

# Relative Persistent Homology

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## Abstract

The alpha complex efficiently computes persistent homology of a point cloud  $X$  in Euclidean space when the dimension  $d$  is low. Given a subset  $A$  of  $X$ , relative persistent homology can be computed as the persistent homology of the relative Čech complex  $\check{C}(X, A)$ . But this is not computationally feasible for larger point clouds  $X$ . The aim of this note is to present a method for efficient computation of relative persistent homology in low dimensional Euclidean space. We introduce the relative Delaunay-Čech complex  $\text{Del}\check{C}(X, A)$  whose homology is the relative persistent homology. It is constructed from the Delaunay complex of an embedding of  $X$  in  $(d+1)$ -dimensional Euclidean space.

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## 1 Introduction

Persistent homology is receiving growing attention in the machine learning community. In that light, the scalability of persistent homology computations is of increasing importance. To date, the alpha complex is the most widely used method to compute persistent homology for large low-dimensional data sets.

Relative persistent homology has been considered several times in recent years. For example Edelsbrunner and Harer [8] have presented an application of relative persistent homology to estimate the dimension of an embedded manifold. Relative persistent homology is also a way to introduce the concept of extended persistence [5]. De Silva and others have shown that the relative persistent homology  $H_*(X, A_t)$  with an increasing family of sets  $A_t$  and a constant  $X = \cup_t A_t$ , and the corresponding relative persistent cohomology have the same barcode [6]. They also show that absolute persistent homology of  $A_t$  can be computed from this particular type of relative persistent homology. More recently, Pokorný and others [9] have used relative persistent homology to cluster two-dimensional trajectories. Some software, such as PHAT [2], even allows for the direct computation of relative persistent homology. For an example see the PHAT github repository.

Despite the fact that relative persistent homology has been considered in many different situations, we are not aware of a relative version of the alpha- or Delaunay-Čech complexes being used.

Our contributions are as follows.

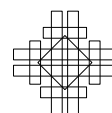
1. We give a new elementary proof that the Delaunay-Čech complex is level homotopy equivalent to the Čech complex. This has previously been shown using discrete Morse theory [1].
2. We extend this proof to the relative versions of the Delaunay-Čech complex and the Čech complex.



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3. We explain how the relative Delaunay-Čech complex can be constructed through embedding in a higher dimension.

Given finite  $A \subseteq X \subseteq \mathbb{R}^d$ , these contributions lead to the construction of a filtered simplicial complex  $\text{Del}\check{C}(X, A)$  with persistent homology isomorphic to the relative persistent homology of Čech persistence modules  $\check{C}_*(X; k)/\check{C}_*(A; k)$ . The underlying simplicial complex of  $\text{Del}\check{C}(X, A)$  is the Delaunay complex of an embedding  $Z$  of  $X$  in  $\mathbb{R}^{d+1}$  with the property that the projection  $\text{pr}: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  takes  $Z$  onto  $X$ . All simplices in the Delaunay complex of  $Z$  projecting to a subset of  $A$  are given filtration value zero. The filtration value of the remaining simplices in the Delaunay complex of  $Z$  is defined to be the Čech filtration value of their projection to  $\mathbb{R}^d$ . This is the content of Theorem 2.

This manuscript is structured as follows. In Section 2, we introduce relative persistent homology, and in Section 3 we construct the relative Delaunay-Čech complex. The rest of the paper serves to prove that the relative Delaunay-Čech complex is level homotopy equivalent to the relative Čech complex. Section 4 introduces Dowker Nerves, the theoretical foundation used in the proof. In Section 5, we introduce the alpha- and Delaunay-Čech complexes using the Dowker Nerve terminology and show that they are level homotopy equivalent to the Čech complex. Section 6 introduces the relative alpha- and Delaunay-Čech dissimilarities, and proves that their nerves are level homotopy equivalent to the relative Čech complex. Finally, in Section 7 we show that the nerve of the relative Delaunay-Čech dissimilarity is level homotopy equivalent to the relative Delaunay-Čech complex.

## 2 Relative persistent homology

Let  $X$  be a finite subset of Euclidean space  $\mathbb{R}^d$ . Given  $t > 0$ , the Čech complex  $\check{C}_t(X)$  of  $X$  is the abstract simplicial complex with vertex set  $X$  and with  $\sigma \subseteq X$  a simplex of  $\check{C}_t(X)$  if and only if there exists a point  $p \in \mathbb{R}^d$  with distance less than  $t$  to every point in  $\sigma$ . Varying  $t$  we obtain the filtered Čech complex  $\check{C}(X)$ .

Given a subset  $A$  of  $X$  we obtain an inclusion  $\check{C}(A) \subseteq \check{C}(X)$  of filtered simplicial complexes and an induced inclusion  $\check{C}_*(A; k) \subseteq \check{C}_*(X; k)$  of associated chain complexes of persistence modules over the field  $k$ . The *relative persistent homology of the pair  $(X, A)$*  is defined as the homology of the factor chain complex of persistence modules  $\check{C}_*(X; k)/\check{C}_*(A; k)$ .

For  $X$  of small cardinality, the relative persistent homology can be calculated as the reduced persistent homology of the relative Čech complex  $\check{C}(X, A)$ , where  $\sigma \subseteq X$  is a simplex of  $\check{C}(X, A)_t$  if either  $\sigma \subseteq A$  or  $\sigma \in \check{C}_t(X)$ . However, as the cardinality of  $X$  grows, this quickly becomes computationally infeasible.

## 3 The relative Delaunay-Čech complex

Before delving into theory we present a filtered simplicial complex that is level homotopy equivalent to the relative Čech complex  $\check{C}(X, A)$  of a pair of finite subsets  $A \subseteq X$  of Euclidean space  $\mathbb{R}^d$ . Two filtered simplicial complexes  $K = (K_t)_{t \geq 0}$  and  $L = (L_t)_{t \geq 0}$  are *level homotopy equivalent* if there exists a filtered simplicial map  $f: K \rightarrow L$  so that the geometric realization of  $f_t: K_t \rightarrow L_t$  is a homotopy equivalence for each  $t$ .

For convenience, we let  $B = X - A$  so that  $X$  is the disjoint union of  $A$  and  $B$ . Choose  $s > 0$  bigger than the maximal filtration values in the alpha complexes of  $A$  and  $B$ . The set

$$Z = A \times \{s\} \cup B \times \{-s\}$$

is an embedding of  $X$  in  $\mathbb{R}^{d+1}$ . Let  $\text{Del}(Z)$  be the Delaunay complex of  $Z$ .

► **Definition 1.** The relative Delaunay-Čech complex of the finite subsets  $A \subseteq X$  of  $\mathbb{R}^d$  is the filtered simplicial complex  $\text{Del}\check{C}(X, A)$  with  $\text{Del}(Z)$  as underlying simplicial complex and with filtration  $R: \text{Del}(Z) \rightarrow \mathbb{R}$  defined as follows: Given  $\sigma \in \text{Del}(Z)$ , let  $\text{pr}(\sigma)$  be the projection of  $\sigma \subseteq \mathbb{R}^{d+1}$  to  $\mathbb{R}^d$ . If  $\text{pr}(\sigma)$  is contained in  $A$  we let  $R(\sigma) = 0$ . Otherwise we let  $R(\sigma)$  be the radius of the smallest enclosing ball of  $\text{pr}(\sigma)$ .

► **Theorem 2.** The filtered simplicial complex  $\text{Del}\check{C}(X, A)$  is level homotopy equivalent to the relative Čech complex  $\check{C}(X, A)$ . In particular, the persistent homology of  $\text{Del}\check{C}(X, A)$  is isomorphic to the relative Čech persistent homology of the pair  $(X, A)$ . If  $X$  is of cardinality  $n$ , then  $\text{Del}\check{C}(X, A)$  contains  $O(n^{\lceil (d+1)/2 \rceil})$  simplices.

The statement about the size of the relative Delaunay-Čech complex is a direct consequence of the result of [10] that the Delaunay triangulation of  $n$  points in  $d + 1$  dimensions contains  $O(n^{\lceil (d+1)/2 \rceil})$  simplices

#### 4 Dowker nerves

A *dissimilarity* is a continuous function of the form  $\Lambda: X \times Y \rightarrow [0, \infty]$ , for topological spaces  $X$  and  $Y$ , where  $[0, \infty]$  is given the order topology. A *morphism*  $f: \Lambda \rightarrow \Lambda'$  of dissimilarities  $\Lambda: X \times Y \rightarrow [0, \infty]$  and  $\Lambda': X' \times Y' \rightarrow [0, \infty]$  consists of a pair  $(f_1, f_2)$  of continuous functions  $f_1: X \rightarrow X'$  and  $f_2: Y \rightarrow Y'$  so that for all  $(x, y) \in X \times Y$  the following inequality holds:

$$\Lambda'(f_1(x), f_2(y)) \leq \Lambda(x, y).$$

This notion of morphism is less general than for example [3, Definition 2.10], but it is simpler and suffices for our purposes.

The *Dowker Nerve*  $N\Lambda$  of  $\Lambda$  is the filtered simplicial complex described as follows: For  $t > 0$ , the simplicial complex  $N\Lambda_t$  consists of the finite subsets  $\sigma$  of  $X$  for which there exists  $y \in Y$  so that  $\Lambda(x, y) < t$  for every  $x \in \sigma$ .

Let  $f: \Lambda \rightarrow \Lambda'$  be a morphism of dissimilarities as above and let  $\sigma \in N\Lambda_t$ . Given  $y \in Y$  with  $\Lambda(x, y) < t$  for every  $x \in \sigma$  we see that

$$\Lambda'(f_1(x), f_2(y)) \leq \Lambda(x, y) < t,$$

so  $f_1(\sigma) \in N\Lambda'_t$ . Thus we have a simplicial map  $f: N\Lambda \rightarrow N\Lambda'$ .

Given  $x \in X$  and  $t > 0$ , the  $\Lambda$ -ball of radius  $t$  centered at  $x$  is the subset of  $Y$  defined as

$$B_\Lambda(x, t) = \{y \in Y, \mid \Lambda(x, y) < t\}.$$

The  $t$ -thickening of  $\Lambda$  is the subset of  $Y$  defined as

$$\Lambda^t = \bigcup_{x \in X} B_\Lambda(x, t).$$

Note that by construction the set of  $\Lambda$ -balls of radius  $t$  is an open cover of the  $t$ -thickening of  $\Lambda$ .

The *geometric realization*  $|K|$  of a simplicial complex  $K$  on the vertex set  $V$  is the subspace of the space  $[0, 1]^V$  of functions  $\alpha: V \rightarrow [0, 1]$  described as follows:

1. The subset  $\alpha^{-1}((0, 1])$  of  $V$  consisting of elements where  $\alpha$  is strictly positive is a simplex in  $K$ . In particular it is finite.
2. The sum of the values of  $\alpha$  is one, that is  $\sum_{v \in V} \alpha(v) = 1$ .

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With respect to the product topology, the subspace topology on  $|K|$  is called the *strong topology* on the geometric realization. It is convenient for construction of functions into  $|K|$ . The *weak topology* on  $|K|$ , which we are not going to use here, is convenient for construction of functions out of  $|K|$ . The homotopy type of  $|K|$  is the same for these two topologies [7, p. 355, Corollary A.2.9]. Given a simplex  $\sigma \in K$ , the simplex  $|\sigma|$  of  $|K|$  is the closure of

$$\{\alpha: V \rightarrow [0, 1] \mid \alpha(v) > 0 \text{ for all } v \in \sigma\}.$$

The simplices of  $|K|$  are the sets of this form.

A *partition of unity subordinate to the dissimilarity*  $\Lambda: X \times Y \rightarrow [0, \infty]$  consists of continuous maps  $\varphi^t: \Lambda^t \rightarrow |N\Lambda_t|$  such that given  $x \in X$ , the closure of the set

$$\{y \in Y \mid \varphi^t(y)(x) > 0\}$$

is contained in  $B_\Lambda(x, t)$ . We say that  $\Lambda$  is *numerable* if a partition of unity subordinate to  $\Lambda$  exists. If  $Y$  is paracompact, then every dissimilarity of the form  $\Lambda: X \times Y \rightarrow [0, \infty]$  is numerable [7, p. 355, paragraph after Definition A.2.10].

Let  $y \in \Lambda^t$  and let  $\{\varphi^t: \Lambda^t \rightarrow |N\Lambda_t|\}$  be a partition of unity subordinate to  $\Lambda$ . If  $x \in X$  with  $\varphi^t(y)(x) > 0$ , then  $\Lambda(x, y) < t$ . Therefore  $\varphi^t(y)$  is contained in a simplex  $|\sigma|$  in  $|N\Lambda_t|$  with  $\sigma$  contained in  $\{x \in X \mid \Lambda(x, y) < t\}$ . Every finite subset of this set is an element of  $N\Lambda_t$ . This implies that for  $s \leq t$  there is a simplex of  $|N\Lambda_t|$  containing both  $\varphi^s(y)$  and  $\varphi^t(y)$ . It also implies that given another partition of unity  $\{\psi^t: \Lambda^t \rightarrow |N\Lambda_t|\}$  subordinate to  $\Lambda$  there is a simplex of  $|N\Lambda_t|$  containing both  $\varphi^t(y)$  and  $\psi^t(y)$ . This is exactly the definition of contiguous maps, so  $\varphi^t$  and  $\psi^t$  are contiguous, and thus homotopic maps [7, Remark 2.22, p. 350]. Similarly, the diagram

$$\begin{array}{ccc} \Lambda^s & \xrightarrow{\varphi^s} & |N\Lambda_s| \\ \downarrow & & \downarrow \\ \Lambda^t & \xrightarrow{\varphi^t} & |N\Lambda_t| \end{array}$$

commutes up to homotopy [7, paragraph on the nerve starting on page 355 and ending on page 356].

Recall that a cover  $\mathcal{U}$  of  $Y$  is *good* if all non-empty finite intersections of members of  $\mathcal{U}$  are contractible. We now state the Nerve Lemma in the context of dissimilarities.

► **Theorem 3.** *If  $Y$  is paracompact and  $\Lambda: X \times Y \rightarrow [0, \infty]$  is a dissimilarity, then there exists a partition of unity  $\{\varphi^t: \Lambda^t \rightarrow |N\Lambda_t|\}$  subordinate to  $\Lambda$ . Moreover, if the cover of  $\Lambda^t$  by  $\Lambda$ -balls of radius  $t$  is a good cover, then  $\varphi^t$  is a homotopy equivalence.*

**Proof.** By the above discussion, we only need to note that the last statement about good covers is [11, Theorem 4.3]. ◀

A functorial version of the Nerve Lemma can be stated as follows:

► **Proposition 4.** *Let  $\Lambda: X \times Y \rightarrow [0, \infty]$  and  $\Lambda': X' \times Y' \rightarrow [0, \infty]$  be dissimilarities and let  $f = f_1 \times f_2: X \times Y \rightarrow X' \times Y'$  be a morphism  $f: \Lambda \rightarrow \Lambda'$  of dissimilarities. If  $\{\varphi^t: \Lambda^t \rightarrow |N\Lambda_t|\}$  is a partition of unity subordinate to  $\Lambda$  and  $\{\psi^t: (\Lambda')^t \rightarrow |N\Lambda'_t|\}$  is a partition of unity subordinate to  $\Lambda'$ , then for every  $t \geq 0$  the diagram*

$$\begin{array}{ccc} \Lambda^t & \xrightarrow{\varphi^t} & |N\Lambda_t| \\ f_2 \downarrow & & \downarrow |f_1| \\ (\Lambda')^t & \xrightarrow{\psi^t} & |N\Lambda'_t| \end{array}$$

commutes up to homotopy.

**Proof.** We show that the two compositions are contiguous. Recall that  $|f_1|$  takes a point  $\alpha: X \rightarrow [0, 1]$  of  $|N\Lambda_t|$  to the point  $|f_1|(\alpha)$  of  $|N\Lambda'_t|$  with  $|f_1|(\alpha)(x') = \sum_{f_1(x)=x'} \alpha(x)$ . Recall further that  $\varphi^t(y)$  is contained in a simplex  $|\sigma|$  in  $|N\Lambda_t|$ , where  $\sigma$  is contained in  $\{x \in X \mid \Lambda(x, y) < t\}$ . Then we have that for  $y \in \Lambda^t$ , the elements  $|f_1|(\varphi^t(y))$  and  $\psi^t(f_2(y))$  of  $|N\Lambda'_t|$  are contained in simplices  $|\sigma|$  and  $|\tau|$  respectively. Both  $\sigma$  and  $\tau$  are subsets of the set  $\{x' \in X' \mid \Lambda'(x', f_2(y)) < t\}$ . However every finite subset of this set is a simplex in  $N\Lambda'_t$ . In particular, so is the union  $\sigma \cup \tau$ . ◀

### 5 The alpha- and Delaunay-Čech complexes

Given a finite subset  $X$  of  $\mathbb{R}^d$  we define the Voronoi cell of  $x \in X$  as

$$\text{Vor}(X, x) = \{p \in \mathbb{R}^d \mid d(x, p) \leq d(y, p) \text{ for all } y \in X\}.$$

Let  $\mathbb{R}_d^d$  be Euclidean space with the discrete topology. The *discrete Delaunay dissimilarity* of  $X$  is defined as

$$\text{del}^X: X \times \mathbb{R}_d^d \rightarrow [0, \infty], \quad \text{del}^X(x, p) = \begin{cases} 0 & \text{if } p \in V(X, x) \\ \infty & \text{if } p \notin V(X, x). \end{cases}$$

The *Delaunay complex*  $\text{Del}(X)$  is the simplicial complex with vertex set  $X$  and with  $\sigma \subseteq X$  a simplex of  $\text{Del}(X)$  if and only if there exists a point in  $\mathbb{R}^d$  belonging to  $\text{Vor}(X, x)$  for every  $x \in \sigma$ . That is,  $\text{Del}(X) = N \text{del}_t^X$  for  $t > 0$ .

Note that with respect to Euclidean topology, the discrete Delaunay dissimilarity is not continuous, and hence  $\text{del}^X: X \times \mathbb{R}_d^d \rightarrow [0, \infty]$  is not a dissimilarity. One way to deal with this is to use the Nerve Lemma for absolute neighbourhood retracts [4, Theorem 8.2.1]. In order to use Theorem 3 and Proposition 4 from above, instead we construct a continuous version of the Delaunay dissimilarity.

Given a subset  $\sigma$  of  $X$  and  $p \in \mathbb{R}^d$ , let

$$d_{\text{Vor}}(p, \sigma) = \max\{d(p, \text{Vor}(X, x)) \mid x \in \sigma\},$$

where for any  $A \subseteq \mathbb{R}^d$ , we define  $d(p, A) = \inf_{a \in A} \{d(p, a)\}$ .

Note that if  $\sigma \notin \text{Del}(X)$ , the infimum  $\varepsilon_\sigma$  of the continuous function  $d_{\text{Vor}}(-, \sigma): \mathbb{R}^d \rightarrow \mathbb{R}$  is strictly positive. Choose  $\varepsilon > 0$  so that  $2\varepsilon < \varepsilon_\sigma$  for every subset  $\sigma$  of  $X$  that is not in  $\text{Del}(X)$ . Given  $x \in X$  we define the  $\varepsilon$ -thickened Voronoi cell  $\text{Vor}(X, x)^\varepsilon$  by

$$\text{Vor}(X, x)^\varepsilon = \{p \in \mathbb{R}^d \mid d(p, \text{Vor}(X, x)) < \varepsilon\}.$$

By construction the nerve of the open cover  $(\text{Vor}(X, x)^\varepsilon)_{x \in X}$  of  $\mathbb{R}^d$  is equal to  $\text{Del}(X)$ .

Let  $h: [0, \infty] \rightarrow [0, \infty]$  be the order preserving map

$$h(t) = \begin{cases} -\ln(1 - t/\varepsilon) & \text{if } t < \varepsilon \\ \infty & \text{if } t \geq \varepsilon. \end{cases} \tag{1}$$

For  $x \in X$ , let  $\text{Del}_x: \mathbb{R}^d \rightarrow [0, \infty]$  be the function defined by  $\text{Del}_x(p) = h(d(p, \text{Vor}(X, x)))$  so that  $\text{Del}_x(\text{Vor}(X, x)) = 0$  and  $\text{Del}_x(\mathbb{R}^d \setminus \text{Vor}(X, x)^\varepsilon) = \infty$ .

The Delaunay dissimilarity of  $X$  is defined as

$$\text{Del}^X: X \times \mathbb{R}^d \rightarrow [0, \infty], \quad \text{Del}^X(x, p) = \text{Del}_x(p).$$

By the above discussion we know that  $N \text{Del}_t^X = N \text{del}_t^X = \text{Del}(X)$  whenever  $t > 0$ .

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The Čech dissimilarity of  $X$  is defined as

$$d^X : X \times \mathbb{R}^d \rightarrow [0, \infty],$$

where  $d^X(x, p)$  is the Euclidean distance between  $x \in X$  and  $p \in \mathbb{R}^d$ .

The alpha dissimilarity of  $X$  is defined as

$$A^X = \max(\text{Del}^X, d^X) : X \times \mathbb{R}^d \rightarrow [0, \infty].$$

The Delaunay-Čech dissimilarity is defined as

$$\text{Del}\check{C}^X : X \times (\mathbb{R}^d \times \mathbb{R}^d) \rightarrow [0, \infty], \quad \text{Del}\check{C}^X(x, (p, q)) = \max(d^X(x, p), \text{Del}^X(x, q)).$$

Note the nerve of the dissimilarity

$$\text{del}\check{C}^X : X \times (\mathbb{R}^d \times \mathbb{R}^d) \rightarrow [0, \infty], \quad \text{del}\check{C}^X(x, (p, q)) = \max(d^X(x, p), \text{del}^X(x, q))$$

is identical to the nerve of  $\text{Del}\check{C}^X$ . Moreover, the Dowker nerves of the Delaunay-, Čech-, alpha- and Delaunay-Čech dissimilarities are the Delaunay-, Čech-, alpha- and Delaunay-Čech complexes respectively. For all these dissimilarities, the corresponding balls are convex, so the geometric realizations are homotopy equivalent to the corresponding thickenings. In order to see that the morphism  $A^X \rightarrow d^X$  of dissimilarities induces homotopy equivalences  $|NA_t^X| \xrightarrow{\cong} |Nd_t^X|$  it suffices to note that the corresponding map  $(A^X)^t \rightarrow (d^X)^t$  is the identity map. This holds because  $B_{A^X}(x, t) = B_{d^X}(x, t) \cap B_{\text{Del}^X}(x, t)$  and given  $y \in B_{d^X}(x, t)$  we have that  $y \in \text{Vor}(X, x')$  for some  $x' \in X$ . Thus,  $d^X(y, x')$  is minimal, so  $d^X(y, x') \leq d^X(y, x) < t$  and  $y \in B_{d^X}(x', t) \cap B_{\text{Del}^X}(x', t)$ .

In order to see that the morphism  $\text{Del}\check{C}^X \rightarrow d^X$  of dissimilarities induces homotopy equivalences  $|N\text{Del}\check{C}_t^X| \xrightarrow{\cong} |Nd_t^X|$  we use the following lemma:

► **Lemma 5.** *For every  $(p, q) \in (\text{Del}\check{C}^X)^t$ , the entire line segment between  $(p, p)$  and  $(p, q)$  is contained in  $(\text{Del}\check{C}^X)^t$ .*

**Proof.** In order not to clutter notation we omit superscript  $X$  on dissimilarities. Let  $\gamma : [0, 1] \rightarrow \mathbb{R}^d$  be the function  $\gamma(s) = (1-s)p + sq$ . We claim that given  $(p, q) \in \text{Del}\check{C}^t$  and  $s \in [0, 1]$  the point  $(p, \gamma(s)) = (p, (1-s)p + sq)$  is in  $\text{Del}\check{C}^t$ .

If  $(p, q) \in \text{Del}\check{C}^t$ , there exists a point  $x \in X$ , such that  $p \in B_d(x, t)$  and  $q \in B_{\text{Del}}(x, t)$ , that is,  $d(q, \text{Vor}(X, x)) < h^\leftarrow(t)$ , where  $h^\leftarrow$  is the generalized inverse of  $h$ . Pick  $q' \in \text{Vor}(X, x)$  so that  $d(q, q') < h^\leftarrow(t)$ . Let  $\gamma' : [0, 1] \rightarrow \mathbb{R}^d$  be the function  $\gamma'(s) = (1-s)p + sq'$ . Given  $s \in [0, 1]$ , suppose that the point  $(p, \gamma'(s)) = (p, (1-s)p + sq')$  is in  $\text{del}\check{C}^t$ . Then there exists  $x' \in X$  so that  $d(x', p) < t$  and  $\gamma'(s) \in V(X, x')$  and  $(p, \gamma(s))$  is in  $\text{Del}\check{C}^t$  since the distance between  $(1-s)p + sq$  and  $(1-s)p + sq'$  is less than  $h^\leftarrow(t)$  and  $d(\gamma'(s), \text{Vor}(X, x')) = 0$ .

We are left to show that, given  $s \in [0, 1]$ , the point  $(p, \gamma'(s)) = (p, (1-s)p + sq')$  is in  $\text{del}\check{C}^t$ . Suppose  $\gamma'(s) \in \text{Vor}(X, y)$  for some  $s \in [0, 1)$  and some  $y \in X$ . We claim that then  $p \in B_d(y, t)$ . To see this, we may without loss of generality assume that  $y \neq x$ . Let  $H$  be the hyperplane in between  $x$  and  $y$ , i.e.

$$H = \{z \in X \mid d(x, z) = d(y, z)\}.$$

Let

$$H_+ = \{z \in X \mid d(x, z) \geq d(y, z)\}$$

and

$$H_- = \{z \in X \mid d(x, z) \leq d(y, z)\}.$$

Since  $\gamma'(s) \in \text{Vor}(X, y)$  we have  $\gamma'(s) \in H_+$ . Since  $q \in \text{Vor}(X, x)$  we have  $q \in H_-$ . Since the line segment between  $p$  and  $q$  either is contained in  $H$  or intersects  $H$  at most once we must have  $p \in H_+$ . That is,  $d(y, p) \leq d(x, p) < t$ , so  $p \in B_d(y, t)$  as claimed. ◀

By Lemma 5, the inclusion

$$(d^X)^t = \cup_{x \in X} B_{d^X}(x, t) \rightarrow \cup_{x \in X} B_{\text{Del}\check{C}^X}(x, t) = (\text{Del}\check{C}^X)^t, \quad p \mapsto (p, p)$$

is a deformation retract. In particular it is a homotopy equivalence.

## 6 The relative Delaunay-Čech dissimilarity

In this section we consider two subsets  $X_1$  and  $X_2$  of  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ .

The *Voronoi diagram* of a finite subset  $X$  of  $\mathbb{R}^d$  is the set of pairs of the form  $(x, \text{Vor}(X, x))$  for  $x \in X$ , that is,

$$\text{Vor}(X) = \{(x, \text{Vor}(X, x)) \mid x \in X\}.$$

This may seem overly formal since the projection on the first factor gives a bijection  $\text{Vor}(X) \rightarrow X$ . However, when we work with Voronoi cells with respect to different subsets  $X_1$  and  $X_2$  of  $\mathbb{R}^d$  it may happen that  $\text{Vor}(X_1, x_1) = \text{Vor}(X_2, x_2)$  even when  $x_1 \neq x_2$ . The *Voronoi diagram* of the pair of subsets  $X_1$  and  $X_2$  of  $\mathbb{R}^d$  is the union

$$\text{Vor}(X_1, X_2) = \text{Vor}(X_1) \cup \text{Vor}(X_2).$$

The *discrete Delaunay dissimilarity* of  $X_1$  and  $X_2$  is defined as

$$\text{del}^{X_1, X_2}: \text{Vor}(X_1, X_2) \times \mathbb{R}_d^d \rightarrow [0, \infty], \quad \text{del}^{X_1, X_2}((x, V), p) = \begin{cases} 0 & \text{if } p \in V \\ \infty & \text{if } p \notin V. \end{cases}$$

The simplicial complex  $N \text{del}_t^{X_1, X_2}$  is independent of  $t > 0$ . It is the *Delaunay complex*  $\text{Del}(X_1, X_2)$  on  $X_1$  and  $X_2$ . In order to describe the homotopy type of this simplicial complex we thicken the Voronoi cells like we did in the previous section:

Given a subset  $\sigma$  of  $\text{Vor}(X_1, X_2)$  and  $p \in \mathbb{R}^d$ , let

$$d_{\text{Vor}}(p, \sigma) = \max\{d(p, V) \mid (x, V) \in \sigma\}.$$

Note that if  $\sigma \notin \text{Del}(X_1, X_2)$ , the infimum  $\varepsilon_\sigma$  of the continuous function  $d_{\text{Vor}}(-, \sigma): \mathbb{R}^d \rightarrow \mathbb{R}$  is strictly positive. Choose  $\varepsilon > 0$  so that  $2\varepsilon < \varepsilon_\sigma$  for every subset  $\sigma$  of  $\text{Vor}(X_1, X_2)$  that is not in  $\text{Del}(X_1, X_2)$ . Given  $(x, V) \in \text{Vor}(X_1, X_2)$  we define the  $\varepsilon$ -thickening  $V^\varepsilon$  of  $V$  by

$$V^\varepsilon = \{p \in \mathbb{R}^d \mid d(p, V) < \varepsilon\}.$$

By construction, the nerve of the open cover  $\{(x, V^\varepsilon)\}_{(x, V) \in \text{Vor}(X_1, X_2)}$  is equal to  $\text{Del}(X_1, X_2)$ . The Delaunay dissimilarity  $\text{Del}^{X_1, X_2}$  of  $X_1$  and  $X_2$  is defined as

$$\text{Vor}(X_1, X_2) \times \mathbb{R}^d \xrightarrow{\text{Del}^{X_1, X_2}} [0, \infty], \quad \text{Del}^{X_1, X_2}((x, V), p) = h(d(p, V))$$

for  $h: [0, \infty] \rightarrow [0, \infty]$  the order preserving map defined in Equation (1).



## 18:8 Relative Persistent Homology

The inclusion  $X_1 \rightarrow \text{Vor}(X_1, X_2)$  taking  $x \in X_1$  to  $(x, \text{Vor}(X_1, x))$  induces a morphism of dissimilarities  $\text{Del}^{X_1} \rightarrow \text{Del}^{X_1, X_2}$  and an inclusion of nerves  $N \text{Del}_t^{X_1} \subseteq N \text{Del}_t^{X_1, X_2}$  for  $t > 0$ .

Next, we construct the dissimilarity  $A^{X_1, X_2}$  as

$$\text{Vor}(X_1, X_2) \times \mathbb{R}^d \xrightarrow{A^{X_1, X_2}} [0, \infty], \quad ((x, V), p) \mapsto \max(d(x, p), \text{Del}^{X_1, X_2}((x, V), p)).$$

Also here we have an obvious inclusion  $NA_t^{X_1} \rightarrow NA_t^{X_1, X_2}$ , and the  $A^{X_1, X_2}$ -balls are convex so the nerve lemma yields a homotopy equivalence

$$|NA_t^{X_1, X_2}| \simeq \bigcup_{(x, V) \in \text{Vor}(X_1, X_2)} B_{A^{X_1, X_2}}((x, V), t) = \bigcup_{x \in X_1 \cup X_2} B_{d^{X_1 \cup X_2}}(x, t) = (X_1 \cup X_2)^t.$$

Finally, we construct the dissimilarity  $\text{Del}\check{C}^{X_1, X_2}$

$$\text{Vor}(X_1, X_2) \times (\mathbb{R}^d \times \mathbb{R}^d) \xrightarrow{\text{Del}\check{C}^{X_1, X_2}} [0, \infty], \\ ((x, V), (p, q)) \mapsto \max(d(x, p), \text{Del}^{X_1, X_2}((x, V), q))$$

Here again we have an obvious inclusion  $N \text{Del}\check{C}_t^{X_1} \rightarrow N \text{Del}\check{C}_t^{X_1, X_2}$ , and the  $\text{Del}\check{C}^{X_1, X_2}$ -balls are convex so the nerve lemma yields a homotopy equivalence

$$|N \text{Del}\check{C}_t^{X_1, X_2}| \simeq (\text{Del}\check{C}^{X_1, X_2})^t$$

The following variant of Lemma 5 implies that  $(\text{Del}\check{C}^{X_1, X_2})^t$  is a deformation retract of  $(X_1 \cup X_2)^t$ .

► **Lemma 6.** *For every  $(p, q) \in (\text{Del}\check{C}^{X_1, X_2})^t$ , the entire line segment between  $(p, p)$  and  $(p, q)$  is contained in  $(\text{Del}\check{C}^{X_1, X_2})^t$ .*

**Proof.** Given  $(p, q) \in (\text{Del}\check{C}^{X_1, X_2})^t = (\text{Del}\check{C}^{X_1})^t \cup (\text{Del}\check{C}^{X_2})^t$ , we have  $(p, q) \in (\text{Del}\check{C}^{X_i})^t$  for some  $i \in \{1, 2\}$ . Then also  $(p, p)$  lies in  $(\text{Del}\check{C}^{X_i})^t$ , and Lemma 5 proves the claim. ◀

## 7 Nerve of the relative Delaunay-Čech dissimilarity

In this section we show that the nerve of the relative Delaunay dissimilarity is level homotopy equivalent to the relative Dealunay-Čech complex.

We fix some notation used in this section:  $X_1 \subseteq \mathbb{R}^d$  and  $X_2 \subseteq \mathbb{R}^d$  are finite subsets. We let  $s$  be a positive real number, we let  $Z = X_1 \times \{s\} \cup X_2 \times \{-s\}$  and we let  $\text{pr}: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  be the projection omitting the last coordinate.

► **Lemma 7.** *The projection  $\text{pr}: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  induces a surjection*

$$\text{Vor}(Z) \xrightarrow{g} \text{Vor}(X_1, X_2), \quad ((x, s), V) \mapsto (x, V(X_1, x)), \quad ((x, -s), V) \mapsto (x, V(X_2, x)),$$

with  $\text{pr}(V) \subseteq V(X_i, x)$  for  $x \in X_i$ . Given  $(x, V) \in \text{Vor}(X_1, X_2)$  the fiber  $g^{-1}((x, V))$  consists of all elements of  $\text{Vor}(Z)$  of the form  $((x, a), V)$  for  $a \in \{\pm s\}$ .

**Proof.** We show that  $\text{pr}(V) \subseteq V(X_1, x_1)$  for  $((x_1, s), V) \in \text{Vor}(Z)$  with  $x_1 \in X_1$ . Given  $(p, r) \in V$  we have for all points of the form  $(x'_1, s)$  for  $x'_1 \in X_1$  that  $d((p, r), (x_1, s)) \leq d((p, r), (x'_1, s))$ . This implies that  $d(p, x_1) \leq d(p, x'_1)$ , and thus  $p \in V(X_1, x_1)$ . We conclude that  $\text{pr}(V) \subseteq V(X_1, x_1)$ . An analogous argument applies for elements of the form  $((x_2, -s), V)$  in  $\text{Vor}(Z)$ . ◀



Let  $s_1$  be larger than the largest filtration value of the alpha complex of  $X_1$ . Then the function  $j_1: \text{Vor}(X_1) \rightarrow \text{Vor}(Z)$  defined by  $j_1(x_1, V) = ((x_1, s), V(Z, (x_1, s)))$  induces a simplicial map of nerves  $\text{del}(X_1) \rightarrow \text{del}(Z)$  for all  $s > s_1$ . Similarly, there is a simplicial map  $\text{del}(X_2) \rightarrow \text{del}(Z)$  for all  $s > s_2$  when  $s_2$  is larger than all filtration values of the alpha complex of  $X_2$ . Let  $s(X_1, X_2) = \max(s_1, s_2)$ .

Recall, from the previous two sections, that  $\varepsilon_\sigma$  is the infimum of the continuous function  $d_{\text{Vor}}(-, \sigma): \mathbb{R}^d \rightarrow \mathbb{R}$ . Choose  $\varepsilon > 0$  satisfying the following two criteria:

1.  $2\varepsilon < \varepsilon_\sigma$  for every subset  $\sigma$  of  $\text{Vor}(X_1, X_2)$  that is not in  $\text{Del}(X_1, X_2)$ .
2.  $2\varepsilon < \varepsilon_\sigma$  for every subset  $\sigma$  of  $\text{Vor}(Z)$  that is not in  $\text{Del}(Z)$ .

Let  $h: [0, \infty) \rightarrow [0, \infty)$  be the order preserving map defined in Equation (1), and let  $\text{Del}^Z$  and  $\text{Del}^{X_1, X_2}$  be constructed using  $h$ . We define a new dissimilarity

$$D: \text{Vor}(Z) \times (\mathbb{R}^d \times \mathbb{R}^{d+1}) \rightarrow [0, \infty], \quad D((z, V), (p, q)) = \max(d(\text{pr}(z), p), \text{Del}^Z((z, V), q)).$$

Note that the underlying simplicial complex  $\bigcup_{t>0} ND_t$  of the nerve of  $D$  is the Delaunay complex  $\text{del}(Z)$ . The filtration value of  $\sigma \in \text{del}(Z)$  in the nerve of  $D$  is the filtration value of  $g(\sigma)$  in the nerve of  $\text{Del}\check{C}^{X_1, X_2}$ .

► **Proposition 8.** *Let  $X_1 \subseteq \mathbb{R}^d$  and  $X_2 \subseteq \mathbb{R}^d$  be finite. Choose  $s > s(X_1, X_2)$ . Then  $\text{Vor}(Z) \xrightarrow{g} \text{Vor}(X_1, X_2)$  and  $\text{id} \times \text{pr}: \mathbb{R}^d \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  form a morphism*

$$f = (g, \text{id} \times \text{pr}): D \rightarrow \text{Del}\check{C}^{X_1, X_2}$$

of dissimilarities inducing a homotopy equivalence

$$g: ND_t \rightarrow N \text{Del}\check{C}_t^{X_1, X_2}$$

for every  $t > 0$ .

**Proof.** For  $i = 1, 2$  the inclusion  $\text{pr}(V) \subseteq V(X_i, x)$  for  $((x, (-1)^{i-1}s), V) \in \text{Vor}(Z)$  implies that

$$\text{Del}^{X_1, X_2}(g(z, V), \text{pr}(q)) \leq \text{Del}^Z((z, V), q)$$

for all  $((z, V), q) \in \text{Vor}(Z)$ . So we have a morphism  $f = (g, \text{id} \times \text{pr}): D \rightarrow \text{Del}\check{C}^{X_1, X_2}$ .

In order to show that  $g$  induces a homotopy equivalence of geometric realizations, by the Nerve Lemma, it suffices to show that given a simplex  $\sigma$  of  $N \text{Del}\check{C}_t^{X_1, X_2}$ , the inverse image  $g^{-1}(\sigma)$  is a simplex of  $ND_t$ . Let  $p$  be a point in the intersection of the Voronoi cells in  $\sigma$ . Write  $g^{-1}(\sigma) = \tau_1 \cup \tau_2$ , where  $\tau_1$  consists of Voronoi cells with centers at height  $s$  and  $\tau_2$  consists of Voronoi cells with centers at height  $-s$ . Let  $\sigma_1 = \{(x_1, s) \mid (x_1, V(X_1, x_1)) \in \sigma\}$  and  $\sigma_2 = \{(x_2, -s) \mid (x_2, V(X_2, x_2)) \in \sigma\}$ .

Suppose that  $\tau_2$  is empty. Then actually  $\sigma \in \text{Del}\check{C}_t^{X_1}$ , and since  $s > s_1$  we know that  $j_1(\sigma) \in \text{del}(Z)$ . Since  $g \circ j_1$  is the inclusion of  $\text{Vor}(X_1)$  in  $\text{Vor}(X_1, X_2) = \text{Vor}(X_1) \cup \text{Vor}(X_2)$  we know that  $j_1(\sigma) \subseteq g^{-1}(\sigma) = \tau_1$  and that  $j_1(\sigma) \in ND_t$ . On the other hand, since  $\tau_2$  is empty, by Lemma 7 we know that  $g^{-1}(\sigma)$  is contained in  $j_1(\sigma)$ , so they must be equal. We conclude that  $g^{-1}(\sigma)$  is a simplex of  $ND_t$ . A similar argument applies when  $\tau_1$  is empty.

In the remaining case where both  $\tau_1$  and  $\tau_2$  are nonempty, the function

$$f: \mathbb{R}^{d+1} \rightarrow \mathbb{R}, \quad f(a) = d_{\text{Vor}}(a, \sigma_1) - d_{\text{Vor}}(a, \sigma_2)$$

has  $f((p, -s)) > 0$  and  $f((p, s)) < 0$ . By the intermediate value theorem there exists  $t \in [-s, s]$  with  $f(p, t) = 0$ . Since  $(p, t)$  has the same distance to all elements of  $\sigma_1$  and also has the same distance to all elements of  $\sigma_2$  we conclude that  $(p, t)$  is in the intersection of the Voronoi cells in  $g^{-1}(\sigma) = \tau_1 \cup \tau_2$ . Thus  $\text{Del}\check{C}^Z((z, V), p) = 0$  and  $d(\text{pr}(z), p) < t$  for all  $(z, V) \in g^{-1}(\sigma)$ . In particular  $g^{-1}(\sigma) \in ND_t$ . ◀

We are now ready to compute persistent homology of  $X_1 \cup X_2$  relative to  $X_1$ . The relative Delaunay-Čech complex  $\text{Del}\check{C}(X_1 \cup X_2, X_1)$  is the filtered simplicial complex with  $\text{Del}\check{C}(X_1 \cup X_2, X_1)_t = j_1(\text{del}(X_1)) \cup ND_t$ . Note that this is consistent with Definition 1.

► **Theorem 9.** *Let  $X_1 \subseteq \mathbb{R}^d$  and  $X_2 \subseteq \mathbb{R}^d$  be finite. Choose  $s > s(X_1, X_2)$ . Then the geometric realization of the filtered simplicial complex  $\text{Del}\check{C}(X_1 \cup X_2, X_1)$  is level homotopy equivalent to the filtered space  $t \mapsto (X_1 \cup X_2)^t / X_1^t$ . In particular, there is an isomorphism*

$$(H_*(\text{Del}\check{C}(X_1 \cup X_2, X_1)_t))_{t>0} \cong (H_*((X_1 \cup X_2)^t, X_1^t))_{t>0}$$

of persistence modules.

**Proof.** Since  $j_1(\text{del}(X_1))$  is contractible, the geometric realization of  $\text{Del}\check{C}(X_1 \cup X_2, X_1)_t$  is homotopy equivalent to the quotient space  $|\text{Del}\check{C}(X_1 \cup X_2, X_1)_t| / |j_1(\text{del}(X_1))|$ . This quotient space is homeomorphic to  $|ND_t| / |ND_t \cap j_1(\text{Del}(X_1))|$ . By Proposition 8 the map  $g: ND_t \rightarrow N\text{Del}\check{C}_t^{X_1, X_2}$  induces a homotopy equivalence of geometric realizations. Moreover  $g$  induces an isomorphism  $ND_t \cap j_1(\text{Del}(X_1)) \rightarrow N\text{Del}\check{C}_t^{X_1}$ . Combining these two statements,  $g$  induces a homotopy equivalence  $|ND_t| / |ND_t \cap j_1(\text{Del}(X_1))| \rightarrow |N\text{Del}\check{C}_t^{X_1, X_2}| / |N\text{Del}\check{C}_t^{X_1}|$ . The space  $|N\text{Del}\check{C}_t^{X_1, X_2}|$  is homotopy equivalent to the Euclidean  $t$ -thickening  $(X_1 \cup X_2)^t$  of  $X_1 \cup X_2$  and  $|N\text{Del}\check{C}_t^{X_1}|$  is homotopy equivalent to the Euclidean  $t$ -thickening  $X_1^t$  of  $X_1$ . ◀

This concludes the proof of Theorem 2.

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