LIGHT STOCKS AND WEALTH ALLOCATION

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Abstract. The aim of this paper is to deal with the problem of wealth allocation. We assume that an investor can share her/his money between consumption, riskless bonds, risky assets frequently traded in the market and illiquid stocks. The financial nature of thin stocks requires the description of their dynamics via jump processes, rather than continuous processes. Therefore, a stochastic control problem in a jump diffusion context is developed. In this paper the dynamic programming approach is adopted, and the optimal investment strategies are derived in closed form.

Keywords: Stochastic control; Portfolio model; Jump diffusion; Viscosity solutions.

1 Introduction

In the field of finance, the problem of optimal wealth allocation attracts the attention both of theorists as well as practitioners. The pioneer of the mathematical formalization of the optimal allocation problems is Markowitz (1952), whose seminal paper contains a first basic uniperiodal model. Markowitz' followers spent much time to discuss and remove the restrictive assumptions of the original model. In particular, the single period setting was extended to a multiperiod framework by Samuelson (1969) and then to continuous time by Merton (1969, 1971). The traditional approach assumes that all assets can be traded at all times. This work deals with the optimal allocation problem, when some assets are rarely traded. The infrequently traded stocks are often denoted as "illiquid", "thin", "light" stocks.

The financial nature of the light assets suggests to model their dynamics as jump diffusion processes. Therefore, a stochastic optimal control problem in a jump diffusion framework is developed. For an excellent survey of this subject, we remind the reader to the monograph of Øksendal and Sulem (2007).

We assume that an agent may choose to allocate her/his money in risky assets that are frequently traded, thin stocks, riskless bonds and consumption. The objective function is the expected utility of the investor, the [illegible] possible choices of [illegible]

Among the possible ways to treat an optimal control problem, we prefer a dynamic programming approach. In particular, we derive the formal integro-differential Hamilton Jacobi Bellman equation (HJB, hereafter) by using a Dynamic Programming principle, and we prove that the value function is its unique classical solution. The stepwise procedure to do this runs as follows: firstly, it is shown that the value function is the unique weak (viscosity) solution of HJB; secondly, the required regularity –twice differentiability- of the value function is derived.

Our work seems to be quite similar to Cretarola et al. (to appear) and Tebaldi and Schwartz (2006). The former paper deals with the development of the model of Pham and Tankov (2008), and describe an optimal consumption/investment model in a market containing only a thin stock. Differently, we insert also a frequently traded stocks. The latter rely on an optimal allocation problem when one asset is illiquid, in the sense that it cannot be traded. In this respect, we consider illiquidity as infrequent trading, that is related to jump diffusions, while Tebaldi and Schwartz (2006) works in a continuous time framework.

The results of this paper are not proved in a detailed manner. We leave the complete proofs and some formal discussions to a longer version of the work.

2 The model

We consider an economic environment in continuous time. The price dynamics are assumed to evolve randomly. All the random quantities defined throughout the paper are assumed to be contained in a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t\}$ is assumed to reflect the whole set of information provided by the market up to time t. In the market the investors can find three assets: a risk free bond, a continuously traded (liquid) risky asset and an infrequently traded (light) stock. For the riskless bond and the frequently traded risky asset we can refer to the usual continuous time deterministic and stochastic models, respectively:

- the riskless bond, B_t, evolves according to the following ordinary differential equation:
$$dB_t = r(t)B_t dt$$
where $r(t)$ is the deterministic continuously compounded risk free interest rate at time t;
- the risky liquid asset S_t evolves stochastically as follows:
$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dW_t^1$$
where, for each $t \geq 0$, μ_1 is greater than $r(t)$ and is the continuously compounded expected rate of return on the risky liquid asset; σ_1 is the continuous standard deviation of the rate of return; W^1 is a standard 1-dimensional Brownian motion.

In order to describe thin stocks' dynamics, it is necessary to rely on jump-type processes. The model proposed by Pham and Tankov (2008) is adopted. Thin stock price H_t evolves as follows:

$$dH_t = \mu_2 H_t dt + \sigma_2 H_t dW_t^2$$

where, for each $t \geq 0$, μ_2 is greater than $r(t)$ and is the risky liquid asset continuously compounded expected rate of return, σ_2 is the continuous standard deviation of the rate of return; W^2 is a standard 1-dimensional brownian motion. By definition of thin stock, it is reasonable to assume $\mu_2 > \mu_1$ and $\sigma_2 > \sigma_1$.

We assume that investors can trade the thin stock only at random times $\{\tau_k\}_{k\geq 0}$, with $\tau_0 = 0 < \tau_1 < \cdots < \tau_k < \cdots$. Moreover, fixed an integer $k \geq 0$, we denote as Z_k the stochastic return of the light stock in the random time interval $\tau_k - \tau_{k-1}$:

$$Z_k = \frac{H_{\tau_k} - H_{\tau_{k-1}}}{H_{\tau_{k-1}}}$$

Assume that an agent take position in this market. She/he holds a capital to be shared among the three assets and the consumption. The agent's choices depend on time. More specifically, we denote as $\gamma, \theta, \varphi, c$ four stochastic processes representing the shares of capital invested on riskless bond, risky asset, thin stock and consumption, respectively. Since the thin stock is characterized by discrete random returns, also φ should have realizations in the discrete times τ's. More precisely, $\varphi_t = 0$, for $t \notin \{\tau_k\}_{k\geq 0}$. We will denote $\varphi_{\tau_k} = \varphi_k$, for $k \in \mathbb{N}$, and $\gamma + \theta + \varphi + c = 1$ is assumed, for each $t \geq 0$.

The agent's portfolio wealth X_t at time $t \geq 0$ changes due to changes in portfolio quantity invested and instantaneous consumption $c_t dt$. We have:

$$X_t = x + \int_0^t X_s[(1-\theta_s-\varphi_s-c_s)\,r(s)+\mu_1\theta_s-c_s]\,ds$$

$$+\int_0^t \sigma_1 X_s\theta_s dW_s^1 + \sum_{i=1}^{\infty} X_{\tau_i}\varphi_i Z_i\,\mathbb{I}_{\{\tau_i\leq t\}}$$

where, $\mathbb{I}$ is the usual characteristic function. The point process $\{\tau_i, Z_i\}_{i\geq 0}$ is assumed to be given by the jumps of a Lévy process Γ_t. We don't lose of generality by assuming that Γ_t is cadlag. In this sense, a jump at time t is described by $\Delta\Gamma_t = \Gamma_t - \Gamma_{t-}$.

Let us assume now that U is a Borel set in $\mathbb{R}$. The number of jumps occurring in the period $[0,t]$ with size in U can be written as follows:

$$N(t,U) = \sum_{i=1}^{n} \#_U\ (\Delta\Gamma_{\tau_i})\mathbb{I}_{\{\tau_i\leq t\}}$$

where, for each $i = 1, \ldots, n$, we define

$$\#_U\left(\Delta\Gamma_{\tau_i}\right) = \begin{cases} 1, & if\ \Delta\Gamma_{\tau_i} \in U \\ 0, & otherwise \end{cases}$$

Substituting the discrete process Z with its continuous version Γ, it is possible to rewrite X_t as follows:

$$X_t = x + \int_0^t X_s[(1-\theta_s-\varphi_s-c_s)\,r(s)+\mu_1\theta_s-c_s]\,ds$$

$$+\int_0^t \sigma_1 X_s\theta_s dW_s^1 + \int_0^t\int_{-1}^{\infty} X_s\,\varphi_s z N(ds,dz)$$

where $N(ds,dz)$ is the differential of $N(t,U)$.

The following assumption holds true hereafter.

Assumption 1. $\{\tau_i\}_{i\geq 0}$ is a sequence of jumps of a Poisson process with intensity λ.

Since the point process $\{\tau_i, Z_i\}_{i\geq 0}$ is a Levy process, Z_k is independent from $\{\tau_i, Z_i\}_{i<k}$, and it has distribution $p(t,dz)$.

Remark 1. Consider a Borel set U in $\mathbb{R}$ and define:

$$Y(t) = \sum_{i=1}^{\infty} Z_i\mathbb{I}_{\{\tau_i\leq t\}}.$$

The Levy measure υ of $Y(t)$ is given by

$$\upsilon(U) = \mathbb{E}[N(1,U)] = \lambda p(t,U)$$

where $\mathbb{E}$ is the expected value operator.

Let us define the admissible control processes.

Definition 1. An admissible control policy is a triplet (θ, φ, c) of continuous time processes $\mathbb{F}_t$ −adapted. We assume that the admissible control policies live in the admissible region, denoted as $\mathcal{A}(x)$.

If (θ, φ, c) is a triple of Markov controls and $\Psi \in C^2(0, +\infty)$, then the generator of the diffusion Levy process X_t is

$$A_{(\theta,\varphi,c)}\Psi(x) = [(1-\theta-\varphi-c)r + \mu_1\theta - c]\Psi'(x) +$$

$$\frac{\sigma_1^2 x^2 \theta^2}{2}\Psi''(x) + \int_{\mathbb{R}} \{\Psi(x + zx\varphi) - \Psi(x) - zx\varphi\Psi'(x)\} \upsilon(dz)$$

Definition 2. The value function of the optimal consumption/portfolio problem is given by

$$V(x) = sup_{(\theta,\varphi,c)\in\mathcal{A}(x)}\, \mathbb{E}[\int_0^\infty e^{-\rho t} U(c_t)dt], \qquad \forall\, x \geq 0$$

where $\rho > 0$ is the discount factor and U is a utility function defined in $[0, \infty)$.

We now state some conditions on the utility function U.

Assumption 2. $U \in C^1(0, +\infty)$ is strictly increasing, strictly concave, $U(0) = 0$, and the Inada condition is satisfied, i.e.

$$\lim_{x\to 0^+} U'(x) = +\infty, \qquad \lim_{x\to +\infty} U'(x) = 0$$

Remark 2. As argued in Cretarola et al. (to appear), Assumption 2 does not provide a strong restriction on the utility function U, since the most commonly used utility functions satisfy it.

The analysis of the above stated optimal stochastic control problem has been carried out by adopting a dynamic programming approach. We enunciate the key result of our study with a stepwise sketch of the proof. The details will be described in a longer version of the work.

Theorem 1. The value function V is the unique classical solution of the following HJB:

$$\rho V(x) = \; sup_{(\theta,\varphi,c)\in\mathbb{R}^2\times[0,+\infty)}\big[U(c) + A_{(\theta,\varphi,c)}\mathrm{V}(x)\big].$$

Sketch of the proof. The proof is articulated in three steps.

- The twice differentiability of the value function is assumed, and the HJB is derived by dynamic programming.
- The uniqueness in viscosity sense of the value function as solution of the HJB is proved.
- The regularity of the value function is analyzed. ■

A Verification Theorem argument guarantees that the optimal strategies can be formalized as follows:

$$c^* = argma_{\;\; c\in[0,+\infty)}[U(c) - cV'(x)] = I\big(V'(x)\big)$$

where I is the inverse of the first derivative of U;

$$\theta^* = argmax_{\theta\in\mathbb{R}}\left[x\theta(\mu_1 - r)V'(x) + \frac{\sigma_1^2 x^2 \theta^2 V''(x)}{2}\right] = \frac{-x(\mu_1 - r)V'(x)}{\sigma_1^2 x^2 V''(x)}$$

$$\varphi^* = argm \quad _{\varphi \in \mathbb{R}} \left[-xr\varphi V'(x) + \int_{\mathbb{R}} \big(V(x + zx\varphi) - zx\varphi V'(x) \big) \upsilon(dz) \right]$$

3 Conclusion

The analyses carried out in this paper allows to write explicitly the optimal strategies to be implemented, in order to obtain the optimal allocation of the wealth in frequently traded risky assets, light stocks, bonds and consumption. A sensitivity analysis of the optimal controls should be implemented, to obtain insights on the relation between optimality and model parameters. A further development of this paper in this direction is already in our research agenda.

References

A. Cretarola, F. Gozzi, H. Pham, P. Tankov. Optimal consumption policies in illiquid markets, Finance and Stochastic, to appear.

H.M. Markowitz. 1952. Portfolio Selection, Journal of Finance 7, 77–91.

R.C. Merton. 1969. Lifetime portfolio selection under uncertainty: the continuous-time case. Review of Economics and Statistics 51, 247–257.

R.C. Merton. 1971. Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory 3, 373–413.

B. Øksendal, A. Sulem. 2007. Applied Stochastic Control of Jump Diffusions. Second Edition, Universitext, Springer Verlag. Berlin, Heidelberg, New York.

H. Pham, P. Tankov. 2008. A model of optimal consumption under liquidity risk with random trading times. Mathematical Finance, 18(4), 613-627.

P.A. Samuelson. 1969. Lifetime Portfolio Selection by Dynamic Stochastic Programming. Review of Economics and Statistics 51, 239–246.

C. Tebaldi, E. Schwartz. 2006. Illiquid assets and Optimal Portfolio Choice. NBER wp 12633.