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Global, decaying solutions of a focusing energy-critical heat equation in $\mathbb{R}^4$

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Abstract

We study solutions of the focusing energy-critical nonlinear heat equation $u_t = \Delta u - |u|^2 u$ in $\mathbb{R}^4$. We show that solutions emanating from initial data with energy and $\dot{H}^1-$norm below those of the stationary solution W are global and decay to zero, via the "concentration-compactness plus rigidity" strategy of Kenig-Merle [35, 36]. First, global such solutions are shown to dissipate to zero, using a refinement of the small data theory and the L^2-dissipation relation. Finite-time blow-up is then ruled out using the backwards-uniqueness of Escauriaza-Seregin-Šverák [17, 18] in an argument similar to that of Kenig-Koch [34] for the Navier-Stokes equations.

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1   Introduction

We consider here the Cauchy problem for the focusing, energy-critical non-linear heat equation in four space dimensions:

$$\begin{cases} u_t = \Delta u + |u|^2 u \\ u(0, x) = u_0(x) \in \dot{H}^1(\mathbb{R}^4) \end{cases} \tag{1.1}$$

for $u(t, x) \in \mathbb{C}$ with initial data in the energy space

$$\dot{H}^1(\mathbb{R}^4) = \{u \in L^4(\mathbb{R}^4; \mathbb{C}) \mid \|u\|_{\dot{H}^1}^2 = \int_{\mathbb{R}^4} |\nabla u(x)|^2 \, dx < \infty\}.$$

This is the L^2 gradient-flow equation for an *energy*, defined for $u \in \dot{H}^1$ as

$$E(u) = \int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{4} |u|^4 \right) dx,$$

and so in particular the energy is (formally) dissipated along solutions of (1.1):

$$\frac{d}{dt} E(u(t)) = - \int_{\mathbb{R}^4} |u_t|^2 \, dx \leq 0. \tag{1.2}$$

We refer to the gradient term in E as the *kinetic energy*, and the second term as the *potential energy*. The fact that the potential energy is negative

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expresses the focusing nature of the nonlinearity. Problem (1.1) is *energy-critical* in the sense that the scaling

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0 \tag{1.3}$$

leaves invariant the equation, the potential energy, and in particular the kinetic energy, which is the square of the *energy norm* $\|\cdot\|_{\dot{H}^1}$.

Static solutions of (1.1), which play a key role here, solve the elliptic equation

$$\Delta W + |W|^2 W = 0. \tag{1.4}$$

The function

$$W = W(x) = \frac{1}{(1 + \frac{|x|^2}{8})} \in \dot{H}^1(\mathbb{R}^4), \ \notin L^2(\mathbb{R}^4)$$

is a well-known solution. Its scalings by (1.3), and spatial translations of these are again static solutions, and multiples of these are well-known [1, 56] to be the unique extremizers of the Sobolev inequality

$$\forall u \in \dot{H}^1, \ \|u\|_{L^4} \leq C\|\nabla u\|_{L^2}, \quad C = \frac{\|W\|_{L^4}}{\|\nabla W\|_{L^2}} = \|\nabla W\|_{L^2}^{-\frac{1}{2}} \text{ the best constant.} \tag{1.5}$$

As for time-dependent solutions, a suitable local existence theory – see Theorem 2.1 for details – ensures the existence of a unique smooth solution $u \in C(I; \dot{H}^1(\mathbb{R}^4))$ on a maximal time interval $I = [0, T_{max}(u_0))$. The main result of this paper states that initial data lying "below" W gives rise to global smooth solutions of (1.1) which decay to zero:

Theorem 1.1. *Let $u_0 \in \dot{H}^1(\mathbb{R}^4)$ satisfy*

$$E(u_0) \leq E(W), \qquad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}. \tag{1.6}$$

Then the solution u of (1.1) is global ($T_{max}(u_0) = \infty$) and satisfies

$$\lim_{t \to \infty} \|u(t)\|_{\dot{H}^1} = 0. \tag{1.7}$$

The conditions (1.6) define a non-empty set, since by the Sobolev inequality (1.5) it includes all initial data of sufficiently small kinetic energy. Moreover, conditions (1.6) are sharp for global existence and decay in several

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senses. Firstly, if the kinetic energy inequality is replaced by equality, W itself provides a non-decaying (though still global) solution. Secondly, if the kinetic energy inequality is reversed, and under the additional assumption $u_0 \in L^2(\mathbb{R}^4)$, by a slight variant of a classical argument [39] we find that the solution blows up in finite time:

Theorem 1.2. *Let $u_0 \in H^1(\mathbb{R}^4)$ with*

$$E(u_0) < E(W), \qquad \|\nabla u_0\|_{L^2} \geq \|\nabla W\|_{L^2}.$$

Then the solution u of (1.1) has finite maximal lifespan: $T_{max}(u_0) < \infty$.

Thirdly, for any $a^* > 0$, [52] constructed finite-time blow-up solutions with initial data $u_0 \in H^1(\mathbb{R}^4)$ satisfying $E(W) < E(u_0) < E(W) + a^*$. See also [20] for formal constructions of blow-up solutions close to W.

It follows from classical variational bounds – see Lemma 2.1 – and energy dissipation (1.2), that any solution u on a time interval $I = [0, T)$ whose initial data satisfies (1.6), necessarily satisfies

$$\sup_{t \in I} \|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}. \tag{1.8}$$

So it will suffice to show that the conclusions of Theorem 1.1 hold for any solution satisfying (1.8). Indeed, we will prove:

1. If $I = [0, \infty)$ and (1.8) holds, then $\lim_{t \to \infty} \|\nabla u(t)\|_{L^2} = 0$. This is given as Theorem 3.1.

2. For any solution satisfying (1.8), $T_{max}(u(0)) = \infty$. This is given as Corollary 1.

That static solutions provide the natural threshold for global existence and decay, as in (1.8), is a classical phenomenon (eg. [53]) for critical equations, particularly well-studied in the setting of parabolic problems, mostly on compact domains, (e.g., [16, 28, 40, 57]) via "blow-up"-type arguments: first, failure of a solution to extend smoothly is shown, by a local regularity estimate, to imply (kinetic) energy concentration; then, near a point of concentration, rescaled subsequences are shown to converge locally to a non-trivial static solution; finally, elliptic/variational considerations prohibit non-trivial static solutions below the threshold.

The main purpose of our work is twofold: first, to establish the global-regularity-below-threshold result Theorem 1.1 on the full space $\mathbb{R}^4$; second, to do so not by way of the classical strategy sketched above, but instead via Kenig-Merle's [35, 36] "concentration-compactness plus rigidity" approach to critical *dispersive* equations, similar to Kenig-Koch's [34] implementation for the Navier-Stokes equations.

The argument is structured as follows. First, in Section 3, we prove the energy-norm decay of *global* solutions which satisfy (1.8). This is the content of Theorem 3.1. The strategy is that employed for the Navier-Stokes equations in [23]: reduce the problem to establishing the decay of *small* solutions (which is a refinement of the local theory) by exploiting the L^2-dissipation relation, using a solution-splitting argument to overcome the fact that the solution fails to lie in L^2.

Second, in Section 4, we prove the existence and compactness (modulo symmetries) of a "critical" element – a counterexample to global existence and decay, which is minimal with respect to $\sup_t \|\nabla u(t)\|_{L^2}$, following closely the work [37]. See Theorem 4.1. The technical tools are a profile decomposition compatible with the heat equation (described in Section 2.2) and a perturbation result for the linear heat equation, based on the local theory (Proposition 2.1).

Finally, in Section 5, we exclude the possibility of a compact solution with finite maximal existence time in Theorem 5.1. In fact this is a much stronger conclusion than required for the proof of Theorem 1.1, since it excludes *compact* finite-time blowup at *any* kinetic energy level – that is, it does not use (1.8). This part is based on classical parabolic tools. We first show that the centre of compactness remains bounded, by exploiting energy dissipation. Then a local small-energy regularity criterion, together with backwards uniqueness and unique continuation theorems of [17, 18], as in [34], imply the triviality of the critical element.

There is a vast literature on the semilinear heat equation $u_t = \Delta u + |u|^{p-1}u$. We content ourselves here with a brief review focused on the case of domain $\mathbb{R}^d$, and refer the reader to the recent book [50] for a more comprehensive review of the literature. For treatments of the Cauchy problem in L^p and Sobolev spaces under various assumptions on the nonlinearity and the initial data, see [59, 60, 6].

Much of the work concerns (energy) subcritical ($p < \frac{d+2}{d-2}$) problems. The seminal papers [29, 30, 31] introduced the study of heat equations through

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similarity variables and characterized blow-up solutions. In continuation of these works, [42] gave a first construction of a solution with arbitrarily given blow-up points, and see [48] (and references therein) for estimates of the blow-up rate, descriptions of the blow-up set, and stability results for the blow-up profile. We remark that blow-up in the subcritical case for $L^\infty-$solutions is known to be of Type I, in the sense that $\limsup_{t \to T_{\max}} (T_{\max} - t)^{\frac{1}{p-1}} \| u(\cdot, t) \|_{L^\infty} <$ $+\infty$, and Type I blow-up solutions are known to behave like self-similar solutions near the blow-up point. For a different set of criteria for global existence/blow-up in terms of the initial data we refer the reader to [9]. For results on the relation between the regularity of the nonlinear term and the regularity of the corresponding solutions, see [10].

For supercritical problems, [43, 44, 45] show that there is no Type II blow-up for $3 \le d \le 10$, while for $d \ge 11$ it is possible if p is large enough. It is also shown that a Type I blow-up solution behaves like a self-similar solution, while a Type II converges (in some sense) to a stationary solution. We also refer to the recent results [11] ($d \ge 11$, bounded domain), and [12] and to the preprint [4] for results in Morrey spaces.

For the critical case, we have already mentioned the finite-time blow-up constructions [20, 52], and we point to recent constructions of infinite-time blowup (bubbling) on bounded domains ($d \ge 5$) [13], and on $\mathbb{R}^3$ [47]. The work [21] deals with the continuation problem for reaction-diffusion equations. We finally mention the recent result [14], where a complete classification of solutions sufficiently close to the stationary solution W is provided for $d \ge 7$: such solutions either exhibit Type-I blow-up, dissipate to zero, or converge to (a slightly rescaled, translated) W. In particular, Type II blow-up is ruled out in $d \ge 7$ near W. We also refer to our work [32] for a critical case of the m-corotational Harmonic Map Heat Flow.

Remark 1.1. *We expect Theorem 1.1 to extend to the energy critical problem for the nonlinear heat equation in general dimension $d \ge 3$:*

$$
\begin{aligned}
u_t &= \Delta u + |u|^{\frac{4}{d-2}} u \\
u(t_0, x) &= u_0(x) \in \dot{H}^1(\mathbb{R}^d).
\end{aligned}
\tag{1.9}
$$

For simplicity of presentation, we will give the proof only for the case $d = 4$. As will be apparent from the proof, the result can be easily transferred to solutions of (1.9) for $d = 3$. The proof should also carry over to $d \ge 5$ with some extra work to estimate the low-power nonlinearity as in [58].

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Remark 1.2. *Our proof makes no use of any parabolic comparison principles, and so applies to complex-valued solutions. That said, for ease of writing some estimates we will sometimes replace the nonlinearity $|u|^2u$ with u^3 though the estimates remain true in the $\mathbb{C}$-valued case.*

2 Some analytical ingredients

2.1 Local theory

We first make precise what we mean by a *solution* in the energy space:

Definition 2.1. *A function $u : I \times \mathbb{R}^4 \to \mathbb{C}$ on a time interval $I = [0, T)$ $(0 < T \leq \infty)$ is a solution of (1.1) if $u \in (C_t \dot{H}_x^1 \cap L_{t,x}^6)([0, t] \times \mathbb{R}^4)$; $\nabla u \in L_{x,t}^3([0, t] \times \mathbb{R}^4)$; D^2u, $u_t \in L_t^2 L_x^2([0, t] \times \mathbb{R}^4)$ for all $t \in I$; and the Duhamel formula*

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}F(u(s))ds, \qquad (2.1)$$

is satisfied for all $t \in I$, where $F(u) = |u|^2u$. We refer to the interval I as the lifespan of u. We say that u is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. We say that u is a global solution if $I = \mathbb{R}^+ := [0, +\infty)$.

We will often measure the space-time size of solutions on a time interval I in $L_{x,t}^6$, denoting

$$S_I(u) := \int_I \int_{\mathbb{R}^4} |u(t,x)|^6 dxdt, \quad \|u\|_{S(I)} := S_I(u)^{\frac{1}{6}} = \left(\int_I \int_{\mathbb{R}^4} |u(t,x)|^6 dxdt \right)^{\frac{1}{6}}.$$

A local wellposedness theory in the energy space $\dot{H}^1(\mathbb{R}^4)$, analogous to that for the corresponding critical nonlinear Schrödinger equation (see e.g. [15]), is easily constructed, based on the Sobolev inequality and space-time estimates for the heat equation on $\mathbb{R}^4$ ([27]),

$$\|e^{t\Delta}\phi\|_{L_x^p(\mathbb{R}^4)} \lesssim t^{-2(1/a-1/p)}\|\phi\|_{L^a}, \qquad 1 \leq a \leq p \leq \infty$$

$$\|e^{t\Delta}\phi\|_{L_t^q L_x^p(\mathbb{R}^+ \times \mathbb{R}^4)} \lesssim \|\phi\|_{L^a}, \qquad \frac{1}{q} + \frac{2}{p} = \frac{2}{a}, \quad 1 < a \leq q$$

$$\|\int_0^t e^{(t-s)\Delta}f(s)ds\|_{L_t^q L_x^p(\mathbb{R}^+ \times \mathbb{R}^4)} \lesssim \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}(\mathbb{R}^+ \times \mathbb{R}^4)},$$

$$\frac{1}{q} + \frac{2}{p} = \frac{1}{\tilde{q}} + \frac{2}{\tilde{p}} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = \frac{1}{p} + \frac{1}{p'} = 1,$$

$$(2.2)$$

and $(\tilde{q}', \tilde{p}')$ the dual to any admissible pair $(\tilde{q}, \tilde{p})$.

We also refer the reader to [5, 59] for a treatment of the Cauchy problem in the critical Lebesque space $L^{\frac{2d}{d-2}}$; the arguments directly adapt to show wellposedness in $\dot{H}^1$. One can use a fixed-point argument to construct local-in-time solutions for arbitrary initial data in $\dot{H}^1(\mathbb{R}^4)$; however, as usual when working in critical scaling spaces, the time of existence depends on the profile of the initial data, not merely on its $\dot{H}^1$-norm. We summarize:

Theorem 2.1. *(Local well-posedness) Assume $u_0 \in \dot{H}^1(\mathbb{R}^4)$.*

1. *(Local existence) There exists a unique, maximal-lifespan solution to the Cauchy Problem (1.1) in $I \times \mathbb{R}^4, I = [0, T_{max}(u_0))$.*

2. *(Continuous dependence) The solution depends continuously on the initial data (in both the $\dot{H}^1$ and the S_I-induced topologies). Furthermore, T_{max} is a lower-semicontinuous function of the initial data.*

3. *(Blow-up criterion) If $T_{max}(u_0) < +\infty$, then $\|u\|_{S([0,T_{max}(u_0)))} = +\infty$.*

4. *(Energy dissipation) The energy $E(u(t))$ is a non-increasing function in time. More precisely, for $0 < t < T_{max}$,*

$$E(u(t)) + \int_0^t \int_{\mathbb{R}^4} |u_t|^2 \ dx \ dt = E(u_0). \tag{2.3}$$

5. *(Small data global existence) There is $\epsilon_0 > 0$ such that if $\|e^{t\Delta} u_0\|_{S(\mathbb{R}^+)} \leq \epsilon_0$, the solution u is global, $T_{max}(u_0) = \infty$, and moreover*

$$\|u\|_{S(\mathbb{R}^+)} + \|\nabla u\|_{(L_t^\infty L_x^2 \cap L_{x,t}^3)(\mathbb{R}^+ \times \mathbb{R}^4)} + \|D^2 u\|_{L_{x,t}^2(\mathbb{R}^+ \times \mathbb{R}^4)} \lesssim \epsilon_0. \tag{2.4}$$

This occurs in particular when $\|u_0\|_{\dot{H}^1(\mathbb{R}^4)}$ is sufficiently small.

An extension of the proof of the local existence theorem implies the following stability result (see, e.g., [38]):

Proposition 2.1. *(Perturbation result)*
For every $E, L > 0$ and $\epsilon > 0$ there exists $\delta > 0$ with the following property: assume $\tilde{u} : I \times \mathbb{R}^4 \to \mathbb{R}, I = [0, T)$, is an approximate solution to (1.1) in the sense that

$$\|\nabla e\|_{L_{t,x}^{\frac{3}{2}}(I \times \mathbb{R}^4)} \leq \delta, \quad e := \tilde{u}_t - \Delta \tilde{u} - |\tilde{u}|^2 \tilde{u},$$

and also
$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^4)} \leq E \quad and \quad \|\tilde{u}\|_{S(I)} \leq L,$$
then if $u_0 \in \dot{H}_x^1(\mathbb{R}^4)$ is such that
$$\|u_0 - \tilde{u}(0)\|_{\dot{H}_x^1(\mathbb{R}^4)} \leq \delta,$$
there exists a solution $u : I \times \mathbb{R}^4 \to \mathbb{R}$ of (1.1) with $u(0) = u_0$, and such that
$$\|u - \tilde{u}\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^4)} + \|u - \tilde{u}\|_{S(I)} \leq \epsilon.$$

2.2 Profile decomposition

The following proposition is the main tool (along with the Perturbation Proposition 2.1) used to establish the existence of a critical element. The idea is to characterize the loss of compactness in some critical embedding; it can be traced back to ideas in [41], [6], [54], [55] and their modern "evolution" counterparts [2], [35] and [36].

Proposition 2.2. *(Profile Decomposition) Let $\{u_n\}_n$ be a bounded sequence of functions in $\dot{H}^1(\mathbb{R}^4)$. Then, after possibly passing to a subsequence (in which case, we rename it u_n), there exists a family of functions $\{\phi^j\}_{j=1}^\infty \subset \dot{H}^1$, scales $\lambda_n^j > 0$ and centers $x_n^j \in \mathbb{R}^4$ such that:*

$$u_n(x) = \sum_{j=1}^J \frac{1}{\lambda_n^j} \phi^j \left(\frac{x - x_n^j}{\lambda_n^j} \right) + w_n^J(x),$$

$w_n^J \in \dot{H}^1(\mathbb{R}^4)$ *is such that:*

$$\lim_{J \to \infty} \limsup_n \|e^{t\Delta} w_n^J\|_{L_{t,x}^6(\mathbb{R}^+ \times \mathbb{R}^4)} = 0, \tag{2.5}$$

$$\lambda_n^j w_n^J(\lambda_n^j x + x_n^j) \rightharpoonup 0, \quad in \ \dot{H}^1(\mathbb{R}^4), \quad \forall j \leq J. \tag{2.6}$$

Moreover, the scales are asymptotically orthogonal, in the sense that

$$\frac{\lambda_n^j}{\lambda_n^i} + \frac{\lambda_n^i}{\lambda_n^j} + \frac{|x_n^i - x_n^j|^2}{\lambda_n^j \lambda_n^i} \to +\infty, \quad \forall i \neq j. \tag{2.7}$$

Furthermore, for all $J \geq 1$ we have the following decoupling properties:

$$\|u_n\|_{\dot{H}^1}^2 = \sum_{j=1}^J \|\phi^j\|_{\dot{H}^1}^2 + \|w_n^J\|_{\dot{H}^1}^2 + o_n(1) \tag{2.8}$$

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and

$$E(u_n) = \sum_{j=1}^{J} E(\phi^j) + E(w_n^J) + o_n(1). \tag{2.9}$$

The ideas behind such results are by now standard so we will skip the proof of the proposition. Let us just remark that the starting point is the static profile decomposition of [26] (also see [7], and Section 4 in [38]), and the main task becomes to show the (asymptotic) "smallness" of the remainder w_n^J, in the space-time sense of (2.5). The proof makes use of a refined Sobolev inequality and standard heat estimates. For details about this step we refer to Chapter 2 in [51].

2.3 Variational estimates

The elementary variational inequalities we use are summarized here:

Lemma 2.1. *(Variational Estimates)*

1. *If*
$$\|\nabla u_0\|_{L^2}^2 \le \|\nabla W\|_{L^2}^2, \quad E(u_0) \le (1 - \delta_0)E(W), \quad \delta_0 > 0,$$
   then there exists $\bar\delta = \bar\delta(\delta_0) > 0$ such that for all $t \in [0, T_{max}(u_0))$, the solution of (1.1) satisfies

$$\int |\nabla u(t)|^2 \le (1 - \bar\delta) \int |\nabla W|^2. \tag{2.10}$$

2. *If (2.10) holds, then*

$$\int (|\nabla u(t)|^2 - |u(t)|^4)dx \ge \bar\delta \int |\nabla u(t)|^2 \tag{2.11}$$

   and moreover $E(u(t)) \ge 0$.

Proof. The second statements are an immediate consequence of the sharp Sobolev inequality (1.5):

$$\int (|\nabla u(t)|^2 - |u(t)|^4)dx \ge \left[1 - \left(\frac{\|\nabla u(t)\|_{L^2}}{\|\nabla W\|_{L^2}}\right)^2\right] \|\nabla u\|_{L^2}^2 \gtrsim \|\nabla u\|_{L^2}^2$$

while the first follows easily from Sobolev and energy dissipation (2.3); see, e.g., Lemma 3.4/Theorem 3.9 in [35]. $\qquad\square$

3 Asymptotic decay of global solutions

In this section we prove the following theorem:

Theorem 3.1. *If $u \in C([0,\infty); \dot{H}^1(\mathbb{R}^4))$ is a solution to equation (1.1) which moreover satisfies*

$$\sup_{t \geq 0} \|\nabla u(t)\|_{L^2} < \|\nabla W\|_{L^2}, \tag{3.1}$$

then

$$S_{\mathbb{R}_+}(u) < \infty \quad and \quad \lim_{t \to \infty} \|u(t)\|_{\dot{H}^1} = 0.$$

Proof. The general strategy, drawn from the techniques of [23] for the Navier-Stokes equations, is as follows. We first show that global solutions for which $S_{\mathbb{R}^+}(u) < \infty$ – which includes *small* solutions by the small data theory (2.4) – decay to zero in the $\dot{H}^1-$norm. Second, we impose the extra assumption of H^1- data, so that we may exploit the L^2-dissipation relation to show finiteness of $\|\nabla u\|_{L^2_{x,t}}$, which in turns allows us to reduce matters to the case of small $\dot{H}^1$ data. Finally, to remove this extra assumption, we split the initial data in frequency, and estimate a perturbed equation.

Proposition 3.1. *If u is a global solution of (1.1) with $S_{\mathbb{R}^+}(u) < \infty$, then*

$$\lim_{t \to \infty} \|\nabla u(t, \cdot)\|_{L^2} = 0. \tag{3.2}$$

Proof. Let $u \in (C_t \dot{H}^1_x \cap L^6_{t,x})(\mathbb{R}_+ \times \mathbb{R}^4)$ be a global solution to (1.1). Just as one proves the blow-up criterion for the local theory Theorem 2.1, we first show:

Claim 3.1. $\|\nabla u\|_{L^3_{t,x}(\mathbb{R}^+ \times \mathbb{R}^4)} < \infty$.

Proof. Since $u \in L^6_{t,x}(\mathbb{R}^+ \times \mathbb{R}^4)$, given $\eta > 0$, we may subdivide $R^+ = [0, \infty)$ into a finite number of subintervals $I_j = [a_j, a_{j+1})$, $j = 0, 1, \ldots, J$, $0 = a_0 < a_1 < \cdots < a_J = \infty$, on which $\|u\|_{L^6_{t,x}}(I_j) \leq \eta$. Taking ∇ in the Duhamel formula (2.1) and using (2.2):

$$\|\nabla u\|_{(L^3_{t,x} \cap L^\infty_t L^2_x)}(I_0) \leq C\|e^{t\Delta}\nabla u_0\|_{L^2} + C\| \int_0^t S(t-s)\nabla(u^3)ds\|_{L^3_{t,x}(I_0)}$$

$$\leq C\|u_0\|_{\dot{H}^1} + C\|u^2\nabla u\|_{L^{3/2}_{t,x}(I_0)}$$

$$\leq C\|u_0\|_{\dot{H}^1} + C\|u\|^2_{L^6_{t,x}(I_0)}\|\nabla u\|_{L^3_{t,x}(I_0)},$$

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so by choosing $\eta < \frac{1}{\sqrt{2C}}$ we ensure

$$\|\nabla u\|_{(L^3_{t,x} \cap L^\infty_t L^2_x)(I_0)} \leq 2C\|u_0\|_{\dot{H}^1}.$$

In particular $\|u(a_1)\|_{\dot{H}^1} \leq 2C\|u_0\|_{\dot{H}^1}$, and so we may repeat this argument on the next interval I_1 to find $\|\nabla u\|_{(L^3_{t,x} \cap L^\infty_t L^2_x)(I_1)} \leq (2C)^2\|u_0\|_{\dot{H}^1}$, and, continuing, $\|\nabla u\|_{(L^3_{t,x} \cap L^\infty_t L^2_x)(I_j)} \leq (2C)^{j+1}\|u_0\|_{\dot{H}^1}$, for $j = 0, 1, \ldots, J$. The claim follows. $\qquad\square$

Now denote the linear evolution by $S(t) = e^{t\Delta}$, so the solution in Duhamel form is written

$$u(t) = S(t)u_0 + \int_0^t S(t-s)u^3(s)ds.$$

Let

$$\mathrm{I} := S(t)u_0, \quad \mathrm{II} := \int_0^\tau S(t-s)u^3(s)ds, \quad \mathrm{III} := \int_\tau^t S(t-s)u^3(s)ds,$$

for some τ to be determined later.

For term I we will take advantage of the decay of the heat propagator. By density, we can approximate ∇u_0 by $v \in L^1 \cap L^2$ and use a standard heat estimate:

$$\begin{aligned}\|\mathrm{I}\|_{\dot{H}^1} = \|S(t)\nabla u_0\|_{L^2} &\leq \|S(t)(\nabla u_0 - v)\|_{L^2} + \|S(t)v\|_{L^2} \\ &\leq \|\nabla u_0 - v\|_{L^2} + \|S(t)v\|_{L^2}.\end{aligned}$$

The first term can be made arbitrary small by the choice of v, while for the second, by (2.2), $\|S(t)v\|_{L^2} \to 0$ as $t \to \infty$, hence

$$\|I\|_{\dot{H}^1} \to 0 \text{ as } t \to \infty.$$

We now treat term III, which will allow us to fix τ. By the claim, for any $\epsilon > 0$, we can find τ such that $\|u\|_{L^6_{t,x}([\tau,\infty)\times\mathbb{R}^4)}, \|\nabla u\|_{L^3_{t,x}([\tau,\infty)\times\mathbb{R}^4)} \leq \epsilon$. Since we are considering the limit $t \to \infty$, we may assume $t > \tau \gg 1$, and so by the same estimate of the nonlinear term as in the proof of the claim,

$$\|III\|_{\dot{H}^1} \lesssim \|u\|^2_{L^6_{t,x}([\tau,t)\times\mathbb{R}^4)}\|\nabla u\|_{L^3_{t,x}([\tau,t)\times\mathbb{R}^4)} \lesssim \epsilon^3.$$

Having fixed τ in this manner, we turn to term II. First notice that

$$II = \int_0^\tau S(t-s)u^3(s)ds = S(t-\tau)\int_0^\tau S(\tau-s)u^3(s)ds.$$

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Since $\int_0^\tau S(\tau - s)u^3(s)ds \in \dot{H}^1$ (by $u \in L^6_{x,t}$ and (2.2)), the same approximation argument used for term I shows

$$\|II\|_{\dot{H}^1} = \|S(t-\tau)\int_0^\tau S(\tau-s)u^3(s)ds\|_{\dot{H}^1} \xrightarrow{t\to\infty} 0.$$

Since ϵ was arbitrary, (3.2) follows. □

Now if we assume $u_0 \in H^1(\mathbb{R}^4)$, multiplying (1.1) by u and integrating over space-time yields the L^2 dissipation relation

$$\|u(t)\|^2_{L^2} = \|u_0\|^2_{L^2} + 2\int_0^t \int_{\mathbb{R}^4} [u^4 - |\nabla u|^2]dxds. \tag{3.3}$$

Because of (3.1), we have the variational estimate (2.11) and so for some $\bar{\delta} > 0$,

$$\sup_{t\geq 0} \|u(t)\|^2_{L^2} + 2\bar{\delta}\|\nabla u\|^2_{L^2_{t,x}(\mathbb{R}_+ \times \mathbb{R}^4)} \leq \|u_0\|^2_{L^2}.$$

This estimate immediately implies that for any $\epsilon_0 > 0$, there is some time t_0 such that $\|u(t_0)\|_{\dot{H}^1} \leq \epsilon_0$, and we can directly apply the small data result (2.4) (with initial time $t = t_0$) to conclude that $S_{\mathbb{R}^+}(u) < \infty$, and so by Proposition 3.1, $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0$, as required.

To remove the extra assumption $u_0 \in L^2$, split

$$u_0 = w_0 + v_0, \qquad \|w_0\|_{\dot{H}^1} \ll 1, \qquad v_0 \in H^1.$$

Define $w(t)$ to be the solution to (1.1) with initial data w_0:

$$w_t = \Delta w + w^3$$
$$w(0,x) = w_0(x) \in \dot{H}^1(\mathbb{R}^4).$$

From the small data theory (2.4), $w \in C_t\dot{H}^1_x(\mathbb{R}_+ \times \mathbb{R}^4)$ is global, with

$$\|w\|_{L^6_{t,x}(\mathbb{R}_+\times\mathbb{R}^4)} + \|\nabla w\|_{(L^\infty_t L^2_x \cap L^3_{t,x})(\mathbb{R}_+\times\mathbb{R}^4)} \lesssim \|\nabla w_0\|_{L^2} \ll 1 \tag{3.4}$$

and by Proposition 3.1, $\|w(t)\|_{\dot{H}^1} \xrightarrow{t\to\infty} 0$.

Defining v by $v := u - w$, it will be a solution of the perturbed equation

$$v_t - \Delta v = v^3 + 3w^2v + 3wv^2.$$

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Just as in the derivation of the L^2-dissipation relation (3.3), multiply by v and integrate in space-time:

$$\|v(t)\|_{L^2}^2 - \|v_0\|_{L^2}^2 + 2\int_0^t \|\nabla v\|_{L^2}^2 = 2\int_0^t \|v\|_{L^4}^4 + 6\int_0^t \int_{\mathbb{R}^4} w^2 v^2 + 6\int_0^t \int_{\mathbb{R}^4} wv^3.$$

By (3.4), picking $\|\nabla w_0\|_{L^2}$ small enough, ensures that condition (3.1) holds also for v: $\sup_{t\geq 0}\|\nabla v(t)\|_{L^2} < \|\nabla W\|_{L^2}$. Hence by (2.11), for some $\bar{\delta} > 0$,

$$\|v(t)\|_{L^2}^2 + \bar{\delta}\int_0^t \|\nabla v\|_{L^2}^2 \lesssim \|v_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^4} w^2 v^2 + 6\int_0^t \int_{\mathbb{R}^4} wv^3,$$

and so by Hölder and Sobolev,

$$\begin{aligned}
\|v(t)\|_{L^2}^2 + \bar{\delta}\|\nabla v\|_{L^2 L^2}^2 &\lesssim \|v_0\|_{L^2}^2 + \|w\|_{L^\infty L^4}^2 \|v\|_{L^2 L^4}^2 + \|w\|_{L^\infty L^4}\|v\|_{L^\infty L^4}\|v\|_{L^2 L^4} \\
&\lesssim \|v_0\|_{L^2}^2 + \|\nabla w\|_{L^\infty L^2}^2\|\nabla v\|_{L^2 L^2}^2 \\
&\quad + \|\nabla w\|_{L^\infty L^2}\|\nabla v\|_{L^\infty L^2}\|\nabla v\|_{L^2 L^2}.
\end{aligned}$$

So by (3.4), choosing $\|\nabla w_0\|_{L^2}$ small enough yields $\int_0^\infty \|\nabla v\|_{L^2}^2 dt < \infty$, and hence there is $T > 0$ for which $\|\nabla v(T)\|_{L^2} < \|w_0\|_{\dot{H}^1}$ and so $\|\nabla u(T)\|_{L^2} \leq 2\|\nabla w_0\|_{L^2}$. Choosing $\|\nabla w_0\|_{L^2}$ smaller still, if necessary, we are able to apply the small data result (2.4) to conclude $S_{\mathbb{R}_+}(u) < \infty$, and moreover by Proposition 3.1,

$$\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0,$$

concluding the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4   Minimal blow-up solution

For any $0 \leq E_0 \leq \|\nabla W\|_2^2$, we define

$$L(E_0) := \sup\{S_I(u) \mid u \text{ a solution of (1.1) on } I \text{ with } \sup_{t\in I}\|\nabla u(t)\|_2^2 \leq E_0\},$$

where $I = [0, T)$ denotes the existence interval of the solution in question. $L : [0, \|\nabla W\|_2^2] \to [0, \infty]$ is a continuous (this follows from Proposition 2.1), non-decreasing function with $L(\|\nabla W\|_2^2) = \infty$. Moreover, from the small-data theory (2.4),

$$L(E_0) \lesssim E_0^3 \text{ for } E_0 \leq \epsilon_0.$$

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Thus, there exists a unique critical kinetic energy $E_c \in (0, \|\nabla W\|_2^2]$ such that

$$L(E_0) < \infty \text{ for } E_0 < E_c, \quad L(E_0) = \infty \text{ for } E_0 \geq E_c.$$

In particular, if $u : I \times \mathbb{R}^4 \to \mathbb{R}$ is a maximal-lifespan solution, then

$$\sup_{t \in I} \|\nabla u(t)\|_2^2 < E_c \implies u \text{ is global, and } \|u\|_{S(\mathbb{R}^+)} \leq L(\sup_{t \in I} \|\nabla u(t)\|_2^2) < \infty.$$

The goal of this section is the proof of the following theorem:

Theorem 4.1. *There is a maximal-lifespan solution $u_c : I \times \mathbb{R}^4 \to \mathbb{R}$ to (1.1) such that $\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 = E_c$, $\|u_c\|_{S(I)} = +\infty$. Moreover, there are $x(t) \in \mathbb{R}^4, \lambda(t) \in \mathbb{R}^+$, such that*

$$K = \left\{ \frac{1}{\lambda(t)} u_c\left(t, \frac{x - x(t)}{\lambda(t)}\right) \mid t \in I \right\} \tag{4.1}$$

is precompact in $\dot{H}^1$.

For the proof of this theorem we closely follow the arguments in [37]. The extraction of this minimal blow-up solution (and its compactness up to scaling and translation) will be a consequence of the following proposition:

Proposition 4.1. *Let $u_n : I_n \times \mathbb{R}^4$ be a sequence of solutions to (1.1) such that*

$$\limsup_n \sup_{t \in I_n} \|\nabla u_n\|_2^2 = E_c \text{ and } \lim_{n \to \infty} \|u_n\|_{S(I_n)} = +\infty. \tag{4.2}$$

where I_n are of the form $[0, T_n)$. Denote the initial data by $u_n(x, 0) = u_{n,0}(x)$. Then the sequence $\{u_{n,0}\}_n$ converges, modulo scaling and translations, in $\dot{H}^1$ (up to an extraction of a subsequence).

Proof. The sequence $\{u_{n,0}\}_n$ is bounded in $\dot{H}^1$ by (4.2) so applying the profile decomposition (up to a further subsequence) we get

$$u_{n,0}(x) = \sum_{j=1}^{J} \frac{1}{\lambda_n^j} \phi^j\left(\frac{x - x_n^j}{\lambda_n^j}\right) + w_n^J(x)$$

with the properties listed in Proposition 2.2.

Define the nonlinear profiles $v^j : I^j \times \mathbb{R}^4 \to \mathbb{R}$, $I^j = [0, T^j_{max})$, associated to ϕ^j by setting them to be the maximal-lifespan solutions of (1.1) with initial data $v^j(0) = \phi^j$. Also, for each $j, n \geq 1$ we introduce $v_n^j : I_n^j \times \mathbb{R}^4 \to \mathbb{R}$ by

$$v_n^j(t) = \frac{1}{\lambda_n^j} v^j \left(\frac{t}{(\lambda_n^j)^2}, \frac{x - x_n^j}{\lambda_n^j} \right), \quad I_n^j := \{ t \in \mathbb{R} : \frac{t}{(\lambda_n^j)^2} \in I^j \}.$$

Each v_n^j is a solution with $v_n^j(0) = \frac{1}{\lambda_n^j} \phi(\frac{x - x_n^j}{\lambda_n^j})$ and maximal lifespan $I_n^j = [0, T^{n,j}_{max})$, $T^{n,j}_{max} = (\lambda_n^j)^2 T^j_{max}$.

For large n, by the asymptotic decoupling of the kinetic energy (property (2.8)), there is a $J_0 \geq 1$ such that $\|\nabla \phi^j\|_2^2 \leq \epsilon_0$ for all $j \geq J_0$, where ϵ_0 is as in Theorem 2.1, 4. Hence, for $j \geq J_0$, the solutions v_n^j are global and decaying to zero, and moreover

$$\sup_{t \in \mathbb{R}^+} \|\nabla v_n^j\|_2^2 + \|v_n^j\|_{S(\mathbb{R}^+ \times \mathbb{R}^4)}^2 \lesssim \|\nabla \phi^j\|_2^2 \tag{4.3}$$

by the small data theory (2.4).

Claim 4.1. *(There is at least one bad profile). There exists $1 \leq j_0 < J_0$ such that $\|v^{j_0}\|_{S(I^{j_0})} = \infty$.*

For contradiction, assume that for all $1 \leq j < J_0$

$$\|v^j\|_{S(I^j)} < \infty \tag{4.4}$$

which by the local theory implies $I^j = I_n^j = [0, \infty)$ for all such j and for all n. The goal is to deduce a bound on $\|u_n\|_{S(I_n)}$ for sufficiently large n. To do so, we will use Proposition 2.1, for which we first need to introduce a good approximate solution.

Define

$$u_n^J(t) = \sum_{j=1}^J v_n^j(t) + e^{t\Delta} w_n^J. \tag{4.5}$$

We will show that for n and J large enough this is a good approximate solution (in the sense of Proposition 2.1) and that $\|u_n^J\|_{S([0,+\infty))}$ is uniformly bounded. The validity of both points implies that the true solutions u_n should not satisfy (4.2), reaching a contradiction.

First observe

$$\sum_{j \geq 1} \|v_n^j\|_{S([0,\infty))}^2 = \sum_{j=1}^{J_0 - 1} \|v_n^j\|_{S([0,\infty))}^2 + \sum_{j \geq J_0} \|v_n^j\|_{S([0,\infty))}^2 \tag{4.6}$$

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$$\lesssim 1 + \sum_{j \geq J_0} \|\nabla \phi^j\|_2^2 \lesssim 1 + E_c \tag{4.7}$$

where we have used (4.4), property (2.8) and (4.2).

Now, using the above and (2.5) in Proposition 2.2:

$$\lim_{J \to \infty} \overline{\lim_n} \|u_n^J\|_{S([0,+\infty))} \lesssim 1 + E_c. \tag{4.8}$$

For convenience, denote

$$\|u\|_{\tilde{S}(I)} := \|\nabla u\|_{L_{x,t}^3(I \times \mathbb{R}^4)}.$$

Under the assumption (4.4), we can also obtain

$$\|v^j\|_{\tilde{S}(I^j)} < \infty,$$

and so similarly we have

$$\lim_{J \to \infty} \overline{\lim_n} \|u_n^J\|_{\tilde{S}([0,+\infty))} < \infty.$$

To apply Proposition 2.1, it suffices to show that u_n^J asymptotically solves (1.1) in the sense that

$$\lim_{J \to \infty} \overline{\lim_n} \left\|\nabla[(\partial_t - \Delta)u_n^J - F(u_n^J)]\right\|_{L_{t,x}^{\frac{3}{2}}([0,+\infty) \times \mathbb{R}^4)} = 0$$

which reduces (adding and subtracting the term $F(\sum_{j=1}^J v_n^j)$ and using the triangle inequality) to proving

$$\lim_{J \to \infty} \overline{\lim_n} \left\|\nabla[\sum_{j=1}^J F(v_n^j) - F(\sum_{j=1}^J v_n^j)]\right\|_{L_{t,x}^{\frac{3}{2}}([0,+\infty) \times \mathbb{R}^4)} = 0 \tag{4.9}$$

and

$$\overline{\lim_n} \left\|\nabla[F(u_n^J - e^{t\Delta}w_n^J) - F(u_n^J)]\right\|_{L_{t,x}^{\frac{3}{2}}([0,+\infty) \times \mathbb{R}^4)} = 0. \tag{4.10}$$

The following easy pointwise estimate will be of use:

$$|\nabla[(\sum_{j=1}^J F(v_j) - F(\sum_{j=1}^J v_j)]| \lesssim_J \sum_{i \neq j} |\nabla v_j||v_i|^2. \tag{4.11}$$

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We have shown that for all $j \geq 1$ and n large enough $v_n^j \in \tilde{S}([0,\infty))$, so using property (2.7)

$$\overline{\lim_n}\||v_n^j|^2 \nabla v_n^i\|_{L_{t,x}^{\frac{3}{2}}([0,\infty)\times\mathbb{R}^4)} = 0$$

for all $i \neq j$; thus

$$\overline{\lim_n}\|\nabla[(\sum_{j=1}^J F(v_j) - F(\sum_{j=1}^J v_j)]\|_{L_{t,x}^{\frac{3}{2}}} \lesssim_J \lim_{n\to\infty}\overline{\lim_n}\sum_{i\neq j}\|\nabla v_n^j |v_n^i|^2\|_{L_{t,x}^{\frac{3}{2}}} = 0$$

settling (4.9).

$$\|\nabla[F(u_n^J - e^{t\Delta}w_n^J) - F(u_n^J)]\|_{L_{t,x}^{\frac{3}{2}}} \lesssim \|\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^3}\|e^{t\Delta}w_n^J\|_{L_{t,x}^6}^2 + \||u_n^J|^2\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^{\frac{3}{2}}} +$$

$$\|\nabla u_n^J\|_{L_{t,x}^3}\|e^{t\Delta}w_n^J\|_{L_{t,x}^6}^2 + \|\nabla u_n^J\|_{L_{t,x}^3}\|e^{t\Delta}w_n^J\|_{L_{t,x}^6}\|u_n^J\|_{L_{t,x}^6}.$$

The first, third and fourth terms are easily seen to converge to zero (using the space-time estimates, the fact that w_n^J is bounded in $\dot{H}^1$ and (2.5)), so (4.10) is reduced to showing

$$\lim_{J\to\infty}\overline{\lim_n}\||u_n^J|^2\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^{\frac{3}{2}}} = 0.$$

By Hölder and the space-time estimates,

$$\||u_n^J|^2\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^{\frac{3}{2}}} \lesssim \|u_n^J\|_{L_{t,x}^6}^{\frac{3}{2}}\|\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^3}^{\frac{1}{2}}\|u_n^J\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^2}^{\frac{1}{2}}$$

$$\lesssim \|(\sum_{j=1}^J v_n^j)\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^2}^{\frac{1}{2}} + \|e^{t\Delta}w_n^J\|_{L_{t,x}^6}^{\frac{1}{2}}\|\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^3}^{\frac{1}{2}}$$

$$\lesssim \|(\sum_{j=1}^J v_n^j)\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^2}^{\frac{1}{2}} + \|e^{t\Delta}w_n^J\|_{L_{t,x}^6}^{\frac{1}{2}}.$$

Again due to (2.5) it suffices to prove

$$\lim_{J\to\infty}\overline{\lim_n}\|(\sum_{j=1}^J v_n^j)\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^2} = 0.$$

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For any $\eta > 0$ by summability, we see that there exists $J' = J'(\eta) \geq 1$ such that $\sum_{j \geq J'} \|v_n^j\|_{S([0,\infty))} \leq \eta$. For this J',

$$\overline{\lim_n}\|\left(\sum_{j=J'}^{J} v_n^j\right)\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^2}^6 \lesssim \overline{\lim_n}\left(\sum_{j \geq J'}\|v_n^j\|_{S([0,\infty))}\right)\|\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^3}^6 \lesssim \eta.$$

As $\eta > 0$ is arbitrary, it suffices to show

$$\lim_{J\to\infty}\overline{\lim_n}\|v_n^j\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^2} = 0, \ 1 \leq j \leq J'.$$

Changing variables and assuming (by density) $v^j \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^4)$, by Hölder and the scale-invariance of the norms, proving (4.10) reduces to proving

$$\lim_{J\to\infty}\overline{\lim_n}\|\nabla e^{t\Delta}w_n^J\|_{L_{t,x}^2(K)} = 0,$$

for any compact $K \in \mathbb{R}^+ \times \mathbb{R}^4$. This result is the direct heat analogue of Lemma 2.5 in [38].

We have verified all the requirements of the stability proposition (2.1), hence we conclude that

$$\|u_n\|_{S([0,\infty))} \lesssim 1 + E_c$$

contradicting (4.2).

The problem now is that the kinetic energy is not conserved. The difficulty arises from the possibility that the S-norm of several profiles is large over short times, while their kinetic energy does not achieve the critical value until later. To finish the proof of proposition we have to prove that only one profile is responsible for the blow-up.

We can now (after possibly rearranging the indices) assume there exists $1 \leq J_1 < J_0$ such that

$$\|v^j\|_{S(I^j)} = \infty, 1 \leq j \leq J_1 \text{ and } \|v^j\|_{S([0,\infty))} < \infty, j > J_1$$

Again, we follow the combinatorial argument of [38]: for each integer $m, n \geq 1$, define an integer $j = j(m,n) \in \{1, ..., J_1\}$ and an interval K_n^m of the form $[0, \tau]$ by

$$\sup_{1 \leq j \leq J_1}\|v_n^j\|_{S(K_n^m)} = \|v_n^{j(m,n)}\|_{S(K_n^m)} = m. \tag{4.12}$$

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By the pigeonhole principle, there is a $1 \leq j \leq J_1$ such that for infinitely many m one has $j(m,n) = j_1$ for infinitely many n. Reordering the indices, if necessary, we may assume $j_1 = 1$. By the definition of the critical kinetic energy

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \sup_{t\in K_n^m} \|\nabla v_n^1(t)\|_2^2 \geq E_c. \tag{4.13}$$

By (4.12), all v_n^j have finite S-norms on K_n^m for each $m \geq 1$. In the same way as before, we check again that the assumptions of Proposition 2.1 are satisfied to conclude that for J and n large enough, u_n^J is a good approximation to u_n on K_n^m. In particular we have for each $m \geq 1$,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|u_n^J - u_n\|_{L_t^\infty \dot{H}_x^1(K_n^m \times \mathbb{R}^4)} = 0. \tag{4.14}$$

Lemma 4.1. *(Kinetic energy decoupling for later times). For all $J \geq 1$ and $m \geq 1$,*

$$\limsup_{n\to\infty} \sup_{t\in K_n^m} \left| \|\nabla u_n^J(t)\|_2^2 - \sum_{j=1}^{J} \|\nabla v_n^j(t)\|_2^2 - \|\nabla w_n^J\|_2^2 \right| = 0 \tag{4.15}$$

Proof. Fix $J \geq 1$ and $m \geq 1$. Then, for all $t \in K_n^m$,

$$\|\nabla u_n^J(t)\|_2^2 = <\nabla u_n^J(t), \nabla u_n^J(t)> = \sum_{j=1}^{J} \|\nabla v_n^j(t)\|_2^2 + \|\nabla w_n^J\|_2^2$$

$$+ \sum_{j\neq j'} <\nabla v_n^j(t), \nabla v_n^{j'}(t)> + 2\sum_{j=1}^{J} <\nabla e^{t\Delta} w_n^J, \nabla v_n^j(t)> .$$

It suffices to prove (for all sequences $t_n \in K_n^m$) that

$$<\nabla v_n^j(t_n), \nabla v_n^{j'}(t_n)> \xrightarrow{n\to\infty} 0 \tag{4.16}$$

and

$$<\nabla e^{t_n\Delta} w_n^J, \nabla v_n^j(t_n)> \xrightarrow{n\to\infty} 0. \tag{4.17}$$

Since $t_n \in K_n^m \subset [0, T_{\max}^{n,j})$, for all $1 \leq j \leq J_1$, we have $t_{n,j} := \frac{t_n}{(\lambda_n^j)^2} \in I^j$ for all $j \geq 1$. For $j > J_1$ the lifespan is $\mathbb{R}^+$. By refining the sequence using the standard diagonalization argument, we can assume that $t_{n,j}$ converges ($+\infty$ is also possible) for every j.

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We deal with (4.16) first. If both $t_{n,j}, t_{n,j'} \to \infty$, necessarily $j, j' > J_1$ and $v^j, v^{j'}$ are global solutions satisfying the kinetic energy bound (3.1), so by Theorem (3.1) $\|v^j\|_{\dot{H}^1}, \|v^{j'}\|_{\dot{H}^1} \xrightarrow{t \to \infty} 0$. Employing Hölder's inequality and the scaling invariance of the $\dot{H}^1$-norm, we get (4.16) for this case. When $t_{n,j} \to \infty$ but $t_{n,j'} \to \tau_{j'}$: using the continuity of the flow in $\dot{H}^1$ we can, for the limit, replace $\nabla\{\frac{1}{\lambda_n^{j'}} v^{j'}(t_{n,j'}, \frac{x - x_n^j}{\lambda_n^{j'}})\}$ with $\nabla\{\frac{1}{\lambda_n^{j'}} v^{j'}(\tau_{j'}, \frac{x - x_n^j}{\lambda_n^{j'}})\}$. By an L^2-approximation, we can also assume we are working with smooth, compactly supported functions. In this case, we can bound $< \nabla v_n^j(t_n), \nabla v_n^{j'}(t_n) >$ by $\|v^j(t_{n,j})\|_{\dot{H}^1}\|v^{j'}(\tau_{j'})\|_{\dot{H}^1} \to 0$, as $n \to \infty$. The remaining case is when both $t_{n,j}$ and $t_{n,j'}$ converge to finite $\tau_j, \tau_{j'}$ in the interior of $I^j, I^{j'}$ respectively. We can replace as above $t_{n,j}, t_{n,j'}$ by $\tau_j, \tau_{j'}$ respectively, and perform a change of variables:

$$< \nabla v_n^j(t_n), \nabla v_n^{j'}(t_n) >= \int (\frac{\lambda_n^j}{\lambda_n^{j'}})^2 \nabla v^j(\tau_j, x), \nabla v^{j'}(\tau_{j'}, \frac{\lambda_n^j}{\lambda_n^{j'}} x + \frac{x_n^j - x_n^{j'}}{\lambda_n^{j'}}) dx$$

which is going to zero assuming, without loss of generality that $\frac{\lambda_n^j}{\lambda_n^{j'}} \to 0$ and the functions in the integrand are compactly supported, thus concluding the case (4.16).

For the case (4.17), perform a change of variable:

$$< \nabla e^{t_n \Delta} w_n^J, \nabla v_n^j(t_n) >=< \nabla e^{t_{n,j} \Delta}[\lambda_n^j w_n^J(\lambda_n^j x + x_n^j)], \nabla v^j(t_{n,j}) > .$$

When $t_{n,j} \to \infty$, using Hölder, the heat estimates (2.2) (and the boundedness of w_n^J in $\dot{H}^1$ coming from the profile decomposition) and Theorem 3.1 as before, we get to the result. For the case $t_{n,j} \to \tau_j < +\infty$, we can, as before, replace $t_{n,j}$ by its limit τ_j in the integral $\int \nabla e^{t_{n,j} \Delta}[\lambda_n^j w_n^J(\lambda_n^j x + x_n^j)] \cdot \nabla v^j(\tau_j, x) dx$. Using (2.5) and (2.2) we can see that $e^{t_{n,j} \Delta}[\lambda_n^j w_n^J(\lambda_n^j x + x_n^j)] \rightharpoonup 0$ in $\dot{H}^1$, which concludes the proof of the case (4.17) and hence the proof of the Lemma. $\qquad\square$

By (4.2), (4.14), (4.15), we get

$$E_c \geq \limsup_{n \to \infty} \sup_{t \in K_n^m} \|\nabla u_n(t)\|_{L^2}^2$$

$$= \lim_{J \to \infty} \limsup_{n \to \infty}\{\|\nabla w_n^J(t)\|_{L^2}^2 + \sup_{t \in K_n^m} \sum_{j=1}^{J} \|\nabla v_n^j(t)\|_{L^2}^2\}.$$

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Taking a limit in m and employing (4.13), we see that we actually have equality everywhere. This implies that $J_1 = 1, v_n^j \equiv 0, \forall j \geq 2, w_n := w_n^1 \xrightarrow{\dot{H}^1}$ 0. So $u_n(0, x) = \frac{1}{\lambda_n}\phi(\frac{x - x_n^1}{\lambda_n^1}) + w_n(x)$, for some functions $\phi, w_n \in \dot{H}^1, w_n \xrightarrow{s}$ 0 in $\dot{H}^1$.

Thus we have shown that for the sequence of initial data $u_{n,0}$ that

$$\lambda_n^1 u_{n,0}(\lambda_n^1 x + x_n^1) \xrightarrow{\dot{H}^1} \phi.$$

This finishes the proof of Proposition 4.1. $\qquad\square$

Now, we are in a position to prove Theorem 4.1.

Proof. By the definition of E_c we can find a sequence of solutions $u_n : I_n \times \mathbb{R}^4 \to \mathbb{R}$, with I_n compact, so that

$$\sup_n \sup_{t \in I_n} \|\nabla u_n(t)\|_{L^2}^2 = E_c \text{ and } \lim_n \|u_n\|_{S(I_n)} = +\infty.$$

An application of Proposition 4.1 shows that the corresponding sequence of initial data converges strongly, modulo symmetries, to some $\phi \in \dot{H}^1$. By rescaling and translating u_n, we may in fact assume $u_{n,0} := u_n(0, \cdot) \xrightarrow{\dot{H}^1} \phi$.

We define $u_c : I \times \mathbb{R}^4 \to \mathbb{R}$ to be the maximal-lifespan solution of (1.1) emanating from initial data ϕ. Since $u_{n,0} \xrightarrow{\dot{H}^1} \phi$, employing the stability Proposition 2.1, $I \subset \liminf I_n$, and $\|u_n - u_c\|_{L_t^\infty \dot{H}_x^1(K \times \mathbb{R}^4)} \xrightarrow{n \to \infty} 0$, for all compact $K \subset I$. Thus, by (4.2):

$$\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 \leq E_c. \tag{4.18}$$

Applying the stability Proposition 2.1 once again we can also see that $\|u_c\|_{S(I)} = \infty$. Hence, by the definition of the critical kinetic energy level, E_c,

$$\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 \geq E_c. \tag{4.19}$$

In conclusion,

$$\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 = E_c \tag{4.20}$$

and

$$\|u_c\|_{S(I)} = +\infty. \tag{4.21}$$

Finally, the compactness modulo symmetries (4.1) follows from another application of Proposition 4.1. We omit the standard proof (see for example [35] or [38]). $\qquad\square$

5 Rigidity

The main result of this section is the following theorem ruling out finite-time blowup of compact (modulo symmetries) solutions. Note this is a considerably stronger statement than we require, since it is not limited to solutions with below-threshold kinetic energy:

Theorem 5.1. *If u is a solution to* (1.1) *on maximal existence interval* $I = [0, T^*)$, *such that* $K := \left\{ \dfrac{1}{\lambda(t)} u(t, \dfrac{x - x(t)}{\lambda(t)}) \mid t \in I \right\}$ *is precompact in* $\dot{H}^1$ *for some* $x(t) \in \mathbb{R}^4$, $\lambda(t) \in \mathbb{R}^+$, *then* $T^* = +\infty$.

As a corollary, we can complete the proof of the main result Theorem 1.1 by showing:

Corollary 1. *For any solution satisfying* (1.8), $T_{max}(u(0)) = \infty$.

Proof. By Theorem 5.1, the solution u_c produced by Theorem 4.1 must be global: $T_{max}(u_c(0)) = \infty$. But since $\|u_c\|_{S(\mathbb{R}^+)} = \infty$, Theorem 3.1 shows $E_c = \|\nabla W\|_2^2$, and the Corollary follows. $\square$

The rest of the section is devoted to the proof of the Theorem 5.1. Our proof is inspired by the work of Kenig and Koch [34] for the Navier-Stokes system, and it's based on classical parabolic tools – local smallness regularity, backwards uniqueness, and unique continuation – though implemented in a somewhat different way. In particular, we will make use of the following two results, proved in [17], [18] (also see [19]):

Theorem 5.2. *(Backwards Uniqueness) Fix any* $R, \delta, M,$ *and* $c_0 > 0$. *Let* $Q_{R,\delta} := (\mathbb{R}^4 \setminus B_R(0)) \times (-\delta, 0)$, *and suppose a vector-valued function v and its distributional derivatives satisfy* $v, \nabla v, \nabla^2 v \in L^2(\Omega)$ *for any bounded subset* $\Omega \subset Q_{R,\delta}$, $|v(x,t)| \leq e^{M|x|^2}$ *for all* $(x,t) \in Q_{R,\delta}$, $|v_t - \Delta v| \leq c_0(|\nabla v| + |v|)$ *on* $Q_{R,\delta}$, *and* $v(x, 0) = 0$ *for all* $x \in \mathbb{R}^4 \setminus B_R(0)$. *Then* $v \equiv 0$ *in* $Q_{R,\delta}$.

Theorem 5.3. *(Unique Continuation) Let* $Q_{r,\delta} := B_r(0) \times (-\delta, 0)$, *for some* $r, \delta > 0$, *and suppose a vector-valued function v and its distributional derivatives satisfy* $v, \nabla v, \nabla^2 v \in L^2(Q_{r,\delta})$ *and there exist* $c_0, C_k > 0, (k \in \mathbb{N})$ *such that* $|v_t - \Delta v| \leq c_0(|\nabla v| + |v|)$ *a.e. on* $Q_{r,\delta}$ *and* $|v(x,t)| \leq C_k(|x| + \sqrt{-t})^k$ *for all* $(x, t) \in Q_{r,\delta}$. *Then* $v(x, 0) \equiv 0$ *for all* $x \in B_r(0)$.

As well, we establish the following:

Lemma 5.1. *(Local Smallness Regularity Criterion) For any $k \in \mathbb{N}$, there are $\epsilon_0 > 0$ and C such that: if u is a solution of equation (1.1) on Q_1, where $Q_r := B_r(0) \times (-r^2, 0)$ for $r > 0$, and satisfies*

$$\epsilon := \|u\|_{L_t^\infty(\dot{H}_x^1 \cap L_x^4)(Q_1)} < \epsilon_0$$

then u is smooth on $\overline{Q_{\frac{1}{2}}}$ with bounds

$$\max_{\overline{Q_{\frac{1}{2}}}} |D^k u| \le C\epsilon.$$

Proof. Assume $\|u\|_{L_t^\infty(\dot{H}_x^1 \cap L_x^4)(Q_1)} < \epsilon$, for ϵ small enough (to be picked). Define

$$\|u\|_{X(Q_1)}^2 := \|\nabla u\|_{L_t^\infty L_x^2 \cap L_t^2 L_x^4 (Q_1)}^2 + \|u\|_{L_t^\infty L^4 (Q_1)}^2 + \|D^2 u\|_{L_t^2 L_x^2 (Q_1)}^2.$$

Assuming for ease of writing that u is real-valued, differentiating (1.1) and defining $\tilde{u} := \nabla u$, we get

$$\tilde{u}_t = \Delta \tilde{u} + 3u^2 \tilde{u}. \tag{5.1}$$

Consider a smooth, compactly supported spatial cut-off function $\phi_0(x)$ such that $\mathrm{supp}(\phi_0) \subset B_1(0)$ and $\phi_0 \equiv 1$ on $B_{\rho_0}(0)$, for some $\frac{1}{2} < \rho_0 < 1$ to be chosen. Multiplying the above equation by $\phi_0^2 \tilde{u}$ and integrating in space-time (from now on, unless otherwise specified, $t \in [-1, 0]$):

$$\int_{-1}^t \int_{|x| \le 1} (\tilde{u}_t - \Delta \tilde{u}) \phi_0^2 \tilde{u} \, dx dt = 3 \int_{-1}^t \int_{|x| \le 1} (u^2 \tilde{u}) \phi_0^2 \tilde{u} \, dx dt$$

$$\Rightarrow \frac{1}{2} \|\phi_0 \tilde{u}(t)\|_{L^2}^2 + \int_{-1}^t \int_{|x| \le 1} \phi_0^2 |\nabla \tilde{u}|^2 \, dx dt$$

$$= \frac{1}{2} \|\phi_0 \tilde{u}(0)\|_{L^2}^2 + 3 \int_{-1}^t \int_{|x| \le 1} \phi_0^2 u^2 \tilde{u}^2 dx dt + 2 \int_{-1}^t \int_{|x| \le 1} \phi_0 \nabla \phi_0 (\tilde{u} \nabla \tilde{u}) dx dt.$$

For the sake of brevity, let us define $v_0 := \phi_0 \tilde{u} = \phi_0 \nabla u$ and thus (always on the same cylinder):

$$\|v_0\|_{L_t^\infty L_x^2}^2 + \|\phi_0 \nabla \tilde{u}\|_{L_t^2 L_x^2}^2 \lesssim \|v_0(0,x)\|_{L^2}^2 + \|u^2\|_{L_t^\infty L_x^2} \|v_0^2\|_{L_t^1 L_x^2} + \|\phi_0 \nabla \tilde{u}\|_{L_t^2 L_x^2} \|\tilde{u}\|_{L_t^2 L_x^2}$$

$$= \|v_0(0,x)\|_{L^2}^2 + \|u\|_{L_t^\infty L_x^4}^2 \|v_0\|_{L_t^2 L_x^4}^2 + \|\phi_0 \nabla \tilde{u}\|_{L_t^2 L_x^2} \|\tilde{u}\|_{L_t^2 L_x^2}.$$

By the smallness assumed on the cylinder Q_1 and an application of Young's inequality, for any $\delta > 0$ (and also using Hölder and the boundedness of the domain):

$$\|v_0\|^2_{L_t^\infty L_x^2} + \|\phi_0 \nabla \tilde{u}\|^2_{L_t^2 L_x^2} \lesssim \epsilon^2 + \epsilon^2 \|v_0\|^2_{L_t^2 L_x^4(Q_1)} + \delta^2 \|\phi_0 \nabla \tilde{u}\|^2_{L_t^2 L_x^2(Q_1)} + \frac{\|\tilde{u}\|^2_{L_t^\infty L_x^4(Q_1)}}{\delta^2}$$

$$\Rightarrow \|v_0\|^2_{L_t^\infty L_x^2} + \|\phi_0 \nabla \tilde{u}\|^2_{L_t^2 L_x^2} \lesssim \epsilon^2 + \frac{\epsilon^2}{\delta^2} + \epsilon^2 \|v_0\|^2_{L_t^2 L_x^4}$$

if δ is chosen small enough. Since $\nabla v_0 = \phi_0 \nabla \tilde{u} + \nabla \phi_0 \, \tilde{u}$:

$$\|\nabla v_0\|_{L^2} \lesssim \|\phi_0 \nabla \tilde{u}\|_{L^2} + \|\nabla \phi_0\|_{L^4} \|\tilde{u}\|_{L^4}$$

and so using the Sobolev inequality,

$$\|v_0\|^2_{L_t^\infty L_x^2} + \|\nabla v_0\|^2_{L_t^2 L_x^2} + \|v_0\|^2_{L_t^2 L_x^4} \lesssim \epsilon^2 + \epsilon^2 \|v_0\|^2_{L_t^2 L_x^4}.$$

Choosing ϵ small enough yields

$$\|u\|_{X(Q_{\rho_0})} \lesssim \epsilon.$$

Define another smooth compactly supported cut-off function $\phi_1(x) \leq \phi_0(x)$, with support in B_{ρ_0}, and $\phi_1 \equiv 1$ on $B_{\rho_1}(0)$, some $\frac{1}{2} < \rho_1 < \rho_0 < 1$ to be chosen. Let $\hat{v} := D^2 u$, and $v_1 := \phi_1 \hat{v}$.

Remark 5.1. *We will be abusing notation from this point onwards. For the pointwise operations and estimates we are actually considering the mixed partial derivatives $\partial_k \partial_j u, j, k = 1, ..., 4$ but we will be writing $D^2 u$ all the same without taking care to specify the matrix element at hand. In the end, we are using standard matrix norms.*

Differentiating (5.1), multiplying by $\phi_1^2 \hat{v}$, and integrating over space gives

$$\frac{1}{2}\partial_t \int \phi_1^2 \hat{v}^2 dx + \int \phi_1^2 |\nabla \hat{v}|^2 dx = 3 \int \phi_1^2 u^2 \hat{v}^2 dx$$
$$+ 6 \int \phi_1^2 u \tilde{u}^2 \hat{v} dx + 2 \int \phi_1 \nabla \phi_1 \cdot \hat{v} \nabla \hat{v} dx. \tag{5.2}$$

Since by the previous step, $\|\nabla v_0\|_{L^2 L^2(Q_{\rho_0})} \lesssim \epsilon$, we can find $-1 < t_1 < -\rho_0^2$ such that $\|\nabla v_0(\cdot, t_1)\|_{L^2(B_{\rho_0})} \lesssim \epsilon$ (where the implied constant may depend on ρ_0), so that

$$\|\phi_1 \hat{v}(\cdot, t_1)\|_{L^2} = \|\phi_1 D^2 u(\cdot, t_1)\|_{L^2} \leq \|\nabla v_0(\cdot, t_1)\|_{L^2(B_{\rho_0})} \lesssim \epsilon.$$

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Integrating (5.2) in t from t_1 to 0, and using the estimates from the previous step:

$$\|v_1\|_{L_t^\infty L^2}^2 + \|\phi_1 \nabla \hat{v}\|_{L_t^2 L_x^2}^2 \lesssim \|\phi_0 u\|_{L^\infty L^4}^2 \|v_1\|_{L^2 L^4}^2 + \|\hat{v} \nabla \hat{v} \phi_1 \nabla \phi_1\|_{L_t^1 L_x^1} + \epsilon^2$$
$$+ \|\phi_0 u\|_{L^\infty L^4} \|v_0\|_{L^\infty L^4} \|v_0\|_{L^2 L^4} \|v_1\|_{L^2 L^4}$$
$$\lesssim \epsilon^2 \|v_1\|_{L^2 L^4}^2 + \epsilon^3 \|v_1\|_{L^2 L^4}$$
$$+ \|\phi_1 \nabla \hat{v}\|_{L^2 L^2} \|\nabla \phi_1 \hat{v}\|_{L^2 L^2} + \epsilon^2$$

where everywhere here the time interval is $[t_1, 0]$. We have

$$\nabla v_0 = \phi_0 D^2 u + \nabla \phi_0 \nabla u = \phi_0 \hat{v} + \nabla \phi_0 \tilde{u}$$

and so

$$|\phi_0 \hat{v}| \lesssim |\nabla v_0| + |\nabla \phi_0 \tilde{u}| \;\Rightarrow\; |\nabla \phi_1 \hat{v}| \lesssim |\frac{\nabla \phi_1}{\phi_0}| \|\phi_0 \hat{v}\| \lesssim |\nabla v_0| + |\nabla \phi_0 \tilde{u}|.$$

Thus

$$\|\nabla \phi_1 \hat{v}\|_{L^2 L^2} \lesssim \|\nabla v_0\|_{L^2 L^2} + \epsilon \lesssim \epsilon.$$

By Young's inequality once more, for some $\delta_1 > 0$ sufficiently small,

$$\|v_1\|_{L^\infty L^2}^2 + \|\phi_1 \nabla \hat{v}\|_{L^2 L^2}^2 \lesssim \epsilon^2 \|v_1\|_{L^2 L^4}^2 + \delta_1^2 \|\phi_1 \nabla \hat{v}\|_{L^2 L^2}^2 + \delta_1^2 \|v_1\|_{L^2 L^4}^2 + \frac{\epsilon^2}{\delta_1^2}.$$

Using Sobolev again as above, $\|v_1\|_{L^\infty L^2}^2 + \|v_1\|_{L^2 L^4}^2 + \|\nabla v_1\|_{L^2 L^2}^2 \lesssim \epsilon$. In particular

$$\|D^2 u\|_{X(Q_{\rho_1})} \lesssim \epsilon.$$

This process can be iterated a given finite number of times, to show that for given $k > 0$, there are $\epsilon_0 = \epsilon_0(k)$, $C = C(k)$, such that if $\|u\|_{L^\infty(\dot{H}^1 \cap L^4)(Q_1)} = \epsilon < \epsilon_0$, then $\|D^k u\|_{X(Q_{1/2})} \leq C\epsilon$. $\qquad \square$

We proceed now with the proof of Theorem 5.1.

Proof. Let us assume that the conclusion is false, i.e., $T^* < +\infty$. Note first that

$$\lambda(t) \to +\infty.$$

In fact, $\liminf\limits_{t \to T^{*-}} \sqrt{T^* - t}\, \lambda(t) > 0$, since if $\sqrt{T^* - t_n}\lambda(t_n) \to 0$ along a sequence $t_n \nearrow T^*$, by the compactness assumption (and up to subsequence)

$$v_n(x) := \frac{1}{\lambda(t_n)} u_c(t_n, \frac{x - x(t_n)}{\lambda(t_n)}) \xrightarrow{\dot{H}^1} \exists\, v(x) \in \dot{H}^1.$$

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Let $\hat{T} > 0$ be the maximal existence time for the solution of the Cauchy problem (1.1) with initial data $v(x)$. Define $w_n(t, x)$ to be the solutions with initial data $w_n(x, t_n) = v_n(x)$ prescribed at time t_n, and denote their maximal lifespans as $[t_n, T_n^{\max})$. By continuous dependence on initial data, $0 < \hat{T} \le \liminf(T_n^{\max} - t_n)$. But from scaling:

$$T_n^{\max} - t_n = T^{\max}\left(\frac{1}{\lambda(t_n)}u_c(t_n, \frac{\cdot - x(t_n)}{\lambda(t_n)})\right) = \lambda^2(t_n)T^{\max}(u_c(t_n, \cdot))$$
$$= \lambda^2(t_n)(T^* - t_n) \to 0,$$

a contradiction.

By compactness in $\dot{H}^1$, and the continuous embedding $\dot{H}^1 \hookrightarrow L^4$, for every $\epsilon > 0$, there is an $R_\epsilon > 0$ such that for all $t \in I := [0, T^*)$:

$$\int_{|x-x(t)| \ge \frac{R_\epsilon}{\lambda(t)}} \left(|\nabla u_c(t, x)|^2 + |u_c(t, x)|^4\right) dx < \epsilon. \tag{5.3}$$

Fix any $\{t_n\} \subset [0, T^*), t_n \nearrow T^*$, and let $\lambda_n = \lambda(t_n) \to \infty$ and $\{x_n\} = \{x(t_n)\} \subset \mathbb{R}^4$, so that (up to subsequence)

$$v_n(x) = \frac{1}{\lambda_n}u_c(\frac{x - x_n}{\lambda_n}, t_n) \xrightarrow{\dot{H}^1} \bar{v}, \text{for some } \bar{v} \in \dot{H}^1,$$

and also in L^4 by Sobolev embedding.

We also make and prove the following claim as in [34]

Claim 5.1. *For any $R > 0$,*

$$\lim_{n \to \infty} \int_{|x| \le R} |u_c(x, t_n)|^2 dx = 0.$$

Proof.

$$\int_{|x| \le R} |u_c(x, t_n)|^2 dx = \int_{|x| \le R} |\lambda_n v_n(\lambda_n x + x_n)|^2 dx$$
$$= \frac{1}{\lambda_n^2}\int_{|y-x_n| \le \lambda_n R} |v_n(y)|^2 dy = \frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n))}.$$

Denoting $B_r := B_r(0)$, for any $\epsilon > 0$,

$$\frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n))} = \frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n)) \cap B_{\epsilon \lambda_n R}} + \frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n)) \cap B^c_{\epsilon \lambda_n R}}.$$

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Using Hölder's inequality, and the compactness, we get

$$\frac{1}{\lambda_n^2}\|v_n\|_{L^2(B_{\lambda_n R}(x_n))}^2 \lesssim \frac{1}{\lambda_n^2}\|v_n\|_{L^4(B_{\lambda_n R}(x_n)\cap B_{\epsilon\lambda_n R}^c)}^2 |B_{\lambda_n R}(x_n)|^{\frac{1}{2}}$$

$$+ \frac{1}{\lambda_n^2}\|v_n\|_{L^4(\mathbb{R}^4)}^2 |B_{\epsilon\lambda_n R}|^{\frac{1}{2}}$$

$$\lesssim \frac{|B_{\lambda_n R}(x_n)|^{\frac{1}{2}}}{\lambda_n^2}\left(\|v_n - \bar{v}\|_{L^4(\mathbb{R}^4)}^2 + \|\bar{v}\|_{L^4(B_{\lambda_n R}(x_n)\cap B_{\epsilon\lambda_n R}^c)}^2\right)$$

$$+ \epsilon^2 R^2\|\bar{v}\|_{L^4(\mathbb{R}^4)}^2$$

$$\lesssim R^2\|v_n - \bar{v}\|_{L^4(\mathbb{R}^4)}^2 + R^2\|\bar{v}\|_{L^4(B_{\epsilon\lambda_n R}^c)}^2 + \epsilon^2 R^2\|\bar{v}\|_{L^4(\mathbb{R}^4)}^2$$

$$\lesssim R^2\|v_n - \bar{v}\|_{L^4(\mathbb{R}^4)}^2 + R^2\|\bar{v}\|_{L^4(B_{\epsilon\lambda_n R}^c)}^2 + \epsilon^2 R^2.$$

The first term goes to zero (as $n \to \infty$) because of the compactness, the second goes to zero (for fixed ϵ) since $\lambda_n \to \infty$, and the last one is arbitrarily small with ϵ. □

We also prove that the center of compactness $x(t)$ is bounded:

Proposition 5.1. $\sup_{0\leq t<T^*} |x(t)| < \infty$.

Proof. We will first make the assumption that

$$\underline{E} := \inf_{t\in[0,T^*)} E(u_c(t)) > 0, \tag{5.4}$$

and later show that this is indeed the case for compact blowing-up solutions, without any size restriction. Note that under the assumptions of our Theorem 1.1, i.e., in the below threshold case, we certainly have that $\underline{E} > 0$. This can be easily deduced by the variational estimates in Lemma 2.1 and the small data theory.

The energy dissipation relation

$$E(u(t_2)) + \int_{t_1}^{t_2} \|u_t\|_{L^2}^2 \, ds = E(u(t_1)) \leq E(u(0)) \tag{5.5}$$

for $t_2 > t_1 > 0$ will be of use. We will assume for contradiction that there is a sequence of times $t_n \nearrow T^* : |x(t_n)| \to \infty$.

Choose a smooth cut-off function ψ such that

$$\psi(r) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 & \text{if } r \geq 2 \end{cases}$$

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and define $\psi_R(x) := \psi(\frac{|x|}{R})$. Choosing any $t_0 \in (0, T^*)$, we can find $R_0 \geq 1$ such that

$$\int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u_c(t_0)|^2 - \frac{1}{4}(u_c(t_0))^4\right)\psi_{R_0}(x)\,dx \leq \frac{1}{4}\underline{E}. \qquad (5.6)$$

Since $|x(t_n)| \to \infty$ and $\lambda(t_n) \to \infty$, for any $\epsilon > 0$, $B_{\frac{R\epsilon}{\lambda(t_n)}}(x(t_n)) \subset B_{2R_0}^c$ for n large enough, and so by (5.3):

$$\lim_{t \nearrow T^*} \int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u_c(t)|^2 - \frac{1}{4}(u_c(t))^4\right)\psi_{R_0}(x)dx = \underline{E},$$

hence we can find a $t_1 \in (t_0, T^*)$ such that

$$\int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u_c(t_1)|^2 - \frac{1}{4}(u_c(t_1))^4\right)\psi_{R_0}(x)\,dx \geq \frac{1}{2}\underline{E}. \qquad (5.7)$$

Combining (5.6) and (5.7):

$$\int_{t_0}^{t_1} \frac{d}{dt}\int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u_c(t)|^2 - \frac{1}{4}(u_c(t))^4\right)\psi_{R_0}(x)\,dxdt \geq \frac{1}{4}\underline{E}. \qquad (5.8)$$

On the other hand:

$$\frac{d}{dt}\int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u_c(t)|^2 - \frac{1}{4}(u_c(t))^4\right)\psi_{R_0}(x)\,dx = \int_{\mathbb{R}^4} \left(\nabla u_c \cdot \nabla(u_c)_t - u_c^3(u_c)_t\right)\psi_{R_0}(x)\,dx$$

$$= \int_{\mathbb{R}^4} \left(\nabla u_c \cdot \nabla(u_c)_t - ((u_c)_t - \Delta u_c)(u_c)_t\right)\psi_{R_0}(x)\,dx$$

$$= -\int_{\mathbb{R}^4} (u_c)_t^2\psi_{R_0}\,dx - \int_{\mathbb{R}^4} (u_c)_t\nabla u_c \cdot \nabla\psi_{R_0}\,dx \lesssim \int_{\mathbb{R}^4} |(u_c)_t||\nabla u_c|\,dx,$$

since $|\nabla\psi_{R_0}(x)| \lesssim \frac{1}{R_0} \leq 1$. So by Hölder,

$$\int_{t_0}^{t_1} \frac{d}{dt}\int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u_c(t)|^2 - \frac{1}{4}(u_c(t))^4\right)\psi_{R_0}(x)\,dxdt$$
$$\lesssim \|\nabla u_c\|_{L_t^\infty L^2}\sqrt{t_1 - t_0}\,\|(u_c)_t\|_{L^2L^2[t_0,t_1]\times\mathbb{R}^4} \qquad (5.9)$$
$$\lesssim \|(u_c)_t\|_{L^2L^2([t_0,T^*)\times\mathbb{R}^4)}$$

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where we have uniformly bounded the kinetic energy of u_c by once more employing the compactness. Combining (5.8) and (5.9) yields:

$$0 < \frac{1}{4}\underline{E} \lesssim \|(u_c)_t\|_{L^2L^2([t_0,T^*)\times\mathbb{R}^4)} \to 0 \text{ as } t_0 \nearrow T^* \qquad (5.10)$$

by the energy dissipation relation (5.5), a contradiction.

Now we show (5.4). Choose a smooth cut-off function ϕ such that

$$\phi(r) = \begin{cases} 1 & \text{if } r \le 1 \\ 0 & \text{if } r \ge 2 \end{cases}$$

and define $\phi_R(x) := \phi(\frac{|x|}{R})$, and

$$I_R(t) := \frac{1}{2}\int (u_c(x,t))^2\phi_R(x)dx, \qquad t \in [0,T^*).$$

We then have

$$I'_R(t) = \int \phi_R((u_c)^4 - |\nabla u_c|^2)dx - \frac{1}{R}\int u_c\nabla u_c \cdot \nabla\phi(\frac{x}{R})dx$$

and by Sobolev, Hardy and the compactness, we can immediately deduce that

$$|I'_R(t)| \le C,$$

C a constant. Integrating from t_0 to $T^* > t > t_0 \ge 0$:

$$|I_R(t) - I_R(t_0)| \le C(t - t_0).$$

By Claim 5.1, we get that $I_R(t) \to 0$ as $t \to T^*$, for all $R > 0$. Hence

$$I_R(t_0) \le C(T^* - t_0).$$

Since this bound is uniform in R, by taking $R \to \infty$, we conclude $u_c(t_0) \in L^2$, and so indeed $u_c(t) \in L^2, t \in [0,T^*)$. Moreover defining

$$I(t) := \frac{1}{2}\int |u_c(t,x)|^2dx,$$

we conclude that

$$I(t) \le C(T^* - t). \qquad (5.11)$$

Now the L^2-dissipation relation (3.3) gives

$$I'(t) = -\int \left(|\nabla u_c|^2 - |u_c|^4 \right) dx = -K(u_c(t)),$$

$$K(u) := \int \left(|\nabla u|^2 - |u|^4 \right) dx = 2E(u) - \frac{1}{2} \int |u|^4 dx.$$

Now for any sequence $\{t_n\}_n \nearrow T^*$, let (up to subsequence)

$$\frac{1}{\lambda(t_k)} u_c \left(\frac{x - x_k}{\lambda(t_k)}, t_k \right) \xrightarrow{\dot{H}^1} \bar{v} \in \dot{H}^1.$$

Proceeding by contradiction, we suppose $\underline{E} \leq 0$. If so,

$$K(\bar{v}) = \lim_{k \to \infty} K(u_c(t_k)) = 2\underline{E} - \frac{1}{2} \int |\bar{v}|^4 \leq -\frac{1}{2} \int |\bar{v}|^4 < 0,$$

since $\bar{v} \equiv 0$ would contradict the assumption $T^* < \infty$. So

$$I'(t_k) = -K(u_c(t_k)) \to -K(\bar{v}) > 0.$$

Thus $I'(t) > 0$ for all t sufficiently close to T^*; otherwise, we could find a subsequence along which $I' \leq 0$, and the preceding argument would provide a contradiction. So $I(t)$ is increasing for t near T^*, which contradicts (5.11).

Thus we have shown that $\underline{E} > 0$, completing the proof that $|x(t)|$ remains bounded. $\qquad \square$

Since $|x(t)|$ remains bounded while $\lambda(t) \xrightarrow{t \to T^*} \infty$, by the compactness we can find an $R_0 > 0$ large enough such that for all $x, |x| \geq R_0$:

$$\|u_c\|_{L_t^\infty \dot{H}_x^1 \cap L_t^\infty L_x^4(\Omega_{T^*})} < \epsilon_0,$$

where $\Omega_{T^*} := (0, T^*) \times B_{\sqrt{T^*}}(x_0)$.

By an appropriate scaling and shifting argument, the Regularity Lemma 5.1 shows that u_c is smooth on $\Omega := (\mathbb{R}^4 \setminus B_{R_0}(0)) \times [\frac{3}{4}T^*, T^*]$, with uniform bounds on derivatives. Since u is continuous up to T^* outside B_{R_0}, Claim 5.1 implies that $u_c(x, T^*) \equiv 0$, in the exterior of this ball. Since u_c is bounded and smooth in Ω, an application of the Backwards Uniqueness Theorem 5.2 implies that $u_c \equiv 0$ in Ω. Define $\tilde{\Omega} := \mathbb{R}^4 \times (\frac{3}{4}T^*, \frac{7}{8}T^*]$. Applying the Unique Continuation Theorem 5.3 on a cylinder of sufficiently large spatial radius, centered at a point of Ω, implies $u_c \equiv 0$ in $\tilde{\Omega}$. By the uniqueness guaranteed by the local wellposedness theory we get that $u_c \equiv 0$, which contradicts (4.21). $\qquad \square$

6 Blow-up

In this section we give criteria on the initial data which ensure that the corresponding solutions blow-up in finite-time.

The following result is well-known [39, 3, 8] but we give the proof for the convenience of the reader.

Proposition 6.1. *Solutions of*

$$u_t = \Delta u + |u|^{p-1}u, \qquad 1 < p \leq 2^* - 1 = \frac{d+2}{d-2} \tag{6.1}$$

$$u(x,0) = u_0(x) \in H^1(\mathbb{R}^d)$$

with

$$E(u_0) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u_0|^2 dx - \frac{1}{p+1}|u_0|^{p+1} \right) dx < 0$$

must blow-up in finite-time, in the sense that there is no global solution $u \in C([0,\infty); H^1(\mathbb{R}^d))$.

Notice that we can always find such initial data, e.g., if $u_0 = \lambda f, f \in H^1(\mathbb{R}^d)$ we can force negative energy by taking λ large.

Proof. We first derive some identities satisfied as long as a solution remains regular.

Multiplying the equation (6.1) first by u and then by u_t and integrating by parts we obtain the L^2-dissipation relation

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^d} |u|^2 dx \right) = \int_{\mathbb{R}^d} |u|^{p+1} dx - \int_{\mathbb{R}^d} |\nabla u|^2 dx =: -K(u) \tag{6.2}$$

and the energy dissipation relation

$$\int_{\mathbb{R}^d} |u_t|^2 dx = \frac{d}{dt}\left(\frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx \right) = -\frac{d}{dt}E(u(t)). \tag{6.3}$$

For convenience we define $J(t) := -E(t)$ and hence by (6.3) we have that $J'(t) := \int_{\mathbb{R}^d} |u_t|^2 dx \geq 0$ and by the assumption on the energy $J(0) > 0$. It will be also useful to write $J(t)$ as

$$J(t) = J(0) + \int_0^t \int_{\mathbb{R}^d} |u_t|^2 dx dt. \tag{6.4}$$

Define

$$I(t) = \int_0^t \int_{\mathbb{R}^d} |u|^2 dx dt + A \qquad (6.5)$$

with $A > 0$, to be chosen later. With this definition

$$I'(t) = \int_{\mathbb{R}^d} |u|^2 dx \qquad (6.6)$$

and

$$I''(t) = 2 \left(\int_{\mathbb{R}^d} |u|^{p+1} dx - \int_{\mathbb{R}^d} |\nabla u|^2 dx \right). \qquad (6.7)$$

Since $p > 1, \delta := \frac{1}{2}(p-1) > 0$; a comparison with the energy functional yields

$$I''(t) \geq 4(1 + \delta)J(t) = 4(1 + \delta) \left(J(0) + \int_0^t \int_{\mathbb{R}^d} |u_t|^2 dx dt \right). \qquad (6.8)$$

We can also rewrite

$$I'(t) = \int_{\mathbb{R}^d} |u|^2 dx = \int_{\mathbb{R}^d} |u_0|^2 dx + 2Re \int_0^t \int_{\mathbb{R}^d} \bar{u} u_t dx dt.$$

For any $\epsilon > 0$ the Young and Hölder inequalities give

$$(I'(t))^2 \leq 4(1+\epsilon) \left(\int_0^t \int_{\mathbb{R}^d} |u|^2 dx dt \right) \left(\int_0^t \int_{\mathbb{R}^d} |u_t|^2 dx dt \right) + (1+\frac{1}{\epsilon}) \left(\int_{\mathbb{R}^d} |u_0|^2 dx \right)^2 \qquad (6.9)$$

Combining (6.8),(6.4),(6.9), for any $\alpha > 0$ we obtain:

$$I''(t)I(t) - (1 + \alpha)(I'(t))^2 \geq 4(1 + \delta) \left[J(0) + \int_0^t \int_{\mathbb{R}^d} |u_t|^2 dx dt \right] \left[\int_0^t \int_{\mathbb{R}^d} |u|^2 dx dt + A \right]$$

$$- 4(1 + \epsilon)(1 + \alpha) \left[\int_0^t \int_{\mathbb{R}^d} |u|^2 dx dt \right] \left[\int_0^t \int_{\mathbb{R}^d} |u_t|^2 dx dt \right]$$

$$- (1 + \frac{1}{\epsilon})(1 + \alpha) \left[\int_{\mathbb{R}^d} |u_0|^2 dx \right]^2. \qquad (6.10)$$

Choose α, ϵ small enough for $1 + \delta \geq (1 + \alpha)(1 + \epsilon)$. Since $J(0) > 0$ picking A large enough we can ensure $I''(t)I(t) - (1 + \alpha)(I'(t))^2 > 0$. But this is

equivalent to $\frac{d}{dt}\left(\frac{I'(t)}{I^{\alpha+1}(t)}\right) > 0$ which in turn implies $\frac{I'(t)}{I^{\alpha+1}(t)} > \frac{I'(0)}{I^{\alpha+1}(0)} =: \tilde{a}$ for all $t > 0$. Integrating $I'(t) > \tilde{a}I^{\alpha+1}$ gives

$$\frac{1}{\alpha}\left(\frac{1}{I^\alpha(0)} - \frac{1}{I^\alpha(t)}\right) > \tilde{a}t \implies I^\alpha(t) > \frac{I^\alpha(0)}{1 - I^\alpha(0)\alpha\tilde{a}t} \to \infty$$

as $t \to \frac{1}{I^\alpha(0)\alpha\tilde{a}} = \frac{1}{A^\alpha\alpha\tilde{a}} =: \hat{t}$. This in turn implies that $\limsup_{t\to\hat{t}-} \|u\|_{L^2} = \infty$, showing that the solution cannot be globally in $C_t H^1$. Note also that (6.2) implies $\limsup_{t\to\hat{t}-} \|u\|_{L^{p+1}} = \infty$. $\square$

We present a refinement in the *critical case* which includes some positive energy data, and in particular establishes Theorem 1.2. So consider now equation (1.9), for which

$$E(u) = \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla u|^2 - \frac{1}{2*}|u|^{2*}\right)dx.$$

Proposition 6.2. *Let $u_0 \in H^1(\mathbb{R}^d)$ such that*

$$E(u_0) < E(W) \quad and \quad \|\nabla u_0\|_{L^2} \geq \|\nabla W\|_{L^2}. \tag{6.11}$$

Then the corresponding solution u to (1.9) blows up in finite time. That is, $T_{max}(u_0)$ (coming from the $\dot{H}^1$ local theory as in Theorem (2.1)) is finite.

Proof. We will give a sketch of the proof, which is largely a modification of the proof of the previous proposition.

By the Sobolev inequality (1.5),

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2 dx - \frac{1}{2*}\int_{\mathbb{R}^d}|u|^{2*}dx \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2*}\frac{\|W\|_{L^{2*}}^{2*}}{\|\nabla W\|_{L^2}^{2*}}\|\nabla u\|_{L^2}^{2*}. \tag{6.12}$$

We define $f(y) := \frac{1}{2}y - \frac{1}{2*}C^{2*}y^{\frac{2*}{2}}$, $\quad C = \frac{\|W\|_{L^{2*}}}{\|\nabla W\|_{L^2}} = \|\nabla W\|_{L^2}^{-\frac{2}{d}}$, so that by energy dissipation and (6.11),

$$f(\|\nabla u\|_{L^2}^2) \leq E(u) \leq E(u_0) < E(W). \tag{6.13}$$

It is straightforward to verify that $f(y)$ is concave for $y \geq 0$ and attains its maximum value $f(\|\nabla W\|_{L^2}^2) = E(W) = \frac{1}{d}\|\nabla W\|_{L^2}^2$ at $y = \|\nabla W\|_{L^2}^2$.

Furthermore, it is strictly increasing on $[0, \|\nabla W\|^2_{L^2}]$ and strictly decreasing on $[\|\nabla W\|^2_{L^2}, +\infty)$. Denote the inverse function of f on $[\|\nabla W\|^2_{L^2}, +\infty)$ as

$$e = f^{-1} : (-\infty, E(W)] \to [\|\nabla W\|^2_{L^2}, +\infty),$$

strictly decreasing. By (6.13) and (6.11) then,

$$\|\nabla u(t)\|^2_{L^2} \geq e(E(u(t))).$$

By the definitions of $K = K(u)$ and the energy $E = E(u)$

$$-K(u) = -\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} |u|^{2^*} dx = \frac{2}{d-2} \int_{\mathbb{R}^d} |\nabla u|^2 dx - 2^* E(u)$$

$$\geq \frac{2}{d-2} \left(e(E) - dE \right) =: g(E).$$

Note that $g(E(W)) = 0$ and for $E < E(W)$, $g(E) > 0$ and $g'(E) = \frac{2}{d-2} e'(E) - 2^* < -2^*$. Defining $I(t)$ as in (6.5):

$$I''(t) = -2K(u) \geq 2g(E(u)) > 0.$$

By the Fundamental Theorem of Calculus and the energy dissipation relation,

$$2g(E(u)) = 2g(E(u_0)) + 2 \int_0^t |g'(E(u(s)))| \int_{\mathbb{R}^d} |u_t|^2 dx ds.$$

One can now repeat the proof of Proposition 6.1 replacing (6.8) by

$$I''(t) \geq 4(1+\delta)J(t) = 4(1+\delta) \left(2g(E(u_0)) + \int_0^t 2|g'(E(u(s)))| \int_{\mathbb{R}^d} |u_t|^2 dx ds \right)$$
(6.14)

Since $g(E(u_0)) > 0$, we can proceed exactly as in the proof of the previous Proposition to conclude that if $T_{max} = \infty$, then we must have $\limsup_{t \to \hat{t}-} \|u(t)\|_{L^2} = \infty$ for some $\hat{t} < \infty$, which by (6.6) implies $\limsup_{t \to \hat{t}-} \|u(t)\|_{L^{2*}} = \infty$, and so by Sobolev, $\limsup_{t \to \hat{t}-} \|\nabla u(t)\|_{L^2} = \infty$, contradicting $T_{max} < \infty$. $\qquad \square$

Acknowledgements

7   References

[1] T.Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. (9), 55, 1976, 3, 269-296.

[2] H.Bahouri, P.Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math. 121, 1999, 131-175.

[3] J.M.Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. Quart. J. Math. Oxford Ser. (2) 28, 1977, no. 112, 473-486.

[4] S.Blatt, M.Struwe, Well-posedness of the supercritical Lane-Emden heat flow in Morrey spaces, arXiv:1511.07685.

[5] H.Brezis, T. Cazenave, A nonlinear heat equation with singular initial data. Journal D' Analyse Mathematique, 68(1), 1996, 277-304.

[6] H.Brézis, J.-M. Coron, Convergence of solutions of H-systems or how to blow bubbles, Arch. Rational Mech. Anal. 89, 1985), 21-56.

[7] A.Bulut, Maximizers for the Strichartz inequalities for the wave equation, Differential Integral Equations (2010) no. 11-12, 1035-1072.

[8] T.Cazenave, Finite-time blowup for a family of nonlinear parabolic equations. Lecture notes prepared for VI ENAMA, Aracaju, Brazil, Nov. 7-9 2012.

[9] T.Cazenave, F.Dickstein, F.B. Weissler, Global existence and blowup for sign-changing solutions of the nonlinear heat equation, J. Differential Equations 344, 2009, no. 7, 2669-2680.

[10] T.Cazenave, F.Dickstein, F.B.Weissler, Non-regularity in Hölder and Sobolev spaces of solutions to the semilinear heat and Schrdinger equations, preprint, arXiv:1503.08954.

[11] C.Collot, Non radial type II blow up for the energy supercritical semilinear heat equation,preprint, arXiv:1604.02856.

[12] C.Collot, P.Raphael, J.Szeftel, On the stability of type I blow up for the energy super critical heat equation,preprint, arXiv:1605.07337.

[13] C.Cortazar, M.del Pino, M.Musso, Green's function and infinitetime bubbling in the critical nonlinear heat equation, preprint, arXiv:1604.07117.

[14] C.Collot, F.Merle, P.Raphael, Dynamics near the ground state for the energy critical nonlinear heat equation in large dimensions, preprint, arXiv:1604.08323.

[15] T. Cazenave, F.B.Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Anal., Theory, Methods and Applications. An International Multidisciplinary Journal. Series A: Theory and Methods, 14, 1990, 10, 807-836.

[16] S.-Z.Du, On partial regularity of the borderline solution of semilinear parabolic equation with critical growth, Adv. Differential Equations 18, 2013, no. 1-2, 147-177.

[17] L.Escauriaza, G.Seregin, V.Šverák. On backward uniqueness for parabolic equations. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 288(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts., 2002, (32):100103, 272.

[18] L.Escauriaza, G.Seregin, V.Šverák. Backward uniqueness for parabolic equations. Arch. Ration. Mech. Anal., 2003, 169(2):147157.

[19] L.Escauriaza, G.Seregin, V.Šverák. $L^{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. Uspekhi Mat. Nauk, 2003, 58(2(350)):344.

[20] S.Filippas, M.A.Herrero, J.J.L.Velázquez, Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical

nonlinearity, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456, 2000, 2957-2982.

[21] V.Galaktionov, J.L.Vazquez, Continuation of blowup solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math. 50 1997, no. 1, 1-67.

[22] I.Gallagher, Profile decomposition for solutions of the Navier-Stokes equations. Bull. Soc. Math. France, 2001 129(2):285-316.

[23] I. Gallagher, D.Iftimie, F.Planchon. Non-explosion en temps grand et stabilite de solutions globales des equations de Navier-Stokes. C. R. Math. Acad. Sci. Paris, 2002, 334(4):289-292.

[24] I.Gallagher, G.S.Koch, F.Planchon, A profile decomposition approach to the $L_t^\infty L_x^3$ Navier-Stokes regularity criterion. Math. Ann. 355, 4, 2013, 1527-1559.

[25] I.Gallagher, G.S.Koch, F.Planchon, Blow-up of critical Besov norms at a potential Navier-Stokes singularity, preprint, arXiv:1407.4156v3.

[26] P.Gérard, Description du défaut de compacité de l'injection de Sobolev. ESAIM Control Optim. Calc. Var. 3, 1998, 213-233.

[27] Y.Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations 62, 1986, 186-212.

[28] M.-H.Giga, Y.Giga, J. Saal, Nonlinear partial differential equations: Asymptotic behavior of solutions and self-similar solutions. Vol. 79. Springer Science & Business Media, 2010

[29] Y.Giga, R.V.Kohn, Asymptotically self-similar blowup of semilinear heat equations, Comm. Pure Appl. Math. 38, 1985, 297-319.

[30] Y.Giga, R.V.Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J. 36, 1987, 1-40.

[31] Y.Giga, R.V.Kohn, Nondegeneracy of blow up for semilinear heat equations, Comm. Pure Appl. Math. 42, 1989, 845-884.

[32] S.Gustafson, D.Roxanas, Global solutions for the critical, higher-degree corotational harmonic map heat flow to $\mathbb{S}^2$, preprint, arXiv:1711.06476.

[33] S.Keraani, On the defect of compactness for the Strichartz estimates for the Schrödinger equations. J. Diff. Eq. 175, 2001, 353-392.

[34] C.E.Kenig, G.S.Koch. An alternative approach to the Navier-Stokes equations in critical spaces. Ann. I. H. Poincare -AN, 2011, 28(2):159-187.

[35] C.E.Kenig, F.Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Inventiones mathematicae, 2006, 166, no 3, 645-675.

[36] C.E.Kenig, F.Merle, Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation, Acta Mathematica, 2008, 201, no 2, 147-212.

[37] R.Killip, M.Visan, The focusing energy-critical nonlinear Schrodinger equation in dimensions five and higher, Amer. J. Math. 132, 2010, 361-424.

[38] R.Killip, M.Visan, Nonlinear Schrödinger Equations at Critical Regularity, "Evolution Equations", 325-437, Clay Math. Proc., 2013, 17, Amer. Math. Soc., Providence, RI.

[39] H.A.Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. Arch. Rational Mech. Anal. 51, 1973, 371-386.

[40] F.Lin, C.Wang, The analysis of harmonic maps and their heat flows, 2008, World Scientific.

[41] P.-L.Lions, The concentration-compactness principle in the calculus of variations. The limit case. I. Rev. Mat. Iberoamericana 1, 1985, 145-201.

[42] F.Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, Communications on Pure and Applied Mathematics, 1992, 263-300.

[43] H.Matano, F. Merle, Classification of type I and type II behaviors for a supercritical nonlinear heat equation. J. Funct. Anal. 256, 2009, no. 4, 9921064.

[44] H.Matano, F.Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation. Comm. Pure Appl. Math. 57, 2004, no. 11, 1494-1541.

[45] H.Matano, F.Merle, Threshold and generic type I behaviors for a supercritical nonlinear heat equation. J. Funct. Anal. 261, no. 3, 2011, 716-748.

[46] F. Merle, L.Vega, Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in $2D$, Internat. Math. Res. Notices 8, 1998, 399-425.

[47] M.del Pino, M.Musso, J.Wei, Infinite time blow-up for the 3-dimensional energy critical heat equation, preprint, arXiv:1705.01672.

[48] F. Merle, H.Zaag, Uniform blow-up estimates for nonlinear heat equations and applications. IMS Conference on Differential Equations from Mechanics, Hong Kong, 1999.

[49] F.Planchon, Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$ Ann. Inst. H. Poincaré Anal. Non Linéaire, 1996, (13):319-336.

[50] P.Quitnner, P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhauser Advanced Texts, 2007, ISBN: 978-3-7643-8441-8.

[51] D.Roxanas, Long-time dynamics for the energy-critical Harmonic Map Heat Flow and Nonlinear Heat Equation, PhD Thesis, 2017, University of British Columbia.

[52] R.Schweyer, Type II blow up for the four dimensional energy critical semi linear heat equation, J. Funct. Anal., 263, 2012, pp. 3922-3983.

[53] M.Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Comment. Math. Helv., 1985, 60, 558-581.

[54] M.Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187, 1984, 511-517.

[55] J.Sacks, K.Uhlenbeck, The existence of minimal immersions of 2-spheres. Ann. of Math. 113, 1981, 1-24.

[56] G.Talenti, Best constant in the Sobolev inequality, Ann. Mat. Pura Appl., (4) 110 1976, 353-372.

[57] P.Topping, Rigidity in the harmonic map heat flow, J. Differential Geometry, 1997, 45, 593-610.

[58] T.Tao, M.Visan, Stability of energy-critical nonlinear Schrödinger equations in high dimensions, Electron. J. Diff. Eqns. 118, 2005, 1-28.

[59] F.B.Weissler, Local existence and nonexistence for semilinear parabolic equations in Lp. Indiana Univ. Math. J, 29(1), 1980, 79-102.

[60] F.B.Weissler, Existence and non-existence of global solutions for a semilinear heat equation, Israel Journal of Mathematics, 1981, Springer.