Maximum Edge-Colorable Subgraph and Strong Triadic Closure Parameterized by Distance to Low-Degree Graphs

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Abstract

Given an undirected graph G and integers c and k, the Maximum Edge-Colorable Subgraph problem asks whether we can delete at most k edges in G to obtain a graph that has a proper edge coloring with at most c colors. We show that Maximum Edge-Colorable Subgraph admits, for every fixed c, a linear-size problem kernel when parameterized by the edge deletion distance of G to a graph with maximum degree $c-1$. This parameterization measures the distance to instances that, due to Vizing's famous theorem, are trivial yes-instances. For $c \leq 4$, we also provide a linear-size kernel for the same parameterization for Multi Strong Triadic Closure, a related edge coloring problem with applications in social network analysis. We provide further results for Maximum Edge-Colorable Subgraph parameterized by the vertex deletion distance to graphs where every component has order at most c and for the list-colored versions of both problems.

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1 Introduction

Edge coloring and its many variants form a fundamental problem family in algorithmic graph theory [3, 12, 13, 14]. In the classic Edge Coloring problem, the input is a graph G and an integer c and the task is to decide whether G has a *proper edge coloring*, that is, an assignment of colors to the edges of a graph such that no pair of incident edges receives the same color, with at most c colors. The number of necessary colors for a proper edge coloring of a graph G is closely related to the degree of G: Vizing's famous theorem states that any graph G with maximum degree Δ can be edge-colored with $\Delta + 1$ colors [27], an early example of an additive approximation algorithm. Later it was shown that Edge Coloring is NP-hard for $c = 3$ [13], and in light of Vizing's result it is clear that the hard instances for $c = 3$ are exactly the subcubic graphs. Not surprisingly, the NP-hardness extends to every fixed $c \geq 3$ [21].

In the more general Maximum Edge-Colorable Subgraph (ECS) problem, we are given an additional integer k and want to decide whether we can delete at most k edges in the input graph G so that the resulting graph has a proper edge coloring with c colors.

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ECS is NP-hard for $c = 2$ [6] and it has received a considerable amount of interest for small constant values of c such as $c = 2$ [6, 18], $c = 3$ [18, 19, 23], and $c \leq 7$ [15]. Feige et al. [6] mention that ECS has applications in call admittance in telecommunication networks. Given the large amount of algorithmic literature on this problem, it is surprising that there is, to the best of our knowledge, no work on fixed-parameter algorithms for ECS. This lack of interest may be rooted in the NP-hardness of Edge Coloring for every fixed $c \geq 3$, which implies that ECS is not fixed-parameter tractable with respect to $k + c$ unless P=NP.

Instead of the parameter k, we consider the parameter ξ_{c-1} which we define as the minimum number of edges that need to be deleted in the input graph to obtain a graph with maximum degree $c - 1$. This is a distance-from-triviality parameterization [11]: Due to Vizing's Theorem, the answer is always yes if the input graph has maximum degree $c - 1$. We parameterize by the edge-deletion distance to this trivial case. Observe that the number of vertices with degree at least c is at most $2\xi_{c-1}$. If we consider Edge Coloring instead of ECS, the instances with maximum degree larger than c are trivial no-instances. Thus, in non-trivial instances, the parameter ξ_{c-1} is essentially the same as the number of vertices that have degree c. This is, arguably, one of the most natural parameterizations for Edge Coloring. We achieve a kernel that has linear size for every fixed c.

▶ **Theorem 1.1.** *ECS admits a problem kernel with at most $4\xi_{c-1} \cdot c$ vertices and $\mathcal{O}(\xi_{c-1} \cdot c^2)$ edges that can be computed in $\mathcal{O}(n + m)$ time.*

Herein, n denotes the number of vertices of the input graph G and m denotes the number of edges. This kernel is obtained by making the following observation about the proof of Vizing's Theorem: When proving that an edge can be safely colored with one of c colors, we only need to consider the closed neighborhood of one endpoint of this edge. This allows us to show that all vertices which have degree at most $c - 1$ and only neighbors of degree at most $c - 1$ can be safely removed.

Next, we consider ECS parameterized by the size λ_c of a smallest vertex set D such that deleting D from G results in a graph where each connected component has at most c vertices. The parameter λ_c presents a different distance-from-triviality parameterization, since a graph with connected components of order at most c can trivially be colored with c edge colors. Moreover, observe that λ_c is never larger than the vertex cover number which is a popular structural parameter. Again, we obtain a linear-vertex kernel for λ_c when c is fixed.

▶ **Theorem 1.2.** *ECS admits a problem kernel with $\mathcal{O}(c^3 \cdot \lambda_c)$ vertices.*

We then consider Multi Strong Triadic Closure (Multi-STC) a closely related edge coloring problem with applications in social network analysis [25]. In Multi-STC, we are given a graph G and two integers k and c and aim to find a coloring of the edges with one *weak* and at most c *strong colors* such that every pair of incident edges that forms an induced path on three vertices does not receive the same strong color and the number of weak edges is at most k. The idea behind this problem is to uncover the different strong relation types in social networks by using the following assumption: if one person has for example two colleagues, then these two people know each other and should also be connected in the social network. In other words, if a vertex has two neighbors that are not adjacent to each other, then this is evidence that either the strong interaction types with these two neighbors are different or one of the interaction types is merely weak.

Combinatorically, there are two crucial differences to ECS: First, two incident edges may receive the same strong color if the subgraph induced by the endpoints is a triangle. Second, instead of deleting edges to obtain a graph that admits such a coloring, we may label edges

Table 1 A summary of our results for the two problems. Herein, ξ_{c-1} denotes the edge-deletion distance to graphs with maximum degree at most $c-1$, and λ_c denotes the vertex-deletion distance to graphs where every connected component has order at most c.

Parameter	ECS	Multi-STC
(ξ_{c-1}, c)	$\mathcal{O}(\xi_{c-1}c)$-vertex kernel (Thm. 1.1)	$\mathcal{O}(\xi_{c-1})$-edge kernel (Thm. 4.14), if $c \leq 4$
(λ_c, c)	$\mathcal{O}(c^3 \cdot \lambda_c)$-vertex kernel (Thm. 1.2)	No poly Kernel, even for $c = 1$ [10]

as weak. In ECS this does not make a difference; in Multi-STC, however, deleting an edge may destroy triangles which would add an additional constraint on the coloring of the two remaining triangle edges.

In contrast to ECS, Multi-STC is NP-hard already for $c = 1$ [25]. This special case is known as Strong Triadic Closure (STC). Not surprisingly, Multi-STC is NP-hard for all fixed $c \geq 2$ [1]. Moreover, for $c \geq 3$ Multi-STC is NP-hard even if $k = 0$, that is, even if every edge has to be colored with a strong color. STC and Multi-STC have received a considerable amount of interest recently [25, 9, 10, 1, 16, 17].

Since the edge coloring for Multi-STC is a relaxed version of a proper edge coloring, we may observe that Vizing's Theorem implies the following: If the input graph G has degree at most $c-1$, then the instance is a yes-instance even for $k = 0$. Hence, it is very natural to apply the parameterization by ξ_{c-1} also for Multi-STC. We succeed to transfer the kernelization result from ECS to Multi-STC for $c \leq 4$. In fact, our result for $c = 3$ and $c = 4$ can be extended to the following more general result.

▶ **Theorem 1.3.** *Multi-STC admits a problem kernel with $\mathcal{O}(\xi_{\lfloor \frac{c}{2} \rfloor+1} \cdot c)$ vertices and $\mathcal{O}(\xi_{\lfloor \frac{c}{2} \rfloor+1} \cdot c^2)$ edges, when limited to instances with $c \geq 3$. The kernel can be computed in $\mathcal{O}(n+m)$ time.*

For $c = 5$, this gives a linear-size kernel for the parameter ξ_3, for $c = 6$, a linear-size kernel for the parameter ξ_4 and so on. Our techniques to prove Theorem 1.3 are very loosely inspired by the proof of Vizing's Theorem but in the context of Multi-STC several obstacles need to be overcome. As a result, the proof differs quite substantially from the one for ECS. Moreover, in contrast do ECS, Multi-STC does not admit a polynomial kernel when parameterized by the vertex cover number [10] which excludes almost all popular structural parameters.

We then show how far our kernelization for ξ_t can be lifted to generalizations of ECS and Multi-STC where each edge may choose its color only from a specified list of colors, denoted as Edge List ECS (EL-ECS) and Edge List Multi-STC (EL-Multi-STC). We show that for ξ_2 we obtain a linear kernel for every fixed c.

▶ **Theorem 1.4.** *For all $c \in \mathbb{N}$, EL-ECS and EL-Multi-STC admit an $11\xi_2$-edge and $10\xi_2$-vertex kernel for EL-ECS that can be computed in $\mathcal{O}(n^2)$ time.*

For $c = 3$, this extends Theorem 1.1 to the list colored version of ECS. For $c > 3$ parameterization by ξ_2 may seem a bit uninteresting compared to the results for ECS and Multi-STC. However, Theorem 1.4 is unlikely to be improved by considering ξ_t for $t > 2$.

▶ **Proposition 1.5.** *EL-ECS and EL-Multi-STC are NP-hard for all $c \geq 3$ on triangle-free cubic graphs even if $\xi_3 = k = 0$.*

A summary of our results is shown in Table 1. Due to space constraints, the proofs of Theorem 1.4 and Proposition 1.5 and further propositions and lemmas needed to show Theorems 1.1–1.3 are deferred to a full version.

2 Preliminaries

Notation. We consider simple undirected graphs $G = (V, E)$. For a vertex $v \in V$, we denote by $N_G(v) := \{u \in V \mid \{u, v\} \in E\}$ the *open neighborhood of* v and by $N_G[v] := N_G(v) \cup \{v\}$ the *closed neighborhood of* v. For a given set $V' \subseteq V$, we define $N_G(V') := \bigcup_{v \in V'} N_G(v)$ as the *neighborhood of* V'. Moreover, let $\deg_G(v) := |N(v)|$ be the *degree* of a vertex v in G and $\Delta_G := \max_{v \in V} \deg_G(v)$ denote the *maximum degree* of G. For any two vertex sets $V_1, V_2 \subseteq V$, we let $E_G(V_1, V_2) := \{\{v_1, v_2\} \in E \mid v_1 \in V_1, v_2 \in V_2\}$ denote the set of edges between V_1 and V_2. For any vertex set $V' \subseteq V$, we let $E_G(V') := E_G(V', V')$ denote the set of edges between the vertices of V'. The *subgraph induced by a vertex set* S is denoted by $G[S] := (S, E_G(S))$. For a given vertex set $V' \subseteq V$, we let $G - V' := G[V \setminus V']$ denote the graph that we obtain after deleting the vertices of V' from G. We may omit the subscript G if the graph is clear from the context.

A *finite sequence* $A = (a_0, a_1, \dots, a_{r-1})$ of length $r \in \mathbb{N}_0$ is an r-tuple of specific elements a_i (for example vertices or numbers). For given $j \in \{0, \dots, r-1\}$, we refer to the jth element of a finite sequence A as $A(j)$. A *path* $P = (v_0, \dots, v_{r-1})$ is a finite sequence of vertices $v_0, \dots, v_{r-1} \in V$, where $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, \dots, r-2\}$. A path P is called *vertex-simple*, if no vertex appears twice on P. A path is called *edge-simple*, if there are no distinct $i, j \in \{0, \dots, r-2\}$ such that $\{P(i), P(i+1)\} = \{P(j), P(j+1)\}$. For a given path $P = (P(0), \dots, P(r-1))$ we define the sets $V(P) := \{P(j) \mid j \in \{0, \dots, r-1\}\}$ and $E(P) := \{\{P(j), P(j+1)\} \mid j \in \{0, \dots, r-2\}\}$ as the set of vertices or edges on P.

For the relevant definitions of parameterized complexit such as parameterized reduction and problem kernelization refer to the standard monographs [4, 5, 7, 22].

Problem Definitions. We now formally define the two main problems considered in this work, ECS and Multi-STC, as well as their extensions to input graphs with edge lists.

▶ **Definition 2.1.** *A* c-colored labeling $L = (S_L^1, \dots, S_L^c, W_L)$ *of an undirected graph* $G = (V, E)$ *is a partition of the edge set* E *into* $c+1$ *color classes. The edges in* S_L^i, $i \in \{1, \dots, c\}$, *are* strong *and the edges in* W_L *are* weak.

1. *A* c-colored labeling L *is a* proper labeling *if there exists no pair of edges* $e_1, e_2 \in S_L^i$ *for some strong color* i, *such that* $e_1 \cap e_2 \neq \emptyset$.
2. *A* c-colored labeling L *is an* STC-labeling *if there exists no pair of edges* $\{u, v\} \in S_L^i$ *and* $\{v, w\} \in S_L^i$ *such that* $\{u, w\} \notin E$.

We consider the following two problems.

Edge-Colorable Subgraph (ECS)
Input: An undirected graph $G = (V, E)$ and integers $c \in \mathbb{N}$ and $k \in \mathbb{N}$.
Question: Is there a c-colored proper labeling L with $|W_L| \leq k$?

Multi Strong Triadic Closure (Multi-STC)
Input: An undirected graph $G = (V, E)$ and integers $c \in \mathbb{N}$ and $k \in \mathbb{N}$.
Question: Is there a c-colored STC-labeling L with $|W_L| \leq k$?

If c is clear from the context, we may call a c-colored labeling just *labeling*. Two labelings $L = (S_L^1, \dots, S_L^c, W_L)$, and $L' = (S_{L'}^1, \dots, S_{L'}^c, W_{L'})$ for the same graph $G = (V, E)$ are called *partially equal* on a set $E' \subseteq E$ if and only if for all $e \in E'$ and $i \in \{1, \dots, c\}$ it holds that $e \in S_L^i \Leftrightarrow e \in S_{L'}^i$. If two labelings L and L' are partially equal on E' we write $L|_{E'} = L'|_{E'}$. For given path $P = (P(0), \dots, P(r-1))$ and labeling $L = (S_L^1, \dots, S_L^c, W_L)$, we define the *color sequence* Q_L^P of P under L as a finite sequence $Q_L^P = (q_0, q_1, \dots, q_{r-2})$ of elements in $\{0, \ldots, c\}$, such that $\{P(i), P(i+1)\} \in S_L^{q_i}$ if $q_i \geq 1$ and $\{P(i), P(i+1)\} \in W_L$ if $q_i = 0$. Throughout this work we call a c-colored STC-labeling L (or proper labeling, respectively) *optimal*if the number of weak edges $|W_L|$ is minimal.

Edge-Deletion Distance to Low-Degree Graphs and Component Order Connectivity. We consider parameters related to the edge deletion-distance ξ_t to low-degree graphs and the vertex-deletion distance λ_t to graphs with small connected components.

First, we define the parameter ξ_t. For a given graph $G = (V, E)$ and a constant $t \in \mathbb{N}$, we call $D_t \subseteq E$ an *edge-deletion set of G and t* if the graph $(V, E \setminus D_t)$ has maximum degree t. We define the parameter ξ_t as the size of the minimum edge-deletion set of G and t. Note that an edge-deletion set of G and t of size ξ_t can be computed in polynomial time [8]. More importantly for our applications, we can compute a 2-approximation D'_t for an edge-deletion set of size ξ_t in linear time as follows: Add for each vertex v of degree at least $t+1$ an arbitrary set of $\deg(v) - t$ incident edges to D'_t. Then $|D'_t| \leq \sum_{v \in V} \max(\deg(v) - t, 0)$. This implies that D'_t is a 2-approximation since $\sum_{v \in V} \max(\deg(v) - t, 0) \leq 2\xi_t$ as every edge deletion decreases the degree of at most two vertices. A given edge-deletion set D_t induces the following important partition of the vertex set V of a graph.

▶ **Definition 2.2.** *Let $t \in \mathbb{N}$, let $G = (V, E)$ be a graph, and let $D_t \subseteq E$ be an edge-deletion set of G and t. We call $\mathscr{C} = \mathscr{C}(D_t) := \{v \in V \mid \exists e \in D_t : v \in e\}$ the set of* core vertices *and $\mathscr{P} = \mathscr{P}(D_t) := V \setminus \mathscr{C}$ the set of* periphery vertices *of G.*

Note that for arbitrary $t \in \mathbb{N}$ and G we have $|\mathscr{C}| \leq 2|D_t|$ and for every $v \in \mathscr{P}$ it holds that $\deg_G(v) \leq t$. Moreover, every vertex in $\mathscr{C}$ is incident with at most t edges in $E \setminus D_t$. In context of ECS and Multi-STC, for a given instance (G, c, k) we consider some fixed edge deletion set D_t of the input graph G and some integer t which depends on the value of c.

Second, we define the parameter λ_t. For a given graph $G = (V, E)$ and a constant $t \in \mathbb{N}$, we call $D \subseteq V$ an *order-t component cover* if every connected component in $G - D$ contains at most t vertices. Then, we define the *component order connectivity* λ_t to be the size of a minimum oder-t component cover. In context of ECS we study λ_c, for the amount of colors c. A $(c+1)$-approximation of the minimal order-c-component cover can be computed in polynomial time [20].

Note that the parameters are incomparable in the following sense: In a path P_n the parameter λ_c can be arbitrarily large when n increases while $\xi_{c-1} = 0$ for all $c \geq 3$. In a star S_n the parameter ξ_{c-1} can be arbitrary large when n increases while $\lambda_c = 1$.

3 Problem Kernelizations for Edge-Colorable Subgraph

In this section, we provide problem kernels for ECS parameterized by the edge deletion distance ξ_{c-1} to graphs with maximum degree $c-1$, and the size λ_c of a minimum order-c component cover. We first show that ECS admits a kernel with $\mathcal{O}(\xi_{c-1} \cdot c)$ vertices and $\mathcal{O}(\xi_{c-1} \cdot c^2)$ edges that can be computed in $\mathcal{O}(n+m)$ time. Afterwards, we consider λ_c and show that ECS admits a problem kernel with $\mathcal{O}(c^3 \lambda_c)$ vertices, which is a linear vertex kernel for every fixed value of c. Note that if $c = 1$ we can solve ECS by computing a maximal matching in polynomial time. Hence, we assume $c \geq 2$ for the rest of this section. In this case the problem is NP-hard [6].

3.1 Edge Deletion-Distance to Low-Degree Graphs

The kernelization presented inhere is based on Vizing's Theorem [27]. Note that Vizing's Theorem implies, that an ECS instance (G, c, k) is always a yes-instance if $\xi_{c-1} = 0$. Our kernelization relies on the following lemma. This lemma is a reformulation of a known fact about edge colorings [26, Theorem 2.3] which, in turn, is based on the so-called *Vizing Fan Equation* [26, Theorem 2.1].

▶ **Lemma 3.1.** *Let $G = (V, E)$ be a graph and let $e := \{u, v\} \in E$. Moreover, let $c := \Delta_G$ and let L be a proper c-colored labeling for the graph $(V, E \setminus \{e\})$ such that $W_L = \emptyset$. If for all $Z \subseteq N_G(u)$ with $|Z| \geq 2$ and $v \in Z$ it holds that $\sum_{z \in Z}(\deg_G(z) + 1 - c) < 2$, then there exists a proper c-colored labeling L' for G such that $W_{L'} = \emptyset$.*

We now use Lemma 3.1 as a plug-in for ECS to prove the next lemma which is the main tool that we need for our kernelization. In the proof, we exploit the fact that, given any proper labeling L for a graph $G = (V, E)$, the labeling $(S_L^1, \ldots, S_L^c, \emptyset)$ is a proper labeling for the graph $(V, E \setminus W_L)$.

▶ **Lemma 3.2.** *Let $L := (S_L^1, S_L^2, \ldots, S_L^c, W_L)$ be a proper labeling with $|W_L| = k$ for a graph $G := (V, E)$. Moreover, let $e := \{u, v\} \subseteq V$ such that $e \notin E$ and let $G' := (V, E \cup \{e\})$ be obtained from G by adding e. If for one endpoint $u \in e$ it holds that every vertex $w \in N_{G'}[u]$ has degree at most $c - 1$ in G', then there exists a proper labeling L' for G' with $|W_{L'}| = k$.*

Proof. Consider the auxiliary graph $G_{\text{aux}} := (V, E \setminus W_L)$. Since L is a proper labeling for G, we conclude that $L_{\text{aux}} := (S_L^1, \ldots, S_L^c, \emptyset)$ is a proper labeling for G_{aux}. Let $H_{\text{aux}} := (V, E_H)$ where $E_H := (E \setminus W_L) \cup \{e\}$. In order to prove the lemma, we show that there exists a proper labeling L'_{aux} for H_{aux} such that $W_{L'_{\text{aux}}} = \emptyset$.

To this end, we first consider the maximum degree of H_{aux}. Observe that $\deg_{H_{\text{aux}}}(w) \leq \deg_{G'}(w)$ for all $w \in V$. Hence, the property that $\deg_{G'}(w) \leq c - 1$ for all $w \in N_{G'}[u]$ implies $\Delta_{H_{\text{aux}}} = \max(\Delta_{G_{\text{aux}}}, c - 1)$. Since L_{aux} is a proper c-colored labeling for G_{aux} we know that $\Delta_{G_{\text{aux}}} \leq c$ and therefore we have $\Delta_{H_{\text{aux}}} \leq c$. So, to find a proper c-colored labeling without weak edges for H_{aux} it suffices to consider the following cases.

Case 1: $\Delta_{H_{\text{aux}}} \leq c - 1$. Then, there exists a proper labeling L'_{aux} for H_{aux} such that $W_{L'_{\text{aux}}} = \emptyset$ due to Vizing's Theorem.

Case 2: $\Delta_{H_{\text{aux}}} = c$. In this case we can apply Lemma 3.1: Observe that $(V, E_H \setminus \{e\}) = G_{\text{aux}}$ and L_{aux} is a proper labeling for G_{aux} such that $W_{L_{\text{aux}}} = \emptyset$. Consider an arbitrary $Z \subseteq N_{H_{\text{aux}}}(u)$ with $|Z| \geq 2$ and $v \in Z$. Note that $Z \subseteq N_{H_{\text{aux}}}(u)$ implies $\deg_{H_{\text{aux}}}(z) \leq c - 1$ for all $z \in Z$. It follows that $\sum_{z \in Z}(\deg_{H_{\text{aux}}}(z) + 1 - c) < 2$. Since Z was arbitrary, Lemma 3.1 implies that there exists a proper labeling L'_{aux} for H_{aux} such that $W_{L'_{\text{aux}}} = \emptyset$.

We now define $L' := (S^1_{L'_{\text{aux}}}, S^2_{L'_{\text{aux}}}, \ldots S^c_{L'_{\text{aux}}}, W_L)$. Note that the edge set $E \cup \{e\}$ of G' can be partitioned into W_L and the edges of G'_{aux}. Together with the fact that L'_{aux} is a labeling for G'_{aux} it follows that every edge of G' belongs to exactly one color class of L'. Moreover, it obviously holds that $|W_{L'}| = |W_L| = k$. Since there is no vertex with two incident edges in the same strong color class $S^i_{L'_{\text{aux}}}$, the labeling L' is a proper labeling for G'. ◀

We now introduce the kernelization rule. Recall that $\mathscr{C}$ is the set of vertices that are incident with at least one of the ξ_{c-1} edge-deletions that transform G into a graph with maximum degree $c - 1$. We make use of the fact that edges that have at least one endpoint u that is not in $\mathscr{C} \cup N(\mathscr{C})$ satisfy $\deg(w) \leq c - 1$ for all $w \in N[u]$. Lemma 3.2 guarantees that these edges are not important to solve an instance of ECS.

▶ **Rule 3.1.** *Remove all vertices in $V \setminus (\mathscr{C} \cup N(\mathscr{C}))$ from G.*

▶ **Proposition 3.3.** *Rule 3.1 is safe.*

Proof. Let $(G' = (V', E'), c, k)$ be the reduced instance after applying Rule 3.1. We prove the safeness of Rule 3.1 by showing that there is a proper labeling with at most k weak edges for G if and only if there is a proper labeling with k weak edges for G'.

($\Rightarrow$) Let $L = (S_L^1, S_L^2, \dots, S_L^c, W_L)$ be a proper labeling with $|W_L| \le k$ for G. Then, obviously $L' := (S_L^1 \cap E', S_L^2 \cap E', \dots, S_L^c \cap E', W_L \cap E')$ is a proper labeling for G' with $|W_{L'}| \le |W_L| \le k$.

($\Leftarrow$) Conversely, let $L' = (S_{L'}^1, S_{L'}^2, \dots, S_{L'}^c, W_{L'})$ be a proper labeling with $|W_{L'}| \le k$ for G'. Let $E \setminus E' = \{e_1, e_2, \dots, e_p\}$. We define $p+1$ graphs $G_0, G_1, G_2, \dots, G_p$ by $G_0 := (V, E')$, and $G_i := (V, E' \cup \{e_1, \dots, e_i\})$ for $i \in \{1, \dots, p\}$. Note that $G_p = G$, $\deg_{G_i}(v) \le \deg_G(v)$, and $N_{G_i}(v) \subseteq N_G(v)$ for every $i \in \{0, 1, \dots, p\}$, and $v \in V$. We prove by induction over i that all G_i have a proper labeling with at most k weak edges.

Base Case: $i = 0$. Then, since G_0 and G' have the exact same edges, L' is a proper labeling for G_0 with at most k weak edges.

Inductive Step: $0 < i \le p$. Then, by the inductive hypothesis, there exists a proper labeling L_{i-1} for $G_{i-1} = (V, E' \cup \{e_1, \dots, e_{i-1}\})$ with at most k weak edges. From $E' = E(\mathscr{C} \cup N(\mathscr{C}))$ we conclude $e_i \in E \setminus E(\mathscr{C} \cup N(\mathscr{C})) = E(\mathscr{P}) \setminus E(N(\mathscr{C}))$. Hence, for at least one of the endpoints u of e it holds that $N_G[u] \subseteq \mathscr{P}$. Therefore $\deg_G(w) \le c - 1$ for all $w \in N_G[u]$. Together with the facts that $\deg_{G_i}(w) \le \deg_G(w)$ and $N_{G_i}(w) \subseteq N_G(w)$ we conclude $\deg_{G_i}(w) \le c - 1$ for all $w \in N_{G_i}[u]$. Then, by Lemma 3.2, there exists a proper labeling L_i for G_i such that $|W_{L_i}| = |W_{L_{i-1}}| \le k$. ◀

▶ **Theorem 1.1.** *ECS admits a problem kernel with at most $4\xi_{c-1} \cdot c$ vertices and $\mathcal{O}(\xi_{c-1} \cdot c^2)$ edges that can be computed in $\mathcal{O}(n + m)$ time.*

Proof. Let (G, c, k) be an instance of ECS. We apply Rule 3.1 on (G, c, k) as follows: First, we compute a 2-approximation D'_{c-1} of the smallest possible edge-deletion set D_{c-1} in $\mathcal{O}(n+m)$ time as described in Section 2. Let $\mathscr{C} := \mathscr{C}(D'_{c-1})$ and note that $|D'_{c-1}| \le 2\xi_{c-1}$. We then remove all vertices in $V \setminus (\mathscr{C} \cup N_G(\mathscr{C}))$ from G which can also be done in $\mathcal{O}(n + m)$ time. Hence, applying Rule 3.1 can be done in $\mathcal{O}(n + m)$ time.

We next show that after this application of Rule 3.1 the graph consists of at most $4\xi_{c-1} \cdot c$ vertices and $\mathcal{O}(\xi_{c-1} \cdot c^2)$ edges. Since D'_{c-1} is a 2-approximation of the smallest possible edge-deletion set we have $|\mathscr{C}| \le 4\xi_{c-1}$. Since every vertex in $\mathscr{C}$ has at most $c-1$ neighbors in $V \setminus \mathscr{C}$, we conclude $|\mathscr{C} \cup N(\mathscr{C})| \le 4\xi_{c-1} \cdot c$. In $E(\mathscr{C} \cup N(\mathscr{C}))$ there are obviously the at most $4\xi_{c-1}$ edges of D'_{c-1}. Moreover, each of the at most $4\xi_{c-1} \cdot c$ vertices might have up to $c-1$ incident edges. Hence, after applying Rule 3.1, the reduced instance has $\mathcal{O}(\xi_{c-1} \cdot c^2)$ edges. ◀

If we consider Edge Coloring instead of ECS, we can immediately reject if one vertex has degree more than c. Then, since there are at most $|\mathscr{C}| \le 2\xi_{c-1}$ vertices that have a degree of at least c, Theorem 1.1 implies the following.

▶ **Corollary 3.4.** *Let h_c be the number of vertices with degree c. Edge Coloring admits a problem kernel with $\mathcal{O}(h_c \cdot c)$ vertices and $\mathcal{O}(h_c \cdot c^2)$ edges that can be computed in $\mathcal{O}(n + m)$ time.*

3.2 Component Order Connectivity

In this section we present a problem kernel for ECS parameterized by the number of strong colors c and the component order connectivity λ_c. We prove that ECS admits a problem kernel with $\mathcal{O}(c^3 \cdot \lambda_c)$ vertices, which is a linear vertex kernel for every fixed value of c. Our kernelization is based on the Expansion Lemma [24], a generalization of the Crown Rule [2]. We use the formulation given by Cygan et al. [4].

► **Lemma 3.5** (Expansion Lemma). *Let q be a positive integer and G be a bipartite graph with partite sets A and B such that $|B| \geq q|A|$ and there are no isolated vertices in B. Then there exist nonempty vertex sets $X \subseteq A$ and $Y \subseteq B$ with $N(Y) \subseteq X$. Moreover, there exist edges $M \subseteq E(X, Y)$ such that*

a) every vertex of X is incident with exactly q edges of M, and

b) $q \cdot |X|$ vertices in Y are endpoints of edges in M.

The sets X and Y can be found in polynomial time.

To apply Lemma 3.5 on an instance of ECS, we need the following definition for technical reasons.

► **Definition 3.6.** *For a given graph $G = (V, E)$, let D be an order-c component cover. We say that D is* saturated *if for every $v \in D$ it holds that $E_G(\{v\}, V \setminus D) \neq \emptyset$.*

Note that every order-c component cover D' can be transformed into a saturated order-c component cover by removing any vertex $v \in D'$ with $N(v) \subseteq D'$ from D' while such a vertex exists. Let $(G = (V, E), c, k)$ be an instance of ECS and let $D \subseteq V$ be a saturated order-c component cover. Furthermore, let $I := V \setminus D$ be the remaining set of vertices.

► **Rule 3.2.** *If there exists a set $J \subseteq I$ such that J is a connected component in G, remove all vertices in J from G.*

Rule 3.2 is safe since $|J| \leq c$ and therefore the graph $G[J]$ has maximum degree $c - 1$ and can be labeled by Vizing's Theorem with c colors. For the rest of this section we assume that (G, c, k) is reduced regarding Rule 3.2. The following proposition is a direct consequence of Lemma 3.5.

► **Proposition 3.7.** *Let $(G = (V, E), c, k)$ be an instance of ECS that is reduced regarding Rule 3.2, let D be a saturated order-c component cover of G, and let $I := V \setminus D$. If $|I| \geq c^2 \cdot |D|$, then there exist nonempty sets $X \subseteq D$ and $Y \subseteq I$ with $N(Y) \subseteq X \cup Y$. Moreover, there exists a set $M \subseteq E(X, Y)$ such that*

a) every vertex of X is incident with exactly c edges of M, and

b) $c \cdot |X|$ vertices in Y are endpoints of edges in M and every connected component in $G[Y]$ contains at most one such vertex.

The sets X and Y can be computed in polynomial time.

Proof. We prove the proposition by applying Lemma 3.5. To this end we define an equivalence relation $\sim$ on the vertices of I: Two vertices $v, u \in I$ are equivalent, denoted $u \sim v$ if and only if u and v belong to the same connected component in $G[I]$. Obviously, $\sim$ is an equivalence relation. For a given vertex $u \in I$, let $[u] := \{v \in I \mid v \sim u\}$ denote the equivalence class of u. Note that $|[u]| \leq c$ since D is an order-c component cover.

We next define the auxiliary graph G_{aux}, on which we will apply Lemma 3.5. Intuitively, we obtain G_{aux} from G by deleting all edges in $E_G(D)$ and merging the at most c vertices in every equivalence class in I. Formally $G_{\text{aux}} := (D \cup I^*, E_{\text{aux}})$, with $I^* := \{[u] \mid u \in I\}$ and

$$E_{\text{aux}} := \{\{[u], v\} \mid [u] \in I^*, v \in \bigcup_{w \in [u]} (N_G(w) \setminus I)\}.$$

Note that G_{aux} can be computed from G in polynomial time and that $|I| \geq |I^*| \geq \frac{1}{c}|I|$.

Observe that G_{aux} is bipartite with partite sets D and I^*. Since G is reduced regarding Rule 3.2, every $[u] \in I^*$ is adjacent to some $v \in D$ in G_{aux}. Furthermore, since D is saturated, every $v \in D$ is adjacent to some $u \in I$ in G and therefore $\{v, [u]\} \in E_{\text{aux}}$. Hence, G_{aux} is a bipartite graph without isolated vertices. Moreover, from $|I| \geq c^2|D|$ and $|I^*| \geq \frac{1}{c}|I|$ we conclude $|I^*| \geq c \cdot |D|$. By applying Lemma 3.5 on G_{aux} we conclude that there exist nonempty vertex sets $X' \subseteq D$ and $Y' \subseteq I^*$ with $N_{G_{\text{aux}}}(Y') \subseteq X'$ that can be computed in polynomial time such that there exists a set $M' \subseteq E_{G_{\text{aux}}}(X', Y')$ of edges, such that every vertex of X' is incident with exactly c edges of M', and $c \cdot |X'|$ vertices in Y' are endpoints of edges in M'.

We now describe how to construct the sets X, Y, and M from X', Y', and M'. We set $X := X' \subseteq D$, and $Y := \bigcup_{[u] \in Y'} [u] \subseteq I$. We prove that $N_G(Y) \subseteq X \cup Y$. Let $y \in Y$. Note that all neighbors of y in I are elements of Y by the definition of the equivalence relation $\sim$ and therefore

$$N_G(y) \subseteq N_{G_{\text{aux}}}([y]) \cup Y \subseteq X' \cup Y = X \cup Y.$$

Next, we construct $M \subseteq E_G(X, Y)$ from M'. To this end we define a mapping $\pi : M' \to E_G(X, Y)$. For every edge $\{[u], v\} \in M'$ with $[u] \in Y'$ and $v \in X'$ we define $\pi(\{[u], v\}) := \{w, v\}$, where w is some fixed vertex in $[u]$. We set $M := \{\pi(e') \mid e' \in M'\}$. It remains to show that the statements *a*) and *b*) hold for M.

a) Observe that $\pi(\{[u_1], v_1\}) = \pi(\{[u_2], v_2\})$ implies $[u_1] = [u_2]$ and $v_1 = v_2$ and therefore, the mapping π is injective. We conclude $|M| = |M'|$. Moreover, observe that the edges of M have the same endpoints in X as the edges of M'. Thus, since every vertex of X' is incident with exactly c edges of M' it follows that statement *a*) holds for M.

b) By the conditions *a*) and *b*) of Lemma 3.5, no two edges in M' have a common endpoint in Y'. Hence, in every connected component in $G[Y]$ there is at most one vertex incident with an edge in M. Moreover, since $|M| = |M'|$ and there are exactly $c \cdot |X'|$ vertices in Y' that are endpoints of edges in M' we conclude that statement *b*) holds for M. ◀

The following rule is the key rule for our kernelization.

▶ **Rule 3.3.** *If $|I| \geq c^2 \cdot |D|$, then compute the sets X and Y from Proposition 3.7, delete all vertices in $X \cup Y$ from G, and decrease k by $|E_G(X, V)| - c \cdot |X|$.*

▶ **Proposition 3.8.** *Rule 3.3 is safe.*

Rules 3.2 and 3.3 together with the fact that we can compute a $(c+1)$-approximation of the minimum order-c component cover in polynomial time [20] give us the following.

▶ **Theorem 1.2.** *ECS admits a problem kernel with $\mathcal{O}(c^3 \cdot \lambda_c)$ vertices.*

Proof. We first consider the running time. We use a $(c+1)$-approximation for the minimum oder-c component cover and compute an order-c component cover D' in polynomial time [20]. Afterwards we remove any vertex $v \in D'$ with $N(v) \subseteq D'$ from D' while such a vertex exists and we end up with a saturated order-c component cover $D \subseteq D'$. Afterwards, consider Rules 3.2 and 3.3. Obviously, one application of Rule 3.2 can be done in polynomial

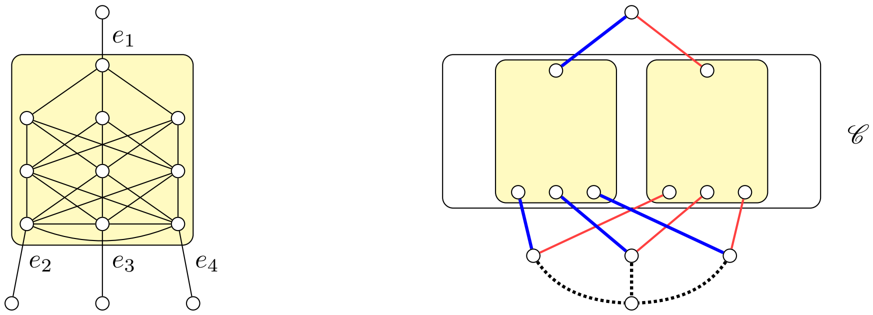

Figure 1 Left: A graph where in any STC-labeling with four strong colors and without weak edges, the edges e_1, e_2, e_3, and e_4 are part of the same strong color class. Right: A no-instance of Multi-STC with $c = 4$ and $k = 0$, where Rule 3.1 does not produce an equivalent instance: The inner rectangles correspond to two copies of the gadget on the left. Observe that all blue edges must have a common strong color, and all red edges must have a common strong color distinct that is not blue. Hence, for any STC-labeling of $G[\mathscr{C} \cup N(\mathscr{C})]$ it is not possible to extend the labeling to the dotted edges without violating STC. However, Rule 3.1 converts this no-instance into a yes-instance.

time if D is known. Moreover, Rule 3.3 can also be applied in polynomial time due to Proposition 3.7. Since every application of one of these two rules removes some vertices, we can compute an instance that is reduced regarding Rules 3.2 and 3.3 from an arbitrary input instance of ECS in polynomial time.

We next consider the size of a reduced instance $(G = (V, E), c, k)$ of ECS regarding Rules 3.2 and 3.3. Let $D \subseteq V$ be a $(c+1)$-approximate saturated order-c component cover, and let $I := V \setminus D$. Since no further application of Rule 3.3 is possible, we conclude $|I| < c^2 \cdot |D|$. Thus, we have $|V| = |I| + |D| < (c^2 + 1) \cdot |D| \leq (c^2 + 1) \cdot (c + 1) \cdot \lambda_c \in \mathcal{O}(c^3 \lambda_c)$. ◀

4 Multi-STC parameterized by Edge Deletion-Distance to Low-Degree Graphs

In this section we provide a problem kernelization for Multi-STC parameterized by ξ_{c-1} when $c \leq 4$. Before we describe the problem kernel, we briefly show that Multi-STC does not admit a polynomial kernel for the component order connectivity ξ_{c-1} even if $c = 1$: If NP $\not\subseteq$ coNP/poly, STC does not admit a polynomial kernel if parameterized by the number of strong edges [10] which – in nontrivial instances – is bigger than the size of a maximal matching M. Since the vertex cover number s is never larger than $2|M|$, this implies that Multi-STC has no polynomial kernel if parameterized by s unless NP $\subseteq$ coNP/poly. Since $\lambda_c \leq s$, we conclude that Multi-STC does not admit a polynomial kernel for λ_c unless NP $\subseteq$ coNP/poly.

Next, consider parameterization by ξ_{c-1}. Observe that Rule 3.1 which gives a problem kernel for ECS does not work for Multi-STC; see Figure 1 for an example. Furthermore, for Multi-STC we need a fundamental new approach: For STC-labelings the maximum degree and the number of colors are not as closely related as in ECS, and therefore, Lemma 3.1 might not be helpful for Multi-STC. Moreover, in the proof of Lemma 3.2 we exploit that in ECS we may remove weak edges from the instance, which does not hold for Multi-STC since removing a weak edge may produce P_3s. However, the results for ECS parameterized by (ξ_{c-1}, c) can be lifted to the seemingly harder Multi-STC for $c \in \{1, 2, 3, 4\}$. We will first discuss the cases $c = 1$ and $c = 2$. For the cases $c \in \{3, 4\}$ we show the more general statement that Multi-STC admits a problem kernel with $\mathcal{O}(\xi_{\lfloor \frac{c}{2} \rfloor + 1} \cdot c)$ vertices.

If $c = 1$, the parameter $\xi_{c-1} = \xi_0$ equals the number m of edges in G. Hence, Multi-STC admits a trivial ξ_{c-1}-edge kernel in this case. If $c = 2$, any input graph consists of core vertices $\mathscr{C}$, periphery vertices in $N(\mathscr{C})$ and isolated vertices and edges. We can compute an equivalent instance in linear time by deleting these isolated components. Afterwards, the graph contains at most $2\xi_{c-1}$ core vertices. Since each of these vertices has at most one neighbor outside $\mathscr{C}$, we have a total number of $4\xi_{c-1}$ vertices.

To extend this result to $c \in \{3, 4\}$, we now provide a problem kernel for Multi-STC parameterized by $(c, \xi_{\lfloor \frac{c}{2} \rfloor + 1})$. Let (G, c, k) be an instance of Multi-STC with edge-deletion set $D := D_{\lfloor \frac{c}{2} \rfloor + 1}$, and let $\mathscr{C}$ and $\mathscr{P}$ be the core and periphery of G. A subset $A \subseteq \mathscr{P}$ is called *periphery component* if it is a connected component in $G[\mathscr{P}]$. Furthermore, for a periphery component $A \subseteq \mathscr{P}$ we define the subset $A^* \subseteq A$ of *close vertices* in A as $A^* := N(\mathscr{C}) \cap A$, that is, the set of vertices of A that are adjacent to core vertices. The key technique of our kernelization is to move weak edges along paths inside periphery components.

▶ **Definition 4.1.** *Let (G, c, k) be an instance of Multi-STC with core vertices $\mathscr{C}$ and periphery vertices $\mathscr{P}$. A periphery component $A \subseteq \mathscr{P}$ is called* good*, if for every STC-labeling $L = (S^1_L, \dots, S^c_L, W_L)$ for G with $E(A) \subseteq W_L$ there exists an STC-labeling $L' = (S^1_{L'}, \dots, S^c_{L'}, W_{L'})$ for G such that 1. $L'|_{E \setminus E(A)} = L|_{E \setminus E(A)}$, and 2. $W_{L'} \cap E(A) = \emptyset$.*

Intuitively, a good periphery component A is a periphery component where the edges in $E(A)$ can always be added to some strong color classes of an STC-labeling, no matter how the other edges of G are labeled. The condition $E(A) \subseteq W_L$ is a technical condition that makes the proof of the next proposition easier.

▶ **Proposition 4.2.** *Let (G, c, k) be an instance of Multi-STC with core vertices $\mathscr{C}$ and periphery vertices $\mathscr{P}$. Furthermore, let $A \subseteq \mathscr{P}$ be a good periphery component. Then, (G, c, k) is a yes-instance if and only if $(G - (A \setminus A^*), c, k)$ is a yes-instance.*

In the following, we show that for instances (G, c, k) with $c \geq 3$ we can compute an equivalent instance of size $\mathcal{O}(\xi_{\lfloor \frac{c}{2} \rfloor + 1} c)$. We first consider all cases where $c \geq 3$ is odd. In this case, we can prove that all periphery components are good.

▶ **Proposition 4.3.** *Let (G, c, k) be an instance of Multi-STC, where $c \geq 3$ is odd. Moreover, let $A \subseteq \mathscr{P}$ be a periphery component. Then, A is good.*

The Propositions 4.2 and 4.3 guarantee the safeness of the following rule:

▶ **Rule 4.1.** *If c is odd, remove $A \setminus A^*$ from all periphery components $A \subseteq \mathscr{P}$.*

▶ **Proposition 4.4.** *Let $(G = (V, E), c, k)$ be an instance of Multi-STC where $c \geq 3$ is odd. Then, we can compute an instance $(G' = (V', E'), c, k)$ in $\mathcal{O}(n + m)$ time such that $|V'| \leq 2 \cdot \xi_{\lfloor \frac{c}{2} \rfloor + 1} \cdot (\lfloor \frac{c}{2} \rfloor + 1)$, and $|E'| \in \mathcal{O}(\xi_{\lfloor \frac{c}{2} \rfloor + 1} \cdot c^2)$.*

It remains to consider instances where c is an even number and $c \geq 4$. In this case, not every periphery component is good (Figure 1 shows an example), so we need to identify good periphery components more carefully. The first rule removes isolated periphery components.

▶ **Rule 4.2.** *Remove periphery components $A \subseteq \mathscr{P}$ with $A^* = \emptyset$ from G.*

▶ **Proposition 4.5.** *Rule 4.2 is safe.*

The intuition for the next lemma is that the small degree of vertices in periphery components can be used to “move” weak edges inside periphery components, the key technique of our kernelization. More precisely, if there is an edge-simple path in a periphery component, that starts with a weak edge, we can either move the weak edge to the end of that path by keeping the same number of weak edges or find a labeling with fewer weak edges.

▶ **Lemma 4.6.** *Let $A \subseteq \mathscr{P}$, let L be an STC-labeling of G, and let $e \in W_L \cap E(A)$ be a weak edge in $E(A)$. Furthermore, let $P = (v_1, v_2, \dots, v_{r-1}, v_r)$ be an edge-simple path in $G[A]$ with $\{v_1, v_2\} = e$ and color sequence $Q_L^P = (q_1 = 0, q_2, q_3, \dots, q_{r-1})$ under L. Then, there exists an STC-labeling L' with $L'|_{E \setminus E(P)} = L|_{E \setminus E(P)}$ such that*

$$Q_{L'}^P = (q_2, q_3, \dots, q_{r-1}, 0) \text{ or } |W_{L'}| < |W_L|.$$

Proof. We prove the statement by induction over the length r of P.

Base Case: $r = 2$. Then, $P = (v_1, v_2)$ and $Q_L^P = (0)$. We can trivially define the labeling L' by setting $L' := L$.

Inductive Step: Let $P = (v_1, \dots, v_r)$ be an edge-simple path with color sequence $Q_L^P = (0, q_2, \dots, q_{r-1})$ under L. Consider the edge-simple subpath $P' = (v_1, \dots, v_{r-1})$. By induction hypothesis there exists an STC-labeling L'' for G with $L''|_{E \setminus E(P')} = L|_{E \setminus E(P')}$, such that $Q_{L''}^{P'} = (q_2, q_3, \dots, q_{r-2}, 0)$ or $|W_{L''}| < |W_L|$.

Case 1: $|W_{L''}| < |W_L|$. Then, we define L' by $L' := L''$.

Case 2: $|W_{L''}| \geq |W_L|$. Then, $Q_{L''}^{P'} = (q_2, q_3, \dots, q_{r-2}, 0)$. Since $Q_{L''}^{P'}$ contains the same elements as $Q_L^{P'}$ and $L''|_{E \setminus E(P')} = L|_{E \setminus E(P')}$, we have $|W_{L''}| = |W_L|$.

Case 2.1: There exists an edge $e \neq \{v_{r-1}, v_r\}$ with $e \in S_{L''}^{q_{r-1}}$ that is incident with $\{v_{r-2}, v_{r-1}\}$. From the fact that $\deg(v_{r-2}) \leq \lfloor \frac{c}{2} \rfloor + 1$ and $\deg(v_{r-1}) \leq \lfloor \frac{c}{2} \rfloor + 1$, we conclude that $\{v_{r-2}, v_{r-1}\}$ is incident with at most c other edges of G. Since two of these incident edges have the same strong color q_{r-1} under L'', the edge $\{v_{r-2}, v_{r-1}\}$ is incident with at most $c-1$ edges of distinct strong colors under L''. Consequently, there exists a strong color $i \in \{1, \dots, c\}$, such that $\{v_{r-2}, v_{r-1}\}$ can safely be added to the strong color class $S_{L''}^i$ and be removed from $W_{L''}$ without producing any strong P_3. This way, we transformed L'' into an STC-labeling L', such that $L'|_{E \setminus E(P')} = L|_{E \setminus E(P')}$ and $|W_{L'}| < |W_L|$.

Case 2.2: No edge $e \neq \{v_{r-1}, v_r\}$ with $e \in S_{L''}^{q_{r-1}}$ is incident with $\{v_{r-2}, v_{r-1}\}$. We then define L' by

$$W_{L'} := W_{L''} \cup \{\{v_{r-1}, v_r\}\} \setminus \{\{v_{r-2}, v_{r-1}\}\}, \text{ and}$$
$$S_{L'}^{q_{r-1}} := S_{L''}^{q_{r-1}} \cup \{\{v_{r-2}, v_{r-1}\}\} \setminus \{\{v_{r-1}, v_r\}\}.$$

Note that $Q_{L'}^P = (q_2, q_3, \dots, q_{r-1}, 0)$ and $L'|_{E \setminus E(P)} = L|_{E \setminus E(P)}$. Moreover, since P is edge-simple, the edge $\{v_{r-1}, v_r\}$ does not lie on P' and since $L''|_{E \setminus E(P')} = L|_{E \setminus E(P')}$, it holds that $\{v_{r-1}, v_r\} \in S_{L''}^{q_{r-1}}$. Therefore, every edge has exactly one color under L'. It remains to show that L' satisfies STC. Assume towards a contradiction, that this is not the case. Then, since L'' satisfies STC, there exists an induced P_3 on $\{v_{r-2}, v_{r-1}\} \in S_{L'}^{q_{r-1}}$ and some edge $e \in S_{L'}^{q_{r-1}}$. Since $\{v_{r-1}, v_r\} \in W_{L'}$ and $L'|_{E \setminus \{\{v_{r-2}, v_{r-1}\}, \{v_{r-1}, v_r\}\}} = L''|_{E \setminus \{\{v_{r-2}, v_{r-1}\}, \{v_{r-1}, v_r\}\}}$, the edge $e \neq \{v_{r-1}, v_r\}$ is incident with $\{v_{r-2}, v_{r-1}\}$ and it holds that $e \in S_{L''}^{q_{r-1}}$. This contradicts the condition of Case 2.2. ◀

We will now use Lemma 4.6 to show useful properties of periphery components. First, if there are two weak edges in one periphery component A, we can make these two weak edges incident, which then helps us to define a new labeling that has fewer weak edges in A:

▶ **Proposition 4.7.** *Let $A \subseteq \mathscr{P}$ be a periphery component and let L be an STC-labeling for G. Then, there exists an STC-labeling L' with $L'|_{E \setminus E(A)} = L|_{E \setminus E(A)}$ and $|W_{L'} \cap E(A)| \leq 1$.*

Proof. If $|W_L \cap E(A)| \leq 1$ the statement already holds for $L' = L$. So, assume there are two distinct edges $e_1, e_2 \in W_L \cap E(A)$. In this case, we construct an STC-labeling which is partially equal to L on $E \setminus E(A)$ and has strictly fewer weak edges in $E(A)$ than L, which then proves the claim.

Since periphery components are connected components in $G[\mathscr{P}]$, there exists an edge-simple path $P = (v_1, \dots, v_r)$ in $G[A]$ such that $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_{r-1}, v_r\}$. Applying Lemma 4.6 on the edge-simple subpath $P' = (v_1, \dots, v_{r-1})$ gives us an STC-labeling L' with $L'|_{E \setminus E(P)} = L|_{E \setminus E(P)}$ such that $|W_{L'}| < |W_L|$ or $Q_{L'}^{P'} = (q_2, q_3, \dots, q_{r-2}, 0)$.

In case of $|W_{L'}| < |W_L|$, nothing more needs to be shown. So, assume $|W_{L'}| = |W_L|$. It follows that $Q_{L'}^{P'} = (q_2, q_3, \dots, q_{r-2}, 0)$ and therefore $Q_{L'}^{P} = (q_2, q_3, \dots, q_{r-2}, 0, 0)$. Then, e_1 and e_2 are weak under L'. Since $\deg(v_{r-1}) \leq \lfloor \frac{c}{2} \rfloor + 1$ and $\deg(v_r) \leq \lfloor \frac{c}{2} \rfloor + 1$, the edge e_2 is incident with at most c edges. Since at least one of these incident edges is weak, e_2 is incident with at most $c-1$ edges of distinct strong colors. Consequently, there exists a strong color color $i \in \{1, \dots, c\}$ such that e_2 can be added to the strong color class $S_{L'}^i$ and deleted from $W_{L'}$ without violating STC. This way, we transformed L' into an STC-labeling L'' such that $L''|_{E \setminus E(A)} = L|_{E \setminus E(A)}$ and $|W_{L''} \cap E(A)| < |W_L \cap E(A)|$. ◀

Next, we use Proposition 4.7 to identify specific good components.

▶ **Proposition 4.8.** *Let $A \subseteq \mathscr{P}$ be a periphery component such that some edge $\{u, v\} \in E(A)$ forms an induced P_3 with less than c other edges in G. Then, A is good.*

Proof. Let L be an arbitrary STC-labeling for G with $E(A) \subseteq W_L$. We prove that there is an STC-labeling which is partially equal to L on $E \setminus E(A)$ and has no weak edges in $E(A)$.

Let L' be an STC-labeling for G with $L'|_{E \setminus E(A)} = L|_{E \setminus E(A)}$. If $W_{L'} \cap E(A) = \emptyset$, nothing more needs to be shown. So, let $W_{L'} \cap E(A) \neq \emptyset$. By Proposition 4.7 we can assume that there is one unique edge $e \in W_{L'} \cap E(A)$. Since A is a connected component in $G[\mathscr{P}]$, there exists an edge-simple path $P = (v_1, \dots, v_r)$ such that $\{v_1, v_2\} = e$, and $\{v_{r-1}, v_r\} = \{u, v\}$ with $Q_{L'}^P = (0, q_2, \dots, q_{r-1})$. By Lemma 4.6, there exists an STC-labeling L'' with $L''|_{E \setminus E(A)} = L|_{E \setminus E(A)}$ such that $|W_{L''}| < |W_L|$ or $Q_{L''}^P = (q_2, \dots, q_{r-1}, 0)$. In case of $|W_{L''}| < |W_L|$, nothing more needs to be shown. Otherwise, the edge e is weak under L''. Since e is part of less than c induced P_3s in G, there exists one strong color $i \in \{1, \dots, c\}$, such that e can safely be added to $S_{L''}^i$ and be removed from $W_{L''}$ without violating STC. This way, we transform L'' into an STC-labeling L''' with $L'''|_{E \setminus E(A)} = L|_{E \setminus E(A)}$ and $W_{L'''} \cap E(A) = \emptyset$.

Since L was arbitrary, the periphery component A is good by definition. ◀

▶ **Proposition 4.9.** *Let $A \subseteq \mathscr{P}$ be a periphery component such that there exists a vertex $v \in A$ with $\deg_G(v) < \lfloor \frac{c}{2} \rfloor + 1$. Then, A is good.*

Proof. If $|A| = 1$, then A is obviously good, since $E(A) = \emptyset$. Let $|A| \geq 2$. Since A contains at least two vertices and forms a connected component in $G[\mathscr{P}]$ there exists a vertex $u \in A$, such that $\{u, v\} \in E(A)$. Since $\deg_G(v) < \lfloor \frac{c}{2} \rfloor + 1$, and $\deg_G(u) \leq \lfloor \frac{c}{2} \rfloor + 1$, the edge $\{u, v\}$ forms induced P_3s with less than c other edges in G. Then, by Proposition 4.8 we conclude that A is good. ◀

Propositions 4.2 and 4.9 guarantee the safeness of the following rule.

▶ **Rule 4.3.** *If there is a periphery component $A \subseteq \mathscr{P}$ with $A \setminus A^* \neq \emptyset$ such that there exists a vertex $v \in A$ with $\deg(v) < \lfloor \frac{c}{2} \rfloor + 1$, then delete $A \setminus A^*$ from G.*

▶ **Proposition 4.10.** *Let $A \subseteq \mathscr{P}$ be a periphery component such that there exists an edge $\{u, v\} \in E(A)$ which is part of a triangle $G[\{u, v, w\}]$ in G. Then, A is good.*

Proof. Since $u, v \in A$, we know $\deg_G(u) \leq \lfloor \frac{c}{2} \rfloor + 1$ and $\deg_G(v) \leq \lfloor \frac{c}{2} \rfloor + 1$. Since u, v are part of a triangle in G, it follows that $\{u, v\}$ forms an induced P_3 with less than c other edges in G. Then, by Proposition 4.8 we conclude that A is good. ◀

Propositions 4.2 and 4.10 guarantee the safeness of the following rule.

▶ **Rule 4.4.** *If there is a periphery component $A \subseteq \mathscr{P}$ with $A \setminus A^* \neq \emptyset$ such that there exists an edge $\{u, v\} \in A$ which is part of a triangle $G[\{u, v, w\}]$ in G, then delete $A \setminus A^*$ from G.*

For the rest of this section we consider instances (G, c, k) for Multi-STC, that are reduced regarding Rules 4.2–4.4. Observe that these instances only contain triangle-free periphery components A where every vertex $v \in A$ has $\deg(v) = \lfloor \frac{c}{2} \rfloor + 1$. Since ECS and Multi-STC are the same on triangle-free graphs one might get the impression that we can use Vizing's Theorem to prove that all periphery components in G are good. Consider the example in Figure 1 to see that this is not necessarily the case.

We now continue with the description of the kernel for Multi-STC. Let (G, c, k) be an instance of Multi-STC that is reduced regarding Rules 4.2–4.4. We analyze the periphery components of G that contain cycles. In this context, a *cycle* (of length r) is an edge-simple path $P = (v_0, v_1, \ldots, v_{r-1}, v_0)$ where the last vertex and the first vertex of P are the same, and all other vertices occur at most once in P. We will see that acyclic periphery components – which are periphery components $A \subseteq \mathscr{P}$ where $G[A]$ is a tree – are already bounded in c and $\xi_{\lfloor \frac{c}{2} \rfloor + 1}$. To remove the other components, we show that periphery components with cycles are always good. To this end we show two lemmas. The intuitive idea behind Lemmas 4.11 and 4.12 is, that we use Lemma 4.6 to rotate weak and strong edge-colors around a cycle.

▶ **Lemma 4.11.** *Let $A \subseteq \mathscr{P}$ be a periphery component, and let L be an STC-labeling for G. Moreover, let $P = (v_0, v_1, \ldots, v_{r-1}, v_0)$ be a cycle in A such that $W_L \cap E(P) \neq \emptyset$ and let $Q_L^P = (q_0, q_1, \ldots, q_{r-1})$ be the color sequence of P under L. Then, there exist STC-labelings $L_0, L_1, L_2, \ldots, L_{r-1}$ for G such that $L_i|_{E \setminus E(P)} = L|_{E \setminus E(P)}$ and*

$$Q_{L_i}^P(j) = q_{(i+j) \bmod r} \text{ or } |W_{L_i}| < |W_L|$$

for all $i, j \in \{0, \ldots, r-1\}$.

Proof. Without loss of generality we assume that $\{v_0, v_1\} \in W_L$ and therefore $q_0 = 0$. We prove the existence of the labelings L_i with $i \in \{0, 1, \ldots, r-1\}$ by induction over i.

Base Case: $i = 0$. In this case we set $L_0 := L$.

Inductive Step: By inductive hypothesis, there is a labeling L_{i-1} with $|W_{L_{i-1}}| < |W_L|$ or

$$Q_{L_{i-1}}^P(j) = q_{(i-1+j) \bmod r}.$$

If $|W_{L_{i-1}}| < |W_L|$, then we define L_i by $L_i := L_{i-1}$ and nothing more needs to be shown. Otherwise, we consider $P' = (v_{r-i+1}, v_{r-i+2}, \ldots, v_{r-1}, v_0, v_1, \ldots, v_{r-i+1})$. Note that P' describes the same cycle as P by rotating the vertices. More precisely,

$$P(j) = P'((j + i - 1) \bmod r).$$

Therefore, P' is edge-simple and has the color sequence $Q_{L_{i-1}}^{P'} = (q_0 = 0, q_1, \ldots, q_{r-1})$. By Lemma 4.6, there exists an STC-labeling L_i with $L_i|_{E \setminus E(P)} = L_{i-1}|_{E \setminus E(P)}$, such that $|W_{L_i}| < |W_{L_{i-1}}|$ or

$$Q_{L_i}^{P'}(j) = q_{(j+1) \bmod r}.$$

In case of $|W_{L_i}| < |W_{L_{i-1}}|$, nothing more needs to be shown. Otherwise, observe that

$$Q^P_{L_i}(j) = Q^{P'}_{L_i}((j+i-1) \bmod r) = q_{(j+i) \bmod r}$$

which completes the inductive step. ◀

▶ **Lemma 4.12.** *Let $A \subseteq \mathscr{P}$ be a periphery component, let L be an STC-labeling. Moreover, let $P = (v_0, v_1, \dots, v_{r-1}, v_0)$ be a cycle in A with $W_L \cap E(P) \neq \emptyset$, and let $e_1, e_2 \in E(P)$ with $e_2 \in S^q_L$ for some strong color $q \in \{1, \dots, c\}$. Then, there exists an STC-labeling L' with $L'|_{E \setminus E(P)} = L|_{E \setminus E(P)}$ such that $e_1 \in S^q_{L'}$, or $|W_{L'}| < |W_L|$.*

Proof. Let $Q^P_L := (q_0, q_1, \dots, q_{r-1})$. Without loss of generality assume that $\{v_0, v_1\} \in W_L$ and $e_2 = \{v_t, v_{t+1}\}$ for some $t \in \{1, \dots, r-1\}$. It then holds, that $q_0 = 0$, and $q = q_t$. Furthermore, since $e_1 \in E(P)$ we have $e_1 = \{P(j), P(j+1)\}$ for some $j \in \{0, 1, \dots, r-1\}$.

Consider the STC-labelings $L_0, L_1, L_2, \dots L_{r-1}$ from Lemma 4.11. If for one such labeling L_i it holds that $|W_{L_i}| < |W_L|$, then nothing more needs to be proven. Otherwise, set $i := (t-j) \bmod r$. We show that $e_1 \in S^{q_t}_{L_i}$ by proving $Q^P_{L_i}(j) = q_t$ as follows:

$$Q^P_{L_i}(j) = q_{(i+j) \bmod r} = q_{((t-j) \bmod r)+j) \bmod r} = q_{(t-j+j) \bmod r} = q_t.$$ ◀

We next use Lemma 4.12 to prove that periphery components with cycles are good.

▶ **Proposition 4.13.** *Let $(G = (V, E), c, k)$ be a reduced instance of Multi-STC regarding rules 4.2–4.4, where $c \geq 4$ is even. Let $A \subseteq \mathscr{P}$ be a periphery component in G such that $A \setminus A^* \neq \emptyset$ and there is a cycle $P = (v_0, v_1, \dots, v_{r-1}, v_0)$ in $G[A]$. Then, A is good.*

Propositions 4.13 and 4.2 imply the safeness of the final rule which together with Rules 4.2–4.4 gives the kernel.

▶ **Rule 4.5.** *If there is a periphery component $A \subseteq \mathscr{P}$ with $A \setminus A^* \neq \emptyset$ such that there exists a cycle P in $G[A]$, then delete $A \setminus A^*$ from G.*

▶ **Theorem 4.14.** *Multi-STC restricted to instances with $c \geq 3$ admits a problem kernel with $\mathcal{O}(\xi_{\lfloor \frac{c}{2} \rfloor + 1} \cdot c)$ vertices and $\mathcal{O}(\xi_{\lfloor \frac{c}{2} \rfloor + 1} \cdot c^2)$ edges that can be computed in $\mathcal{O}(n+m)$ time.*

Proof. Throughout this proof let $\xi := 2\xi_{\lfloor \frac{c}{2} \rfloor + 1}$ denote the size of a 2-approximate edge-deletion set $D_{\lfloor \frac{c}{2} \rfloor + 1}$ of G and $\lfloor \frac{c}{2} \rfloor + 1$. We defer the proof of the running time and show that $|V'| \leq (c+7) \cdot \xi$. Let $\mathscr{C}$ be the set of core vertices of G' and $\mathscr{P}$ be the set of periphery vertices of G'. Since $|\mathscr{C}| \leq 2\xi$, and every $v \in \mathscr{C}$ is incident with at most $\frac{c}{2} + 1$ edges, there are $2\xi + 2\xi(\frac{c}{2} + 1) = \xi c + 4\xi$ vertices in $\mathscr{C} \cup N(\mathscr{C})$. It remains to show that there are at most 3ξ non-close vertices in $\mathscr{P}$. Consider the family $\mathcal{A} := \{A \subseteq \mathscr{P} \mid A \text{ is periphery component with } A \setminus A^* \neq \emptyset\}$ of periphery components.

Since G' is reduced regarding Rules 4.3, 4.4, and 4.5, every $G[A]$ with $A \in \mathcal{A}$ is a tree, where every vertex $v \in A$ has degree $\deg_G(v) = \frac{c}{2} + 1$ in G. We define a *leaf vertex* as a vertex $v \in \bigcup_{A \in \mathcal{A}} A$ with $\deg_{G[\mathscr{P}]}(v) = 1$. Note that these vertices are exactly the leaves of a tree $G[A]$ for some $A \in \mathcal{A}$, and all leaf vertices are close vertices in $\mathscr{P}$. Let p be the number of leaf vertices. We show that $p \leq 3\xi$. Since (G', c, k) is reduced regarding Rule 4.3, every vertex $v \in \bigcup_{A \in \mathcal{A}} A$ has a degree of $\deg_G(v) = \frac{c}{2} + 1$, hence every leaf vertex has exactly $\frac{c}{2}$ neighbors in $\mathscr{C}$. We thus have $p \cdot \frac{c}{2} \leq |E(\mathscr{C}, N(\mathscr{C}))| \leq 2\xi(\frac{c}{2} + 1)$, and therefore $p \leq 2\xi + \frac{4\xi}{c} \leq 3\xi$, since $c \geq 4$. Recall that every non-close vertex v in some tree $G[A]$ satisfies $\deg_{G[A]}(v) = \frac{c}{2} + 1 > 2$. Since a tree has at most as many vertices with degree at least three as it has leaves, we conclude $|(\bigcup_{A \in \mathcal{A}} A) \setminus (\bigcup_{A \in \mathcal{A}} A^*)| \leq 3\xi$. Hence, there are at most 3ξ non-close vertices in $\mathscr{P}$. Then, G' contains of at most $(c+7) \cdot \xi \in \mathcal{O}(\xi c)$ vertices, as claimed. Since each vertex is incident with at most $\frac{c}{2} + 1$ edges, G' has $\mathcal{O}(\xi c^2)$ edges. ◀

5 Conclusion

In this work, we showed that Maximum Edge-Colorable Subgraph with c colors is tractable on instances that have small edge-deletion distance to graphs whose maximum degree is $c-1$. This result implies that Edge Coloring with c colors is fixed-parameter tractable with respect to the combination of c and the number of vertices that have degree c. For Multi Strong Triadic Closure with c colors, we obtain fixed-parameter algorithms for the same parameter for $c \leq 4$. For Multi Strong Triadic Closure with $c \geq 5$, the parameter in our fixed-parameter algorithms is the edge-deletion distance to graphs with maximum degree c' for some $c' < c-1$.

There are several ways of extending our results that seem interesting topics for future research. First, in our fixed-parameter algorithms the value of c is always part of the parameter and it would be very interesting to understand whether this is necessary. For example, is Edge Coloring fixed-parameter tractable with respect to the number of vertices that have degree at least c alone? Second, our results are obtained via kernelizations. Are there any direct fixed-parameter algorithms that achieve a better running time than using kernelization and brute-force on the kernels? Third, can our results for Multi Strong Triadic Closure and $c \geq 5$ be improved to fixed-parameter algorithms for the edge-deletion distance to graphs with maximum degree $c-1$? Moreover, our parameters use the *edge-deletion* distance to tractable special cases. Can one improve these results by obtaining fixed-parameter algorithms for the *vertex-deletion* distance? Finally, in our parameterization for Multi Strong Triadic Closure we use the fact that Multi Strong Triadic Closure is polynomial-time solvable on graphs with maximum degree $c-1$. This is a simple corollary of Vizing's theorem and the fact that every proper edge coloring is a valid coloring for Multi Strong Triadic Closure. It would be nice to extend the class of tractable instances further in the following sense: For which superclasses of the graphs with maximum degree $c-1$ does Multi Strong Triadic Closure remain polynomial-time solvable? Surely, our fixed-parameter algorithms give such superclasses but are there some that can be described without the use of parameters, for example via a characterization of forbidden induced subgraphs of size at most $f(c)$?

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