SHARP REVERSE HÖLDER INEQUALITY FOR C_p WEIGHTS AND APPLICATIONS

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ABSTRACT. We prove an appropriate sharp quantitative reverse Hölder inequality for the C_p class of weights from which we obtain as a limiting case the sharp reverse Hölder inequality for the A_∞ class of weights [13, 14]. We use this result to provide a quantitative weighted norm inequality between Calderón–Zygmund operators and the Hardy–Littlewood maximal function, precisely

$$\|Tf\|_{L^p(w)} \lesssim_{T,n,p,q} [w]_{C_q}(1 + \log^+[w]_{C_q}) \|Mf\|_{L^p(w)},$$

for $w \in C_q$ and $q > p > 1$, quantifying Sawyer's theorem [26].

1. INTRODUCTION AND MAIN RESULTS

One of the many forms of the classical Hölder inequality is the following one. For any non-negative function f and $\delta > 0$, we have

$$(1.1) \qquad \frac{1}{|Q|}\int_Q f(x)dx \le \left(\frac{1}{|Q|}\int_Q f(x)^{1+\delta}dx\right)^{\frac{1}{1+\delta}},$$

where $Q \subset \mathbb{R}^n$ is a cube and $|\cdot|$ denotes the Lebesgue measure. Reverse Hölder inequalities (RHI) are the same as (1.1) but with the inequality in the opposite direction. More precisely,

$$(1.2) \qquad \left(\frac{1}{|Q|}\int_Q f(x)^r dx\right)^{\frac{1}{r}} \le \frac{C}{|Q|}\int_Q f(x)dx,$$

for some $r > 1$. There has to be some constant $C \ge 1$, since otherwise it would be an equality. Weights satisfying (1.2) with uniform C for all cubes belong to the class RH_r.

The characterization of weights satisfying a RHI is a classical result: a weight satisfies a RHI if and only if it is contained in the class A_∞. In other words,

$$A_\infty = \bigcup_{r>1} RH_r.$$

A sharp quantitative RHI for A_∞ was given by Hytönen, Pérez and Rela [13, 14], see Section 2 for details. This quantification has two main properties: the constant

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on the right hand side is uniform for all weights and the dependence on the A_∞ constant of the exponent is sharp. These sharp RHI inequalities have been used in different applications: for obtaining quantitative estimates for norms of some singular integral operators [9, 15, 19], or for obtaining sharp estimates for solutions of certain PDE [16], among many others.

The main aim of this article is to, mimicking the A_∞ RHI of Hytönen–Pérez–Rela, give a sharp RHI in the context the C_p class of weights. This class was introduced by Muckenhoupt in [22] and it is intrinsically related to the Coifman–Fefferman inequality (CFI). In our context, CFI is a weighted norm inequality between a Calderón–Zygmund operator and the Hardy–Littlewood maximal function. More precisely,

$$(\text{CFI}-p) \qquad \int_{\mathbb{R}^n} (T^* f(x))^p w(x)dx \le c \int_{\mathbb{R}^n} (Mf(x))^p w(x)dx.$$

See Section 5 for the precise definitions and for an exposition on the inequality. Recently, inequality (CFI$-p$) has been shown to hold for a wider variety of operators [5, 7].

This inequality was first proved by Coifman and Fefferman in [8] for A_∞ weights, but Muckenhoupt showed in [22] that A_∞ is not a necessary condition for (CFI$-p$). In that article, he gave a necessary condition which he named the C_p condition. Note how the class depends on the exponent p. Later on, Sawyer [26] proved that $w \in C_{p+\eta}$ for some $\eta > 0$ is a sufficient condition in the range $p \in (1, \infty)$. It is still an open conjecture if C_p is a sufficient condition.

Since the C_p class is strictly bigger than A_∞, one cannot expect a true RHI for these weights. Nevertheless, there is a weaker RHI for these weights. Indeed, for $1 < p < \infty$, a weight w belongs to C_p if and only if there exist $\delta, C > 0$ such that

$$\left(\fint_Q w(x)^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \le \frac{C}{|Q|} \int_{\mathbb{R}^n} \left(M\chi_Q(x) \right)^p w(x) dx$$

for every cube Q, where $M\chi_Q$ denotes the Hardy–Littlewood maximal function of the characteristic function of the cube Q. Since $M\chi_Q \ge \chi_Q$ a.e., this is weaker than (1.2). Abusing slightly the language, we shall also call this weaker reverse Hölder inequality a reverse Hölder inequality.

As stated before, the aim of this article is to give a quantitative RHI for C_p weights, with a sharp dependence of the exponent on the weight. In order to do that we define the C_p constant of a weight w as

$$[w]_{C_p} := \sup_Q \frac{1}{\int_{\mathbb{R}^n} (M\chi_Q)^p w} \int_Q M(w\,\chi_Q).$$

See Section 2 for the motivation behind this definition.

Theorem 1.1 (Sharp quantitative RHI for C_p weights). *Let $1 < p < \infty$ and let w be a weight such that $0 < [w]_{C_p} < \infty$. Then $w \in C_p$ and w satisfies, for $\delta = \frac{1}{B_{n,p} \max([w]_{C_p}, 1)}$,*

$$\left(\fint_Q w^{1+\delta} \right)^{\frac{1}{1+\delta}} \le \frac{4}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w.$$

We emphasize that, even though the result is very similar to the sharp A_∞ RHI, the proof is by completely different methods.

Taking advantage of the connection between the classes A_∞ and C_p, we are able to obtain the sharp RHI for A_∞ weights as a consequence of the RHI for C_p weights. In this way, we know that the dependence of the C_p constant is sharp.

As it is intrinsically related to the C_p class, the last part of this article is devoted to the CFI. We give a quantification on the weighted inequalities between the Hardy–Littlewood maximal operator and Calderón–Zygmund operators. See Section 5 for precise definitions.

Theorem 1.2. *Let T be a Calderón–Zygmund operator and let $q > p > 1$. Then, if $w \in C_q$ and $f \in C_c^\infty(\mathbb{R}^n)$, then the following estimate holds*

$$(1.3) \qquad \|T^* f_{L^p(w)}\| \le c_{n,T,p,q}([w]_{C_q} + 1) \log(e + [w]_{C_q}) \|Mf\|_{L^p(w)}.$$

The proof of this theorem follows the original article of Sawyer [26] with some variants. In particular, the quantitative RHI for C_p weights above and the use of the good-λ inequality with exponential decay of Buckley [4] rather than the linear decay of Coifman-Fefferman [8] will play a main role in the argument.

For A_∞ weights, the following quantification of the CFI is known:

$$(1.4) \qquad \|T^* f\|_{L^p(w)} \le c_p [w]_{A_\infty} \|Mf\|_{L^p(w)}.$$

We note that the logarithm on (1.3) appears as a consequence of the non-local nature of the C_p condition, but based on (1.4) and the discussion on Section 4, we conjecture that the correct dependence should be linear:

Conjecture 1.3. *Let T and q, p as in the theorem. Then*

$$\left\| T^* f \right\|_{L^p(w)} \le c_{n,T,p,q}([w]_{C_q} + 1) \|Mf\|_{L^p(w)}.$$

2. Preliminaries

We start by fixing the basic notation. By a weight we mean a non-negative locally integrable function in $\mathbb{R}^n$. Weights will be denoted by the symbol w. For a measurable set E, χ_E denotes the characteristic function of E. M will denote the Hardy–Littlewood maximal operator

$$Mf(x) := \sup_Q \frac{\chi_Q(x)}{|Q|} \int_Q |f|,$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes. For a weight w and a measurable set E, $w(E)$ denotes $\int_E w(x)dx$. Also we will be using the notation, $\fint_E w = \frac{1}{|E|}\int_E w$ when E is of finite measure.

We present the definition of C_p as given in [22] and [26].

Definition 2.1 (C_p weights)**.** Let $1 < p < \infty$. We say that a weight w is of class C_p, and we write $w \in C_p$, if there exist $C, \varepsilon > 0$ such that for every cube Q and every measurable $E \subset Q$ we have

$$(2.2) \qquad w(E) \le C \left(\frac{|E|}{|Q|} \right)^\varepsilon \int_{\mathbb{R}^n} (M\chi_Q(x))^p w(x)dx.$$

It is clear, and this is a key point, that the A_∞ class of weights is contained in C_p for any $p \in (1, \infty)$.

We call the quantity $\int_{\mathbb{R}^n} (M\chi_Q)^p w$ the C_p-tail of w at Q. A weight has either finite C_p-tails at every cube or infinite C_p-tails at every cube.

Example 2.3 ([3], Chapter 7)**.** Let $w \in A_p$ and g a non-negative bounded convexly contoured function. Then $gw \in C_p$. The weights in C_p are non-doubling, and they may even vanish in a set of positive measure.

The weights in this class also satisfy a non-local weak Reverse Hölder Inequality, as stated in the following proposition. We shall call this property Reverse Hölder Inequality (RHI) for C_p weights, though it is not actually a proper RHI.

Proposition 2.1 (Reverse Hölder Inequality for C_p weights)**.** *A weight w belongs to the class C_p if and only if there exist $C, \delta > 0$ such that for every cube Q*

$$
(2.4) \qquad \left(\fint_Q w^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq \frac{C}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w.
$$

Moreover, we have that δ in (2.4) and ε in (2.2) are equivalent up to a dimensional constant.

We present the sharp reverse Hölder inequality for A_∞ weights. Using the notation in [14], we define for a positive weight w

$$
[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q),
$$

where the supremum is taken over all cubes with sides parallel to the axes. It is known that $w \in A_\infty$ if and only if $[w]_{A_\infty} < \infty$.

Theorem 2.2 (Sharp Reverse Hölder Inequality for A_∞ weights, [14])**.** *Let $w \in A_\infty$ and let Q be a cube. Then*

$$
(2.5) \qquad \left(\fint_Q w^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq 2 \fint_Q w,
$$

for any $\delta > 0$ such that $0 < \delta \leq \frac{1}{2^{n+1}[w]_{A_\infty}-1}$.

When we compare Proposition 2.1 and Theorem 2.2, we notice that $\int_{\mathbb{R}^n} (M\chi_Q)^p w$ in (2.4) plays the role of $w(Q)$ in (2.5). Keeping this similarity in mind, we define the C_p constant.

Definition 2.6 (C_p constant)**.** For an arbitrary non-zero weight w, we define

$$
[w]_{C_p} := \sup_Q \frac{1}{\int_{\mathbb{R}^n} (M\chi_Q)^p w} \int_Q M(\chi_Q w),
$$

where the supremum is taken over all cubes Q with sides parallel to the axes.

Notice that if w is not identically zero, the quantity on the denominator is always strictly greater than zero.

Remark 2.3. A weight w has infinite C_p-tails if and only if $[w]_{C_p} = 0$. Indeed, if w has infinite C_p-tails then the denominator equals infinity and we have $[w]_{C_p} = 0$. Conversely, if $[w]_{C_p} = 0$ we have that for every cube Q,

$$\frac{1}{\int_{\mathbb{R}^n}(M\chi_Q)^p w} \int_Q M(\chi_Q w) = 0.$$

This means that either $\int_Q (M\chi_Q w) = 0$ or $\int_{\mathbb{R}^n} M(\chi_Q)^p w = \infty$ for every cube Q. In the latter case, w has infinite C_p-tails. If $\int_Q (M\chi_Q w) = 0$ for every cube, then w must be zero almost everywhere.

By Proposition 2.1 we have that a weight w is in the class C_p if and only if $0 \leq [w]_{C_p} < \infty$.

Example 2.7. For $p > 1$ and small ε, for $w_\varepsilon(x) = |x|^{n(p-1-\varepsilon)}$ we have $[w_\varepsilon]_{C_p} \lesssim \varepsilon$. This can be shown by direct computation.

This is the main difference between the A_∞ and C_p constants, since $[w]_{A_\infty} \geq 1$ for an arbitrary weight w.

Remark 2.4. For any weight w we have the following relation between the different constants for $q \leq p$, $[w]_{C_q} \leq [w]_{C_p} \leq [w]_{A_\infty}$.

We now restate the quantitative RHI for C_p weights we mentioned on the introduction.

Theorem 2.5 (Quantitative RHI for C_p weights). *Let* $1 < p < \infty$ *and let* w *be a weight such that* $0 \leq [w]_{C_p} < \infty$. *Then* $w \in C_p$ *and* w *satisfies, for* $\delta = \frac{1}{B\max([w]_{C_p},1)}$, *with*

$$B = \frac{2^{1+4np+3n}(20)^n}{1 - 2^{-n(p-1)}},$$

(2.8) $$\left(\fint_Q w^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq \frac{4}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w.$$

Remark 2.6. Notice that B depends on the dimension and on p. Moreover, we have $B \to \infty$ whenever p tends to either ∞ or 1.

Remark 2.7. The quantification in terms of the parameters ε and C in (2.2) is $C = 2$ and

$$\varepsilon = \frac{1 - 2^{-n(p-1)}}{2^{2np+3n}(20)^n} \min(1, [w]_{C_p}^{-1}).$$

In particular, we have that both ε and δ are smaller than one.

3. Proof of the RHI

We may assume that w has finite C_p-tails, that is, $[w]_{C_p} > 0$. Indeed, if $[w]_{C_p} = 0$ then the right side of (2.8) equals infinity and the theorem is trivially true.

The proof follows a remark from [2], section 8.1, keeping track of the dependence on the constant of the weight combined with the proof given in [14] of the RHI for A_∞ weights.

We now introduce a functional over cubes that serves as a discrete analogue for the C_p-tail. Define, for a cube Q

$$(3.1) \qquad a_{C_p}(Q) := \sum_{k=0}^{\infty} 2^{-n(p-1)k} \fint_{2^k Q} w.$$

We note that $\alpha = \sum_{k \geq 0} 2^{-n(p-1)k} = (2^{n(p-1)})' < \infty$ only depends on n and p. In the following lemma we prove that the discrete and continuous C_p-tails are equivalent.

Lemma 3.1. *Let* $\beta = \sum_{l=0}^{\infty} 2^{-npl}$. *Then, for every weight w and every cube Q, we have*

$$(3.2) \qquad \frac{1}{\beta} a_{C_p}(Q) \leq \frac{1}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w \leq \frac{4^{np}}{\beta} a_{C_p}(Q).$$

As a corollary of this, we have that $a_{C_p}(Q) < \infty$ for every cube Q whenever w has finite C_p-tails.

Proof. Observe that $\beta = \sum_{l=0}^{\infty} 2^{-npl} = (2^{np})'$ and hence $\beta < 2$. Note that for $x \in 2^k Q \setminus 2^{k-1} Q$ we have $2^{-kn} \leq M\chi_Q(x) \leq 2^{-n(k-2)}$. Then

$$\frac{1}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w = \fint_Q w + \sum_{k=1}^{\infty} \frac{1}{|Q|} \int_{2^k Q \setminus 2^{k-1} Q} (M\chi_Q)^p w,$$

so we actually have

$$\fint_Q w + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} w(2^k Q \setminus 2^{k-1} Q) \leq \frac{1}{|Q|} \int_{\mathbb{R}^n} (M\chi_Q)^p w$$

$$\leq \fint_Q w + \sum_{k=1}^{\infty} \frac{2^{-np(k-2)}}{|Q|} w(2^k Q \setminus 2^{k-1} Q)$$

$$\leq 4^{np} \left(\fint_Q w + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} w(2^k Q \setminus 2^{k-1} Q) \right)$$

Now we rewrite (3.1) in the following way

$$\sum_{k=0}^{\infty} 2^{-n(p-1)k} \fint_{2^k Q} w = \fint_Q w + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} \left(\int_Q w + \sum_{j=1}^{k} \int_{2^j Q \setminus 2^{j-1} Q} w \right)$$

$$= \beta \fint_Q w + \frac{1}{|Q|} \sum_{j=1}^{\infty} \left(\sum_{k=j}^{\infty} 2^{-npk} \right) \int_{2^j Q \setminus 2^{j-1} Q} w$$

$$= \beta \left(\fint_Q w + \frac{1}{|Q|} \sum_{j=1}^{\infty} 2^{-pnj} \int_{2^j Q \setminus 2^{j-1} Q} w \right).$$

This finishes the proof of (3.2).   □

Proposition 3.2. *Let w be a weight and $p > 1$. Suppose that there exists a constant $0 < \gamma < \infty$ such that for every cube Q*

$$(3.3) \qquad \fint_Q M(\chi_Q w) \leq \gamma \, a_{C_p}(Q) < \infty.$$

Then there exists $0 < \delta \leq \frac{1}{A \max(\gamma, 1)}$, with

$$A = 20^n \frac{2^{1+3n}}{1 - 2^{-n(p-1)}},$$

such that for every cube Q,

$$\fint_Q M(\chi_Q w)^{1+\delta} \leq 2^{1+n(2p+3)} \gamma \, a_{C_p}(Q)^{1+\delta}.$$

Note that the infimum of the constants γ such that (3.3) holds is equivalent to the C_p constant of w, because of Lemma 3.1. In this case we will have $0 < [w]_{C_p} < \infty$.

Proof. Fix a cube $Q = Q(x_0, R)$, that is, the cube centered at the point x_0 and with side length $2R$ ($Q(x, R)$ is just a ball with the l^∞ distance in $\mathbb{R}^n$). The proof will be carried out following some steps.

Step 1. Let $r, \rho > 0$ and $l \in \mathbb{Z}$ be numbers that satisfy $R \leq r < \rho \leq 2R$ and $2^l(\rho - r) = R$. This in particular implies $l \geq 0$.

We define a new maximal operator

$$\tilde{M}v(x) := \sup_{k \in \mathbb{Z}} \fint_{Q(x, 2^k(\rho - r))} |v|.$$

We have the following pointwise bounds between the different maximal functions

$$\tilde{M}v \leq Mv \leq \kappa \tilde{M}v,$$

where κ does not depend on $\rho - r$. In particular, we can choose $\kappa = 4^n$. For $t \geq 0$ and a function F we define $F_t = \min(F, t)$. Now fix $m > 0$ with the intention of letting $m \to \infty$ in the end. Call $Q_r = Q(x_0, r)$ and $Q_\rho = Q(x_0, \rho)$.

We then have

$$\int_{Q_r} (M(\chi_{Q_r} w))_m^{1+\delta} \leq \kappa^{1+\delta} \int_{Q_r} (\tilde{M}(\chi_{Q_r} w))_m^\delta \, \tilde{M}(\chi_{Q_r} w)$$

$$\leq \kappa^{1+\delta} \int_{Q_r} (\tilde{M}(\chi_{Q_\rho} w))_m^\delta \, \tilde{M}(\chi_{Q_\rho} w)$$

$$\leq \kappa^{1+\delta} \delta \int_0^m \lambda^{\delta-1} u(Q_r \cap \{u > \lambda\}) d\lambda,$$

where $u = \tilde{M}(\chi_{Q_\rho} w)$. To state it in a separate line, we have

$$(3.4) \qquad \int_{Q_r} (M(\chi_{Q_r} w))_m^{1+\delta} \leq \kappa^{1+\delta} \delta \int_0^m \lambda^{\delta-1} u(Q_r \cap \{u > \lambda\}) d\lambda.$$

Step 2. Now we pick $\lambda_0 := 2^{n(l+1)} a_{C_p}(2Q)$ (which is finite by hypothesis). It is easy to see that for $x \in Q_r$ and $k \geq 0$, by the choice of λ_0 we have

$$(3.5) \qquad \fint_{Q(x, 2^k(\rho - r))} \chi_{Q_\rho} w \leq \lambda_0.$$

Indeed, we have that $Q_\rho \subset 2Q$, so we can make

$$\fint_{Q(x,2^k(\rho-r))} \chi_{Q_\rho} w \le \fint_{Q(x,2^k(\rho-r))} \chi_{2Q} w$$
$$= \frac{|2Q|}{|Q(x, 2^k(\rho - r))|}\fint_{2Q} w$$
$$\le 2^{n(l+1-k)}a_{C_p}(2Q) \le 2^{n(l+1)}a_{C_p}(2Q).$$

This completes the proof of (3.5) when $x \in Q_r$ and $k \ge 0$.

Let $\lambda > \lambda_0$ and $x \in Q_r \cap \{u > \lambda\}$. As $u(x) = \tilde{M}(\chi_{Q_\rho}w)(x) > \lambda > \lambda_0$, (3.5) and the fact $Q(x, 2^k(\rho - r)) \subset Q_\rho$ when $k < 0$ imply

$$u(x) = \sup_{k<0}\fint_{Q(x,2^k(\rho-r))} \chi_{Q_\rho} w = \sup_{k<0}\fint_{Q(x,2^k(\rho-r))} w.$$

For such an x, let $k_x = \max\{k : \fint_{Q(x,2^k(\rho-r))} w > \lambda\}$. Trivially, we have

$$Q_r \cap \{u > \lambda\} \subset \bigcup_{x \in Q_r \cap \{u>\lambda\}} Q(x, \frac{1}{5}2^{k_x}(\rho - r)).$$

We use the Vitali covering lemma for infinite sets and choose a countable collection of $x_i \in Q_r \cap \{u > \lambda\}$ so that the family of cubes $Q_i = Q(x_i, 2^{k_{x_i}}(\rho - r))$ satisfy the following properties:

- $Q_r \cap \{u > \lambda\} \subset \cup_i Q_i$,
- the cubes $\frac{1}{5}Q_i$ are pairwise disjoint,
- $\fint_{Q_i} w > \lambda$,
- $\fint_{2^k Q_i} w \le \lambda$, for any $k \ge 1$
- $Q_i \subset Q_\rho$.

We make the following claim. If we denote $Q_i^* = 2Q_i$ then for all $x \in Q_i \cap Q_r$,

$$u(x) \le 2^n M(\chi_{Q_i^*}w)(x).$$

Indeed, fix $x \in Q_i \cap Q_r$ and $k < 0$. If $k \ge k_{x_i}$ then by the stopping time we get

$$\fint_{Q(x,2^k(\rho-r))} w \le \frac{|Q(x_i, 2^{k+1}(\rho - r))|}{|Q(x, 2^k(\rho - r))|}\fint_{Q(x_i,2^{k+1}(\rho-r))} w$$
$$\le 2^n \lambda \le 2^n\fint_{Q_i} w \le 2^n M(\chi_{Q_i^*}w)(x).$$

In the other case, namely $k < k_{x_i}$ we have $Q(x, 2^k(\rho - r)) \subset Q_i^* \cap Q_\rho$ and hence

$$\fint_{Q(x,2^k(\rho-r))} w \le M(\chi_{Q_i^*}w)(x),$$

and thus the claim is proved.

Step 3. We use now this claim together with the stopping time and the hypothesis (3.3) to see

$$u(Q_r \cap \{u > \lambda\}) \le \sum_i u(Q_i \cap Q_r) \le \sum_i \int_{Q_i \cap Q_r} u \le 2^n \sum_i \int_{Q_i \cap Q_r} M(\chi_{Q_i^*}w)$$

$$\le 2^n \sum_i |Q_i^*| \fint_{Q_i^*} M(\chi_{Q_i^*} w) \le 2^n \gamma \sum_i |Q_i^*| a_{C_p}(Q_i^*)$$

But, using the properties of Q_i we get

$$a_{C_p}(Q_i^*) = \sum_{k=0}^{\infty} 2^{-nk(p-1)} \fint_{2^{k+1}Q_i} w \le \lambda\alpha,$$

so we have

$$u(Q_r \cap \{u > \lambda\}) \le 2^n \gamma \sum_i |Q_i^*| \alpha\lambda \le (20)^n \gamma\alpha \, |\cup_i Q_i| \lambda,$$

where in the last inequality we have used that $\frac{1}{5} Q_i$ are disjoint. Since each one of the cubes $Q_i \subset Q_\rho$ and $\lambda < \fint_{Q_i} w$ we have $\cup_i Q_i \subset Q_\rho \cap \{M(\chi_{Q_\rho} w) > \lambda\}$ so we have obtained for $\lambda > \lambda_0$

$$u(Q_r \cap \{u > \lambda\}) \le (20)^n \alpha\gamma\lambda |Q_\rho \cap \{M(\chi_{Q_\rho} w) > \lambda\}|.$$

Plugging everything on what we had in (3.4) we have

$$\int_{Q_r} (M(\chi_{Q_r}))_m^{1+\delta} \le \kappa^{1+\delta} \lambda_0^\delta u(Q_r) + \kappa^{\delta+1} (20)^n \gamma\alpha\delta \int_{\lambda_0}^m \lambda^\delta |Q_\rho \cap \{M(\chi_{Q_\rho} w) > \lambda\}| d\lambda.$$

Step 4. We define

$$\varphi(t) = \int_{Q_t} (M(\chi_{Q_t} w))_m^{1+\delta} \qquad t > 0.$$

Observe that $\varphi(t) < \infty$ for any $t > 0$. We claim that,

(3.6) $$\varphi(r) \le c_1 \gamma |Q| 2^{nl\delta} \left(a_{C_p}(Q) \right)^{1+\delta} + \delta \kappa^{\delta+1} (20)^n \gamma\alpha\varphi(\rho).$$

Indeed, combining what we obtained before in the following way:

$$\varphi(r) \le c_1 \gamma |Q| 2^{nl\delta} \left(a_{C_p}(Q) \right)^{1+\delta} + \kappa^{\delta+1} (20)^n \gamma\alpha \frac{\delta}{\delta+1} \int_{Q_\rho} M(\chi_{Q_\rho} w)_m^{\delta+1}$$

$$\le c_1 \gamma |Q| 2^{nl\delta} \left(a_{C_p}(Q) \right)^{1+\delta} + (\kappa^{\delta+1} (20)^n \gamma\alpha)\delta\varphi(\rho),$$

where $c_1 = 2^{n(p+1)(\delta+1)}$, and where we have used

$$u(Q_r) = \int_{Q_r} \tilde{M}(\chi_{Q_\rho} w) \le |2Q| \fint_{2Q} M(\chi_{2Q} w) \le 2^n |Q| \gamma a_{C_p}(2Q) \le 2^{np} |Q| \gamma a_{C_p}(Q),$$

since

$$a_{C_p}(2Q) \le 2^{n(p-1)} a_{C_p}(Q).$$

This yields the claim.

Step 5. Now we present an iteration scheme starting from claim (3.6). Remember that $l \ge 0$ was an integer such that $2^l(\rho - r) = R$. Set

$$t_0 = R,$$

$$t_{i+1} = t_i + 2^{-(i+1)} R = \sum_{j=0}^{i+1} 2^{-j} R, \quad i \ge 0.$$

Clearly, $t_i \to 2R$ as $i \to \infty$. This way, $2^{i+1}(t_{i+1} - t_i) = R$ and we can use them as $\rho = t_{i+1}$, $t_i = r$, and $l = i + 1$ in (3.6).

In other words, we have the estimate for $\varphi(t_i)$ in terms of $\varphi(t_{i+1})$:

$$\varphi(t_i) \leq c_2 2^{n\delta i} + c_3 \varphi(t_{i+1}),$$

where $c_2 = c_1 2^{n\delta}\gamma|Q|(a_{C_p}(Q))^{1+\delta}$, $c_3 = \kappa^{\delta+1}20^n\alpha\gamma\delta$. So, iterating this last inequality i_0 times we get

$$\varphi(R) = \varphi(t_0) \leq c_2 \sum_{j=0}^{i_0-1}(c_3 2^{n\delta})^j + c_3^{i_0}\varphi(t_{i_0}) \leq c_2 \sum_{j=0}^{i_0-1}(c_3 2^{n\delta})^j + (c_3)^{i_0}\varphi(2R)$$

We have to choose $\delta > 0$ so that we have the relation

$$(3.7) \qquad\qquad c_3 2^{n\delta} = 20^n \kappa^{\delta+1}\gamma\alpha\delta 2^{n\delta} < 1/2.$$

We may suppose $\delta < 1$. Once we have (3.7), we can take the limit $i_0 \to \infty$ and the sum is bounded by 2 and the second term goes to zero since $\varphi(2R) < \infty$. Hence

$$\varphi(R) \leq 2c_2 = 2^{1+n\delta+n(\delta+1)(p+1)}\gamma|Q|(a_{C_p}(Q))^{1+\delta}$$
$$< 2^{1+n(2p+3)}\gamma|Q|(a_{C_p}(Q))^{1+\delta},$$

and then

$$\frac{1}{|Q|}\int_Q M(\chi_Q w)_m^{1+\delta} \leq 2^{1+n(2p+3)}\gamma\left(a_{C_p}(Q)\right)^{1+\delta}.$$

Now, letting $m \to \infty$ and using the Fatou lemma we can conclude the proof.

To finish the proof, we make the choice of δ as follows. Coming back to (3.7) we see that, since we have δ in the exponent and γ can be arbitrarily small, we have to choose $\delta = \frac{1}{A\max(1,\gamma)}$ with

$$A = 2\kappa^2(20)^n 2^n \alpha = (20)^n \frac{2^{1+3n}}{1-2^{-n(p-1)}}. \qquad\qquad \square$$

We are ready to finally prove the theorem.

Proof of Theorem 2.5. Fix a cube Q. Let $M_{d,Q}$ denote the maximal operator with respect to the dyadic children of Q, that is

$$M_{d,Q}v(x) = \sup_{\substack{R\in\mathcal{D}(Q)\\x\in R}} \frac{1}{|R|}\int_R |v|, \quad x \in Q.$$

We argue as in [14], Theorem 2.3. By the Lebesgue differentiation theorem,

$$\int_Q w^{1+\delta} \leq \int_Q (M_{d,Q}w)^\delta w.$$

Call now $\Omega_\lambda = \{x \in Q : M_{d,Q}w(x) > \lambda\}$. For $\lambda \geq w_Q$ we make the Calderón–Zygmund decomposition of w at height λ to obtain $\Omega_\lambda = \cup_j Q_j$ with Q_j pairwise disjoint and

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} w \leq 2^n\lambda.$$

Multiplying by $|Q_j|$ and summing on j this inequality chain becomes

$$\lambda|\Omega_\lambda| \leq w(\Omega_\lambda) \leq 2^n\lambda|\Omega_\lambda|.$$

Then we have

$$\fint_Q (M_{d,Q}w)^\delta w = \frac{1}{|Q|} \int_0^\infty \delta\lambda^{\delta-1} w(\Omega_\lambda) d\lambda$$

$$\leq w_Q^{\delta+1} + \frac{1}{|Q|} \int_{w_Q}^\infty \delta\lambda^{\delta-1} w(\Omega_\lambda) d\lambda$$

$$\leq w_Q^{\delta+1} + \delta 2^n \frac{1}{|Q|} \int_{w_Q}^\infty \lambda^\delta |\Omega_\lambda| d\lambda$$

$$\leq w_Q^{\delta+1} + 2^n \frac{\delta}{\delta+1} \frac{1}{|Q|} \int_Q (M_{d,Q}w)^{1+\delta}.$$

Now we apply Proposition 3.2. We have $[w]_{C_p} \leq \beta\gamma \leq 4^{np}[w]_{C_p}$, so we need $\delta \leq \beta/A(\max(1, [w]_{C_p}))$, with β as in Lemma 3.1. So we get

$$\fint_Q (M_{d,Q}w)^\delta w \leq (1 + 2^{1+n(2p+4)} \frac{\delta}{\delta+1}\gamma) \left(a_{C_p}(Q)\right)^{1+\delta}$$

$$\leq (1 + 2^{1+n(2p+4)} \frac{\delta}{\delta+1} [w]_{C_p} \frac{4^{np}}{\beta}) \left(\frac{\beta}{|Q|} \int (M\chi_Q)^p w\right)^{1+\delta},$$

where we have used Lemma 3.1. Now, since we have $2^{4np}/\beta$ multiplying δ, we have to change the choice of δ slightly and make

$$\delta \leq \frac{2^{-4np}}{\beta} \frac{\beta}{A \max(1, [w]_{C_p})} = \frac{1}{B \max(1, [w]_{C_p})}.$$

This finishes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Sharpness of the exponent

For a cube Q, it is clear that $M\chi_Q$ equals 1 on the cube and is smaller than 1 outside the cube. Therefore $(M\chi_Q)^p$ converges to χ_Q a.e. when $p \to \infty$. Moreover, for a weight w with finite C_{p_0}-tails, by the dominated convergence theorem we have

$$\lim_{p\to\infty} \int_{\mathbb{R}^n} (M\chi_Q)^p w = w(Q).$$

For any weight $w \in A_\infty$, we have by the definition of the constant $[w]_{A_\infty}$ that for any cube Q

$$\int_Q M(w\chi_Q) \leq [w]_{A_\infty} w(Q) \leq [w]_{A_\infty} a_{C_p}(Q),$$

where $a_{C_p}(Q) = \sum_{k\geq 0} 2^{-n(p-1)k} \fint_{2^k Q} w$ is the discrete C_p-tail introduced in the previous section.

If we modify slightly the proof of Proposition 3.2 and Theorem 2.5 and add some extra hypothesis, we can recover the RHI for A_∞ weights. We explain how to do this in this section.

Fix a number $s > 1$. This will be the dilation parameter, which was $s = 2$ in the previous section. We plan on letting t tend to one in the end. We introduce the

corresponding discrete C_p-tail with respect to t,

$$a_{C_p,s}(Q) = \sum_{k \geq 0} s^{-n(p-1)k} \fint_{s^k Q} w.$$

Note that for any weight $w \in C_{p_0}$ we have $\lim_{p \to \infty} a_{C_p,s}(Q) = w_Q$ for any $s > 1$. Also, for a fixed $s > 1$ we introduce the corresponding discrete C_p constant

$$[w]_{C_p,s} := \sup_Q \frac{\int_Q M(\chi_Q w)}{a_{C_p,s}(Q)}$$

Remark 4.1. For a weight $w \in A_\infty$ and any $s > 1$ we have $\lim_{p \to \infty} [w]_{C_p,s} \leq [w]_{A_\infty}$.

Theorem 4.2. *Fix* $2 \geq s > 1$ *and* $1 < p < \infty$. *For a weight w in C_p and $\delta = \frac{1}{A_{t,p} \max(1,[w]_{C_p,s})}$ and every cube Q, with*

$$A_{s,p} = \frac{5^n 2^{1+5n}}{1 - s^{-n(p-1)}},$$

we have

(4.1)
$$\left(\frac{1}{|Q|} \int_Q w^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq (2^n + 1)\, a_{C_p,s}(sQ).$$

Before we prove this theorem, we give a proof of Theorem 2.2 as a corollary. Let $w \in A_\infty$. By Remark 4.1, we can let $p \to \infty$ in equation (4.1) and we obtain

(4.2)
$$\left(\frac{1}{|Q|} \int_Q w^{1+\delta_\infty} \right)^{\frac{1}{1+\delta_\infty}} \leq (2^n + 1)\, w_{sQ},$$

where

$$\delta_\infty = \lim_{p \to \infty} \frac{1 - s^{-n(p-1)}}{c_n \max(1, [w]_{C_p,s})} = \frac{1}{c_n [w]_{A_\infty}}.$$

Now we let $s \to 1$ in (4.2) and obtain

$$\left(\frac{1}{|Q|} \int_Q w^{1+\delta_\infty} \right)^{\frac{1}{1+\delta_\infty}} \leq (2^n + 1)\, w_Q,$$

which is in fact the reverse Hölder inequality for A_∞ weights.

Remark 4.3. The dimensional constants are bigger from those in Theorem 2.2, but the dependence on the weight is essentially the same. Because of this, we obtain that the dependence on w in Theorem 2.5 is sharp.

Proof of Theorem 4.2. We repeat the first three steps of the proof of Proposition 3.2, with the following modifications. This time, r, ρ, l will satisfy $s^l(\rho - r) = R$ and $R \leq r < \rho \leq R$. Also, now we will use the maximal operator $\tilde{M}v(x) = \sup_{k \in \mathbb{Z}} \fint_{Q(x,s^k(\rho-r))} u$, and some other trivial changes. For the fourth step, we leave $a_{C_p,s}(sQ)$ in the equation, so we get

$$\varphi(r) \leq s^{n(\delta+1)} \gamma |Q| s^{n\delta l} \left(a_{C_p,s}(sQ) \right)^{1+\delta} + (\kappa^{1+\delta}(5s^2)^n \gamma \alpha_s)\, \delta\, \varphi(\rho),$$

where $\alpha_s = \sum_{k \geq 0} s^{-nk(p-1)} = (1 - s^{-n(p-1)})^{-1}$. We make a similiar iteration scheme, namely $t_0 = R$ and $t_{i+1} = t_i + s^{-(i+1)}R \leq sR$. Now the condition for δ translates to $\delta \leq \frac{1}{A_{s,p} \max(1,\gamma)}$ where

$$A_{s,p} = \frac{5^n 2^{1+5n}}{1 - s^- n(p-1)}.$$

The main difference is that now we get

$$\frac{1}{|Q|} \int_Q (M(\chi_Q w)_m)^{1+\delta} \leq 2^{1+5n} \gamma \left(a_{C_p,s}(sQ)\right)^{1+\delta},$$

where the right part stays bounded whenever $p \to \infty$. Now we use Fatou lemma and make $m \to \infty$ to get

(4.3)
$$\frac{1}{|Q|} \int_Q M(\chi_Q w)^{1+\delta} \leq 2^{1+5n} \gamma \left(a_{C_p,s}(sQ)\right)^{1+\delta}.$$

Now we make the argument in the proof of Theorem 2.5 and combine it with (4.3). We get,

$$
\begin{aligned}
\fint_Q w^{1+\delta} &\leq (w_Q)^{1+\delta} + 2^n \frac{\delta}{1+\delta} \frac{1}{|Q|} \int_Q (M_{d,Q} w)^{1+\delta} \\
&\leq (w_Q)^{1+\delta} + 2^n \frac{\delta}{1+\delta} 2^{1+5n} \gamma \left(a_{C_p,s}(sQ)\right)^{1+\delta} \\
&\leq (2^n + \delta 2^{1+6n} \gamma) \left(a_{C_p,s}(sQ)\right)^{1+\delta} \\
&\leq (2^n + 1) \left(a_{C_p,s}(sQ)\right)^{1+\delta},
\end{aligned}
$$

whenever $\delta \leq \frac{1}{2^{1+6n}\gamma}$, which is true by the choice of δ. This finishes the proof. $\square$

5. C_p WEIGHTS AND THE COIFMAN–FEFFERMAN INEQUALITY

Let T^* denote a maximally truncated Calderón–Zygmund operator and M the Hardy–Littlewood maximal operator. Then for $w \in A_\infty$ and any $f \in L_c^\infty$, we have for any $0 < p < \infty$

(5.1)
$$\|T^* f\|_{L^p(w)} \leq C \|Mf\|_{L^p(w)},$$

where the constant depends only on w, T and p.

The classical proof of inequality (5.1) in [8] uses a good-λ inequality between the operators T^* and M. If the kernel of T is not regular enough, there is in general no good-λ inequality and even inequality (5.1) can be false, as is shown in [20].

There are ways of proving inequality (5.1) without using the good-λ inequality. For example, the proof given in [1] uses a pointwise estimate involving the sharp maximal function. Another proof can be found in [10], where the main tool is an extrapolation result that allows to obtain estimates like (5.1) for any A_∞ weight from the smaller class A_1 (see also [12]).

Inequality (5.1) is very important in the classical theory of Calderón–Zygmund operators, as it is used in the proof of many other weighted norm inequalities. The

first, and probably most important consequence of (5.1) is the boundedness of T^* in $L^p(w)$ for any weight $w \in A_p$, $1 < p < \infty$, namely

$$\int_{\mathbb{R}^n} (T^*f)^p w \le c \int_{\mathbb{R}^n} |f|^p w.$$

This comes as a direct corollary of Muckenhoupt's theorem [21].

Another consequence of inequality (5.1), though not as direct as the previous one, is the following inequality, obtained in [24]. For any weight w it holds

$$\left\| T^*f \right\|_{L^p(w)} \le c \left\| f \right\|_{L^p(M^{[p]+1}w)},$$

where $[p]$ denotes the integer part of p and M^k denotes the $k-$fold composition of M. This result is sharp since $[p] + 1$ cannot be replaced by $[p] + 1$. This is saying that inequality (5.1) encodes a lot of information. Very recently, this result was extended in [19] to the non-smooth case kernels, more precisely to the case case of rough singular operators T_Ω with $\Omega \in L^\infty(\mathbb{S}^{n-1})$, by proving inequality (5.1) for these operators. The proof of this result is quite different from the classical situation since there is no good-λ estimate involving these operators and it is a consequence of a sparse domination result for T_Ω obtained in [9] combined with the A_∞ extrapolation theorem mentioned above in [10].

Norm inequalities similar to (5.1) are true for other operators, for instance in [23] (fractional integrals) or [28] (square functions). Also, in the context of multilinear harmonic analysis one can find other examples, for example, it was shown in [18] an analogue for multilinear Calderón–Zygmund operators T, namely

$$\|T(f_1, ..., f_m)\|_{L^p(w)} \le c \|\mathcal{M}(f_1, ..., f_m)\|_{L^p(w)},$$

for $w \in A_\infty$ extending (5.1). We refer to [18] for the definition of the operator $\mathcal{M}$. The proof for the multilinear setting is in the spirit of the proof of inequality (5.1) given in [1]. There are also inequalities for (5.1) for more singular operators like the case of commutators of Calderón–Zygmund operators with BMO functions, as was proved in [25]. In this case, the result is, for $w \in A_\infty$,

$$\|[b, T]f\|_{L^p(w)} \le c \left\| M^2 f \right\|_{L^p(w)},$$

where $[b, T]f = bTf - T(bf)$ and $M^2 = M \circ M$. The result is false for M, because the commutator is not of weak type (1,1) and it would then contradict the extrapolation result from [10].

All of the inequalities mentioned above are true for the class A_∞ of weights, but A_∞ is not the whole picture for some of them. The correct class of weights is, in some sense, the C_p class. Muckenhoupt showed in [22] that A_∞ is not necessary for the CFI (5.1), and that the correct necesary condition is C_p. About sufficiency, Sawyer [26] proved that $w \in C_{p+\eta}$ for some $\eta > 0$ is sufficient for (5.1) in the range $p \in (1, \infty)$. It is still an open conjecture if C_p is a sufficient condition.

Although C_p weights were introduced in the context of the CFI, other inequalities have been proved to hold for these weights. For example, the Fefferman–Stein inequality, between the maximal operators of Hardy–Littlewood and of Fefferman–Stein, as can be found in [27], [6] for a quantified version, [17] in the weak-type context. In [7], the authors extended Sawyer's result to a wider class of operators than Calderón–Zygmund operators, including some pseudo-differential operators

and oscillatory integrals. Finally, in [5], Sawyer's result was extended to rough singular integrals and sparse forms.

The rest of this Section is devoted to the quantification of Sawyer's result. We define now the Calderón–Zygmund operators in a similar way as in [8]. We will need a kernel K defined away from the diagonal $x = y$ of $(\mathbb{R}^n)^2$ that satisfies the size condition

$$|K(x, y)| \leq \frac{A}{|x - y|^n}$$

for some $A > 0$ and every $x \neq y$. Furthermore, we require the following regularity conditions for some $\varepsilon > 0$

$$|K(x, y) - K(x', y)| \leq A \frac{|x - x'|^\varepsilon}{|x - y|^{n+\varepsilon}}$$

whenever $2|x - x'| \leq |x - y|$, and the symmetric condition

$$|K(x, y) - K(x, y')| \leq A \frac{|y - y'|^\varepsilon}{|x - y|^{n+\varepsilon}}$$

whenever $2|y - y'| \leq |x - y|$.

A Calderón–Zygmund operator associated to a kernel K satisfying the above conditions is a linear operator $T : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n)$ that satisfies

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

for $f \in C_c^\infty(\mathbb{R}^n)$ and $x \notin \mathrm{supp}(f)$. Additionally, we will require that T is bounded in L^2.

Now we define the maximal truncated singular integral operator T^* as follows

$$T^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} K(x, y) f(y) dy \right|.$$

We state the quantification of Theorem B from [26] and Theorem 16 from [7].

Theorem 5.1. *Fix $q > p > 1$. For all Calderón–Zygmund operator T, all bounded f with compact support and all weights $w \in C_q$ we have*

$$\left\| T^* f \right\|_{L^p(w)} \leq c_{n,T} \left(q + \frac{qp^2}{q - p} \right) \Phi(\max([w]_{C_p}, 1)) \|Mf\|_{L^p(w)},$$

where $\Phi(t) = t \log(e + t)$.

We begin with a few lemmas, which correspond to Lemmas 2-4 in [26]. We include most of the details concerning the quantification of the weight for the sake of completion.

Lemma 5.2. *Let $w \in C_q$. Fix $R \geq 2$ and $\delta > 0$. Then for every cube Q and any collection of pairwise disjoint cubes $Q_j \subset Q$ we have*

$$(5.2) \quad \int_{RQ} \sum_j (M\chi_{Q_j}(x))^p w(x) dx \leq \frac{1}{a\varepsilon} \log \frac{cR^{nq}}{\varepsilon\delta} w(RQ) + \delta \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx,$$

where a, c are dimensional constants and ε is the parameter for w in (2.2). Hence, we have

(5.3) $$\int \sum_j (M\chi_{Q_j}(x))^q w(x)dx \le c_n 4^{nq} \frac{1}{\varepsilon} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx.$$

Proof. For $\lambda > 0$, we will call $E_\lambda = \{x \in RQ : \sum_j M\chi_{Q_j}(x)^q > \lambda\}$. Since the cubes are pairwise disjoint, we have $\sum_j \chi_{Q_j} \in L^\infty$. Then by the exponential inequality from [11] we have $|E_\lambda| \le c_n e^{-a\lambda}|RQ|$, where c_n and a are positive dimensional constants. Then, applying the C_q condition (2.2) we get

$$w(E_\lambda) \le 2 \left(\frac{|E_\lambda|}{|RQ|} \right)^\varepsilon \int_{\mathbb{R}^n} (M\chi_{RQ}(x))^q w(x)dx$$

$$\le c_n e^{-\varepsilon a\lambda} R^{nq} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx.$$

Now we compute

$$\int_{RQ} \sum_j (M\chi_{Q_j}(x))^q w(x)dx = \int_0^\infty w(E_t)dt = \lambda w(E_\lambda) + \int_\lambda^\infty w(E_t)dt$$

$$\le \lambda w(RQ) + c_n R^{qn} \frac{1}{a\varepsilon} e^{-a\varepsilon\lambda} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx.$$

We can choose λ big enough so that

$$c_n R^{qn} \frac{1}{a\varepsilon} e^{-a\varepsilon\lambda} \le \delta,$$

and we get (5.2). In order to get (5.3), choose $R = 2$, $\delta = \frac{1}{\varepsilon}$ and use $\sum M\chi_{Q_j}^q \le 2^{nq} M\chi_Q$ almost everywhere outside of $2Q$. $\qquad\square$

Lemma 5.3 (Whitney covering lemma)**.** *Given $R \ge 1$, there is $C = C(n, R)$ such that if Ω is an open subset in $\mathbb{R}^n$, then $\Omega = \cup_j Q_j$ where the Q_j are disjoint cubes satisfying*

$$5R \le \frac{\text{dist}(Q_j, \mathbb{R}^n \setminus \Omega)}{\text{diam } Q_j} \le 15R,$$

$$\sum_j \chi_{RQ_j} \le C\chi_Q.$$

We now define an auxiliary function considered in [26]. This operator will be used to intuitively represent the integral of the function h to the power p after we apply the C_q condition.

Definition 5.4. Let h be a positive lower-semicontinuous function on $\mathbb{R}^n$ and k an integer. Let $\mathcal{W}(k)$ be the Whitney decomposition of the level set $\Omega_k = \{h(x) > 2^k\}$, that is, $\Omega_k = \cup_{Q \in \mathcal{W}(k)} Q$. We define the function

(5.5) $$M_{p,q}h(x)^p = \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{W}(k)} 2^{kp}(M\chi_Q(x))^q.$$

We need lower-semicontinuity in this definition to ensure that we can apply Whitney's decomposition theorem. In the practice, we will apply this operator to Mf and to T^*f, which are always lower-semicontinuous.

Lemma 5.4. *For a bounded, compactly supported function f and a weight $w \in C_q$ with $q > p$, we have*

$$(5.6) \qquad \int_{\mathbb{R}^n} (M_{p,q} Mf(x))^p w(x) dx \leq \left(c_n 2^{c_n \frac{pq}{q-p}} \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \right) \int_{\mathbb{R}^n} (Mf(x))^p w(x) dx,$$

where $M_{p,q}$ denotes the Marcinkiewicz integral operator as defined in (5.5).

Proof. Let $\mathcal{W}(k)$ be the Whitney decomposition of $\Omega_k = \{Mf > 2^k\}$, for any integer k. Let N be a positive integer to be chosen later and fix a cube P from the $k - N$ generation. We have, as in [26],

$$(5.7) \qquad |\Omega_k \cap 5P| \leq C 2^{-N} |P|,$$

where C depends only on the dimension n.

Now define the partial sums

$$S(k) = 2^{kp} \sum_{Q \in \mathcal{W}(k)} \int_{\mathbb{R}^n} (M\chi_Q)^q w$$

and

$$S(k; N, P) = 2^{kp} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \cap P \neq \emptyset}} \int_{\mathbb{R}^n} (M\chi_Q)^q w,$$

where in the last sum $P \in \mathcal{W}(k - N)$ is fixed. Because of the Whitney decomposition, $Q \cap R \neq \emptyset$ implies $Q \subset 5P$ for large N, so we have

$$S(k; N, P) \leq \int_{\mathbb{R}^n} 2^{kp} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} (M\chi_R)^q w$$

$$= \int_{10P} + \int_{(10P)^c} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} (M\chi_R)^q w = I + II \quad \text{for large } N.$$

Now, by (5.2), for any $\eta > 0$, which will be chosen chosen later, and for $R = 10$ we get

$$I \leq 2^{kp} \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\eta \varepsilon} w(10P) + \eta 2^{kp} \int_{\mathbb{R}^n} (M\chi_P)^q w.$$

Standard estimates for the maximal function of characteristics of cubes show that if x_P is the center of the cube Q_P then

$$II \leq c_n^q 2^{kp} \int_{(10P)^c} \frac{\sum |Q|^q}{|x - x_P|^{nq}} w(x) dx$$

$$\leq c_n^q 2^{kp} \int_{(10P)^c} \frac{|\Omega_k \cap P|^q}{|x - P|^{nq}} w(x) dx$$

$$\leq c_n^q 2^{kp} \int_{(10P)^c} \frac{2^{-qN} |P|^q}{|x - x_P|^{nq}} w(x) dx$$

$$\leq c_n^q 2^{N(p-q)+(k-N)p} \int_{\mathbb{R}^n} (M\chi_P)^q w,$$

where we have used (5.7) on the third inequality. Thus we have, by the Whitney decomposition theorem, for N large,

$$S(k) \leq \sum_{P \in \mathcal{W}(k-N)} S(k; N, P)$$

$$\leq \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\eta\varepsilon} 2^{kp} \int_{\mathbb{R}^n} \sum_{P \in \mathcal{W}(k-N)} (\chi_{10P}) w + (\eta 2^{Np} + c_n^q 2^{N(p-q)}) S(k-N)$$

$$\leq c_n \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\eta\varepsilon} 2^{kp} w(\Omega_{k-N}) + (\eta 2^{Np} + c_n^q 2^{N(p-q)}) S(k-N)$$

$$= c_n 2^{Np} \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\eta\varepsilon} 2^{p(k-N)} w(\Omega_{k-N}) + (\eta 2^{Np} + c_n^q 2^{N(p-q)}) S(k-N).$$

Now, since $q > p$, we can chose N so that $c_{n,q} 2^{N(p-q)} < \frac{1}{4}$, that is, $N \geq c_n \frac{q}{q-p}$; and η so that $\eta 2^{Np} < \frac{1}{4}$.

$$S(k) \leq c_n 2^{c_n \frac{pq}{q-p}} \frac{1}{a\varepsilon} (qc_n + \log\frac{1}{\varepsilon} + c_n \frac{pq}{q-p}) 2^{p(k-N)} w(\Omega_{k-N}) + \frac{1}{2} S(k-N)$$

$$\leq c_n 2^{c_n \frac{qp}{q-p}} \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} 2^{p(k-N)} w(\Omega_{k-N}) + \frac{1}{2} S(k-N).$$

Thus, with $S_M = \sum_{k \leq M} S(k)$ we get

$$S_M \leq \frac{1}{2} S_M + c_n 2^{c_n \frac{qp}{q-p}} \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \int_{\mathbb{R}^n} (Mf)^p w.$$

Now, exactly as in [26], p. 260, we have that $S_M < \infty$ and since it is clear that

$$\sup_M S_M = \int_{\mathbb{R}^n} (M_{p,q}(Mf))^p w,$$

we conclude the proof of the lemma. □

Remark 5.5. The important part of the dependence of the constant on the exponents p and q is that the lemma will fail to be true for $p = q$, with this kind of blowup.

Lemma 5.6. *Under the same assumptions of Theorem 5.1 we have*

$$\int_{\mathbb{R}^n} (M_{p,q} T^* f(x))^p w(x) dx \leq \left(c_n 2^p \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq} 2^{p+2}}{\varepsilon} \right) \int_{\mathbb{R}^n} (T^* f(x))^p w(x) dx$$

$$+ \left(c_n^q 2^{c_n \frac{p^2 q}{q-p}} \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon} \right) \int_{\mathbb{R}^n} (Mf(x))^p w(x) dx.$$

Proof. Let $\mathcal{W}(k)$ be the Whitney decomposition of the level set $\Omega_k = \{x \in \mathbb{R}^n : T^* f(x) > 2^k\}$ for integer k. One can prove as in [8] the following inequality: if $Q \in \mathcal{W}(k-1)$ and $5Q \not\subset \{Mf > 2^{k-N}\}$ for some $N \geq 1$, then

(5.8) $$|\{x \in Q; T^* f > 2^k\}| \leq C_T 2^{-N} |Q|.$$

Let $\mathcal{V}(k)$ be the Whitney decomposition of the set $\{Mf > 2^k\}$. We observe that for each cube $Q \in \mathcal{W}(k-1)$ there are two cases (for a fixed N that we will chose later).

Case (a). $5Q \subset \{Mf > 2^{k-N}\}$ in which case $5Q \subset c_n I$ for some $I \in \mathcal{V}(k-N)$.

Case (b). $5Q \not\subset \{Mf > 2^{k-N}\}$ in which case (5.8) implies

$$\sum_{\substack{P \in \mathcal{W}(k) \\ P \subset 5Q}} |P| \leq c_T 2^{-N} |Q|.$$

Now define the partial sums

$$S(k) = \sum_{Q \in \mathcal{W}(k)} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q)^q w$$

and

$$S(k; P) = \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \cap P \neq \emptyset}} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q)^q w \leq \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q)^q w.$$

Here, $P \in \mathcal{W}(k-1)$ and the last inequality follows from the Whitney decomposition. Thus,

$$S(k; P) \leq \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q)^q w = \int_{10P} + \int_{(10P)^c} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} 2^{kp} (M\chi_Q)^q w = I + II.$$

By (5.2) with $R = 10$ we have

$$I \leq c_n \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\varepsilon \eta} 2^{kp} w(5P) + \eta 2^{kp} \int_{\mathbb{R}^n} (M\chi_P)^q w,$$

where $\eta > 0$ is a positive number at our disposal. As in the previous lemma one can show

$$II \leq c_n^q 2^{kp - Nq} \int_{\mathbb{R}^n} (M\chi_P)^q w.$$

Combining estimates for I and II we obtain, for every case (b) cube $P \in \mathcal{W}(k-1)$,

(5.9) $$S(k; P) \leq c_n \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\varepsilon \eta} 2^{kp} w(5P) + (\eta + c_n^q 2^{-Nq}) 2^{kp} \int_{\mathbb{R}^n} (M\chi_P)^q w.$$

Thus

$$S(k) \leq \sum_{\substack{P \in \mathcal{W}(k-1) \\ P \text{ is } (a)}} S(k; P) + \sum_{\substack{P \in \mathcal{W}(k-1) \\ P \text{ is } (b)}} S(k; P) = III + IV.$$

Now, since each of the $Q \in \mathcal{W}(k)$ of type (a) intersects at most c of the $P \in \mathcal{W}(k-1)$, (yet again due to the Whitney decomposition), we have

$$III \leq c \sum_{I \in \mathcal{V}(k-N)} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset c_n I}} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q)^q w \leq c_n^q \frac{1}{\varepsilon} \sum_{I \in \mathcal{V}(k-N)} 2^{kp} \int_{\mathbb{R}^n} (M\chi_I)^q w,$$

where we have used (5.3) and $M\chi_{c_n I} \leq c_n M\chi_I$ (for two different c_n of course). For the remaining part we have by (5.9)

$$IV \leq c_n \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\varepsilon \eta} 2^{kp} \int_{\mathbb{R}^n} w(\Omega_{k-1}) + (\eta 2^p + c_n^q 2^{p-Nq}) 2^{(k-1)p} \sum_{P \in \mathcal{W}(k-1)} \int_{\mathbb{R}^n} (M\chi_P)^q w$$

$$\leq c_n 2^p \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}}{\varepsilon \eta} 2^{(k-1)p} w(\Omega_{k-1}) + \frac{1}{2} S(k-1),$$

if we choose η small enough and N big enough. This means $\eta = 2^{-(p+2)}$ and $N \geq c_n \frac{p+q}{q}$. Combining now estimates for III and IV we get

$$S(k) \leq \frac{1}{2}S(k-1) + \left(c_n 2^p \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}2^{p+2}}{\varepsilon}\right) 2^{(k-1)p}w(\Omega_{k-1})$$

$$+ \left(c_n^q 2^{c_n \frac{p}{q}(p+q)}\frac{1}{\varepsilon}\right) \sum_{I \in \mathcal{V}(k-N)} 2^{(k-N)p} \int_{\mathbb{R}^n} (M\chi_I)^q w.$$

Set $S_M = \sum_{k \leq M} S(k)$ and sum the previous inequality over $k \leq M$ to obtain

$$S_M \leq \frac{1}{2}S_M + \left(c_n 2^p \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}2^{p+2}}{\varepsilon}\right) \int_{\mathbb{R}^n} (T^*f)^p w$$

$$+ \left(c_n^q 2^{c_n \frac{p}{q}(p+q)}\frac{1}{\varepsilon}\right) \int_{\mathbb{R}^n} (M_{p,q}(Mf))^p w$$

$$\leq \frac{1}{2}S_M + \left(c_n 2^p \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq}2^{p+2}}{\varepsilon}\right) \int_{\mathbb{R}^n} (T^*f)^p w$$

$$+ \left(c_n^q 2^{c_n \frac{p}{q}(p+q)}\frac{1}{\varepsilon}\right)\left(c_n 2^{c_n \frac{pq}{q-p}}\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right) \int_{\mathbb{R}^n} (Mf)^p w,$$

by (5.6). It can be shown (cf. [26], p.262) that $S_M < \infty$, so taking it to the left and then taking the supremum over all M we obtain the desired result. □

Proof of theorem 5.1. Using the exponential decay from [4], we know that if we write $\{T^*f > 2^k\} = \cup_j Q_j$ as in the Whitney decomposition theorem, we have

(5.10) $$|\{x \in Q_j : T^*f(x) > 2\lambda, Mf(x) \leq \gamma\lambda\}| \leq ce^{-\frac{c}{\gamma}}|Q_j|,$$

for any $\gamma > 0$. We call E_j to the set in the left side of (5.10). Then, if we call r to the exponent $1 + \delta$ in Theorem 2.5, we get

$$w(E_j) = |E_j|\frac{1}{|E_j|}\int_{E_j} w \leq |E_j|\left(\frac{1}{|E_j|}\int_{E_j} w^r\right)^{\frac{1}{r}} \leq |E_j|^{\frac{1}{r'}}|Q_j|^{\frac{1}{r}}\left(\frac{1}{|Q_j|}\int_{Q_j} w^r\right)^{\frac{1}{r}}$$

$$\leq |E_j|^{\frac{1}{r'}}|Q_j|^{\frac{1}{r}}\frac{2}{|Q_j|}\int_{\mathbb{R}^n}(M\chi_{Q_j})^q w \leq ce^{-\frac{c}{\gamma r'}}\int_{\mathbb{R}^n}(M\chi_{Q_j})^q w.$$

We use the standard good-λ techniques as in [26] combined with Lemma 5.6 to get

$$\int_{\mathbb{R}^n}(T^*f)^p w \leq \left(\frac{2}{\gamma}\right)^p \int_{\mathbb{R}^n}(Mf)^p w + ce^{-\frac{c}{\gamma r'}}\int_{\mathbb{R}^n}(M_{p,q}T^*f)^p w$$

$$\leq \left(2^p\gamma^{-p} + e^{-\frac{c}{\gamma r'}}\left(c_n^q 2^{c_n \frac{p^2 q}{q-p}}\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon}\right)\right)\int_{\mathbb{R}^n}(Mf)^p w$$

$$+ ce^{-\frac{c}{\gamma r'}}\left(c_n 2^p \frac{1}{a\varepsilon}\log\frac{c_n 10^{nq}2^{p+2}}{\varepsilon}\right)\int_{\mathbb{R}^n}(T^*f)^p w$$

Choosing $\gamma^{-1} \sim c_n(q + \frac{p^2 q}{q-p})\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ we can make

$$e^{-\frac{c}{\gamma r'}}\left(c_n^q 2^{c_n \frac{p^2 q}{q-p}}\frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon}\right) < \frac{1}{2}$$

and

$$ce^{-\frac{c}{\gamma r'}} \left(c_n 2^p \frac{1}{a\varepsilon} \log \frac{c_n 10^{nq} 2^{p+2}}{\varepsilon} \right) < \frac{1}{2}$$

and taking the term to the left side (which is possible since it is finite, see [26]) we obtain

$$\int_{\mathbb{R}^n} (T^*f)^p w \le c_n^p \left(c_n(q + \frac{p^2 q}{q-p}) \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \right)^p \int_{\mathbb{R}^n} (Mf)^p w. \qquad \square$$

Remark 5.7. We conjecture that the first q in the constant should not be there. That way $\lim_{q\to\infty} c_q < \infty$. We think this should be the case because whenever $w \in C_q$ and q is bigger, we have more information. This way we could recover a weighted inequality for the A_∞ class, though it would be a worse one than the one we mention in the introduction. For this very reason, we conjecture that the dependence on the C_q constant is not sharp in this sense.

REFERENCES

[1] J. Álvarez, C. Pérez, *Estimates with A_∞ weights for various singular integral operators*, Boll. U. M. I. (7) 8-A (1994), 123-133.

[2] P. Auscher, S. Bortz, M. Egert, O. Saari, *Non-local Gehring lemmas in spaces of homogeneous type and applications*, J Geom Anal (2019).

[3] S. M. Buckley, *Harmonic analysis on weighted spaces.* ProQuest LLC, Ann Arbor, MI, 1990. Thesis (Ph.D.)–The University of Chicago.

[4] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc, 340 (1993), no. 1, 253-272.

[5] J. Canto, K. Li, L. Roncal, O. Tapiola, *C_p estimates for rough singular integrals and sparse forms*, preprint, arXiv:1909.08344.

[6] J. Canto, C. Pérez, *Extensions of the John–Nirenberg theorem and applications*, preprint, arXiv:1910.12957.

[7] E. Cejas, K. Li, C. Pérez, I. P. Rivera-Ríos, *Vector-valued operators, optimal weighted estimates and the C_p condition*, to appear Science China Mathematics.

[8] R. R. Coifman, C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Mathematica, 51 (1974), 241-250.

[9] J. M. Conde-Alonso, A. Culiuc, F. Di Plinio, and Y. Ou, A sparse domination principle for rough singular integrals, Anal. PDE **10** (2017), no. 5, 1255–1284.

[10] D. Cruz-Uribe, J. M. Martell, C. Pérez, *Extrapolation from A_∞ weights and applications*, J. Func. Anal., 213 (2004), 412-439.

[11] C. Fefferman and E. M. Stein, *Some maximal inequalities* Amer. J. Math., 93 (1971), no. 1, 107-115.

[12] G.P. Curbera, J. García-Cuerva, J.M. Martell and C. Pérez, *Extrapolation with weights, Rearrangement Invariant Function Spaces, modular inequalities and applications to Singular Integrals,* Advances in Mathematics, **203** (2006) 256-318.

[13] T. Hytönen, C. Pérez, *Sharp weighted bounds involving A_∞*, Anal. PDE, 6 (2013), 777-818.

[14] T. Hytönen, C. Pérez and E. Rela, *Sharp reverse Hölder property for A_∞ wieghts on spaces of homogeneous type*, Jour. of Func. Anal. 263 (2012) no. 12, 3883-3899.

[15] T. Hytönen, L. Roncal, O. Tapiola, *Quantitative weighted estimates for rough homogeneous singular integrals*, Israel J. Math. 218 (2017), no. 1, 133–164.

[16] R. Jiang, K. Li, J. Xiao, *Flow with A_∞ density and transport equation in $BMO(\mathbb{R}^n)$*, Forum of Math. Sigma (2019), vol 7, e43.

[17] A. Lerner, *A characterization of the weighted weak type Coifman–Fefferman and Fefferman–Stein inequalities*, Math. Ann. (2020). https://doi.org/10.1007/s00208-020-01980-z

[18] A. Lerner, S. Ombrosi, C. Pérez, R. Torres, R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory,* Adv. Math., 220 (2009) 1222-1264.

[19] K. Li, C. Pérez, L. Roncal, I. P. Rivera-Ríos, *Weighted norm inequalities for rough singular integral operators*, J. Geom. Anal., 2018.

[20] J. M. Martell, C. Pérez, R. Trujillo-González, *Lack of natural weighted estimates for some singular integral operators*, Trans. Amer. Math. Soc., 357 (2004), no. 1, 385-396.

[21] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., 165 (1972), 207-226.

[22] B. Muckenhoupt, *Norm inequalities relating the Hilbert transform to the Hardy–Littlewood maximal function*, in: Functional analysis and approximation (Oberwolfach 1980), Internat. Ser. Number. Math. 60 (Birkhäuser, Basel-Boston, Mass., 1981) pp. 219–231.

[23] B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **191** (1974), 261–274.

[24] C. Pérez, *Weighted norm inequalities for singular integral operators*, J. London. Math. Soc., 49 (1994), 296-308.

[25] C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy–Littlewood maximal function*, J. Fourier Anal. App., 3 (1997) no. 6, 743-756.

[26] E. T. Sawyer, *Norm inequalities relating singular integrals and the maximal function*, Studia Math. **75** (1983), no. 3, 253–263.

[27] K. Yabuta, *Sharp maximal function and C_p condition*, Arch. Math. 55 (1990), 151-155.

[28] J. M. Wilson, *Weighted Littlewood-Paley Theory and Exponential-Square Integrability* (Springer-Verlag, Lecture Notes in Mathematics) **1924** (2007).

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