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# Hybrid algorithm with perturbations for total asymptotically non-expansive mappings in CAT(0) space

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## ABSTRACT

In this paper, we establish strong and  $\Delta$ -convergence theorems of the modified hybrid-CR three steps iteration with perturbations for total asymptotically non-expansive mapping in CAT(0) spaces. Our results improve and extend the corresponding results from the current literature. We also provide three examples to illustrate the convergence behaviour of the proposed algorithm and numerically compare the convergence of the proposed iteration scheme with the existing schemes.

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## 1. Introduction and basic definitions

In a metric space  $(X, d)$ , a geodesic path joining  $x \in X$  and  $y \in X$  is a map  $c$  from a closed interval  $[0, r] \subset \mathbb{R}$  to  $X$  such that  $c(0) = x$ ,  $c(r) = y$  and  $d(c(t), c(s)) = |s - t|$  for all  $s, t \in [0, r]$ . In particular, the mapping  $c$  is an isometry and  $d(x, y) = r$ . The image of geodesic path (joining  $x$  and  $y$ ) under  $c$  is called a geodesic segment joining  $x$  and  $y$  which is denoted by  $[x, y]$  whenever such a segment exists uniquely. For any  $x, y \in X$ , we denote the point  $z \in [x, y]$  by  $z = (1 - \alpha)x \oplus \alpha y$ , where  $0 \leq \alpha \leq 1$  if  $d(x, z) = \alpha d(x, y)$  and  $d(z, y) = (1 - \alpha)d(x, y)$ .

A metric space  $(X, d)$  is called a geodesic space if any two points of  $X$  are joined by a geodesic, and  $X$  is said to be uniquely geodesic if there is exactly one geodesic joining  $x$  and  $y$  for each  $x, y \in X$ . A subset  $C$  of  $X$  is called convex if  $C$  contains every geodesic segment joining any two points in  $C$ .

A geodesic triangle  $\Delta(x_1, x_2, x_3)$  in a geodesic metric space  $(X, d)$  consists of three points of  $X$  (as the vertices of  $\Delta$ ) and a geodesic segment between each pair of points (as the edges of  $\Delta$ ). A comparison triangle for  $\Delta(x_1, x_2, x_3)$  (denoted by  $\bar{\Delta}$ ) is a triangle  $\bar{\Delta}(x_1, x_2, x_3) := (\bar{x}_1, \bar{x}_2, \bar{x}_3)$  in Euclidean plane  $\mathbb{R}^2$  such that  $d_{\mathbb{R}^2}(x_i, x_j) = d(x_i, x_j)$  for  $i, j \in 1, 2, 3$ . A point  $x \in [x_1, x_2]$  is said to be comparison point for  $x \in [x_1, x_2]$  if  $d(x_1, x) = d_{\mathbb{R}^2}(x_1, x)$ . The comparison points on  $[x_2, x_3]$  and  $[x_3, x_1]$  are defined in same way. Let  $\Delta$  be a geodesic triangle in  $X$  and  $\bar{\Delta}$  its comparison triangle in  $\mathbb{R}^2$ . Then  $\Delta$  is said to satisfy CAT(0) inequality if for all  $x, y \in \Delta$  and all comparison points  $\bar{x}, \bar{y} \in \bar{\Delta}$ ,  $d(x, y) \leq d_{\mathbb{R}^2}(x, y)$ .

A geodesic metric space  $X$  is called a CAT(0) space if all geodesic triangles satisfy the above comparison axiom (i.e. CAT(0) inequality). Some well known examples of CAT(0) spaces are complete

and simply connected Riemannian manifold having non-positive sectional curvature, pre-Hilbert spaces, R-trees, Euclidean buildings and the complex Hilbert ball with a hyperbolic metric.

Let  $x, y \in X$  in a CAT(0) space  $X$ . For each  $t \in [0, 1]$ , there exists a unique point  $z \in [x, y]$  such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \tag{1}$$

For convenience, from now on, we will use the notation  $(1 - t) \oplus ty$  for unique point  $z$  satisfying (1). For details as regards spaces please see [9,11,12]. Some results are recalled here for CAT(0) space  $X$ .

**Lemma 1.1 ([6]):** *Let  $X$  be a CAT(0) space. Then, for all  $x, y, z \in X$  and  $t \in [0, 1]$ ,*

- (i)  $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$ ,
- (ii)  $d^2((1 - t)x \oplus y, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y)$ .

Let  $x_i \in X$  and  $\lambda_i \in [0, 1]$  for  $i = 1, 2, \dots, n$  such that  $\sum_{i=1}^n \lambda_i = 1$ . Following the definition of unique point  $(1 - \alpha)x \oplus \alpha y$  on a geodesic segment  $[x, y]$ , we build the following notations:  $\bigoplus_{i=1}^2 \lambda_i x_i = \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 \oplus \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2$ . By induction, we can write

$$\bigoplus_{i=1}^n \lambda_i x_i = (1 - \lambda_n) \left( \frac{\lambda_1}{1 - \lambda_n} x_1 \oplus \frac{\lambda_2}{1 - \lambda_n} x_2 \oplus \dots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} x_{n-1} \right) \oplus \lambda_n x_n.$$

In view of Lemma 1.1, we can have the following:

**Lemma 1.2 ([20]):** *Let  $X$  be a CAT(0) space with  $x, x_i \in X$  and  $\lambda_i \in [0, 1]$  for  $i = 1, 2, \dots, n$  such that  $\sum_{i=1}^n \lambda_i = 1$ . Then*

- (i)  $d(\bigoplus_{i=1}^n \lambda_i x_i, x) \leq \sum_{i=1}^n d(x_i, x)$ ;
- (ii)  $d(\bigoplus_{i=1}^n \lambda_i x_i, x)^2 \leq \sum_{i=1}^n \lambda_i d^2(x_i, x) - \lambda_i \lambda_j d^2(x_i, x_j)$  for  $i, j \in \{1, 2, \dots, n\}$ .

Kirk and Panyanak [8] specialized the concept of  $\Delta$ -convergence to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting.

Let  $\{x_n\}$  be a bounded sequence in a CAT(0) space  $X$ . For  $x \in X$ , we set  $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n)$ . The asymptotic radius  $r\{x_n\}$  of  $\{x_n\}$  is given by  $r\{x_n\} = \inf\{r(x, \{x_n\}) : x \in X\}$  and the asymptotic center  $A(\{x_n\})$  of  $\{x_n\}$  is the set  $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r\{x_n\}\}$ .

**Lemma 1.3 ([6]):** *If  $\{x_n\}$  is a bounded sequence in a CAT(0) space  $X$  with  $A(\{x_n\}) = \{x\}$  and  $\{u_n\}$  is a subsequence of  $\{x_n\}$  with  $A(\{u_n\}) = \{u\}$  and the sequence  $\{d(x_n, u)\}$  converges, then  $x = u$ .*

**Lemma 1.4 ([5]):** *If  $K$  is a closed convex subset of a CAT(0) space  $X$  and if  $\{x_n\}$  is a bounded sequence in  $K$ , then the asymptotic center of  $\{x_n\}$  is in  $K$ .*

**Lemma 1.5 ([10]):** *Let  $K$  be a closed convex subset of a complete CAT(0) space  $X$  and  $T : K \rightarrow K$  be a total asymptotically non-expansive mapping. If  $\{x_n\}$  is a bounded sequence in  $K$   $\Delta$ -converging to  $x$  and  $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ , then  $x \in K$  and  $Tx = x$ .*

**Lemma 1.6 ([17]):** *Suppose that  $\{a_n\}, \{b_n\}$  and  $\{\delta_n\}$  are sequence of non-negative real numbers such that  $a_{n+1} \leq (1 + \delta_n)a_n + b_n$  for all  $n \geq 1$ . If  $\sum_{n=1}^{\infty} \delta_n < \infty$  and  $\sum_{n=1}^{\infty} b_n < \infty$ , then  $\lim_{n \rightarrow \infty} a_n$  exists.*

**Definition 1.1 ([8]):** A sequence  $\{x_n\}$  in CAT(0) space is said to  $\Delta$ -convergent to  $x \in X$  if  $x$  is the unique asymptotic center of  $\{x_n\}$  for every subsequence  $\{u_n\}$  of  $\{x_n\}$ . In this case, we write  $\Delta - \lim_n x_n = x$  and call  $x$  is the  $\Delta$ -limit of  $\{x_n\}$ .

**Definition 1.2** ([7]): Let  $K$  be a closed convex subset of a complete CAT(0) space  $X$ . A bounded sequence  $\{x_n\}$  in  $K$  is said to *converge weakly* to  $q \in K$  if and only if  $\Phi(q) = \inf_{x \in K} \Phi(x)$ , where  $\Phi(x) = \limsup_{n \rightarrow \infty} d(x_n, x)$ .

Let us recall some basics for nonlinear mappings on metric spaces. Let  $(X, d)$  be a metric space and  $K$  be its non-empty subset. Then  $T : K \rightarrow K$  is said a:

- (a) **Contraction** if there exists a  $k \in [0, 1)$  such that  $d(Tx, Ty) \leq kd(x, y)$ , for all  $x, y \in K$ .
- (b) **Non-expansive** if  $d(Tx, Ty) \leq d(x, y)$ , for all  $x, y \in K$ .
- (c) **Asymptotically non-expansive** if for a sequence  $u_n \subset [0, \infty)$  with  $\lim_{n \rightarrow \infty} u_n = 0$  such that  $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$  for all  $x, y \in K$  and  $n \geq 1$ .
- (d) **Uniformly L-lipschitzian** if there exists  $L > 0$  such that  $d(T^n x, T^n y) \leq Ld(x, y)$  for all  $x, y \in K$  and  $n \geq 1$ .
- (e) **Semi-compact** if for sequence  $\{x_n\}$  in  $K$  with  $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ , there exists a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  such that  $\{x_{n_j}\} \rightarrow p \in K$ .

Every contraction mapping in non-expansive and every non-expansive mapping is asymptotically non-expansive mapping. But the converse of each may not be true.

In 2007, Agarwal *et al.* [1] introduced a new iteration process for non-expansive mappings, which they called 'S-iteration process'. The sequence  $\{x_n\}$  in this process is given by

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_nTy_n \\y_n &= (1 - \beta_n)x_n + \beta_nTx_n \quad n \geq 1,\end{aligned}\tag{2}$$

where  $\{\alpha_n\}_{n=1}^{\infty}$  and  $\{\beta_n\}_{n=1}^{\infty}$  are real sequences in  $[0, 1]$ . They showed that S-iteration procedure is independent of Ishikawa (and hence of Mann) but converges faster than both of these iterative procedures.

In 2012 R. Chugh *et al.* [4] introduce the following iterative process:

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n, \\y_n &= (1 - \beta_n)Tx_n + \beta_nTz_n \quad n \geq 1, \\z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n,\end{aligned}\tag{3}$$

where  $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}$  are real sequences in  $[0, 1]$ . It is called the CR iterative scheme. If we take  $\alpha_n = 0$  the iterative process (3) reduces to S-iteration (2).

On the other hand, Schu [16] defined the modified Mann iterative procedure which is a generalization of Mann iterative procedure. For CAT(0) spaces it can be written as

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_nT^n x_n \quad n \geq 1,\tag{4}$$

Tan and Xu [18] generalized Ishikawa iteration procedure in a similar way.

Chang *et al.* [3] defined the concept of total asymptotically non-expansive mapping in a framework CAT(0) space as follows:

**Definition 1.3:** Let  $(X, d)$  be a CAT(0) space,  $K$  be a non-empty closed convex subset and let  $T : K \rightarrow K$  be a mapping.  $T$  is said to be *total asymptotically non-expansive mapping* if there exist non-negative real sequences  $\{\mu_n\}, \{v_n\}$ , with  $\mu_n \rightarrow 0, v_n \rightarrow 0$  and strictly increasing continuous function  $\zeta : [0, \infty) \rightarrow [0, \infty)$  with  $\zeta(0) = 0$  such that

$$d(T^n x, T^n y) \leq d(x, y) + v_n \zeta(d(x, y)) + \mu_n\tag{5}$$

for all  $x, y \in K$  and  $n \geq 1$ .

It follows from the above definitions that each non-expansive mapping is an asymptotically non-expansive mapping with  $k_n = 1 \forall n > 1$  and that each asymptotically non-expansive mapping is a total asymptotically non-expansive mapping with  $\nu_n = k_n - 1, \mu_n = 0, \nu_n \geq 1, \zeta(t) = t \forall t \geq 0$ . Moreover, each asymptotically non-expansive mapping is a uniformly L-Lipschitzian mapping with  $L = \sup_{n \geq 1} \{k_n\}$  However, the converse of these statements is not true, in general.

They [3] further studied the iterative approximation of fixed point for total asymptotically non-expansive mappings using a modified Krasnoselskii–Mann iteration process. Iterative approximation of fixed points of total asymptotically non-expansive mappings has also been studied by [2,13,14,19].

Thakur *et al.* [19] proposed a modified hybrid-Mann iteration process for iterative approximation of fixed points of total asymptotically non-expansive mappings in CAT(0) spaces. The sequence  $\{x_n\}$  in this iteration is given by

$$\begin{aligned} x_{n+1} &= T^n y_n \\ y_n &= (1 - \alpha_n)x_n \oplus \alpha_n T^n x_n \quad n \geq 1, \end{aligned} \tag{6}$$

where  $\{\alpha_n\}_{n=1}^\infty$  is a real sequence in  $[0, 1]$ .

Pansuwan and Sintunavarat [13], propose a modified hybrid-Ishikawa iteration process as follows:

$$\begin{aligned} x_{n+1} &= T^n y_n, \\ y_n &= (1 - \alpha_n)z_n \oplus \alpha_n T^n z_n \quad n \geq 1, \\ z_n &= (1 - \beta_n)x_n \oplus \beta_n T^n x_n, \end{aligned} \tag{7}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}_{n=1}^\infty$  and  $\{\beta_n\}_{n=1}^\infty$  are real sequences in  $[0, 1]$ .

Motived and inspired by Chugh *et al.* [4], Kumam *et al.* [10], Pansuwan and Sintunavarat [13] and Saluja and Postolache [15] and some others, we propose: Consider  $K$  to be a non-empty closed convex subset of a complete CAT(0) space  $X$  and  $T : K \rightarrow K$  be a total asymptotically non-expansive and uniformly L-lipschitzian mapping with  $F(T) \neq \emptyset$ . We introduce and study a three-step scheme with perturbations. The scheme is defined as follows:

$$\begin{aligned} x_{n+1} &= T^n((1 - \alpha_n - \alpha'_n)y_n \oplus \alpha_n T^n y_n \oplus \alpha'_n \epsilon_n), \\ y_n &= (1 - \beta_n - \beta'_n)T^n x_n \oplus \beta_n T^n z_n \oplus \beta'_n \epsilon'_n \quad n \geq 1, \\ z_n &= (1 - \gamma_n - \gamma'_n)x_n \oplus \gamma_n T^n x_n \oplus \gamma'_n \epsilon''_n, \end{aligned} \tag{8}$$

where  $\{\epsilon_n\}, \{\epsilon'_n\}$  and  $\{\epsilon''_n\}$  are bounded sequences in  $K$  and  $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty, \{\alpha'_n\}_{n=1}^\infty, \{\beta'_n\}_{n=1}^\infty$  and  $\{\gamma'_n\}_{n=1}^\infty$  are appropriate real sequences in  $[0, 1]$ .

In this paper, we will establish strong and  $\Delta$ -convergence theorems of modified three-step sequence with perturbation for total asymptotically non-expansive mapping in the frameworks of CAT(0) spaces. Our results can be particularize to case  $\alpha_n = 0$  and we obtain results for modified hybrid-S-iteration process [10].

## 2. Main results

We will prove the following useful lemmas of proposed iteration with perturbations (8) for a total asymptotically non-expansive mapping in CAT(0) spaces. Moreover, we also prove strong and  $\Delta$ -convergence theorems under mild conditions.

**Lemma 2.1:** *Let  $K$  be a non-empty closed convex subset of a complete CAT(0) space  $X$ . Let  $T : K \rightarrow K$  be a total asymptotically non-expansive mapping with  $\sum_{n=1}^\infty \mu_n < \infty, \sum_{n=1}^\infty \nu_n < \infty$  and  $F(T) \neq \emptyset$ .*

Let  $\{x_n\}$  be the sequence defined by (8) such that

- (C1)  $\sum_{n=1}^{\infty} \alpha'_n < \infty$ ,  $\sum_{n=1}^{\infty} \beta'_n < \infty$  and  $\sum_{n=1}^{\infty} \gamma'_n < \infty$ .  
 (C2) There exists a constant  $M_1 > 0$  such that  $\zeta(r) \leq M_1 r$ ,  $r \geq 0$ .

Then  $\lim_{n \rightarrow \infty} d(x_n, p)$  and  $\lim_{n \rightarrow \infty} d(x_n, F(T))$  exist for all  $p \in F(T)$ .

**Proof:** Let  $p \in F(T)$ . Trivially from (C2), for all  $x \in K$ ,

$$d(T^n x, p) \leq (1 + v_n M_1) d(x, p) + \mu_n. \quad (9)$$

From (8), (9) and Lemma 1.1 we have

$$\begin{aligned} d(z_n, p) &= d((1 - \gamma_n - \gamma'_n)x_n \oplus \gamma_n T^n x_n \oplus \gamma'_n \epsilon''_n, p) \\ &\leq (1 - \gamma_n - \gamma'_n) d(x_n, p) + \gamma_n [(1 + v_n M_1) d(x_n, p) + \mu_n] + \gamma'_n d(\epsilon''_n, p) \\ &\leq (1 - \gamma_n - \gamma'_n)(1 + v_n M_1) d(x_n, p) + \gamma_n [(1 + v_n M_1) d(x_n, p) + \mu_n] + \gamma'_n d(\epsilon''_n, p) \\ &= (1 - \gamma'_n)(1 + v_n M_1) d(x_n, p) + \gamma_n \mu_n + \gamma'_n d(\epsilon''_n, p) \\ &\leq (1 + v_n M_1) d(x_n, p) + \mu_n + \gamma'_n d(\epsilon''_n, p). \end{aligned} \quad (10)$$

Again using (8), (10) and Lemma 1.1, we have

$$\begin{aligned} d(y_n, p) &= d((1 - \beta_n - \beta'_n)T^n x_n \oplus \beta_n T^n z_n \oplus \beta'_n \epsilon'_n, p) \\ &\leq (1 - \beta_n - \beta'_n) [(1 + v_n M_1) d(x_n, p) + \mu_n] \\ &\quad + \beta_n [(1 + v_n M_1) d(z_n, p) + \mu_n] + \beta'_n d(\epsilon'_n, p) \\ &\leq (1 - \beta_n - \beta'_n) [(1 + v_n M_1) d(x_n, p) + \mu_n] \\ &\quad + \beta_n [(1 + v_n M_1) [(1 + v_n M_1) d(x_n, p) + \mu_n + \gamma'_n d(\epsilon''_n, p)] + \mu_n] \\ &\quad + \beta'_n d(\epsilon'_n, p) \end{aligned} \quad (11)$$

i.e.

$$\begin{aligned} d(y_n, p) &\leq (1 - \beta_n - \beta'_n)(1 + v_n M_1)^2 d(x_n, p) + (1 + v_n M_1)(1 - \beta_n - \beta'_n) \mu_n \\ &\quad + \beta_n (1 + v_n M_1)^2 d(x_n, p) + \beta_n \mu_n (1 + v_n M_1) + \beta_n \mu_n \\ &\quad + \beta_n \gamma'_n (1 + v_n M_1) d(\epsilon''_n, p) + \beta'_n d(\epsilon'_n, p) \\ &= (1 - \beta'_n)(1 + v_n M_1)^2 d(x_n, p) + (1 + v_n M_1)(1 - \beta'_n) \mu_n \\ &\quad + \beta_n \mu_n + \beta_n (1 + v_n M_1) \gamma'_n d(\epsilon''_n, p) + \beta'_n d(\epsilon'_n, p) \\ &\leq (1 + v_n M_1)^2 d(x_n, p) + (2 + v_n M_1) \mu_n + (1 + v_n M_1) \gamma'_n d(\epsilon''_n, p) + \beta'_n d(\epsilon'_n, p). \end{aligned} \quad (12)$$

Set  $w_n = (1 - \alpha_n - \alpha'_n)y_n \oplus \alpha_n T^n y_n \oplus \alpha'_n \epsilon_n$ . Now using (12) and Lemma 1.1, we get

$$\begin{aligned} d(w_n, p) &= d((1 - \alpha_n - \alpha'_n)y_n \oplus \alpha_n T^n y_n \oplus \alpha'_n \epsilon_n, p) \\ &\leq (1 - \alpha'_n + \alpha_n v_n M_1) [(1 + v_n M_1)^2 d(x_n, p) + (2 + v_n M_1) \mu_n \\ &\quad + (1 + v_n M_1) \gamma'_n d(\epsilon''_n, p) + \beta'_n d(\epsilon'_n, p)] + \alpha_n \mu_n + \alpha'_n d(\epsilon_n, p) \\ &\leq (1 + v_n M_1)^3 d(x_n, p) + [(1 + v_n M_1)(2 + v_n M_1) + \alpha_n] \mu_n \\ &\quad + (1 + v_n M_1)^2 \gamma'_n d(\epsilon''_n, p) + (1 + v_n M_1) \beta'_n d(\epsilon'_n, p) + \alpha'_n d(\epsilon_n, p). \end{aligned} \quad (13)$$

From (5), (8) and (13) we have

$$d(x_{n+1}, p) = d(T^n(w_n), p) \leq (1 + v_n M_1) d(w_n, p) + \mu_n \leq (1 + A_n) d(x_n, p) + B_n + \varepsilon_n,$$

where  $A_n = 4M_1 v_n + 6M_1^2 v_n^2 + 4M_1^3 v_n^3 + M_1^4 v_n^4$ ,  $B_n = (3 + 5v_n M_1 + 4v_n^2 M_1^2 + v_n^3 M_1^3 + (1 + v_n M_1)) \mu_n$  and  $\varepsilon_n = (1 + v_n M_1)^3 \gamma'_n d(\epsilon''_n, p) + (1 + v_n M_1)^2 \beta'_n d(\epsilon'_n, p) + (1 + v_n M_1) \alpha'_n d(\epsilon_n, p)$ . Since  $\{\epsilon_n\}$ ,  $\{\epsilon'_n\}$  and  $\{\epsilon''_n\}$  are bounded sequences in  $K$ , we can put

$$M_2 = \sup_{n \geq 1, x \in K} \{d(\epsilon_n, x), d(\epsilon'_n, x), d(\epsilon''_n, x)\}.$$

Then  $M_2$  is a finite number and there exists some positive constant  $D$  such that  $\varepsilon_n \leq D(\gamma'_n + \beta'_n + \alpha'_n)$ . Taking infimum over all  $p \in F(T)$ , we get

$$d(x_{n+1}, p) \leq (1 + A_n) d(x_n, F(T)) + B_n + D(\gamma'_n + \beta'_n + \alpha'_n).$$

Since  $\sum_{n=1}^\infty \mu_n < \infty$ ,  $\sum_{n=1}^\infty v_n < \infty$ , it follows that  $\sum_{n=1}^\infty A_n < \infty$ ,  $\sum_{n=1}^\infty B_n < \infty$ . From (C1) and Lemma 1.6,  $\lim_{n \rightarrow \infty} d(x_n, p)$  and  $\lim_{n \rightarrow \infty} d(x_n, F(T))$  exist. ■

**Theorem 2.1:** *Let  $X, K, T, \{x_n\}$  satisfy the hypothesis of Lemma 2.1. Then the sequence  $\{x_n\}$  defined by (8) converges strongly to a fixed point of  $T$  if and only if  $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ , where  $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$ .*

**Proof:** The necessity is obvious. To prove the converse, suppose that  $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ . Thus by hypothesis  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ . Next, we show that  $\{x_n\}$  is Cauchy sequence in  $K$ . with help of inequality  $1 + x \leq e^x, x \geq 0$ . From Lemma 2.1,  $d(x_{n+1}, p) \leq (1 + A_n) d(x_n, p) + B_n + \varepsilon_n$ , i.e. for any integer  $m \geq 1$ ,

$$\begin{aligned} d(x_{n+m}, p) &\leq (1 + A_{n+m-1}) d(x_{n+m-1}, p) + B_{n+m-1} + \varepsilon_{n+m-1} \\ &\leq e^{A_{n+m-1}} d(x_{n+m-1}, p) + B_{n+m-1} + \varepsilon_{n+m-1} \\ &\leq e^{A_{n+m-1}} \left[ e^{A_{n+m-2}} d(x_{n+m-2}, p) + B_{n+m-2} + \varepsilon_{n+m-2} \right] \\ &\quad + B_{n+m-1} + \varepsilon_{n+m-1} \\ &\leq \dots \\ &\leq \left( e^{\sum_{k=n}^\infty A_k} \right) d(x_n, p) + \left( e^{\sum_{k=n}^\infty A_k} \right) \sum_{k=n}^{n+m-1} B_k \\ &\quad + \left( e^{\sum_{k=n}^\infty A_k} \right) \sum_{k=n}^{n+m-1} \varepsilon_k \\ &= G d(x_n, p) + G \sum_{k=n}^{n+m-1} B_k + G \sum_{k=n}^{n+m-1} \varepsilon_k, \end{aligned} \tag{14}$$

where  $G = e^{\sum_{k=n}^\infty A_k}$ .

Since  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ , without loss of generality, we may assume that a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and subsequence  $\{p_{n_k}\} \subset F(T)$  such that  $d(x_{n_k}, p_{n_k}) \rightarrow 0$  as  $k \rightarrow \infty$ . Then for any  $\epsilon > 0$ , there exists  $k_\epsilon > 0$  such that

$$d(x_{n_k}, p_{n_k}) < \frac{\epsilon}{6G}, \quad \sum_{k=n}^\infty B_k < \frac{\epsilon}{6G} \quad \text{and} \quad \sum_{k=n}^\infty \varepsilon_k < \frac{\epsilon}{6G} \tag{15}$$

for all  $k \geq k_\epsilon$ .

For any  $m \geq 1$  and for all  $n \geq n_k$ , by (14), we have

$$\begin{aligned}
 d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_{nk}) + d(x_n, p_{nk}) \\
 &\leq Gd(x_n, p_{nk}) + G \sum_{k=n_{k\epsilon}}^{\infty} B_k + G \sum_{k=n}^{\infty} \varepsilon_k \\
 &\quad + Gd(x_n, p_{nk}) + G \sum_{k=n_{k\epsilon}}^{\infty} B_k + G \sum_{k=n}^{\infty} \varepsilon_k \\
 &\leq 2Gd(x_n, p_{nk}) + 2G \sum_{k=n_{k\epsilon}}^{\infty} B_k + 2G \sum_{k=n}^{\infty} \varepsilon_k \\
 &\leq 2G \frac{\epsilon}{6G} + 2G \frac{\epsilon}{6G} + 2G \frac{\epsilon}{6G} = \epsilon.
 \end{aligned} \tag{16}$$

This proves that  $\{x_n\}$  is a Cauchy sequence in  $K$ . Thus, the completeness of  $X$  implies that  $\{x_n\}$  must be convergent. Assume that  $\lim_{n \rightarrow \infty} x_n = q$ . Since  $K$  is closed, therefore  $q \in K$ . Next, we show that  $q \in F(T)$ . Since  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ , we get  $d(q, F(T)) = 0$ , closedness of  $F(T)$  gives that  $q \in F(T)$ . Thus  $\{x_n\}$  converges strongly to a point in  $F(T)$ . This completes the proof.  $\blacksquare$

**Lemma 2.2:** Let  $K$  be a non-empty closed convex subset of a complete CAT(0) space  $X$ . Let  $T : K \rightarrow K$  be a uniformly continuous and total asymptotically non-expansive mapping with  $\sum_{n=1}^{\infty} \mu_n < \infty$ ,  $\sum_{n=1}^{\infty} \nu_n < \infty$  and  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  be sequences in  $(0, 1)$  such that,  $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n - \alpha'_n) > 0$ ,  $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n - \beta'_n) > 0$  and  $\liminf_{n \rightarrow \infty} \gamma_n(1 - \gamma_n - \gamma'_n) > 0$ . Let  $\{x_n\}$  be the sequence defined by (8) such that

- (C1)  $\sum_{n=1}^{\infty} \alpha'_n < \infty$ ,  $\sum_{n=1}^{\infty} \beta'_n < \infty$  and  $\sum_{n=1}^{\infty} \gamma'_n < \infty$ .  
 (C2) There exists a constant  $M_1 > 0$  such that  $\zeta(r) \leq M_1 r$ ,  $r \geq 0$ .

Then  $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$ .

**Proof:** For any  $p \in F(T)$ , it follows from Lemma 2.1 that  $\lim_{n \rightarrow \infty} d(x_n, p)$  exists. Let  $\lim_{n \rightarrow \infty} d(x_n, p) = q$ , for some  $q \geq 0$ . We claim that  $\lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0$ .

Since  $\{x_n\}$  is bounded, there exists  $R > 0$  such that  $\{x_n\}, \{y_n\}, \{z_n\} \subset B_R(p)$  for all  $n \geq 1$  Using (8) and Lemma 1.2,

$$\begin{aligned}
 d^2(z_n, p) &= d^2((1 - \gamma_n - \gamma'_n)x_n \oplus \gamma_n T^n x_n \oplus \gamma'_n \epsilon''_n, p) \\
 &\leq \gamma_n d^2(T^n x_n, p) + (1 - \gamma_n - \gamma'_n) d^2(x_n, p) + \gamma'_n d^2(\epsilon''_n, p) \\
 &\quad - \gamma_n(1 - \gamma_n - \gamma'_n) d^2(T^n x_n, x_n) \\
 &\leq \gamma_n [(1 + \nu_n M_1) d(x_n, p) + \mu_n]^2 + (1 - \gamma_n - \gamma'_n) d^2(x_n, p) + \gamma'_n d^2(\epsilon''_n, p) \\
 &\quad - \gamma_n(1 - \gamma_n - \gamma'_n) d^2(T^n x_n, x_n) \\
 &\leq \gamma_n(1 + \nu_n M_1)^2 d^2(x_n, p) + \gamma_n \mu_n (2(1 + \nu_n M_1) d(x_n, p) + \mu_n) \\
 &\quad + (1 - \gamma_n - \gamma'_n) (1 + \nu_n M_1)^2 d^2(x_n, p) + \gamma'_n d^2(\epsilon''_n, p) \\
 &\quad - \gamma_n(1 - \gamma_n - \gamma'_n) d^2(T^n x_n, x_n)
 \end{aligned}$$



$$\begin{aligned}
 &= (1 - \gamma'_n)(1 + \nu_n M_1)^2 d^2(x_n, p) + \gamma_n \mu_n (2(1 + \nu_n M_1)d(x_n, p) + \mu_n) + \gamma'_n d^2(\epsilon''_n, p) \\
 &\quad - \gamma_n (1 - \gamma_n - \gamma'_n) d^2(T^n x_n, x_n) \\
 &\leq d^2(x_n, p) + P_n \nu_n + Q_n \mu_n + \gamma'_n d^2(\epsilon''_n, p) - \gamma_n (1 - \gamma_n - \gamma'_n) d^2(T^n x_n, x_n), \tag{17}
 \end{aligned}$$

where  $P_n = M_1(2 + \nu_n M_1)d^2(x_n, p)$  and  $Q_n = 2(1 + \nu_n M_1)d(x_n, p) + \mu_n$ . This implies that

$$d^2(z_n, p) \leq d^2(x_n, p) + P_n \nu_n + Q_n \mu_n + \gamma'_n d^2(\epsilon''_n, p). \tag{18}$$

From (17) we have,

$$\gamma_n (1 - \gamma_n - \gamma'_n) d^2(T^n x_n, x_n) \leq d^2(x_n, p) + P_n \nu_n + Q_n \mu_n - d^2(z_n, p) + \gamma'_n M_2.$$

Since  $\sum_{n=1}^\infty \mu_n < \infty$ ,  $\sum_{n=1}^\infty \nu_n < \infty$ ,  $\sum_{n=1}^\infty \gamma'_n < \infty$  and  $d(x_n, p) \leq R$  for all  $n$ , we have  $\gamma_n (1 - \gamma_n - \gamma'_n) d^2(T^n x_n, x_n) < \infty$ . By  $\liminf_{n \rightarrow \infty} \gamma_n (1 - \gamma_n - \gamma'_n) > 0$ , it follows that

$$\lim_{n \rightarrow \infty} d(T^n x_n, x_n) = 0. \tag{19}$$

Again from (8) and Lemma 1.1, we have

$$\begin{aligned}
 d^2(y_n, p) &\leq d^2((1 - \beta_n - \beta'_n)T^n x_n \oplus \beta_n T^n z_n \oplus \beta'_n \epsilon'_n, p) \\
 &\leq \beta_n d^2(T^n z_n, p) + (1 - \beta_n - \beta'_n) d^2(T^n x_n, p) + \beta'_n d^2(\epsilon'_n, p) \\
 &\quad - \beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n) \\
 &\leq \beta_n [(1 + \nu_n M_1)d(z_n, p) + \mu_n]^2 \\
 &\quad + (1 - \beta_n - \beta'_n) [(1 + \nu_n M_1)d(x_n, p) + \mu_n]^2 + \beta'_n d^2(\epsilon'_n, p) \\
 &\quad - \beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n) \\
 &= \beta_n (1 + \nu_n M_1)^2 d^2(z_n, p) + \beta_n [2\mu_n d(z_n, p) + \mu_n^2] \\
 &\quad + (1 - \beta_n - \beta'_n) [(1 + \nu_n M_1)^2 d^2(x_n, p) + Q'_n \mu_n + \mu_n^2] \\
 &\quad + \beta'_n d^2(\epsilon'_n, p) - \beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n), \tag{20}
 \end{aligned}$$

where  $Q'_n = Q_n - \mu_n$ .

Substituting (18) into (20), we have

$$\begin{aligned}
 d^2(y_n, p) &\leq \beta_n (1 + \nu_n M_1)^2 [d^2(x_n, p) + P_n \nu_n + Q'_n \mu_n + \gamma'_n d^2(\epsilon''_n, p)] \\
 &\quad + \beta_n [2\mu_n d(z_n, p) + \mu_n^2] \\
 &\quad + (1 - \beta_n - \beta'_n) [(1 + \nu_n M_1)^2 d^2(x_n, p) + Q'_n \mu_n + \mu_n^2] \\
 &\quad + \beta'_n d^2(\epsilon'_n, p) - \beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n) \\
 &\leq (1 + \nu_n M_1)^2 d^2(x_n, p)
 \end{aligned}$$

$$\begin{aligned}
 & + (1 + v_n M_1)^2 [P_n v_n + Q'_n \mu_n + \gamma'_n d^2(\epsilon''_n, p)] \\
 & + 2\mu_n d(z_n, p) + Q'_n \mu_n + \mu_n^2 \\
 & + \beta'_n d^2(\epsilon'_n, p) - \beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n) \\
 = & (1 + v_n M_1)^2 d^2(x_n, p) + (1 + v_n M_1)^2 P_n v_n \\
 & + (1 + v_n M_1)^2 Q'_n \mu_n + 2\mu_n d(z_n, p) + Q'_n \mu_n + \mu_n^2 \\
 & + \gamma'_n (1 + v_n M_1)^2 d^2(\epsilon''_n, p) + \beta'_n d^2(\epsilon'_n, p) \\
 & - \beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n) \\
 \leq & d^2(x_n, p) + R_n v_n + S_n \mu_n \\
 & + \gamma'_n (1 + v_n M_1)^2 d^2(\epsilon''_n, p) + \beta'_n d^2(\epsilon'_n, p) \\
 & - \beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n), \tag{21}
 \end{aligned}$$

where  $R_n = P_n + (1 + v_n M_1)^2 P_n$  and  $S_n = Q_n + (1 + v_n M_1)^2 Q'_n + 2d(z_n, p)$ . This implies that

$$\beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n) \leq d^2(x_n, p) - d^2(y_n, p) + R_n v_n + S_n \mu_n + \lambda_n M_2,$$

where  $\lambda_n = \beta'_n + \gamma'_n (1 + v_n M_1)^2$ .

Since  $\sum_{n=1}^\infty \mu_n < \infty$ ,  $\sum_{n=1}^\infty v_n < \infty$ ,  $\sum_{n=1}^\infty \beta'_n < \infty$ ,  $\sum_{n=1}^\infty \gamma'_n < \infty$  and  $d(x_n, p) \leq R$  and  $d(z_n, p) \leq R$  for all  $n$ , we have  $\beta_n (1 - \beta_n - \beta'_n) d^2(T^n x_n, T^n z_n) < \infty$ . This implies by  $\liminf_{n \rightarrow \infty} \beta_n (1 - \beta_n - \beta'_n) > 0$  that

$$\lim_{n \rightarrow \infty} d(T^n x_n, T^n z_n) = 0. \tag{22}$$

In a similar way, we can obtain that

$$\lim_{n \rightarrow \infty} d(T^n y_n, y_n) = 0. \tag{23}$$

By using (5) and (8), we get

$$\begin{aligned}
 d(T^n y_n, x_{n+1}) & = d(T^n y_n, T^n w_n) \leq (1 + v_n M_1) d(y_n, w_n) + \mu_n \\
 & \leq (1 + v_n M_1) [\alpha_n d(y_n, T^n y_n) + \alpha'_n d(y_n, \epsilon_n)] + \mu_n. \tag{24}
 \end{aligned}$$

From (23), we get  $\lim_{n \rightarrow \infty} d(T^n y_n, x_{n+1}) = 0$ . Note that

$$\begin{aligned}
 d(y_n, T^n x_n) & \leq d((1 - \beta_n - \beta'_n) T^n x_n \oplus \beta_n T^n z_n \oplus \beta'_n \epsilon'_n, T^n x_n) \\
 & \leq \beta_n d(T^n z_n, T^n x_n) + \beta'_n M_2 \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{25}
 \end{aligned}$$

By (19) and (25), we have  $d(x_n, y_n) \leq d(x_n, T^n x_n) + d(T^n x_n, y_n) \rightarrow 0$  as  $n \rightarrow \infty$ . By triangular inequality,  $d(x_n, x_{n+1}) \leq d(x_n, y_n) + d(y_n, T^n y_n) + d(T^n y_n, x_{n+1}) \rightarrow 0$ , as  $n \rightarrow \infty$ . By (19) and uniform continuity of  $T$ , we have

$$\begin{aligned}
 d(x_n, Tx_n) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) \\
 & \quad + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\
 & \leq (2 + v_{n+1} M_1) d(x_{n+1}, x_n) + d(x_{n+1}, T^{n+1} x_{n+1}) \\
 & \quad + d(T^{n+1} x_n, Tx_n) + \mu_{n+1} \\
 & \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{26}
 \end{aligned}$$

This completes the proof. ■

**Theorem 2.2:** *Let  $X, K, T, \{x_n\}$  satisfy the hypothesis of Lemma 2.2. Then the sequence  $\{x_n\}$  defined by (8)  $\Delta$ -converges to a fixed point of  $T$ .*

**Proof:** We first show that  $w_\Delta(\{x_n\}) \subseteq F(T)$ . We denote  $w_\Delta(\{x_n\}) := \cup(A\{u_n\})$  where the union is taken over all subsequence  $\{u_n\}$  of  $\{x_n\}$ . Let  $u \in w_\Delta(\{x_n\})$  then there exists a subsequence  $\{u_n\}$  of  $\{x_n\}$  such that  $A(\{u_n\}) = \{u\}$ . By Lemma 1.4 there exists a subsequence  $\{v_n\}$  of  $\{u_n\}$  such that  $\Delta - \lim_n v_n = v \in K$ . By Lemma 1.5,  $\{v_n\} \in F(T)$ . Now by Lemma 2.1  $\lim_{n \rightarrow \infty} d(x_n, F(T))$  exists, so by Lemma 1.3, we have  $u = v$ , i.e.  $w_\Delta(\{x_n\}) \subseteq F(T)$ .

To show that  $\{x_n\}$   $\Delta -$  convergent to point in  $F(T)$ , it is sufficient to show that  $w_\Delta(\{x_n\})$  consists of exactly one point.

Let  $\{u_n\}$  be subsequence of  $\{x_n\}$  with  $A(\{u_n\}) = \{u\}$ , and let  $A(\{x_n\}) = \{x\}$  for some  $u \in w_\Delta(\{x_n\}) \subseteq F(T)$  and  $\{d(x_n, v)\}$  converge. By Lemma 1.3, we have  $x = v \in F(T)$ . Thus  $w_\Delta(\{x_n\}) = \{x\}$ . This shows that  $\{x_n\}$  is  $\Delta -$  convergent to a point of  $F(T)$ . ■

**Theorem 2.3:** *Let  $X, K, T, \{x_n\}$  satisfy the hypothesis of Lemma 2.2. If  $T^m$  is semi-compact for some  $m \in \mathbb{N}$ , then the sequence  $\{x_n\}$  converges strongly to a point of  $F(T)$ .*

**Proof:** By Lemma 2.2 and the uniform continuity of  $T$ , we have  $\lim_{n \rightarrow \infty} d(x_n, T^m x_n) = 0$ . By the semi-compactness of  $T^m$ , there exists a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  such that  $x_{n_j} \rightarrow p \in K$ . Lemma 2.2 guarantees that  $\lim_{n_j \rightarrow \infty} d(x_{n_j}, T x_{n_j}) = 0$  and so  $d(p, T p) = 0$ . By Lemma 2.1,  $\lim_{n \rightarrow \infty} d(x_n, p)$ , thus  $p$  is the strong limit of the sequence  $\{x_n\}$  itself. ■

### 3. Numerical examples

In this section, we provide the numerical examples to illustrate its performance and to compare propose iteration with existing methods.

**Example 3.1:** Let  $X := \mathbb{R}$  be a usual metric space with the metric  $d$ , which is also a complete CAT(0) space, and  $C = [1, 999]$ . We see that  $C$  is a bounded closed convex subset of  $X$ . Define a mapping

$$Tx = \sqrt{x^2 - 8x + 40}.$$

It easy see that  $T$  is a continuous uniformly  $L$ -Lipschitzian and a total asymptotically non-expansive mapping with  $F(T) = \{5\}$ .

Let  $\alpha_n = \beta_n = \frac{n}{n+1}$  and  $\gamma_n = 1$  for all  $n \in \mathbb{N}$  and without perturbations. By using Wolfram Mahtematica 10, we computed the iterates of (4), (6)–(8) (and moreover (8) with  $\alpha_n = 0$ , hybrid S-iteration) for an initial point  $x_1 = 999$ . The convergence behaviour of all iterations for approximating the fixed point 5 are given in Figures 1 and 2.

In Figures 3, and 4, we give the convergence behaviour of the iterates of and for some initial point under the different control conditions.

**Example 3.2:** Consider  $X = \mathbb{R}^2$  equipped with the Euclidean norm. Let  $x = (x_1, x_2) \in \mathbb{R}^2$ , then the squared distance of  $x$  from the origin is

$$\|x\|^2 = x_1^2 + x_2^2.$$

Consider  $K$  as the closed unit disk:

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\},$$

which is bounded, closed, and convex in  $X$ . We define mapping  $\text{Rot}_\theta : K \rightarrow K$  by:

$$\text{Rot}_\theta(x_1, x_2) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

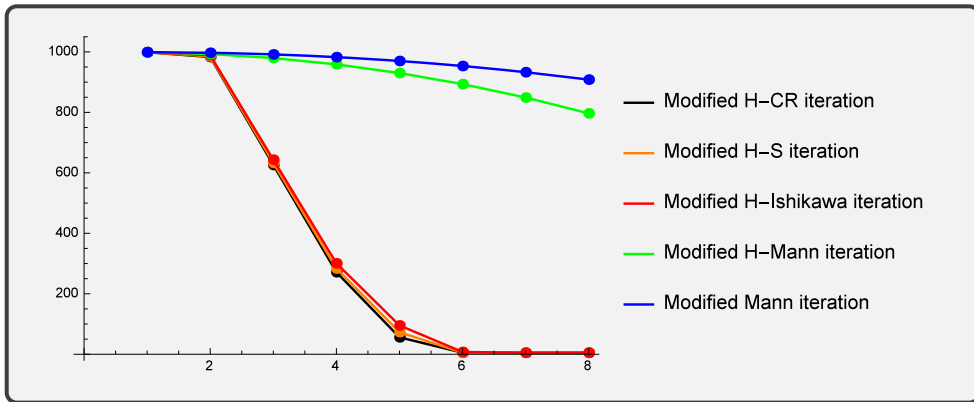


Figure 1. Convergence behaviour for Ex. 1 in case  $\alpha_n = \beta_n = \frac{n}{n+1}$  and  $\gamma_n = 1$  for all  $n \in \mathbb{N}$ .

Iterate	Mann	H-Mann	H-Ishikawa	H-S	H-CR
$x_1$	999.	999.	999.	999.	999.
$x_2$	997.006	993.018	989.03	985.042	983.049
$x_3$	991.689	979.725	643.415	632.765	626.158
$x_4$	982.716	958.791	300.868	283.686	271.47
$x_5$	969.956	930.082	95.324	73.1034	56.1376
$x_6$	953.341	893.537	7.66672	5.00009	5.
$x_7$	932.836	849.119	5.00023	5	5.
$x_8$	908.416	796.811	5.00006	5	5.

Figure 2. Convergence behaviour for Example 3.1 in case  $\alpha_n = \beta_n = \frac{n}{n+1}$  and  $\gamma_n = 1$  for all  $n \in \mathbb{N}$ .

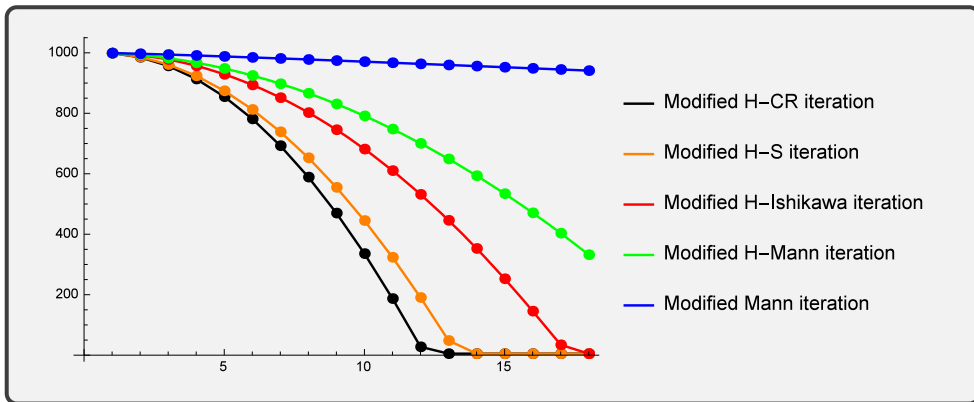
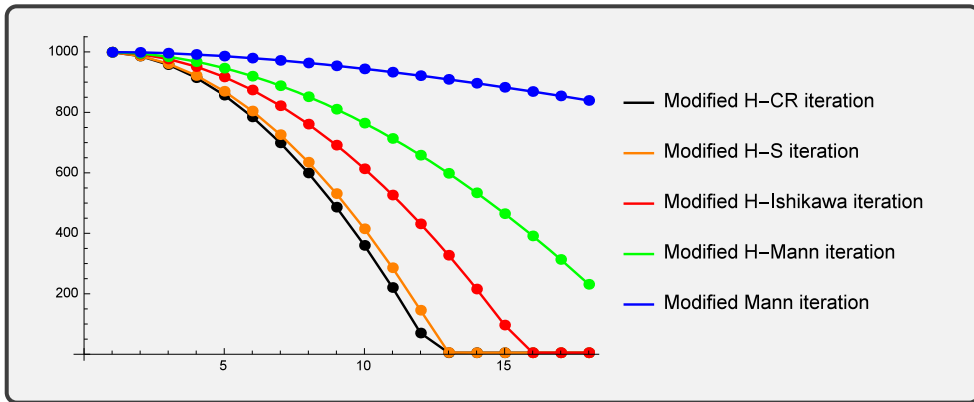


Figure 3. Convergence behaviour for Example 3.1 in case  $\alpha_n = \beta_n = 1 - \frac{1}{\sqrt{n+1}}$  and  $\gamma_n = \frac{1}{n+1}$  for all  $n \in \mathbb{N}$ .

For  $\theta = \frac{\pi}{4}$ , our algorithm is the following:

$$x_{(n+1)} = \begin{pmatrix} 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \\ (-1)^n 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \end{pmatrix} w_{(n)}$$

$$w_{(n)} = (1 - \alpha_n)y_{(n)} + \alpha_n \begin{pmatrix} 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \\ (-1)^n 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \end{pmatrix} y_{(n)}$$



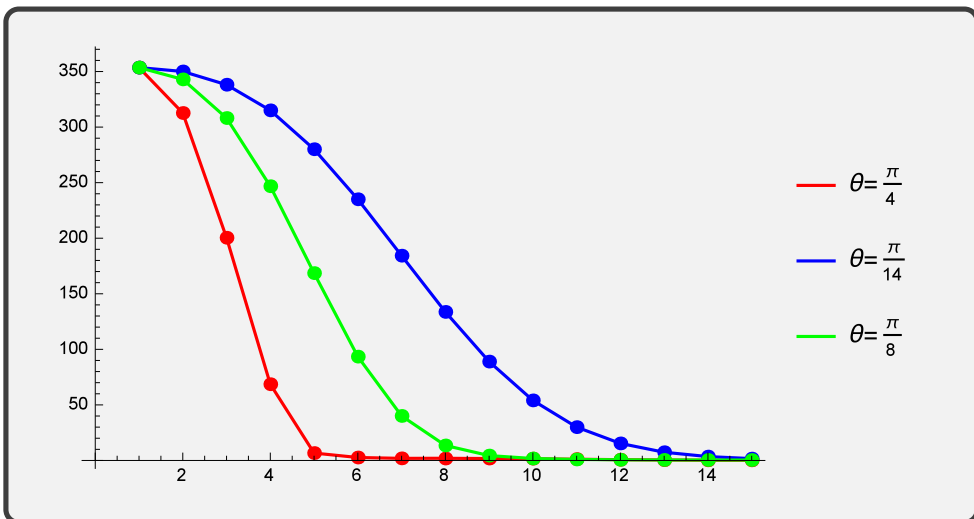
**Figure 4.** Convergence behaviour for Example 3.1 in case  $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}} - \frac{1}{n\sqrt{n+2}}$  and  $\gamma_n = \frac{n}{n+1}$  for all  $n \in \mathbb{N}$ .

$$y(n) = (1 - \beta_n) \begin{pmatrix} 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \\ (-1)^n 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \end{pmatrix} x(n) + \beta_n \begin{pmatrix} 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \\ (-1)^n 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \end{pmatrix} z(n)$$

$$z(n) = (1 - \gamma_n) x(n) + \gamma_n \begin{pmatrix} 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \\ (-1)^n 2^{-\frac{n}{2}} & 2^{-\frac{n}{2}} \end{pmatrix} x(n).$$

It is easy to see that  $\text{Rot}_\theta$  is non-expansive, since for all  $(x_1, x_2), (y_1, y_2) \in K$  and  $R_\theta$  is total asymptotically non-expansive mapping. Clearly, zero is the only fixed point of the mapping  $\text{Rot}_\theta$ .

Let  $\alpha_n = 1 - \frac{1}{\sqrt{n+1}}$  and  $\beta_n = \gamma_n = \frac{1}{n+1}$  for all  $n \in \mathbb{N}$  and without perturbations. By using Wolfram Mathematica 10, we computed the iterates of (8) for an initial point  $x_{(1)} = (250, 250)$  for  $\theta = \frac{\pi}{4}, \frac{\pi}{8}$  and  $\frac{\pi}{14}$ . The convergence behaviour of all iterations for approximating the fixed point  $(0, 0)$  are given in Figures 5 and 6.



**Figure 5.** Convergence behaviour for Ex. 2.

Error	$\theta = \frac{\pi}{4}$	$\theta = \frac{\pi}{8}$	$\theta = \frac{\pi}{14}$
$\ x_{(1)}^2\ $	353.553	353.553	353.553
$\ x_{(2)}^2\ $	312.684	342.933	350.055
$\ x_{(3)}^2\ $	200.42	308.158	338.135
$\ x_{(4)}^2\ $	68.5979	246.782	315.054
$\ x_{(5)}^2\ $	6.66271	168.598	280.081
$\ x_{(6)}^2\ $	2.65704	93.4389	235.044
$\ x_{(7)}^2\ $	1.8949	40.062	184.281
$\ x_{(8)}^2\ $	1.75554	13.4935	133.676
$\ x_{(9)}^2\ $	1.75554	4.38678	88.9765
$\ x_{(10)}^2\ $	1.63583	1.76351	54.0581
$\ x_{(11)}^2\ $	1.23427	0.921586	29.9896
$\ x_{(12)}^2\ $	0.666829	0.600032	15.3399
$\ x_{(13)}^2\ $	0.293425	0.461712	7.39926
$\ x_{(14)}^2\ $	0.167443	0.400777	3.48589
$\ x_{(15)}^2\ $	0.130935	0.377136	1.6708

Figure 6. Convergence behaviour for Ex. 2.

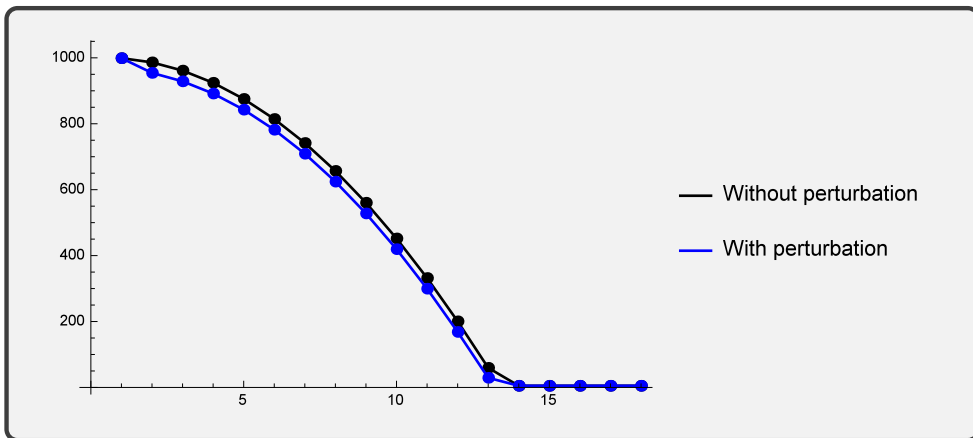


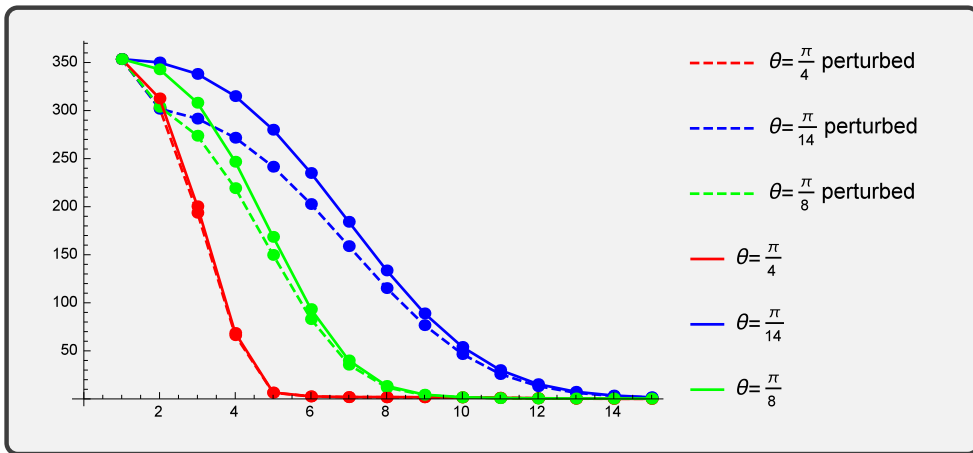
Figure 7. Convergence behaviour for Example 3.3.

**Example 3.3:** Here we reconsider Example 3.1, in case  $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$  and  $\gamma_n = \frac{n}{n+1}$  for all  $n \in \mathbb{N}$  (see Figure 9), where we consider a version with perturbation given by  $\alpha'_n = \beta'_n = \gamma'_n = n^{-5}$  and  $\epsilon_n = \epsilon'_n = \epsilon''_n = \frac{(n+1)n^3}{2^n} - 16$  (Figures 7 and 8).

**Example 3.4:** Here we reconsider Example 3.2, with same  $\alpha_n = 1 - \frac{1}{\sqrt{n+1}}$  and  $\beta_n = \gamma_n = \frac{1}{n+1}$  for all  $n \in \mathbb{N}$  and with perturbations given by  $\alpha'_n = \beta'_n = \gamma'_n = n^{-5}$  and  $\epsilon_n = \epsilon'_n = \epsilon''_n = \frac{(n+1)n^3}{2^n} - 16$ . By using Wolfram Mathematica 10, we computed the iterates of (8) for an initial point  $x_{(1)} = (250, 250)$  for  $\theta = \frac{\pi}{4}, \frac{\pi}{8}$  and  $\frac{\pi}{14}$ . The convergence behaviour of all iterations for approximating the fixed point  $(0, 0)$  are given in Figure 9, where we compare perturbed and no perturbed ones.

Iterate	Hybrid CR	Perturbed Hybrid CR
$x_1$	999.	999.
$x_2$	986.26	954.314
$x_3$	961.306	928.714
$x_4$	924.316	891.707
$\vdots$	$\vdots$	$\vdots$
$x_{11}$	332.015	299.626
$x_{12}$	200.602	168.44
$x_{13}$	59.5073	29.2372
$x_{14}$	5.	5.

**Figure 8.** Convergence behaviour for Example 3.3.



**Figure 9.** Convergence behaviour for Example 3.4.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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