges such that 1 and 2 share an edge, 2 and 3 share an edge, and 3 and 1 share an edge. In fact it is easy to see this forms a triangle. Two vertices are said to be **adjacent** if they share an edge. As in the example above we can say that 1 is adjacent to 2. I will now give a small list of some important definitions that will be needed for the rest of this paper.

DEFINITION 1.1 A graph $G'(V', E')$ is a **subgraph** of $G(V, E)$ if $V' \subset V$ and $E' \subset E$.

DEFINITION 1.2 A **path** is a set of vertices $x_1, x_2, \ldots, x_n$ such that each vertex is distinct from the others and x_i is adjacent to x_{i+1} for all $1 \leq i \leq n$. Furthermore, the **length** of a path is the number of edges between x_1 and x_n.

DEFINITION 1.3 A path is also a **cycle** if x_n is adjacent to x_1.

DEFINITION 1.4 The **distance** between $u, v \in V(G)$ is the length of the shortest path connecting u and v, or defined to be ∞ if there is no path connecting u to v.

This will be denoted as $d(u, v)$.

DEFINITION 1.5 The **girth** of a graph G is the length of the smallest cycle in G.

DEFINITION 1.6 A graph is said to be **connected** if there exists a path from any vertex to any other vertex. If there exists at least two vertices where this is not the case, then the graph is said to be **disconnected**.

Finally, the **degree** of a vertex is the number of other vertices it is adjacent to. It is also standard notation to denote the minimum degree of the vertices of a graph G as $\delta(G)$.

1.2. The Game of Cops and Robbers

For the remainder of this paper all graphs are assumed to be connected and have a finite number of vertecies. It is important to note that these are not required assumptions for the game of cops and robbers, however these are common assumptions in the field and dealing with infinite graphs is outside the realm of this paper. Moreover, studying disconnected graphs collapses down to studying the connected subgraphs that make up the larger graph. We also need to assume that the robber uses optimal strategies. So that if there is a move for the robber to escape they will take it. This removes the problem of either player making an unpredictable error.

The main goal in the study of cops and robbers is to determine the minimum number of cops needed to catch a robber on a given graph. This minimum is defined to be the **cop number** of the graph and denoted $c(G)$ for graph G. Thus it is not sufficient to find a strategy so that a given number of cops can catch the robber, but it needs to be shown that less cops cannot catch the robber. This is often done by finding upper and lower bounds such that the upper bound equals the lower bound. To this end, finding a strategy so that n cops can catch the robber only shows that $c(G) \leq n$ because there might be a better strategy using less cops.

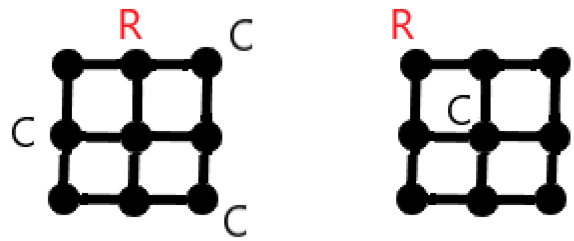
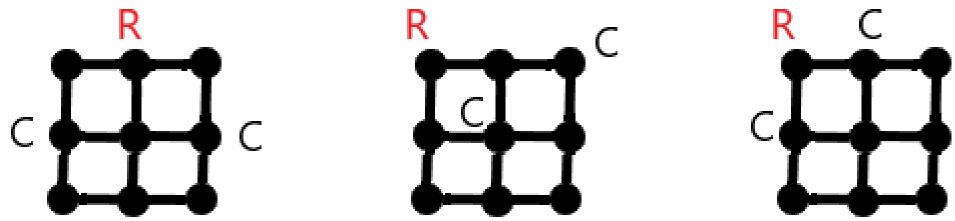


Figure 1: (left) 3 cops on G and 1 cop on G (right).

Next, I will present an example of the game of cops and robbers using a simple graph. The graph I will present is a three by three grid of vertices. Notice in the left image in Figure 1 that three cops can be arranged so that every vertex is adjacent to a vertex with a cop, so no matter where the robber starts the cops will win after the first move. Hence, we know $c(G) \leq 3$ and we only need to check the cases with one or two cops. Now assume there is only one cop as in the right image in Figure 1. In lieu of going through every possible starting position and move, I believe it is easy to see that wherever the cop moves the robber will have a safe move to evade the cop. This is because of the four cycle subgraphs that comprise the graph. Lastly, assume that there are two cops, then they can start on each side as in Figure 2a. Therefore, the robber must start on the top middle or bottom middle vertex to not lose on the first turn. Since the graph is symmetric both positions are equivalent so assume the robber starts at the top as shown. For their first turn C_1 can move right and C_2 can move up; this forces the robber to move to the corner to escape. Finally, C_1 can move back to the side and C_2 can move left to pin the robber in as in Figure 2c. Now the robber can not move and so on the next turn the cops have caught the robber. Therefore we have shown that $1 < c(G) \leq 2$ and thus $c(G) = 2$. Note that this might not be the strategy using the fewest moves, nonetheless, it is adequate for our purposes.

The prior example allows the reader to build an understanding of the basic rules of the game, but for more complicated graphs stronger tools will be needed. Es-



a b c

Figure 2: (a) starting positions (b) an intermediate step, and (c) catching the robber.

pecially as graphs get much larger and less symmetric, playing through every possible game becomes very computationally hard. That is why we will start from the simplest types of graphs, those that only need one cop, and build up some theorems to help find the cop number of bigger graphs.

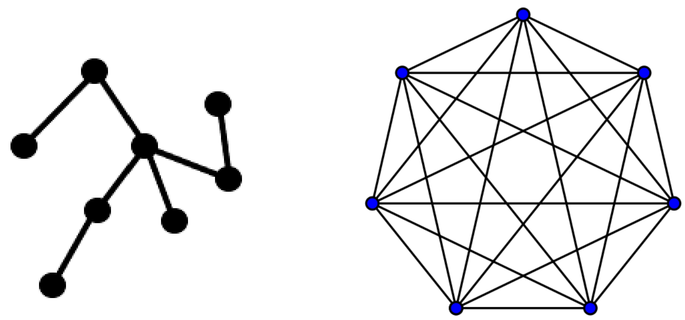


Figure 3: (left) a tree and (right) complete graph K_7

2. COP WIN GRAPHS

First consider which kinds of graphs only require one cop to be able to catch a robber. These are referred to as **cop-win** graphs. One example is any simple path of length n. Let $P = \{x_1, x_2, \ldots, x_n\}$ be a path and start the cop at x_1. Then no matter where the robber starts the cop moves along the path towards x_n and will eventually catch the robber. This same strategy works for a more general family of graphs called trees.

DEFINITION 2.1 A **tree** is a graph that does not contain any cycles.

Since there are no cycles, in each round there exists a unique shortest path from the cop to the robber which the cop can take. Eventually the robber will be on a vertex of degree one because the graph is finite and the cop will be guaranteed a win.

Another important type of graph that are cop-win are called complete graphs. These graphs are defined by the property that each vertex is connected to every other vertex and are often denoted as K_n, where n is the number of vertices. Only one cop is needed because no matter where the robber starts in the first move the cop can catch them because there is an edge connecting them. These simple examples lead us to an important definition.

DEFINITION 2.2 A **pitfall** is a vertex, u, that has an adjacent vertex, v, such that all the neighbors of u are also neighbors of v.

In some literature, a pitfall could also be called a corner or a dominated vertex and its corresponding neighbor is called the dominating vertex. Going back to the examples, we can find pitfalls in each case. For the complete graphs, every vertex is simultaneously a pitfall and dominating vertex because all the vertices are connected to each other. The trees also have pitfalls in the form of the vertices with degree one. We can see this because if they have only one edge, then the one adjacent vertex to it is dominating it. The reason pitfalls are important is that they can be removed from the graph without changing the cop number. This becomes obvious when seen from the robber's perspective.

LEMMA 2.3 Let $H(V_0, E_0) \subset G(V, E)$ be the induced subgraph created by removing a pitfall from G, then $c(H) = c(G)$.

Proof. Assume $c(G) = n$ and there exists a pitfall on G, then there exists a strategy for n cops to catch the robber, but not for $n - 1$ cops. If the robber does not move to the dominating vertex or the pitfall, then removing the pitfall will not change the cop number. Thus assume the robber moves to the dominating vertex. Since all the vertices adjacent to the pitfall are also adjacent to the dominating vertex, the robber will not need to move to the pitfall. If the robber does move to the pitfall, then the cops will have the opportunity to decrease the distance between them and the robber by moving toward the dominating vertex as if the robber was on it instead. Hence moving to the pitfall will only hinder the robber so they will not move there. Therefore the robber will only move on vertices in H, which implies the strategy to catch the robber on G is also the strategy for H. $\square$

Since pitfalls can be removed without changing the cop number, a graph can be decomposed into a smaller graph by removing all the pitfalls.

DEFINITION 2.4 If a graph can be decomposed by successively removing pitfalls until there is only a single vertex remaining, then the graph is said to be **decompos-**

able.

THEOREM 2.5 A graph G is cop-win if and only if it is decomposable.

Proof. I will first prove the sufficiency condition, if a graph is decomposable then it is cop-win. Since the graph G is decomposable, pitfalls can be successively removed until only one vertex remains. By Lemma 2.3, $c(G)$ is the same as the cop number of a single vertex, which is trivially one. Thus G is cop-win.

Now assume that G is cop-win and I will prove that it is necessary that G is decomposable. Because G is cop-win, after a finite number of moves the cop will catch the robber. Now consider the robber's turn before the cop wins, I claim the robber is on a pitfall. Suppose, for contradiction, that the robber is not on a pitfall. Then there is no vertex that is adjacent to all the neighbors of the robber's vertex. Hence the robber has a safe move away from the cop no matter where the cop is. But this contradicts the fact that on the cop's turn the cop will catch the robber, therefore the robber must be on a pitfall. Let H_1 be the induced subgraph of G defined by removing the previous pitfall. By Lemma 2.3, $c(H_1) = c(G) = 1$ and so the cop catches the robber after a finite number of moves. Once again the robber must be on a pitfall on their last turn, and so we can define H_2 to be the induced subgraph of H_1 with the pitfall removed. Notice that this pattern continues and so, by induction, pitfalls can be successively removed from G until some H_n which is only a single vertex. Thus G is decomposable and the theorem holds. □

3. RETRACTIONS

Theorem 2.5 gives a characterization of all the cop-win graphs, but for graphs with higher cop number the idea of a pitfall will need to be generalized. The generalization defined below comes from [1], which is one of the earliest papers on the cop number.

DEFINITION 3.1 Let H be a subgraph of G. Then a **retraction** of G onto H is a mapping $f : G \to H$ such that $f(v) = v$ for all v in H and every edge $(vw) \in G$ has a corresponding edge $(f(v)f(w)) \in H$. This subgraph H is called a **retract** of G.

Less formally a retraction sends vertices to vertices and edges to edges with the convention that if adjacent vertices are sent to the same vertex then there is an implied loop. Hence distances between vertices are either preserved or decrease. Now we see how this generalizes the idea of pitfalls since removing a pitfall is a type of retraction. For example, in Figure 4 there are two graphs G on the left and G_0 on the right both with their vertices labeled. Notice that vertex 7 in G is a pitfall because it is dominated by both 5 and 4. Therefore we can define the mapping $f(v) = v$ if $v \neq 7$ and $f(v) = 5$ if $v = 7$. This maps G to G_0 so that every vertex except 7 goes to itself and 7 goes to 5, furthermore each edge goes to itself except (47) which goes to (45) and (57) which goes to the loop (55). In this case $f(v)$ is a retraction by the definition and G_0 is the retract. There are other types of retractions as well, in particular, two useful types are retracting a graph onto a path and retracting to a cut vertex.

The path retraction is given by the following mapping of $\phi : G \to P$ given by:

$$\phi(v) = \begin{cases} v_i & d(v_0, v) \leq n \\ v_n & d(v_0, v) > n \end{cases}$$

where P is a given path $\{v_0, v_1, \ldots, v_n\}$ of length n, v_i is the i-th element of the path, and $d(v_0, v)$ is the distance between v and the start of the path v_0. This is a retraction

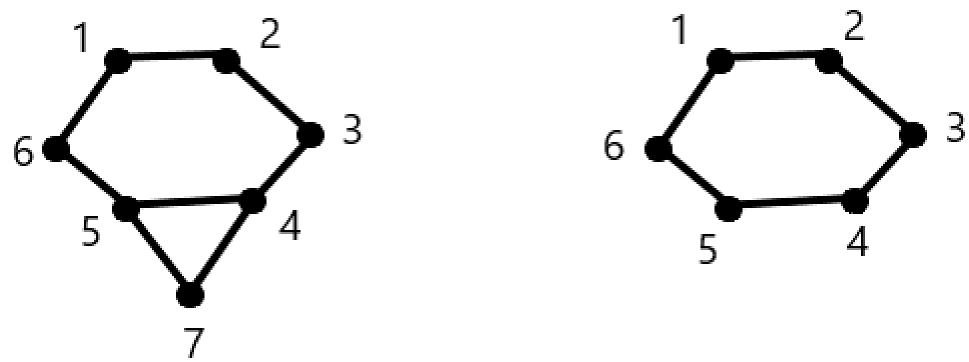


Figure 4: G (left) is a graph with a pitfall and G_0 (right) is the retract

since every element of the path is sent to itself. The path retraction is the main motivation behind the following theorem.

THEOREM 3.2 One cop can guard a shortest path between two vertices.

Proof. To show that a set is guarded, it suffices to show that after a finite number of turns, the robber cannot move to a vertex in the path without being caught. First let the cop start at v_0 on the given path while the robber may start anywhere. Now consider the path retraction from before. Notice that wherever the robber is, that vertex has an image on the retract. We will call this the robber's shadow. Therefore the cop can move along the path and after at most n turns catch the robber's shadow. If the robber was on the path then it is caught. Otherwise we need to show that the cop can stay on the robber's shadow. Once the cop moves onto the shadow it is the robber's turn to move. Assume that the robber is a distance j from v_0, then after the robber moves they will be $j-1$, j, or $j+1$ from v_0. We know this is true since they can only move one step and so the distance can only change by at most one. If after moving the robber is still j away, then the cop remains on vertex v_j. If the robber is now $j-1$ or $j+1$ away, then the cop moves to v_{j-1} or v_{j+1} respectively. Since the cop can remain on the shadow wherever the robber moves, if the robber were to move onto the path then it would be on its shadow and the cop would catch it. Thus the path is guarded by one cop. $\square$

Before defining the next retraction, we need to define a cut vertex.

DEFINITION 3.3 A **cut vertex** is a vertex such that if it is removed the graph becomes disconnected.

Another way to think about this is that it splits the graph into two components, H_1 and H_2, and every path from H_1 to H_2 contains the cut vertex. If v_0 is a cut vertex of G, we can define the retraction to a cut vertex $\psi : G \to H_2$ as follows:

$$\psi(v) = \begin{cases} v & v \in H_2 \\ v_0 & v \in H_1 \end{cases}$$

It is important to note that this can be extended to include multiple cut vertices as needed.

We will conclude our study of retractions with two theorems that give bounds on the cop number for a graph. The first of which seems relatively straight forward, and the second gives an upper bound for $c(G)$ provided a retract can be found. Both of these can be found in [3].

THEOREM 3.4 If H is a retract of G, then $c(H) \leq c(G)$.

Before the proof of Theorem 3.4, notice that every graph is a retract of itself under the identity mapping and so it is possible for $c(H) = c(G)$. Also it has already been shown that a path can be guarded by one cop and so every graph also has a retract that is cop-win.

Proof. We will rely heavily on the shadow strategy that was used in the previous proof. Since H is a retract of G, there is a retraction f such that every $v \in G$ has a corresponding $f(v) \in H$. Therefore, as the cops and robber move on G, their shadows move along the corresponding vertices $f(C), f(R) \in H$. Now consider right before the robber is caught on G. Every vertex adjacent to the robber R either has a cop or is adjacent to a cop. This means that every vertex adjacent to $f(R)$ either has a cop's shadow or is adjacent to a cop's shadow. Thus on the next move the cops will catch the robbers shadow on H and so $c(H) \leq c(G)$. $\square$

THEOREM 3.5 If H is a retract of G, then $c(G) \leq \max\{c(H), c(G \setminus H) + 1\}$.

Proof. Let $n = \max\{c(H), c(G \setminus H) + 1\}$ and assume that n cops play the game. Since $n \geq c(H)$, there is a strategy such that n cops can catch a robber on H. Because H is a retract of G, the cops catch the robber's shadow on H. If the robber was on their shadow, then the cops win. Otherwise, it only takes one cop to remain on the shadow while the other $n-1$ cops chase the robber on the subgraph $G \setminus H$. Notice that $n \geq c(G \setminus H) + 1$ implies $n - 1 \geq c(G \setminus H)$. Therefore, there is a strategy so that $n-1$ cops can catch a robber on $G \setminus H$. Since the robber cannot move onto H because of the remaining cop, the cops have caught the robber. $\square$

Figure 5: K_4 drawn with and without an edge crossing

4. PLANAR GRAPHS

The notion of planar graphs is about as old as graph theory itself, but first we will need to define what it means for a graph to be planar.

DEFINITION 4.1 A graph is said to be **planar** if there is some drawing of the graph embedded in a plane where no edges cross except at a vertex.

Notice that this definition talks about how a graph is drawn on a flat surface. In particular, there can be many ways to draw a graph since graph theory generally cares more about the ways vertices are connected than what a drawing of a graph looks like. This means that sometimes two different drawings of a graph might look different, but still represent the same graph. Figure 5 shows two different drawings of K_4 where one drawing crosses itself while the other does not. However, since there exists some drawing with no edges crossing, K_4 is a planar graph.

Unlike the cop-win graphs seen earlier, planar graphs do not have a complete characterization if $c(G) > 1$. However, it has been proven that there is an upper bound on the cop number if the graph is planar. Specifically, the cop number for any planar graph must be less than or equal to three. This was originally proven by Aigner and Fromme in [1]. The converse of this does not hold true; just because a graph only needs three or fewer cops does not imply it is planar. For example, any complete graph only needs one cop as already shown, but every complete graph with more than four

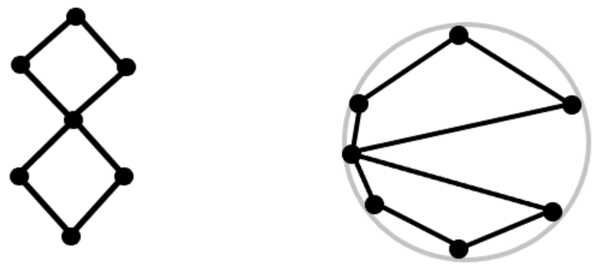


Figure 6: two drawings of an outerplanar graph.

vertices is not planar.

4.1. Outerplanar Graphs

Before the proof for all planar graphs it will be helpful to consider a slightly simpler family of graphs inside the planar graphs. These are the outerplanar graphs, which are planar graphs that have the added property that they can be drawn with the vertices arranged around a circle without any edges crossing. Consider the graph shown in Figure 6 which shows how an outerplanar graph can be redrawn to highlight this added property. The reason to start with outerplanar graphs is because there is a similar upper bound theorem which has a similar proof.

THEOREM 4.2 If G is outerplanar, then $c(G) \leq 2$.

The below proof of Theorem 4.2 follows an induction type argument where after each round the cops have gained territory until the whole graph is their territory, which means that the robber is caught.

Proof. First, consider the case when there are no cut vertices. Then I claim that the graph can be labeled so that every vertex v_j is adjacent to both v_{j-1} and v_{j+1} modulo the number of vertices. Suppose this is not the case, then there is some vertex v_i that is not adjacent to v_{i+1}. Continue in labeling order to the next vertex that is adjacent to v_i and call it v_k. The edge $(v_i v_k)$ prevents any vertex in $\{v_{i+1}, \ldots, v_{k-1}\}$ from being adjacent to any vertex in $\{v_{k+1}, \ldots, v_{i-1}\}$. Also by how we found v_k, none of $\{v_{i+1}, \ldots, v_{k-1}\}$ are adjacent to v_i. So v_k must be a cut vertex, but that contradicts our assumption and the claim holds.

Now choose a vertex of degree 2, relabel it v_0, and start the cops there. On their first turn move one cop to vertex v_1 and the other to v_{n-1}, notice that the cops have gained territory since the robber cannot reach v_0 without going through v_1 or v_{n-1}. Each turn continue moving both cops opposite ways around the circle until one of them reaches a vertex v_a of degree ≥ 3. Let v_b be the vertex with the shortest distance to the other cop such that $(v_a v_b)$ is a cord of the circle. Note that $(v_a v_b)$ separates G into two parts $\{v_a, v_{a+1}, \ldots, v_b\}$ and $\{v_a, v_{a-1}, \ldots, v_b\}$. Without loss of generality assume that the other cop is on $\{v_a, v_{a+1}, \ldots, v_b\}$. If the robber is on this section, then the first cop should move to v_b. The cops have now gained more territory because now the robber cannot move off of $\{v_a, v_{a+1}, \ldots, v_b\}$ without going through v_b or the second cops position. If the robber is on the other section, $\{v_a, v_{a-1}, \ldots, v_b\}$, then the cop should wait on v_a until the other cop reaches v_b. Again the cops have gained territory because the robber cannot leave $\{v_a, v_{a-1}, \ldots, v_b\}$ without going through either v_a or v_b. Eventually the cops territory is the entire graph which means the robber must have been caught.

Finally, assume there are cut vertices, then the graph can be broken down into subgraph components $B_1, B_2, \ldots, B_m$ each separated by a cut vertex. For G to be outerplanar overall, each component must be outerplanar and so the above strategy works for each component. The cops can use the cut vertex retraction to map the robber's shadow onto their current component and catch the shadow. If the cops start on B_1 they can catch the robber's shadow on B_1. If the robber was on the shadow, then they win. Otherwise, they are now on the next component and have gained territory because the robber cannot reach B_1. It follows that by component B_m the robber will be caught. $\square$

4.2. Planar Graphs Require at Most Three Cops

Before we get to the main theorem of this section we will first need a lemma.

LEMMA 4.3 Two cops can guard a pair of internally disjoint, shortest paths between two vertices.

In other words, given two points $v, w \in G$ there are two paths P_1 and P_2 from v to w such that $P_1 \cap P_2 = \{v, w\}$ and after finitely many turns the robber can not cross either P_1 or P_2. Notice that this gives the cops a way to divide any planar graph into two sections, inside and outside of the paths.

Proof. Without loss of generality assume that the length of P_1 is less than or equal to P_2. By Theorem 3.2, we know that one cop can guard a shortest path between two vertices therefore assign C_1 to guard path P_1. Now consider the graph $G \setminus (P_1 \setminus \{v, w\})$ and notice that P_2 is the shortest path in this subgraph from v to w. Thus C_2 can guard P_2 in $G \setminus (P_1 \setminus \{v, w\})$. Since P_2 can be guarded in the subgraph, the only way for the robber to enter P_2 without being captured is to cross P_1, but then they would be caught by C_1. □

And with that we now have enough tools to prove the theorem.

THEOREM 4.4 If G is planar, then $c(G) \leq 3$.

The proof of this theorem relies on the claim that the cops are, at any time, in one of three cases:

1. Some cop is guarding a shortest path between two vertices
2. Two cops are guarding internally disjoint, shortest paths between two vertices
3. Some cop is guarding a cut vertex.

Therefore I will call these case 1, 2, and 3 respectively for convenience.

Proof. Let G be drawn in the plane with no edges crossing, and denote the subgraph of G in which the robber has safe moves as the robber's territory R_i. We will show that after a finite amount of turns one of the three cases will be reached and the robber's new territory will have decreased so that $R_{i+1} \subsetneq R_i$. Let the three cops start together on vertex v, then there is a vertex w and a shortest path P_1 from v to w such that one cop can guard P_1. This is case 1 and $R_i = G \setminus P_1$.

If P_1 is adjacent to the component of R_i containing the robber by only one vertex v_0, then v_0 is a cut vertex. Move one of the free cops to v_0 and we are now in case 3. If there are more than one vertices connecting P_1 and the component of R_i containing the robber, then label these vertices $v_0, v_1, \ldots, v_j$. Now consider the vertices $u_0, u_1, \ldots, u_k$ in R_i that are adjacent to $v_0, \ldots, v_j$. Let P_2 be the shortest path between u_0 and u_k in R_i. While one cop guards P_1, a free cop can move to guard P_2. Notice we are now in case 2 and the robber is either on the interior or exterior of the area defined by the two paths. Either way R_i has decreased.

Now assume that one of the cops is on a cut vertex v_0 as in case 3. Then there exists a vertex w_0 in R_i such that there is a shortest path P_3 from v_0 to w_0. While this cop remains guarding v_0, a free cop can move to guard P_3. We are now back to case 1 and $R_{i+1} = R_i \setminus P_3$.

Finally assume we are in case 2, so that two cops are guarding edge disjoint paths between vertices v and w. Without loss of generality, assume that the robber is on the interior of the two paths. If both of the paths are adjacent to the interior by only one vertex each, then those are cut vertices. We can now move the third cop to guard the shortest path between the two cut vertices and we are in case 1 having gained more territory. If at least one of the paths has more than one vertex adjacent to the interior, then we can label the outermost two x_1 and x_2. Also label their corresponding vertices in the interior y_1 and y_2. Notice that we can find a shortest path P between y_1 and y_2 and move the free cop to guard $P \cup \{x_1, x_2\}$. This puts us back into

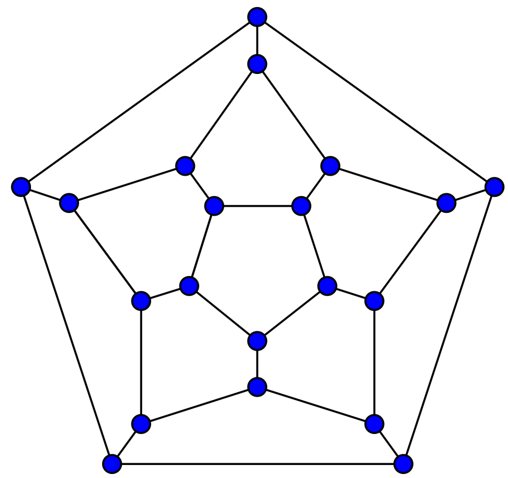


Figure 7: Dodecahedron

case 2 again, but we have gained territory while freeing up the first cop. This covers all cases, thus any planar graph only needs at most three cops. □

4.3. Dodecahedron

All of this leads up to the motivating question of my research, is the Dodecahedron the smallest planar graph with cop number three? This is currently an open conjecture and sadly will not be answered in this paper. Instead, the first part of this section will be devoted to showing that the cop number of the dodecahedron is three and then the rest will be a collection of results supporting the conjecture.

THEOREM 4.5 If G has girth at least 5, then $c(G) \geq \delta(G)$.

Proof. Assume that $\delta(G) = d$ and $d - 1$ cops are in play. At round 0 the cops choose their locations, we will denote the set of vertices the cops occupy as $\mathcal{C}$. Observe that the robber survives the first round if and only if there is some vertex in G that is not adjacent to any vertex in $\mathcal{C}$. We first show that the robber can survive round 1 and then by induction every subsequent round.

Suppose, by way of contradiction, that $\mathcal{C}$ dominates G, and so every vertex in G is adjacent to a vertex in $\mathcal{C}$. Now choose some $u \in V(G \setminus \mathcal{C})$ and partition its neighbors as $X \cup Y$, where $X \subset \mathcal{C}$ and $Y \cap \mathcal{C} = \emptyset$. Then $d = \delta(G) \leq deg(u) = |X| + |Y|$. Since $\mathcal{C}$ dominates G, every $y \in Y$ is adjacent to some $c_y \in \mathcal{C}$, but $c_y \notin X$ because there are no 3-cycles in G. Moreover, no two distinct vertices in Y are adjacent to a common vertex in $\mathcal{C}$ since G has no 4-cycles either. This means the function $C : X \cup Y \to \mathcal{C}$ defined by $\begin{cases} x & \mapsto x \\ y & \mapsto c_y \end{cases}$ is injective. Thus $|X \cup Y| \leq |\mathcal{C}| = d - 1$. However, that means $d \leq |X| + |Y| \leq d - 1$ which is a contradiction. Therefore, there exists some vertex $u_0 \in G$ that is not adjacent to any vertices in $\mathcal{C}$. Hence the robber can start on u_0 and will survive the first round.

Notice that each cop is adjacent to at most one neighbor of u_0 otherwise there would be a 4-cycle. Because the degree of u_0 is at least d, there exists a u_1 adjacent to u_0 but not adjacent to $\mathcal{C}$ that the robber can move to safely. Similarly, for every u_n there is a safe u_{n+1} the robber can move onto by induction. Hence $d - 1$ cops cannot catch the robber. $\square$

If we combine Theorem 4.5 and Theorem 4.4, then we see that any planar graph with girth at least five and has $\delta(G) = 3$ has cop number three. With this result and the observation that the dodecahedron has girth five and $\delta(G) = 3$, we have that the cop number of the dodecahedron is three.

We can use Theorem 4.5 similarly to show that the Peterson graph also has cop number three. However, the Peterson graph is not planar and so the theorem only tells that the cop number is greater or equal to three. But there is a strategy for three cops to catch a robber as shown by Figure 8 where the red vertices represent the three cops. Notice that each cop is adjacent to three other vertices in such a way that every vertex is guarded. The reason this is important is because Beveridge et al. proved that it is in fact the smallest graph that needs three cops [2]. This relates to the conjecture for planar graphs because there exists a function that maps the dodecahedron onto the Peterson graph. Each vertex in the dodecahedron graph has a unique vertex that is of length five away, which is the maximum distance for the graph. The function maps these vertex pairs to the corresponding vertex on the Peterson graph.

Another result that may be useful is Theorem 4.8. Before the proof however two well know results must be stated. The first is Euler's Formula and the second is the Handshaking Lemma. Since both of these are well know in graph theory, proofs for them will not be included in this paper.

THEOREM 4.6 (Euler's Formula) If G is a connected planar graph drawn without any edges crossing, then $|V(G)| + |F| = |E(G)| + 2$, where F is the set of faces in the drawing.

LEMMA 4.7 (Handshaking Lemma) For any graph $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

THEOREM 4.8 If G is planar, has $\delta(G) = 3$, and girth 5, then $|V(G)| \geq 20$.

Proof. Let G be a planar graph drawn on the plane such that no edges cross. Therefore Euler's Formula applies and we see $|V(G)| + |F| = |E(G)| + 2$. The girth of G being 5 means that any face in G has at least 5 edges. Furthermore, each edge touches at most 2 faces, so $2|E(G)| \geq 5|F|$. This gives us $5|V(G)| + 2|E(G)| \geq 5|V(G)| + 5|F| = 5|E(G)| + 10$. This simplifies to $5|V(G)| \geq 3|E(G)| + 10$. Since $\delta(G) = 3$, every vertex has at least 3 edges. Hence by the Handshaking Lemma, $3|V(G)| \leq \sum_{v \in V(G)} \deg(v) = 2|E(G)|$. Therefore,

$$\begin{aligned} 5|V(G)| &\geq 3|E(G)| + 10 \\ 10|V(G)| &\geq 6|E(G)| + 20 \\ 10|V(G)| &\geq 9|V(G)| + 20 \end{aligned}$$

Thus $|V(G)| \geq 20$. $\square$

Notice that the contra-positive of the theorem is $|V(G)| < 20$ implies that $\delta(G) \leq 2$ or the girth is less than five. Recall that a vertex of degree one is a pitfall

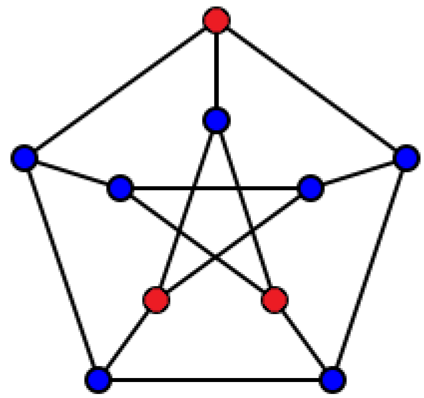


Figure 8: Peterson Graph

and that pitfalls can be removed without changing the cop number. Hence, the existence of a degree one vertex implies that there is a smaller graph with the same cop number. Since the conjecture is about the smallest 3-cop graph, this case can be ignored. Furthermore, the smallest non-trivial cycle has three vertices and so the only cases to be considered are when there is a vertex of degree two or a cycle of three or four. Pisantechakool and Tan prove that there are what they call winning vertices on such graphs, but not that there must be a 2-cop strategy for all graphs with $|V(G)| < 20$ [4].

4.4. Tutte Graph

The last thing I want to discuss is a novel proof that the cop number of the Tutte graph is two. The Tutte graph, shown in Figure 9 , is made by connecting three smaller graphs known as Tutte fragments. Although it is not considered to be an important graph to the game of cops and robbers, it has some nice properties that prompted me to find the cop number. First, it is planar and symmetric and second, it is 3-regular and contains mostly 5-cycles. However, it does contain a few 4-cycles so we can not use Theorem 4.5 to show it has cop number three.

As you will see later in the proof, the symmetry and the 4-cycles are precisely

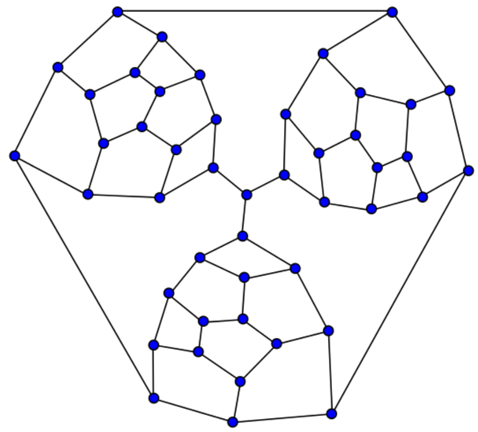


Figure 9: Tutte graph

why the Tutte graph only needs two cops. The proof is broken into two parts first a lemma showing that the Tutte fragments have a two cop strategy and then finishing by showing that two cops can contain the robber onto a single fragment.

LEMMA 4.9 The Tutte fragment has a strategy such that two cops can catch a robber without the robber reaching a cut vertex.

To simplify notation and aid the proof of the lemma, I will refer to the labeling system given by Figure 10.

Proof. Suppose that the three degree 1 vertices e_1, e_2, e_3 are escape hatches such that the robber can not start on them, but if the robber moves onto one of them then the robber immediately wins. What I want to show is that two cops can catch a robber without the robber reaching any of these escape hatches. Now start by placing C_1 on v_1 and C_2 on v_{14} so that it is two vertices away from both e_2 and e_3. Notice that this guards all the escape hatches so the robber must be on the interior of the Tutte fragment.

We will now construct a series of guardable paths according to Theorem 3.2 that each restrict the robber's territory until the robber is caught. Let $P_1 = \{v_2, v_1, v_4, v_9, v_{15}\}$ and $P_2 = \{v_{13}, v_{14}, v_{12}, v_8, v_7\}$; then C_1 can move to guard P_1 and C_2 can guard P_2. Note that if C_2 stays on v_{14} until C_1 is guarding P_1 and then moves to guard P_2 this

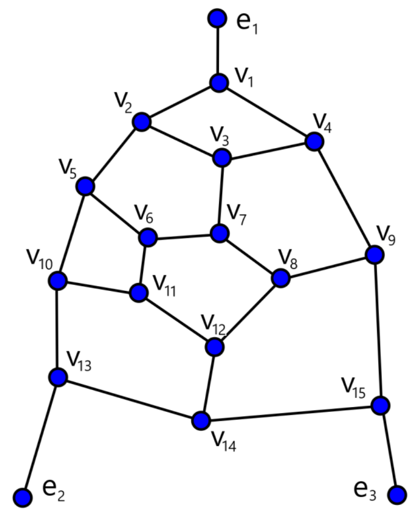


Figure 10: Tutte fragment labeled for the proof of Lemma 4.9

insures that all escape hatches remain guarded during the time it takes to move into position. Notice that there are only five vertices where the robber could be v_3, v_5, v_6, v_{10}, and v_{11}. Once both paths are guarded either the robber is on v_3 or the 4-cycle v_5, v_6, v_{11}, v_{10}. If the robber is on v_3, then C_1 should be on v_1 and C_2 should be on v_7 based on how the path retractions are defined. Now the robber is adjacent to C_2, but can not move to a safe vertex since v_2 and v_4 are both in P_1. Hence the robber will be caught next turn. Now assume the robber is on the 4-cycle. In this case, C_2 should continue to guard P_2 while C_1 moves to v_5. There are now only three possibilities of where the robber is. If the robber is on v_{11}, then C_2 should be on v_{12} and the robber is captured. If the robber is on v_6 or v_{10}, then C_2 should be on v_8 or v_{14} respectively. Which means the only safe move for the robber is to v_{11} in which case C_2 moves to v_{12} and once again the robber is captured. Thus we have a strategy such that two cops can catch the robber without the robber being able to reach any of the escape hatches e_1, e_2, e_3.

□

THEOREM 4.10 The cop number of the Tutte graph is two.

Proof. Begin by labeling the Tutte fragments T_1, T_2, and T_3 counter-clockwise and start the cops such that C_1 is on the center vertex and C_2 is on the outermost vertex of T_1. If the robber is on T_1, then the cops are in a position to use the strategy from Lemma 4.9 to catch the robber and we are done. Otherwise notice that the shortest path from the center to the outermost vertex of T_3 contains the vertex that connects T_1 and T_3. Move C_1 to guard this path. Since the robber is not on T_1, move C_2 to the outermost vertex of T_2. If the robber is on T_2 then we are in position so that we can catch the robber by Lemma 4.9. If not we know that the robber must be on T_3 because C_1 is guarding a path that contains two of the vertices connecting T_1 to the rest of the graph and C_2 was guarding the third. Therefore, we can move C_2 to the outermost vertex of T_3 and because the robber must be on T_3 we know by the lemma that the cops can catch the robber. $\square$

5. CONCLUSION

One of the most intriguing parts of math is that simple puzzles and childish games can lead to profound and often complex questions. The game of cops and robbers is a good example of this. Based on a set of rules that can be explained to an elementary school student, cops and robbers has lead to many papers, conjectures, and theorems.

Beginning with a basic notion of how the game is played and some definitions, we managed to build a characterization for all the graphs which are cop-win. Theorem 2.5 tells us that any graph that can be completely decomposed to a single vertex by successively removing pitfalls must be cop-win. Moreover, pitfalls could be generalized into retractions, which is one of the most powerful tools in the study of cops and robbers. In fact, retractions were used in many of the main proofs throughout later sections.

Even restricting our attention to only planar graphs, many answers have been found, but there are still many questions left to answer. For example, it has been shown that planar graphs have cop number less than or equal to three and yet 2-cop and 3-cop planar graphs still have no characterizations. It is conjectured that the dodecahedron is the smallest planar graph with cop number three, but the proof still eludes mathematicians. Hopefully the recent proof that the Peterson graph is the smallest non-planar 3-cop graph will lead to answering this conjecture, as well as the results brought together during my research. Even though we did not prove that the dodecahedron is the smallest planar graph with cop number three, we did manage to find a novel proof that the Tutte graph has cop number two.

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