Hypergeometric Expansions of Solutions of the Degenerating Model Parabolic Equations of the Third Order

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Abstract—In investigation of boundary-value problems for certain partial differential equations arising in applied mathematics, we often need to study the solution of system of partial differential equations satisfied by hypergeometric functions and find explicit linearly independent solutions for the system. In this investigation, we build private solutions for a certain class of degenerating differential equations of parabolic type of a high order. These special solutions are expressed in terms of hypergeometric functions of one variable.

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1. INTRODUCTION AND PRELIMINARIES

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of one and several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [? , p. 47 et seq., Section 1.7]). For instance, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [?]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well (cf. [? ?]). Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [?]. We note that RiemannпїЅs functions [?] and the fundamental solutions [? ? ? ?] of the degenerate second order partial differential equations are expressible by means of hypergeometric functions of several variables [?].

Further, many problems of modern mathematics and theoretical physics lead to the investigation of hypergeometric functions of one and several variables. In particular, problems of super-string theory [?], analytical continuations of Mellin–Barnes integrals [?] and algebraic geometry [?]. Systems of hypergeometric type differential equations have numerous applications as nontrivial model examples in realization of algorithms for symbolic calculations, which are used in modern systems of computer algebra [?]. Hypergeometric functions of many variables appear in quantum field theory as solutions of Knizhnik–Zamolodchikov equation [?]. These equations can be considered as generalized hypergeometric type equations and their solutions have integral representations, which generalize classic Euler integrals for hypergeometric functions of one variable. This approach allows us to link the special functions of hypergeometric type and challenges the

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theory of representations of Lie algebras and quantum groups [?]. Initially hypergeometric functions introduced by many authors with different methods, which are not related with each other. Their occurrence is determined, as a rule, by the need to solve problems, that led to a differential equation (or system of equations), insoluble in the class of elementary functions. Many problems of gas dynamics are reduced to boundary value problems for degenerate equations of mixed type.

We define the generalized hypergeometric series ${}_1F_2$ [? , p.182, (1)] with one numerator parameter and two denominator parameters by

$$ {}_1F_2(a;b,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n(c)_n}\frac{x^n}{n!}, \tag{1}$$

where $(\lambda)_m$ is the Pochhammer symbol defined (for $\lambda \in C$) by

$$(\lambda)_m=\frac{\Gamma(\lambda+m)}{\Gamma(\lambda)},$$

Γ being the well-known Gamma function. For $a=b$, equation (??) reduces to the hypergeometric function

$$ {}_0F_1(-;c;x)=\sum_{n=0}^{\infty}\frac{1}{(c)_n}\frac{x^n}{n!}. \tag{2}$$

An alternative representation for the function ${}_1F_2$ is given by the following integral formula:

$$ {}_1F_2(a;b,c;x)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1\xi^{a-1}(1-\xi)^{b-a-1}{}_0F_1(-;c;x\xi)d\xi, \Re(b)>\Re(a)>0.$$

In view of the operator $H(\alpha,\beta)$ [?], it is easily seen that

$$ {}_1F_2(a;b,c;x)=\sum_{n=0}^{\infty}\frac{(-1)^n(b-a)_nx^n}{(b)_n(c)_nn!}{}_0F_1(-;c+n;x),$$

and the inverse is given by the following formula:

$$ {}_0F_1(-;c;x)=\sum_{n=0}^{\infty}\frac{(b-a)_nx^n}{(b)_n(c)_nn!}{}_1F_2(a;b+n,c+n;x).$$

Motivated by the important role of the degenerating model parabolic equations of the third order and the hypergeometric functions, this work aims at finding and than expanding the solutions of the following degenerating model parabolic equations of the third order

$$Lu\equiv x^nu_t-t^ku_{xxx}=0, \tag{3}$$

in the domain $D=\{(x,t):x>0,t>0\}$, where $n>0$ and $k>0$ are constants.

In Section 2 we construct the differential equation of the hypergeometric function ${}_1F_2$ and then find the linearly independent solutions of the differential equation of ${}_1F_2$. Section 3 aims at finding the solutions of the degenerating model parabolic equations of the third order defined in (??) and then based upon the solutions of the differential equation of ${}_1F_2$, we expand the solutions of (??) in terms of ${}_1F_2$.

2. LINEARLY INDEPENDENT SOLUTIONS OF THE DIFFERENTIAL EQUATION OF ${}_1F_2$

According to the theory of hypergeometric functions (see [? ?]), the differential equation for the hypergeometric function ${}_1F_2(a;b,c;x)$ is readily seen to be given as follows:

$$\left(b+x\frac{d}{dx}\right)\left(c+x\frac{d}{dx}\right)\left(x\frac{d}{dx}+1\right)\left(\frac{\nu}{x}\right)-\left(a+x\frac{d}{dx}\right)\nu=0, \tag{4}$$

where $\nu(x) = {}_1F_2(a; b, c; x)$. Now, by making use of some elementary calculations, we find that

$$x^2\nu'''(x) + (b + c + 1)x\nu''(x) + (bc - x)\nu'(x) - a\nu(x) = 0.$$

Now, in order to find the linearly independent solutions of equation (**??**), we will search the solutions in the form $\nu(x) = x^\lambda\omega(x)$, where $\omega(x)$ is an unknown function, and λ is constant, which are to be determined. Next, substituting $\nu(x) = x^\lambda\omega(x)$ into equation (**??**), we get

$$x^2\omega'''(x) + (3\lambda - b - c + 1)x\omega''(x) + [3\lambda(\lambda - 1) + 2\lambda(b + c + 1) + bc - x]\,\omega'(x)$$

$$- \left[-\lambda(\lambda + b - 1)(\lambda + c - 1)x^{-1} + (\lambda + a)\right]\omega(x) = 0. \tag{5}$$

We note that equation (**??**) is analogical to equation (**??**), therefore we require that the conditions

$$\lambda(\lambda + b - 1)(\lambda + c - 1) = 0 \tag{6}$$

should be satisfied. It is evident that equation (**??**) has the following solutions:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline \lambda & 0 & 1-b & 1-c \end{array}. \tag{7}$$

Substituting all solutions (**??**) into (**??**), we find the following linearly independent solutions of equation (**??**):

$$\nu_1 = {}_1F_2(a; b, c; x), \tag{8}$$

$$\nu_2 = x^{1-b}{}_1F_2(1 - b + a; 2 - b, 1 - c + b; x), \tag{9}$$

$$\nu_3 = x^{1-c}{}_1F_2(1 - c + a; 1 + b - c, 2 - c; x). \tag{10}$$

3. SOLUTIONS OF THE DEGENERATING MODEL EQUATION (**??**)

In this section we establish the solutions of the degenerating model equation (**??**) in terms of the hypergeometric function ${}_1F_2(a; b, c; x)$. First, consider the equation

$$u(x, t) = P\omega(\sigma), \tag{11}$$

where

$$P = \left(\frac{1}{k+1}t^{k+1}\right)^{-1}, \quad \sigma = -\frac{1}{(n+3)^3}x^{n+3}\left(\frac{1}{k+1}t^{k+1}\right)^{-1}$$

$$(0 < x < \infty, t > 0 \quad \text{is any fixed point}).$$

On substituting (**??**) into (**??**), we get

$$A\omega'''(\sigma) + B\omega''(\sigma) + C\omega'(\sigma) + D\omega(\sigma) = 0, \tag{12}$$

where

$$A = t^k P\sigma_x^3, \quad B = 3t^k\sigma_x\left(P_x\sigma_x + P\sigma_{xx}\right),$$

$$C = 3t^kP_{xx}\sigma_x + 3t^kP_x\sigma_{xx} + t^kP\sigma_{xxx} - x^nP\sigma_t, \quad D = t^kP_{xxx} - x^nP_t.$$

After several evaluations, we find that

$$A = -\frac{x^nP(k+1)}{t}\sigma^2, \tag{13}$$

$$B = -3\frac{x^nP(k+1)}{t}\frac{(n+2)}{n+3}\sigma, \tag{14}$$

$$C=-\frac{x^nP(k+1)}{t}\frac{(n+2)(n+1)}{(n+3)^2}+\frac{x^nP(k+1)}{t}\sigma, \tag{15}$$

$$D=\frac{x^nP(k+1)}{t}. \tag{16}$$

Upon substituting equalities (??), (??), (??) and (??) into (??), we are led finally to the following hypergeometric differential equation

$$\sigma^2\omega'''(\sigma)+\left(\frac{\alpha+2}{3}+\frac{2\alpha+1}{3}+1\right)\sigma\omega''(\sigma)+\left(\frac{2\alpha+1}{3}+\frac{2\alpha+1}{3}-\sigma\right)\omega'(\sigma)-\omega(\sigma)=0, \tag{17}$$

where $\alpha=n/(n+2)$. Now, if we consider the solutions (??), (??) and (??) of equation (??), we get from (??) that

$$\omega_1={}_1F_2\left[1;\frac{\alpha+2}{3},\frac{2\alpha+1}{3};\sigma\right], \tag{18}$$

$$\omega_2=\sigma^{\frac{1-\alpha}{3}}{}_1F_2\left[\frac{4-\alpha}{3};\frac{4-\alpha}{3},\frac{2+\alpha}{3};\sigma\right]=\sigma^{\frac{1-\alpha}{3}}{}_0F_1\left[-;\frac{2+\alpha}{3};\sigma\right], \tag{19}$$

$$\omega_3=\sigma^{\frac{2-2\alpha}{3}}{}_1F_2\left[\frac{5-2\alpha}{3};\frac{5-2\alpha}{3},\frac{4+\alpha}{3};\sigma\right]=\sigma^{\frac{2-2\alpha}{3}}{}_0F_1\left[-;\frac{4-\alpha}{3};\sigma\right].$$

where ${}_1F_2$ and ${}_0F_1$ are defined by formulae (??) and (??), respectively.

Finally, substituting (??), (??) and (??) into (??), we find the following linearly independent solutions of the degenerating model parabolic equation of the third about (??):

$$u_1(x,t)={}_1F_2\left[1;\frac{\alpha+2}{3},\frac{2\alpha+1}{3};-\frac{1}{(n+3)^3}x^{n+3}\left(\frac{1}{k+1}t^{k+1}\right)^{-1}\right], \tag{20}$$

$$u_2(x,t)=\left(\frac{1}{k+1}t^{k+1}\right)^{-1}\left(-\frac{1}{(n+3)^3}x^{n+3}\left(\frac{1}{k+1}t^{k+1}\right)^{-1}\right)^{\frac{1-\alpha}{3}}$$

$$\times{}_0F_1\left[-;\frac{2+\alpha}{3};-\frac{1}{(n+3)^3}x^{n+3}\left(\frac{1}{k+1}t^{k+1}\right)^{-1}\right], \tag{21}$$

$$u_3(x,t)=\left(\frac{1}{k+1}t^{k+1}\right)^{-1}\left(-\frac{1}{(n+3)^3}x^{n+3}\left(\frac{1}{k+1}t^{k+1}\right)^{-1}\right)^{\frac{2-2\alpha}{3}}$$

$$\times{}_0F_1\left[-;\frac{4-\alpha}{3};-\frac{1}{(n+3)^3}x^{n+3}\left(\frac{1}{k+1}t^{k+1}\right)^{-1}\right]. \tag{22}$$

According to the series representation of the Bessel functions $J_\nu(z)$ [? , p.44, (11)]

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1)}{}_0F_1\left[-;\nu+1;-z^2/4\right],$$

it may of interest to point out that the solutions in (??) and (??) can be rewritten in forms of Bessel functions. Also, it is importance to note that the received solutions (??), (??) and (??) of the degenerating model parabolic equation of the third (??), are very useful for the solution of boundary value problems.

Note that, in the indicated way, the fundamental solutions for the generalized Helmholtz equation were constructed in [? ?]. Found fundamental solutions are expressed by hypergeometric functions of many variables.

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