Drawing Graphs with Circular Arcs and Right-Angle Crossings

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Abstract

In a RAC drawing of a graph, vertices are represented by points in the plane, adjacent vertices are connected by line segments, and crossings must form right angles. Graphs that admit such drawings are RAC graphs. RAC graphs are beyond-planar graphs and have been studied extensively. In particular, it is known that a RAC graph with n vertices has at most $4n - 10$ edges.

We introduce a superclass of RAC graphs, which we call *arc-RAC* graphs. A graph is arc-RAC if it admits a drawing where edges are represented by circular arcs and crossings form right angles. We provide a Turán-type result showing that an arc-RAC graph with n vertices has at most $14n - 12$ edges and that there are n-vertex arc-RAC graphs with $4.5n - O(\sqrt{n})$ edges.

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1 Introduction

A *drawing* of a graph in the plane is a mapping of its vertices to distinct points and each edge uv to a curve whose endpoints are u and v. Planar graphs, which admit crossing-free drawings, have been studied extensively. They have many nice properties and several algorithms for drawing them are known, see, e.g., [19, 20]. However, in practice we must also draw non-planar graphs and crossings make it difficult to understand a drawing. For this reason, graph classes with restrictions on crossings are studied, e.g., graphs that can be drawn with at most k crossings per edge (known as *k-planar graphs*) or where the angles formed by each crossing are "large". These classes are categorized as *beyond-planar* graphs and have experienced increasing interest in recent years [13].

As introduced by Didimo et al. [12], a prominent beyond-planar graph class that concerns the crossing angles is the class of k-bend right-angle-crossing graphs, or RAC_k graphs for short, that admit a drawing where all crossings form 90° angles and each edge is a polygonal

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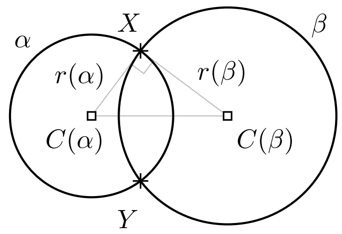


Figure 1 Circles α and β are orthogonal if and only if $\triangle XC(\alpha)C(\beta)$ is right-angled.

chain with at most k bends. Using right-angle crossings and few bends is motivated by several cognitive studies suggesting a positive correlation between large crossing angles or small curve complexity and the readability of a graph drawing [16–18]. Didimo et al. [12] studied the edge density of RAC$_k$ graphs. They showed that RAC$_0$ graphs with n vertices have at most $4n-10$ edges (which is tight), that RAC$_1$ graphs have at most $O(n^{\frac{4}{3}})$ edges, that RAC$_2$ graphs have at most $O(n^{\frac{7}{4}})$ edges and that all graphs are RAC$_3$. Dujmović et al. [14] gave an alternative simple proof of the $4n-10$ bound for RAC$_0$ graphs using charging arguments similar to those of Ackerman and Tardos [2] and Ackerman [1]. Arikushi et al. [5] improved the upper bounds to $6.5n-13$ for RAC$_1$ graphs and to $74.2n$ for RAC$_2$ graphs. The bound of $6.5n-13$ for RAC$_1$ graphs was also obtained by charging arguments. They also provided a RAC$_1$ graph with $4.5n-O(\sqrt{n})$ edges. The best known lower and upper bound for the maximum edge density of RAC$_1$ graphs of $5n-10$ and $5.5n-11$, respectively, are due to Angelini et al. [4].

We extend the class of RAC$_0$ graphs by allowing edges to be drawn as circular arcs but still requiring 90° crossings. An angle at which two circles intersect is the angle between the two tangents to each of the circles at an intersection point. Two circles intersecting at a right angle are called *orthogonal*. For any circle γ, let $C(\gamma)$ be its center and let $r(\gamma)$ be its radius. The following observation follows from the Pythagorean theorem.

▶ **Observation 1.1.** *Let α and β be two circles. Then α and β are orthogonal if and only if $r(\alpha)^2 + r(\beta)^2 = |C(\alpha)C(\beta)|^2$; see Figure 1.*

In addition we note the following.

▶ **Observation 1.2.** *Given a pair of orthogonal circles, the tangent to one circle at one of the intersection points goes through the center of the other circle; see Figure 1. In particular, a line is orthogonal to a circle if the line goes through the center of the circle.*

Similarly, two circular arcs α and β are orthogonal if they intersect properly (that is, ignoring intersections at endpoints) and the underlying circles (that contain the arcs) are orthogonal. For the remainder of this paper, all arcs will be circular arcs. We consider any straight-line segment to be an arc with infinite radius. Note, though, that the above observations do not hold for (pairs of) circles of infinite radius. As in the case of circles, for any arc γ of finite radius, let $C(\gamma)$ be its center.

We call a drawing of a graph an *arc-RAC drawing* if the edges are drawn as arcs and any pair of intersecting arcs is orthogonal; see Figure 2. A graph that admits an arc-RAC drawings is called an *arc-RAC graph*.

The idea of drawing graphs with arcs dates back to at least the work of the artist Mark Lombardi who drew social networks, featuring players from the political and financial sector [22]. Indeed, user studies [25, 27] state that users prefer edges drawn with curves of small curvature; not necessarily for performance (that is, tasks such as finding shortest paths,

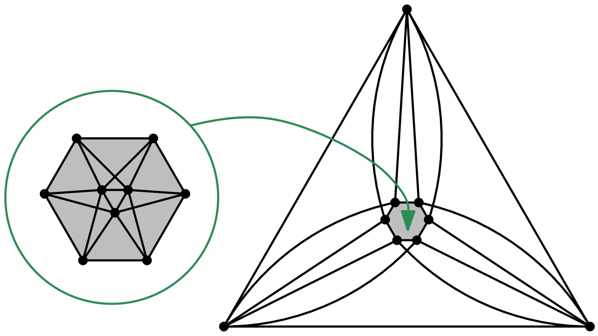


Figure 2 An arc-RAC drawing of a graph. This graph is not RAC_0 [6].

identifying common neighbors, or determining vertex degrees) but for aesthetics. Drawing graphs with arcs can help to improve certain quality measures of a drawing such as angular resolution [3, 11] or visual complexity [21, 26].

An immediate restriction on the edge density of arc-RAC graphs is imposed by the following known result.

▶ **Lemma 1.3** ([23]). *In an arc-RAC drawing, there cannot be four pairwise orthogonal arcs.*

It follows from Lemma 1.3 that arc-RAC graphs are 4-*quasi-planar*, that is, an arc-RAC drawing cannot have four edges that pairwise cross. This implies that an arc-RAC graph with n vertices can have at most $72(n-2)$ edges [1].

Our main contribution is that we reduce this bound to $14n-12$ using charging arguments similar to those of Ackerman [1] and Dujmović et al. [14]; see Section 2. For us, the main challenge was to apply these charging arguments to a modification of an arc-RAC drawing and to exploit, at the same time, geometric properties of the original arc-RAC drawing to derive the bound. We also provide a lower bound of $4.5n - O(\sqrt{n})$ on the maximum edge density of arc-RAC graphs based on the construction of Arikushi et al. [5]; see Section 3. We conclude with some open problems in Section 4. Throughout the paper our notation won't distinguish between the entities (vertices and edges) of an abstract graph and the geometric objects (points and curves) representing them in a drawing.

As usual for topological drawings, we forbid vertices to lie in the relative interior of an edge and we do not allow edges to *touch*, that is, to have a common point in their relative interiors without crossing each other at this point. Hence an *intersection point* of two edges is always a *crossing*. When we say that two edges *share a point*, we mean that they either cross each other or have a common endpoint.

2 An Upper Bound for the Maximum Edge Density

Let G be a 4-quasi-planar graph, and let D be a 4-quasi-planar drawing of G. In his proof of the upper bound on the edge density of 4-quasi-planar graphs, Ackerman [1] first modified the given drawing so as to remove faces of small degree. We use a similar modification that we now describe.

Consider two edges e_1 and e_2 in D that intersect multiple times. A region in D bounded by pieces of e_1 and e_2 that connect two consecutive crossings or a crossing and a vertex of G is called a *lens*. If a lens is adjacent to a crossing and a vertex of G, then we call such a lens a *1-lens*, otherwise a *0-lens*. A lens that does not contain a vertex of G is *empty*. Every drawing with 0-lenses has a *smallest* empty 0-lens, that is, an empty 0-lens that does not

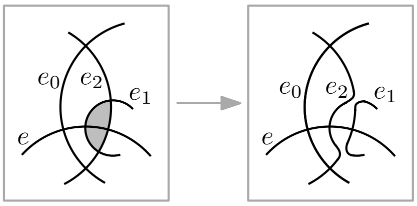


Figure 3 A simplification step resolves a smallest empty 0-lens; if two edges e_1 and e_2 change the order in which they cross the edge e, they form an empty 0-lens intersecting e before the step, and thus, in the original 4-quasi-planar drawing.

contain any other empty 0-lenses in its interior. We can swap [1, 24] the two curves that bound a smallest empty 0-lens; see Figure 3. We call such a swap a *simplification step*. Since a simplification step resolves a smallest empty lens, we observe the following.

▶ **Observation 2.1.** *A simplification step does not introduce any new pairs of crossing edges or any new empty lenses.*

We exhaustively apply simplification steps to our drawing and refer to this as the *simplification process*. Observation 2.1 guarantees that applying the *simplification process* to a drawing D terminates, that is, it results in an empty-0-lens-free drawing D' of G. We call the resulting drawing D' *simplified*; it is a *simplification* of D. Observation 2.1 implies the following important property of any simplification step.

▶ **Observation 2.2.** *Applying a simplification step to a* 4*-quasi-planar drawing yields a* 4*-quasi-planar drawing.*

As mentioned above, Ackerman [1] used a similar modification to prepare a 4-quasi-planar drawing for his charging arguments; note, that unlike Ackerman, we do not resolve 1-lenses. We look at the simplification process in more detail, in particular, we consider how it changes the order in which edges cross.

▶ **Lemma 2.3.** *Let D be an arc-RAC drawing, and let D' be a simplification of D. If two edges e_1 and e_2 cross another edge e in D' in an order different from that in D, then e_1 and e_2 form an empty 0-lens intersecting e in D.*

Proof. Let e_1 and e_2 be two edges as in the statement of the lemma. Then there is a simplification step i where the order in which e_1 and e_2 cross e changes. Let D_i be the drawing immediately before simplification step i, and let D_{i+1} be the drawing right after step i. By construction, the order in which e_1 and e_2 cross e is different in D_i and in D_{i+1}. Since D_i is 4-quasi-planar (see Observation 2.2) and since we always resolve a smallest empty 0-lens, the edges e_1 and e_2 form a smallest empty 0-lens in D_i; see Figure 3. Given that the simplification process does not introduce new empty lenses (see Observation 2.1), e_1 and e_2 form an empty 0-lens in the original 4-quasi-planar drawing. ◀

We now focus on the special type of 4-quasi-planar drawings we are interested in. Suppose that G is an arc-RAC graph, D is an arc-RAC drawing of G, and D' is a simplification of D. Note that, in general, D' is not an arc-RAC drawing. If two edges e_1 and e_2 cross in D', then they do not form an empty 0-lens in D. This holds because for any two edges forming an empty 0-lens in D, the simplification process removes both of their crossings; therefore, in D' the two edges do not have any crossings. If e_1 and e_2 are incident to the same vertex, they also do not form an empty 0-lens in D, as otherwise they would share three points in D (the two crossing points of the lens and the common vertex of G). Thus, we have the following observation.

▶ **Observation 2.4.** *Let D be an arc-RAC drawing, and let D' be a simplification of D. If two edges e_1 and e_2 share a point in D', then they do not form an empty 0-lens in D.*

In the following, we first state the main theorem of this section and provide the structure of its proof (deferring one small lemma and the main technical lemma until later). Then, we prove the remaining technical details in Lemmas 2.6 to 2.10 to establish the result.

▶ **Theorem 2.5.** *An arc-RAC graph with n vertices can have at most $14n - 12$ edges.*

Proof. Let $G = (V, E)$ be an arc-RAC graph, let D be an arc-RAC drawing of G, let D' be a simplification of D, and let $G' = (V', E')$ be the planarization of D'. Our charging argument consists of three steps.

First, each face f of G' is assigned an initial charge $ch(f) = |f| + v(f) - 4$, where $|f|$ is the degree of f in the planarization and $v(f)$ is the number of vertices of G on the boundary of f. Applying Euler's formula several times, Ackerman and Tardos [2] showed that $\sum_{f \in G'} ch(f) = 4n - 8$, where n is the number of vertices of G. In addition, we set the charge $ch(v)$ of a vertex v of G to $16/3$. Hence the total charge of the system is $4n - 8 + 16n/3 = 28n/3 - 8$.

In the next two steps (described below), similarly to Dujmović et al. [14], we redistribute the charges among faces of G' and vertices of G so that, for every face f, the final charge $ch_{\text{fin}}(f)$ is at least $v(f)/3$ and the final charge of each vertex is non-negative. Observing that

$$28n/3 - 8 \geq \sum_{f \in G'} ch_{\text{fin}}(f) \geq \sum_{f \in G'} v(f)/3 = \sum_{v \in G} \deg(v)/3 = 2|E|/3$$

yields that the number of edges of G is at most $14n - 12$ as claimed. (The second-last equality holds since both sides count the number of vertex–face incidences in G'.)

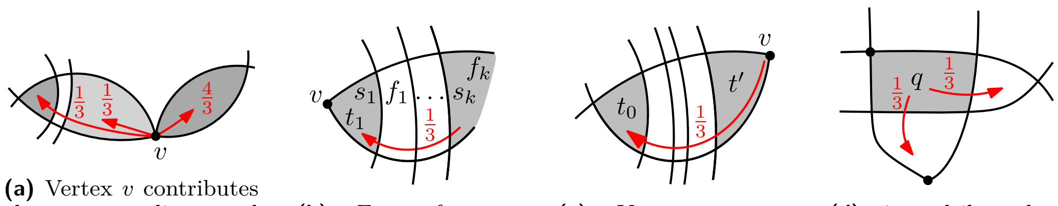
After the first charging step above, it is easy to see that $ch(f) \geq v(f)/3$ holds if $|f| \geq 4$. We call a face f of G' a k-triangle, k-quadrilateral, or k-pentagon if f has the corresponding shape and $v(f) = k$. Similarly, we call a face of degree two a *digon*. Note that any digon is a 1-digon since all empty 0-lenses have been simplified.

After the first charging step, each digon and each 0-triangle has a charge of -1, and each 1-triangle has a charge of 0. Thus, in the second charging step, we need to find $4/3$ units of charge for each digon, one unit of charge for each 0-triangle, and $1/3$ unit of charge for each 1-triangle. Note that all other faces including 2- and 3-triangles already have sufficient charge.

To charge a digon d incident to a vertex v of G, we decrease $ch(v)$ by $4/3$ and increase $ch(d)$ by $4/3$; see Figure 4a. We say that v *contributes* charge to d.

To charge triangles, we proceed similarly to Ackerman [1] and Dujmović et al. [14, Theorem 7].

Consider a 1-triangle t_1. Let v be the unique vertex incident to t_1, and let $s_1 \in E'$ be the edge of t_1 opposite of v; see Figure 4b. Note that the endpoints of s_1 are intersection points in D'. Let f_1 be the face on the other side of s_1. If f_1 is a 0-quadrilateral, then we consider its edge $s_2 \in E'$ opposite to s_1 and the face f_2 on the other side of s_2. We continue iteratively until we meet a face f_k that is not a 0-quadrilateral. If f_k is a triangle, then all the faces $t_1, f_1, f_2, \ldots, f_k$ belong to the same empty 1-lens l incident to the vertex v of t_1. In this case, we decrease $ch(v)$ by $1/3$ and increase $ch(t_1)$ by $1/3$; see Figure 4a. Otherwise, f_k is not a triangle and $|f_k| + v(f_k) - 4 \geq 1$ (see Figure 4b). In this case, we decrease $ch(f_k)$ by $1/3$ and increase $ch(t_1)$ by $1/3$. We say that the face f_k contributes charge to the triangle t_1 *over* its side s_k.



(a) Vertex v contributes charge to a digon and two triangles contained in empty 1-lenses.

(b) Face f_k contributes charge to the 1-triangle t_1.

(c) Vertex v contributes charge to the 0-triangle t_0.

(d) 1-quadrilateral q contributes charge to at most two triangles.

Figure 4 Transferring charge from vertices and high-degree faces to small-degree faces.

For a 0-triangle t_0, we repeat the above charging over each side. If the last face on our path is a triangle t', then t_0 and t' are contained in an empty 1-lens (recall that D' does not contain empty 0-lenses) and t' is a 1-triangle incident to a vertex v of G. In this case, we decrease $ch(v)$ by 1/3 and increase $ch(t_0)$ by 1/3; see Figure 4c.

Thus, at the end of the second step, the charge of each digon and triangle f is at least $v(f)/3$. Note that the charge of f comes either from a higher-degree face or from a vertex v incident to an empty 1-lens containing f.

In the third step, we do not modify the charging any more, but we need to ensure that

(i) $ch(f) \geq v(f)/3$ still holds for each face f of G' with $|f| \geq 4$ and

(ii) $ch(v) \geq 0$ for each v of G.

We first show statement (i). Ackerman [1] noted that a face f with $|f| \geq 4$ can contribute charges over each of its edges at most once. Moreover, f can contribute at most one third unit of charge over each of its edges. Therefore, if $|f| + v(f) \geq 6$, then in the worst case (that is, f contributes charge over each of its edges) f still has a charge of $|f| + v(f) - 4 - |f|/3 \geq v(f)/3$. Thus, it remains to verify that 1-quadrilaterals and 0-pentagons, which initially had only one unit of charge, have a charge of at least 1/3 unit or zero, respectively, at the end of the second step.

A 1-quadrilateral q can contribute charge to at most two triangles since the endpoints of any edge of G' over which a face contributes charge must be intersection points in D'; see Figure 4d and recall that q now plays the role of f_k in Figure 4b.

A 0-pentagon cannot contribute charge to more than three triangles; see Lemma 2.10.

Now we show statement (ii). Recall that a vertex v can contribute charge to a digon incident to v or to at most two triangles contained in an empty 1-lens incident to v. Observe that two empty 1-lenses with either triangles or a digon taking charge from v cannot overlap; see Figure 4a. We show in Lemma 2.6 that v cannot be incident to more than four such empty 1-lenses. In the worst case, v contributes 4/3 units of charge to each of the at most four incident digons representing these empty 1-lenses. Thus, v has non-negative charge at the end of the second step. ◀

▶ **Lemma 2.6.** *In any simplified arc-RAC drawing, each vertex is incident to at most four non-overlapping empty 1-lenses.*

Proof. Let v be a vertex incident to some non-overlapping empty 1-lenses. Consider a small neighborhood of the vertex v in the simplified drawing and notice that in this neighborhood the simplified drawing is the same as the original arc-RAC drawing. Let l be one of the

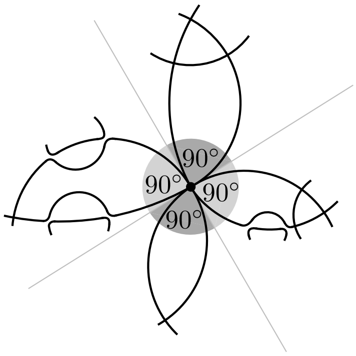


Figure 5 The edges of an empty 1-lens form a $\pi/2$ angle at the vertex of the lens.

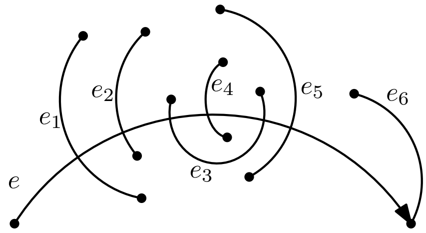


Figure 6 The relation $\Pi(\cdot\,;\cdot)$ does not necessarily describe *all* intersection points along an edge. Here, e.g., $\Pi(e; e_1, e_2, e_3, e_4, e_3, e_5, e_6)$ and $\Pi(e; e_1, e_3, e_4, e_5)$ both hold.

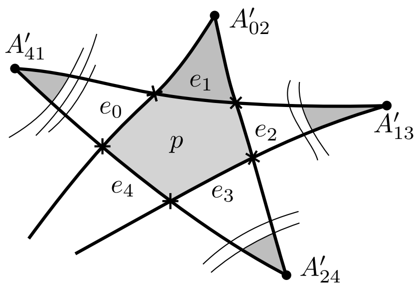
non-overlapping empty 1-lenses incident to v. Then l forms an angle of 90° between the two edges incident to v that form l; see Figure 5. This is due to the fact that the other "endpoint" of l is an intersection point where the two edges must meet at 90°. Thus v is incident to at most four non-overlapping empty 1-lenses. ◀

We now set the stage for proving Lemma 2.10, which shows that a 0-pentagon in a simplified drawing does not contribute charge to more than three triangles. The proof goes by a contradiction. Consider a 0-pentagon that contributes charge to at least four triangles in the simplified drawing. First, we examine which edges of this 0-pentagon cross; see Lemma 2.7. We then describe the order in which these edges share points in the simplified drawing and show that the original arc-RAC drawing must adhere to the same order; see Lemma 2.8. Finally, we use geometric arguments to show that, under these order constraints, an arc-RAC drawing of the edges does not exist; see Lemma 2.9.

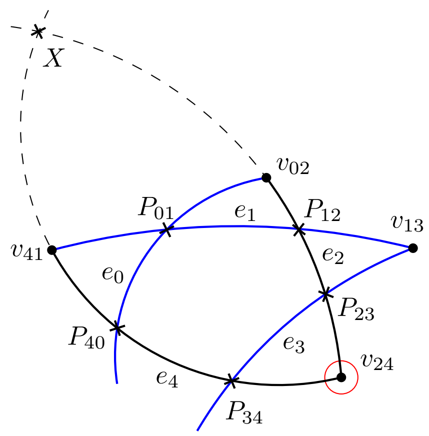
Let D be an arc-RAC drawing of some arc-RAC graph $G = (V, E)$, let D' be its simplification, and let p be a 0-pentagon that contributes charge to at least four triangles. Let $s_0, s_1, \ldots, s_4$ be the sides of p in clockwise order and denote the edges of G that contain these sides as $e_0, e_1, \ldots, e_4$ so that edge e_0 contains side s_0 etc. Since p contributes charge over at least four sides, these sides are consecutive around p. Without loss of generality, we assume that s_4 is the side over which p does not necessarily contribute charge.

For $i \in \{0, 1, 2, 3\}$, let t_i be the triangle that gets charge from p over the side s_i. The triangle t_i is bounded by the edges e_{i-1} and e_{i+1}. (Indices are taken modulo 5.) Note that all faces bounded by e_{i-1} and e_{i+1} that are between t_i and p must be 0-quadrilaterals. If t_i is a 1-triangle, then e_{i-1} and e_{i+1} are incident to the same vertex of the triangle. Otherwise, t_i is a 0-triangle and e_{i-1} and e_{i+1} cross at a vertex of the triangle. Let $A'_{i-1,i+1}$ denote this common point of e_{i-1} and e_{i+1}, and let $E_p = \{e_0, \ldots, e_4\}$; see Figure 7a.

We now describe the order in which the edges in E_p share points in D'. To this end, we orient the edges in E_p so that this orientation conforms with the orientation of a clockwise walk around the boundary of p in D'. In addition, we write $\Pi(e_k; e_{i_1}, e_{i_2}, \ldots, e_{i_l})$ if the edge e_k shares points (either crossing points or vertices of the graph) with the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_l}$ in this order with respect to the orientation of e_k; see Figure 6. (Note that we can have $\Pi(e_k; e_i, e_j, e_i)$ as edges may intersect twice. We will not consider more than two edges sharing the same endpoint.) Due to the order in which we numbered the edges in E_p, it holds in D' that $\Pi(e_0; e_4, e_1, e_2)$, $\Pi(e_3; e_1, e_2, e_4)$, and, for $i \in \{1, 2, 4\}$, $\Pi(e_i; e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2})$; see Figure 7a. Now we show that in D the order is the same. Obviously every pair of edges



(a) Notation: A 0-pentagon p in D' and the edges in E_p. The points of type $A'_{i-1,i+1}$ are either intersection points or vertices of G.



(b) Also in D, it holds that $\Pi(e_0; e_4, e_1, e_2)$, $\Pi(e_3; e_1, e_2, e_4)$, and, for $i \in \{1, 2, 4\}$, $\Pi(e_i; e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2})$.

Figure 7 A 0-pentagon cannot contribute charge to more than three triangles.

(e_{i-1}, e_{i+1}) that shares an endpoint in D' also shares an endpoint in D. Furthermore, every pair (e_i, e_{i+1}) or (e_{i-1}, e_{i+1}) of crossing edges crosses in D, too, because the simplification process does not introduce new pairs of crossing edges; see Observation 2.1.

▶ **Lemma 2.7.** *In the drawing D, the edges e_0 and e_3 do not cross.*

Proof. Assume that the edges e_0 and e_3 cross in D and notice that each of the pairs of edges (e_0, e_1), (e_1, e_2), and (e_2, e_3) forms a crossing in D' (see Figure 7a), and hence in D, too. For any arc e, let $\bar{e}$ denote the circle containing e. Recall that a family of *Apollonian circles* [9, 23] consists of two sets of circles such that each circle in one set is orthogonal to each circle in the other set. Thus, the pairs of circles $(\bar{e}_1, \bar{e}_3)$ and $(\bar{e}_0, \bar{e}_2)$ belong to such a family; the pair $(\bar{e}_1, \bar{e}_3)$ belongs to one set of the family and $(\bar{e}_0, \bar{e}_2)$ belongs to the other set. If not all of the circles in the family share the same point, which is the case for the circles $\bar{e}_0$, $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$, then one such set consists of disjoint circles. So either the pair $(\bar{e}_0, \bar{e}_2)$ or the pair $(\bar{e}_1, \bar{e}_3)$ must consist of disjoint circles. This is a contradiction because each of the two pairs shares a point in D' (see Figure 7a), and thus, in D. ◀

▶ **Lemma 2.8.** *In the drawing D, it holds that $\Pi(e_0; e_4, e_1, e_2)$, $\Pi(e_3; e_1, e_2, e_4)$, and, for each $i \in \{1, 2, 4\}$, $\Pi(e_i; e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2})$.*

Proof. Recall that in the drawing D', it holds that $\Pi(e_0; e_4, e_1, e_2)$, $\Pi(e_3; e_1, e_2, e_4)$, and, for each $i \in \{1, 2, 4\}$, $\Pi(e_i; e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2})$; see Figure 7a. Consider distinct indices $i, j, k \in \{0, 1, 2, 3, 4\}$ so that the edges e_i and e_j share points with e_k in this order in D', that is, $\Pi(e_k; e_i, e_j)$ in D'. We will show that the edges e_i and e_j share points with e_k in the same order in D, that is, $\Pi(e_k; e_i, e_j)$ in D. In other words, the order in which the edges in E_p share points in D is the same as in D'.

First, note that if the edge e_i or the edge e_j shares an endpoint with e_k, then e_i and e_j do not change the order in which they share points with e_k. This is due to the fact that the simplification process does not modify the graph. Therefore, e_i and e_j share points with e_k in the same order in D as in D', that is, $\Pi(e_k; e_i, e_j)$ in D.

Assume now that both e_i and e_j cross e_k.

If $(i, j) \in \{(0, 3), (3, 0)\}$, then, according to Lemma 2.7, the edges e_i and e_j do not cross in D, so they do not form an empty 0-lens in D, and thus, by Lemma 2.3, e_i and e_j cross e_k in the same order in D as in D', that is, $\Pi(e_k; e_i, e_j)$ in D.

Otherwise, the edges e_i and e_j share a point in D'; see Figure 7a. Therefore, by Observation 2.4, e_i and e_j do not form an empty 0-lens in D, and thus, by Lemma 2.3, e_i and e_j cross e_k in the same order in D as in D', that is, $\Pi(e_k; e_i, e_j)$ in D. ◀

Thus, we have shown that the order in which the edges in E_p share points in D is the same as in D', see Figure 7b. We show now that an arc-RAC drawing with this order does not exist; see Lemma 2.9. This is the main ingredient to prove Lemma 2.10, which says that a 0-pentagon in a simplified arc-RAC drawing contributes charge to at most three triangles.

For simplicity of presentation and without loss of generality, we assume that the points $A'_{i-1,i+1}$ are vertices of G, which we denote by $v_{i-1,i+1}$.

▶ **Lemma 2.9.** *The edges in E_p do not admit an arc-RAC drawing where it holds that $\Pi(e_0; e_4, e_1, e_2)$, $\Pi(e_3; e_1, e_2, e_4)$, and, for $i \in \{1, 2, 4\}$, $\Pi(e_i; e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2})$.*

Proof. Assume that the edges in E_p admit an arc-RAC drawing where they share points in the order indicated above. For $i \in \{0, \ldots, 4\}$, let $P_{i,i+1}$ be the intersection point of e_i and e_{i+1}; see Figure 7b. Note that on e_i, the point $P_{i-1,i}$ is before the point $P_{i,i+1}$ (due to $\Pi(e_i; e_{i-1}, e_{i+1})$).

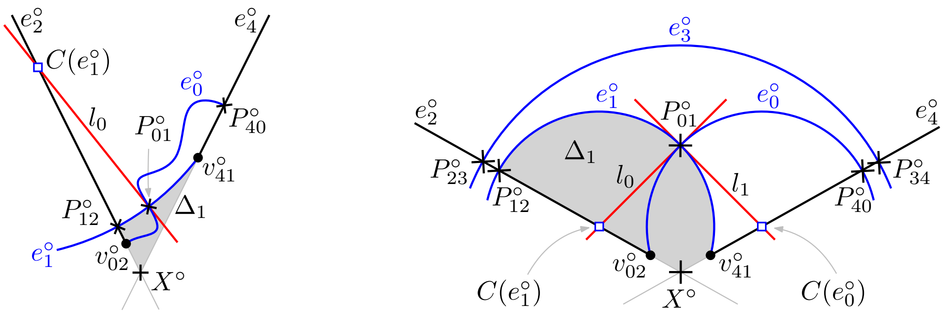
Recall that an *inversion* [23] with respect to a circle α, the *inversion circle*, is a mapping that takes any point $P \neq C(\alpha)$ to a point P' on the straight-line ray from $C(\alpha)$ through P so that $|C(\alpha)P'| \cdot |C(\alpha)P| = r(\alpha)^2$. Inversion maps each circle not passing through $C(\alpha)$ to another circle and each circle passing through $C(\alpha)$ to a line. The center of the inversion circle is mapped to the "point at infinity". It is known that inversion preserves angles.

We invert the drawing of the edges in E_p with respect to a small inversion circle centered at v_{24}. Let e_i° be the image of e_i, $v_{i-1,i+1}^\circ$ be the image of $v_{i-1,i+1}$ (v_{24}° is the point at infinity), and $P_{i,i+1}^\circ$ be the image of $P_{i,i+1}$. Because in the pre-image the arcs e_2 and e_4 pass through v_{24}, in the image e_2° and e_4° are straight-line rays. We assume that in the image e_2° meets e_4° at the point at infinity, that is, at v_{24}°. Then, taking into account that inversion is a continuous and injective mapping, the order in which the edges in E_p share points is the same in the image.

We consider two cases regarding whether the edges e_2 and e_4 belong to two different circles or not.

Case I: e_2 and e_4 belong to two different circles.
One of the intersection points of their circles is v_{24}, and we let X denote the other intersection point. Here we have that e_2° and e_4° are two straight-line rays meeting at infinity at v_{24}°. Their supporting lines are different and intersect at X°, which is the image of X; see Figure 8.

We now assume for a contradiction that the arc e_1° forms a concave side of the triangle $\Delta_1 = P_{12}^\circ v_{41}^\circ X^\circ$; see Figure 8a where the triangle is filled gray. (Symmetrically, we can show that the arc e_0° cannot form a concave side of the triangle $\Delta_0 = P_{40}^\circ v_{02}^\circ X^\circ$.) By Observation 1.2, $C(e_1^\circ)$ must lie on the ray e_2°. Since we assume that the arc e_1° forms a concave side of the triangle Δ_1, $C(e_1^\circ)$ and v_{02}° are separated by P_{12}° on e_2°. Consider the tangent l_0 to e_0° at P_{01}°. Again in light of Observation 1.2, l_0 has to go through $C(e_1^\circ)$ because e_0° and e_1° are orthogonal. On the one hand, v_{02}° is to the same side of l_0 as P_{12}°; see Figure 8a. On the other hand, l_0 separates P_{12}° and v_{41}° due to $\Pi(e_1; e_4, e_0, e_2)$. Moreover, l_0 does not separate v_{41}° and P_{40}° since it intersects the line of e_4° when leaving the gray triangle Δ_1. So the two points v_{02}° and P_{40}° of the same arc e_0° are separated by l_0, which is a tangent of this arc; contradiction.



(a) **(b)**

Figure 8 Illustration for the proof of Lemma 2.9 when e_2 and e_4 belong to two different circles. Image of the inversion with respect to the red circle in Figure 7b.

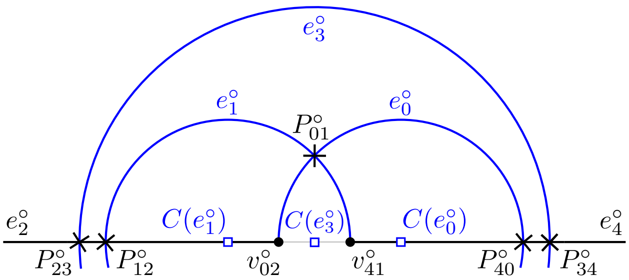


Figure 9 Illustration to the proof of Lemma 2.9 when e_2 and e_4 belong to the same circle. Image of the inversion with respect to the red circle in Figure 7b.

Thus, the arc e_1° forms a convex side of the triangle Δ_1, and e_0° forms a convex side of Δ_0; see Figure 8b. Now, due to Observation 1.2, $C(e_0^\circ)$ is between v_{41}° and P_{40}°, and $C(e_1^\circ)$ is between v_{02}° and P_{12}°, because that is where the tangents l_1 of e_1° and l_0 of e_0° in P_{01}° intersect the lines of e_4° and e_2°, respectively. Taking into account that $C(e_3^\circ) = X^\circ$, because e_3° is orthogonal to both e_2° and e_4°, we obtain that the points $C(e_3^\circ)$, $C(e_1^\circ)$, P_{12}°, P_{23}° appear on the line of e_2° in this order. Thus, the circle of e_1° is contained within the circle of e_3°. This is a contradiction because e_3° and e_1° must share a point; namely v_{13}°.

Case II: e_2 and e_4 belong to the same circle.
Here e_2° and e_4° are two disjoint straight-line rays on the same line l (meeting at infinity at v_{24}°); see Figure 9. We direct l as e_4° and e_2° (from right to left in Figure 9). Because e_0°, e_1°, and e_3° are orthogonal to l, their centers have to be on l. Due to our initial assumption, we have $\Pi(e_4; e_2, e_3, e_0, e_1)$ and $\Pi(e_2; e_0, e_1, e_3, e_4)$. Hence, along l, we have P_{34}°, P_{40}°, v_{41}°, (on e_4°) and then v_{02}°, P_{12}°, P_{23}° (on e_2°). Therefore, the circle of e_1° is contained in that of e_3°. Hence, e_1° does not share a point with e_3°; a contradiction. ◀

▶ **Lemma 2.10.** *A 0-pentagon in a simplified arc-RAC drawing contributes charge to at most three triangles.*

Proof. As discussed above, if a 0-pentagon formed by edges $e_0, e_1, \dots, e_4$ contributes charge to more than three triangles in a simplified drawing (see Figure 7a), then this implies the existence of an arc-RAC drawing where it holds that $\Pi(e_0; e_4, e_1, e_2)$, $\Pi(e_3; e_1, e_2, e_4)$ and, for $i \in \{1, 2, 4\}$, $\Pi(e_i; e_{i-2}, e_{i-1}, e_{i+1}, e_{i+2})$; see Figure 7b. This, however, contradicts Lemma 2.9. ◀

With the proofs of Lemmas 2.6 and 2.10 now in place, the proof of Theorem 2.5 is complete.

3 A Lower Bound for the Maximum Edge Density

In this section, we construct a family of arc-RAC graphs with high edge density. Our construction is based on a family of RAC_1 graphs of high edge density that Arikushi et al. [5] constructed. Let G be an embedded graph whose vertices are the vertices of the hexagonal lattice clipped inside a rectangle; see Figure 10a. The edges of G are the edges of the lattice and, inside each hexagon that is bounded by the cycle $(P_0, \dots, P_5)$, six additional edges $(P_i, P_{i+2 \bmod 6})$ for $i \in \{0, 1, \dots, 5\}$; see Figure 10b. We refer to a part of the drawing made up of a single hexagon and its diagonals as a *tile*. In Theorem 3.3 below, we show that each hexagon can be drawn as a regular hexagon and its diagonals can be drawn as two sets of arcs $A = \{\alpha_0, \alpha_1, \alpha_2\}$ and $B = \{\beta_0, \beta_1, \beta_2\}$, so that the arcs in A are pairwise orthogonal, the arcs in B are pairwise non-crossing, and for each arc in B intersecting another arc in A the two arcs are orthogonal; we use this construction to establish the theorem. In particular, the arcs in A form the 3-cycle (P_0, P_2, P_4), and the arcs in B form the 3-cycle (P_1, P_3, P_5).

We first define the radii and centers of the arcs in a tile and show that they form only orthogonal crossings. We use the geometric center of the tile as the origin of our coordinate system in the following analysis. We now discuss the arcs in A; then we turn to the arcs in B. For each $j \in \{0, 1, 2\}$, the arc α_j has radius $r_A = 1$ and center $C(\alpha_j) = (d_A \cos(\pi/6 + j\frac{2\pi}{3}), d_A \sin(\pi/6 + j\frac{2\pi}{3}))$, where $d_A = \sqrt{2/3}$ is the distance of the centers from the origin; see Figure 11a.

▶ **Lemma 3.1.** *The arcs in A are pairwise orthogonal.*

Proof. Consider the equilateral triangle $\triangle C(\alpha_0)C(\alpha_1)C(\alpha_2)$ formed by the centers of the three arcs in A. Because the origin is in the center of the triangle, the edge length of the triangle is $2d_A \cos \pi/6 = \sqrt{2}$, and so the distance between the centers of any two arcs is $\sqrt{2}$. The radii of the arcs are 1, hence by Observation 1.1, every two arcs are orthogonal. ◀

As in Figure 11b, for each $j \in \{0, 1, 2\}$, the arc β_j has radius $r_B = \sqrt{\frac{70+40\sqrt{3}}{6}}$ and center $C(\beta_j) = (d_B \cos(\frac{\pi}{2} + \frac{(j+1)2\pi}{3}), d_B \sin(\frac{\pi}{2} + \frac{(j+1)2\pi}{3}))$, where $d_B = \sqrt{\frac{1}{6}} + \sqrt{\frac{73+40\sqrt{3}}{6}}$ is the distance of the centers from the origin.

▶ **Lemma 3.2.** *If an arc in B intersects an arc in A, then the two arcs are orthogonal.*

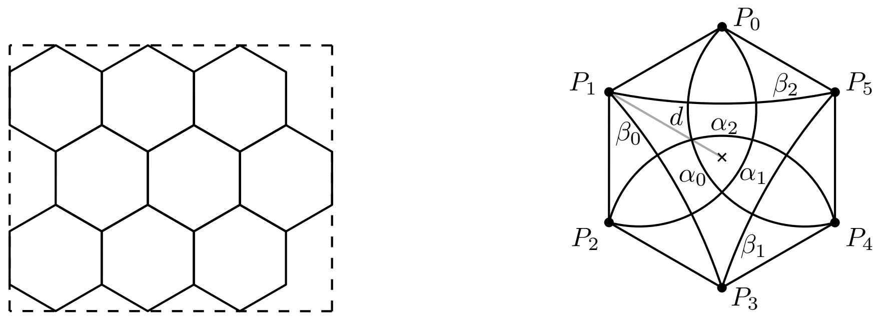
Proof. Let $i, j \in \{0, 1, 2\}$. If $j \neq i$, $\|C(\alpha_i) - C(\beta_j)\|^2 = \frac{76+40\sqrt{3}}{6} = 1 + \frac{70+40\sqrt{3}}{6} = r_A^2 + r_B^2$, so by Observation 1.1 α_i and β_j are orthogonal. Otherwise, for $i \in \{0, 1, 2\}$, $\|C(\alpha_i) - C(\beta_i)\| = \sqrt{\frac{112+64\sqrt{3}}{6}} > 1 + \sqrt{\frac{70+40\sqrt{3}}{6}} = r_A + r_B$, so α_i and β_i do not intersect. ◀

▶ **Theorem 3.3.** *For infinitely many values of n, there exists an n-vertex arc-RAC graph with $4.5n - O(\sqrt{n})$ edges.*

Proof. We first construct a tile and show that its drawing is indeed a valid arc-RAC drawing. Then it is easy to draw an embedded graph G with the claimed edge density.

Consider two circles α and β that intersect in two points of different distance from the origin. Let $X^-_{\alpha\beta}$ be the intersection point that is closer to the origin, and let $X^+_{\alpha\beta}$ be the intersection point further from the origin.

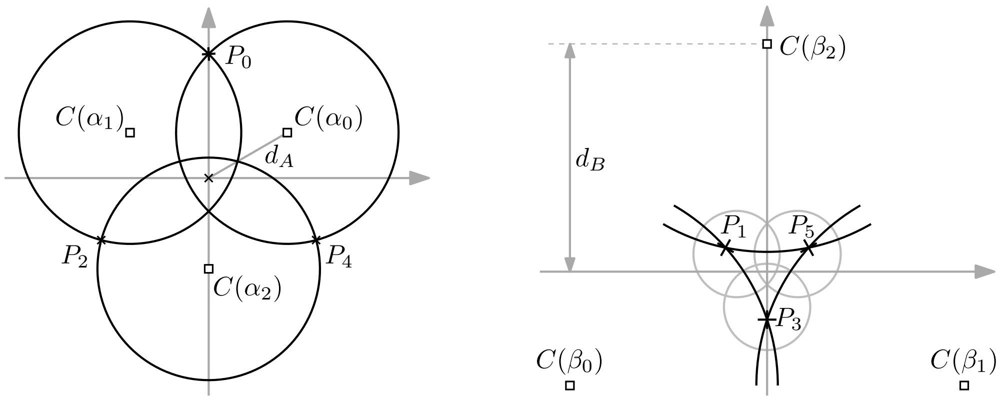
Let the vertices of the hexagon in a tile be $P_0 = X^+_{\alpha_0\alpha_1}$, $P_1 = X^-_{\beta_2\beta_0}$, $P_2 = X^+_{\alpha_1\alpha_2}$, $P_3 = X^-_{\beta_0\beta_1}$, $P_4 = X^+_{\alpha_2\alpha_0}$, and $P_5 = X^-_{\beta_1\beta_2}$. Due to the symmetric definitions of the arcs, the angle between two consecutive vertices of the hexagon is $\pi/3$. Moreover, by a simple



(a) The hexagonal lattice.

(b) A tile.

Figure 10 Tiling used for the lower-bound construction.



(a) Circles covering the arcs in A.

(b) Circles covering the arcs in B.

Figure 11 Construction for the lower bound on the maximum edge density of arc-RAC graphs.

computation, we see that for each $j \in \{0, 1, 2\}$ and with $d = \sqrt{1/2} + \sqrt{1/6}$ being the distance of the vertices of the hexagon from the origin, we have:

$$\begin{aligned} P_{2j} &= X^+_{\alpha_j \alpha_{j+1 \bmod 3}} = \left(d \cos(\tfrac{\pi}{2} + j\tfrac{2\pi}{3}),\, d \sin(\tfrac{\pi}{2} + j\tfrac{2\pi}{3})\right) \\ P_{2j+3 \bmod 6} &= X^-_{\beta_j \beta_{j+1 \bmod 3}} = \left(d \cos(\tfrac{\pi}{6} + (j+2)\tfrac{2\pi}{3}),\, d \sin(\tfrac{\pi}{6} + (j+2)\tfrac{2\pi}{3})\right). \end{aligned}$$

Thus, all the vertices of the hexagon are equidistant from its center, so the hexagon is regular. According to Lemmas 3.1 and 3.2 all crossings of the arcs that belong to the same tile are orthogonal. Now we argue that the arcs in A and B are contained in the regular hexagon. To this end, we show that the arcs do not intersect the relative interior of the edges of the hexagon. To see this, take, for example, the arc α_2, which connects P_2 and P_4. The line segment P_2P_4 is orthogonal to the side P_1P_2 of the hexagon. As the center of α_2 is below P_2P_4, the tangent of α_2 in P_2 enters the interior of the hexagon in P_2. Thus, α_2 does not intersect the relative interior of the edge P_1P_2 (or of any other edge) of the hexagon. Similarly we can show that the arcs in B do not intersect the relative interior of an edge of the hexagon. Therefore, each tile is an arc-RAC drawing, and G is an arc-RAC graph.

Almost all vertices of the lattice with the exception of at most $O(\sqrt{n})$ vertices at the lattice's boundary have degree 9 [5]. Hence G has $4.5n - O(\sqrt{n})$ edges. ◀

As any n-vertex RAC graph has at most $4n - 10$ edges [12], we obtain the following.

▶ **Corollary 3.4.** *The arc-RAC graphs are a proper superclass of the RAC_0 graphs.*

4 Open Problems and Conjectures

An obvious open problem is to tighten the bounds on the edge density of arc-RAC graphs in Theorems 2.5 and 3.3.

Another immediate question is the relation to RAC_1 graphs, which also extend the class of RAC_0 graphs. This is especially intriguing as the best known lower bound for the maximum edge density of RAC_1 graphs is indeed larger than our lower bound for arc-RAC graphs whereas there may be arc-RAC graphs that are denser than the densest RAC_1 graphs.

The relation between RAC_k graphs and 1-planar graphs is well understood [5–8, 10, 15]. What about the relation between arc-RAC graphs and 1-planar graphs? In particular, is there a 1-planar graph which is not arc-RAC?

We are also interested in the area required by arc-RAC drawings. Are there arc-RAC graphs that need exponential area to admit an arc-RAC drawing? (A way to measure this off the grid is to consider the ratio between the longest and the shortest edge in a drawing.)

Finally, the complexity of recognizing arc-RAC graphs is open, but likely NP-hard.

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