GLOBAL SOLUTIONS FOR THE CRITICAL, HIGHER-DEGREE COROTATIONAL HARMONIC MAP HEAT FLOW TO $\mathbb{S}^2$

STEPHEN GUSTAFSON AND DIMITRIOS ROXANAS

ABSTRACT. We study m-corotational solutions to the Harmonic Map Heat Flow from $\mathbb{R}^2$ to $\mathbb{S}^2$. We first consider maps of zero topological degree, with initial energy below the threshold given by twice the energy of the harmonic map solutions. For $m \geq 2$, we establish the smooth global existence and decay of such solutions via the *concentration-compactness* approach of Kenig-Merle, recovering classical results of Struwe by this alternate method. The proof relies on a profile decomposition, and the energy dissipation relation. We then consider maps of degree m and initial energy *above* the harmonic map threshold energy, but below three times this energy. For $m \geq 4$, we establish the smooth global existence of such solutions, and their decay to a harmonic map (stability), extending results of Gustafson-Nakanishi-Tsai to higher energies. The proof rests on a stability-type argument used to rule out finite-time bubbling.

CONTENTS

2010 *Mathematics Subject Classification.* 35K05, 35K58.

Key words and phrases. Harmonic Map Heat Flow, Concentration Compactness, Global Existence, Stability.

1. Introduction and Results

The harmonic map heat flow into $\mathbb{S}^2$ is given by the equation

$$\mathbf{u}_t = \Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u}, \quad \mathbf{u}(0,x) = \mathbf{u}_0(x) \tag{1.1}$$

where for $t \geq 0$,

$$\mathbf{u}(t,\cdot) : \mathbb{R}^2 \to \mathbb{S}^2,$$

$$\mathbb{S}^2 := \{\mathbf{u} = (u_1, u_2, u_3) : |\mathbf{u}| = 1\} \subset \mathbb{R}^3,$$

is the unit 2-sphere, Δ denotes the Laplace operator in $\mathbb{R}^2$, and $|\nabla\mathbf{u}|^2 = \sum_{j=1}^{2}\sum_{i=1}^{3}(\frac{\partial u_i}{\partial x_j})^2$. Equation (1.1) is the L^2-gradient flow of the energy functional

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\mathbf{u}|^2 dx$$

for such maps. Taking formally the scalar product of the PDE with $\mathbf{u}_t$ and integrating over $[0,t) \times \mathbb{R}^2$, we obtain

$$\mathcal{E}(\mathbf{u}(t,\cdot)) + \int_0^t \int_{\mathbb{R}^2} |\mathbf{u}_t|^2 = \mathcal{E}(\mathbf{u}_0)$$

which implies that the energy is non-increasing. A more geometric way to write (1.1) is

$$\mathbf{u}_t = \sum_{j=1}^{2} D_j\partial_j\mathbf{u} = P^{\mathbf{u}}\Delta\mathbf{u},$$

where $P^{\mathbf{u}}$ denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane

$$T_{\mathbf{u}}\mathbb{S}^2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 : \boldsymbol{\xi}\cdot\mathbf{u} = 0\}$$

to $\mathbb{S}^2$ at $\mathbf{u}$, $\partial_j = \frac{\partial}{\partial x_j}$ is the usual partial derivative and D_j the covariant derivative acting on vector fields $\xi(x) \in T_{\mathbf{u}(x)}\mathbb{S}^2$:

$$D_j\xi := P^{\mathbf{u}}\partial_j\xi = \partial_j\xi - (\partial_j\xi\cdot\mathbf{u})\mathbf{u} = \partial_j\xi + (\partial_j\mathbf{u}\cdot\xi)\mathbf{u}.$$

The harmonic map heat flow between Riemannian manifolds was introduced by Eells-Sampson [16] to study harmonic maps, which are its static solutions. Equation (1.1), where the target manifold is $\mathbb{S}^2$, is, in addition, physically relevant as the purely diffusive case of the

Landau-Lifshitz equations of ferromagnetism [35]. The setting of a two-dimensional domain is therefore of physical importance, but is also analytically interesting as the *energy-critical* one: the scaling $u(x) \mapsto u(\lambda x)$ leaves both the equation and the energy invariant

$$\mathcal{E}(\mathbf{u}(\cdot)) = \mathcal{E}(\mathbf{u}(\frac{\cdot}{\lambda})),$$

and so this is the borderline case for smooth global existence versus possible singularity formation.

The question of singularity formation and characterization of possible blow-up has attracted a lot of attention. On a compact manifold domain, Struwe [46] constructed a global weak solution whose singularities occur through energy concentration at a finite number of space-time points, at each of which a non-trivial harmonic map bubbles off: for $t_n \nearrow T$,

$$\mathbf{u}(t_n, a_n + \lambda(t_n)x) \to \mathbf{Q}(x), \;\; \lambda(t_n) \to 0, \; a_n \to a, \quad \mathbf{Q} \text{ harmonic}$$

locally in space. Later work [38, 15, 39, 48, 49] (see also the book [36]) showed that at a singularity, all the energy is accounted for by the bubbles and the the weak limit (*body map*), and therefore that the solution converges strongly to the body map, after all the bubbles are removed.

Working in the subclass of the m-corotational solutions with $m = 1$, on a disk, [10] showed that, indeed, finite time blow-up does occur in some situations, using the sub-solution method. Formal analysis [4], and later rigorous constructions [40, 41], show that for 1-corotational maps described by the azimuthal angle $u(r,t)$, approaching a blow-up time $t \nearrow T$,

$$u(t,r) - Q(\frac{r}{\lambda(t)}) \to u^* \quad \text{in} \quad \dot{H}^1,$$
$$\lambda(t) = c(u_0)(1 + o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{\frac{2L}{2L-1}}}, c(u_0) > 0,$$

where Q corresponds to the unique (up to scaling) harmonic map in this class, and $L \in \mathbb{Z}^+$, with $L = 1$ providing the generic blow-up rate. See also [14] for a related recent result, and [5, 6, 20, 21] concerning the breakdown of solutions in higher (supercritical) dimensions.

On the other hand, Grotowski-Shatah [23], using maximum principle methods, showed that on the unit disc in $\mathbb{R}^2$, m-corotational solutions will not blow-up in finite-time for degrees $m \geq 2$, given certain pointwise bounds on the initial data. One of our goals is to extend this result to the domain $\mathbb{R}^2$, and, more importantly, to give a *maximum principle-free proof*, which one can therefore hope might extend to systems such as the Landau-Lifshitz equations.

In this work we specialize to *m-co-rotational* maps: in polar coordinates,

$$\mathbf{u}(t,(r,\theta)) = (\cos(m\theta)\sin(u(t,r)), \sin(m\theta)\sin(u(t,r)), \cos(u(t,r))).$$

for which (1.1) reduces to the problem

$$u_t = u_{rr} + \frac{1}{r}u_r - m^2\frac{\sin 2u}{2r^2}, \qquad u(0,r) = u_0(r) \tag{1.2}$$

for the angle $u(t,r)$. Without loss of generality we assume $m > 0$. Defining

$$\Delta_r u = u_{rr} + \frac{1}{r}u_r, \quad \text{the radial Laplacian in } \mathbb{R}^2,$$
$$\Delta_m u = (\Delta_r - \frac{m^2}{r^2})u,$$

we may write (1.2) as

$$\begin{aligned} u_t &= (\Delta_r - \frac{m^2}{r^2})u + \frac{m^2}{r^2}(u - \frac{\sin 2u}{2}) \\ &= \Delta_m u + F(u), \qquad F(u) = \frac{m^2}{r^2}(u - \frac{\sin 2u}{2}). \end{aligned}$$

The energy for these maps is given by

$$\mathcal{E}(\mathbf{u}) = 2\pi E(u), \quad E(u) := \frac{1}{2}\int_0^\infty (u_r^2 + m^2\frac{\sin^2(u)}{r^2})rdr.$$

Note that finite energy requires

$$\lim_{r\to 0,\,\infty} u(r) \in \pi\mathbb{Z},$$

and indeed the assumption of finite energy is sufficient to guarantee the existence of these above limits (e.g.,[25]). For m-corotational maps, the classical energy lower-bound by the topological degree reads

$$\begin{aligned} E(u) &= \frac{1}{2}\int_0^\infty \left(u_r \pm \frac{m}{r}\sin(u)\right)^2 rdr \pm m\int_0^\infty (\cos(u))_r\, dr \\ &\geq \frac{1}{2}\int_0^\infty \left(u_r \pm \frac{m}{r}\sin(u)\right)^2 rdr + m|\cos(u(\infty)) - \cos(u(0))| \\ &\geq 2\,|\text{degree}(\mathbf{u})| \end{aligned} \tag{1.3}$$

for the appropriate choice of $\pm$ sign. The m-corotational stationary solutions – corresponding to the harmonic maps – are the functions saturating this inequality, given by

$$Q(r) = \pi - 2\arctan(r^m), \quad Q_r + \frac{m}{r}\sin(Q) = 0, \;\; Q(0) = \pi, \; Q(\infty) = 0 \tag{1.4}$$

and their scalings $Q(\frac{r}{s})$, $s > 0$, as well as the negatives and shifts by $\pi\mathbb{Z}$ of these. Since these harmonic maps each minimize the energy within their topological class, they provide natural thresholds for global smoothness and decay vs. singularity formation.

In light of the above considerations, we make the following definitions:

$$E_0 := \{u : [0,\infty) \to \mathbb{R} \mid E(u) < 2E(Q), \;\; \lim_{r\to 0+} u(r) = 0, \; \lim_{r\to\infty} u(r) = 0\},$$

$$E_1 := \{u : [0,\infty) \to \mathbb{R} \mid E(Q) \le E(u) \le 3E(Q), \lim_{r\to 0+} u(r) = \pi, \lim_{r\to\infty} u(r) = 0\}.$$

and note that

$$u \in E_0 \implies \mathit{degree}(\mathbf{u}) = 0, \quad \min_{u\in E_0} E(u) = E(0) = 0,$$

$$u \in E_1 \implies \mathit{degree}(\mathbf{u}) = m, \quad \min_{u\in E_1} E(u) = E(Q) = 2m.$$

Our first result concerns solutions in the "below-threshold" class E_0:

Theorem 1.1. *Assuming $u_0 \in E_0$, and $m \ge 2$,* (1.2) *has a unique solution $u(t,r)$, which is global in time, smooth, and decays: $E(u(t,\cdot)) \to 0$ and $\sup_r |u(t,r)| \to 0$, as $t \to \infty$.*

The main purpose here is to give a proof which follows Kenig-Merle's *concentration-compactness* strategy [31], originally developed for (and widely applied to) *dispersive* problems, but relevant also to certain *diffusive* ones [17, 18, 30, 29]. The method is well-suited to the noncompact domain, and, more pertinently, provides an alternative approach to the classical theory of Struwe and successors.

We first establish a local well-posedness theory for solutions of (1.2) in E_0 which parallels that for (say) the energy-critical nonlinear Schrödinger equation, and differs from that appearing in the classical parabolic literature. Then the key tools for the concentration-compactness strategy are a stability-under-small-perturbations variant of the local theory, and a profile decomposition for an $\dot{H}^1$-like space, adapted to the heat flow. A profile decomposition directly applicable to our setting was not readily available, stemming from the absence of some Sobolev embeddings in dimension two. So we take an indirect approach, first establishing estimates on the linear evolution in higher dimensions, which then connect back to our problem through a change of variable.

For applications of the concentration-compactness approach to other (below threshold) geometric problems we refer, for instance, to [12] in the context of Wave Maps, and to [2, 3, 27] for Schrödinger Maps. A more comprehensive review of the literature can be found in [42].

To put our results in the "above-threshold" class E_1 into context, we first recall results from the series of papers [25, 26, 24, 28] which apply to the m-corotational heat-flow (1.2), but more generally to solutions of the *Landau-Lifshitz* family of equations

$$\mathbf{u}_t = a\left(\Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u}\right) + b\,\mathbf{u}\times\Delta\mathbf{u}, \quad \mathbf{u}(0,x) = \mathbf{u}_0(x), \quad a \geq 0,\ b \in \mathbb{R} \tag{1.5}$$

of degree m, with *equivariant* symmetry. For higher degrees, the m-equivariant harmonic maps are shown to be asymptotically stable in the strong sense that if the initial data has near-minimal (harmonic) energy given the degree, the solution is globally smooth and asymptotically converges to a nearby harmonic map:

Theorem 1.2. *([28]). Assume $\boldsymbol{u}_0$ is of degree $m \geq 3$ with equivariant symmetry, and*

$$\mathcal{E}(\boldsymbol{u}_0) - 4\pi m \ll 1.$$

Then the solution of (1.5) is globally regular (continuous into the energy space) and there exists a harmonic map $\boldsymbol{Q}$ close to $\boldsymbol{u}_0$ (in the energy norm) such that

$$\|\boldsymbol{u}(t,\cdot) - \boldsymbol{Q}\|_{L^\infty} + a\mathcal{E}(\boldsymbol{u}(t,\cdot) - \boldsymbol{Q}) \to 0, \text{ as } t \to \infty.$$

Remarks:

- in the dissipative case ($a > 0$) these solutions are converging to a harmonic map in the energy norm, while this is impossible for the conservative case $a = 0$, known as the *Schrödinger flow*;
- the case $m = 3$ is significantly more complex that $m \geq 4$, in particular requiring a normal form-type argument ([28]) to establish the asymptotic behaviour – for this reason we consider only $m \geq 4$ here;
- for the $m = 2$ corotational heat-flow, the above conclusion is false: solutions are still global, but may exhibit blow-up in *infinite time*, or other complex behaviours ([28]);
- for $m = 1$, near-minimal energy solutions may exhibit *finite-time* blow-up ([40, 41]).

Our main result is to extend this theorem, for the corotational heat-flow, beyond the perturbative regime to the higher energy maps in E_1:

Theorem 1.3. *Assuming $u_0 \in E_1$ and $m \geq 4$, (1.2) has a unique solution, which is global in time, smooth, and converges to a harmonic map:*

for some $s > 0$, $\mathcal{E}(u(t,\cdot) - Q(r/s)) \to 0$ *and* $\sup_r |u(t,r) - Q(r/s)| \to 0$, *as* $t \to \infty$.

We would like to emphasize here that solutions in this class *are not prohibited from forming a singularity by either energetic or topological constraints* – that these solutions remain globally smooth *does not follow from any classical theory.*

The main point is to exclude the possibility of finite-time blowup. As in [46], if the solution blows-up in finite time, it does so by bubbling off a non-trivial harmonic map. The corotational symmetry (and finite energy) ensures the only possible concentration points are $r = 0$ and $r = \infty$. Finite-time energy concentration at spatial infinity is ruled out using the energy dissipation relation. The condition $E(u_0) \leq 3E(Q)$ ensures only one bubble may form, and so following [38], if $u(t,\cdot) \in E_1$ is a solution blowing up at time T, there exists a sequence of times $t_n \nearrow T$, scales $s_n \searrow 0$, and a function $w_0 \in E_0$ such that $\xi(t_n,\cdot) := u(t_n,\cdot) - Q(\frac{\cdot}{s_n}) - w_0 \to 0$ in the energy norm. This is contradicted by adapting the modulation theory and linearized (about Q) evolution estimates of [28] to estimate $\xi(t_n,\cdot)$, and show $s_n \not\to 0$. We exploit the fact that in certain space-time norms, the nonlinear interaction of $Q(\frac{r}{s_n})$ and the (smooth, global) solution emanating from data w_0 is small on small time intervals.

The remaining impediment is possible infinite-time concentration. But in that scenario, the energy must approach the minimal energy $E(Q)$, and so it is excluded by [28].

We emphasize that *the proof does not in any way rely on the maximum principle*. Therefore, the result may be extended beyond the corotational class to the (larger) *equivariant* class, and even to the Landau-Lifshitz equations (1.5) (work in progress).

2. Heat-Flow Below Threshold

In this section we prove Theorem 1.1 on the "below-threshold" solutions of the corotational heat-flow.

2.1. **Analytical ingredients.**

2.1.1. *Energy properties of maps in* E_0. We begin by showing the function class E_0 is naturally endowed with the energy-space norm

$$\|u\|_{X^2}^2 = \int_0^\infty \left(u_r^2 + m^2 \frac{u^2}{r^2} \right) r dr,$$

in the following sense: given $\delta_1 > 0$, there is $C = C(\delta_1) > 0$ such that

$$u \in E_0,\ E(u) \leq 2E(Q) - \delta_1 \implies \frac{1}{C}\|u\|_{X^2}^2 \leq E(u) \leq C\|u\|_{X^2}^2.$$

This follows directly from a version of the topological lower bound (1.3) localized to intervals:

Lemma 2.1. *If $u \in E_0$, with $E(u) \leq 2E(Q) - \delta_1$, for some $\delta_1 > 0$, there is $\delta_2 = \delta_2(\delta_1) > 0$ such that*

$$|u(r)| \leq \pi - \delta_2.$$

Proof. As in [44, 12] we define

$$G(u) := \int_0^u m|\sin(s)|ds$$

and

$$E_{r_1}^{r_2}(u) := \frac{1}{2}\int_{r_1}^{r_2} (u_r^2 + \frac{m^2 \sin^2(u)}{r^2})rdr.$$

Then for all $0 \leq r_1 < r_2 < \infty$, by the Fundamental Theorem of Calculus and Young's inequality:

$$\begin{aligned} |G(u(r_2)) - G(u(r_1))| &= \left|\int_{r_1}^{r_2} \frac{\partial}{\partial r}G(u(r))dr\right| = \left|\int_{r_1}^{r_2} m|\sin u|u_r dr\right| \\ &\leq \frac{1}{2}\int_{r_1}^{r_2} (\frac{m^2\sin^2(u)}{r^2} + u_r^2)rdr \leq E_{r_1}^{r_2}(u) \end{aligned} \tag{2.1}$$

For $u \in E_0$, $G(u(\infty)) = G(0) = 0$ and $G(u(0)) = G(\pi) = 0$. From (2.1) for any $r > 0$:

$$|G(u(r))| = |G(u(r)) - G(u(0))| \leq E_0^r(u)$$

and

$$|G(u(r))| = |G(u(\infty)) - G(u(r))| \leq E_r^\infty(u).$$

Thus

$$|G(u(r))| \leq \frac{1}{2}E(u) \leq \frac{1}{2}(2E(Q) - \delta_1) = 2m - \frac{\delta_1}{2}.$$

G is odd, increasing on $[-\pi, \pi]$, and $G(\pi) = 2m$, so

$$|u(r)| \leq G^{-1}(2m - \frac{\delta_1}{2}) =: \pi - \delta_2, \quad \delta_2 > 0.$$

□

We remark that due to the boundary conditions, E_0 contains no non-trivial static solutions (corresponding to harmonic maps) since these are all monotone (1.4).

2.1.2. *Local well-posedness for maps in E_0.* From now on, unless otherwise specified, all norms will be for functions defined on $(0,\infty)$, with the measure rdr We define spaces rL^p via norm

$$\|u\|_{rL^p} := \left\|\frac{u}{r}\right\|_{L^p},$$

and X^p via norms

$$\|u\|_{X^p}^p := \|u_r\|_{L^p}^p + m^p\|u\|_{rL^p}^p, \qquad \|u\|_{X^\infty} = \|u_r\|_{L^\infty} + \|u\|_{rL^\infty}.$$

Recall we may write the m-corotational heat-flow (1.2) for initial data $u_0 \in E_0$ as

$$\begin{cases} u_t = \Delta_m u + F(u) \\ u(0) = u_0 \in X^2 \end{cases} \tag{2.2}$$

where $\Delta_m = \Delta_r - \frac{m^2}{r^2}$ and $F(u) = \frac{m^2}{r^2}(u - \frac{\sin 2u}{2})$. We will say that a function $u : I \times \mathbb{R} \to \mathbb{R}$, $I = [0,T)$, is a *solution* to (2.2) on I if $u \in C_t X_r^2 \cap L_t^4 rL_r^4(K)$ for every compact $K \subset I$, and for every $t \in I$

$$u(t) = e^{t\Delta_m}u_0 + \int_0^t e^{(t-s)\Delta_m}F(u(s))ds.$$

We summarize the local theory:

Theorem 2.1. *(Local well-posedness)*

(1) *(Local Existence) Let $u_0 \in X^2$ There exists an $\epsilon > 0$ such that if $I = [0,T)$ and $\|e^{t\Delta_m}u_0\|_{L_t^4(I;rL^4)} < \epsilon$, then there exists a unique solution to (2.2), which moreover satisfies $\|u\|_{L_t^4(I;rL^4)} \leq 2\epsilon$. To each initial datum u_0 we can associate a maximal time interval $I = [0, T_{max}(u_0))$ on which there is a solution ($T_{max}(u_0)$ may be $+\infty$).*

(2) *(Blow-up Criterion) $T_{max}(u_0) < +\infty \implies \|u\|_{L_t^4([0,T_{max}(u_0));rL_r^4)} = +\infty$.*

(3) *(Energy dissipation) $u_t \in L_t^2([0,T_{max}(u_0)); L^2)$ and for each $t < T_{max}(u_0)$,*

$$E(u(t)) + \int_0^t \int_0^\infty u_t^2(s) rdr\, ds = E(u_0).$$

(4) *(Decay) If $T_{max}(u_0) = +\infty$ and $\|u\|_{L_t^4([0,\infty);rL_r^4)} < +\infty$, then $\|u(t)\|_{X^2} \to 0$ as $t \to +\infty$.*

(5) *(Small data) If $\|u_0\|_{X^2}$ is sufficiently small, then $T_{max}(u_0) = \infty$ and $\|u\|_{L_t^4([0,\infty);rL_r^4)} \lesssim \|u_0\|_{X^2}$; in particular $\|u(t)\|_{X^2} \to 0$.*

(6) *(Continuous Dependence) T_{max} is a lower semi-continuous function of $u_0 \in X^2$, and $u_0 \to u(t,\cdot)$ is continuous on X_2.*

This local well-posedness rests on space-time estimates for the linear evolution $e^{t\Delta_m}$. The decay estimate

$$\|e^{t\Delta_m}\phi\|_{L^p} \lesssim t^{-(1/a-1/p)}\|\phi\|_{L^a}, \quad 1 \leq a \leq p \leq \infty \tag{2.3}$$

is an immediate consequence of Young's inequality and the explicit heat kernel.

For $\phi = \phi(r)$, $f = f(t,r)$, the space-time estimates

$$m \geq 2: \quad \|e^{t\Delta_m}\phi\|_S \lesssim \|\phi\|_{X^2} \qquad \|\int_0^t e^{(t-s)\Delta_m} f(s)ds\|_S \lesssim \|f\|_{L_t^1 X^2 + L_t^2 X^1} \tag{2.4}$$

where

$$\|u\|_S := \|u\|_{L_t^\infty X^2} + \|u\|_{L_t^2 X^\infty} + \|u\|_{L_t^2\, r^2 L^2} + \|u_r\|_{L_t^2 X^2}$$

follow from the standard heat equation energy estimate for the derived function $\nabla\left(e^{i\theta}u(t,r)\right)$, and an interpolation estimate ([24]). We remark that this procedure only yields the *endpoint* spaces $u \in L_t^2 X^\infty \cap L_t^2 r^2 L^2$, $u_r \in L_t^2 X^2$ under the restriction $m \geq 2$. For $m = 1$, the estimate still holds if these spaces are replaced by some $L_t^r X^p$ with $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, $p < \infty$, and this suffices for the local well-posedness. However, for $m = 1$, the endpoint space would be required below in the profile decomposition argument. That is the reason we impose $m \geq 2$ here, and in Theorem 1.1.

Given the space-time estimates (2.4), together will the elementary pointwise inequalities

$$|\frac{1}{r}F(u)| \lesssim \frac{1}{r^3}|u|^3, \quad |\partial_r F(u)| \lesssim \frac{1}{r^3}|u|^3 + \frac{1}{r^2}u^2|u_r| \tag{2.5}$$

on the nonlinearity, the proof of the local well-posedness is a standard variant of the corresponding proof for the critical NLS, based on the Banach fixed-point theorem (see [9]). So we will omit most of the details (which can be found in [42]), and just indicate how to establish decay of global solutions with finite space-time norm.

Proof of the decay of global solutions: assuming $\|u\|_{L^4 r L^4([0,\infty))} < \infty$, we first show that also $\|u\|_{S([0,\infty))} < \infty$. For a given $\tilde{\epsilon}$ (to be chosen small), subdivide the interval $[0,\infty)$ into a finite number of intervals $I_j = [t_j, t_{j+1})$ so that $\|u\|_{L^4 r L^4(I_j)} \leq \tilde{\epsilon}$. For $t \in I_j$, by the Duhamel formula,

$$u(t) = e^{(t-t_j)\Delta_m}u(t_j) + \int_{t_j}^t e^{(t-s)\Delta_m}F(u(s))ds.$$

Using (2.4), (2.5), and Hölder's inequality, we arrive at

$$\|u\|_{S(I_j)} \leq C\|u(t_j)\|_{X^2} + C\|u\|_{S(I_j)}\|u\|^2_{L^4 r L^4(I_j)} \leq C\|u(t_j)\|_{X^2} + C\tilde{\epsilon}^2\|u\|_{S(I_j)}.$$

Choosing $\tilde{\epsilon}$ such that $C\tilde{\epsilon}^2 \leq \frac{1}{2}$ yields

$$\|u\|_{S(I_j)} \leq 2C\|u(t_j)\|_{X^2}.$$

Since there are only finitely many I_j, it follows that $\|u\|_{S([0,\infty))} < \infty$. So in particular, for any $\epsilon > 0$, there is a $T > 0$ such that

$$\|u\|^3_{S([T,+\infty))} \leq \epsilon.$$

By the Duhamel formula, for $t \geq T$,

$$u(t) = e^{(t-T)\Delta_m}u(T) + \int_T^t e^{(t-s)\Delta_m}F(u)ds.$$

By (2.4) and (2.5) as above,

$$\left\| \int_T^t e^{(t-s)\Delta_m}F(u)ds \right\|_{S([T,\infty))} \lesssim \|F(u)\|_{L^{4/3}([T,\infty);X^{4/3})} \lesssim \|u\|^3_{S([T,\infty))} \lesssim \epsilon.$$

As well,

$$\|e^{(t-T)\Delta_m}u(T)\|_{X^2} \to 0 \text{ as } t \to \infty,$$

by (2.3) and the density of $X^1 \cap X^2$ in X^2. Therefore

$$\limsup_{t\to\infty} \|u\|_{X^2} \lesssim \epsilon.$$

Since ϵ was arbitrary, the result follows. □

Remark 2.1. *Uniform decay follows from energy space decay by the elementary embedding $X^2 \subset L^\infty$:*

$$\|u(t)\|_{X^2} \to 0 \implies \|u(t)\|_{L^\infty} \lesssim \|u(t)\|_{X^2} \to 0.$$

Notice that our local theory combined with the previous section on the equivalence of the X^2 and the energy topology implies that if $u_0 \in E_0$, the boundary conditions persist in time, i.e. $u(t,\cdot) \in E_0$ throughout its lifespan.

2.1.3. *Stability under perturbations.* An important extension of the local existence theory is the following "perturbation" or "stability" theorem. This type of result, which establishes the existence of a solution to (2.2) nearby a given approximate one, goes back to [11, 47], and is by now standard. We use the following version (for a proof see [42]):

Theorem 2.2. *(Stability) Let $I = [0,T)$ and let $\tilde{u}$ be defined on $I \times [0,\infty)$ with*

$$\|\tilde{u}\|_{L^\infty(I;X^2)} \leq M, \qquad \|\tilde{u}\|_{L^4(I;rL^4)} \leq L$$

for some $M, L > 0$, and set

$$e := \tilde{u}_t - \Delta_m\tilde{u} - F(\tilde{u})$$

Let $u_0 \in X$ be such that

$$\|u_0 - \tilde{u}(\cdot, 0)\|_{X^2} \leq M'$$

for some $M' > 0$. There is $\epsilon_1 = \epsilon_1(M, M', L) > 0$ such that if for some $0 < \epsilon \leq \epsilon_1$ the smallness conditions

$$\|e^{t\Delta_m}(u_0 - \tilde{u}(0, \cdot)\|_{L^4(I;rL^4)} \leq \epsilon$$
$$\|e\|_{L^{4/3}(I;rL^{4/3})} \leq \epsilon$$

hold, then there exists a solution of (2.2) with $u(0) = u_0$ satisfying

$$\|u - \tilde{u}\|_{L^4 rL^4(I)} \leq C(M', M, L)M'.$$

2.2. **Profile decomposition.** The following proposition is the main tool (together with the stability theorem above) in the concentration-compactness approach to establishing global existence and decay. The idea of a profile decomposition is to characterize the loss of compactness in some embedding, and to recover some compactness. It can be traced back to [37, 7, 45, 43] and their modern "evolution" counterparts [1, 31, 32].

Proposition 2.1. *Let $\{u_n\}_n$ be a bounded sequence of radial functions in X^2. Then, after possibly passing to a subsequence (in which case, we rename it u_n again), there exist a family of radial functions $\{\phi^j\}_{j=1}^{\infty} \subset X^2$ and scales $\lambda_n^j > 0$ such that for each $J \geq 1$,*

$$u_n(x) = \sum_{j=1}^{J} \phi^j(\frac{x}{\lambda_n^j}) + w_n^J(x),$$

where $w_n^J \in X^2$ is such that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e^{t\Delta_m} w_n^J\|_{L_t^4 rL_r^4} = 0, \tag{2.8}$$

$$w_n^J(\lambda_n^j x) \rightharpoonup 0 \ \ in \ \ X^2, \ \ \forall j \leq J. \tag{2.9}$$

Moreover, the scales are asymptotically orthogonal in the sense that

$$\frac{\lambda_n^j}{\lambda_n^i} + \frac{\lambda_n^i}{\lambda_n^j} \to +\infty, \ \ \forall i \neq j. \tag{2.10}$$

Furthermore, for all $J \geq 1$ we have the following decoupling properties:

$$\|u_n\|_{X^2}^2 = \sum_{j=1}^{J} \|\phi^j\|_{X^2}^2 + \|w_n^J\|_{X^2}^2 + o_n(1)$$

$$E(u_n) = \sum_{j=1}^{J} E(\phi^j) + E(w_n^J) + o_n(1). \tag{2.11}$$

The procedure through which one establishes such a decomposition has become standard by now (e.g. see [1, 33]), thus we will only present the equation-specific parts of the argument.

There are two general roadmaps to follow in establishing such a decomposition. To get the convergence of the error w_n^J in the appropriate space-time norm, one can either use a refinement of the space-time estimates on the linear propagator, or a refinement of a Sobolev inequality through which the refinement of the space-time estimates will follow via interpolation arguments. The first approach would require more work in our case: arguments used in the Schrödinger case cannot be applied directly due to the lack of an analogue of the restriction theorems used. For the second approach, dimension two is very special due to the lack of the usual embeddings.

Our strategy is to first establish (2.8) for the homogeneous linear heat equation for radial functions in higher dimensions. We make use of a refined Sobolev inequality, first proved in [1] for $d = 3$ and later generalized to $d > 3$ in [8]. Then we convert this estimate to our 2d spaces by a change of variable, and use interpolation again to obtain the desired convergence.

Definition 2.1. *An exponent pair (q, p) is L^2-admissible in dimension d if*

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2},$$

and $\dot{H}^1$-admissible if

$$\frac{2}{q} + \frac{d}{p} = \frac{d-2}{2}.$$

We define the following Besov norm on L^2 :

$$I_k(f) := \left(\int_{2^k \le |\xi| \le 2^{k+1}} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \qquad \|f\|_B := \sup_{k \in \mathbb{Z}} I_k(f).$$

The following refinement of the Sobolev inequality is from [8] (Lemma 3.1):

Lemma 2.2. *(Refined Sobolev) For $d \geq 3$ there is a constant $C = C(d) > 0$ such that for every $u \in \dot{H}^1(\mathbb{R}^d)$, we have*

$$\|u\|_{L^p} \le C \|\nabla u\|_{L^2}^{\frac{2}{p}} \|\nabla u\|_B^{1-\frac{2}{p}}, \tag{2.12}$$

where $p = 2^* = \frac{2d}{d-2}$.

The next result, from [19], provides a decomposition of bounded sequences in $L^2(\mathbb{R}^d)$ (for a different, but equivalent Besov norm). Here, we specialize to radial functions.

Proposition 2.2. *Let* $\{f_n\}_n$ *be a bounded sequence of radially symmetric functions in* $L^2(\mathbb{R}^d), d \geq 3$. *Then there exist a subsequence (still denoted by* $\{f_n\}_n$*), a sequence of scales* $\{\lambda_n^j\}_n \subset (0,\infty)$ *satisfying (2.10), and bounded radial* $\{g^j\}_j, \{r_n^J\}_n \subset L^2(\mathbb{R}^d)$, *such that for every* $J \geq 1$, $x \in \mathbb{R}^d$,

$$f_n(x) = \sum_{j=1}^{J} \frac{1}{(\lambda_n^j)^{d/2}} g^j(\frac{x}{\lambda_n^j}) + r_n^J(x),$$

$$\|f_n\|_{L^2}^2 = \sum_{j=1}^{J} \|g^j\|_{L^2}^2 + \|r_n^J\|_{L^2}^2 + o_n(1),$$

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|r_n^J\|_B = 0.$$

Applying the above result to a bounded radially symmetric sequence $\{v_n\} \subset \dot{H}^1(\mathbb{R}^d)$, we conclude that there is a subsequence (again denoted by $\{v_n\}$), a family of scales λ_n^j satisfying (2.10)), and radial $\psi^j \in \dot{H}^1$ such that for every $J \geq 1$,

$$v_n(x) = \sum_{j=1}^{J} \psi_n^j(x) + \tilde{w}_n^J(x) \tag{2.13}$$

where $\psi_n^j(x) := \frac{1}{(\lambda_n^j)^{\frac{d}{2}-1}} \psi^j(\frac{x}{\lambda_n^j})$,

$$\|v_n\|_{\dot{H}^1}^2 = \sum_{j=1}^{J} \|\psi^j\|_{\dot{H}^1}^2 + \|\tilde{w}_n^J\|_{\dot{H}^1}^2 + o_n(1)$$

and

$$\limsup_n \|\nabla \tilde{w}_n^J\|_B \xrightarrow{J\to\infty} 0. \tag{2.14}$$

Denote the linear propagator for the homogeneous heat equation

$$v_t = \Delta v \tag{2.15}$$

on $\mathbb{R}^d$ by $S(t) := e^{t\Delta}$. Evolving (2.13) by the linear propagator we get

$$S(t)v_n = \sum_{i=1}^{J} S(t)\psi_n^j + S(t)\tilde{w}_n^J.$$

Our first goal is to estimate $S(t)\tilde{w}_n^J$ in an appropriate space-time norm. If σ is a function on $\mathbb{R}^d$, we define $\sigma(D)$ by

$$\widehat{\sigma(D)f}(\xi) := \sigma(\xi)\hat{f}(\xi).$$

Also define $\sigma_k(\xi) := \chi_{2^k \le |\xi| \le 2^{k+1}}(\xi), k \in \mathbb{Z}$. Then if v solves (2.15), by commutation of Fourier multipliers with derivatives, $\sigma_k(\xi)v$ is also solves (2.15). By standard decay estimate (2.3) for (2.15), for any k

$$\|\nabla(\sigma_k(\xi)w(t,\cdot)\|_{L^2} \le \|\nabla(\sigma_k(\xi)w(0,\cdot)\|_{L^2}.$$

Using Plancherel's identity and properties of the Fourier transform and the definition of multipliers:

$$\|\nabla(\sigma_k v)\|_{L^2} = \|\widehat{\nabla(\sigma_k v)}\|_{L^2} = \||\xi|\sigma_k\hat{v}\|_{L^2} = I_k(\nabla v).$$

Hence, by taking supremum in k,

$$\|\nabla v(t,\cdot)\|_B \le \|\nabla v(0,\cdot)\|_B.$$

This general observation implies

$$\|\nabla(S(t)\tilde{w}_n^J)\|_{L_t^\infty B_x} \le \|\nabla \tilde{w}_n^J\|_B.$$

Then, due to (2.14) we conclude

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\nabla(S(t)\tilde{w}_n^J)\|_{L_t^\infty B_x} = 0. \tag{2.16}$$

For every $t > 0$, (2.12) gives:

$$\begin{aligned}\|S(t)\tilde{w}_n^J\|_{L_x^{\frac{2d}{d-2}}} &\lesssim \|\nabla(S(t)\tilde{w}_n^J)\|_{L^2}^{\frac{2(d-2)}{2d}} \cdot \|\nabla(S(t)\tilde{w}_n^J)\|_B^{1-\frac{2(d-2)}{2d}} \\ &\lesssim \|\nabla\tilde{w}_n^J\|_{L^2}^{\frac{2(d-2)}{2d}} \cdot \|\nabla(S(t)\tilde{w}_n^J)\|_B^{1-\frac{2(d-2)}{2d}}\end{aligned}$$

and so

$$\|S(t)\tilde{w}_n^J\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \lesssim \|\nabla\tilde{w}_n^J\|_{L^2}^{\frac{2(d-2)}{2d}} \cdot \|\nabla(S(t)\tilde{w}_n^J)\|_{L_t^\infty B_x}^{1-\frac{2(d-2)}{2d}}.$$

Since $\|\nabla\tilde{w}_n^J\|_{L^2}$ is uniformly bounded, using (2.16):

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|S(t)\tilde{w}_n^J\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} = 0. \tag{2.17}$$

It is straightforward to verify, using the Hardy inequality in dimension $d = 2m+2$, that the map

$$X^2 \ni u = u(r) \mapsto v(r) = \frac{u(r)}{r^m} \in \dot{H}^1_{rad}(\mathbb{R}^d)$$

is an isomorphism (e.g., see Lemma 4 in [13]) and moreover

$$u_t = \Delta_m u \quad \Longleftrightarrow \quad v_t = \Delta_{\mathbb{R}^d} v.$$

Moreover, if (r,p) is an L^2-admissible pair for $d=2$, i.e. $\frac{1}{r}+\frac{1}{p}=\frac{1}{2}$, then

$$\|\frac{u}{r}\|^r_{L^r_t(I;L^p_r(rdr))} = \int_I\left(\int_0^\infty \frac{r^{1-p}}{r^{2m+2}}|u|^p r^{2m+1}dr\right)^{r/p}$$
$$= \int_I\left(\int_0^\infty \frac{r^{1-p}}{r^{2m+2}}r^{mp}|v|^p r^{2m+1}dr\right)^{r/p}$$
$$= \int_I\left(\int_0^\infty r^{p(m-1)-2m}|v|^p r^{2m+1}dr\right)^{r/p}.$$

Taking $p=\frac{2m}{m-1}$, and thus $r=2m$, we observe that

$$\|\frac{u}{r}\|_{L^r_t(I;L^p_r(rdr))} = \|v\|_{L^r_t(I;L^p(\mathbb{R}^{2m+2})},$$

and this choice of (r,p) is an $\dot{H}^1$-admissible pair in dimension $2m+2$.

These observations are the connecting link between the two-dimensional problem and the higher-dimensional estimates. So for u_n bounded in X^2, $v_n=\frac{u_n}{r^m}$ is bounded in $\dot{H}^1(\mathbb{R}^d)$ and we have (2.13). First, one has to show that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|S(t)\tilde{w}^J_n\|_{L^{2m}(I;L^{\frac{2m}{m-1}}(\mathbb{R}^{2m+2})}=0. \tag{2.18}$$

For this we use interpolation and (2.17) :

$$\|S(t)\tilde{w}^J_n\|_{L^{2m}_t L^{\frac{2m}{m-1}}} \le \|S(t)\tilde{w}^J_n\|^{\frac{m-1}{m}}_{L^\infty_t L_r^{\frac{2m+2}{m}}} \cdot \|S(t)\tilde{w}^J_n\|^{\frac{1}{m}}_{L^2_t L_r^{\frac{2(m+1)}{m-1}}}.$$

Taking $\lim_{J\to\infty}\limsup_{n\to\infty}$, and noting that the second term is uniformly bounded (by the standard space-time estimates for the heat equation – see e.g., [22]), the claim (2.18) follows. Undoing the transformation $u_n=r^m v_n$ in (2.13) yields

$$u_n(x)=\sum_{j=1}^J \phi^j(\frac{r}{\lambda^j_n})+w^J_n,\ w^J_n=r^m\tilde{w}^J_n.$$

Now, again by interpolation

$$\|\frac{u}{r}\|_{L^4_tL^4_r} \le \|\frac{u}{r}\|^{\frac{m}{2(m-1)}}_{L^{2m}_t L_r^{\frac{2m}{m-1}}} \cdot \|\frac{u}{r}\|^{\frac{m-2}{2(m-1)}}_{L^2_tL^\infty_r}.$$

Invoking (2.18), we get (2.8), i.e.,

$$\lim_{J\to\infty}\limsup_{n}\|e^{t\Delta_m}w^J_n\|_{L^4_t rL^4_r}=0.$$

The rest of the proof of the profile decomposition follows the same arguments as in the references cited at the beginning of this section, and is thus omitted, with the exception of the asymptotic energy splitting (2.11) which we now demonstrate.

Expanding using the definition,

$$E(u_n) = \int_0^\infty [\sum_{j=1}^J (\frac{1}{\lambda_n^j})^2 (\phi_r^j(\frac{r}{\lambda_n^j}))^2 + \sum_{j=1, i<j}^J \frac{1}{\lambda_n^j \lambda_n^i} \phi_r^j(\frac{r}{\lambda_n^j}) \phi_r^i(\frac{r}{\lambda_n^i}) + (w_{n,r}^J(r))^2$$
$$+ 2\sum_{j=1}^J w_{n,r}^J(r) \frac{1}{\lambda_n^j} \phi_r^j(\frac{r}{\lambda_n^j}) + \frac{m^2}{r^2} \sin^2(\sum_{j=1}^J \phi^j(\frac{r}{\lambda_n^j}) + w_{n,r}^J(r))] r dr,$$

We need to show that $E(u_n) - \sum_{j=1}^J E(\phi^j) - E(w_n^J) = o_n(1)$. Expanding this out,

$$E(u_n) - \sum_{j=1}^J E(\phi^j) - E(w_n^J)$$
$$= \int_0^\infty \sum_{j=1, i<j}^J \frac{1}{\lambda_n^j \lambda_n^i} \phi_r^j(\frac{r}{\lambda_n^j}) \phi_r^i(\frac{r}{\lambda_n^i}) r dr + 2\int_0^\infty [\sum_{j=1}^J w_{n,r}^J(r) \frac{1}{\lambda_n^j} \phi_r^j(\frac{r}{\lambda_n^j})] r dr$$
$$+ \int_0^\infty \frac{m^2}{r^2} [\sin^2(\sum_{j=1}^J \phi^j(\frac{r}{\lambda_n^j}) + w_n^J(r)) - \sum_{j=1}^J \sin^2(\phi^j(\frac{r}{\lambda_n^j})) - \sin^2(w_n^J(r))] r dr.$$

For the first two sums it suffices to look at single pairs and show they all are $o_n(1)$. Using an approximation argument, we can assume every function involved is in C_c^∞ and that all the supports lie in some ball $B(0, R)$. The argument is standard so we only give a sketch: for the first sum, we just change variables, assuming without loss of generality $s_n^{i,j} := \frac{\lambda_n^j}{\lambda_n^i}$ goes to zero. Then, by Hölder's inequality, each term in the sum is bounded by $\|\phi_r^i\|_{X^2} \int_0^{s_n^{i,j} R} (\phi_r^j(r))^2 r dr = o_n(1)$. For the second one, change variables again and employ the weak convergence of $w_n^J(\lambda_n^j r)$ to zero, for all $j \leq J$, i.e., (2.9).

For the rest we will use the trigonometric identity

$$\sin^2(a+b) - \sin^2(a) - \sin^2(b) = \frac{1}{2}\sin(2a)\sin(2b) - 2\sin^2(a)\sin^2(b)$$

and the inequality derived from it,

$$|\sin^2(a+b) - \sin^2(a) - \sin^2(b)| \leq C|a||b|$$

for some $C > 0$. We want to show that

$$\left|\int_0^\infty \frac{m^2}{r^2}[\sin^2(\sum_{j=1}^J \phi^j(\frac{r}{\lambda_n^j}) + w_n^J(r)) - \sum_{j=1}^J \sin^2(\phi^j(\frac{r}{\lambda_n^j})) - \sin^2(w_n^J(r))]rdr\right| = o_n(1).$$

Using (2.2) $J-1$ times, this can be reduced to showing the following two estimates:

$$\int_0^\infty \frac{|\phi^j(\frac{r}{\lambda_n^j})||\phi^i(\frac{r}{\lambda_n^i})|}{r^2} rdr = o_n(1), \quad i \neq j$$

$$\int_0^\infty \frac{|w_n^J(r)||\phi^j(\frac{r}{\lambda_n^j})|}{r^2} rdr = o_n(1) \text{ for any } j \leq J. \tag{2.19}$$

The proof of (2.2) follows the same rescaling argument as before.
As for (2.19), a change of variables gives

$$\int_0^\infty |w_n^J(\lambda_n^j r')||\phi^j(r')|\frac{r'dr'}{(r')^2}$$

which suggests that we should use the weak convergence; however, because of the absolute value, we cannot directly obtain the result. Since $\|\frac{\phi^j}{r}\|_{L^2(rdr)} < +\infty$, for every $\epsilon > 0$ we can find an $R = R(\epsilon) > 1$ such that

$$\left(\int_{r\geq R} |\frac{\phi^j(r)}{r}|^2 rdr\right)^{1/2} + \left(\int_{r\leq \frac{1}{R}} |\frac{\phi^j(r)}{r}|^2 rdr\right)^{1/2} < \frac{\epsilon}{2M},$$

where

$$M := \sup_n \|w_n^J\|_{X^2} < +\infty.$$

So if we split the integral at hand into the obvious three regions, then the inner and outer contributions, by Hölder's inequality, are $< \frac{\epsilon}{2}$, while for the one in the middle we have

$$\int_{\frac{1}{R}\leq r\leq R} |w_n^J(\lambda_n^j r)||\phi^j(r)|\frac{rdr}{r^2} < \int_{\frac{1}{R}\leq r\leq R} |w_n^J(\lambda_n^j r)||\phi^j(r)|dr.$$

But, on a fixed interval $\Omega = [a,b]$, $0 < a < b < \infty$, $\|u\|_{X^2([a,b])}$ and $\|u\|_{H^1([a,b])}$ are equivalent, so by the compact embedding of $H^1([a,b])$ in $L^2([a,b])$, $w_n^J(\lambda_n^j r) \to 0$ in $L^2([a,b])$. So by Hölder again, we conclude that for n sufficiently large, the integral in (2.19) is $< \epsilon$, and the result follows. $\square$

2.3. **Minimal blow-up solution.** For $u_0 \in E_0$, define

$$E_c = \inf\{E(u_0) \mid u \text{ solves (2.2) with } u(0) = u_0, \|u\|_{L^4 rL^4([0,T_{\max}))} = +\infty\},$$

the infimum of the energies of initial data leading to solutions which fail either to be global or to decay to zero, in the sense of the local well-posedness theory. Note that $T_{\max}$ can be finite (blow-up), or infinite (corresponding to a global but not decaying solution).

Observe that Theorem 1.1 is equivalent to $E_c \geq 2E(Q)$. Note also that $E_c > 0$, since for $u_0 \in E_0$, $E(u_0)$ small $\implies$ $\|u_0\|_{X^2}$ small, and by the local theory, such solutions are global and decay.

We will follow the contradiction approach of Kenig-Merle: under the assumption

$$0 < E_c < 2E(Q)$$

we will first show existence of a *critical element* – a datum with energy E_c giving rise to a solution that that either fails to exist globally or decay to zero. Then, as an immediate consequence of energy dissipation, we show that such a critical element cannot exist, reaching a contradiction.

Proposition 2.3. *Assume $E_c < 2E(Q)$. There exists $u_{0,c} \in X^2$ with $E(u_{0,c}) = E_c$ such that if $u_c(t,r)$ is the solution of (2.2) with initial data $u_{0,c}$ and maximal interval of existence $I = [0, T_{max}(u_{0,c}))$, then $\|u_c\|_{L^4 rL^4(I)} = +\infty$.*

For the proof of this proposition we follow the same strategy as in [31, 34].

Proof. Let $\{u_{0,n}\}_n \subset X^2$ such that $E(u_{0,n}) \searrow E_c, n \to \infty$, and the corresponding solutions u_n of (2.2) with maximal intervals of existence $I_n = [0, T_{\max}(u_{0,n}))$ satisfy $\|u_n\|_{L^4 rL^4(I_n)} = +\infty$. By the comparability of the energy and the X^2−norm, the sequence $\{u_{0,n}\}_n$ is bounded in X^2. Thus, passing to a subsequence, if necessary, we have the profile decomposition

$$u_{0,n}(r) = \sum_{j=1}^{J} \phi^j(\frac{r}{\lambda_n^j}) + w_n^J(r)$$

with the stated properties in Proposition 2.1.

Define the *nonlinear profile* $v^j : I^j \times [0, \infty) \to \mathbb{R}$ associated to ϕ^j to be the maximal-lifespan solution to (2.2) with initial data ϕ^j, and for each $j, n \geq 1$, define $v_n^j : I_n^j \times [0, \infty) \to \mathbb{R}$ by

$$v_n^j(t, r) = v^j(\frac{t}{(\lambda_n^j)^2}, \frac{r}{\lambda_n^j}), \qquad I_n^j := \{t \in \mathbb{R}^+ : \frac{t}{(\lambda_n^j)^2} \in I^j\},$$

the solution to (2.2) with initial data $v_n^j(0) = \phi^j(\frac{r}{\lambda_n^j})$. The energy decoupling reads

$$E(u_{0,n}) = \sum_{j=1}^{J} E(\phi^j) + E(w_n^J) + o_n(1) \quad \forall J.$$

Taking $\overline{\lim}_n$, we get

$$E_c = \sum_{j=1}^{J} E(\phi^j) + \overline{\lim_n} E(w_n^J)$$

which by the positivity of every term implies $\sum_{j=1}^{J} E(\phi^j) \le E_c$, for any J, and so

$$\sup_j E(\phi^j) \le E_c.$$

The goal is to show that $\phi^j = 0, j \ge 2$ and $E(\phi^1) = E_c$.

We consider the following possibilities:

Case 1: $\sup_j E(\phi^j) < E_c$. Then by the definition of E_c, each v^j (hence also v_n^j) is global ($I^j = [0, \infty)$) and decaying: $\|v^j\|_{L^4([0,\infty);rL^4)} < \infty$. Define an approximate solution (to $u_n(t)$) of the nonlinear equation by

$$u_n^J(t) = \sum_{j=1}^{J} v_n^j(t) + e^{t\Delta_m} w_n^J.$$

What we want to show is that u_n^J is a good approximate solution to u_n (for n, J sufficiently large) in the sense of the Stability Theorem 2.2. This would imply that $u_n(t)$ is global, a contradiction.

First, to see that $\sup_{n,J} \|u_n^J\|_{L^4rL^4(\mathbb{R}^+)} < +\infty$: for any $\epsilon > 0$, (2.8) provides a J such that

$$\overline{\lim_n} \|u_n^J\|_{L^4rL^4(\mathbb{R}^+)} \le \overline{\lim_n} \| \sum_{j=1}^{J} v_n^j \|_{L^4rL^4(\mathbb{R}^+)} + \overline{\lim_n} \|e^{t\Delta_m} w_n^J\|_{L^4rL^4(\mathbb{R}^+)}$$
$$\le \sum_{j=1}^{J} \|v^j\|_{L^4rL^4(\mathbb{R}^+)} + \epsilon.$$

To conclude the claim, we will show that the latter norms are bounded uniformly in J. We can split the sum into two parts (for every fixed J); one over $1 \le j \le J_0$, and the rest. Let ϵ_0 be such that Theorem

2.1 guarantees that, if $\|u_0\|_{X^2} \leq \epsilon_0$, then the corresponding solution u is global with $\|u\|_{L_t^4 r L^4} \leq C\epsilon_0$. Pick J_0 such that

$$\sum_{j \geq J_0} E(\phi^j) \leq \epsilon_0.$$

Then for $j \geq J_0$, the $\|v^j\|_{L_t^4 r L^4}$ are uniformly bounded, and the claim follows.

By construction, we have $\|u_n^J(0) - u_n(0)\|_{X^2} = 0, \ \forall J, n$, and also, $\|e^{t\Delta_m}(u_n^J(0) - u_n(0))\|_{L^4 r L^4(\mathbb{R}^+)} = 0, \ \forall J, n$.

The perturbed PDE for $u_n^J(t)$ is

$$\partial_t u_n^J - \Delta_m u_n^J = \sum_{j=1}^{J} F(v_n^j),$$

hence the error is given by

$$e_n^J = F(u_n^J) - \sum_{j=1}^{J} F(v_n^j),$$

where F is the nonlinear term $F(u) = \frac{m^2}{r^2}(u - \frac{\sin 2u}{2})$. We will show that the error is small in the dual norm $\|\cdot\|_{L_t^{4/3} r L_r^{4/3}}$ for sufficiently large n and J. Explicitly,

$$e_n^J = \frac{m^2}{2r^2}\left(2u_n^J - \sin(2u_n^J) - \sum_{j=1}^{J}(2v_n^j - \sin(2v_n^j)\right).$$

For simplicity, denote $W_n^J(r,t) := e^{t\Delta_m} w_n^J(r)$. We will make use of the following trigonometric relation:

$$\begin{aligned} |\sin(2u) + \sin(2v) - \sin(2u+2v)| &= |2\sin(2u)\sin^2(v) + 2\sin(2v)\sin^2(u)| \\ &\lesssim |u||v|^2 + |v||u|^2. \end{aligned}$$

Using this,

$$\begin{aligned} &|\sum_{j=1}^{J} \sin(v_n^j) + \sin(W_n^J) - \sin(\sum_{j=1}^{J} v_n^j + W_n^J) \pm \sin(\sum_{j=1}^{J} v_n^j)| \\ &\qquad \lesssim |\sum_{j=1}^{J} \sin(v_n^j) - \sin(\sum_{j=1}^{J} v_n^j)| + |W_n^J||\sum_{j=1}^{J} v_n^j|^2 + |W_n^J|^2|\sum_{j=1}^{J} v_n^j| \end{aligned}$$

Define $A := |W_n^J| \, |\sum_{j=1}^J v_n^j|^2 + |W_n^J|^2 \, |\sum_{j=1}^J v_n^j|$, $B := |\sum_{j=1}^J \sin(v_n^j) - \sin(\sum_{j=1}^J v_n^j)|$. By Hölder's inequality:

$$\begin{aligned}\|\frac{1}{r^2}A\|_{L^{4/3}rL^{4/3}} &\leq \|W_n^J\|_{L^4rL^4} \, \|\sum_{j=1}^J v_n^j\|^2_{L^4rL^4} + \|W_n^J\|^2_{L^4rL^4} \, \|\sum_{j=1}^J v_n^j\|_{L^4rL^4} \\ &\leq \|W_n^J\|_{L^4rL^4}(\sum_{j=1}^J \|v_n^j\|_{L^4rL^4})^2 + \|W_n^J\|^2_{L^4rL^4}(\sum_{j=1}^J \|v_n^j\|_{L^4rL^4}).\end{aligned}$$

But by (2.8),

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|W_n^J\|_{L^4rL^4} = 0,$$

and hence, by the scaling invariance of the L^4rL^4-norm,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|A\|_{L^{4/3}rL^{4/3}} = 0.$$

For term B, again by adding and subtracting $\sin(\sum_{j=1}^{J-1} v_n^j)$ we get, using the trigonometric inequality:

$$B \lesssim |\sum_{j=1}^{J-1} \sin(v_n^j) - \sin(\sum_{j=1}^{J-1} v_n^j)| + |v_n^J| |\sum_{j=1}^{J-1} v_n^j|^2 + |v_n^J|^2 |\sum_{j=1}^{J-1} v_n^j|.$$

We will show how to treat the second term, and after that the procedure can be easily iterated. It consists of terms of the form $|v_n^J||v_n^j|^2$ and $|v_n^J|^2|v_n^j|$. We treat terms of the first type, namely $\|\frac{|v_n^J|^2}{r^2} \frac{|v_n^j|}{r}\|_{L^{4/3}rL^{4/3}}$ (and the others follow in the same way). We may employ an approximation argument to assume the functions are smooth and compactly supported in space-time, say on $[0,T] \times [0,R]$. Without loss of generality, assume $s_n^{J,j} := \frac{\lambda_n^J}{\lambda_n^j} \to 0$. Changing variables (in space and time), and using Hölder's inequality, we find that the above norm is controlled by

$$\|v^J\|^2_{L^4rL^4} \, \|v^j\|_{L^4rL^4([0,(s_n^{J,j})^2T]\times[0,(s_n^{J,j})R])} \xrightarrow{n\to\infty} 0.$$

The other terms may be treated similarly, proving

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e_n^J\|_{L^{4/3}rL^{4/3}} = 0.$$

Thus, we have shown that for sufficiently large J and n, $u_n^J(t)$ is a good approximate solution in the sense of Stability Theorem 2.2, from which it follows that $u_n(t)$ is global, with $\|u_n\|_{L^4_t r L^4} < \infty$, a contradiction.

Case 2: $\sup_j E(\phi^j) = E_c$. This immediately implies (possibly after a relabeling) that $\phi^j = 0$ for $j \geq 2$, and the profile decomposition simplifies to

$$u_{0,n}(r) = \phi^1(\frac{r}{\lambda_n^1}) + w_n^1(r).$$

By the energy splitting and the fact that $E(u_{0,n}) \to E_c$, we get

$$\overline{\lim_n} E(w_n^1) = 0,$$

from where, by the comparability of the energy and the X^2-norm, it follows that

$$\tilde{u}_{0,n}(r) := u_{0,n}(\lambda_n^1 r) \to \phi^1(r)$$

strongly in X^2. We also get that $E(\phi^1) = E_c$.

Define our critical element u_c to be the solution of (2.2) emanating from initial data ϕ_1. To complete the proof of the Proposition, we must conclude that $\|u_c\|_{L^4([0,T_{max}(\phi^1));rL^4)} = \infty$. To see that, assume it is false, and again employ the Stability Theorem 2.2 as above to reach a contradiction. □

2.4. **Rigidity.** In this short section, we complete the proof of Theorem 1.1 by showing, as an immediate consequence of energy dissipation, that:

Proposition 2.4. *The critical element u_c found in Proposition 2.3 cannot exist.*

Proof. By the energy dissipation relation, for $0 < t < T_{max}(\phi_1)$,

$$E(u_c(t)) < E(u_c(0)) = E_c \in (0, 2E(Q))$$

unless u_c is a stationary solution. There are no non-zero stationary solutions in E_1, so this strict inequality holds. Thus $E(u_c(t)) < E_c$ for some $t > 0$, whence it follows from the definition of E_c that u_c is global with $\|u_c\|_{L^4_t r L^4} < \infty$, a contradiction. □

3. Heat Flow Above Threshold

In this section we prove Theorem 1.3 on the "above-threshold" solutions of the corotational heat-flow.

3.1. **Corotational maps in E_1.** Recall, we consider here solutions $u(r,t)$ of

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{m^2}{2r^2}\sin(2u) \tag{3.1}$$

in the class

$$E_1 := \{u \mid E(Q) \leq E(u_0) \leq 3E(Q), \quad u(0) = \pi, \lim_{r\to\infty} u(r) = 0\},$$

where the energy is given by

$$E(u) = \frac{1}{2}\int_0^\infty \left(u_r^2 + \frac{m^2}{r^2}\sin^2(u)\right) rdr.$$

Recall the unique (up to scaling) static solution with these boundary conditions is

$$Q(r) = \pi - 2\tan^{-1}(r^m),$$

and define the following quantities

$$h(r) := \sin(Q(r)) = \frac{2r^m}{1+r^{2m}}, \quad \hat{h}(r) := \cos(Q(r)) = \frac{r^{2m}-1}{r^{2m}+1}.$$

For later use, we record the easy computations

$$h_r = -\frac{m}{r}h\hat{h}, \qquad \hat{h}_r = \frac{m}{r}h^2.$$

We will denote scalings by

$$Q^s(r) := Q(r/s), \quad h^s(r) = h(r/s), \quad etc., \quad s > 0.$$

Recall that the energy space (for maps with trivial topology) is:

$$X^2 = \{w : [0,\infty) \mapsto \mathbb{R} \mid \int_0^\infty \left(w_r^2 + \frac{w^2}{r^2}\right) r\,dr < \infty\}.$$

3.2. **No concentration at spatial infinity.** As discussed in the introduction, the mechanism of possible singularity formation is well-known: energy concentration by bubbling off static solutions (harmonic maps). By the corotational symmetry and finite energy, a concentration may a priori occur only at the spatial origin or infinity. The latter cannot happen in finite time:

Lemma 3.1. *Let u be a finite energy smooth solution on (3.1) on $(0,T)$. No energy concentration at spatial infinity is possible:*

$$\lim_{R\to\infty} \limsup_{t\to T} E(u(t); B_R^c) = 0.$$

Proof. The energy dissipation relation

$$E(u(t_2)) + \int_{t_1}^{t_2} \|u_t\|_{L^2}^2 ds = E(u(t_1))$$

for $0 \leq t_1 < t_2 < T$ will be used. First choose a smooth, radial cut-off function ψ such that

$$\psi(r) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 & \text{if } r \geq 2 \end{cases}$$

and define $\psi_R(r) := \psi(\frac{r}{R})$.

If there was energy concentration at spatial infinity at time $t = T$, for some $\delta > 0$, we would have

$$\limsup_{t \nearrow T} E(u(t); B_R^c) \geq \delta > 0, \quad \forall R > 0,$$

and we could find sequences of radii $R_n \nearrow \infty$ and times $t_n \nearrow T$ such that $\lim_n E(u(t_n), B_{R_n}^c) \geq \delta > 0$. Define the "exterior" energy

$$\hat{E}_R(t) := \frac{1}{2} \int_0^\infty \psi_R(r) \left(u_r^2 + \frac{m^2}{r^2} \sin^2(u) \right) r dr.$$

By the finiteness of the energy, for any $t_0 < T$ there is an $R_0 > 1$, such that $\hat{E}_{R_0}(t_0) \leq \frac{\delta}{4}$. By assumption, there is $T > t_1 > t_0$ such that $\hat{E}_{R_0}(t_1) \geq \frac{\delta}{2}$.

By direct calculation

$$\frac{d}{dt} \hat{E}_{R_0}(t) = - \int_0^\infty \psi_{R_0} u_t^2 r dr - \int_0^\infty u_r u_t \frac{d\psi_{R_0}}{dr} r dr.$$

Using $\frac{d}{dr}\psi_R(r) = \frac{1}{R}\psi'(\frac{r}{R})$ and (3.2):

$$\begin{aligned}
\frac{\delta}{4} &\leq \int_{t_0}^{t_1} \frac{d}{dt} E(u(t); B_{R_0}^c) dt = - \int_{t_0}^{t_1} \int_0^\infty \psi_{R_0} u_t^2 r dr - \int_{t_0}^{t_1} \int_0^\infty u_r u_t \psi'_{R_0} r dr. \\
&\leq \left(\int_{t_0}^{t_1} \int_0^\infty u_t^2 r dr \right)^{1/2} \cdot \left(\int_{t_0}^{t_1} \int_0^\infty u_r^2 \left(\psi'_{R_0} \right)^2 \right)^{1/2} \\
&\lesssim \frac{1}{R_0} (t_1 - t_0)^{1/2} E(u_0),
\end{aligned}$$

which yields a contradiction taking $t_0 \nearrow T$. □

3.3. **Bubbling description.** Having ruled out energy concentration at infinity, and since the energy bound and boundary conditions in E_1 prohibit the formation of more than one bubble, the following proposition giving the strong convergence of the solution at a blow-up time, after removal of the bubble, is a direct adaptation of Theorem 1.1 in [38]:

Proposition 3.1. *Let $u_0 \in E_1$ and $u(t)$ the corresponding solution to (3.1) blowing up at time $t = T > 0$. Then there exists a sequence of times $t_j \nearrow T$, a sequence of scales $s_j = o(\sqrt{T - t_j})$, a map $w_0 \in E_0$, and a decomposition*

$$u(t_j, r) = Q(\frac{r}{s_j}) + w_0(r) + \xi(t_j, r)$$

such that $\xi(t_j, 0) = \lim_{r\to\infty} \xi(t_j, r) = 0$ and $\xi(t_j) \to 0$ in X^2 as $j \to \infty$.

So to exclude finite-time singularity formation, it suffices to show:

Proposition 3.2. *Assume $m \geq 4$. Suppose $u(t, r)$ is a smooth solution of (3.1) on $[0, T)$ such that along some sequence $t_j \to T-$, there are $s_j > 0$ such that*

$$u(t_j, \cdot) - Q^{s_j} \to w_0 \text{ in } X^2 \tag{3.2}$$

for some $w_0 \in X^2$ with $E(w_0) < 2E(Q)$. Then $s_j \not\to 0$.

The next three subsections build up to to a proof of this.

3.4. **Approximate solution.** Introduce the solution $w(t, r)$ of (3.1) with initial data at $t = t_j$ given by w_0:

$$\begin{array}{c} w_t - w_{rr} - \frac{1}{r} w_r - \frac{m^2}{2r^2} \sin(2w) = 0 \\ w(t_j, r) = w_0(r) \in X^2, \quad E(w_0) < 2E(Q) \end{array} . \tag{3.3}$$

By Theorem 1.1, we know that w is a global, smooth solution with

$$\|w\|_{L_t^\infty X^2 \cap L_t^2 (X^\infty \cap r X^2)([t_j,\infty))} < \infty. \tag{3.4}$$

For later use, we record one consequence of the higher regularity gained after the initial time:

$$\frac{w}{r^2} \in L_t^2 X^2([t^*, \infty)) \quad \text{for every } t^* > t_j.$$

This follows from the observations that by standard parabolic regularity estimates (for example by performing energy-type estimates on the differentiated PDE), the function $v(x, t) = w(r, t)e^{im\theta}$ satisfies $D^2 v \in L_t^2 L_x^2([t^*, \infty))$, and that w/r^2 and w_r/r are controlled pointwise by $|D^2 v|$ (for any $m \geq 2$).

For fixed $s > 0$, Q^s is also a (static) solution of (3.1). Since the PDE is nonlinear, of course the sum $Q^s + w$ is *not* a solution:

$$\left(\partial_t - \partial_r^2 - \frac{1}{r}\partial_r - \frac{m^2}{2r^2}\sin(2\,\cdot\,)\right)(Q^s + w)$$
$$= \frac{m^2}{2r^2}\left(\sin(2Q^s) + \sin(2w) - \sin(2Q^s + 2w)\right)$$
$$= \frac{m^2}{2r^2}\left(\sin(2Q^s)(1 - \cos(2w)) + \sin(2w)(1 - \cos(2Q^s)\right) =: Eqn(Q^s + w).$$

However, $Q^{s(t)} + w$ is a good *approximate solution* over short time intervals in the sense:

Lemma 3.2. $\|Eqn(Q^{s(t)} + w)\|_{L_t^2 X^1} \lesssim \|w\|_{L_t^2 X^\infty} + \|w\|^2_{L_t^4 X^4}$ *and therefore by* (3.4),

$$\|Eqn(Q^{s(t)} + w)\|_{L_t^2 X^1[t_j,T)} \to 0 \quad as \quad t_j \to T-\,. \tag{3.5}$$

Remark: We do not need it here, but if $0 < s(t) \ll 1$, then $Q^{s(t)} + w$ is a good approximate solution *globally*, in the sense that $\|Eqn(Q^{s(t)} + w)\|_{L_t^2 X^1[t_j,\infty)} \to 0$ as $\sup_{t\in[t_j,\infty)} s(t) \to 0$.

Proof. This is an easy consequence of the elementary pointwise estimates

$$|Eqn(Q^s + w)| \lesssim \frac{1}{r^2}\left(h^s w^2 + (h^s)^2 w\right)$$
$$|\partial_r Eqn(Q^s + w)| \lesssim \frac{1}{r^3}\left(h^s w^2 + (h^s)^2 w\right) + \frac{1}{r^2}\left(h^s|w||w_r| + (h^s)^2|w_r|\right).$$

Then using $\|\frac{h^s}{r}\|_{L^2} \lesssim 1$, $\|\frac{h^s}{r^2}\|_{L^1} \lesssim 1$, and Hölder's inequality, the Lemma follows. □

3.5. **Linearized evolution estimates.**

Lemma 3.3. *Assume $m \geq 4$. Let $\xi(\cdot, t) \in X^2$ be a solution of the inhomogeneous linearized equation about Q^s, where $s = s(t) > 0$ is a differentiable function of time,*

$$\left\{\begin{array}{c} \partial_t\xi + H^s\xi = f(r,t) \\ \xi(r,0) = \xi_0(r) \end{array}\right\} \qquad H^s := -\partial_r^2 - \frac{1}{r}\partial_r - \frac{m^2}{r^2}\cos(2Q^s),$$

which also satisfies the orthogonality condition

$$\left(\xi(\cdot,t),\ h^{s(t)}\right)_{L^2_{rdr}} \equiv 0. \tag{3.6}$$

Then we have the estimates

$$\|\xi\|_{L_t^\infty X^2 \cap L_t^2 X^\infty} \lesssim \|\xi_0\|_{X^2} + \|f\|_{L_t^1 X^2 + L_t^2 X^1} + \|\dot{s}\|_{L_t^2}. \tag{3.7}$$

Proof. The idea comes from [28] where it appeared as a linearization of a *generalized Hasimoto transformation*, while here we apply it directly at the linear level: exploit the factorized form of the linearized operator

$$H^s = (L^s)^* L^s, \quad L^s = \partial_r + \frac{m}{r}\cos(Q^s) = h^s \partial_r (h^s)^{-1},$$

and the fact that the reverse factorization is positive,

$$(L^s)(L^s)^* = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\left(1 + m^2 - 2m\cos(Q^s)\right) \geq -\partial_r^2 - \frac{1}{r}\partial_r + \frac{(m-1)^2}{r^2}. \tag{3.8}$$

Applying L^s to the linearized equation produces

$$\partial_t \eta + L^s(L^s)^*\eta = L^s f + (\partial_t L^s)\eta \qquad \eta := L^s \xi, \;\; \partial_t L^s = \frac{m^2}{r}\frac{1}{s}(h^s)^2 \dot{s}.$$

Multiplying this equation by η, integrating over space and time, and using (3.8) gives

$$\|\eta\|^2_{L^\infty_t L^2} + \|\eta\|^2_{L^2_t X^2} \lesssim \|L^s \xi_0\|^2_{L^2} + \|(L^s f)\eta\|_{L^1_t L^1} + \|\frac{1}{s}(h^s)^2 \frac{\eta}{r}\|_{L^2_t L^1} \|\dot{s}\|_{L^2_t}.$$

Using Hölder's inequality on the right, then Young's, as well as $\|\frac{1}{s}(h^s)^2\|_{L^2} \lesssim 1$, yields

$$\|\eta\|_{L^\infty_t L^2 \cap L^2_t X^2} \lesssim \|\xi_0\|_{X^2} + \|f\|_{L^1_t X^2 + L^2_t X^1} + \|\dot{s}\|_{L^2_t}.$$

Finally, in [28] it was shown that we can invert L^s under the orthogonality condition (3.6) to bound ξ:

$$m \geq 4, \;\; (\xi, \, h^s)_{L^2} = 0 \quad \Longrightarrow \quad \|\xi\|_{X^p} \lesssim \|L^s \xi\|_{L^p}, \;\; 2 \leq p \leq \infty.$$

Together with the standard embedding $\|\eta\|_{L^\infty} \lesssim \|\eta\|_{X^2}$ this completes the proof. □

3.6. **Modulation argument.**

Proof. (of Proposition 3.2) Let $w(r,t)$ be as in (3.3). For $t \in [t_j, T)$, the idea is to write the solution $u(r,t)$ in the form

$$u(r,t) = Q^{s(t)}(r) + w(r,t) + \xi(r,t), \tag{3.9}$$

where $s(t) > 0$ is chosen so that the orthogonality condition (3.6) holds. The fact that we can make such a choice follows from a standard implicit function theorem argument:

Lemma 3.4. *There is $\epsilon_0 > 0$ such that for any $s_0 > 0$ and any $\xi \in X^2$ with $\|\xi\|_{X^2} \leq \epsilon_0$, there is $0 < s = s(\xi, s_0)$ such that*

$$Q^{s_0} + \xi = Q^s + \tilde{\xi} \quad \textit{with} \;\; \left(\tilde{\xi}, \, h^s\right)_{L^2_{rdr}} = 0, \quad \left|\frac{s}{s_0} - 1\right| + \|\tilde{\xi}\|_{X_2} \lesssim \|\xi\|_{X^2} \leq \epsilon_0.$$

Proof. First take $s_0 = 1$. For $s > 0$ and $\xi \in X^2$ define

$$g(s;\xi) := (Q - Q^s + \xi,\ h^s)_{L^2_{rdr}},$$

a smooth function of s and ξ because the spatial decay of $h(r)$ implies $\|rh(r)\|_{L^2_{rdr}} < \infty$ (provided $m > 2$). We observe that $g(1;0) = 0$, and

$$\partial_s g(1;0) = \Big((-\frac{m}{s} h^s,\ h^s) + (Q - Q^s + \xi,\ \partial_s h^s) \Big) |_{s=1,\xi=0} = -m\|h\|^2_{L^2_{rdr}} \neq 0,$$

so by the Implicit Function Theorem there is $\epsilon_0 > 0$ such that for all ξ with $\|\xi\|_{X^2} \leq \epsilon_0$, there is $s = s(\xi)$ with $|s-1| \lesssim \|\xi\|_{X^2}$ such that $g(s;\xi) = 0$. Then also $\tilde{\xi} := \xi + Q - Q^s \implies \|\tilde{\xi}\|_{X^2} \lesssim \|\xi\|_{X^2} + |s-1| \lesssim \|\xi\|_{X^2}$. The case of general $s_0 > 0$ follows from simple rescaling, and the scale invariance of the X^2-norm. □

This lemma shows that as long as

$$\inf_{s>0} \|u(\cdot,t) - w(\cdot,t) - Q^s\|_{X^2} < \epsilon_0, \tag{3.10}$$

we may write u in the form (3.9), with orthogonality (3.6) holding.

In particular, (3.2) implies that for any $0 < \delta_0 < \epsilon_0$, by taking j large enough, and therefore $\|u(\cdot,t_j) - Q^{s(t_j)} - w_0\|_{X^2}$ small enough, we may write

$$u(t_j,r) = Q^{s(0)} + w_0(r) + \xi(r,0), \quad \big(\xi(\cdot,0),\ h^{s(0)}\big) = 0, \quad \|\xi(\cdot,0)\|_{X^2} \leq \delta_0. \tag{3.11}$$

So by continuity, (3.10) holds on some non-empty time interval $I = [t_j,\tau)$, $t_j < \tau \leq T$, on which we may write $u(r,t)$ as in (3.9) with orthogonality (3.6).

Moreover by regularity of $u(r,t)$, by shrinking τ even more if needed, we may also assume

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty([t_j,\tau))} \leq \delta_0^{\frac{2}{3}}, \tag{3.12}$$

which in particular implies (3.10) for δ_0 sufficiently small.

We will use a standard "continuity argument". That is, we will carry out all our estimates over the time interval $I = [t_j,\tau)$ under the assumption (3.12), and then conclude that we may take $\tau = T$ provided δ_0 is chosen sufficiently small.

Inserting (3.9) into the PDE and using standard trigonometric identities yields the following equation for ξ:

$$(\partial_t + H^s)\xi = -m\frac{\dot{s}}{s} h^s + Eqn(Q^s + w) + \frac{m^2}{2r^2}\left(V\sin(2\xi) + N\right), \tag{3.13}$$

where

$$V = \cos(2Q^s)(\cos(2w) - 1)) - \sin(2Q^s)\sin(2w),$$

and N contains only terms super-linear in ξ coming from the terms

$$\cos(2Q^s)[2(w+\xi) - \sin(2(w+\xi))] \text{ and}$$
$$\sin(2Q^s)[1 - \cos(2(w+\xi))].$$

Rather than write out all the terms of N explicitly, we just record the elementary estimates

$$|N| \lesssim (h^s + |w|)\xi^2 + |\xi|^3$$
$$|N_r| \lesssim (1+|w|)(|w_r| + \frac{h^s}{r})\xi^2 + \frac{1}{r}|\xi|^3 + (h^s + |w|)|\xi||\xi_r| + \xi^2|\xi_r|. \tag{3.14}$$

Our goal is to estimate all the terms on the right side of (3.13) in appropriate space-time norms, so that we may apply the linear estimates (3.7).

For the first term, using $\|\frac{1}{s}h^s\|_{X^1} \lesssim 1$, we have

$$\| - m\frac{\dot{s}}{s}h^s\|_{L_t^2 X^1} \lesssim \|\dot{s}\|_{L_t^2}. \tag{3.15}$$

The main estimates for V are

$$|V| \lesssim w^2 + h^s|w| \implies \|\frac{1}{r^2}V\|_{L^2} \lesssim \|\frac{w}{r}\|_{L^4}^2 + \|\frac{w}{r}\|_{L^\infty},$$

using $\|\frac{h^s}{r}\|_{L^2} \lesssim 1$, and

$$|V_r| \lesssim |w||w_r| + \frac{1}{r}(h^s)^2 w^2 + h^s|w_r| + \frac{1}{r}h^s|w|$$
$$\implies \|\frac{1}{r}V_r\|_{L^2} \lesssim \|\frac{w}{r}\|_{L^4}\|w_r\|_{L^4} + \|\frac{w}{r}\|_{L^4}^2 + \|w_r\|_{L^\infty} + \|\frac{w}{r}\|_{L^\infty},$$

using $\|h^s\|_{L^\infty} \lesssim 1$ and $\|\frac{h^s}{r}\|_{L^2} \lesssim 1$. Combining these, we obtain a spatial-norm estimate on the linear term on the right side of (3.13),

$$\|\frac{m^2}{2r^2}V\sin(2\xi)\|_{X^1} \lesssim \left(\|w\|_{X^4}^2 + \|w\|_{X^\infty}\right)\|\xi\|_{X^2},$$

and from there a space-time estimate:

$$\|\frac{m^2}{2r^2}V\sin(2\xi)\|_{L_t^2 X^1} \lesssim \left(\|w\|_{L_t^4 X^4}^2 + \|w\|_{L_t^2 X^\infty}\right)\|\xi\|_{L_t^\infty X^2}, \tag{3.16}$$

where, recall, the time interval over which these norms are taken is $I = [t_j, \tau)$.

Finally, from (3.14), we estimate the nonlinear terms:

$$\|\frac{1}{r^3}N\|_{L^1} \lesssim \left(\|\frac{h^s}{r}\|_{L^2} + \|\frac{w}{r}\|_{L^2}\right)\|\frac{\xi}{r}\|_{L^4}^2 + \|\frac{\xi}{r}\|_{L^3}^3 \lesssim \|\frac{\xi}{r}\|_{L^4}^2 + \|\frac{\xi}{r}\|_{L^3}^3,$$

and using $\|w\|_{L^\infty} \lesssim \|w\|_{X^2} \lesssim 1$, and $\|h_s\|_{X^2} \lesssim 1$,

$$\|\frac{1}{r^2}N_r\|_{L^1} \lesssim \|\frac{\xi}{r}\|^2_{L^4} + \|\frac{\xi}{r}\|^3_{L^3} + \|\frac{\xi}{r}\|_{L^4}\|\xi_r\|_{L^4} + \|\frac{\xi}{r}\|^2_{L^4}\|\xi_r\|_{L^2}.$$

These last two give

$$\|\frac{1}{r^2}N\|_{X^1} \lesssim \|\xi\|^2_{X^4} + \|\xi\|^3_{X^3} + \|\xi\|^2_{X^4}\|\xi\|_{X^2},$$

and then the space-time estimate:

$$\|\frac{m^2}{2r^2}N\|_{L^2_tX^1} \lesssim \|\xi\|^2_{L^4_tX^4}(1 + \|\xi\|_{L^\infty_t X^2}) + \|\xi\|^3_{L^6_tX^3}. \tag{3.17}$$

Now applying the linear estimates (3.7) to (3.13), using (3.11), (3.15), (3.5) (taking j larger as needed), (3.16), and (3.17), as well as (3.4), we get

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty} \leq C\left(\delta_0 + \|\dot{s}\|_{L^2_t} + \left(\|w\|^2_{L^4_tX^4} + \|w\|_{L^2_tX^\infty}\right)\|\xi\|_{L^\infty_t X^2} \right.$$
$$\left. + \|\xi\|^2_{L^4_tX^4}(1 + \|\xi\|_{L^\infty_t X^2}) + \|\xi\|^3_{L^6_tX^3}\right). \tag{3.18}$$

By (3.4), by choosing j larger still, if needed, we can ensure that on the interval $[t_j, T) \supset I$,

$$C\left(\|w\|^2_{L^4_tX^4([t_j,T)} + \|w\|_{L^2_tX^\infty([t_j,T)}\right) < \frac{1}{2}, \tag{3.19}$$

so that the estimate (3.18) becomes

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty} \lesssim \delta_0 + \|\dot{s}\|_{L^2_t} + \|\xi\|^2_{L^\infty_t X^2 \cap L^2_t X^\infty} + \|\xi\|^3_{L^\infty_t X^2 \cap L^2_t X^\infty},$$

and then by using (3.12),

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty} \lesssim \delta_0 + \|\dot{s}\|_{L^2_t}. \tag{3.20}$$

It remains to estimate $\dot{s}$. For this, we differentiate the orthogonality relation (3.6), rewritten as

$$\left(\xi, \frac{1}{s}h^s\right)_{L^2_{rdr}}$$

for convenience of calculation, with respect to t, and use the equation (3.13) for ξ:

$$0 = \left(\xi,\ -\dot{s}\frac{1}{r^2}(r(r^2h)')^s\right)$$
$$+\left(-m\frac{\dot{s}}{s}h^s + Eqn(Q^s + w) + \frac{m^2}{2r^2}\left(V\sin(2\xi) + N\right),\ \frac{1}{s}h^s\right)$$

where we used $H^s h^s = 0$. The first term is bounded by

$$|\dot{s}| \|\frac{\xi}{r}\|_{L^2} \|\frac{1}{r}(r(r^2 h)')^s\|_{L^2} \lesssim |\dot{s}| \|\xi\|_{X^2},$$

while

$$\left(-m\frac{\dot{s}}{s}h^s, \frac{1}{s}h^s\right) = -m\dot{s}\|\frac{1}{r}(rh)^s\|_{L^2}^2 = -m\dot{s}\|h\|_{L^2}^2,$$

so

$$\left(m\|h\|_{L^2}^2 + O(\|\xi\|_{X^2})\right)\dot{s} = \left(Eqn(Q^s + w) + \frac{m^2}{2r^2}\left(V\sin(2\xi) + N\right),\ \frac{1}{s}h^s\right). \tag{3.21}$$

Then by (3.12) and $\|r\frac{1}{s}h^s\|_{L^\infty} = \|rh\|_{L^\infty} \lesssim 1$,

$$|\dot{s}| \lesssim \|Eqn(Q^s + w)\|_{X^1} + \|\frac{1}{r^2}V\sin(2\xi)\|_{X^1} + \|\frac{1}{r^2}N\|_{X^1},$$

and so by (3.5), (3.16), (3.4), (3.14) and (3.12):

$$\|\dot{s}\|_{L_t^2} \lesssim \delta_0 + \left(\|w\|_{L_t^4X^4}^2 + \|w\|_{L_t^2X^\infty}\right)\|\xi\|_{L_t^\infty X^2}.$$

Using (3.20) then shows

$$\|\dot{s}\|_{L_t^2} \le C\left(\delta_0 + \left(\|w\|_{L_t^4X^4}^2 + \|w\|_{L_t^2X^\infty}\right)\|\dot{s}\|_{L_t^2}\right).$$

As above, by taking j larger if needed we can ensure (3.19) and so (using again (3.12))

$$\|\dot{s}\|_{L_t^2} + \|\xi\|_{L_t^\infty X^2 \cap L_t^2 \cap X^\infty} \lesssim \delta_0.$$

This now shows that in our bootstrap assumption (3.12), since we take $\delta_0 \ll \delta_0^{2/3}$, we may indeed take $\tau = T$, and all of our previous estimates hold on the full time interval $[t_j, T)$.

It remains to show that $s(t)$ stays bounded away from zero. Recall the pointwise bounds used above

$$\begin{aligned} |V| &\lesssim w^2 + h^s|w|, \\ |N| &\lesssim (h^s + |w|)\xi^2 + |\xi|^3, \\ Eqn(Q^s + w) &\lesssim \frac{1}{r^2}\left(h^s w^2 + (h^s)^2 w\right). \end{aligned}$$

We isolate the term in the equation (3.21) for s coming from the part of $Eqn(Q^s + w)$ which behaves linearly in w, and write:

$$\frac{\dot{s}}{s} = v_1 + v_2 + v_3$$

where

$$|v_1| \lesssim \|\frac{w}{r^3}\|_{L^2} \|\frac{1}{s}(rh^3)^s\|_{L^2} \lesssim \|\frac{|w|}{r^2}\|_{X^2} \in L_t^2,$$

$$|v_2| \lesssim \|\frac{w^2}{r^2}\|_{L^\infty}\|\frac{1}{s^2}(h^2)^s\|_{L^1} \lesssim \|w\|_{X^\infty}^2 \in L_t^1$$

and

$$\begin{aligned}|v_3| \lesssim &\|\frac{w^2}{r^2}\|_{L^2}\|\frac{\xi}{r}\|_{L^\infty}\|\frac{1}{s}(rh)^s\|_{L^2} + \|\frac{w}{r}\|_{L^\infty}\|\frac{\xi}{r}\|_{L^\infty}\|\frac{1}{s^2}h^s\|_{L^1} \\ &+ \|\frac{\xi^2}{r^2}\|_{L^\infty}\|\frac{1}{s^2}(h^s)^3\|_{L^1} + \|\frac{\xi^2}{r^2}\|_{L^2}(\|\frac{w}{r}\|_{L^\infty} + \|\frac{\xi}{r}\|_{L^\infty})\|\frac{1}{s}(rh)^s\|_{L^2} \\ \lesssim &\|w\|_{X^4}^2\|\xi\|_{X^\infty} + \|w\|_{X^\infty}\|\xi\|_{X^\infty} + \|\xi\|_{X^\infty}^2 + \|\xi\|_{X^4}^2(\|w\|_{X^\infty} + \|\xi\|_{X^\infty}) \\ \in &L_t^1.\end{aligned}$$

So $\frac{\dot{s}}{s} \in L_t^1 + L_t^2$ over $[t^*, T)$, and by the Fundamental Theorem of Calculus, and Cauchy-Schwartz,

$$\sup_{t^* \le t < T}\left|\log\left(\frac{s(t)}{s(t^*)}\right)\right| \le \|\frac{\dot{s}}{s}\|_{L_t^1([t^*,T))} + \sqrt{t-t^*}\|\frac{\dot{s}}{s}\|_{L_t^2([t^*,T))} < \infty,$$

so that $s(t)$ remains bounded away from zero, as required. □

3.7. **Completion of the proof.** Proposition 3.2 shows that a solution of (3.1), with $m \ge 4$ and $u_0 \in E_1$ cannot form a finite-time singularity. Hence such a solution is global. Moreover, it cannot form a singularity at infinite time $t = \infty$, since such this would produce a sequence $t_j \to \infty$, with $0 < s_j \to 0$ or ∞, along which $u(\cdot, t_j) - Q^{s_j} \to v_0$ with $X^2 \ni v_0$ a *static solution*, hence $v_0 \equiv 0$. This is however prohibited by the asymptotic stability result of [28]. Hence we must have $E(u(\cdot,t) - Q^{s_\infty}) \to 0$ for some $s_\infty > 0$. Uniform convergence then follows from the embedding $X^2 \subset L^\infty$. This completes the proof of Theorem 1.3. □

Acknowledgements

The first author's research is supported through NSERC Discovery Grant 22124-12. The second author is supported by the European Research Council (grant no. 637995 "ProbDynDispEq"). The results reported in this article were obtained as part of the second author's PhD thesis at the University of British Columbia.

References

[1] H.Bahouri, P.Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math. 121 (1999), 131-175

[2] I.Bejenaru, A.Ionescu, C.E.Kenig, D.Tataru, Equivariant Schrödinger Maps in two spatial dimensions, Duke Mathematical Journal 162, (2013), 1967-2025

[3] I.Bejenaru, A.Ionescu, C.E.Kenig, D.Tataru, Equivariant Schrödinger Maps in two spatial dimensions: the $\mathbb{H}^2$ target, Kyoto J. Math. 56 (2016), no. 2, 283-323

[4] G.J.B.Van den Berg, J. Hulshof, J. King, Formal asymptotics of bubbling in the harmonic map heat flow, SIAM J. Appl. Math. vol 63, no5, (2003), 1682-1717
[5] P.Biernat, R.Donninger, Construction of a spectrally stable self-similar blowup solution to the supercritical harmonic map heat flow, preprint, arXiv:1610.09496
[6] P.Biernat, R.Donninger, B. Schörkhuber, Stable self-similar blowup in the supercritical heat flow of harmonic maps, Calc. Var. Partial Differential Equations, 56 (2017), no. 6, 56-171
[7] H.Brézis and J.-M.Coron, Convergence of solutions of H-systems or how to blow bubbles, Arch. Rational Mech. Anal. 89 (1985), 21-56
[8] A.Bulut, Maximizers for the Strichartz inequalities for the wave equation, Differential Integral Equations (2010), no. 11-12, 1035-1072
[9] T.Cazenave, F.B.Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Anal., Theory, Methods and Applications, Series A: Theory and Methods, 14, 1990, 10, 807-836
[10] K.-C.Chang, W.-Y.Ding, R.Ye, Finite-time blow-up of the heat flow of harmonic maps from surfaces, J. Diff. Geom. 36 (1992), 507-515
[11] J.Colliander, M.Keel, G.Staffilani, H.Takaoka, T.Tao, Global well-posedness and scattering in the energy space for the critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Annals of Mathematics, 167 (2008), 767-865
[12] R.Côte, C.E.Kenig, A.Lawrie, W.Schlag, Characterization of large energy solutions of the equivariant wave map problem:I, Amer. J. Math. 137 (2015), no. 1, 139-207
[13] R.Côte, C.E.Kenig, F.Merle, Scattering below critical energy for the radial 4D Yang-Mills equation and for the 2D corotational wave map system, Communications in Mathematical Physics 284 (2008), no. 1, 203-225
[14] J.Davila, M.del Pino, J.Wei, Singularity formation for the two-dimensional harmonic map flow into $\mathbb{S}^2$, preprint, arXiv:1702.05801
[15] W.-Y.Ding, G.Tian, Energy identity for a class of approximate harmonic maps from surfaces, Comm. Anal. Geom. 3 (1995), 543-554
[16] J.Eells, J.H.Sampson, Harmonic mappings of Riemannian manifolds. Amer. J. Math. 86 (1964), 109-169
[17] I.Gallagher, G.S.Koch, F. Planchon, A profile decomposition approach to the $L^\infty_t L^3_x$ Navier-Stokes regularity criterion. Math. Ann. 355, 4 (2013), 1527-1559
[18] I.Gallagher, G.S.Koch, F.Planchon, Blow-up of critical Besov norms at a potential Navier-Stokes singularity, Comm. Math. Phys. 343 (2016), no. 1, 39-82
[19] P. Gérard, Description du défaut de compacité de l'injection de Sobolev. ESAIM Control Optim. Calc. Var. 3 (1998), 213-233
[20] T.-E.Ghoul, S.Ibrahim, V.T.Nguyen, On the stability of type II blowup for the 1-corotational energy supercritical harmonic heat flow, preprint, arXiv:1611.08877
[21] T.-E.Ghoul, S.Ibrahim, V.T.Nguyen, Construction of type II blowup solutions for the 1-corotational energy supercritical wave maps, preprint, arXiv:1704.05685
[22] Y.Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations 62, (1986) 186-212

[23] J.Grotowski, J.Shatah, Geometric evolution equations in critical dimensions, Calc. Var. Partial Differential Equations 30 (2007), no. 4, 499-512
[24] M.Guan, S.Gustafson, T.-P.Tsai, Global existence and blow-up for harmonic map heat flow, J. Diff. Equations 246 (2009), 1-20
[25] S.Gustafson, K.Kang, T.-P.Tsai, Schrödinger flow near harmonic maps, Comm. Pure Appl. Math. 60 (2007) no. 4, 463-499
[26] S.Gustafson, K.Kang, T.-P.Tsai, Asymptotic stability of harmonic maps under the Schrödinger flow, Duke Math. J. 145, No. 3 (2008), 537-583
[27] S.Gustafson, E.Koo, Global well-posedness for 2D radial Schrdinger maps into the sphere, preprint, arXiv:1105.5659
[28] S.Gustafson, K.Nakanishi, T.-P.Tsai, Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on $\mathbb{R}^2$, Comm. Math. Phys. 300, 205-242 (2010)
[29] S.Gustafson, D.Roxanas, Global, decaying solutions of a focusing energy-critical heat equation in $\mathbb{R}^4$, preprint,arXiv:1707.07644
[30] C.E.Kenig, G.S.Koch, An alternative approach to the Navier-Stokes equations in critical spaces, Ann. I. H. Poincare, 28(2): 159-187, 2011
[31] C.E.Kenig, F.Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Inventiones mathematicae (2006), 166, no 3, 645-675
[32] C.E.Kenig, F.Merle, Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation, Acta Mathematica (2008), 201, no 2, 147-212
[33] S.Keraani, On the defect of compactness for the Strichartz estimates for the Schrödinger equations. J. Diff. Eqns. 175 (2001), 353-392
[34] R.Killip, M.Visan,, Nonlinear Schrödinger Equations at Critical Regularity, "Evolution Equations", 325-437, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013
[35] A.Kosevich, B.Ivanov, A.Kovalev, Magnetic Solitons, Phys. Rep. 194 (1990), 117-238
[36] F.Lin, C.Wang, The analysis of harmonic maps and their heat flows, World Scientific, 2008
[37] P.-L.Lions, The concentration-compactness principle in the calculus of variations. The limit case. I. Rev. Mat. Iberoamericana 1 (1985), 145-201
[38] J.Qing, On singularities of the heat flow for harmonic maps from surfaces into spheres, Comm. Anal. Geom. 3 (1995), 297-315
[39] J.Qing, G.Tian, Bubbling of the heat flows for harmonic maps from surfaces, Comm. Pure Appl. Math. 50 (1997), 295-310
[40] P.Raphael, R.Schweyer, Stable blow up dynamics for the co-rotational energy critical harmonic heat flow, Comm. Pure Appl. Math. 66 (2013), no. 3, 414-480
[41] P.Raphael, R.Schweyer, Quantized slow blow up dynamics for the co-rotational energy critical harmonic heat flow, Anal. PDE 7 (2014), no. 8, 1713-1805
[42] D.Roxanas, Long-time dynamics for the energy-critical Harmonic Map Heat Flow and Nonlinear Heat Equation, PhD Thesis, University of British Columbia, May 2017
[43] J.Sacks, K.Uhlenbeck, The existence of minimal immersions of 2-spheres. Ann. of Math. 113 (1981), 1-24

[44] J.Shatah, M.Struwe, Geometric wave equations. Courant Lecture Notes in Mathematics, 2, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998
[45] M.Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187, 511-517 (1984)
[46] M.Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Comment. Math. Helv. (1985), 60, 558-581
[47] T.Tao, M.Visan, Stability of energy-critical nonlinear Schrödinger equations in high dimensions, Electron. J. Diff. Eqns. 2005 (2005), 1-28
[48] P.Topping, Rigidity in the harmonic map heat flow, J. Differential Geometry (1997), 45, 593-610
[49] P.Topping, Winding behaviour of finite-time singularities of the harmonic map heat flow, Math. Zeit., 247 (2004), 279-302

Stephen Gustafson
Department of Mathematics, University of British Columbia
Vancouver, V6T 1Z2, Canada
gustaf@math.ubc.ca

Dimitrios Roxanas
Department of Mathematics, The University of Edinburgh
James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road
Edinburgh, EH9 3FD,United Kingdom
dimitrios.roxanas@ed.ac.uk