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On quadratic stability of state-dependent planar switching systems*

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Abstract

In this paper we consider the stability of a class of switched systems. Specifically, we ask the following question. Given a partition of the state space, and a set of linear dynamics, each of which are active in parts of the state space in a manner governed by the partition, does there exist a quadratic Lyapunov function for the resulting system? For planar systems with conic partitions and two dynamics, necessary and sufficient conditions are given for such a Lyapunov function. Examples are given to illustrate our results.

1 Introduction

Recent years have witnessed great interest in stability problems arising in the study of switched and hybrid systems; see [1] for a review of recent results and open problems in this area. While most authors have focused on linear switched systems, the dynamics of which are constructed by switching between a finite number of vector fields arbitrarily quickly, a growing number of researchers are studying non-linear problems. Among these, the problem of state-dependent switching between linear vector fields represents an important problem (of the Popov type) that arises frequently in practice.

This latter problem arises if the rule for switching between the constituent linear systems of a switched system is determined by the state vector of the system; consequently, we say that the switching is *state dependent*. Loosely speaking, the

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stability problems associated with this type of switching regime can be divided into two classes. In the first of these, the state space is partitioned by a number of hyper-surfaces that determine the mode switches in the system dynamics, and the problems is to analyse the stability of the time-varying system defined in this way. In the second class of problem we are concerned with finding state-dependent rules for switching between a family of unstable systems that result in stability. Thus, in the former case, a partition of the state space is specified and the problem is to determine the stability of the piecewise linear system defined by that partition, while in the latter case the aim is to find stabilizing state-dependent rules for switching between potentially unstable systems.

Before proceeding it is worth pointing out that problems in this latter category have been the subject of some discussion in the hybrid system community. Well known papers by Feron [2], DeCarlo [3] and others, and the more recent book by Sun and Ge [4], have all dealt with this problem with some success. However, aside from some notable LMI based numerical approaches [5], the former problem, despite its evident practical significance, has received considerably less attention. Our objective in writing this paper is to begin the task of addressing this problem from a more theoretical perspective. As this is our initial thrust in this direction, our study begins with a somewhat simplified version of the afore mentioned general problem. Specifically, we consider planar systems where the state space partition is constructed using rays passing through the origin, and different linear dynamics are active in the regions between these rays. Given this basic set-up, we then ask for, and obtain, necessary and sufficient conditions for the existence of a quadratic Lyapunov function for the resulting nonlinear system.

While this paper considers planar systems, we believe the results to be of general significance. Firstly, a thorough understanding of the second order case is usually very revealing and is an important preliminary step for understanding higher dimensional cases. Secondly, the techniques that we use are not restricted to two dimensions and can be used to study higher order systems as well (see our work on the common quadratic Lyapunov function problem [6, 7]). Finally, as we shall see, our results are a natural extension of our previous work on the CQLF existence problem, and of classical results such as the Circle Criterion, and hence open up many exciting research directions for future study.

The structure of this note is as follows. We begin by presenting some basic facts about quadratic Lyapunov functions. We then present our main result and discuss its relationship with known results in the area. Finally, we conclude our discussion with examples that illustrate the pertinent features of our results.

2 Preliminary remarks and basic results

We begin our discussion by recalling facts that shall be useful in deriving the main result in this paper.

(i) Lyapunov functions : A real $n \times n$ matrix A is called Hurwitz if its spectrum lies in the open left half of the complex plane. For a symmetric positive definite matrix P, the function $x^T Px$ defines a quadratic Lyapunov function (QLF) for the dynamic system $\Sigma_A : \dot{x} = Ax$, if $PA + A^T P$ is negative definite. By allowing a small abuse of notation we say that P is a QLF for A, meaning that the function $V(x) = x^T Px$ is a QLF for the dynamic system Σ_A.

(ii) The cone of Lyapunov functions $\mathcal{L}(A)$: It is well known that A is Hurwitz if and only if there exists a QLF for A. Define the set of all such QLF matrices

$$\mathcal{L}(A) = \{P = P^T > 0 \,|\, PA + A^T P < 0\}. \tag{1}$$

Then $\mathcal{L}(A) \subset \mathcal{P}_n(\mathbb{R}) \subset \mathcal{S}_n(\mathbb{R})$, where $\mathcal{S}_n(\mathbb{R})$ is the space of $n \times n$ symmetric matrices, and $\mathcal{P}_n(\mathbb{R})$ is the set of symmetric positive definite matrices. The set $\mathcal{L}(A)$ is an open convex cone in $\mathcal{S}_n(\mathbb{R})$. Also $\mathcal{S}_n(\mathbb{R})$ is isomorphic to $n(n+1)/2$-dimensional Euclidean space with the inner product $\langle A, B\rangle = \mathrm{Tr}AB$. This identification of symmetric matrices in $\mathcal{S}_n(\mathbb{R})$ as "vectors" in $\mathbb{R}^{n(n+1)/2}$ will play an important role in our analysis.

(iii) Common quadratic Lyapunov functions (CQLF's) : The CQLF problem is to find conditions which guarantee the existence of a common QLF for a set of Hurwitz matrices $\{A_1, \dots, A_N\}$. The existence of a CQLF implies stability of the dynamic system $\dot{x} = A_{\tau(t)}x$ where $\tau : \mathbb{R}^+ \mapsto \{1, \dots, N\}$ is any switching function. Referring to the QLF cones defined above, an equivalent formulation of the CQLF problem is to find conditions for a nonempty intersection of the cones $\mathcal{L}(A_1), \dots, \mathcal{L}(A_N)$. This problem has a long and interesting history [1] and is of consequence for the problem considered in this paper.

(iv) The cones $\mathcal{L}(A)$ and $\mathcal{L}(A^{-1})$: As Loewy ([8]) originally observed, $\mathcal{L}(A) = \mathcal{L}(A^{-1})$, that is any quadratic Lyapunov function for Σ_A is also a quadratic Lyapunov function for $\Sigma_{A^{-1}}$. This congruence between Σ_A and $\Sigma_{A^{-1}}$ means that asking the question whether $\dot{x} = Ax$ is quadratically stable, is identical to asking the question whether $\dot{x} = A^{-1}x$ is quadratically stable.

Suppose now that $x^T(A^T P + PA)x < 0$ in some region of the state space Ω. Then it follows from congruence that $x^T({A^T}^{-1}P + PA^{-1})x < 0$ in a region of the state space $A(\Omega)$, obtained by applying A to all vectors in Ω. The significance of this observation is that asking the question whether there exists a matrix $P = P^T > 0$ such that $x^T PAx < 0$ for all x in some Ω, is equivalent to asking whether there is a $P = P^T > 0$ such that $y^T PA^{-1}y < 0$ for all y in $A(\Omega)$.

(v) The boundary of $\mathcal{L}(A)$: The boundary of $\mathcal{L}(A)$, denoted $\partial\mathcal{L}(A)$, consists of positive semidefinite matrices P for which $PA + A^TP$ is negative semidefinite. If x_0 is in the kernel of $P_0A + A^TP_0$ for some $P_0 \in \partial\mathcal{L}(A)$ then

$$0 = x_0^T(P_0A + A^TP_0)x_0 = \mathrm{Tr}P_0(Ax_0x_0^T + x_0x_0^TA^T) \tag{2}$$

The set of all symmetric matrices M for which $\mathrm{Tr}M(Ax_0x_0^T + x_0x_0^TA^T) = 0$ defines a hyperplane in $\mathcal{S}_n(\mathbb{R})$, and it is easy to see that this hyperplane is tangent to the QLF set $\mathcal{L}(A)$. In terms of the geometry of $\mathcal{S}_n(\mathbb{R})$ regarded as a $n(n+1)/2$-dimensional vector space, this hyperplane is defined by its normal vector, which is the matrix $Ax_0x_0^T + x_0x_0^TA^T$. This normal vector is directed outward from the set $\mathcal{L}(A)$. Furthermore every hyperplane of this form is tangent to $\mathcal{L}(A)$, for every $x \in \mathbb{R}^n$.

(vi) Separating hyperplanes and Lyapunov sets $\mathcal{L}(A)$: We will make use of the following important fact, first proven in [6]. Let x, y, u, v be four non-zero vectors in $\mathbb{R}^n$ such that for all $M \in \mathcal{S}_n(\mathbb{R})$, $x^TMy = -k^2u^TMv$ for some real k. Then either

$$x = \alpha u \ \text{ and } \ y = -\left(\frac{k^2}{\alpha}\right)v \tag{3}$$

for some real scalar α or

$$x = \beta v \ \text{ and } \ y = -\left(\frac{k^2}{\beta}\right)u, \tag{4}$$

for some real scalar β. To see how this result is of consequence in this paper suppose we have that $x^TMA_1x = -k^2y^TMA_2y$ for all $M \in \mathcal{S}_n(\mathbb{R})$, and for some invertible A_1 and A_2. Then the above result implies that either some convex combination of A_1 and A_2 is singular or some convex combination of A_1 and A_2^{-1} is singular. This situation will arise if the QLF sets $\mathcal{L}(A_1)$ and $\mathcal{L}(A_2)$ have a common tangent hyperplane, and the hyperplane is determined in the form given above in **(v)**. In this case the normal vectors which define the hyperplanes in $\mathcal{S}_n(\mathbb{R})$ must be anti-parallel, implying that there is some real scalar k and vectors $x, y \in \mathbb{R}^n$ for which

$$A_1xx^T + xx^TA_1^T = -k^2\Big(A_2yy^T + yy^TA_2^T\Big). \tag{5}$$

This equation is equivalent to $x^TMA_1x = -k^2y^TMA_2y$ for all $M \in \mathcal{S}_n(\mathbb{R})$, and hence leads to the singularity conditions described above.

(vii) The general joint QLF problem:

Turning to the subject of this paper, suppose now that Ω is a closed double cone in $\mathbb{R}^n$, so that if $x \in \Omega$ then also $\lambda x \in \Omega$ for all $\lambda \in \mathbb{R}$. Given a real $n \times n$ matrix A, define the QLF set for the pair (A, Ω) as follows:

$$\mathcal{L}(A, \Omega) = \{P = P^T > 0 \,|\, x^T(PA + A^TP)x < 0 \text{ all } x \in \Omega,\ x \neq 0\} \tag{6}$$

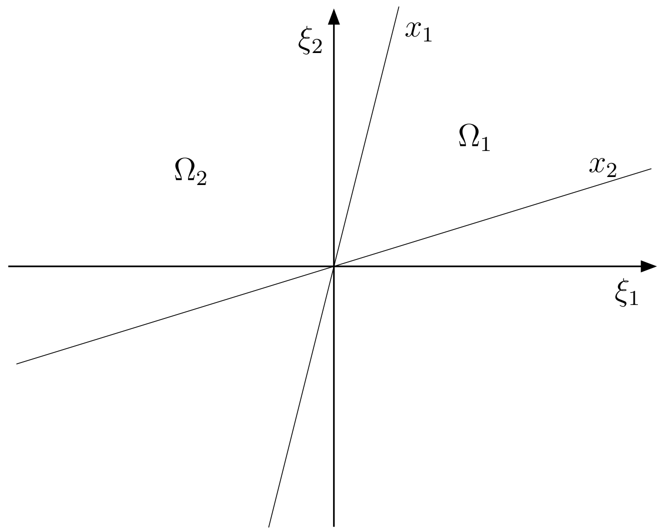

Figure 1: Partition of the state space

Note that if the matrix A is Hurwitz then $\mathcal{L}(A, \Omega)$ is non-empty. The joint QLF problem for a collection of matrices A_i and regions Ω_i is to find conditions for a non-empty intersection of the sets $\{\mathcal{L}(A_i, \Omega_i)\}$. If $\cup_i \Omega_i = \mathbb{R}^n$, then the existence of a joint QLF implies exponential stability of the state-dependent switching system $\dot{x} = A(x)x$, where $A(x) \in \{A_1, \ldots, A_N\}$, with $A(x) = A_i$ implying $x \in \Omega_i$.

3 Main result

We now present our main results. As mentioned in the Introduction, our focus in this paper is to consider perhaps the most basic version of the joint QLF problem described in Item **(vii)** in the previous section. In our specific problem we shall consider $A_1, A_2 \in \mathbb{R}^{2\times 2}$ and let (Ω_1, Ω_2) be subsets of $\mathbb{R}^2$ with $\Omega_1 \cup \Omega_2 = \mathbb{R}^2$ (although the solution can be readily extended to situations when this is not the case). For definiteness we shall assume that Ω_1 is characterised as follows. Let x_1, x_2 be two vectors in $\mathbb{R}^2$. Then $x \in \mathbb{R}^2$ belongs to Ω_1 if and only if $x = \alpha x_1 + \beta x_2$ with $\alpha\beta \geq 0$. We shall describe such a region as a *closed double wedge* in the plane. Thus, the boundary of Ω_1 is the pair of lines parallel to x_1 and x_2, each passing through the origin. This setup is depicted in Figure 1. We shall consider several different possibilities for the region Ω_2.

Comment 1: Before proceeding it is worth noting that this is not an artificial construction and that such problems arise frequently in practice. For example, in the design of rollover prevention systems, one often designs switched control systems where the vehicle roll angle and the load transfer ratio are used to control the currently active controller [9].

Comment 2: Given this basic formulation two problems of practical importance are immediately evident.

- **Problem 1 :** Ω_1 is a closed double wedge in the plane, and $\Omega_2 = \mathbb{R}^2$. Here, $A(x)$ may be either A_1 or A_2 in Ω_1 and is equal to A_2 everywhere else.
- **Problem 2 :** Ω_1 and Ω_2 are both closed double wedges in the plane and their union is $\mathbb{R}^2$. Hence Ω_2 is the closure of $\mathbb{R}^2 \setminus \Omega_1$, and vice versa.

Note that while this problem statement does not preclude some of the matrices having eigenvalues in the closed right half of the complex plane, we shall assume in what follows that both A_1 and A_2 are indeed Hurwitz, noting that this corresponds to most practical cases of interest, and furthermore that our results are easily generalised to the case where some of the A_i are non-Hurwitz.

Given this basic setup we are interested in determining necessary and sufficient conditions for the existence of a $P \in \mathcal{P}_2(\mathbb{R})$ such that the following inequalities

$$x^T\left(A_1^T P + P A_1\right)x \;<\; 0,\; x \in \Omega_1 \tag{7}$$

$$y^T\left(A_2^T P + P A_2\right)y \;<\; 0,\; y \in \Omega_2 \tag{8}$$

are simultaneously satisfied. The following two Theorems provide a complete solution to this question.

3.1 Solution to Problem 1

Theorem 3.1 *Let A_1 and A_2 be 2×2 Hurwitz matrices. Let $\Omega_1 \subset \mathbb{R}^2$ be the closed double wedge region defined by the vectors x_1, x_2. Denote by $A_1(\Omega_1)$ the image of Ω_1 under the map A_1. Then there is a joint QLF for the pairs (A_1, Ω_1) and $(A_2, \mathbb{R}^2)$ if and only if the following conditions are satisfied:*

(a) there is no convex combination of A_1 and A_2, or of A_1 and A_2^{-1}, which has an eigenvector in Ω_1 with non-negative eigenvalue

(b) there is no convex combination of A_1^{-1} and A_2, or of A_1^{-1} and A_2^{-1}, which has an eigenvector in $A_1(\Omega_1)$ with non-negative eigenvalue

(c) there is no nonzero vector y satisfying both equations

$$(aA_1 + bA_1^{-1} + cA_2)y \;=\; 0, \tag{9}$$

$$ayy^T + bA_1^{-1}yy^T(A_1^{-1})^T \;=\; d_1x_1x_1^T + d_2x_2x_2^T \tag{10}$$

for some nonnegative coefficients $a, b, c, d_1, d_2 \geq 0$

Comment 3: Although the conditions (a), (b), (c) of Theorem 3.1 are necessary and sufficient for the existence of a QLF, some of these conditions are redundant. For example Condition (a) with a positive eigenvalue is activated only when $x = x_1$ or $x = x_2$. However we have written the conditions in this more general form because it allows a clear and unified statement of the results.

Comment 4: In the case where x_1 and x_2 are anti-parallel, so that Ω_1 is itself $\mathbb{R}^2$, Problem 1 reduces to the standard CQLF problem for a pair of 2×2 matrices, the solution to which is given in [10, 11]. For this case, the minimal necessary conditions of Theorem 3.1 can be combined into the single condition that $\Big(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_1^{-1} + \lambda_4 A_2^{-1}\Big)$ is nowhere singular for all non-negative $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which is equivalent to the solution given in [10, 11].

Comment 5: It is evident that the conditions given in Theorem 3.1 are necessary for the existence of a joint QLF. For example if $(A_1 + \lambda A_2)x = k^2 x$ for some $x \in \Omega_1$, and some $\lambda > 0$, then $x^T P A_1 x$ and $x^T P A_2 x$ cannot both be negative for any positive definite P, meaning that $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2)$ must be disjoint. Similar reasoning applies to the other conditions in Theorem 3.1.

Comment 6: Other necessary conditions for the existence of P can be derived in a similar fashion. For example, it follows from the congruence argument in Item **(iv)**, and from the previous comment, that if a positive matrix $P = P^T$ of the required form exists, then $x^T P \Big(A_1 + k^2 A_2\Big)^{-1} x < 0$ for all x in some region Θ. In the case where Θ and Ω_1 overlap, then this immediately implies that $A_1 + k^2 A_2 + m^2 A_1^{-1}$ may not be singular in some region. Conditions similar to (c) arise in this way.

3.2 Solution to Problem 2

Our second result concerns Problem 2; namely, the two pairs (A_1, Ω_1) and (A_2, Ω_2), where A_1, A_2 are 2×2 Hurwitz matrices, and Ω_1, Ω_2 are the complementary closed double wedge regions defined by two vectors $x_1, x_2 \in \mathbb{R}^2$. Note that Ω_1 and Ω_2 are closed by definition, so their intersection is the two rays parallel to the vectors x_1 and x_2. Define $\mathcal{C}_{12}$ to be the set of all positive combinations of the projection matrices $x_1 x_1^T$ and $x_2 x_2^T$, that is

$$\mathcal{C}_{12} = \{a x_1 x_1^T + b x_2 x_2^T \;:\; a, b \geq 0\} \tag{11}$$

As before we denote by $A(\Omega)$, the image of the region Ω under the action of the matrix A.

Theorem 3.2 *Let A_1 and A_2 be 2×2 Hurwitz matrices. Let $\Omega_1 \subset \mathbb{R}^2$ be the closed double wedge region between two vectors x_1, x_2, and let $\Omega_2 \subset \mathbb{R}^2$ be the closure of $\mathbb{R}^2 \setminus \Omega_1$. Then there is a joint QLF for the pairs (A_1, Ω_1) and (A_2, Ω_2) if and only if the following conditions are satisfied:*

(a) there is no convex combination of A_1 and A_2 which has an eigenvector in $\Omega_1 \cap \Omega_2$ with non-negative eigenvalue

(b) there is no convex combination of A_1 and A_2^{-1} which has an eigenvector in $\Omega_1 \cap A_2(\Omega_2)$ with non-negative eigenvalue

(c) there is no convex combination of A_1^{-1} and A_2 which has an eigenvector in $A_1(\Omega_1) \cap \Omega_2$ with non-negative eigenvalue

(d) there is no convex combination of A_1^{-1} and A_2^{-1} which has an eigenvector in $A_1(\Omega_1) \cap A_2(\Omega_2)$ with non-negative eigenvalue

(e) there is no nonzero vector $y \in \Omega_2$ satisfying both equations

$$(aA_1 + bA_2 + cA_1^{-1} - dI_n)y = 0, \tag{12}$$
$$ayy^T + cA_1^{-1}yy^T(A_1^{-1})^T \in \mathcal{C}_{12} \tag{13}$$

for some nonnegative coefficients $a, b, c, d \geq 0$

(f) there is no nonzero vector $x \in \Omega_1$ satisfying both equations

$$(aA_1 + bA_2 + cA_2^{-1} - dI_n)x = 0, \tag{14}$$
$$bxx^T + cA_2^{-1}xx^T(A_2^{-1})^T \in \mathcal{C}_{12} \tag{15}$$

for some nonnegative coefficients $a, b, c, d \geq 0$

(g) there is no nonzero vector $z \in A_2(\Omega_2)$ satisfying both equations

$$(aA_1 + bA_1^{-1} + cA_2^{-1} - dI_n)z = 0, \tag{16}$$
$$azz^T + bA_1^{-1}zz^T(A_1^{-1})^T \in \mathcal{C}_{12} \tag{17}$$

for some nonnegative coefficients $a, b, c, d \geq 0$

(h) there is no nonzero vector $w \in A_1(\Omega_1)$ satisfying both equations

$$(aA_1^{-1} + bA_2 + cA_2^{-1} - I_n)w = 0, \tag{18}$$
$$bww^T + cA_2^{-1}ww^T(A_2^{-1})^T \in \mathcal{C}_{12} \tag{19}$$

for some nonnegative coefficients $a, b, c, d \geq 0$

(i) denote by S a 2×2 matrix of the form

$$S = \begin{bmatrix} s_{11} & s_{12} \\ -s_{12} & s_{22} \end{bmatrix},$$

where s_{11}, s_{22} are nonnegative. Define $w_i = A_1 x_i$, $z_i = A_2 x_i$ for $i = 1, 2$. Then there are no relations of the following forms

$$\begin{bmatrix} a x_i \\ b z_j \end{bmatrix} = S \begin{bmatrix} x_i \\ x_j \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} a w_i \\ b z_j \end{bmatrix} = S \begin{bmatrix} w_i \\ x_j \end{bmatrix}, \tag{21}$$

$$\begin{bmatrix} a x_i \\ b w_j \end{bmatrix} = S \begin{bmatrix} z_i \\ x_j \end{bmatrix}, \tag{22}$$

$$\begin{bmatrix} a x_i \\ b x_j \end{bmatrix} = S \begin{bmatrix} w_i \\ z_j \end{bmatrix}, \tag{23}$$

where $a, b, c, d \geq 0$.

4 Proof of main results

In this Section we present the proof of the main results given in the previous section. To aid exposition we first present a mathematical preamble, in order to give the geometric ideas behind the proofs and to explain the intuition behind their logic.

4.1 Preamble

The set of positive definite symmetric matrices: The basic idea in our proof is to exploit the geometry of the convex cones that are generated by the Lyapunov equation. To this end it is convenient to represent matrices $M \in \mathcal{S}_2(\mathbb{R})$ as points in the plane. If we label the coordinates in the plane (m_{12}, m_{22}), then each point defines a symmetric matrix of the form

$$M = \begin{bmatrix} 1 & m_{12} \\ m_{12} & m_{22} \end{bmatrix}. \tag{24}$$

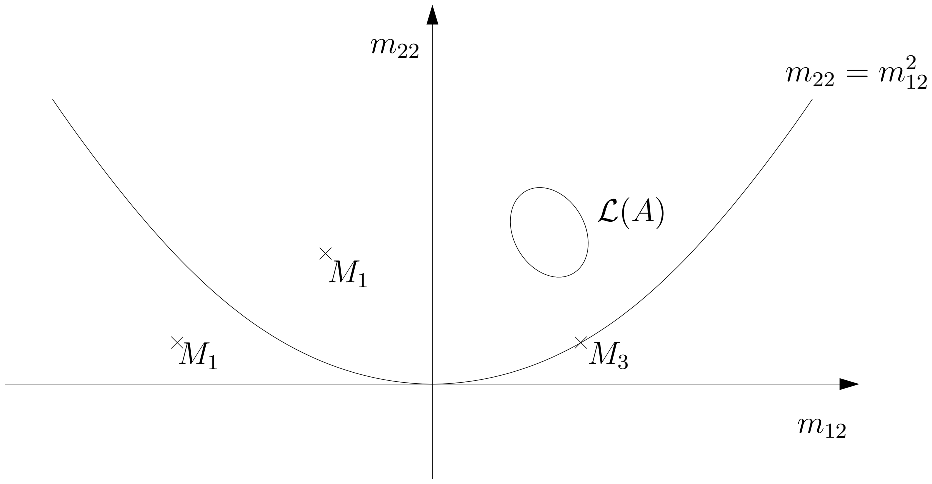

Figure 2: The closed set $\mathcal{L}(A)$

Figure 2 depicts three such points, and the parabola $m_{22} = m_{12}^2$. All points on this parabola correspond to positive semi-definite matrices. Points on the positive side of this locus (for example M_1) correspond to positive definite matrices, while points on the negative side of the locus correspond to indefinite matrices (for example M_2). It is evident that the set of all positive semi-definite symmetric matrices is convex.

The set $\mathcal{L}(A)$: Figure 2 depicts the set $\mathcal{L}(A)$. This is the set of all quadratic Lyapunov functions for the dynamic system Σ_A, namely all $P = P^T > 0$ such that $A^T P + PA$ is negative definite. Under the assumption that A is not a triangular matrix, this set is the interior of an ellipse [10]. It is also evident that this set is convex.

The set $\mathcal{L}(A, \Omega)$: Now we consider the set $\mathcal{L}(A, \Omega)$. Recall that this is the set of matrices $P = P^T > 0$ for which $x^T PAx$ is negative for all $x \in \Omega$. It immediately follows that this set is convex and that $\mathcal{L}(A)$ is a subset of $\mathcal{L}(A, \Omega)$. It also follows that $\mathcal{L}(A, \Omega)$ lies between the hyperplanes $H_1 : \{P : x_1^T PAx_1 = 0\}$ and $H_2 : \{P : x_2^T PAx_2 = 0\}$, and that these hyperplanes are tangent to the set. These hyperplanes define lines in the plane, and the possible configurations for $\mathcal{L}(A, \Omega)$ are shown in Figure 3, Figure 4, and Figure 5.

Tangent planes to $\mathcal{L}(A, \Omega)$: The basic problem considered in this paper is to find verifiable conditions which determine if two cones $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \Omega_2)$ intersect. We address this problem by considering tangent planes to these sets. Clearly, the boundary of $\mathcal{L}(A, \Omega)$ may have three parts. These are represented in Figure 7 by the points (a), (b), and (c) respectively. Point (a) denotes a point on the boundary where H_1 and H_2 intersect, that is where both $x_1^T PAx_1 = 0$ and

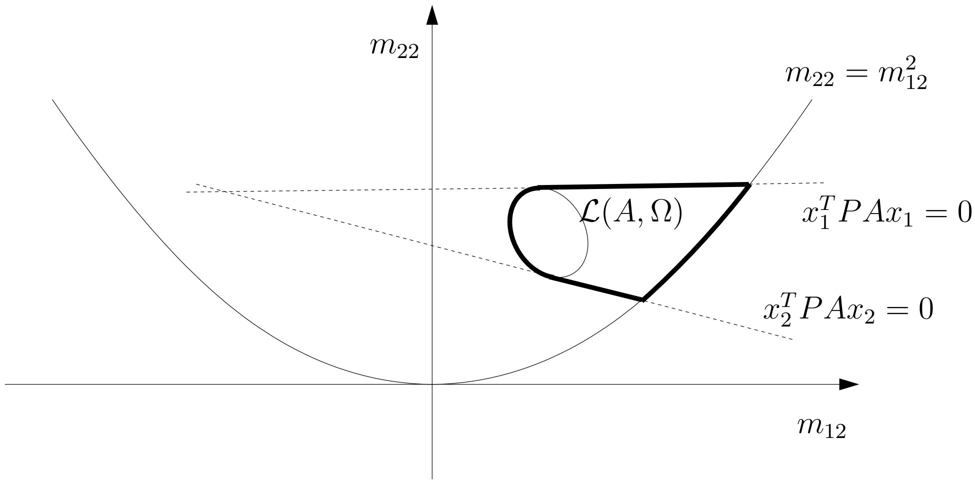

Figure 3: The closed set $\mathcal{L}(A, \Omega)$

$x_2^T P A x_2 = 0$. Point (b) corresponds to a point on the boundary where either H_1 or H_2 intersects the parabola of semi-definite matrices. Finally, point (c) corresponds to a point that also lies on the boundary of $\mathcal{L}(A)$.

The representation (24) defines a one-to-one correspondence between lines in the plane and hyperplanes in $\mathcal{S}_2(\mathbb{R})$, and every hyperplane in $\mathcal{S}_2(\mathbb{R})$ is uniquely determined by its normal "vector", which is itself a symmetric matrix. For example the tangent line $H_1 : \{P : x_1^T P A x_1 = 0\}$ is described by its normal vector $A x_1 x_1^T + x_1 x_1^T A^T$. Using this representation we get the following descriptions of tangent lines to the boundary of $\mathcal{L}(A, \Omega)$ at the points (a), (b), (c):

(a) the tangent is a convex combination of H_1 and H_2, so its normal is

$$k^2(A x_1 x_1^T + x_1 x_1^T A^T) + m^2(A x_2 x_2^T + x_2 x_2^T A^T) \tag{25}$$

for some real k, m.

(b) the tangent is a convex combination of H_i and the tangent to the parabola at this point, for $i \in \{1, 2\}$. The tangent to the parabola is either $x_1^T P x_1 = 0$ or $(A x_1)^T P A x_1 = 0$, which corresponds to the normal vectors $x_1 x_1^T$ or $A x_1 x_1^T A^T$ directed toward the positive definite matrix side of the parabola. Hence the tangent is one of the following:

$$\begin{array}{lcl} k^2(A x_1 x_1^T + x_1 x_1^T A^T) & - & m^2 x_1 x_1^T, \\ k^2(A x_1 x_1^T + x_1 x_1^T A^T) & - & m^2 A x_1 x_1^T A^T, \\ k^2(A x_2 x_2^T + x_2 x_2^T A^T) & - & m^2 x_2 x_2^T, \\ k^2(A x_2 x_2^T + x_2 x_2^T A^T) & - & m^2 A x_2 x_2^T A^T \end{array} \tag{26}$$

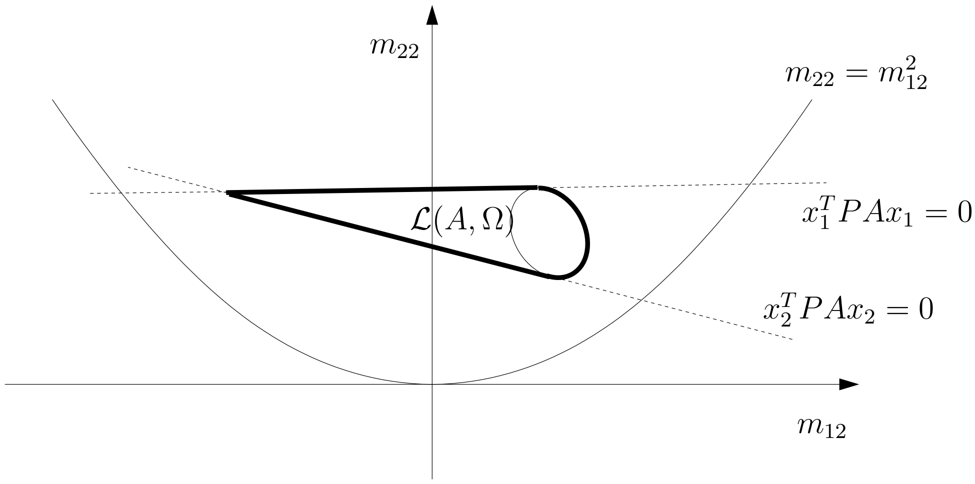

Figure 4: The closed set $\mathcal{L}(A, \Omega)$

(c) at this point $A^T P + PA$ is negative semi-definite, hence the tangent is

$$Axx^T + xx^TA^T, \qquad x \in \Omega \tag{27}$$

Simultaneous tangents : The basic idea that we exploit is that given two non-intersecting sets $\mathcal{L}(A_1, \Omega_1)$, $\mathcal{L}(A_2, \Omega_2)$, we can find a separating hyperplane between them which is simultaneously tangent to both convex sets. By exploiting our knowledge of the nature of the form of the tangents, we reveal verifiable conditions on the matrices A_1 and A_2. One such case is depicted in Figure 7.

4.2 Proof of Theorem 3.1

As discussed in Comment 5, any nonzero solution of the conditions (a), (b), (c) would imply that a QLF cannot exist. So it is sufficient to show that non-existence of a QLF implies that at least one of these conditions must hold. Accordingly we will assume that the sets $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2)$ are disjoint. These are open convex sets, and we will assume initially that their closures are also disjoint. At the end of the proof we will consider the case where their closures may intersect. We denote their closures by $\overline{\mathcal{L}(A_1, \Omega_1)}$ and $\overline{\mathcal{L}(A_2)}$.

Using the two-dimensional representation (24), and the fact that $\overline{\mathcal{L}(A_1, \Omega_1)}$ and $\overline{\mathcal{L}(A_2)}$ are disjoint, closed convex sets and one of them ($\overline{\mathcal{L}(A_2)}$) is bounded, it follows that there are infinitely many lines in the plane which separate these sets. Among these separating lines there are two extreme cases which are simultaneously tangential to both sets. A line which is simultaneously tangential to the two sets $\overline{\mathcal{L}(A_1, \Omega_1)}$ and $\overline{\mathcal{L}(A_2)}$ has two descriptions in terms of normal vectors

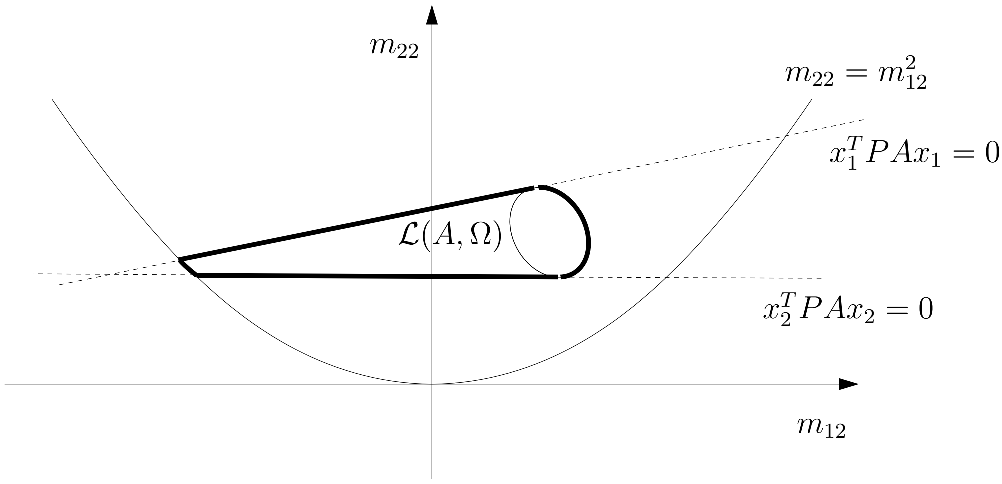

Figure 5: The closed set $\mathcal{L}(A, \Omega)$

in the space of symmetric matrices. These normal vectors must be oppositely directed, since by assumption there is no joint QLF for the sets. The six possible tangents for $\mathcal{L}(A_1, \Omega_1)$ are listed in (25), (26) and (27). Every tangent for $\mathcal{L}(A_2)$ has the form $A_2yy^T + yy^TA_2^T$ for some $y \in \mathbb{R}^2$. Setting a convex combination of these vectors to zero leads to the six possible cases listed below:

(i)

$$A_2yy^T + yy^TA_2^T + k^2(A_1x_1x_1^T + x_1x_1^TA_1^T) + m^2(A_1x_2x_2^T + x_2x_2^TA_1^T) = 0 \quad (28)$$

(ii)

$$A_2yy^T + yy^TA_2^T + k^2(A_1x_1x_1^T + x_1x_1^TA_1^T) - m^2x_1x_1^T = 0 \quad (29)$$

(iii)

$$A_2yy^T + yy^TA_2^T + k^2(A_1x_1x_1^T + x_1x_1^TA_1^T) - m^2A_1x_1x_1^TA_1^T = 0 \quad (30)$$

(iv)

$$A_2yy^T + yy^TA_2^T + k^2(A_1x_2x_2^T + x_2x_2^TA_1^T) - m^2x_2x_2^T = 0 \quad (31)$$

(v)

$$A_2yy^T + yy^TA_2^T + k^2(A_1x_2x_2^T + x_2x_2^TA_1^T) - m^2A_1x_2x_2^TA_1^T = 0 \quad (32)$$

(vi)

$$A_2yy^T + yy^TA_2^T + A_1xx^T + xx^TA_1^T = 0 \quad (33)$$

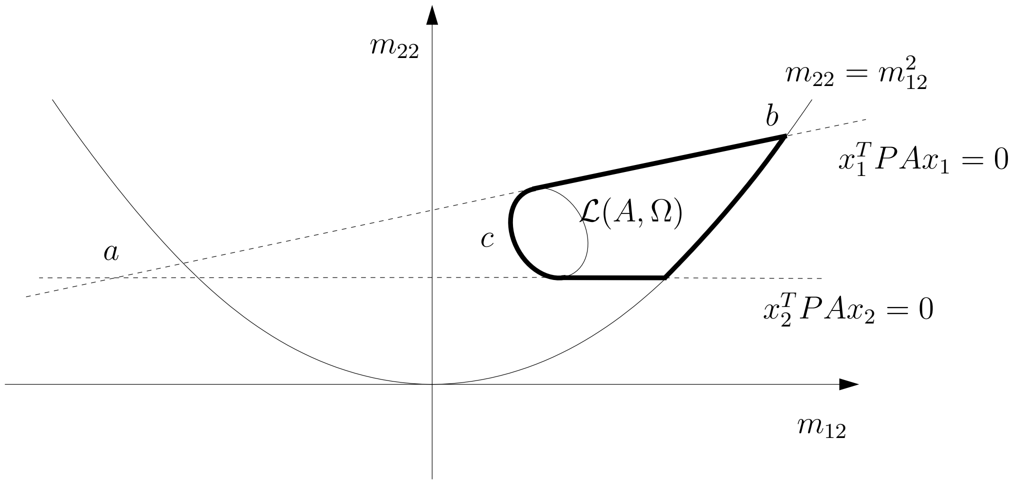

Figure 6: The closed set $\mathcal{L}(A, \Omega)$ with possible tangent points a, b, c

These six equations lead to the singularity conditions of Theorem (3.1), as we now explain.

Equation (i) can be solved by first writing $X = k^2 x_1 x_1^T + m^2 x_2 x_2^T$ so that it becomes

$$A_2 yy^T + yy^T A_2^T + A_1 X + XA_1^T = 0 \tag{34}$$

If X is semidefinite then this is a special case of Equation (vi), which we discuss shortly. If X is positive definite then there is a unique $\lambda > 0$ and vector w such that

$$X = \lambda yy^T + ww^T \tag{35}$$

Inserting this into (34) gives

$$(A_2 + \lambda A_1) yy^T + yy^T (A_2 + \lambda A_1)^T + A_1 ww^T + ww^T A_1^T = 0 \tag{36}$$

Applying (3) and (4) (and noting that $y \neq w$) yields

$$(A_2 + \lambda A_1 + \alpha A_1^{-1}) y = 0 \tag{37}$$

for some $\lambda, \alpha > 0$. Together with (35) this leads to Condition (c) of Theorem 3.1.

Equations (ii) and (iv) are alike, and lead to similar conditions. Equation (ii) can be written as

$$A_2 yy^T + yy^T A_2^T + \left((k^2 A_1 - \frac{m^2}{2} I_n) x_1 x_1^T + x_1 x_1^T (k^2 A_1 - \frac{m^2}{2} I_n)^T \right) = 0 \tag{38}$$

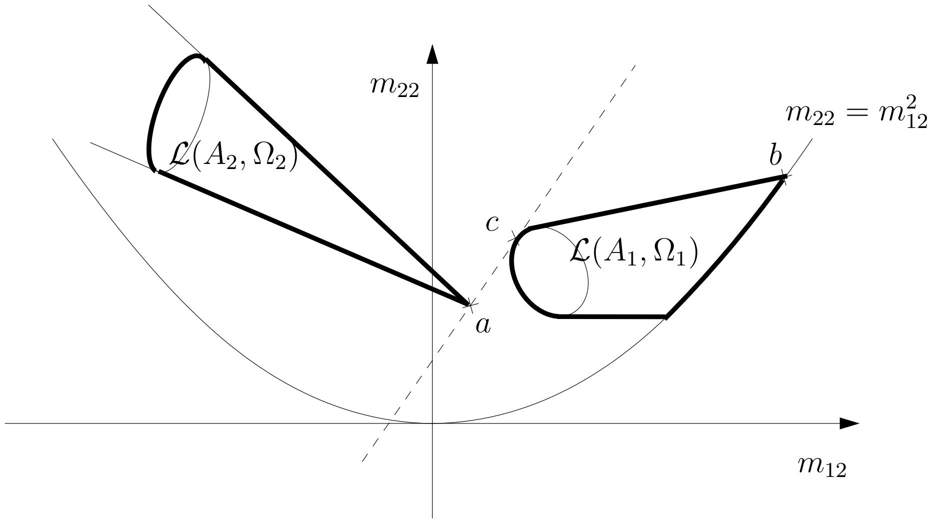

Figure 7: Two sets $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \Omega_2)$

where I_n is the $n \times n$ identity matrix. If y is parallel to x_1 this leads to $(k^2 A_1 - \frac{m^2}{2} I_n + \alpha A_2)x_1 = 0$ which is a special case of Condition (a). If y and x_1 are not parallel this leads to

$$(k^2 A_1 - \frac{m^2}{2} I_n + \alpha A_2^{-1})x_1 = 0 \tag{39}$$

which is again a special case of Condition (a). Equation (iv) leads to identical conclusions with x_1 replaced by x_2, and so also leads to Condition (a).

Equations (iii) and (v) are also alike. Equation (iii) can be written as

$$\begin{aligned} A_2 y y^T + y y^T A_2^T + (k^2 A_1^{-1} - \frac{1}{2} m^2 I_n) A_1 x_1 x_1^T A_1 T \\ + A_1 x_1 x_1^T A_1^T (k^2 A_1^{-1} - \frac{1}{2} m^2 I_n)^T = 0 \end{aligned} \tag{40}$$

If y and x_1 are parallel this leads to $(k^2 A_1^{-1} - \frac{m^2}{2} I_n + \alpha A_2) A_1 x_1 = 0$ which is a special case of Condition (b). If y and x_1 are not parallel it leads to

$$(k^2 A_1^{-1} - \frac{m^2}{2} I_n + \alpha A_2^{-1}) A_1 x_1 = 0 \tag{41}$$

and this again is a special case of Condition (b). Similar reasoning applies to Equation (v).

Equation (vi) can be solved using (3) and (4) leading to the two possibilities $(A_1 + \lambda A_2)x = 0$, or $(A_1 + \lambda A_2^{-1})x = 0$ with $\lambda > 0$. These are covered by Condition (a) of the Theorem. This concludes the argument by showing that all possible cases of simultaneous tangent lines are covered by the Conditions (a), (b), (c). Since the existence of these lines is equivalent to the disjointness of the sets $\overline{\mathcal{L}(A_1, \Omega_1)}$ and $\overline{(A_2)}$, this shows that the conditions are sufficient to distinguish the two sets.

In the case where $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2)$ are disjoint but with intersecting boundaries again there must be at least one simultaneous tangent hyperplane. So the above reasoning applies again and leads to the same conclusions.

4.3 Proof of Theorem 3.2

The strategy is the same as for Theorem 3.1, that is we identify lines in the plane which can be simultaneous tangent vectors for both sets $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \Omega_2)$. For these lines we equate the normal vectors arising from the tangential conditions at both sets, and these lead to the spectral conditions in the Theorem. There are now many possibilities, corresponding to the cases (25), (26) and (27) for both sets. We can exclude the case where (25) occurs for both sets: there are always two simultaneous tangent vectors, and at most one of these can pass through both vertices where the hyperplanes H_i intersect, so in such a case we choose the alternative. All other cases must be considered, and this leads to the many conditions of the Theorem.

We omit the reasoning that leads from a simultaneous tangent line to a singularity condition, as this is the same as in the proof of Theorem 3.1. Instead we indicate which conditions correspond to which tangent vectors.

(25) and (26): this leads to conditions (e), (f), (g), (h).

(25) and (27): this leads to conditions (a), (e), (f).

(26) and (26): this leads to condition (i).

(26) and (27): this leads to conditions (a) (b) (c), (d).

(27) and (27): this leads to conditions (a), (b), (c).

5 Examples and some implications of results

In this section we present two examples to illustrate our results. The first example is to determine whether or not a given system is quadratically stable. The second example is more abstract and is given to illustrate the power of our methodology.

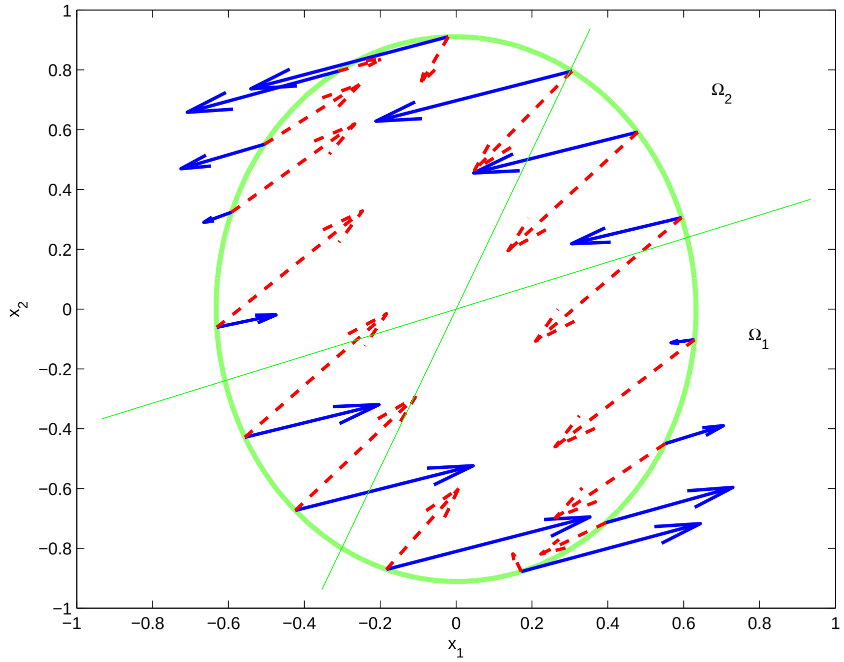

Figure 8: Joint QLF for the systems A_1 and A_2 for the partition Ω_1, Ω_2.

Example 1 *Consider the System with*

$$A_1 = \begin{pmatrix} -0.4 & -1.2 \\ -0.1 & -0.4 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1.3 & -0.2 \\ -1.3 & -0.4 \end{pmatrix} \tag{42}$$

where A_1 is active in the whole state-space, i.e. $\Omega_1 = \mathbb{R}^{2\times 2}$ and A_2 is active in the Region $\Omega_2 = \{x \,|\, x = \alpha x_1 + \beta x_2,\, \alpha, \beta \geq 0\}$ with $x_1 = [0.1 \;\; 1]^T$ and $x_2 = [1 \;\; 0.1]^T$.

We can verify that the conditions of Theorem 3.1 are satisfied and that

$$P = \begin{pmatrix} 1.0 & 0.0027 \\ 0.0027 & 0.4817 \end{pmatrix} \tag{43}$$

satisfies conditions given in the theorem. Figure 8 shows a levelset of the Lyapunov function $V(x) = x^T P x$. The dashed arrows indicate the flow of system A_1 on this level set and the solid arrows are the flow of system A_2. Note, that the flow of A_2 only is negative with respect to V in the region Ω_2.

The approach advocated here extends to other stability problems. For example, consider the following common piecewise quadratic stability problem defined in the following example.

Example 2 *The techniques developed in this paper can be applied to various stability problems for switched linear systems. Consider a partition of* $\mathbb{R}^2$ *defined by* x_1 *and* x_2 *as in Figure 1. This partition defines two closed double wedged regions* Ω_1 *and* Ω_2*. Consider the case where* A_1 *and* A_2 *are be* ***active in the whole state-space*** *and* ***switching occurs in an arbitrary fashion****. We shall determine necessary and sufficient conditions for the existence of a common piecewise quadratic Lyapunov function, i.e. for the existence of* $P_1 = P_1^T > 0$ *and* $P_2 = P_2^T > 0$*, such that*

$$x^T\Big(A_i^T P_1 + P_1 A_i\Big)x \quad < \quad 0, \ x \in \Omega_1, \tag{44}$$

$$x^T\Big(A_i^T P_2 + P_2 A_i\Big)x \quad < \quad 0, \ x \in \Omega_2. \tag{45}$$

are simultaneously satisfied for $i = 1, 2$ *where* A_1 *and* A_2 *are both assumed to be Hurwitz matrices. Variations on this problem arise in different forms in the study of piecewise quadratic Lyapunov functions.*

The geometry of this problem is depicted in Figure 9. Certainly there exists a common piecewise quadratic Lyapunov function of A_1 *and* A_2 *if* $\mathcal{L}(A_1, \Omega_1) \cap \mathcal{L}(A_2, \Omega_2) \neq \varnothing$ *and* $\mathcal{L}(A_1, \Omega_2) \cap \mathcal{L}(A_2, \Omega_1) \neq \varnothing$*; or if* $\mathcal{L}(A_i) \cap \mathcal{L}(A_j, \Omega_1) \cap \mathcal{L}(A_j, \Omega_2) \neq \varnothing$*,* $i \neq j$*. By appealing to simple geometric arguments it follows that former condition implies by the latter relation. But the latter relation is the condition that we obtained for the existence of a simple quadratic Lyapunov function for Problem 1, i.e. where* A_i *active in the whole state-space and* A_j *is active in either* Ω_1 *or* Ω_2*. The implication of this interesting observation is that the existence of a common piecewise quadratic Lyapunov function implies the existence of a simpel QLF of the aforementioned form. In other words, certain stability problems may be replaced by related ones for the purpose of determining whether certain types of Lyapunov functions exist. This observation is consistent with celebrated stability criteria such as the Popov criterion and off-axis circle criterion.*

6 Conclusions

In this paper we presented stability results for a class of two dimensional state dependent switched systems. These results are important for a number of reasons. Firstly, as we have argued, a thorough understanding of the second order case is usually very revealing and is an important preliminary step for understanding higher dimensional cases. Given the nature of the discussion here, and complexity of the second order problem, it is clear that studying this case has been a worthwhile exercise, and is likely to provide important insights into studying higher dimensional systems. Secondly, the techniques that we use are not restricted to

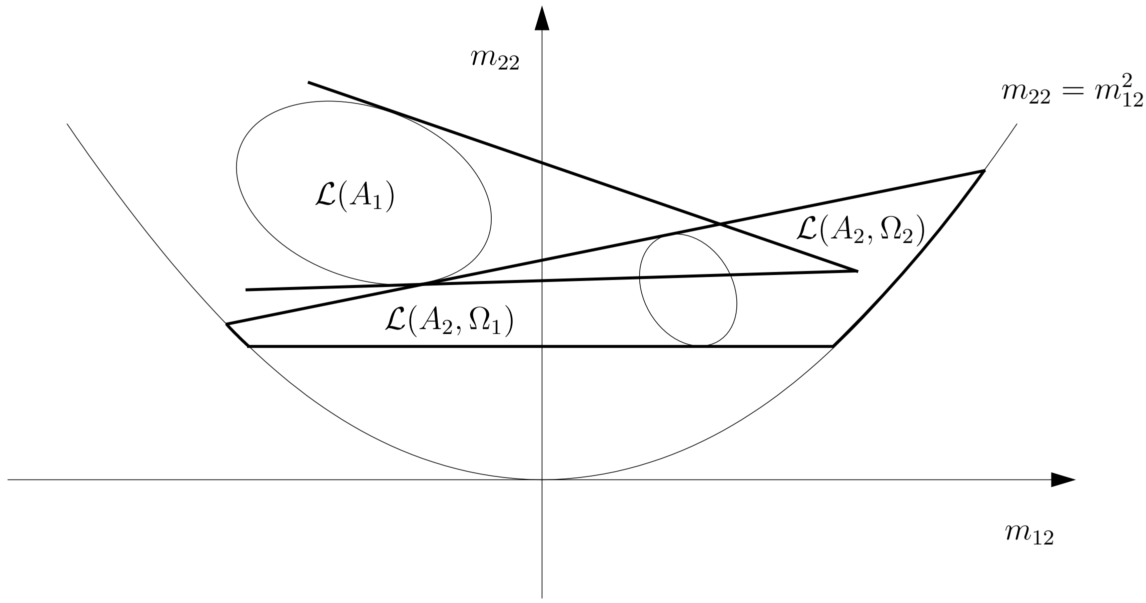

Figure 9: Marginal case for Example 2, where $\mathcal{L}(A_1)$ is just touching $\mathcal{L}(A_2, \Omega_1)$.

two dimensions and can be used to study higher order systems as well. Our approach is based on a detailed knowledge of the sets generated by the Lyapunov equation, and the implications of separating hyper-planes that are parameterised in a certain manner. Finally, our results, which are a natural extension of our previous work on the CQLF existence problem, and of classical results such as the Circle Criterion, reveal connections between many different stability problems, and hence open up many exciting research directions for future study.

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