Cut Vertices in Random Planar Maps

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Abstract

The main goal of this paper is to determine the asymptotic behavior of the number X_n of cut-vertices in random planar maps with n edges. It is shown that $X_n/n \to c$ in probability (for some explicit $c > 0$). For so-called subcritial subclasses of planar maps like outerplanar maps we obtain a central limit theorem, too.

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1 Introduction

A *planar map* is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane. A map is *rooted* if a vertex v and an edge e incident with v are distinguished, and are called the *root-vertex* and *root-edge*, respectively. Sometimes the root-edge is considered as directed away from the root-vertex. In this sense, the face to the right of e is called the *root-face* and is usually taken as the outer face. All maps in this paper are rooted.

The enumeration of rooted maps is a classical subject, initiated by Tutte in the 1960's. Tutte (and Brown) introduced the technique now called "the quadratic method" in order to compute the number M_n, $n \in \mathbb{N}$, of rooted maps with n edges, proving the formula

$$M_n = \frac{2(2n)!}{(n+2)!n!}3^n.$$

This was later extended by Tutte and his school to several classes of planar maps: 2-connected, 3-connected, bipartite, Eulerian, triangulations, quadrangulations, etc.

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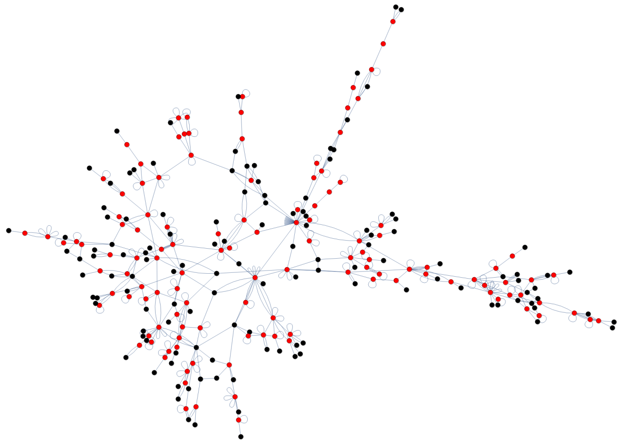

Figure 1 A randomly generated planar map with 500 edges, embedded using a spring-electrical method. Cut vertices are coloured red.

The standard random model is to assume that every map with n edges appears with the same probabiltiy $1/M_n$. Within this random setting several shape parameters of random planar maps have been studied so far, see for example [2, 8, 10, 9]. However, the number of cut vertices does not appear to have been studied. (A cut vertex is a vertex that disconnects a graph when it is removed). Figure 1 displays a randomly generated planar map with cut vertices coloured red. It is natural to expect that the number of cut vertices is asymptotically linear – and this is in fact true.

▶ **Theorem 1.** *Let X_n denote the number of cut vertices in random planar maps with n edges. Then we have*

$$\frac{X_n}{n} \xrightarrow{p} \frac{5-\sqrt{17}}{4} \approx 0.219223594.$$

Moreover, we have $\mathbb{E}[X_n] = (5-\sqrt{17})/4 \cdot n + O(1)$.

We provide two different approaches for Theorem 1. First, by a probabilistic approach, that makes use of the local convergence of random planar maps re-rooted at a uniformly selected vertex (see Section 3). Second, by a combinatorial approach based on generating functions and singularity analysis (see Section 4). The combinatorial approach yields additional information on related generating functions and on error terms, and one obtains more precise information on the expected value (see Section 4).

We conjecture that the number X_n additionally satisfies a normal central limit theorem. The intuition behind this is that X_n may be written as the sum of n seemingly weakly dependent indicator variables. The conjecture is backed up by numerical simulations we carried out, see the histogram in Figure 2. Sampling over $2 \cdot 10^5$ planar maps with $n = 5 \cdot 10^5$ edges, we obtained an average value of approximately $\mathbf{0.219223}677 \cdot n$ cut vertices. This value is already very close to the exact asymptotic value obtained in Theorem 1. The variance was approximately $0.082788 \cdot n$. It is actually possible to extend our combinatorial approach and the corresponding asymptotic analysis to second moments that leads to the precise asympotic behavior of the variance (details will be given in the journal version of this Extended Abstract).

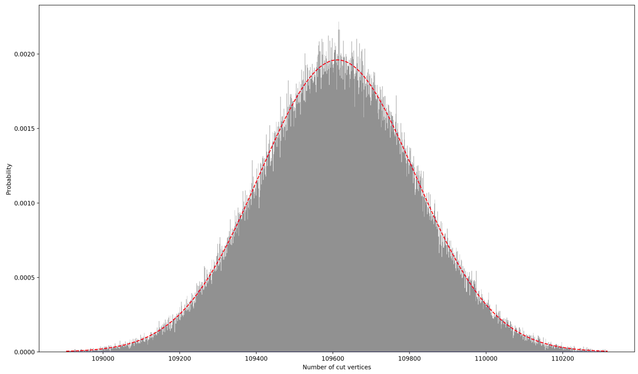

Figure 2 Histogram for the number of cut vertices in more than $2 \cdot 10^5$ randomly generated planar maps with $n = 5 \cdot 10^5$ edges each.

One important property of random planar maps that we will use in the proof of Theorem 1 is that it has a *giant 2-connected component* of linear size. There are, however, several interesting subclasses of planar maps, for example outerplanar maps (that is, all vertices are on the outer face), where all 2-connected components are (in expectation) of bounded size. Informally this means that on a global scale the map looks more or less like a tree. Such classes of maps are called subcritical – we will give a precise definition in Section 2.

▶ **Theorem 2.** *Let X_n denote the number of cut vertices in random outerplanar (or bipartite outerplanar) maps of size n. Then X_n satisfies a central limit theorem of the form*

$$\frac{X_n - cn}{\sqrt{\sigma^2 n}} \xrightarrow{d} N(0,1)$$

where $c = 1/4$ and $\sigma^2 = 5/32$ in the outerplanar case and $c = (\sqrt{3}-1)/2$ and $\sigma^2 = (11\sqrt{3}-17)/12$ in the bipartite outerplanar case.

We will give a generating function based proof for the case of outerplanar graphs in Section 5. (The proof for the bipartite outerplanar case is very similar to that.)

2 Generating Functions for Planar Maps

The generating function of planar maps is given by

$$M(z) = \sum_{n\ge 0} M_n z^n = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2} = 1 + 2z + 9z^2 + 54z^3 + \cdots, \tag{1}$$

This can be shown in various ways, for example by the so-called quadratic method, where it is necessary to use an additional *catalytic variable* u that takes care of the root face valency. The corresponding generating function $M(z,u)$ (u takes care of the root face valency or equivalently by duality of the root degree) satisfies then

$$M(z,u) = 1 + zu^2 M(z,u)^2 + uz\frac{uM(z,u) - M(z)}{u-1} \tag{2}$$

which follows from a combinatorial consideration (removal of the root edge). Then this relation can be used to obtain (1) and to solve the counting problem. We refer to [11, Sec. VII. 8.2.].

Similarly it is possible to count also the number of non-root faces (with an additional variable x) which leads to the relation[1]

$$M(z,x,u) = 1 + zu^2 M(z,x,u)^2 + uzx\frac{uM(z,x,u) - M(z,x,1)}{u-1}.$$

Note that by duality $M(z,x,1)$ can be also seen as the generating function that is related to edges and non-root vertices of planar maps.

A planar map is 2-connected if it does not contain cut vertices. There are various ways to obtain relations for the corresponding generating function $B(z,x,u)$ of 2-connected planar maps – as above z takes care of the number of edges, x of the number of non-root faces, and u of the valency of the root face. By using the fact, that a 2-connected planar map, where we delete the root edge, decomposes into a sequence of 2-connected maps or single edges, we obtain the relation

$$B(z,x,u) = zxu\frac{\frac{uB(z,x,1)-B(z,x,u)}{1-u} + zu}{1 - \frac{uB(z,x,1)-B(z,x,u)}{1-u} - zu}. \tag{3}$$

We can use, for example, the quadratic method to solve this equation or we just check that we have

$$\begin{aligned} B(z,x,u) = &-\frac{1}{2}\left(1-(1+U-V+UV-2U^2V)u + U(1-V)^2u^2\right) \\ &+\frac{1}{2}(1-(1-V)u)\sqrt{1-2U(1+V-2UV)u+U^2(1-V)^2u^2}, \end{aligned} \tag{4}$$

where $U = U(x,y)$ and $V = V(x,y)$ are given by the algebraic equations

$$z = U(1-V)^2, \qquad xz = V(1-U)^2. \tag{5}$$

Note that in the above counting procedure we do not take the one-edge map (nor the one-edge loop) into account. Therefore we have to add the term zu on the right hand side in order to cover the case of a single edges that might occur in the above mentioned decomposition into a sequence of 2-connected maps or single edges.

Sometimes it is more convenient to include the one-edge map as well as the one-edge loop to 2-connected maps (since they have no cut-points) which leads us to the alternative generating function

$$A(z,x,u) = B(z,x,u) + zxu + zu^2.$$

Now a general rooted planar map can be obtained from a 2-connected rooted map (including the one-edge map as well as the one-edge loop) by adding to every corner a rooted planar map (a *corner* of a planar map is the angle region between two adjacent half-edges of the same vertex – note that there are $2n$ corners if there are n edges):

$$M(z,x,u) = 1 + A\left(zM(z,x,1)^2, x, \frac{uM(z,x,u)}{M(z,x,1)}\right). \tag{6}$$

[1] By abuse of notation we will use for simplicity for $M(z)$, $M(z,u)$, $M(z,x,u)$ the same symbol.

If $x = 1$ then $V(z, 1)$ (and $U(z, 1)$) satisfies the equation $z = V(1 - V)^2$ and, thus, the dominant singularity of $V(z, 1)$ (and $U(z, 1)$) is $z_0 = \frac{4}{27}$, and we also have $V(z_0, 1) = \frac{1}{3}$ (as well as $U(z_0, 1) = \frac{1}{3}$). Hence, from (4) it follows that the function $A(z, 1, 1)$ has its dominant singularity at $z_0 = \frac{4}{27}$, too. On the other hand, by (1) $M(z)$ has its dominant singularity at $z_1 = \frac{1}{12}$ and we also have $M(z_1) = \frac{4}{3}$. Since $z_1 M(z_1)^2 = \frac{4}{27} = z_0$, the singularities of $M(z)$ and $A(z, 1, 1)$ interact. We call such a situation *critical*.

The relation (6) can also be seen as a way how all planar maps can be constructed (recursively) from 2-connected planar maps – which reflects the block-decomposition of a connected graph into its 2-connected components. Actually this principle holds, too, for several sub-classes of planar maps. As an example we consider outerplanar maps – these are maps, in which all vertices are on the outer face. Here the generating function $M_O(z)$ of outerplanar (rooted) maps satisfies

$$M_O(z) = \frac{z}{1 - A_O(M(z))}, \tag{7}$$

where $A_O(z)$ is the generating function for polygon dissections (plus a single edge) where z marks non-root vertices, which satisfies

$$2A_O(z)^2 - (1 + z)A_O(z) + z = 0. \tag{8}$$

Note that the dominant singularity of $A_O(z)$ is $z_{0,O} = 3 - 2\sqrt{2}$, whereas the dominant singularity of $M_O(z)$ is $z_{1,O} = \frac{1}{8}$ and we have $M_O(z_{1,O}) = \frac{1}{18}$. So we clearly have

$$M_O(z_{1,O}) < z_{0,O}, \tag{9}$$

so that the singularities of $M_O(z)$ and $A_O(z)$ do not interact. Such a situation is called *subcritical*.

3 A probabilistic approach to cut vertices of random planar maps

We let M_n denote the uniform random planar map with n edges. It is known that M_n and related models of random planar maps admit local limits that describe the asymptotic vicinity of a typical corner, see [16, 1, 13, 4, 6, 15].

In a recent work by Drmota and Stufler [9, Thm. 2.1], a related limit object M_∞ was constructed that describes the asymptotic vicinity of a uniformly selected *vertex* v_n of M_n instead. That is, M_∞ is a random infinite but locally finite planar map with a marked vertex such that

$$(\mathsf{M}_n, v_n) \xrightarrow{d} \mathsf{M}_\infty \tag{10}$$

in the local topology.

In the present section we provide a probabilistic proof of Theorem 1. There are two steps. The first proves a law of large numbers for the number X_n of cut vertices in M_n without determining the limiting constant explicitly:

▶ **Lemma 3.** *We have $X_n/n \xrightarrow{p} p/2$, with $p > 0$ the probability that the root of M_∞ is a cut vertex.*

The factor 1/2 originates from the fact that the number of vertices in the random map M_n has order $n/2$. We prove Lemma 3 in Section 3.4 below. In the second step, we determine this limiting probability (the proof is given in Section 3.6),

▶ **Lemma 4.** *It holds that $p = \frac{5-\sqrt{17}}{2}$.*

3.1 The local topology

We briefly recall the background related to local limits. Consider the collection $\mathfrak{M}$ of vertex-rooted locally finite planar maps. For all integers $k \geq 0$ we may consider the projection $U_k : \mathfrak{M} \to \mathfrak{M}$ that sends a map from $\mathfrak{M}$ to the submap obtained by restricting to all vertices with graph distance at most k from the root vertex. The local topology is induced by the metric

$$d_{\mathfrak{M}}(M_1, M_2) = \frac{1}{1 + \sup\{k \geq 0 \mid U_k(M_1) = U_k(M_2)\}}, \qquad M_1, M_2 \in \mathfrak{M}.$$

It is well-known that the metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ is a Polish space. A limit of a sequence of vertex rooted maps in $\mathfrak{M}$ is called a local limit. The vertex rooted map (M_n, v_n) is a random point of the space of $\mathfrak{M}$, and hence the standard probabilistic notions for different types of convergence (such as distributional convergence in (10)) of random points in Polish spaces apply.

3.2 Continuity on a subset

We consider the indicator variable $f : \mathfrak{M} \to \{0, 1\}$ for the property, that the root vertex is a cut vertex.

Note that f is not continuous on $\mathfrak{M}$. Therefore we consider the subset $\Omega \subset \mathfrak{M}$ of all locally finite vertex-rooted maps with the property, that either the root is not a cut vertex, or it is a cut vertex and deleting it creates at least one finite connected component.

▶ **Lemma 5.** *The indicator variable f is continuous on Ω.*

Proof. Let $(M_n)_{n \geq 1}$ denote a sequence in $\mathfrak{M}$ with a local limit $M = \lim_{n\to\infty} M_n$ that satisfies $M \in \Omega$. If the root of M is not a cut vertex, then there is a finite cycle containing it, and this cycle must then be already present in M_n for all sufficiently large n. Hence in this case $\lim_{n\to\infty} f(M_n) = 0 = f(M)$. If the root of M is a cut vertex, then $M \in \Omega$ implies that removing it creates a finite connected component, and this component must then also be separated from the remaining graph when removing the root vertex of M_n for all sufficiently large n. Thus, $\lim_{n\to\infty} f(M_n) = 1 = f(M)$. This shows that f is continuous on Ω. ◀

Note that by similar arguments it follows that the subset Ω is closed.

3.3 Random probability measures

The collection $\mathbb{M}_1(\mathfrak{M})$ of probability measures on the Borel sigma algebra of $\mathfrak{M}$ is a Polish space with respect to the weak convergence topology.

For any finite planar map M with k vertices we may consider the uniform distribution on the k different rooted versions of M. If the map M is random, then this is a random probability measure, and hence a random point in the space $\mathbb{M}_1(\mathfrak{M})$. In particular, the conditional law $\mathbb{P}((\mathsf{M}_n, v_n) \mid \mathsf{M}_n)$ is a random point of $\mathbb{M}_1(\mathfrak{M})$. Let $\mathfrak{L}(\mathsf{M}_\infty) \in \mathbb{M}_1(\mathfrak{M})$ denote the law of the random map M_∞. It follows from [18, Thm. 4.1] that

$$\mathbb{P}((\mathsf{M}_n, v_n) \mid \mathsf{M}_n) \xrightarrow{p} \mathfrak{L}(\mathsf{M}_\infty). \tag{11}$$

The explicit construction of the limit M_∞ also entails that among the connected components created when removing any single vertex of M_∞ at most one is infinite. In particular,

$$\mathbb{P}(\mathsf{M}_\infty \in \Omega) = 1. \tag{12}$$

3.4 Proving Lemma 3 using the continuous mapping theorem

Let us recall the continuous mapping theorem (see, for example, the book by Billingsley [3, Thm. 2.7]) that says that random variables $X, X_1, X_2, \ldots$ that take values in a Polish space $\mathfrak{X}$ have the property that $X_n \xrightarrow{d} X$ implies $g(X_n) \xrightarrow{d} g(X)$, where $g : \mathfrak{X} \to \mathfrak{Y}$ is a measurable map to a Polish space $\mathfrak{Y}$ and X almost surely takes values on the subset of $\mathfrak{X}$, where g is continuous.

Hence, by combining the convergence (10) with Lemma 5 and Equation (12) allows us to apply the continuous mapping theorem with $\mathfrak{X} = \mathfrak{M}$ and $\mathfrak{Y} = \{0, 1\}$ to deduce

$$f(\mathsf{M}_n, v_n) \xrightarrow{d} f(\mathsf{M}_\infty).$$

In other words, the probability for v_n to be a cut vertex of M_n converges toward the probability $p = \mathbb{E}[f(\mathsf{M}_\infty)]$ that the root of M_∞ is a cut vertex. Equivalently, the number of vertices $\mathrm{v}(\mathsf{M}_n)$ in the map M_n satisfies

$$\mathbb{E}[X_n/\mathrm{v}(\mathsf{M}_n)] \to p.$$

Of course, it follows by the same arguments that in general for any sequence of probability measures $P_1, P_2, \ldots \in \mathbb{M}_1(\mathfrak{M})$ satisfying the weak convergence $P_n \Rightarrow \mathfrak{L}(\mathsf{M}_\infty)$, the push-forward measures satisfy

$$P_n f^{-1} \Rightarrow \mathfrak{L}(\mathsf{M}_\infty) f^{-1}. \tag{13}$$

Let us now consider the setting $\mathfrak{X} = \mathbb{M}_1(\mathfrak{M})$, $\mathfrak{Y} = \mathbb{R}$, and

$$g : \mathbb{M}_1(\mathfrak{M}) \to \mathbb{R}, \quad P \mapsto \int f \, \mathrm{d}P = P(f = 1). \tag{14}$$

That is, a probability measure $P \in \mathbb{M}_1(\mathfrak{M})$ gets mapped to the expectation of f with respect to P. In other words, to the P-probability that the root is a cut vertex. It follows from (13) that g is continuous at the point $\mathfrak{L}(\mathsf{M}_\infty)$. Hence, using (11) and again the continuous mapping theorem, it follows that

$$\mathbb{E}[f(\mathsf{M}_n, v_n) \mid \mathsf{M}_n] \xrightarrow{d} p. \tag{15}$$

As p is a constant, this convergence actually holds in probability. Moreover,

$$\mathbb{E}[f(\mathsf{M}_n, v_n) \mid \mathsf{M}_n] = X_n/\mathrm{v}(\mathsf{M}_n). \tag{16}$$

The number $\mathrm{v}(\mathsf{M}_n)$ is known to satisfy $\mathrm{v}(\mathsf{M}_n)/n \xrightarrow{p} 1/2$. In fact, a normal central limit theorem is known to hold (see, for example, [9, Lem. 4.1]). This allows us to apply Slutsky's theorem, yielding $X_n/n \xrightarrow{p} p/2$. We have thus completed the proof of Lemma 3.

3.5 Structural properties of the local limit

We let M denote a random map following a Boltzmann distribution with parameter $z_1 = \frac{1}{12}$. That is, M attains a finite planar map M with $\mathrm{c}(M)$ corners with probability

$$\mathbb{P}(\mathsf{M} = M) = \frac{z_1^{\mathrm{c}(M)}}{M(z_1)} = \frac{3}{4}\left(\frac{1}{12}\right)^{\mathrm{c}(M)}. \tag{17}$$

The local limit M_∞ exhibits a random number of independent copies of M close to its root. This can be made more precise by the following property.

▶ **Lemma 6.** *There is an infinite random planar map M_∞^* with a root vertex u^* that is not a cut vertex of M_∞^*, such that M_∞ is distributed like the result of attaching an independent copy of M to each corner incident to u^*.*

Here we use the term *attach* in the sense that the origin of the root-edge of the independent copy of M gets identified with the vertex u^*. In what follows we will only use the fact that such a (random) map M_∞^* exists. The proof of Lemma 6 (that is given in Appendix A) provides additional information about the distribution of M_∞ and M_∞^*.

3.6 Proving Lemma 4 via the asymptotic degree distribution

Let $q(z) = \sum_{k\geq 1} q_k z^k$ denote the probability generating function of the root-degree of the map M_∞^*. If we attach an independent copy of M to each corner incident to the vertex u^* in the map M_∞^*, then u^* becomes a cut vertex if and only if at least one of these copies has at least one edge. The probability for M to have no edges, that is, to consist only of a single vertex, is given by $1/M(z_1) = 3/4$. Hence the probability p for the root of M_∞ to be a cut vertex may be expressed by

$$p = \sum_{k\geq 1} q_k \left(1 - \left(\frac{3}{4}\right)^k\right) = 1 - q\left(\frac{3}{4}\right). \tag{18}$$

Hence, in order to determine p we need to determine $q(z)$. Surprisingly, we may do so without concerning ourselves with the precise construction of M_∞^*.

It was shown in [12] that the degree of the origin of the root-edge of the random planar map M_n admits a limiting distribution with a generating series $d(z)$ given by

$$d(z) = \frac{z\sqrt{3}}{\sqrt{(2+z)(6-5z)^3}}. \tag{19}$$

That is, $d_k := [z^k]d(z)$ is the asymptotic probability for the origin of the root-edge of M_n to have degree k. Let s_k denote the limit of the probability for a uniformly selected vertex of M_n to have degree k. It follows from [14, Prop. 2.6] that

$$s_k = 4d_k/k \tag{20}$$

for all integers $k \geq 1$. Setting $s(z) = \sum_{k\geq 1} s_k z^k$, Equation (20) may be rephrased by

$$zs'(z) = 4d(z). \tag{21}$$

Via integration, this yields the expression

$$s(z) = \frac{1}{2}\left(-1 + \frac{\sqrt{2+z}}{\sqrt{2 - \frac{5z}{3}}}\right) \tag{22}$$

As M_∞ is the local limit of M_n rooted at a uniformly chosen vertex, it follows that for each $k \geq 1$ the limit s_k equals the probability for the root of M_∞ to have degree k. Let $r(z)$ denote the probability generating series of the degree distribution of the origin of the root-edge of the Boltzmann map M. It follows from Lemma 6 that

$$s(z) = q(zr(z)). \tag{23}$$

We are going to compute $r(z)$. To this end, let $M(z,v)$ denote the generating series of planar maps with z marking edges and v marking the degree of the root vertex. By duality, $M(z,v)$ coincides with the bivariate generating series where the second variable marks the degree of the outer face. The quadratic method (see [11, p. 515] or compare with (1) and (2)) hence yields the known expression

$$M(z_1,u) = \frac{-3u^2+36u-36+\sqrt{3(u+2)(6-5u)^3}}{6u^2(u-1)}. \tag{24}$$

The series $r(z)$ is related to $M(z,u)$ via

$$r(u) = M(z_1,u)/M(z_1,1) = \frac{3}{4}M(z_1,u). \tag{25}$$

Forming the compositional inverse of $zr(z)$ and plugging it into Equation (23) yields the involved expression

$$q(z) = \frac{1}{2}\left(\frac{\sqrt{\frac{20z^2+48z-\sqrt{2z-27}(2z-3)^{3/2}+123}{z(4z+3)+24}}}{2\sqrt{\frac{6-4z}{-14z+5\sqrt{2z-27}\sqrt{2z-3}+51}}}-1\right). \tag{26}$$

Equation (26) allows us to evaluate the constant $q(3/4)$ in the expression for p given in Equation (18), yielding

$$p = 1-q(3/4) = \frac{5-\sqrt{17}}{2}. \tag{27}$$

This concludes the proof of Lemma 4.

4 A combinatorial approach to cut vertices of planar maps

The goal of this section is to re-derive the constant $(5-\sqrt{17})/4 = p/2$ in Theorem 1 with the help of a combinatorial approach by deriving an asymptotic expansion for the expected value $\mathbb{E}[X_n]$. We want to emphasize again that an extension of this approach (that will be given in the journal version of this paper) provides the asymptotic expansion of the second moment $\mathbb{E}[X_n^2]$ and consequently of the variance.

4.1 Generating function for the expected number of cut vertices

By extending the combinatorial approach that relates all planar maps with 2-connected maps (see (6)) it is possible to derive the following explicit formula for the generating function

$$E_a(z) = \sum_{n\ge 0} M_n \mathbb{E}[X_n] z^n.$$

▶ **Lemma 7.** *Let $u_1(z)$ denote the function $u_1(z) = 1/(1-V(z,1)$, where $V(z,x)$ (and $U(z,x)$) is given by (5). Then we have*

$$\begin{aligned} E_a(z) = {} & \frac{1}{1-2zM(z)A_z(zM(z)^2,1,1)} \\ & \times \Big[A(zM(z)^2,1,1)+A_x(zM(z)^2,1,1) \\ & \quad -2zM(z)-z-B(zM(z)^2,1,1/M(z))-B^\bullet(zM(z)^2,1/M(z)) \\ & \quad +2zM(z)A_z(zM(z)^2,1,1)\left(B(zM(z)^2,1,1/M(z))-M(z)+zM(z)+z+1\right)\Big], \end{aligned} \tag{28}$$

where

$$B^\bullet(z,w) = zw \frac{\frac{u_1(z)B(z,1,w)-wB(z,1,u_1(z))}{w-u_1(z)} + zwu_1(z)}{1-\frac{u_1(z)B(z,1,w)-wB(z,1,u_1(z))}{w-u_1(z)} - zwu_1(z)}. \tag{29}$$

The proof is given in Appendix B. Note that all involved functions are algebraic, which shows that the generating function $E_a(z)$ is algebraic, too.

4.2 Asymptotics

We start with a proper representation of $B_x(z,1,1)$ and $B_z(z,1,1)$.

▶ **Lemma 8.** *Let $B(z,x,u)$ be given by (4) and $u_1(z) = 1/(1-V(z,1))$ as in Lemma 11. Then we have*

$$B_x(z,1,1) = \frac{u_1(z)-1}{u_1(z)} Q(z)(1-Q(z)) \tag{30}$$

and

$$B_z(z,1,1) = \frac{u_1(z)-1}{z\,u_1(z)} Q(z)(1-Q(z)) + u_1(z) - 1 \tag{31}$$

where $Q(z)$ abbreviates

$$Q(z) = \frac{V(z,1)^2}{u_1(z)-1} - \frac{u_1(z)B(z,1,1)}{u_1(z)-1} + z\,u_1(z).$$

The proof is an easy application of the kernel method applied to the derivative of the defining relation (3).

▶ **Lemma 9.** *We have the following local expansions in powers of $\left(1-\frac{27}{4}z\right)$:*

$$B_x(z,1,1) = \frac{2}{27} - \frac{2\sqrt{3}}{27}\sqrt{1-\frac{27}{4}z} + \frac{2}{81}\left(1-\frac{27}{4}z\right) + \frac{19\sqrt{3}}{729}\left(1-\frac{27}{4}z\right)^{3/2} + \cdots \tag{32}$$

$$B_z(z,1,1) = 1 - \sqrt{3}\left(1-\frac{27}{4}z\right)^{1/2} + \frac{4}{3}\left(1-\frac{27}{4}z\right) - \frac{35\sqrt{3}}{54}\left(1-\frac{27}{4}z\right)^{3/2} + \cdots \tag{33}$$

$$\begin{aligned} B^\bullet(z,w) &= -4\,\frac{w\left(-2\,w+\sqrt{4\,w^2-60\,w+81}-9\right)}{243-54\,w+27\,\sqrt{4\,w^2-60\,w+81}} \\ &\quad + \frac{16\,\sqrt{3}w^2\left(-2\,w+\sqrt{4\,w^2-60\,w+81}+3\right)}{9\,\left(9-2\,w+\sqrt{4\,w^2-60\,w+81}\right)^2(2\,w-3)}\sqrt{1-\frac{27}{4}z} + \cdots \end{aligned} \tag{34}$$

Proof. By inverting the equation $z = V(1-V)^2$ it follows that $V(z,1)$ has the local expansion

$$V(z,1) = \frac{1}{3} - \frac{2}{3\sqrt{3}}Z + \frac{2}{27}Z^2 - \frac{5}{81\sqrt{3}}Z^3 + \cdots,$$

where Z abbreviates

$$Z = \sqrt{1-\frac{27}{4}z}.$$

Consequently $u_1(z) = 1/(1-V(z,1))$ is given by

$$u_1(z) = \frac{3}{2} - \frac{\sqrt{3}}{2}Z + \frac{2}{3}Z^2 - \frac{35\sqrt{3}}{108}Z^3 \cdots$$

We already know that

$$B(z,1,u_1(z)) = V(z,1)^2 = \frac{1}{9} - \frac{4\sqrt{3}}{27}Z + \frac{16}{81}Z^2 - \frac{34\sqrt{3}}{729}Z^3 + \cdots$$

and from (4) we directly obtain

$$B(z,1,1) = \frac{1}{27} - \frac{4}{27}Z^2 + \frac{8\sqrt{3}}{81}Z^3 + \cdots$$

Hence, the local expansion of $Q(z) = Q_0(z,1,u_1(z))$ can be easily calculated:

$$Q(z) = \frac{1}{3} - \frac{2\sqrt{3}}{9}Z + \frac{2}{27}Z^2 - \frac{5\sqrt{3}}{243}Z^3 + \cdots,$$

and, thus, (32) and (33) follow from this expansion and from (30) and (31).

Finally we have to use (29) and the expansion for $B(x,1,w)$ to obtain (34). ◀

This leads us to the following local expansion for $E_a(z)$ and a corresponding asymptotic relation.

▶ **Lemma 10.** *The function $E_a(z)$ has the following local expansion*

$$E_a(z) = \frac{11\sqrt{17}-37}{24} - (5-\sqrt{17})\sqrt{1-12z} + \cdots \tag{35}$$

which implies

$$\mathbb{E}[X_n] = \frac{[z^n]\,E_a(z)}{[z^n]\,M(z)} = \frac{(5-\sqrt{17})}{4}n + O(1).$$

Proof. We note that several parts of (28) have a dominant singulartiy of the form $(1-12z)^{3/2}$. For those parts only the value at $z_1 = 1/12$ influences the constant term and coefficient of $\sqrt{1-12z}$ in the local expansion of $E_a(z)$. In particular we have

$$\begin{aligned} M(z_1) &= \frac{4}{3}, \\ A(z_1M(z_1)^2,1,1) &= \frac{1}{3}, \\ B(z_1M(z_1)^2,1,1/M(z_1)) &= \frac{3\sqrt{17}-11}{72}. \end{aligned}$$

The other appearing function will have a non-zero coefficient at the $\sqrt{1-12z}$–term. Note also that we have

$$\sqrt{1-\frac{27}{4}zM(z)^2} = \sqrt{3}\sqrt{1-12z} - \frac{2}{3}\sqrt{3}(1-12z) + O((1-12z)^{3/2}).$$

Hence we get

$$\begin{aligned} A_z(zM(z)^2,1,1) &= 3 - 3\sqrt{1-12z} + \cdots, \\ A_x(zM(z)^2,1,1) &= \frac{2}{9} - \frac{2}{9}\sqrt{1-12z} + \cdots, \\ B^\bullet(zM(z)^2,1,1,1/M(z)) &= \frac{\left(7-\sqrt{17}\right)\left(5-\sqrt{17}\right)}{72} - \frac{\left(1+\sqrt{17}\right)\left(-5+\sqrt{17}\right)^2}{48}\sqrt{1-12z} + \cdots \end{aligned}$$

and so (35) follows.

From (35) it directly follows that

$$[z^n]\,E_a(z) = \frac{5-\sqrt{17}}{2\sqrt{\pi}}n^{-3/2}12^n \cdot (1+O(1/n))$$

By dividing that by $M_n = [z^n]M(z) = (2/\sqrt{\pi})n^{-5/2}12^n \cdot (1+O(1/n))$ the final result follows. ◀

5 Outerplanar Maps

We give a proof of Theorem 2 for the case of (all) outerplanar maps. (The proof in the bipartite case is very similiar.)

We recall that the generating function $M_O(z)$ of outerplanar maps satisfies (7), where the function

$$A_O(z) = \frac{1}{4}\left(1 + z - \sqrt{1 - 6z + z^2}\right)$$

is the generating function for polygon dissections (plus a single edge) has radius of convergence $z_{0,O} = 3 - 2\sqrt{2}$. From this we obtain

$$M_O(z) = \frac{z\left(3 - \sqrt{1-8z}\right)}{2(1+z)}.$$

The radius of convergence of $M_O(z)$ is $z_{1,O} = \frac{1}{8}$ so that $M_O(z_{1,O}) = \frac{1}{18} < z_{0,O}$. Note that $M_O(z)$ has a squareroot singularity (as it has to be). Now let $M_O(z,y)$ denote the generating function of outerplanar maps, where y takes care of the number of cut-vertices. We already mentioned that $M_O(z,y)$ satisfies the functional equation

$$M_O(z,y) = \frac{z}{1 - A_O(z + y(M_O(z,y) - z))}$$

which gives

$$M_O(z,y) = \frac{z\left(3 - z + yz - \sqrt{(y-1)z^2 - (6+2y)z + 1}\right)}{2(1+yz)}.$$

Clearly, if y is sufficiently close to 1 then the singularities of $M_O(z,y)$ and $A_O(z)$ do not interact and so we obtain a squareroot singularity

$$\rho(y) = \frac{3 + y - 2\sqrt{2+2y}}{(y-1)^2}.$$

for the mapping $z \mapsto M_O(z,y)$. Note that $\rho(y)$ is actually regular at $y = 1$ and satisfies $\rho(1) = 1/8$.

By [7, Theorem 2.25] we immediately obtain a central limit theorem with $\mathbb{E}[X_n] = c\,n + O(1)$ and variance $\mathbb{V}\mathrm{ar}[X_n] = \sigma^2 n + O(1)$, where

$$c = -\frac{\rho'(1)}{\rho(1)} = \frac{1}{4} \quad \text{and} \quad \sigma^2 = -\frac{\rho''(1)}{\rho(1)} + \mu + \mu^2 = \frac{5}{32}.$$

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A Proof of Lemma 6

A direct description of the limit M_∞ that uses a generalization of the Bouttier, Di Francesco and Guitter bijection [5] was given in [18, Thm. 4.1]. Although the structure of M_∞ may be studied in this way, it will be easier to show that M_∞ has the desired shape via a construction related to limits of the 2-connected core within M_n.

Let $\mathcal{B}(\mathsf{M}_n) \subset \mathsf{M}_n$ denote the largest (meaning, having a maximal number of edges) 2-connected block in the map M_n. Typically $\mathcal{B}(\mathsf{M}_n)$ is uniquely determined, as the number $c(n)$ of corners of $\mathcal{B}(\mathsf{M}_n)$ is known to have order $2n/3$, and the number of corners in the second largest block has order $n^{2/3}$.

Consider the random planar map $\bar{\mathsf{M}}_n$ constructed from the core $\mathsf{C}_n := \mathcal{B}(\mathsf{M}_n)$ by attaching for each integer $1 \le i \le c(n)$ an independent copy $\mathsf{M}(i)$ of M at the ith corner of C_n. We use the notation C_n instead of $\mathcal{B}(\mathsf{M}_n)$ from now on to emphasize that we consider C_n always as a part of $\bar{\mathsf{M}}_n$ (as opposed to M_n).

Clearly, the two models M_n and $\bar{\mathsf{M}}_n$ are not identically distributed. For example, the number of edges in $\bar{\mathsf{M}}_n$ is a random quantity that fluctuates around n. However, analogously as in the proof of [17, Lem. 9.2], local convergence of $\bar{\mathsf{M}}_n$ is equivalent to local convergence of M_n, implying that M_∞ is also the local limit of $\bar{\mathsf{M}}_n$ with respect to a uniformly selected vertex u_n.

The random 2-connected planar map B_n with n edges was shown to admit a local limit $\hat{\mathsf{B}}$ that describes the asymptotic vicinity of a typical corner (equivalently, the root-edge of B_n), see [17, Thm. 1.3]. Arguing entirely analogously as in [9], it follows that there is also a local limit B_∞ that describes the asymptotic vicinity of a typical vertex.

The number of vertices of $\bar{\mathsf{M}}_n$ has order $n/2$, and the number of vertices in C_n is known to have order $n/6$. Let u_n^{B} denote the result of conditioning the random vertex u_n to belong to C_n. The probability for this to happen tends to $1/3$. As u_n^{B} is uniformly distributed among all vertices of C_n, it follows that $(\mathsf{C}_n, u_n^{\mathrm{B}}) \xrightarrow{d} \mathsf{B}_\infty$ in the local topology. This implies that $(\bar{\mathsf{M}}_n, u_n^{\mathrm{B}})$ converges in distribution towards the result $\mathsf{M}_\infty^{\mathrm{B}}$ of attaching an independent copy of M to each corner of B_∞. The limit $\mathsf{M}_\infty^{\mathrm{B}}$ has the desired shape.

Let u_n^{c} denote the result of conditioning the random vertex u_n to lie outside of C_n. It remains to show that the limit $\mathsf{M}_\infty^{\mathrm{c}}$ of $(\bar{\mathsf{M}}_n, u_n^{\mathrm{c}})$ has the desired shape as well. Let $1 \le i_n \le c(n)$ denote the index of the corner where the component containing u_n^{c} is attached. It is important to note that given the maps $\mathsf{M}(1), \ldots, \mathsf{M}(c(n))$, the random integer i_n need not be uniform, as it is more likely to correspond to a map with an above average number of vertices. This well-known waiting time paradox implies that *asymptotically* the component containing u_n^{c} follows a size-biased distribution $\mathsf{M}^\bullet$. That is, $\mathsf{M}^\bullet$ is a random finite planar map with a marked non-root vertex, such that for any planar map M with a marked non-root vertex v it holds that

$$\mathbb{P}(\mathsf{M}^\bullet = (M, v)) = \mathbb{P}(\mathsf{M} = M)/(\mathbb{E}[\mathrm{v}(\mathsf{M})] - 1),$$

with $\mathrm{v}(\mathsf{M})$ denoting the number of vertices in the Boltzmann planar map M.

In detail: Given the random number $c(n)$, let i_n^* be uniformly selected among the integers from 1 to $c(n)$. For each $1 \le i \le c(n)$ with $i \ne i_n^*$ let $\bar{\mathsf{M}}(i)$ denote an independent copy of M, and let $\bar{\mathsf{M}}(i_n^*)$ denote an independent copy of $\mathsf{M}^\bullet$. Likewise, for each $1 \le i \le c(n)$ with $i \ne i_n$ set $\mathsf{M}^*(i) = \mathsf{M}(i)$, and let $\mathsf{M}^*(i_n) = (\mathsf{M}(i_n), u_n^{\mathrm{c}})$. Analogously as in the proof of [17, Lem. 9.2], it follows that

$$(\mathsf{M}^*(i))_{1 \le i \le c(n)} \overset{d}{\approx} (\bar{\mathsf{M}}(i))_{1 \le i \le c(n)}.$$

This entails that the core C_n rooted at the corner with index i_n admits $\hat{\mathsf{B}}$ (and not B_∞) as local limit. Moreover, the local limit $\mathsf{M}_\infty^{\mathrm{c}}$ of $\bar{\mathsf{M}}_n$ rooted at u_n^{c} may be constructed by attaching an independent copy of M to each corner of $\hat{\mathsf{B}}$, except for the root-corner of $\hat{\mathsf{B}}$, which receives an independent copy of $\mathsf{M}^\bullet$. The marked vertex of the limit object $\mathsf{M}_\infty^{\mathrm{c}}$ is then given by the marked vertex of this component.

To proceed, we need information on the shape of $\mathsf{M}^\bullet$. Consider the ordinary generating functions $M(v, w)$ and $A(v, w)$ of planar maps and 2-connected planar maps, with v marking corners, and w marking non-root vertices. The block-decomposition yields

$$M(v, w) = A(vM(v, w), w). \tag{36}$$

That is, a planar map consists of a uniquely determined block containing the root-edge, with uniquely determined components attached to each of its corners. Let us call this block the *root block*. For the trivial map consisting of a single vertex and no edges, this block is identical to the trivial map, with nothing attached to it as it has no corners.

Marking a non-root vertex (and no longer counting it) corresponds to taking the partial derivative with respect to w. It follows from (36) that

$$\frac{\partial M}{\partial w}(v, w) = \frac{\partial A}{\partial w}(vM(v, w), w) + \frac{\partial A}{\partial v}(vM(v, w), w)v\frac{\partial M}{\partial w}(v, w).$$

The combinatorial interpretation is that either the marked non-root vertex is part of the root block (accounting for the first summand), or there is a uniquely determined corner of the root block such that the component attached to this corner contains it. This is a recursive decomposition, as in the second case we could proceed with this component, considering whether the marked vertex belongs to its root block or not. We may do so a finite number of times, until it finally happens that the marked vertex belongs to the root-block of the component under consideration. That is, if we follow this decomposition until encountering the marked non-root vertex, we have to pass through a uniquely determined sequence of blocks, always proceeding along uniquely determined (and hence marked) corners, until arriving at a block with a marked non-root vertex. On a generating function level, this is expressed by

$$\frac{\partial M}{\partial w}(v,w) = \frac{1}{1-\frac{\partial A}{\partial v}(vM(v,w),w)v}\frac{\partial A}{\partial w}(vM(v,w),w).$$

This allows us to apply Boltzmann principles, yielding that the random map $\mathsf{M}^\bullet$ may be sampled in two steps, that may be described as follows: First, generate this sequence of blocks by linking a geometrically distributed random number N of random independent Boltzmann distributed blocks $\mathsf{B}_1^\circ, \ldots, \mathsf{B}_N^\circ$ with marked corners into a chain, and attach an extra random Boltzmann distributed block $\mathsf{B}^\bullet$ with a marked non-root vertex to the end of the chain. The random number N has generating function

$$\mathbb{E}[u^N] = \frac{1-\frac{\partial A}{\partial v}(z_1 M(z_1,1),1)z_1}{1-u\frac{\partial A}{\partial v}(z_1 M(z_1,1),1)z_1}.$$

The corner-rooted blocks are independent copies of a Boltzmann distributed block B°, whose number of corners $\mathrm{c}(\mathsf{B}^\circ)$ has generating function

$$\mathbb{E}[u^{\mathrm{c}(\mathsf{B}^\circ)}] = \frac{\frac{\partial A}{\partial v}(uz_1 M(z_1,1),1)}{\frac{\partial A}{\partial v}(z_1 M(z_1,1),1)}.$$

The distribution of B° is fully characterized by the fact that, when conditioning on the number of corners, B° is conditionally uniformly distributed among the corner-rooted blocks with that number of corners. The distribution of $\mathsf{B}^\bullet$ is defined analogously. If we attach a block $\tilde{B}$ to the marked corner c of some block B, we say the resulting corner "to the right" of $\tilde{B}$ *corresponds* to c. Hence the map obtained by linking $(\mathsf{B}_1^\circ, \ldots, \mathsf{B}_N^\circ, \mathsf{B}^\bullet)$ has precisely N corners that correspond to marked corners. We call these corners *closed*, and all other corners *open*. The second and final step in the sampling procedure of $\mathsf{M}^\bullet$ is to attach an independent copy of M to each open corner of the map corresponding to $(\mathsf{B}_1^\circ, \ldots, \mathsf{B}_N^\circ, \mathsf{B}^\bullet)$. Note that since the marked vertex of $\mathsf{B}^\bullet$ is a non-root vertex, all corners incident to the marked vertex are open. Consequently, the limit $\mathsf{M}_\infty^{\mathrm{c}}$ has the desired shape, and the proof is complete.

B Proof of Lemma 7

First we introduce (formally) a generating function that takes care of all vertex degrees in 2-connected planar maps (including the one-edge map and the one-edge loop)

$$\overline{A}(z; w_1, w_2, w_3, w_4, \ldots; u),$$

where w_k, $k \geq 1$, corresponds to vertices of degree k and we also take the root vertex into account. As usual, u corresponds to the root degree.

Similarly we introduce a variant of this generating function that takes care of all vertex degrees in 2-connected planar maps (without the one-edge map and one-edge loop) and does not take the root vertex into account:

$$\overline{B}(z; w_2, w_3, w_4, \ldots; u).$$

It seems to be impossible to work directly with $\overline{A}(z; w_1, w_2, w_3, \ldots)$ or with $\overline{B}(z; w_2, w_3, w_4, \ldots; u)$, however, we have the following easy relations:

$$\overline{A}(z; xv, xv^2, xv^3, \ldots; u) = xA(zv^2, x, u), \ \overline{B}(z; xv^2, xv^3, \ldots; u) = B(zv^2, x, u/v).$$

This follows from the fact that every vertex of degree k corresponds to k half-edges. So summing up these half-edges we get twice the number of edges. In particular by taking derivatives with respect to x and v it follows that

$$\sum_{k\ge 1} \overline{A}_{w_k}(z; v, v^2, v^3, \ldots) v^k = A(zv^2, 1, 1) + A_x(zv^2, 1, 1) \tag{37}$$

and

$$\sum_{k\ge 1} k\overline{A}_{w_k}(z; v, v^2, v^3, \ldots) v^{k-1} = 2zvA_z(zv^2, 1, 1). \tag{38}$$

It turns out that we will also have to deal with the sum of all derivatives which is slightly more difficult to understand.

▶ **Lemma 11.** *Let $u_1(z)$ denote the function $u_1(z) = 1/(1 - V(z, 1)$, where $V(z, x)$ (and $U(z, x)$) is given by (5). Then we have*

$$\sum_{k\ge 1} \overline{A}_{w_k}(z; v, v^2, v^3, \ldots) = 2zv + z + B(zv^2, 1, 1/v) \tag{39}$$
$$+ zv \frac{\frac{u_1(zv^2)B(zv^2,1,1/v) - B(zv^2,1,u_1(zv^2))/v}{1/v - u_1(zv^2)} + zvu_1(zv^2)}{1 - \frac{u_1(zv^2)B(zv^2,1,1/v) - B(zv^2,1,u_1(zv^2))/v}{1/v - u_1(zv^2)} - zvu_1(zv^2)}$$

Proof. We note that the derivative with respect to w_k marks a vertex of degree k and discounts it. By substituting w_k by v^k we, thus, see that the resulting exponent of v is twice the number of edges minus the degree of the marked vertex. Hence we have to cover the situation, where we mark a vertex and keep track of the degree of the marked vertex.

Let $B^\bullet(z, x, u, w)$ be the generating function of vertex marked 2-connected planar maps, where the marked vertex is different from the root and where u takes care of the root degree and w of the degree of the pointed vertex. By duality this is also the generating function of face marked 2-connected planar maps where u takes care of the root face valency and w of the valency of the marked face (that is different from the root face). Then we have

$$\sum_{k\ge 1} \overline{A}_{w_k}(z; v, v^2, v^3, \ldots) = 2zv + z + B(zv^2, 1, 1/v) + B^\bullet(zv^2, 1, 1, 1/v).$$

The term $2zv$ corresponds to the one-edge map, the term z to the one-edge loop, the term $B(zv^2, 1/v)$ to the case where the root vertex is marked and the third term $B^\bullet(zv^2, 1, 1, 1/v)$ to the case where a vertex different from the root is marked. Note that the substitution $u = 1/v$ (or $w = 1/v$) discounts the degree of the marked vertex in the exponent of v as needed.

Thus, it remains to get an expression for $B^\bullet(z,1,u,w)$. For this purpose we start with the generating function $B(z,1,u)$ and determine first the generating function $\tilde{B}(z,x,u,w)$ (for $x=1$), where the additional variable w takes care of the valency of the second face incident to the root edge. By using the same construction as above we have

$$\tilde{B}(z,1,u,w) = zuw\frac{\frac{uB(z,1,w)-wB(z,1,u)}{w-u} + zuw}{1-\frac{uB(z,1,w)-wB(z,1,u)}{w-u} - zuw}.$$

This gives (by again applying this construction)

$$B^\bullet(z,1,u,w) = \tilde{B}(z,1,u,w) + zu\frac{\frac{uB^\bullet(z,1,1,w)-B^\bullet(z,1,u,w)}{1-u}}{\left(1-\frac{uB(z,1,1)-B(z,1,u)}{1-u} - zu\right)^2}.$$

This equation can be solved with the help of the kernel method. By rewriting it to

$$B^\bullet(z,1,u,w)\left(1+\frac{zu}{1-u}\frac{1}{\left(1-\frac{uB(z,1,1)-B(z,1,u)}{1-u} - zu\right)^2}\right)$$
$$= B(z,1,u,w) + \frac{zu^2B^\bullet(z,1,1,w)}{1-u}\frac{1}{\left(1-\frac{uB(z,1,1)-B(z,1,u)}{1-u} - zu\right)^2}$$

we observe that by setting $u_1(z) = 1/(1-V(z,1))$ the left hand side cancels. This implies

$$B(z,1,u_1(z),w) + \frac{zu_1(z)^2B^\bullet(z,1,1,w)}{1-u_1(z)}\frac{1}{\left(1-\frac{u_1(z)B(z,1,1)-B(z,1,u_1(z))}{1-u_1(z)} - zu_1(z)\right)^2} = 0$$

and leads (after some simple algebra) finally to (39). ◀

Let $M_0(z,y)$ denote the generating function of planar maps with at least one edge, where the root vertex is not a cut point and where z takes care of the number of edges and y of the number of cut-points (that are then different from the root vertex). Next let $M_r(z,y)$ denote the generating function of (all) planar maps, where z takes care of the number of edges and y of the number of non-root cut-points. Finally let $M_a(z,y)$ denote the generating function of (all) planar maps, where z takes care of the number of edges and y of the number of (all) cut-points. Obviously we have the following relation between these three generating functions:

$$M_a(z,y) = yM_r(z,y) - (y-1)(1+M_0(z,y)). \tag{40}$$

Note that $M_0(z,1) = \overline{B}\left(z; M(z)^2, M(z)^3, \ldots; 1\right) = B(zM(z)^2, 1, 1/M(z)) + zM(z) + z$.

Furthermore we set

$$E_a(z) = \left.\frac{\partial M_a(z,y)}{\partial y}\right|_{y=1} = \sum_{n\ge 0} M_n \mathbb{E}[X_n] z^n \quad \text{and} \quad E_r(z) = \left.\frac{\partial M_r(z,y)}{\partial y}\right|_{y=1}.$$

By differentiating (40) with respect to y and setting $y=1$ we obtain

$$E_a(z) = E_r(z) + M(z) - 1 - M_0(z,1). \tag{41}$$

With the help of the above notions we obtain the following (formal relation):

$$M_a(z,y) = 1 + \overline{A}\left(z; yM_r(z,y) - y + 1, yM_r(z,y)^2 - y + 1, \ldots; 1\right). \tag{42}$$

The right hand side is based on the block-decompostion (similarly to (6)) and takes care whether the vertices of the block that contains the root edge become cut-vertices or not. By differentiating (42) with respect to y and setting $y = 1$ we, thus, obtain

$$\begin{aligned} E_a(z) &= \sum_{k\geq 1} \overline{A}_{w_k}\left(z; M(z), M(z)^2, \ldots; 1\right)\left(M(z)^k - 1 + kM(z)^{k-1}E_r(z)\right) \\ &= \sum_{k\geq 1} \overline{A}_{w_k}\left(z; M(z), M(z)^2, \ldots\right) M(z)^k - \sum_{k\geq 1} \overline{A}_{w_k}\left(z; M(z), M(z)^2, \ldots\right) \\ &+ E_r(z) \sum_{k\geq 1} k\overline{A}_{w_k}\left(z; M(z), M(z)^2, \ldots\right) M(z)^{k-1}. \end{aligned}$$

By using (41) we get a proper expression for $E_a(z)$. At this stage we can apply (37) and (38) with $v = M(z)$. Furthermore Lemma 11 gives

$$\begin{aligned} &\sum_{k\geq 1} \overline{A}_{w_k}(z; M(z), M(z)^2, \ldots) \\ &= 2zM(z) + z + B(zM(z)^2, 1, 1/M(z)) + B^{\bullet}(zM(z)^2, 1, 1, 1/M(z)) \\ &= 2zM(z) + z + B(zM(z)^2, 1, 1/M(z)) \\ &+ zM(z)\frac{\frac{u_1(zM(z)^2)B(zM(z)^2,1,1/M(z)) - B(zM(z)^2,1,u_1(zM(z)^2))/M/z)}{1/M(z) - u_1(zM(z)^2)} + zM(z)u_1(zM(z)^2)}{1 - \frac{u_1(zM(z)^2)B(zM(z)^2,1,1/M(z)) - B(zM(z)^2,1,u_1(zM(z)^2))/M(z)}{1/M(z) - u_1(zM(z)^2)} - zM(z)u_1(zM(z)^2)} \end{aligned}$$

This finally leads to the proposed explicit formula for $E_a(z)$.