Shear Wave Models in Linear and Nonlinear Elastic Materials

A Thesis Submitted to the
College of Graduate and Postdoctoral Studies
in Partial Fulfillment of the Requirements
for the degree of Master of Science
in the Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon

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Abstract

Nonlinear shear wave models are of significant importance in a large number of areas, including engineering and seismology. The study of such wave propagation models has helped in the prediction and exploration of hidden resources in the Earth. Also, the frequent occurrences of earthquakes and the damage they cause to lives and properties are of more significant concern to the society. Augustus Edward Hough Love studied horizontally polarized shear waves (Love surface waves) in homogeneous elastic media. In the current thesis, after presenting some basic concepts of linear and nonlinear elasticity, we discuss linear Love waves in both isotropic and anisotropic elastic media, and consider extended linear and nonlinear wave propagation models in elastic media, including models of nonlinear Love-type surface waves. A new general partial differential equation model describing the propagation of one- and two-dimensional Love-type shear waves in incompressible hyperelastic materials is derived, holding for an arbitrary form of the stored energy function. The results can be further generalized to include an arbitrary viscoelastic contribution. We also discuss aspects of Hamiltonian mechanics in finite- and infinite-dimensional systems and present Hamiltonian formulations of some nonlinear wave models discussed in this thesis.

Acknowledgements

I owe my heartfelt and deepest gratitude to my supervisor Professor Alexey F. Shevyakov, for his support, frankness, patience, guidance, and valuable comments throughout my program and in the preparation of this work. I am also grateful to Professor Jacek Szmigielski for his insightful discussions and contributions. I am highly indebted to my family, friends, and past teachers for their love and immense contributions in my academic life. Finally, I would also like to acknowledge and thank the University of Saskatchewan and the Mathematics and Statistics Department for financially supporting me during my program.

To my family.

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1 INTRODUCTION

In the study of models arising from physical problems in various areas of applied mathematics, geophysics, geodynamics, biomechanics, continuum mechanics, solid mechanics, civil engineering, earthquake engineering and so on, an important class of equations involving the derivatives of a function or functions (differential equations) appears. The fundamental goal of many researchers is to analyze and find solutions to these differential equations arising from such physical models. Among the numerous methods of analyzing and seeking solutions to these differential equations are numerical simulations, Lagrangian and Hamiltonian analyses [87, 88, 126], conservation laws [8, 70], Lie symmetries [21, 22, 70] and so on. Among these physical models are wave equations describing wave propagation during earthquakes.

The study of wave propagation is essential for the exploration and predicting of hidden natural resources in the Earth. The Earth's crust is highly anisotropic and heterogeneous; certain materials in the crust exhibit elastic and viscoelastic behaviors. Nowadays, the damage caused by earthquakes to lives and properties has raised notable concerns. Surface waves have become a fascinating study area. Models obtained in this area of seismology are solved numerically or analytically. Because the Earth's crust exhibits elastic properties, it is relevant to discuss what these materials (elastic materials) are.

Elastic materials are generally classified as hyperelastic materials, Cauchy elastic materials, and hypoelastic materials. These types of elastic materials are defined based on the notions of stresses, strains, and strain rates. The prefixes 'hypo -' and 'hyper -' mean a reduction in something versus some norm and a strong increase in something versus some norm, respectively. Hypoelastic constitutive relations are needed when modeling materials that exhibit nonlinear, but reversible stress-strain behavior, even under small strains. Cauchy elastic constitutive relations are used to model materials that exhibit linear, but reversible, stress-strain relation when subjected to small deformations. On the other hand, hyperelastic constitutive laws are required when modeling materials that undergo large strains when subjected to external stress. These types of materials are of importance in this work.

Three kinds of elastic waves, namely, the shear vertical (SV) waves, compressional (P) waves, and the shear horizontal (SH) waves arise when a continuum moves in three planes with respect to the direction of propagation of wave [3]. P waves are longitudinal waves that involve the compression and rarefaction of the material as the waves propagate through it, but not rotation. SV waves are vertically polarized shear waves, and the particle motion is orthogonal to the direction of propagation of waves. SH waves are also horizontally polarized shear waves. These definitions of P, SV, and SH waves are used in seismology. In a homogeneous and isotropic medium, the velocities of longitudinal waves, β and transversely polarized waves [26], α are represented in the formulae

$$\alpha = \sqrt{\frac{\mu}{\rho}}, \quad \beta = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \tag{1.1}$$

where λ and μ are the Lamé parameters, and ρ is the density. The parameter λ appears in the Hooke's law, it is a linear combination of the bulk modulus K and the shear modulus μ. The parameter μ is the shear modulus (modulus of rigidity), which measures the material's resistance to shear.

Rayleigh [115], studied compressional waves (P-waves) and vertically polarized waves (SV-waves) propagating in infinite homogeneous isotropic elastic solids. Based on seismograph earthquake observations, A. E. Love [81, 82] studied horizontally polarized shear waves (SH-waves).

In this chapter, we present the basic theory of linear and nonlinear elasticity in continuum mechanics, based on the texts by Ciarlet [38], Marsden and Hughes [87], and Rushchitsky [121]. We also briefly describe the types of seismic waves [121]. We begin in Section 1.1 by introducing the concept of linear elasticity and discuss anisotropic and isotropic elastic materials. Besides, we present figures showing various examples of elastic materials. In Section 1.2, we give a brief description of seismic waves. In Section 1.3, we present the principles of the nonlinear theory of elasticity in continuum mechanics. The derivation of equations of motion in continuum mechanics is presented in Section 1.4. Constitutive laws of elastic materials, including models of isotropic and anisotropic hyperelastic materials, are discussed in Section 1.5. We conclude with a general overview of the entire chapter in Section 1.6.

The notation used within the current thesis is as follows. Vectors are indicated with boldface characters (e.g., $\mathbf{X}$). Subscripts on scalar functions and vector quantities denote partial differentiation (e.g. $\partial v^i/\partial X_j = v^i_{,j} = v^i_j$ denotes partial differentiation with respect to X_j). All functions discussed in the current thesis are assumed to be sufficiently smooth on their domains. We also use the Cartesian coordinates and flat space metrics $g^{ij} = \delta^{ij}$, where δ^{ij} is the Kronecker delta-symbol; therefore, indices of all tensors can be lowered or raised freely as needed throughout this current thesis. The main goal of this thesis is to consider earthquake-type elastic waves using the apparatus of nonlinear hyperelasticity theory. In Chapter 2, we discuss linear Love waves in isotropic and anisotropic media. Several shear wave propagation models are considered in Chapter 3. We discuss the finite and infinite-Hamiltonian systems in Chapter 4. Finally, discussions and conclusions of the entire thesis will be presented in Chapter 5.

1.1 Linear Elasticity

An elastic solid is any material that goes through deformation (small or large) when a force is applied to it externally and returns to its original shape when such a force is withdrawn. Materials such as rubbers, fabrics, and biological tissues are elastic materials. The external force deforms the material either by squeezing, stretching, or shearing. The study of the mathematical framework, the characteristics, and properties of such deformed elastic solids is termed as elasticity theory [135, 136]. Elasticity theory has multiple applications in earthquake engineering, material science, geodynamics, and many other relevant fields, as seen in the literature [26, 121, 139]. In linear elasticity, one usually assumes small strains (deformations), and the constitutive relation is giving by a generalized Hooke's law. Hooke's law states that the stress of an elastic solid is proportional to its strain. In elastic materials, the strain (the deformation) is infinitesimally small, and Hooke's law holds. Most engineering materials, such as concrete and metals, undergo small strains in practical applications, ranging from 10^{-6} to 10^{-2} in relative terms. Linear elasticity is the simplest constitutive model with the strain energy density [26, 7] given by

$$W = \frac{1}{2}\lambda(\mathrm{Tr}(\varepsilon))^2 + \mu\mathrm{Tr}(\varepsilon^2), \tag{1.2}$$

where ε denotes the strain tensor, and λ and μ are the Lamé constants. The strain tensor ε is a symmetric second-order ranked tensor used to quantify the strain of an object undergoing a small deformation in three dimensions. The Lamé constants λ and μ are two material-dependent quantities that appear in constitutive (stress-stress) relations. The Lamé constants are elastic constants derived from the Young modulus, κ, and the Poisson ratio, ν. Young modulus, κ, is a proportionality constant between the uniaxial stress and strain. The Poisson's ratio is the negative ratio of the transverse strain to the axial strain. The Lamé parameters are given in terms of the Young modulus and the Poisson's ratio [7, 38, 62] by

$$\mu = \frac{\kappa}{2\,(\nu+1)}, \qquad \lambda = \frac{\kappa\nu}{(\nu+1)\,(-2\nu+1)}. \tag{1.3}$$

Another type of elastic modulus is the bulk modulus, which is an extension of the Young modulus in 3D, describes the tendency of a material to deform in all directions when uniformly loaded in all directions. The bulk modulus [7], K, in elasticity can be expressed as a linear combination of the two Lamé constants as

$$K = \lambda + \frac{2}{3}\mu.$$

In what follows, anisotropic and isotropic elastic materials are briefly discussed.

1.1.1 Anisotropic Elastic Materials

Mathematically, the generalized Hooke's law in Cartesian coordinates which shows a linear relationship between the strain and stress in an elastic body is expressed in conventional symbols as

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\epsilon}, \qquad \sigma_{ik} = E_{iklm}\epsilon_{lm}, \quad i,k,l,m = 1,2,3, \tag{1.4}$$

where σ_{ik}, ϵ_{lm}, and E_{iklm} represent the Cauchy stress tensor, the strain tensor, and the elastic stiffness tensor respectively. We use the Einstein summation convention where needed (i.e., summation in repeated indices). In matrix notation (1.4) is given by

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1113} & E_{1121} & E_{1131} & E_{1132} & E_{1123} \\ E_{2211} & E_{2222} & E_{2233} & E_{2212} & E_{2213} & E_{2221} & E_{2231} & E_{2232} & E_{2223} \\ E_{3311} & E_{3322} & E_{3333} & E_{3312} & E_{3313} & E_{3321} & E_{3331} & E_{3332} & E_{3323} \\ E_{1211} & E_{1222} & E_{1233} & E_{1212} & E_{1213} & E_{1221} & E_{1231} & E_{1232} & E_{1223} \\ E_{1311} & E_{1322} & E_{1333} & E_{1312} & E_{1313} & E_{1321} & E_{1331} & E_{1332} & E_{1323} \\ E_{2111} & E_{2122} & E_{2133} & E_{2112} & E_{2113} & E_{2321} & E_{2131} & E_{2132} & E_{2123} \\ E_{3111} & E_{3122} & E_{3133} & E_{3112} & E_{3113} & E_{3121} & E_{3131} & E_{3132} & E_{3123} \\ E_{3211} & E_{3222} & E_{3233} & E_{3212} & E_{3213} & E_{3221} & E_{3231} & E_{3232} & E_{3223} \\ E_{2311} & E_{2322} & E_{2333} & E_{2312} & E_{2313} & E_{2321} & E_{2331} & E_{2332} & E_{2323} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{21} \\ \epsilon_{31} \\ \epsilon_{32} \\ \epsilon_{23} \end{bmatrix} . \tag{1.5}
$$

Figure 1.1 shows a one-dimensional stress-strain relation of an elastic material. When a force is applied to an elastic material, there is a deformation proportional to the applied force. The proportionality existing between the deformation and stress is characterized by the elastic constants of the medium defined by the elastic stiffness tensor. Physical parameters such as the Young modulus, bulk modulus, shear modulus, and Poisson's ratio are all examples of the elastic coefficients. Both the stress tensor and strain tensor are second-order, and the stiffness tensor is a fourth-order tensor.

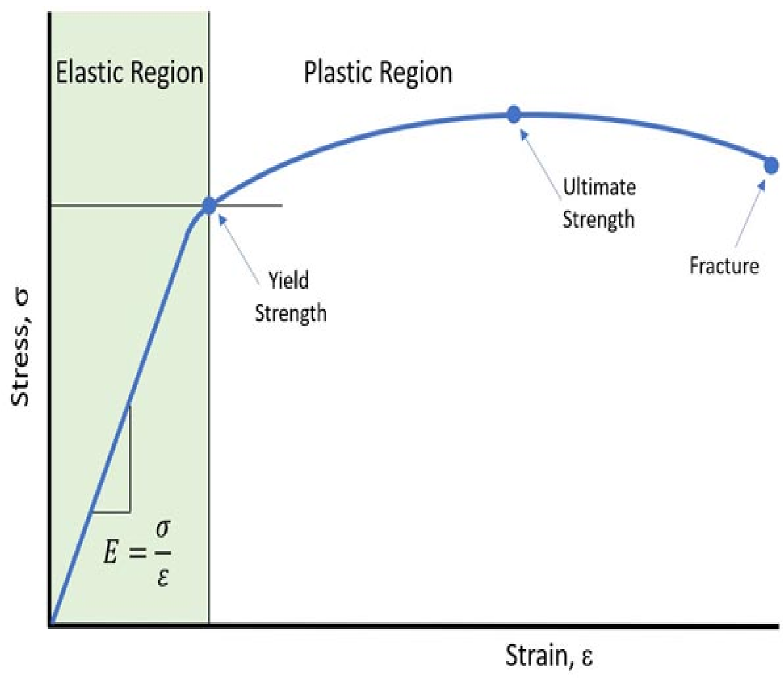

Figure 1.1: A typical stress-strain curve for an elastic material [127].

The relation between the stress and deformation of a body in continuum mechanics is termed as the constitutive relation.

The actual (Eulerian) positions of material points in the body are given by

$$
\mathbf{x} = \mathbf{X} + \mathbf{u}, \tag{1.6}
$$

where $\mathbf{X}$ are the material coordinates (the labels of the material points, or the Lagrangian coordinates), and $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ denotes the displacement of the corresponding material point. Thus, the displacement of the material point is given by $\mathbf{u} = \mathbf{x} - \mathbf{X}$. The (small) strain which shows a linear correspondence between the strain tensor $\boldsymbol{\epsilon} = (\epsilon_{lm})$ and components of the displacement vector $\mathbf{u} = (u_m) = (u_1,\ u_2,\ u_3)$ in linear elasticity is given by the classical relation

$$\epsilon_{lm} = \frac{1}{2}\left(u_{m,l} + u_{l,m}\right), \tag{1.7}$$

where $u_{m,l} = \partial u_m / \partial X_l$ and $u_{l,m} = \partial u_l / \partial X_m$ are the components of the displacement gradients. For large deformations, the Lagrange strain tensor defined by

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right), \quad \epsilon_{lm} = \frac{1}{2}\left(u_{m,l} + u_{l,m} + u_{k,l}u_{k,m}\right) \tag{1.8}$$

is used to describe the deformation in nonlinear hyperelasticity framework [26]. In (1.8), $\mathbf{C}$ is the right Cauchy stress tensor (see Section 1.5.4 below), $\boldsymbol{\epsilon}$ in (1.8) is the Lagrangian strain tensor, $\mathbf{I}$ is the identity matrix, $u_{m,l},\ u_{l,m},\ u_{k,l}$, and $u_{k,m}$ are components of the displacement gradient. The stiffness tensor E_{iklm} of materials in (1.4) and (1.5) connecting the second-order stress and strain tensors involves 81 material parameters for 3D-problems and 16 material parameters for 2D-problems. The elasticity term (stiffness tensor) is the amount of stress (σ_{ik}) relating to the deformation or strain (ε_{lm}). The inverse of the stiffness tensor is called the compliance matrix. The compliance matrix is also a symmetric matrix measuring the amount of strain relating to the stress. Since the stress tensor and strain tensor are symmetric (i.e. $\sigma_{ik} = \sigma_{ki}$ and $\epsilon_{lm} = \epsilon_{ml}$), the 81 material constants reduce to 36 parameters. And from the differentiability of the strain energy density functions, the elastic stiffness tensor must satisfy the relations

$$E_{iklm} = E_{kilm} = E_{ikml} = E_{kiml}, \quad i, k, l, m = 1, 2, 3,$$

and as a result, the 36 parameters scale down to 21 independent parameters. The constitutive equation (1.4) for the most general linear elastic material is given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & E_{1113} & E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & E_{2223} & E_{2213} & E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & E_{3323} & E_{3313} & E_{3312} \\ E_{2311} & E_{2322} & E_{2333} & E_{2323} & E_{2313} & E_{2312} \\ E_{1311} & E_{1322} & E_{1333} & E_{1323} & E_{1313} & E_{1312} \\ E_{1211} & E_{1222} & E_{1233} & E_{1223} & E_{1213} & E_{1212} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix},$$

where $2\epsilon_{23} = \epsilon_{23}+\epsilon_{32}$, $2\epsilon_{12} = \epsilon_{12}+\epsilon_{21}$, and $2\epsilon_{13} = \epsilon_{13}+\epsilon_{31}$ are the engineering shear strains written as a sum of the symmetric components. Different kinds of models of material anisotropy arise as a result of the presence of various symmetries in the internal structure of the materials considered. These internal symmetries determine how the stiffness tensor should be in terms of structure. Many other symmetries that define the anisotropy of materials are studied extensively in the framework of crystallography [26, 83, 121]. Each type of symmetry results in the invariance of the stiffness tensor with respect to a unique symmetry transformation, for example, rotation about specific axes, and reflection on particular planes [83]. Orthotropic, transversely isotropic, and isotropic material models defined below have multiple applications in seismology, engineering, and building construction [119, 121]. In what follows, we discuss orthotropic, transversely isotropic, and isotropic elastic materials.

Orthotropic Materials

Materials such as wood, rolled metals, and composites are orthotropic. These materials have symmetric elastic properties with respect to three perpendicular axes and possess nine independent elastic stiffness parameters [61]. Figure 1.2 shows an example of an orthotropic elastic material. The orthotropic elastic stiffness tensor is of the form

$$E_{iklm} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{3131} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{1212} \end{bmatrix}.$$

Figure 1.2: Example of an orthotropic material [96].

Transversely Isotropic Materials

Transversely isotropic material is a type of orthotropic elastic material with only one isotropic plane. Typical examples of such materials are the unidirectional fiber composite, layered materials, and plywoods. They possess three mutually perpendicular planes of reflection symmetry and an axial symmetry with respect to the z-axis. They are defined with five independent elastic stiffness coefficients [121]. The transversely isotropic elastic stiffness tensor is of the form

$$E_{iklm} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{1111} & E_{1133} & 0 & 0 & 0 \\ E_{1133} & E_{1111} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/2)\,(E_{1111} - E_{1212}) \end{bmatrix}.$$

Isotropic Materials

Isotropic materials are invariant under all rotations, and their elastic properties are the same in all directions. They are defined by the two Lamé independent elastic coefficients. These coefficients are independent of the choice of the coordinate system. Most alloys, glasses, and metals are isotropic [121]. The most general isotropic elastic stiffness tensor is represented by

$$E_{iklm} = \lambda\delta_{ik}\delta_{lm} + \mu(\delta_{kl}\delta_{im} + \delta_{im}\delta_{kl}), \tag{1.9}$$

where

$$E_{1111} = E_{2222} = E_{3333} = \lambda + 2\mu, \quad E_{1212} = E_{1313} = E_{2323} = \lambda, \quad \frac{1}{2}\,(E_{1111 - E_{1212}}) = \mu.$$

Here μ and λ are the Lamé constants. The isotropic elastic stiffness tensor is of the form:

$$E_{iklm} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}.$$

Or

$$E_{iklm} = \begin{bmatrix} E_{1111} & E_{1212} & E_{1212} & 0 & 0 & 0 \\ E_{1212} & E_{1111} & E_{1212} & 0 & 0 & 0 \\ E_{1212} & E_{1212} & E_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E_{1111} - E_{1212}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(E_{1111} - E_{1212}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(E_{1111} - E_{1212}) \end{bmatrix}.$$

The generalized Hooke's law for the isotropic materials with two elastic constants becomes

$$\sigma_{ik} = \lambda \epsilon_{ll} \delta_{ik} + 2\mu \epsilon_{ik}. \tag{1.10}$$

In the next section, earthquakes and seismic waves are discussed. The two main types of seismic waves, namely, body and surface waves, are also discussed briefly.

1.2 Earthquakes and Seismic Waves

Earthquakes occur when there is a sudden release of potential energy in the crust of the Earth, generating seismic waves. Earthquakes happen as a result of faulting, explosions from mines or nuclear bombs, volcanic eruptions, etc., involving displacements of rocks along with fractures. These motions along faults are well studied in the framework of plate tectonic theory [90]. Earthquakes start from a hypocenter (focus). A point on the surface of the Earth, which is directly above the hypocenter of an earthquake where the vibration is most potent, and most damages happen, is called the epicenter. When an earthquake occurs, the structural destruction associated with earthquake vibrations depends mainly on the nature of materials on which the edifice rests, the intensity, the architectural design, distance from the epicenter, and the duration of the vibration. Below is a picture of the structure of an earthquake.

Earthquakes have been recorded in all the seven continents of the world. However, most earthquakes occur in just three regions of the world, namely, the alpide belt (Mediterranean to South Asia), the circum-pacific belt (along the edges of pacific ocean where about 80% of the world's quakes occur), and the middle Atlantic ridge (underwater line that runs through the ocean). Countries such as Indonesia, Nepal, the Philippines in the circum-pacific belt or 'the Pacific Ring of Fire' have about 80 percent likelihood of experiencing earthquakes.

The size of an earthquake can be determined based on these related measurements; shaking intensity, energy released, and magnitude. The shaking intensity is the measure of the strength of shaking at each location and varies from place to place, depending mostly on the distance from the fault rupture area. The intensity also measures the damage caused by an earthquake. Another way of measuring an earthquake is to compute how much energy is released after an earthquake. This amount of energy released by an earthquake is also potential damage to (human-made) structures. Earthquake's magnitude is recorded on a seismograph.

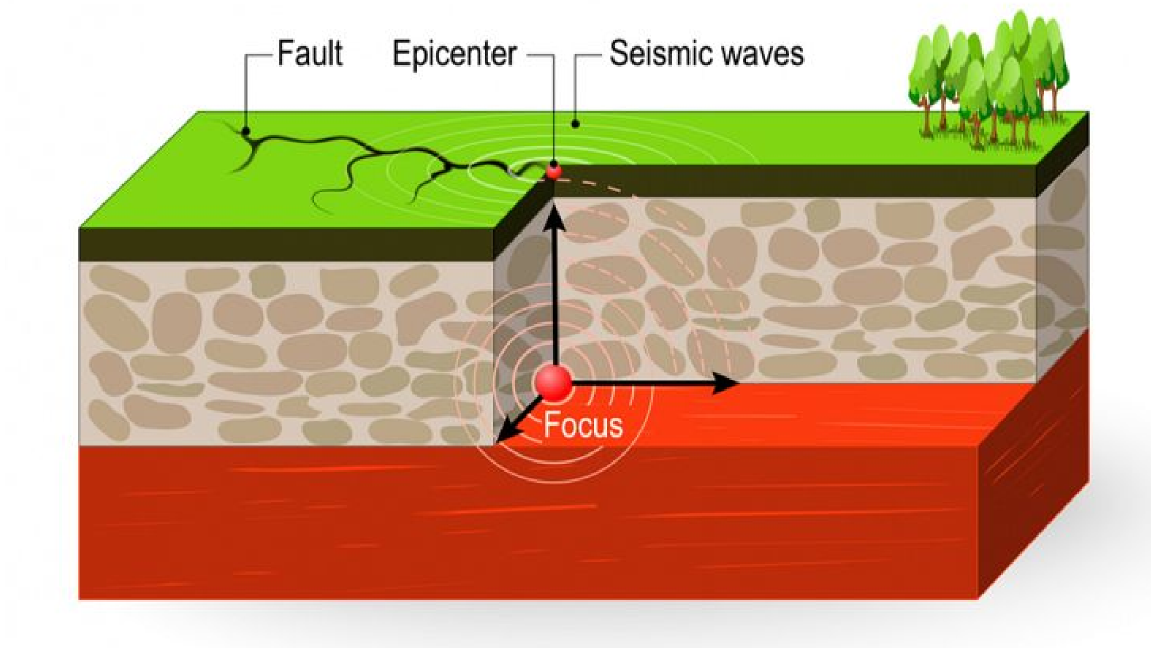

Figure 1.3: Earthquake structure [109].

The magnitude according to Richter scale of an earthquake is related to the amount of energy released at the source of an earthquake.

The range of the Richter scale is from 0 to 9 and the scale has no upper limit. The Richter scale is a logarithmic (base-10) scale, meaning that each level of magnitude is 10 times more intensive than the previous one. That is, a two is 10 times more severe than a one, and a three is 100 times greater. Also, on the Richter scale, the rise is in wave amplitude. That is, the wave amplitude in a level 5 earthquake is 10 times greater than in a level 4 earthquake. On a Richter scale, a level 0- to level 2-magnitude earthquakes are called microearthquakes, which are not commonly felt by people but can be recorded on seismographs. A level 9-magnitude earthquake is not common but would cause unimaginable damage.

Recently, the moment magnitude scale is widely used to measure the magnitude of earthquakes. This scale is also a logarithmic scale with no upper limit. The moment magnitude scale can be used to measure earthquakes of the highest magnitude globally, unlike the Richter scale. The largest ever recorded earthquake with a magnitude of about 9.5 occurred on May 22, 1960, near Valdivia in Chile. Aside the tremendous damage the Valdivia quake had on the land, it also caused a great tsunami reaching up to about 80 feet high which affected part of Japan and the Philippines. Due to the emergence of more and accurate seismic instruments for measuring more earthquakes, quakes are recorded almost everyday in the world.

According to long-term records (since about 1900), one usually anticipates about 16 major earthquakes in any calendar year, which includes 15 earthquakes in the magnitude 7 range and one earthquake magnitude 8.0 or greater. In the past 46 years, from 1973 through 2019, records reveal that we have passed the long-term average number of larger earthquakes only 11 times, in 1976, 1990, 1995, 1999, 2007, 2009, 2010, 2011, 2013, 2015, 2016, 2017, 2018, and 2019. The year with the highest total was 2010, with 24 earthquakes greater than or equal to magnitude 7.0. In other years the sum was well under the 16 per year anticipated based on the long-term average: 1989 only saw 6, while 1988 saw only 7 major earthquakes.

The magnitude is expressed in decimal fraction or whole numbers. For example, a magnitude of 7.8 is a strong earthquake, and a magnitude 5.1 is a moderate earthquake. In April 2015, Nepal experienced a strong earthquake of magnitude 7.8 on a Richter scale, and just a few days later had an aftershock of magnitude 7.3. The quake occurred less than 50 miles northwest of the country's capital, with a shallow depth of 11 km (contributing to its strength and massive damage). The earthquake affected nearly 8 million people, killed about 8,900 people, and injured around 22,000 people. This earthquake was the second strongest earthquake after an 8.4 magnitude earthquake in 1934 that killed about 10,000 people [93, 131]. A 5.1−magnitude moderate earthquake also happened in South California in 2014 that had a hypocenter recorded at a depth of 2 km, and aftershock of magnitude 4.8. There were no casualties, but there were only limited material damages [99]. Below are the pictures showing the aftermath of the earthquake that occurred in Nepal in 2015 and that of South California in 2014.

(a) Magnitude 7.8 earthquake in Nepal in 2015 [93].

(b) Magnitude 7.8 earthquake in Nepal in 2015 [131].

(c) Magnitude 5.1 earthquake [99].

(d) Magnitude 5.1 earthquake [99].

Figure 1.4: The Earthquake of 7.8 magnitude that occured in Nepal in 2015 [93] and Magnitude 5.1 earthquake in Southern California in 2014 [99].

Seismic waves travel through the Earth as a result of tectonic earthquakes or explosions from nuclear bombs. These waves are of two main kinds, namely, surface waves and body waves, and are distinguished based on the medium and direction of propagation. While body waves travel through the interior layers of the Earth, surface waves travel near the surface of the Earth. The two types of seismic waves are described briefly, and a picture of seismic waves and the Earth structure is shown in Figure 1.5.

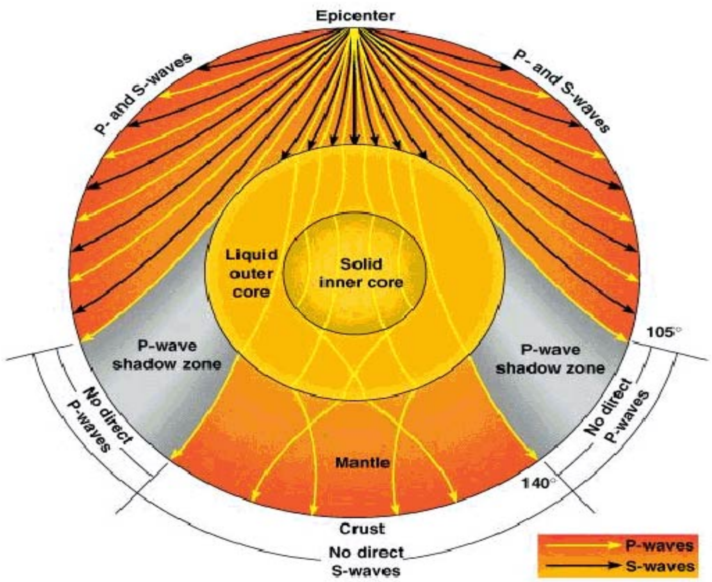

Figure 1.5: Seimic waves and the Earth structure [132].

1.2.1 Body waves

During earthquakes, the waves that propagate through the inner layers of the Earth are referred to as body waves. Body waves arrive first on the seismograph after the earthquake before surface waves, and they have higher frequencies than surface waves. These kinds of waves are categorized as P waves, and S waves, and each wave is identified by the direction and medium in which it travels. P waves and S waves are briefly discussed, and pictures of the nature of propagation of the body waves are shown below.

Primary (P) waves

The primary waves are the first kind of body waves, also called compressional, because of the pulling and pushing they do as they change the volume of the intervening material. They are the swiftest seismic waves because they are the first to be recorded on the seismograph after earthquakes. The exact speed of P waves changes depending on the region of the Earth's interior, from less than 6 km/s in the Earth's crust to about 13.5 km/s in the lower mantle, and 11 km/s through the inner core. Typical velocities of P waves ranges from about 5 km/s to 8 km/s. P waves exist in solids as well as fluids, gases, and plasmas.

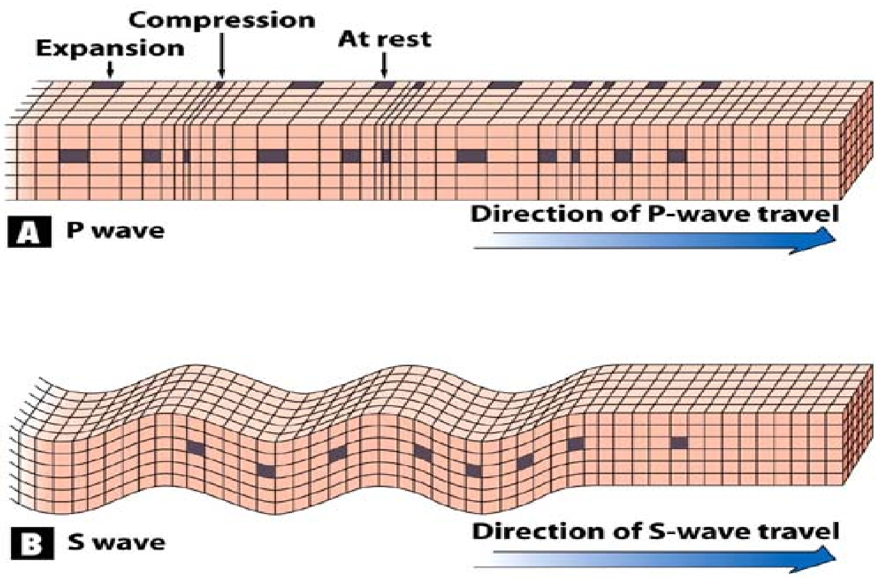

Figure 1.6: Primary and Secondary waves [79].

Secondary (S) waves

The secondary (S) waves are also called shear waves, and propagate only in solids. Because secondary waves travel only in solids, seismologists can conclude that the outer core of the Earth contains fluids. S waves are transverse waves that move particles orthogonal to the direction of the waves, and they have slower velocities than the P waves. Typical velocities of S waves within the Earth range from 3.5 km/s to 6 km/s. Moreover, these velocities depend on the properties of materials through which the wave is propagating.

1.2.2 Surface Waves

Surface waves are the second kind of seismic waves. These are waves where the movement in the medium occurs near the interface or surface. Moreover, the motion of particles is in the horizontal or vertical direction, transverse to the direction of propagation of waves. An elastic surface wave moves freely along the surface and a different medium that has maximum amplitude on the surface (interface) but sharply decays inside the solid medium [121]. Surface waves are sometimes called seismic waves when they propagate freely along the Earth's surface. Surface waves have lower time frequencies as compared to body waves when both propagate through the Earth's crust. They cause the most damage associated with earthquakes even though they arrive after the body waves. Surface waves have frequencies ranging from 1 to 0.1 Hertz. The speed of surface waves is calculated in terms of the wavelength by

$$C_s = f\lambda, \tag{1.11}$$

C_s is the speed of surface wave in km/s, f is the frequency in cycles/s, and λ is the wavelength in km/cycle. Among the numerous kinds of surface waves, Rayleigh and Love waves are the two most studied in literature due to their numerous applications and their destructive nature. Surface waves also have the most significant periods, exhibit the most significant amplitude, and lowest velocities as compared to body waves. Love waves and Rayleigh waves are discussed in what follows with much emphasis on Love waves because Love waves are of greater interest to this work. The picture below depicts an effect of Love surface waves on the railway line.

Figure 1.7: An effect of Love waves after earthquakes [76]

Rayleigh Surface Waves

Rayleigh surface waves are waves that travel both longitudinally and transversely in a vertical plane containing the direction of wave propagation (Figure 1.8). Rayleigh surface waves were initially found as a theoretical solution to the wave equation. Lord Rayleigh [115], in his early work, predicted the existence of this type of surface wave in the linear theory of elasticity framework. The Rayleigh waves are the slowest seismic wave, and they are the most complex of all the seismic waves. Typical Rayleigh wave speed in the metals ranges from 2 km/s to 5 km/s, and 0.05 km/s to 0.3 km/s in the ground. They are also dispersive, so the particular speed at which they travel depends on the wave period and the near-surface geologic structure, and they also decrease in amplitude with depth. Rayleigh waves are classified into three main types, namely, classical, spherical, and cylindrical Rayleigh waves. The classical Rayleigh waves travel near the interface of vacuum and elastic half-space. Spherical and cylindrical Rayleigh waves propagate along the surfaces of elastic bodies with spherical and cylindrical boundaries, respectively [121].

Love Surface waves

Love surface waves have various applications in earthquake engineering and seismology. They are dispersive waves that travel along the interface of the elastic layers and elastic half-spaces with different elastic moduli [119, 121]. Love surface waves propagate horizontally and travel faster in a horizontal plane than in the

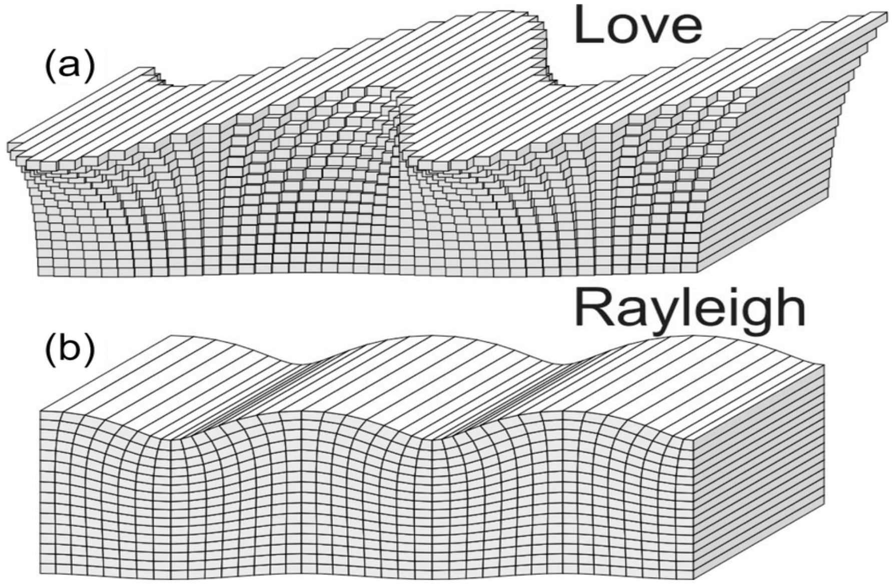

Figure 1.8: Love and Rayleigh waves [28].

vertical direction, and this is termed as seismic anisotropy. Typical Love wave speed in the ground ranges from 2 km/s to 5 km/s. Love waves are formed by the interaction of a free surface and a horizontally polarized SH waves, and they are the fastest surface waves compared to Rayleigh waves since they have higher velocities. Hence in seismology, Love waves arrive first to a seismograph before Rayleigh waves. They can be considered plane waves with low velocities along the surface of the Earth.

Love waves may arise in an isotropic layer overlying an isotropic elastic half-space (when both the layer are in welded contact) if and only if the phase velocities satisfy the condition

$$v_T^2 > v_{\text{Love}} > v_T^1, \tag{1.12}$$

where v_{Love} denotes Love wave velocity, v_T^1 is the shear wave phase velocity in the elastic layer, and v_T^2 is the shear wave phase velocity in the elastic half-space. Equation (1.12) is called the condition of the existence of Love waves. Love waves do not exist if the shear wave phase velocity in the elastic half-space is less than the shear wave phase velocity in the elastic layer ($v_T^2 < v_T^1$). See Section 2.2 below.

Much research has been conducted on both linear and nonlinear Love waves. Love waves in materials with homogeneous initial stress were studied in [46]. (Also see [44, 46, 124] and references therein for details.) The method of separation of variables and Whittaker's functions were employed in [85] to derive a dispersion equation of a Love wave propagating in a piezoelectric layer lying over an inhomogeneous elastic half-space. Dispersion relation of Love waves travelling in a three-layer isotropic elastic half-space was presented in [74]. The area of nonlinear surface waves has also gained notable attention from researchers. The theoretical study of nonlinear elastic waves, in general, was considered in [89]. In [120, 121], a nonlinear Love model based on the Murnaghan model [97] was derived, and the resulting nonlinear equation (considering only cubic nonlinearity) was solved using the method of successive approximations. Several works on nonlinear elastic surface waves can be found in [16, 47, 95, 107, 128, 129]. In what follows, some basic concepts in continuum mechanics are reviewed.

1.3 Nonlinear Elasticity

In this section, some basic definitions, theorems, and concepts in continuum mechanics are reviewed. The equations of motion presented here are given in terms of both actual and reference coordinates.

1.3.1 Basic Notions of the Nonlinear Theory of Elasticity

Continuum solid mechanics is concerned with describing the motion of media represented by physical continua, rather than discrete connected particles. Consider a solid body that occupies at $t = 0$ the spatial region $\overline{\Omega}_0 \subset \mathbb{R}^3$ in the Lagrangian coordinates. Suppose that the open subset $\overline{\Omega}_0$ is connected and has a continuous boundary (having a Lipschitz boundary). The actual region occupied at time t by that body is $\overline{\Omega} \subset \mathbb{R}^3$. The deformation of a body is defined as the change in dimensions of a material body, and it is represented by the transformation $\xi : \overline{\Omega}_0 \to \mathbb{R}^3$ that is invertible, smooth, and preserves orientation. The material (Lagrangian) points are the points in $\overline{\Omega}_0$ represented using Lagrangian coordinates $\mathbf{X} = (X, Y, Z) \in \overline{\Omega}_0$, and the spatial (Eulerian) points are represented Eulerian coordinates $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$;

$$\mathbf{x} = \boldsymbol{\xi}\left(t, \mathbf{X}\right) = \mathbf{X} + \mathbf{u}\left(t, \mathbf{X}\right), \tag{1.13}$$

where $\mathbf{u}$ is the displacement.

The motion of a body in the material coordinates $\overline{\Omega}_0$ consists of a family of configurations which is a function of time t, denoted by $\mathbf{x} = \boldsymbol{\xi}(t, \mathbf{X})$. The velocity of a spatial point $\mathbf{v}(t, \mathbf{X})$ is the derivative of displacement with respect to time.

$$\mathbf{v}(t, \mathbf{X}) = \frac{\partial \boldsymbol{\xi}}{\partial t}(t, \mathbf{X}) = \frac{\partial \mathbf{u}}{\partial t}.$$

The material acceleration $\mathbf{a}(t, \mathbf{X})$ of motion is defined as the derivative of material velocity with respect to time. Thus,

$$\mathbf{a}(t, \mathbf{X}) = \frac{\partial \mathbf{v}}{\partial t}(t, \mathbf{X}) = \frac{\partial^2 \mathbf{x}}{\partial t^2}(t, \mathbf{X}).$$

The material derivative or total derivative is the derivative of a scalar quantity $\Phi(t, \vec{x})$ with respect to t for a fixed $\mathbf{X}$ is represented as

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + (\vec{v} \cdot \nabla)\Phi,$$

where ∇ is the gradient with respect to the Eulerian coordinates $\vec{x}$. The deformation gradient for the motion described by (1.13) is defined by a Jacobian matrix of the motion:

$$\mathbf{F}(t, \mathbf{X}) = \mathrm{grad}_{\mathbf{X}} \boldsymbol{\xi} = \nabla_{\mathbf{X}} \boldsymbol{\xi}(t, \mathbf{X}), \qquad F^i_j = \frac{\partial \xi^i}{\partial X^j} = F_{ij}. \tag{1.14}$$

The components of the deformation gradient $\mathbf{F}$ are partial derivatives of $\boldsymbol{\xi}$. $\mathbf{F}$ satisfies the orientation-preserving condition $J = \det \mathbf{F} > 0$, for all material points in the Lagrangian configuration (i.e., $\mathbf{X} \in \overline{\Omega}_0$).

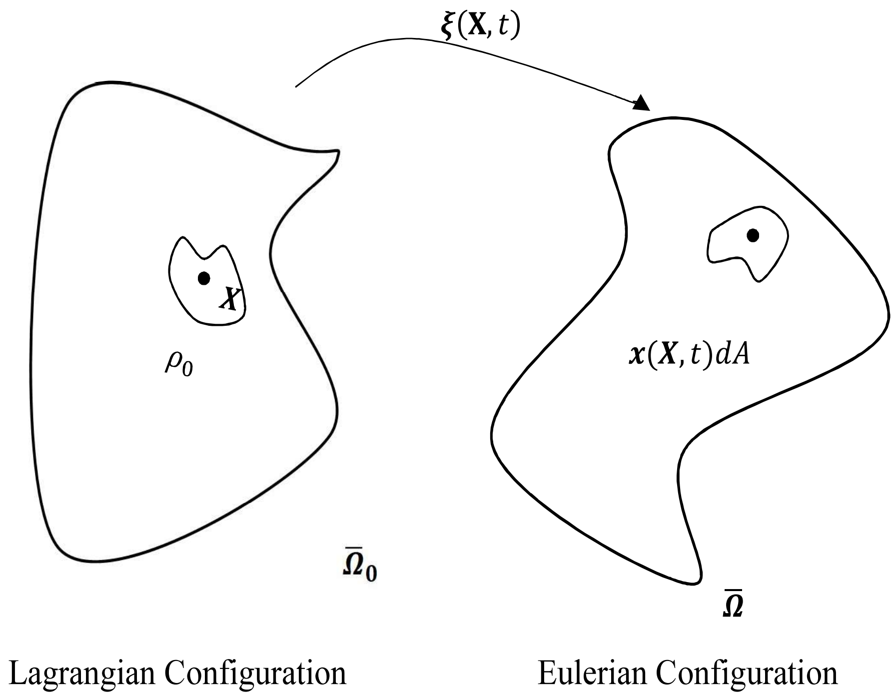

Figure 1.9: Reference (Lagrangian) and Actual (Eulerian) Configurations.

1.4 Equations of Motion in Continuum Mechanics

In this section, systems of PDEs governing the motion of solids are derived. These PDEs are usually written in the actual coordinates or sometimes with respect to the reference configuration. PDEs in the reference configuration would be of interest in our study, although systems of PDEs in both material and spatial frames would be presented in subsequent sections. The governing equations in both linear and nonlinear elasticity theory are derived (in reference configuration) from the law of balance of conservation of linear momentum, the law of mass conservation, the balance of angular momentum, and the balance of energy.

1.4.1 Equations of Motion in the Actual Coordinates

We now present PDEs describing the solid motions with respect to the actual configuration. These PDEs are derived from the conservation laws (mass, linear, and angular momenta) and Cauchy stress principle. Some fundamental theorems in the study of balance laws are the divergence (Gauss- Ostrogradski) theorem and the transport theorem.

The Divergence Theorem

Let $G \in \mathbb{R}^3$ be a connected bounded domain with piecewise-smooth boundary, and S the boundary surface of G. Let $\mathbf{E}$ be a vector field whose components have continuous first-order partial derivatives. Then,

$$\iint_{\mathbf{S}} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \iiint_{\mathbf{G}} \mathrm{div}\, \mathbf{E}\, \mathrm{d}V. \tag{1.15}$$

Leibniz Rule

Let a spatial quantity $\phi(t, \mathbf{x})$ (a function of time and position) be C^1 for $\mathbf{x} \in B \subset \overline{\Omega}$. Then one has

$$\frac{d}{dt}\iiint_{B} \phi\,(t,\mathbf{x})\ \mathrm{d}\mathbf{x} = \iiint_{B} \left(\frac{d}{dt}\phi\,(t,\mathbf{x}) + \phi\ \mathrm{div}_{(\mathbf{x})}\mathbf{v}\right)\ \mathrm{d}\mathbf{x} = \iiint_{B} \left(\frac{\partial \phi}{\partial t} +\ \mathrm{div}_{(\mathbf{x})}(\phi\mathbf{v})\right)\ \mathrm{d}\mathbf{x}, \tag{1.16}$$

where $\mathbf{v}(t, \mathbf{x})$ is the spatial velocity, $B \subset \overline{\Omega}$ is an open set with smooth boundary, $\mathrm{div}_{\mathbf{x}}$ is the divergence with respect to $\mathbf{x}$, $d\mathbf{x}$ represents the infinitesimal volume element, d/dt is the material derivative and $d\phi/dt = \partial\phi/\partial t + \nabla\phi \cdot v$ is the material derivative of ϕ. Here the spatial integral and the time derivative can be interchanged. The relation (1.16) is also called *the transport theorem.*

The Balance Law of Mass Conservation

For an open set $B \subset \overline{\Omega}$ having a piecewise-smooth boundary, the classical equation of conservation of mass is given by

$$0 = \frac{d}{dt}\iiint_{B} \rho(t,\mathbf{x})\ \mathrm{d}\mathbf{x}, \tag{1.17}$$

where $\rho(t, \mathbf{x})$ is the mass density in the actual configuration $\overline{\Omega}$. Applying the transport theorem (1.16) to (1.17), the law of conservation of mass becomes

$$0 = \iiint_{B} \left(\frac{\partial \rho}{\partial t} +\ \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v})\right)\mathrm{d}\mathbf{x}. \tag{1.18}$$

Here, since (1.18) holds for every $B \subset \overline{\Omega}$, the integrand must be equal to zero identically. Thus the law of mass conservation in the actual configuration frame has a PDE form

$$\frac{\partial \rho}{\partial t} = -\ \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v}),\ \ \mathbf{x} \in \overline{\Omega}. \tag{1.19}$$

Before we consider the next conservation law, we will define the Cauchy stress vector.

Cauchy Stress

The stress principle of Cauchy states that, for any smooth and closed surface $S^1 \subset \overline{\Omega}$, there exists a stress vector field $\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n})$ of a deformed solid $\overline{\Omega}$ at time t, where $\mathbf{x}$ represents the spatial points, the unit normal vector is represented by $\mathbf{n}$, and t denotes time. Thus $\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n})$ depends on time, $\mathbf{x}$ and $\mathbf{n}$ and also depends implicitly on $\boldsymbol{\xi}(\mathbf{X}, t)$. The Cauchy stress vector $\mathbf{t} = \mathbf{t}(t, \mathbf{x}, \mathbf{n})$ is defined in the actual framework as the force per unit area employed on a surface element with unit normal vector $\mathbf{n}$ [29, 87, 121]: i. e., $d\mathbf{f} = \mathbf{t}(t, \mathbf{x}, \mathbf{n})da$.

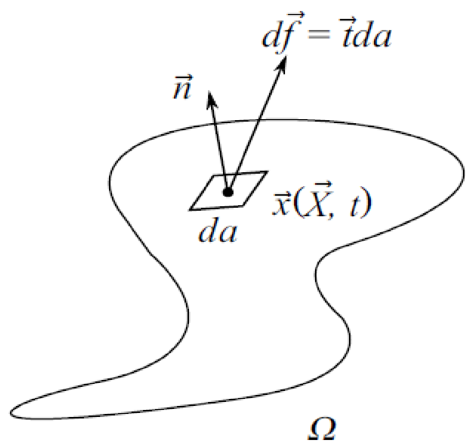

Figure 1.10: Force acting on an elementary area da in the actual configuration [70].

The Balance Law of Momentum

This balance law is based on Newton's second law of motion. The law takes into consideration the presence of external forces. Let $\mathbf{D}_f$ denote the acceleration due to body forces. Then the second law of Newton states

$$\frac{d}{dt}\iiint\limits_{B} \rho\mathbf{v}\ \mathrm{d}\mathbf{x} = \iint\limits_{\partial B} \mathbf{t}(\mathbf{x}, t, \mathbf{n}) \cdot \mathrm{d}\mathbf{S} + \iiint\limits_{B} \rho\ \mathbf{D}_f\ \mathrm{d}\mathbf{x}, \tag{1.20}$$

where the material derivative (total derivative) is represented by d/dt and an open set $B \subset \overline{\Omega}$ with smooth boundary ∂B. The equation (1.20) is represented in the form where the Cauchy stress vector depends explicitly on the normal vector $\mathbf{n}$ and this helps derive the PDE describing the conservation of linear momentum. The following well-known theorem concerning the Cauchy stress tensor holds.

Theorem 1.4.1 *Let $\boldsymbol{\xi}(x,t)$ be C^1. If the Cauchy stress vector $\mathbf{t}(\mathbf{x}, \mathbf{n}, t)$ is a continuously differentiable function in terms of $\mathbf{x} \in \overline{\Omega}$ for any $\mathbf{n} \in S^1$ and $\mathbf{t}(\mathbf{x}, \mathbf{n}, t)$ is a continuous function of its arguments, and assuming the balance law of momentum holds, then there exists a tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$ (Cauchy symmetric stress tensor) which is unique, such that*

$$\boldsymbol{\sigma}(\mathbf{x}, t)\ \mathbf{n} = \mathbf{t}(\mathbf{x}, \mathbf{n}, t), \quad \forall\ \mathbf{n} \in S^1,\ \ \mathbf{x} \in \overline{\Omega}. \tag{1.21}$$

The proof of the theorem can be found in [87]. Equation (1.20) is therefore written after applying the equation (1.21) as

$$\frac{d}{dt}\iiint\limits_{B} \rho\mathbf{v}\ \mathrm{d}\mathbf{x} = \iint\limits_{\partial B} \boldsymbol{\sigma}\cdot\mathbf{n}\ \mathrm{d}S + \iiint\limits_{B} \rho\ \mathbf{D}_f\ \mathrm{d}\mathbf{x}. \tag{1.22}$$

Now applying the divergence theorem (1.15) to (1.22) yields

$$\frac{d}{dt}\iiint\limits_{B} \rho\mathbf{v}\ \mathrm{d}\mathbf{x} = \iiint\limits_{B} \Big(\rho\mathbf{D}_f + \mathrm{div}_{(\mathbf{x})}\boldsymbol{\sigma}\Big)\mathrm{d}\mathbf{x}. \tag{1.23}$$

By applying the transport theorem (1.16) and inserting the balance law of conservation of mass (1.20) into (1.23) yield

$$0 = \iiint\limits_{B} \Big(\mathrm{div}_{\mathbf{x}}\boldsymbol{\sigma} + \rho\,\mathbf{D}_f - \rho\frac{d\mathbf{v}}{dt}\Big)\mathrm{d}\mathbf{x}. \tag{1.24}$$

The equation (1.24) must hold for any subset $B \subset \overline{\Omega}$. We note that all terms are supposed to be continuously differentiable and that the integrand must be zero identically. The balance law of linear momentum is thus written below as

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{D}_f + \mathrm{div}_{(\mathbf{x})}\boldsymbol{\sigma}. \tag{1.25}$$

Equation (1.25) in component form is denoted by

$$\rho\frac{dv_i}{dt} = \sigma_{ik,k} + D_{f\,i}, \qquad i = 1, 2, 3. \tag{1.26}$$

where $D_{f\,i}$ denotes the body forces and $\sigma_{ik,k}$ represents the divergence of $\boldsymbol{\sigma}$ (Cauchy stress) with reference to the Eulerian coordinates x_k (i. e., $\sigma_{ik,k} = \mathrm{div}_{(\mathbf{x})}\sigma_{ik} = \partial\sigma_{ik}/\partial x_k$ where summation in k is assumed).

The Balance Law of Angular Momentum

This law is presented in an integral form as

$$\frac{d}{dt}\iiint\limits_{B} (\mathbf{x}\times\rho\mathbf{v})\ \mathrm{d}\mathbf{x} = \iint\limits_{\partial B} \big(\mathbf{x}\times\mathbf{t}(\mathbf{x}, t, \mathbf{n})\big)\cdot\mathrm{d}\mathbf{S} + \iiint\limits_{B} \big(\mathbf{x}\times\rho\,\mathbf{D}_f\big)\,\mathrm{d}\mathbf{x}. \tag{1.27}$$

We state a theorem that shows the equivalence of (1.27) to the symmetric nature of the Cauchy stress tensor.

Theorem 1.4.2 *Suppose the balance laws of mass (1.19) and momentum (1.25) hold. Then the conservation of angular momentum holds if and only if* $\boldsymbol{\sigma}$ *is symmetric.*

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \ \ \sigma_{ik} = \sigma_{ki}, \ \ i, k = 1, 2, 3. \tag{1.28}$$

The above theorem and its proof can be found in [87]. Thus from henceforth, we consider the conservation of the angular momentum in the actual coordinates as in equation (1.28).

Equations of Motion in the Eulerian Configuration

In summary, the equations of motion in the actual configuration consist of the conservation of mass (1.19), the conservation of linear momentum (1.25) and the conservation of angular momentum (1.28):

$$0 = \frac{\partial\rho}{\partial t} + \ \mathrm{div}_{(\mathbf{x})}(\rho\mathbf{v}), \tag{1.29a}$$

$$\rho\frac{d\mathbf{v}}{dt} = \mathrm{div}_{(\mathbf{x})}\boldsymbol{\sigma} + \rho\mathbf{D}_f, \tag{1.29b}$$

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}. \tag{1.29c}$$

Here the dependent variables are the mass density $\rho(t, \mathbf{x})$ and the spatial velocity $\mathbf{v}(t, \mathbf{x})$. The stress $\boldsymbol{\sigma}$ is obtained from the stress-strain relationship. In what follows, we present the balance laws in the reference coordinates.

1.4.2 Balance Laws in the Reference Configuration

The equations of motion obtained in the actual configuration are now written in terms of the material coordinates. Thus, equations (1.29) are re-written interms of material points $\mathbf{X}$ in the reference confguration [15, 64, 70, 80, 87].

The Balance Law of Mass Conservation

Here, we consider the mass density of the Lagrangian configuration as the density of the initial moment $\rho_0(\mathbf{X})$ which is independent of time. Therefore the conservation of mass can be written equivalently in the material picture as

$$\iiint_{B_0} \rho_0(\mathbf{X})\mathrm{d}\mathbf{X} = \iiint_{B} \rho(t, \mathbf{x})\mathrm{d}\mathbf{x}, \tag{1.30}$$

where $B \in \overline{\Omega}_0$ is the domain such that $\boldsymbol{\xi} : B_0 \to B$. The equation (1.30) is re-written in another way through change of variables

$$\iiint_{B_0} J(t, \mathbf{X})\rho(\mathbf{x}(t, \mathbf{X}), t)\mathrm{d}\mathbf{X} = \iiint_{B} \rho(\mathbf{x}, t)\mathrm{d}\mathbf{x}, \tag{1.31}$$

where $d\mathbf{x} = J(t, \mathbf{X})\mathrm{d}\mathbf{X}$, $J = \det \mathbf{F}$ is the determinant of the deformation gradient of the change of variable. Comparing equations (1.30) and (1.31) yields

$$\iiint_{B_0} J(t, \mathbf{X})\rho(t, \mathbf{x}(t, \mathbf{X}))\mathrm{d}\mathbf{X} = \iiint_{B_0} \rho_0(\mathbf{X})\mathrm{d}\mathbf{X},$$

$$0 = \iiint_{B_0} \Big(J(t, \mathbf{X})\rho(t, \mathbf{x}(t, \mathbf{X})) - \rho_0(\mathbf{X}) \Big)\mathrm{d}\mathbf{X}. \tag{1.32}$$

Equation (1.32) holds for all open sets $B_0 \in \overline{\Omega}_0$, since $\mathbf{X}$ is an arbitrary region, the integrand must vanish everywhere so that

$$\rho_0 = J\rho. \tag{1.33}$$

We could see that the material form of the continuity equation (1.33) is an algebraic equation as compared to the PDE (1.19). However, the two are equivalent, and one can be deduced from the other.

Stresses and Forces

The force per unit area, acting on the area element with normal vector $\mathbf{n}$ of the solid body is given in the Eulerian (actual) configuration by $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$, where $\mathbf{n}$ is a unit normal, and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ is a symmetric Cauchy stress tensor. The first Piola-Kirchhoff stress vector $\mathbf{T} = \mathbf{P}\mathbf{N}$ is the force d$\mathbf{f}$ per unit undeformed area d$\mathbf{A}$ with unit normal vector (see the Figure 1.11) [38, 87]. The force acting on a surface element S_0 in the Lagrangian (reference) configuration is given by the stress vector $\mathbf{T} = \mathbf{P}\mathbf{N}$, where $\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$ denotes the asymmetric first Piola-Kirchhoff tensor, related to the Cauchy stress through

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}. \tag{1.34}$$

where $\mathbf{F}^{-T}$ represents the transpose of the inverse of the deformation gradient $\mathbf{F}$. Force is preserved during the coordinate transformation between the actual and reference frames. Thus, $\mathbf{T}\,\mathrm{d}\mathbf{A} = \mathbf{t}\,\mathrm{d}\mathbf{a}$, here d$\mathbf{A}$ denotes the infinitesimal area in the Lagrangian configuration and da represents the infinitesimal area in the Eulerian configuration.

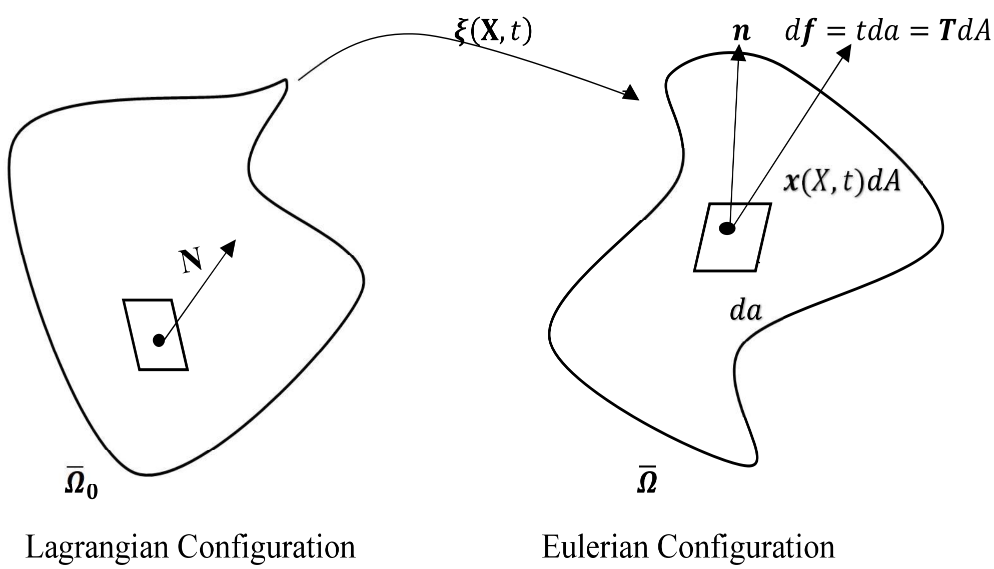

Figure 1.11: First Piola-Kirchhoff stress vector

In actual configuration, Eulerian variables are used, and the stresses in the reference framework have a similar notion to that of the Cauchy stress vector. The second Piola-Kirchhoff stress vector is obtained by using the stress vector $\mathbf{T} = \boldsymbol{P}\mathbf{N}$ and by setting $\Xi = \mathbf{F}^{-1}\mathbf{T}$ such that $\Xi = \mathbf{S}\mathbf{N}$, where $\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{P}$. Therefore, the second Piola-Kirchhoff stress tensor is given in terms of the inverse of the deformation gradient $\mathbf{F}^{-1}$ and the first Piola-Kirchhoff stress tensor $\mathbf{P}$ by

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}. \tag{1.35}$$

Here the divergence of the stress tensors is related by the Piola transform as follows [38, 70, 87].

Theorem 1.4.3 *Let* $\boldsymbol{P}$ *and* $\boldsymbol{\sigma}$ *be the first Piola-Kirchhoff stress and Cauchy stress tensors respectively associated to the Piola transform (1.34). Then*

$$\mathrm{div}_{\mathbf{X}}\mathbf{P} = J\ \mathrm{div}_{\mathbf{x}}\boldsymbol{\sigma}, \tag{1.36}$$

where $\mathrm{div}_{\mathbf{X}}$ *is the divergence with reference to the material point* $\mathbf{X}$ *and* $\mathrm{div}_{\mathbf{x}}$ *is the divergence in terms of the spatial point* $\mathbf{x}$.

The proof of the theorem can be found in [38].

The Balance Law of Momentum

The law of balance of linear momentum in the actual coordinates (1.25) can be written with respect to the reference coordinates. This is achieved by substituting $d\mathbf{v}/dt = \mathbf{v}_t \equiv \mathbf{x}_{tt}$, (1.33), (1.34), (1.36) into (1.29c) and simplifying gives

$$\rho_0 \frac{\partial^2 \mathbf{x}}{\partial t^2} = \mathrm{div}_{\mathbf{X}}\mathbf{P} + \rho_0 \mathbf{D}_f, \tag{1.37}$$

where $\mathbf{D}_f = \mathbf{D}_f(t, \mathbf{X})$ is the total body force per unit mass (the acceleration) with respect to the material coordinates, and the divergence of $\mathbf{P}$ with respect to the material coordinates is $(\mathrm{div}_{\mathbf{X}}\mathbf{P})^i = \partial P^{ij}/\partial X^j$. The form (1.37) of Cauchy's equation of motion is suitable for studying Hamiltonian and variational principles [87].

The Balance Law of Angular Momentum

The balance of angular momentum can be expressed in the reference configuration by applying the Piola transform (1.34) to (1.29c) provides

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T. \tag{1.38}$$

Equations of Motion in the Lagrangian Configuration

In summary, all equations of motion describing motion of a continuous object with respect to the reference configuration are given by

$$\rho_0 \frac{\partial^2 \mathbf{x}}{\partial t^2} = \mathrm{div}_{\mathbf{X}}\mathbf{P} + \rho_0 \mathbf{D}_f, \quad \mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T, \quad \rho_0 = J\rho. \tag{1.39}$$

We note that the dependent variables are the actual coordinates $\mathbf{x}(\mathbf{X}, t)$ and the independent variables are the reference coordinates $\mathbf{X}$ and time t. There are six governing equations of the motion from the balance law of angular momentum and the balance law of momentum. Also, the stress tensor $\mathbf{P}$ depends on the motion of the solid, for which the specific relationship is determined from properties of the material under consideration.

1.4.3 Equations of Motion for Incompressible models

The full system of equations of motion of an incompressible elastic material is given by

$$\rho_0 \frac{\partial^2 \mathbf{x}}{\partial t^2} = \mathrm{div}_{\mathbf{X}} \mathbf{P} + \rho_0 \mathbf{D}_f, \tag{1.40a}$$

$$J = 1, \tag{1.40b}$$

where $J = 1$ is the incompressibility condition, $\mathbf{D}_f = \mathbf{D}_f(t, \mathbf{X})$ is the total body force per unit mass and the divergence of $\mathbf{P}$ given with respect to the material coordinates is given by $(\mathrm{div}_{\mathbf{X}} \mathbf{P})^i = \partial P^{ij} / \partial X^j$. For incompressible models, there is one extra unknown (the hydrostatic pressure $p(t, X)$ that is a part of the form of the tensor $\mathbf{P}$), so that there are four PDEs and four unknowns. We discuss the particular forms of the stress-strain relations for elastic solids in the next section.

1.5 Constitutive Laws for Elastic Materials

An important property when it comes to studying the PDE systems (1.29) and (1.39) is the relationship existing between the strain and stress of the elastic materials, termed as constitutive relation. We now present in detail examples of such constitutive models, the relationship between the strain and stress for elastic and hyperelastic materials, and the first Piola-Kirchhoff stress tensor. We discuss the isotropy, material frame-indifference, and anisotropy of materials. Also, we study compressible, incompressible, and homogeneous materials. Afterwards, we present several examples of strain energy densities for both isotropic and anisotropic hyperelastic materials that appear in literature.

1.5.1 Elasticity and Hyperelasticity

The generalized Hooke's law which provides a linear relation between stress and strain of elastic solids is well known in continuum mechanics (the stress is a function of the strain (deformation)). Any material in which the first Piola-Kirchhoff stress $\mathbf{P}$ can be written as a function of material points $\mathbf{X} \in \overline{\Omega}$ and the deformation gradient (1.14) such that, $\mathbf{P}(t, \mathbf{X}) = \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{F}(t, \mathbf{X}))$ is termed as an elastic material [87]. A material is hyperelastic if the potential energy is an analytic function of the strain tensor components with reference to the natural state [121]. A material is said to be hyperelastic if there exists a strain energy function $W(\mathbf{X}, \mathbf{F})$ such that

$$\mathbf{P}(t, \mathbf{X}) = \frac{\partial W}{\partial \mathbf{F}}(t, \mathbf{X}), \quad P^{ij} = \frac{\partial W}{\partial F_{ij}}(t, \mathbf{X}). \tag{1.41}$$

Here, W is called the constitutive function [38]. Marsden and Hughes [87] also defined a hyperelastic material in terms of the strain energy function W as

$$\mathbf{P}(t, \mathbf{X}) = \rho_0(\mathbf{X}) \frac{\partial W}{\partial \mathbf{F}}(t, \mathbf{X}), \quad P^{ij} = \rho_0(\mathbf{X}) \frac{\partial W}{\partial F_{ij}}(t, \mathbf{X}). \tag{1.42}$$

Equation (1.42) differs from equation (1.41) by a multiplier $\rho_0(\mathbf{X})$ (the mass density). Also the constitutive function in (1.41) is mass energy density [J/kg] and the constitutive function in (1.42) is a volumetric energy density [J/m^3]. In this work we will use the constitutive function given in terms of the mass energy density (1.41).

1.5.2 Material Properties of Isotropy, Material Frame Indifference, Incompressibility and Homogeneity

Here we present definitions of some relevant class of materials in view of the dependence of the strain energy density W on the gradient of deformation $\mathbf{F}$ (material frame indifference). The principle of frame indifference states that the response of the material is independent of the observer [38]. In the theory of elasticity, the observable quantity is the stress. Material frame indifference (objectivity) simply mean that if we view the same configuration from a rotated point of view, then the stress transforms by the same rotation [87]. (i. e. $\mathbf{P}(\mathbf{X}, \mathbf{QF}) = \mathbf{QP}(\mathbf{X}, \mathbf{F})$ for any orthogonal matrix $\mathbf{Q} \in SO(n)$). The frame indifference also implies the conservation of angular momentum [87].

The materials defined here are isotropic materials, incompressible materials, homogeneous materials, and homogeneous hyperelastic materials. *Isotropic materials* have elastic material properties being the same in all directions [38]. Thus an isotropic elastic material is any material in which at a material point $\mathbf{X} \in \overline{\Omega}_0$,

$$\mathbf{P}(\mathbf{X}, \mathbf{FQ}) = \mathbf{P}(\mathbf{X}, \mathbf{F})\mathbf{Q}. \tag{1.43}$$

That is, it is a material which is invariant under rotations. For *hyperelastic* materials,

$$W(\mathbf{X}, \mathbf{FQ}) = W(\mathbf{X}, \mathbf{F}), \;\; \forall \mathbf{X} \in \overline{\Omega}_0. \tag{1.44}$$

We now define an important class of materials called *homogeneous* materials. If the first Piola-Kirchhoff stress $\mathbf{P}$ of an elastic material does not depend explicitly on the material points $\mathbf{X} \in \overline{\Omega}_0$, then such material is homogeneous.

$$\mathbf{P}(\mathbf{X}, \mathbf{F}) = \mathbf{P}(\mathbf{F}). \tag{1.45}$$

Also, a material is homogeneous hyperelastic if

$$W(\mathbf{X}, \mathbf{F}) = W(\mathbf{F}). \tag{1.46}$$

An *incompressible* elastic material is a material for which there is no change in volume in the deformed body. That is, for all open set $B_0 \in \overline{\Omega}$ with smooth boundary and B such that $\boldsymbol{\xi} : B_0 \to B$,

$$\iiint_B \mathrm{d}\mathbf{x} = \iiint_{B_0} \mathrm{d}\mathbf{X}. \tag{1.47}$$

Here, by change of variables, it is observed that

$$\iiint_B \mathrm{d}\mathbf{x} = \iiint_{B_0} J \, \mathrm{d}\mathbf{X}. \tag{1.48}$$

Incompressible materials are easily identified when the determinant of the deformation gradient is unity. Thus, the incompressibility condition (1.47) is equivalent to

$$J = 1, \tag{1.49}$$

where J is the determinant of the deformation gradient. Indeed, comparing (1.47) and (1.48), gives $J = 1$. The first Piola-Kirchhoff assumes a different form for incompressible materials, and it is given by

$$\mathbf{P}(t,\mathbf{X}) = \mathbf{P}(\mathbf{X},\mathbf{F}(t,\mathbf{X})) - p(t,\mathbf{X})\mathbf{F}^{-T}(t,\mathbf{X}), \quad P^{ij} = \frac{\partial W}{\partial F_{ij}} - p\left(F^{-1}\right)^{ji}, \tag{1.50}$$

where $p = p(t,\mathbf{X})$ is the scalar *hydrostatic pressure* [38]. For compressible materials, the first Piola-Kirchhoff stress tensor is given by

$$\mathbf{P}(t,\mathbf{X}) = \mathbf{P}(\mathbf{X},\mathbf{F}(t,\mathbf{X})). \tag{1.51}$$

1.5.3 Hyperelastic Materials

A *hyperelastic material* is a material for which the potential energy W is an analytic function of the strain tensor components with reference to the natural state (1.46). The stress-strain relationship follows from the strain energy density function. For these types of elastic materials, the forms of the first and second Piola-Kirchhoff stress tensors $\mathbf{P}$ and $\mathbf{S}$ follow from a postulated form of a strain energy function W. Isotropic hyperelastic materials normally assume the existence of a volumetric strain energy function $W = W(\mathbf{F})$ and the mass strain energy density $\rho_0 W$ both in the reference configuration. Thus, the isotropic model (material properties are the same in all directions) have the strain energy density of the form

$$W = W_{\text{iso}}\left(I_1, I_2, I_3\right), \tag{1.52}$$

where I_1, I_2, I_3 are the principal invariants. The first Piola-Kirchhoff stress tensor can be written in two ways. For nearly incompressible models, the first Piola-Kirchhoff stress tensor is related to the scalar valued stored energy function W through (1.42) or (1.41). Here, the mass density $\rho_0(\mathbf{X})$ which is the only difference between (1.42) and (1.41) does not depend on time in the Lagrangian configuration. The actual density in the Eulerian configuration $\rho = \rho(t,\mathbf{X})$ depends on time t and is given by $J\rho(t,\mathbf{X}) = \rho_0(\mathbf{X})$. For incompressible models, the first and second Piola-Kirchhoff tensors are obtained from the stored energy function through the formulas (1.50) and

$$\mathbf{S} = 2\rho_0 \frac{\partial W}{\partial \mathbf{C}} - p\,\mathbf{C}^{-1}, \quad S^{ij} = 2\rho_0 \frac{\partial W}{\partial C_{ij}} - p\left(C^{-1}\right)^{ij}, \tag{1.53}$$

where $p = p\left(t,\mathbf{X}\right)$ is the hydrostatic pressure. From (1.53), $\mathbf{C}$ is known to be symmetric and (1.53) is understood as

$$\frac{\partial W}{\partial \mathbf{C}} = \frac{1}{2}\Big(\frac{\partial W}{\partial \mathbf{C}} + \frac{\partial W}{\partial \mathbf{C}^T}\Big). \tag{1.54}$$

If $W = W(\mathbf{C})$, which means the strain energy density W is given in terms of the right Cauchy-Green tensor $\mathbf{C}$ (defined below in (1.55)), then W can be symmetrized by using the expression $C_{qr} \rightarrow \dfrac{1}{2}\big(C_{qr} + C_{rq}\big)$.

1.5.4 Constitutive Models in Isotropic Hyperelasticity

For an isotropic homogeneous hyperelastic material, the strain energy density is written as a function of the principal invariants of the Cauchy-Green tensors. Principal invariants are the coefficients of the characteristic polynomial of a second-order Cauchy-Green tensor. Based on these principal invariants, various constitutive models of hyperelastic materials have been studied in the literature. Constitutive relation for an isotropic homogeneous hyperelastic material is given in terms of the strain energy density $W = W(I_1, I_2, I_3)$ where I_1, I_2, and I_3 are the principal invariants of the left and right Cauchy-Green strain tensors $\mathbf{B}$ and $\mathbf{C}$. The left and right Cauchy-Green strain tensors are derived from the polar decomposition of the gradient of deformation $\mathbf{F}$. Since $\mathbf{F}$ is invertible, we denote the polar decomposition by $\mathbf{F} = \mathbf{VQ} = \mathbf{QU}$, where $\mathbf{V}$ and $\mathbf{U}$ are the left and right stretch tensors respectively which are symmetric positive-definite matrices, and $\mathbf{Q}$ is a proper orthogonal matrix (that is, $\det(\mathbf{Q}) = 1$) [45]. The eigenvalues of the right stretch tensor $\mathbf{U}$ are called the principal stretches. The left Cauchy-Green tensor $\mathbf{B} = \mathbf{FF}^T = \mathbf{V}^2$ [143]. These Cauchy-Green tensors $\mathbf{B}$ and $\mathbf{C}$ are defined by

$$\mathbf{B} = \mathbf{FF}^T, \quad B^{ij} = F^i_l F^j_l, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad C^{ij} = F^l_i F^l_j. \tag{1.55}$$

The strain tensors $\mathbf{B}$ and $\mathbf{C}$ have the same positive eigenvalues given by the squares of the principal stretches $\lambda_1,\ \lambda_2,$ and λ_3. The three principal invariants I_1, I_2, and I_3 are given by

$$I_1 = \mathrm{Tr}\,\mathbf{B} = F^i_l F^j_l, \tag{1.56a}$$

$$I_2 = \frac{1}{2}\left[I_1^2 - \mathrm{Tr}(\mathbf{B}^2)\right] = \frac{1}{2}\left(I_1^2 - B^{li}B^{il}\right), \tag{1.56b}$$

$$I_3 = \det\mathbf{B} = J^2, \tag{1.56c}$$

where Tr abbreviates trace (trace operator). The invariants in equations (1.56) can also be written using the principal stretches, and thus the above equations (1.56) can be re-written as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{1.57a}$$

$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \tag{1.57b}$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{1.57c}$$

In (1.56), $\mathbf{B}$ and $\mathbf{C}$ can be interchanged since they both have the same positive eigenvalues. For incompressible models, where deformations are volume-preserving, the condition (1.49) holds, and the formula for the strain energy density is written in terms of only two principals invariants of the Cauchy-Green strain tensors, that is, $W = W(I_1, I_2)$. In the natural state where all displacements are absent then $\mathbf{x} = \mathbf{X}$, and the first two principal invariants $I_1 = I_2 = 3$. For non-prestressed configurations, the strain energy density function $W(I_1, I_2)$ must satisfy the differentiability condition

$$\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2} = 0. \tag{1.58}$$

The condition (1.58) may not hold in situations where prestressed configurations are allowed. (See [35] and references therein for details.) In multiple elasticity problems, the Lagrangian (reference) configuration $\bar{\Omega}_0$ has zero displacement which implies zero stress. In other words, the Lagrangian configuration is a natural state. In such case, the constitutive equation $W = W(\mathbf{F})$ to be used should be compatible with natural state in order for a well-posed boundary value problem to be formulated. Thus, when $\mathbf{x} = \mathbf{X}$, $\mathbf{F} = \mathbf{I}$, $J = 1$, and the Cauchy stress $\boldsymbol{\sigma}$ should vanish, (i. e., $\boldsymbol{\sigma} = 0$). Neo-Hookean and the Mooney-Rivlin constitutive models do not correspond to a natural state [35] and also do not satisfy condition (1.58).

Nearly Incompressible Materials

According to [4, 101, 102], the total deformation of a nearly incompressible material is decomposed into two parts, namely, the volumetric and the isochoric parts where

$$\bar{\mathbf{F}} = \left(J^{-1/3}\right)\mathbf{F}, \tag{1.59}$$

and $\bar{\mathbf{F}}$ is the isochoric part of the deformation which has determinant one (det $\bar{\mathbf{F}} = 1$). The left and right Cauchy Green tensors for the isochoric part are given as

$$\bar{\mathbf{B}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^T, \quad \bar{B}^{ij} = \bar{F}^i_l \bar{F}^j_l, \qquad \bar{\mathbf{C}} = \bar{\mathbf{F}}^T\bar{\mathbf{F}}, \quad C^{ij} = \bar{F}^l_i \bar{F}^l_j. \tag{1.60}$$

The three principal invariants associated with the isochoric part are thus given below as

$$\bar{I}_1 = \mathrm{Tr}\ \bar{\mathbf{B}}, \quad \bar{I}_2 = \frac{1}{2}\left[\left(\mathrm{Tr}\ \bar{\mathbf{B}}\right)^2 - \mathrm{Tr}\ \left(\bar{\mathbf{B}}\right)^2\right], \quad \bar{I}_3 \equiv 1. \tag{1.61}$$

Also, the invariants in (1.61) can be expressed in terms of the modified principal stretches $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ as

$$\bar{I}_1 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2, \quad \bar{I}_2 = \frac{1}{2}\left[\bar{\lambda}_1^{-2} + \bar{\lambda}_2^{-2} + \bar{\lambda}_3^{-2}\right], \quad \bar{I}_3 = \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 \equiv 1. \tag{1.62}$$

where $\bar{\lambda}_1 = \lambda_1 J^{-1/3}$, $\bar{\lambda}_2 = \lambda_2 J^{-1/3}$, and $\bar{\lambda}_3 = \lambda_3 J^{-1/3}$.

For nearly incompressible materials where $J \neq 1$, the strain energy density of an isotropic homogeneous hyperelastic material is given by [26, 64, 78, 141]

$$W = W(\bar{I}_1, \bar{I}_2, \bar{I}_3), \tag{1.63}$$

where a new set of invariants (1.62) can be written as

$$\bar{I}_1 = J^{-2/3} I_1, \quad \bar{I}_2 = J^{-4/3} I_2, \quad \bar{I}_3 = J. \tag{1.64}$$

Here, $(J)^{-1} = 1/J$, and these alternative invariants (1.64) are used to resolve computational challenges that arise in numerical simulations of nearly incompressible materials [23, 24, 64, 67, 78, 141].

1.5.5 Equations of Motion for Frame-Indifferent, Isotropic, and Homogeneous Hyperelastic Materials

The equations of motion of homogeneous hyperelastic materials in the reference configuration where the unknown quantities to be determined are positions $\mathbf{x} = \boldsymbol{\xi}(t, \mathbf{X})$ of materials coordinates $\mathbf{X}$ are given by

$$\rho_0 \frac{\partial^2 \mathbf{x}}{\partial t^2} = \text{div}_{\mathbf{X}} \mathbf{P} + \rho_0 \mathbf{D}_f, \tag{1.65a}$$

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T, \tag{1.65b}$$

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \tag{1.65c}$$

where $\mathbf{P} = \mathbf{P}(\mathbf{t}, \mathbf{X})$ represents the first Piola-Kirchhoff stress tensor, $\rho_0 = \rho_0(\mathbf{X})$ represents the mass density (variable or constant), $\mathbf{x} = \mathbf{x}(t, \mathbf{X})$ are the actual coordinates, $\mathbf{D}_f = \mathbf{D}_f(t, \mathbf{X})$ represents body force, $\mathbf{F} = \mathbf{F}(t, \mathbf{X})$ denotes the deformation gradient, $W = W(I_1, I_2, I_3)$ is the strain energy density, and (I_1, I_2, I_3) denote principal invariants [38]. In Ciarlet's model, equation (1.65) can be written by replacing (1.65c) with $\mathbf{P} = \rho_0 \partial W / \partial \mathbf{F}$ [87]. The full system of equations of motion for incompressible materials in the reference coordinates are given by

$$\rho_0 \frac{\partial^2 \mathbf{x}}{\partial t^2} = \text{div}_{\mathbf{X}} \mathbf{P} + \rho_0 \mathbf{D}_f, \tag{1.66a}$$

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T, \tag{1.66b}$$

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-T}, \tag{1.66c}$$

$$J = 1, \tag{1.66d}$$

where $p = p(t, \mathbf{X})$ denotes the hydrostatic pressure. Equation (1.66c) may be replaced by $\mathbf{P} = \rho_0 \dfrac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-T}$ for incompressible materials [87]. The pressure term appearing in (1.66c) for incompressible materials is as a result of writing the strain energy density function as a sum of the volumetric part and the isochoric part of the deformation. That is,

$$W = W(\mathbf{F}) - p(J - 1). \tag{1.67}$$

Differentiating (1.67) with respect to $\mathbf{F}$ and noting

$$\frac{\partial J}{\partial \mathbf{F}} = \frac{\partial (\det \mathbf{F})}{\partial \mathbf{F}} = (\det \mathbf{F})\mathbf{F}^{-T}, \quad J = \det \mathbf{F} = 1,$$

yields equation (1.66c). In particular, the first Piola-Kirchhoff stress tensors are given by the formulas for incompressible and compressible materials. Both cases will be of interest and will be considered in subsequent sections.

1.5.6 Constitutive Relations for Some Nonlinear Hyperelastic Models

In this section, some constitutive models for isotropic hyperelastic materials and their applications are reviewed. Main attention is given to the Hadamard-type constitutive models since they are of greater interest to our study. Hyperelastic material models are generally formulated in three ways. The first kind of model is called phenomenological models, and they are issued from mathematical developments of the strain energy density [100, 118]. Other researchers [58, 117] determined the material functions using experimental data. Hyperelastic models such as the Gent and Thomas model, the Gent model, the Rivlin and Saunders model, are developed using experimental data. The last kind of models is developed just from physical motivation. These models are based on statistical methods and the physics of polymer chain networks. Examples of such models are the neo-Hookean model, the 3−chain model, the Van der Waals model, and so on.

Saint Venant-Kirchhoff model

The Saint Venant-Kirchhoff model is the simplest nonlinear hyperelastic model. It has a stored energy density as

$$W = \frac{1}{2}\lambda(\mathrm{Tr}(\boldsymbol{\epsilon}))^2 + \mu\mathrm{Tr}(\boldsymbol{\epsilon}^2), \tag{1.68}$$

where $\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{C} - 1)$ is the Green-Saint Venant tensor, $\mathbf{C}$ is given by (1.55), and λ and μ are the two Lamé constants. The relation (1.68) differs from (1.2) because $\boldsymbol{\epsilon}$ denotes the Lagrangian strain tensor. The first Piola-Kirchhoff stress tensor for this model is given by [38]

$$\mathbf{P} = \lambda\mathrm{Tr}(\boldsymbol{\epsilon})\mathbf{I} + 2\mu\boldsymbol{\epsilon}. \tag{1.69}$$

The stored energy density function (1.68) can be written as

$$\begin{aligned} W &= -\frac{(2\mu+3\lambda)}{4}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right) + \frac{(2\mu+\lambda)}{8}\left(\lambda_1^4+\lambda_2^4+\lambda_3^4\right) + \frac{\lambda}{4}\left(\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2+\lambda_1^2\lambda_2^2\right) + \frac{6\mu+9\lambda}{8}, \\ &= -\frac{(2\mu+3\lambda)}{4}\mathrm{Tr}(\mathbf{C}) + \frac{(2\mu+\lambda)}{8}\mathrm{Tr}(\mathbf{C}^2) + \frac{\lambda}{4}\mathrm{Tr}\ \mathrm{Cof}(\mathbf{C}) + \frac{6\mu+9\lambda}{8}, \end{aligned} \tag{1.70}$$

where $\lambda_j = \lambda_j(\mathbf{F}),\ j = 1, 2, 3$, and $\mathrm{Cof}(\mathbf{C}) = (\det\ \mathbf{C})\mathbf{C}^{-1}$. One can observe that the stored energy function of the Saint Venant-Kirchhoff material is not polyconvex. Thus the coefficient of Tr $\mathbf{C}$ is negative. This however is not enough to show the non-polyconvexity of the stored energy density of the Saint Venant-kirchhoff material (see [38, 114] for details). According to [111], the Saint Venant-Kirchhoff material model is not polyconvex because of lack of positive definiteness of the Hessian operator. The notion of polyconvexity of the stored energy density function serves as a theoretical basis for the development of the constitutive law resulting in guaranteeing the existence of solutions of boundary value problems in linear elasticity [73]. The polyconvexity of strain energy function involves the growing of energy (rather than sometimes decreasing) when there is an increase in strain. So the natural state (zero stress) would correspond to the minimal stored energy function and therefore be a stable equilibrium. It implies the existence for all boundary conditions and body forces which might be somewhat unrealistic [12]. A function $W : \mathcal{F} \to \mathbb{R}$ defined on an arbitrary subset $\mathcal{F}$ of the set $\mathbb{M}^3$ is polyconvex if there exists a convex function $\hat{W} : V \to \mathbb{R}$ where

$$V := \{(\mathbf{F}, \mathrm{Cof}\ \mathbf{F}, \det\ \mathbf{F}) \in \mathbb{M}^3 \times \mathbb{M}^3 \times \mathbb{R}; \quad \mathbf{F} \in \mathcal{F}\}$$

such that

$$W(\mathbf{F}) = \hat{W}\left(\mathbf{F}, \mathrm{Cof}\ \mathbf{F}, \det \mathbf{F}\right),\ \forall \mathbf{F} \in \mathcal{F}.$$

A stored energy function $W : \bar{\Omega}_0 \times \mathbb{M}^3_+ \to \mathbb{R}$ is polyconvex [12] if for each $Y \in \bar{\Omega}_0$, there exist a convex function

$$\tilde{W}\left(Y, \cdot\right) : \mathbb{M}^3 \times \mathbb{M}^3 \times (0, +\infty) \to \mathbb{R},$$

such that

$$W(Y, \mathbf{F}) = \tilde{W}\left(Y, \mathbf{F}, \mathrm{Cof}\ \mathbf{F}, \det \mathbf{F}\right),\ \forall \mathbf{F} \in \mathbb{M}^3_+,$$

where $\mathbb{M}^3$, $\mathbb{M}^3_+$ are sets of matrices, and Y is in the reference (Lagrangian) configuration $\bar{\Omega}_0$. The idea of polyconvexity imposed on the stored energy density function is an important condition in mathematics and nonlinear elasticity. It is a generalization of the notion of convexity for functions defined on spaces of matrices. It is also sufficient for the ellipticity of the constitutive relation and for the material stability. In compressible and incompressible hyperelastic models, one requires the strain energy functions to be polyconvex functions of the deformation gradient tensor in order to meet the materials stability criteria so that the numerical stability of computational solutions can be achieved. Polyconvexity guarantees the existence of local minimizers of the strain energy functions when subjected to appropriate boundary conditions. Neither Saint Venant-Kirchhoff nor Murnaghan models based on the Lagrangian or Hencky strain tensors lead to an elliptic equilibrium condition. Due to this reason, one requires a strain energy function which is polyconvex such as Ogden, Mooney-Rivlin, neo-Hookean, and Yeoh strain energy function in the description of material properties [60].

The Ogden model

The stored energy density of the Ogden model is given by

$$W = \sum_{i=0}^{r} a_i (\mathrm{Tr}(\mathbf{B})^{\frac{\gamma_i}{2}}) + \sum_{i=0}^{s} b_j (\mathrm{Tr}(\mathrm{Cof}\ \mathbf{B})^{\frac{\delta_j}{2}}) + \Upsilon(I_3), \tag{1.71}$$

where $a_i > 0$, $\delta_j \geq 1$, $b_j > 0$ and $\gamma_i \geq 1$ are material parameters, $\mathbf{B}$ is given by (1.55), Cof $\mathbf{B}$ is the matrix of cofactors of $\mathbf{B}$, and $\Upsilon : (0, +\infty) \to \mathbb{R}$ is a convex function that $\lim\limits_{\delta \to 0+} \Upsilon(\delta) = +\infty$ and subjected to suitable growth conditions as $\delta \to +\infty$ [38]. The stored energy function for Ogden material model can be written with respect to the principal stretches $\lambda_1,\ \lambda_2$ and λ_3 as

$$W = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3), \tag{1.72}$$

where N, μ_i and α_i are the model parameters. The normalization constant '-3' can be dropped without loss of generality and for incompressible materials the constant $\Upsilon(I_3)$ can also be dropped. The Ogden model is used to describe rubber-like materials.

Hadamard, Mooney-Rivlin, and neo-Hookean models

When we put $r = 0$, $s = 0$, $a_0 = a$, $b_0 = b$, $\delta_0 = 2$, and $\gamma_0 = 2$ in (1.71), the constitutive relation for the Mooney-Rivlin materials is commonly written in terms of the principal invariants (1.56a)-(1.56c) as

$$W = a(I_1 - 3) + b(I_2 - 3) + \Upsilon(I_3), \quad a,\ b > 0, \tag{1.73}$$

where the constant '-3' can be ignored without loss of generality. The Mooney-Rivlin, Ciarlet-Mooney-Rivlin and neo-Hookean models are the most commonly used form of the Hadamard-type constitutive relations. These models are given in Table 1.1 [35]. The constitutive relation of the two-parameter Mooney-Rivlin model that usually describes a wide class of rubber-like materials is given by

$$W = aI_1 + bI_2, \quad a,\ b > 0. \tag{1.74}$$

The simplest case of the Hadamard materials are called the neo-Hookean models and they are obtained when $b = 0$ in (1.74) [35]. The compressible Mooney-Rivlin model is a sub-class of the Ogden model used in modeling nonlinear dynamic (see [38] for details). Models such as Ogden, Mooney-Rivlin, or Neo-Hookean constitutive functions of isotropic, rubber-like materials have polyconvex stored energy densities [38].

Type	Neo-Hookean, $(a, d, c > 0)$	Mooney-Rivlin, $(a, b, c, d > 0)$
Incompressible	$W = aI_1$	$W = aI_1 + bI_2$
Compressible	$W = a(\bar{I}_1 - 3) + c(J - 1)^2$	$W = a\bar{I}_1 + b\bar{I}_2 + c(J - 1)^2$
Generalized (Ciarlet)	$W = aI_1 + cp^2 - d \log p$	$W = aI_1 + bI_2 + \Upsilon(J),\ \Upsilon(p) = cp^2 - d \log p.$

Table 1.1: Neo-Hookean and Mooney-Rivlin constitutive models [35].

Some other types of isotropic constitutive relations for some nonlinear hyperelastic models in literature are reviewed below.

The generalized Mooney-Rivlin model

The Mooney-Rivlin model (1.73) is extended to a generalized Mooney-Rivlin model which is sometimes called the polymonial-type material [59, 86]. This model has a stored energy function defined by

$$W = \sum_{i=0}^{k} \sum_{j=0}^{l} c_{ij} (I_1 - 3)^i (I_2 - 3)^j + \Upsilon(I_3), \tag{1.75}$$

where c_{ij} are material coefficients, $c_{00} = 0$, and (I_1, I_2, I_3) represent principal invariants (1.56). The stored energy function (1.75) is truncated to the second and third order, with the third order having nine material constants [59, 86]. The '-3' can be ignored without loss of generality. For incompressible materials, the constant $\Upsilon(I_3)$ may also be dropped.

The Biderman model

The Birderman material model is a sub-class of the generalized Mooney-Rivlin models. The stored energy density for this model is given by

$$W = c_{10}(I_1 - 3) + c_{01}(I_2 - 3) + c_{20}(I_1 - 3)^2 + c_{30}(I_1 - 3)^3, \tag{1.76}$$

where c_{10}, c_{01}, c_{20}, and, c_{30} are the elastic coefficients. Biderman [17] computed the elastic constants c_{10}, c_{01}, c_{20}, and, c_{30} in an experiment and he obtained the values; $c_{10} = 1.9\text{kg/cm}^2$, $c_{01} = -0.019\text{kg/cm}^2$, $c_{20} = 4.6 \times 10^{-4}\text{kg/cm}^2$, and $c_{30} = 0.1\text{kg/cm}^2$. From the strain energy density in equation (1.75), Biderman [17] retained terms for which $i = 0$ or $j = 0$ and he considered one term for I_2 and the first three terms for I_1. The Biderman model was successfully used in [6] in an experiment that involved 8 percent sulfur rubber (material already used by [133]). (see [6, 133] and references therein for details.)

The Gent and Thomas model

Gent and Thomas [53] provided a two material constant empirical stored energy density as

$$W = C_1(I_1 - 3) + C_2 \ln\frac{I_2}{3}, \tag{1.77}$$

where C_1 and C_2 are the two elastic parameters. This model is also used to model rubber-like materials but it is not as efficient as the Mooney-Rivlin model (1.73).

The Gent model

Gent [52] used the idea of limiting polymeric chain extensibility, defined as the final chain length divided by the initial chain length. He proposed that at the maximum chain length, the invariant I_1 has the maximum length I_n. The stored energy density Gent proposed is given by

$$W = -\frac{A}{6}(I_n - 3)\ln\Big(1 - \frac{I_1 - 3}{I_n - 3}\Big), \tag{1.78}$$

where A denotes the small strain tensile modulus and that, for incompressible model, A is associated with the shear modulus μ through the expression $A = 3\mu$ [69]. The constitutive equation (1.78) has a singularity as $I_1 \to \eta + 3$. The strain energy density associated with the Gent's model depends only on the first principal invariant I_1 of (1.55) (i.e., $W = W(I_1)$). Thus, the Gent model (1.78) is a sub-class of the generalized neo-Hookean materials (see [113] and references therein for details).

The Yeoh model

The Yeoh model is a phenomenological model for the deformation of nearly incompressible, nonlinear elastic materials such as rubber and foam [142]. The Yeoh model is also useful in describing isotropic incompressible rubber-like materials [116]. For incompressible materials, the strain energy potential is written in terms of only the first principal invariant I_1 as

$$W(I_1) = \sum_{j=1}^{3} C_{j0} (I_1 - 3)^j . \tag{1.79}$$

Nowadays, the Yeoh's model for incompressible hyperelastic materials has strain energy function given by

$$W(I_1) = \sum_{j=1}^{n} C_{j0} (I_1 - 3)^j . \tag{1.80}$$

When $n = 1$, the Yeoh model (1.80) simplifies to the neo-Hookean model for incompressible materials. For numerical purposes, the Yeoh model proves important in separating the deformation into the isochoric and volumetric parts usually through a multiplicative split of the deformation gradient [116]. For compressible materials one replaces the first strain invariant of $\mathbf{C}_{iso}$ by $\bar{I}_1 = J^{-2/3} I_1$, which replaces I_1 in $W(I_1)$. The strain energy potential for Yeoh's compressible hyperelastic materials is given by

$$W(\bar{I}_1) = \sum_{j=1}^{n} C_{j0} \left(\bar{I}_1 - 3\right)^j + \sum_{l=1}^{n} C_{l1} (J - 1)^{2l} , \tag{1.81}$$

where C_{j0} and C_{l1} are the elastic coefficients, and the second term on the right hand side enforces the incompressibility constraint. The condition $2C_{10} = \mu$ is called the consistency condition of the Yeoh model. Other constitutive models used in describing the hyperelastic behavior of foam-like or rubber-like materials in the nonlinear elasticity framework can be found in [10, 125, 142].

The Murnaghan model

The Murnaghan model (Potential) involves a cubic potential that is used to describe isotropic hyperelastic deformation. The Murnaghan model is a five-constant model proposed by Murnaghan [97] for the Cauchy-Green strain tensor. The model has a strain density function in terms of the invariants of the form

$$W(I_I, I_{II}, I_{III}) = \frac{1}{2}\lambda I_I^2 + \mu I_{II} + \frac{1}{3} A I_{III} + B I_I I_{II} + \frac{1}{3} C I_I^3, \tag{1.82}$$

where λ, μ are second order Lamé constants, A, B, and C are the third-order Murnaghan elastic constants, and the invariants I_I, I_{II}, and I_{III} are given by

$$I_I = Tr\, \boldsymbol{\epsilon} \equiv \epsilon_{ll}, \quad I_{II} = Tr\, \boldsymbol{\epsilon}^2 \equiv \epsilon_{lk}\epsilon_{kl}, \quad I_{III} = Tr\, \boldsymbol{\epsilon}^3 \equiv \epsilon_{lr}\epsilon_{rk}\epsilon_{kl}, \tag{1.83}$$

where Tr abbreviates trace, and $\boldsymbol{\epsilon} = \epsilon_{lm}$ denotes the Lagrangian strain tensor given in (1.8). The invariants (1.83) are related to the principal invariants (1.57) by the relations

$$I_I = I_1, \quad I_{II} = I_1^2 - 2I_2, \quad I_{III} = I_1^3 - 3I_1 I_2 + 3I_3. \tag{1.84}$$

The Murnaghan model is commonly used to describe nonlinear acoustic waves because it has a third order algebraic invariants that makes it possible to take into consideration numerous essential wave effects. The potential (1.82) is sometimes written in terms of the Lagrangian strain tensor (1.8) as

$$W(\epsilon_{lm}) = \frac{1}{2}\lambda (\epsilon_{ii})^2 + \mu (\epsilon_{lm})^2 + \frac{1}{3} A \epsilon_{lm}\epsilon_{li}\epsilon_{mi} + B (\epsilon_{lm})^2 \epsilon_{ii} + \frac{1}{3} C (\epsilon_{ii})^3 , \tag{1.85}$$

where for three-dimensional problems with displacement $u_k = u_k(x_1, x_2, x_3, t)$, one can write

$$\varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \quad \varepsilon_{lm}^2 = \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{21}^2 + \varepsilon_{31}^2 + \varepsilon_{32}^2 + \varepsilon_{23}^2,$$
$$\varepsilon_{lm}\varepsilon_{li}\varepsilon_{mi} = \varepsilon_{11}^3 + \varepsilon_{22}^3 + \varepsilon_{33}^3 + 3\varepsilon_{11}^2\varepsilon_{12} + 3\varepsilon_{13}^2\varepsilon_{11} + 3\varepsilon_{23}^2\varepsilon_{22} + 3\varepsilon_{13}^2\varepsilon_{33} + 3\varepsilon_{23}^2\varepsilon_{33} + 3\varepsilon_{21}^2\varepsilon_{22} + 3\varepsilon_{12}\varepsilon_{13}\varepsilon_{23}. \tag{1.86}$$

From (1.86), the Lagrangian strain tensor for large deformation can be written in terms of the displacement gradients in (1.8). According to Bland [20], the Murnaghan model for an isotropic solid comes from a generalised Taylor series expansion of the dynamic strain energy density W correct to the third-order in displacement gradients which can be written in terms of the strain invariants (1.85). (See [121, 122] and references therein for details on the Murnaghan model). One can observe that, the Murnaghan model is an extension of the Saint Venant-Kirchhoff model. Thus, the first two terms of the Murnaghan model is the Saint Venant-Kirchhoff model. Hence the non-polyconvexity of the Murnaghan model follows from the Saint Venant-Kirchhoff model.

1.5.7 Anisotropic Hyperelastic Materials

Anisotropic materials such as wood, fiber-reinforced composites, soft biological tissues (arterial walls, heart tissue, etc.) are of interest in multiple applications including earthquake engineering, geodynamics, biomechanics, mathematical biology, medical, industrial and technological world. These materials display anisotropic elastic properties because of the preferred directions in their microstructure. Hyperelastic materials are expressed in terms of a stored energy density, which provides a definition of the strain energy as stored energy in the material per unit of volume as a function of the deformation at that point in the material. In anisotropic hyperelastic materials, the stored energy density involves the interaction of the strain energy density for the isotropic and the anisotropic contributions. Consequently, anisotropic models have the strain energy density defined as the sum of isotropic and anisotropic contributions. The anisotropic contribution is assumed to depend on the pseudo-invariants which comprise of fiber strength, interaction parameter, and fiber directions [15, 40, 65]:

$$W = W_{\text{iso}}(I_1, I_2, I_3) + W_{\text{aniso}}, \tag{1.87}$$

where W_{iso} is the stored energy density for the isotropic material and W_{aniso} is the stored energy density of the anisotropic material. An invariant is a property of a mathematical object which remains unaltered, after the transformations of a certain type of class applied to the object. In isotropic hyperelastic materials, the strain energy density functions are written as functions of the material points and the deformation gradient or just as a function of the second-order Cauchy-Green strain tensors. Thus the strain energy density function $W = W(\mathbf{C})$ for isotropic hyperelastic materials must be independent of the coordinate system. Three principal invariants of $\mathbf{C}$ are required to fully describe the deformation. In anisotropic hyperelastic materials, the strain energy density function is written to involve fibers and $\mathbf{C}$ given by

$$W = W(\mathbf{C}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_k). \tag{1.88}$$

This strain energy density function must also be independent of the coordinate system. Again since W in (1.88) is defined in the Lagrangian configuration, a rigid body motion will not affect W. Therefore, W must remain unaltered under a rotation described by a proper orthogonal rotation tensor $\mathbf{Q}$. The invariants associated with the new W are termed as pseudo-invariants, where 'Pseudo' means artificial. Pseudo-invariants are invariants of the Cauchy-Green strain tensors that are used to represent the anisotropic contribution of the strain energy function. For anisotropic materials involving fibers (reinforced composites) $W = W(\mathbf{X}, \mathbf{F}, \mathbf{A_1}, \mathbf{A_2}, ..., \mathbf{A_n})$, where $\mathbf{A_i}$, $i = 1, ..., n$ are the unit vectors and define in the material configuration the direction fields of n independent, possibly interacting fiber families. The strain energy density of equation (1.87) describes the behavior of elastic materials. Figure 1.12 shows an anisotropic elastic artery with three layers.

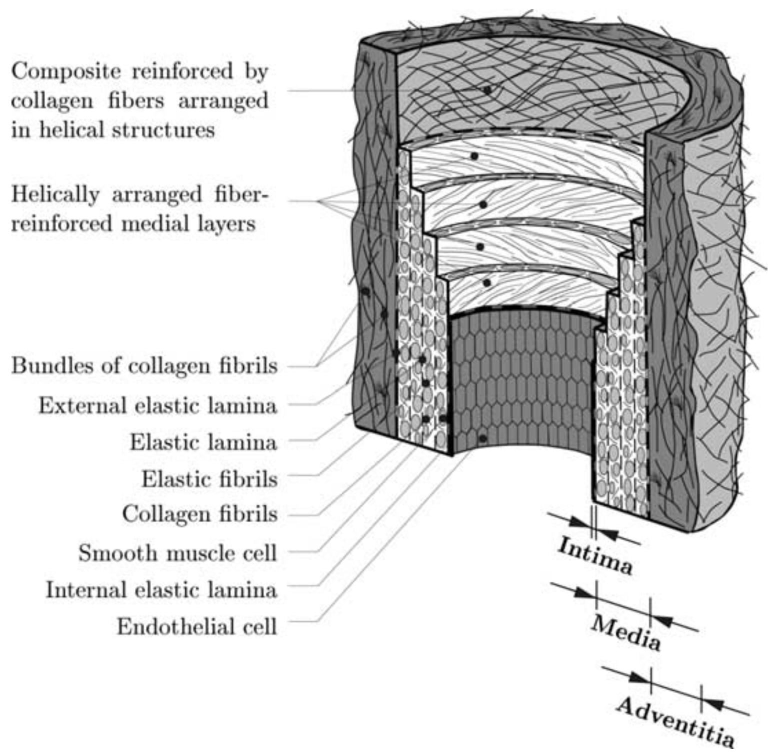

Figure 1.12: Diagram of an elastic arterial tissue which is composed of three layers [64].

1.5.8 Constitutive Models in Anisotropic Hyperelasticity

In this section, we discuss the theory of fiber-reinforced isotropic hyperelastic materials. Each fiber bundle is given a vector field $\mathbf{A}_i = \mathbf{A}_i(\mathbf{X})$, $|\mathbf{A}_i| = 1$. Fiber bundles are categorized as single ($n = 1$) and multiple ($n \geq 2$) which are presented in the vector direction fields given as $\{\mathbf{A}_{\mathrm{i}}\}_{i=1}^{n}$ (Figure 1.12). The anisotropic stored energy derived from new invariants (pseudo-invariants) for a single fiber bundle (family) are given by

$$I_4 = \mathbf{A_1}^T \mathbf{C} \mathbf{A}_1, \tag{1.89a}$$

$$I_5 = \mathbf{A_1}^T \mathbf{C}^2 \mathbf{A}_1, \tag{1.89b}$$

where $\mathbf{C}$ is the right Cauchy-Green tensor (1.55), $\mathbf{C}^2{=}\mathbf{CC}$ is the square of (1.55), $I_4 = \tilde{\lambda}_1^2$ is the squared fiber strength factor [34] which represents the deformations that controls the fiber length, and I_5 accounts for the effect of the fiber on shear response in the material [40, 36, 91, 112]. The most general form of the stored energy density for the anisotropic component is given by

$$W_{\text{aniso}} = W_{\text{aniso}}(I_4, I_5). \tag{1.90}$$

We note that for anisotropic materials $\mathbf{PF}^T = \mathbf{FP}^T$ may not hold identically and that for an anisotropic strain energy density involving contribution from both the isotropic and anisotropic settings, we obtain equations of motion as in (1.65) and (1.66) (depending on which first Piola-Kirchhoff stress tensor used and whether we have compressible and incompressible materials). For multiple fiber bundles, we get a general form of the anisotropic stored energy function by taking into consideration pseudo-invariants specific to each fiber family and on the interaction of other new invariants. In particular, for $n = 2$ (two fiber bundles), the most general stored energy function depends on the above two pseudo-invariants for each fiber bundle and the two interaction new invariants [63, 65] is

$$W_{\text{aniso}} = W_{\text{aniso}}(I_4, I_5, I_6, I_7, I_8, I_9), \tag{1.91}$$

where $I_6 = \mathbf{A}_2^T\mathbf{C}\mathbf{A}_2$ and $I_7 = \mathbf{A}_2^T\mathbf{C}^2\mathbf{A}_2$ the new invariants for the second fiber bundle, and the other fiber interaction pseudo-invariants are $I_8 = (\mathbf{A}^T{}_1\mathbf{A_2})\mathbf{A}^T{}_1\mathbf{C}\mathbf{A_2}$ and $I_9 = (\mathbf{A}^T{}_1\mathbf{A_2})^2$. Some particular stored energy densities used in the study of fiber-reinforced materials are the standard reinforced model, arterial models, limited fiber extensive models, and the model of shear response in fiber-reinforced materials [41, 63, 69, 70, 98].

1.6 Discussion

In this chapter, we presented the key ideas of the theory of elastodynamics that would be used in subsequent chapters. We discussed the theory of linear elasticity, anisotropic materials, and isotropic materials. We also discussed the earthquake and seismic waves. Moreover, we discussed the second type of seismic waves (surface waves) and in particular, Love surface waves in detail. Additionally, we discussed the coordinates of continuum mechanics, gradient of deformation, and material derivative. Also, we presented derivation of equations of motion in Eulerian coordinates (in terms of the Cauchy stress) and Lagrangian coordinates (in terms of the first Piola-Kirchhoff stress) from the balance laws of mass, linear momentum, and angular momentum. The first Piola-Kirchhoff stress tensor was described according to Ciarlet [38] and Marsden and Hughes [87] and presented for isotropic, homogeneous, hyperelastic materials under the assumption of material framework indifference. Finally, examples of some constitutive relations from hyperelastic isotropic materials such as the Hadamard, Mooney-Rivlin, neo-Hookean, and Ogden models have been presented.

Regarding the existence and uniqueness of solutions in elastodynamics and elastostatics, we employed the results based on the work of Marsden and Hughes [87]. Also, in the nonlinear theory of elasticity (elastostatics of hyperelastic materials), global existence and uniqueness have been shown for a polyconvex strain energy density [38]. Marsden and Hughes [87] derived conditions for global existence and uniqueness for a compressible semilinear system of equations and for local existence and uniqueness in time of solutions to compressible quasilinear systems. (See [13] and references therein for details in existence and uniqueness of open problems in elasticity.)

2 LINEAR LOVE SURFACE WAVES IN ELASTIC MEDIA

2.1 Introduction

The Earth's inner layers cannot be studied directly, and as a result, Earth scientists employ seismic waves to explore the interior of the Earth. Vital information on the constituents of the Earth's inner layers has been obtained from the study of seismic waves. One particular type of seismic wave (Love surface wave) which can propagate along the surface of the Earth is of great importance given its multiple applications in geophysics, and in understanding possible processes that happen during earthquakes as well as for predicting the possible damage.

In what follows, we discuss linear Love surface waves propagating in isotropic and anisotropic elastic media.

2.2 Linear Love waves in Isotropic Elastic Media

In this section, let us consider a linear Love surface wave in isotropic elastic material layer lying over isotropic elastic half-space with different elastic coefficients. The solutions to the equations corresponding to both the layer and the elastic half-space are computed. The dispersion relation of the Love wave is presented.

2.2.1 Problem Formulation

Let us consider a two-dimensional wave traveling along a free surface. Also, let us consider an isotropic upper elastic half-space $(x,\ y,\ z)$ with density ρ_2 and the Lamé constants $\lambda_2;\ \mu_2$. Let $(u_1,\ u_2,\ u_3)$ be the components of displacement vector in (1.13) along $(x,\ y,\ z)$ directions respectively, and the u_2^1 and u_2^2 are the displacement in the layer and the half-space respectively [3, 81, 119, 121] (Figure 2.1). The following mechanical assumptions are made:

1. Assume that there are no stresses on the boundary plane $z = -H$, (i. e., the boundary is free).

2. Assume also that the elastic layer has constant thickness H $(-H \leq z \leq 0)$ with mechanical properties $\lambda_1,\ \mu_1,$ and ρ_1 overlies the half-space $z = 0$. (That is, the elastic layer is in welded contact with the elastic half-space.)

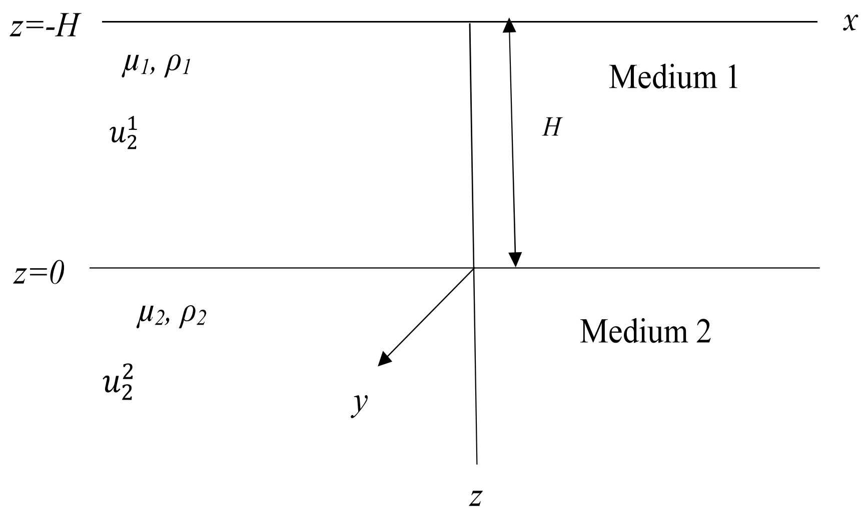

Figure 2.1: Geometric Picture I.

2.2.2 Solution of the Wave Problem in the Layer and the Half-space

The equations of motion (1.26) are written as

$$\rho \frac{d^2 u_k^j}{dt^2} = \sigma_{ki,i}^j + D_{f\,k}, \quad k = 1, 2, 3, \tag{2.1}$$

where σ_{ki} and $D_{f\,k}$ are the stresses and body forces respectively, $j = 1$ is for layer and $j = 2$ is for half-space. If the body forces $D_{f\,k}$ are neglected, the three scalar equations of motion in the layer and the half-space are given by

$$\rho_j \frac{\partial^2 u_1^j}{\partial t^2} = \frac{\partial \sigma_{11}^j}{\partial x} + \frac{\partial \sigma_{12}^j}{\partial y} + \frac{\partial \sigma_{13}^j}{\partial z}, \tag{2.2a}$$

$$\rho_j \frac{\partial^2 u_2^j}{\partial t^2} = \frac{\partial \sigma_{21}^j}{\partial x} + \frac{\partial \sigma_{22}^j}{\partial y} + \frac{\partial \sigma_{23}^j}{\partial z}, \tag{2.2b}$$

$$\rho_j \frac{\partial^2 u_3^j}{\partial t^2} = \frac{\partial \sigma_{31}^j}{\partial x} + \frac{\partial \sigma_{32}^j}{\partial y} + \frac{\partial \sigma_{33}^j}{\partial z}. \tag{2.2c}$$

(See, e. g., Biot [18, 19].) Now consider the situation of propagation of a transversely polarized harmonic wave in such a way that both longitudinal and horizontal displacements u_1 and u_3 respectively are absent. That is, we assume that $u_1 = u_3 = 0$, and u_2 is a function of x, z, and t as given by

$$u_1 = 0,\ u_3 = 0,\ u_2 = u_2(x, z, t) \ \text{ and } \frac{\partial}{\partial y} \equiv 0. \tag{2.3}$$

From the assumptions of Love waves in (2.3) [81], two of the three equations in (2.2) turn out to be identically zero. The remaining equation is

$$\rho_j \frac{\partial^2 u_2^j}{\partial t^2} = \frac{\partial \sigma_{21}^j}{\partial x} + \frac{\partial \sigma_{23}^j}{\partial z}. \tag{2.4}$$

Using the stress-strain relations for a general isotropic elastic medium (1.10), one can write the stresses as

$$\begin{aligned}
\sigma_{11}^j &= \lambda_j \left(\epsilon_{22}^j + \epsilon_{33}^j\right) + \left(\lambda_j + 2\mu_j\right) \epsilon_{11}^j, \\
\sigma_{22}^j &= \lambda_j \left(\epsilon_{11}^j + \epsilon_{33}^j\right) + \left(\lambda_j + 2\mu_j\right) \epsilon_{22}^j, \\
\sigma_{33}^j &= \lambda_j \left(\epsilon_{11}^j + \epsilon_{22}^j\right) + \left(\lambda_j + 2\mu_j\right) \epsilon_{33}^j, \\
\sigma_{12}^j &= 2\mu_j \epsilon_{12}^j, \\
\sigma_{23}^j &= 2\mu_j \epsilon_{23}^j, \\
\sigma_{31}^j &= 2\mu_j \epsilon_{13}^j.
\end{aligned} \tag{2.5}$$

In the case of small deformation, the linear strain is given by (1.7). The strain-displacement relations after applying (2.3) to (1.7) are given by

$$\begin{aligned}
\epsilon_{11}^j &= \frac{1}{2}\left(\frac{\partial u_1^j}{\partial x} + \frac{\partial u_1^j}{\partial x}\right) = \frac{\partial u_1^j}{\partial x} = 0,\ \epsilon_{22}^j = \frac{1}{2}\left(\frac{\partial u_2^j}{\partial y} + \frac{\partial u_2^j}{\partial y}\right) = \frac{\partial u_2^j}{\partial y} = 0, \\
\epsilon_{33}^j &= \frac{1}{2}\Big(\frac{\partial u_3^j}{\partial z} + \frac{\partial u_3^j}{\partial z}\Big) = \frac{\partial u_3^j}{\partial z} = 0,\ \epsilon_{23}^j = \frac{1}{2}\Big(\frac{\partial u_2^j}{\partial z} + \frac{\partial u_2^j}{\partial y}\Big) = \frac{1}{2}\left(\frac{\partial u_2^j}{\partial z}\right), \\
\epsilon_{13}^j &= \frac{1}{2}\left(\frac{\partial u_1^j}{\partial z} + \frac{\partial u_3^j}{\partial x}\right) = 0,\ \ \epsilon_{12}^j = \frac{1}{2}\left(\frac{\partial u_1^j}{\partial y} + \frac{\partial u_2^j}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial u_2^j}{\partial x}\right).
\end{aligned} \tag{2.6}$$

It is observed from (2.6) that only two components of strain, ϵ_{12} and ϵ_{23}, are nonzero, and thus the corresponding shear stresses σ_{12} and σ_{23} are nonzero. Thus substitution of (2.6) into (2.5) yields

$$\sigma_{11}^j = \sigma_{13}^j = \sigma_{22}^j = \sigma_{33}^j = 0, \quad \sigma_{23}^j = \mu_j \frac{\partial u_2^j}{\partial z}, \quad \sigma_{12}^j = \mu_j \frac{\partial u_2^j}{\partial x}. \tag{2.7}$$

From relation (2.3), the only possibility of traveling of the wave is along the $x-$ axis direction near the interface between the layer and half-space. The displacement corresponding to Love wave can be represented as

$$u_2^j = U_2^j(z) \exp\left(i(kx - \omega t)\right), \tag{2.8}$$

where the functions $U_2^j(z)$ describe the amplitudes of the Love wave as functions of the depth, $k = \omega/v_L$ denotes the wave number of the Love wave, $\omega = 2\pi f$ is its angular frequency, and v_L is the phase velocity of Love wave. Now substituting (2.7) into (2.4) yields a simple motion consisting of only one displacement component (u_2) in each layer. These displacements satisfy the wave equation

$$\left[\mu_j \Big(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\Big) - \rho_j \frac{\partial^2}{\partial t^2}\right] u_2^j(x, z, t) = 0, \quad j = 1, 2. \tag{2.9}$$

Substituting (2.8) into (2.9) results in transforming partial differential equations (2.9) into ordinary differential equations:

$$\left(U_2^j\right)''_{,11} + k^2 \left[\left(\frac{v}{v_T^j}\right)^2 - 1\right] U_2^j(z) = 0, \quad v_T^j = \sqrt{\frac{\mu_j}{\rho_j}}, \quad j = 1, 2, \tag{2.10}$$

where v_T^j, $j = 1, 2$, are the S wave phase velocities in the layer and half-space and v is the velocity of Love wave. The equations (2.10) can be written as

$$\left[\frac{d^2}{dz^2} + k^2 \left(\left(\frac{v}{v_T^j}\right)^2 - 1\right)\right] U_2^j(z) = 0, \quad j = 1, 2. \tag{2.11}$$

2.2.3 Solution to the Linear Wave Equation in the Elastic Half-space

The solution to the wave equation corresponding to the half-space is obtained by considering the ODE in equation (2.11) where $j = 2$:

$$\left[\frac{d^2}{dz^2} + k^2 \left(\left(\frac{v}{v_T^2}\right)^2 - 1\right)\right] U_2^2(z) = 0. \tag{2.12}$$

The solution is given by

$$U_2^2 = l_2 \exp\left(-\sqrt{\left[1 - \left(v_L/v_T^2\right)^2\right]} kz\right) + B_2 \exp\left(\sqrt{\left[1 - \left(v_L/v_T^2\right)^2\right]} kz\right), \tag{2.13}$$

where l_2 and B_2 are unknown amplitude constants, $(\beta_2)^2 = 1 - \left(v/v_T^{(2)}\right)^2$, and $k_L = \omega/v_{\mathrm{L}}$. The solution of (2.12) in the half-space is chosen as an exponentially decreasing one and it is given by

$$U_2^2 = l_2 \exp\left(-\sqrt{\left[1 - \left(v_L/v_T^2\right)^2\right]} kz\right), \tag{2.14}$$

in which a condition that

$$(\beta_2)^2 = 1 - \left(v_L/v_T^2\right)^2 > 0, \tag{2.15}$$

is imposed. The exponentially decreasing solution is chosen for the equation in the elastic half-space. Otherwise Love wave will not be the surface wave because it will not be localized at the near-the-elastic layer from the side of half-space [119]. The condition (2.15) means that the term under the square root should be positive, and that the Love wave phase velocity must be less than the phase velocity of the shear wave in the half-space. Here l_2 is an unknown constant amplitude factor.

2.2.4 Solution of the Linear Wave Equation (2.11) in the Elastic Layer

For the linear wave equation (2.11) in the elastic layer, the solution can be sought in terms of harmonic functions. From (2.11), the linear wave equation in the layer is given by

$$\frac{d^2 U_2^1(z)}{dz^2} + k^2 (\beta_1)^2 U_2^2 = 0, \tag{2.16}$$

and the solution to (2.16) is given by

$$U_2^1(z) = A_{1l}\sin(\beta_1 kz) + A_{2L}\cos(\beta_1 kz),$$

where $\beta_1 = \sqrt{\left(v_L/v_T^2\right)^2 - 1}$, and A_{1l} and A_{2l} are constant amplitude factors. One has

$$U_2^1(z) = A_{1l}\,\sin\left(\sqrt{\left[\left(v/v_T^2\right)^2 - 1\right]}kz\right) + A_{2l}\,\cos\left(\sqrt{\left[\left(v/v_T^2\right)^2 - 1\right]}kz\right). \tag{2.17}$$

Here, a condition that

$$(\beta_1)^2 = (v_L/v_T^1)^2 - 1 > 0 \tag{2.18}$$

must be imposed [121]. The condition (2.18) means the term under the square root must be positive and that the Love wave phase velocity must be higher than the phase velocity of the shear wave in the layer. Hence, the form of linear solutions (2.14) and (2.17) corresponds to the Love wave if the condition

$$v_T^2 > v_{\mathrm{L}} > v_T^1 \tag{2.19}$$

is satisfied. The condition (2.19) means the phase velocity in the half-space is greater than the phase velocity in the layer. The relation (2.19) is the condition of existence of Love waves, which restricts the ratio of physical properties of the system [121]. Hence the solutions to the linear equations (2.9) in both layer and half-space with three unknown amplitude factors ($l_2,\ A_{1l},$ and A_{2l}) are

$$u_2^2(x,z,t) = l_2\exp\left(-\sqrt{\left[1-\left(v_L/v_T^2\right)^2\right]}kz\right)\exp\left(i(kx-\omega t)\right), \quad x\in(-\infty,\infty), \quad z\in[0,\infty), \tag{2.20}$$

$$u_2^1 = \left\{ A_{1l}\,\sin\left(\sqrt{\left[\left(v/v_T^1\right)^2 - 1\right]}kz\right) + A_{2l}\,\cos\left(\sqrt{\left[\left(v/v_T^1\right)^2 - 1\right]}kz\right)\right\}\exp\left(i(kx-\omega t)\right), \quad x\in(-\infty,\infty), \quad z\in[-H,0]. \tag{2.21}$$

In most cases, Love waves are observed as traveling waves (on a thin layer overlying a thick foundation). And in the problem statement on linear Love waves above, the base (foundation) is modelled as one of infinite thickness.

2.2.5 Boundary Conditions

The full solution to (2.9) (for both the layer and the half-space) is obtained by considering three boundary conditions. The three unknown amplitude factors l_2, A_{1l} and A_{2l} are solved using the boundary conditions. For the first boundary condition, we know that on the plane $z = -H$, there are no stresses, therefore $\left(\partial U_2^1/\partial z\right)_{z=-H} = 0$. (the normal component of the stress vanishes). Thus, after using this boundary condition we have

$$\left(\partial U_2^1/\partial z\right)_{z=-H} = A_{1l}\cos\sqrt{\left[\left(v_L/v_T^1\right)^2 - 1\right]}kH + A_{2l}\sin\sqrt{\left[\left(v_L/v_T^1\right)^2 - 1\right]}kH$$

$$= A_{1l}\cos(k\beta_1 H) + A_{2l}\sin(k\beta_1 H) = 0, \tag{2.22}$$

Again on the plane $z = 0$, the condition of full mechanical contact at the interface should be satisfied:

$$U_2^1(0) = U_2^2(0), \quad \mu_1\left(\partial U_2^1/\partial z\right)_{z=-0} = \mu_2\left(\partial U_2^2/\partial z\right)_{z=+0}, \tag{2.23}$$

One consequently obtains

$$U_2^1(0) = A_{2l},\ U_2^2(0) = l_2.$$

This implies that

$$U_2^1(0) = A_{2l} = U_2^2(0) = l_2,\ A_{2l} = l_2. \tag{2.24}$$

Now using the last boundary condition, $\mu_1\left(\partial U_2^1/\partial z\right)_{z=-0} = \mu_2\left(\partial U_2^2/\partial z\right)_{z=+0}$, one has

$$\mu_1 k\sqrt{\left[\left(v_L/v_T^1\right)^2 - 1\right]}A_{1l} = -\mu_2 l_2 k\sqrt{\left[1 - \left(v_L/v_T^2\right)^2\right]},$$

that is,

$$\mu_1\beta_1 A_{1l} = -\mu_2\beta_2 A_{2l}.$$

Doing the necessary simplifications, one obtains the relation

$$\frac{\mu_2\sqrt{\left[1 - \left(v_L/v_T^2\right)^2\right]}}{\mu_1\sqrt{\left[\left(v_L/v_T^1\right)^2 - 1\right]}} = \tan\left\{\sqrt{\left[\left(v_L/v_T^1\right)^2 - 1\right]}kH\right\}. \tag{2.25}$$

It is observed that one of the amplitudes (l_2) is arbitrary, which represents the fact that the Love wave is a traveling surface wave, and the transcendental relation (2.25) above is solved to determine the phase velocity and the wave number. Therefore (2.25) is re-written in terms of the wave number as

$$\frac{\mu_2\sqrt{\left[1 - \left(v_L/v_T^2\right)^2\right]}}{\mu_1\sqrt{\left[\left(v_L/v_T^1\right)^2 - 1\right]}} = \tan\left\{\frac{\omega H}{v_L}\sqrt{\left[\left(v_L/v_T^1\right)^2 - 1\right]}\right\}, \tag{2.26}$$

where $\omega = kv_L$. The presence in the transcendental equation (2.25) of the frequency under the sign of tangent is an important feature of dispersion of Love waves, and it shows the nonlinear dependence of Love wave phase velocity on frequency. The transcendental equation (2.25) has a countable set of roots $k_n = k_0 + n\pi$ (k_0 is the first root, defining the zero mode, and $n \in \mathbb{N}$). The infinite number of wave modes and wave numbers are generated by the infinite number of roots. The full solution to (2.9) in the layer and the half space are therefore given by

$$u_2^2(x,z,t) = l_2\exp\left(-\sqrt{\left[1 - \left(v_L/v_T^2\right)^2\right]}kz\right)\exp\left(i(kx - \omega t)\right), \\ x \in (-\infty,\infty), \quad x \in (-\infty,\infty), \quad z \in [0,\infty), \tag{2.27}$$

$$u_2^1 = l_2 \left\{ -\frac{\mu_2\sqrt{\left[1-\left(v_L/v_T^2\right)^2\right]}}{\mu_1\sqrt{\left[\left(v_L/v_T^1\right)^2-1\right]}} \sin\sqrt{\left[\left(v_L/v_T^1\right)^2-1\right]}kz + \cos\sqrt{\left[\left(v_L/v_T^1\right)^2-1\right]}kz \right\} \exp\left(i(kx-\omega t)\right),$$

$$x \in (-\infty,\infty), \quad x \in (-\infty,\infty), \quad z \in [-H,0). \tag{2.28}$$

This shows that linear Love waves are dispersive waves (which can be verified from the nonlinear dependence of phase velocity v_L and wave number k as provided in (2.26)). It is observed that $v_L \to v_T^1$ as

$$kH\sqrt{\left[\left(v_L/v_T^1\right)^2-1\right]} \to n\pi, \quad (n \in \mathbb{N}). \tag{2.29}$$

Thus when the wave number increases, the velocity decreases, and the maximum values of velocity for higher modes may be determined by the values from (2.29). This is associated with the speed of a plane shear wave propagating in a half-space. Similarly, when $v_L \to v_T^2$, the wavelength increases to the thickness of the layer in the medium 1, and the surface wave approaches the body shear wave speed.

2.3 Linear Love Waves Propagating in Anisotropic Elastic Materials

In this section, we briefly consider a Love surface wave propagating in a general homogeneous anisotropic elastic layer lying over an anisotropic elastic half-space with each medium been described by different elastic properties. We then proceed to obtain a solution to the linear wave equations corresponding to the layer and the half-space, and present the dispersion relation for Love waves in the most general anisotropic setting.

2.3.1 Problem Formulation

Consider a homogeneous anisotropic layer of elastic substance of constant thickness H, which lies over an anisotropic elastic half-space. Also consider a rectangular coordinate system $(x,\ y,\ z)$ in such a way that the z-axis points vertically downward the half-space and the x-axis points in the direction of propagation of the wave. The elastic properties in the layer are the mass density ρ_1 and elastic stiffness coefficients a_{ik}. Also, in the elastic half-space, the elastic properties taken are the mass density ρ_2 and elastic moduli c_{ik}. Let (u,v,w) be the components of displacement vectors along $x, y,$ and z directions, respectively [110]. The geometry of the problem is shown in Figure 2.2.

2.3.2 Solution of the Linear Wave Problem in the Anisotropic Half-space

The equation of motion (1.26) corresponding to the half-space is also given by

$$\rho\frac{d^2u_k}{dt^2} = \eta_{ki,i} + G_k, \quad k = 1,2,3.$$

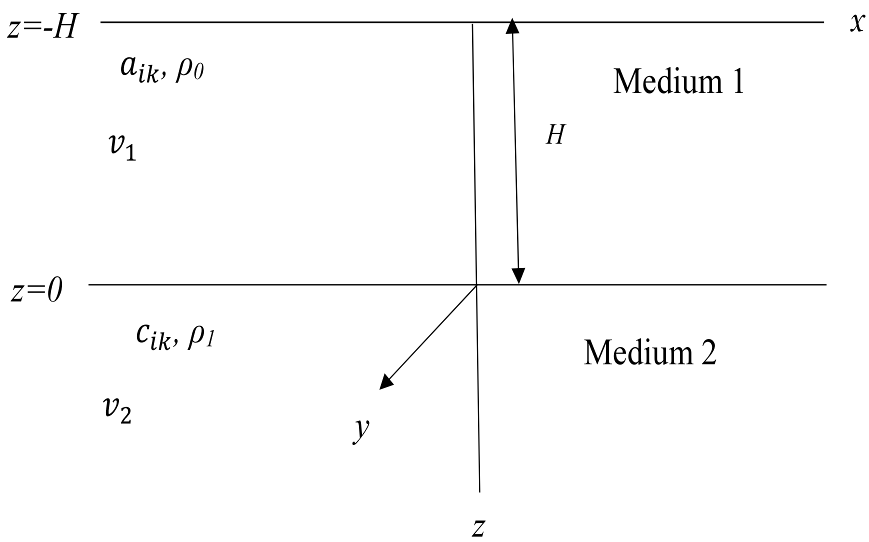

Figure 2.2: Geometric picture of the linear Love waves in anisotropic media.

Neglecting body forces, the equations of motion in the elastic half-space is written as

$$\rho_1 \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial \eta_{11}}{\partial x} + \frac{\partial \eta_{12}}{\partial y} + \frac{\partial \eta_{13}}{\partial z}, \tag{2.30a}$$

$$\rho_1 \frac{\partial^2 v_2}{\partial t^2} = \frac{\partial \eta_{21}}{\partial x} + \frac{\partial \eta_{22}}{\partial y} + \frac{\partial \eta_{23}}{\partial z}, \tag{2.30b}$$

$$\rho_1 \frac{\partial^2 w_2}{\partial t^2} = \frac{\partial \eta_{31}}{\partial x} + \frac{\partial \eta_{32}}{\partial y} + \frac{\partial \eta_{33}}{\partial z}, \tag{2.30c}$$

where η_{11}, η_{12} η_{13}, η_{21}, η_{22}, η_{23}, η_{31}, η_{32}, and η_{33} are the stress components in the elastic half-space, u_2, v_2, and w_2 represent the components of the displacement vector in the half-space. From the assumptions of Love waves (2.3), there is only one equation out of the three equations of motion in (2.45) which is non-zero. That is, $u = w = 0$, y is independent of body forces, and v is the function of x, z and t. Thus, the only non-zero equation of the three equations of motion (2.30) is given by

$$\rho_1 \frac{\partial^2 v_2}{\partial t^2} = \frac{\partial \eta_{21}}{\partial x} + \frac{\partial \eta_{23}}{\partial z}. \tag{2.31}$$

Using the constitutive relation (1.4), the stress-strain relation for the half-space is given by

$$\begin{aligned}
\eta_{11} &= c_{11}\epsilon_{11} + c_{12}\epsilon_{22} + c_{13}\epsilon_{33} + 2c_{14}\epsilon_{23} + 2c_{15}\epsilon_{13} + 2c_{16}\epsilon_{12},\\
\eta_{22} &= c_{12}\epsilon_{11} + c_{22}\epsilon_{22} + c_{23}\epsilon_{33} + 2c_{24}\epsilon_{23} + 2c_{25}\epsilon_{13} + 2c_{26}\epsilon_{12},\\
\eta_{33} &= c_{13}\epsilon_{11} + c_{23}\epsilon_{22} + c_{33}\epsilon_{33} + 2c_{34}\epsilon_{23} + 2c_{35}\epsilon_{13} + 2c_{36}\epsilon_{12},\\
\eta_{12} &= c_{14}\epsilon_{11} + c_{24}\epsilon_{22} + c_{34}\epsilon_{33} + 2c_{44}\epsilon_{23} + 2c_{45}\epsilon_{13} + 2c_{46}\epsilon_{12},\\
\eta_{23} &= c_{15}\epsilon_{11} + c_{25}\epsilon_{22} + c_{35}\epsilon_{33} + 2c_{45}\epsilon_{23} + 2c_{55}\epsilon_{13} + 2c_{56}\epsilon_{12},\\
\eta_{31} &= c_{16}\epsilon_{11} + c_{26}\epsilon_{22} + c_{36}\epsilon_{33} + 2c_{46}\epsilon_{23} + 2c_{56}\epsilon_{13} + 2c_{66}\epsilon_{12}.
\end{aligned} \tag{2.32}$$

Now assume that for small deformation, the linear strain is given by (1.7). Therefore, the strain-displacement relations after applying (2.3) to (2.32) are given by

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial x}\right) = \frac{\partial u_2}{\partial x} = 0, \quad \epsilon_{22} = \frac{1}{2}\left(\frac{\partial v_2}{\partial y} + \frac{\partial v_2}{\partial y}\right) = \frac{\partial v_2}{\partial y} = 0,\\
\epsilon_{33} &= \frac{1}{2}\Big(\frac{\partial w_2}{\partial z} + \frac{\partial w_2}{\partial z}\Big) = \frac{\partial w_2}{\partial z} = 0, \quad \epsilon_{23} = \frac{1}{2}\Big(\frac{\partial v_2}{\partial z} + \frac{\partial w_2}{\partial y}\Big) = \frac{1}{2}\left(\frac{\partial v_2}{\partial z}\right),\\
\epsilon_{13} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right) = 0, \quad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial v_2}{\partial x}\right).
\end{aligned} \tag{2.33}$$

Here from (2.33), only two components of strain tensor are non-zero and thus their corresponding two shear stresses are also non-zero. Substituting (2.33) into (2.32) yields the stresses

$$\eta_{11} = \eta_{13} = \eta_{22} = \eta_{33} = 0, \quad \eta_{23} = c_{44}\frac{\partial v_1}{\partial z} + c_{46}\frac{\partial v_1}{\partial x}, \qquad \eta_{12} = c_{46}\frac{\partial v_1}{\partial z} + c_{66}\frac{\partial v_1}{\partial x}, \tag{2.34}$$

where c_{44}, c_{46}, and c_{46} are elastic constants in the half-space. Inserting equations (2.34) into equation (2.31), yields

$$\rho_1\frac{\partial^2 v_2}{\partial t^2} = \frac{\partial}{\partial x}\Big[c_{46}\frac{\partial v_2}{\partial z} + c_{66}\frac{\partial v_1}{\partial x}\Big] + \frac{\partial}{\partial z}\Big[c_{44}\frac{\partial v_2}{\partial z} + c_{46}\frac{\partial v_2}{\partial x}\Big]. \tag{2.35}$$

Simplifying and dividing equation (2.35) by a_{44} gives,

$$\frac{\rho_1}{c_{44}}\frac{\partial^2 v_2}{\partial t^2} = \frac{\partial^2 v_2}{\partial z^2} + 2\frac{c_{46}}{c_{44}}\frac{\partial^2 v_2}{\partial x \partial z} + \frac{c_{66}}{c_{44}}\frac{\partial^2 v_2}{\partial x^2}. \tag{2.36}$$

Denoting

$$\psi = \frac{c_{66}}{c_{44}}, \quad \gamma = \frac{c_{46}}{c_{44}}, \quad c_2^2 = \frac{c_{66}}{\rho_1}, \tag{2.37}$$

yield

$$\frac{\rho_1}{c_{44}}\frac{\partial^2 v_2}{\partial t^2} = \frac{\partial^2 v_2}{\partial z^2} + 2\gamma\frac{\partial^2 v_2}{\partial z \partial x} + \psi\frac{\partial^2 v_2}{\partial x^2}. \tag{2.38}$$

The wave displacement in the half-space is given by

$$v_2(x, z, t) = V_2(z)\exp\big(i(kx - \omega t)\big), \tag{2.39}$$

where the function $V_2(z)$ describes the amplitude factor of the Love wave as a function of the depth, $k = \omega/c$ denotes the wave number of the Love wave, $\omega = 2\pi f$ is its angular frequency, and c is the phase velocity of the Love wave. Putting (2.39) and its derivatives (with respect to x, z and t) into the PDE (2.38) transforms it into ODE given by

$$\frac{d^2V_2}{dz^2} + 2\gamma ik\frac{dV_2}{dz} - \psi k^2\Big(1 - \frac{c^2}{c_2^2}\Big)V_2 = 0. \tag{2.40}$$

Here, note from (2.37) that, the ratio $\rho_1/c_{44} = \psi\rho_1/c_{66}$. Solving the equation (2.40) yields

$$V_2(z) = A_3 \exp\left(-ik\Upsilon_1 z\right) + A_4 \exp\left(ik\Upsilon_2 z\right), \tag{2.41}$$

where $\Upsilon_1 = \gamma + \sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_2^2} - 1\Big)}$ and $\Upsilon_2 = -\gamma + \sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_2^2} - 1\Big)}$. Hence the displacement and stress component for the lower anisotropic half-space are given by

$$v_2(x,z,t) = \big[A_3 \exp\left(-ik\Upsilon_1 z\right)\big] \exp\big(ik(x-ct)\big), \tag{2.42}$$

$$\eta_{23} = c_{44}\frac{\partial v_2}{\partial z} + c_{46}\frac{\partial v_2}{\partial x}. \tag{2.43}$$

The displacement (2.42) and stress component (2.43) in the anisotropic half-space has many elastic stiffness constants (c_{44}, c_{c6}, etc.) as compared to only one elastic constant (shear modulus μ_2) in the displacement (2.20) and stress component (2.7) in the isotropic layer.

2.3.3 Solution of the Linear Wave Problem in the Anisotropic Layer

The solution to the wave equation in the anisotropic elastic layer (medium 1) is sought for in a similar fashion to that of the half-space. The equation of motion (1.26) is re-written as

$$\rho\frac{d^2u_k}{dt^2} = \sigma_{ki,i} + D_{f\,k}, \quad k = 1,2,3. \tag{2.44}$$

We write the equations of motion in the layer ignoring body forces, following Biot [18, 19]:

$$\rho_0\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{12}}{\partial y} + \frac{\partial\sigma_{13}}{\partial z}, \tag{2.45a}$$

$$\rho_0\frac{\partial^2 v_1}{\partial t^2} = \frac{\partial\sigma_{21}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} + \frac{\partial\sigma_{23}}{\partial z}, \tag{2.45b}$$

$$\rho_0\frac{\partial^2 w_1}{\partial t^2} = \frac{\partial\sigma_{31}}{\partial x} + \frac{\partial\sigma_{32}}{\partial y} + \frac{\partial\sigma_{33}}{\partial z}, \tag{2.45c}$$

where σ_{ki} are the stress, ρ_0 is the density in the upper layer, and (u_1, u_2, u_3) are the components of displacement in the layer. From the assumptions for the Love waves (2.3), we have only one equation out of the three equations in (2.45) as non-zero. That is, $u = w = 0$, y is independent of body forces and v is the function of x, z and t. Thus, the only non-zero equation of the three equations of motion (2.45) is given by

$$\rho_0\frac{\partial^2 v_1}{\partial t^2} = \frac{\partial\sigma_{21}}{\partial x} + \frac{\partial\sigma_{23}}{\partial z}. \tag{2.46}$$

Using the constitutive relation (1.4), the stress-strain relation for the layer is given by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix},$$

or in components,

$$\begin{aligned}
\sigma_{11} &= a_{11}\epsilon_{11} + a_{12}\epsilon_{22} + a_{13}\epsilon_{33} + 2a_{14}\epsilon_{23} + 2a_{15}\epsilon_{13} + 2a_{16}\epsilon_{12}, \\
\sigma_{22} &= a_{12}\epsilon_{11} + a_{22}\epsilon_{22} + a_{23}\epsilon_{33} + 2a_{24}\epsilon_{23} + 2a_{25}\epsilon_{13} + 2a_{26}\epsilon_{12}, \\
\sigma_{33} &= a_{13}\epsilon_{11} + a_{23}\epsilon_{22} + a_{33}\epsilon_{33} + 2a_{34}\epsilon_{23} + 2a_{35}\epsilon_{13} + 2a_{36}\epsilon_{12}, \\
\sigma_{12} &= a_{14}\epsilon_{11} + a_{24}\epsilon_{22} + a_{34}\epsilon_{33} + 2a_{44}\epsilon_{23} + 2a_{45}\epsilon_{13} + 2a_{46}\epsilon_{12}, \\
\sigma_{23} &= a_{15}\epsilon_{11} + a_{25}\epsilon_{22} + a_{35}\epsilon_{33} + 2a_{45}\epsilon_{23} + 2a_{55}\epsilon_{13} + 2a_{56}\epsilon_{12}, \\
\sigma_{31} &= a_{16}\epsilon_{11} + a_{26}\epsilon_{22} + a_{36}\epsilon_{33} + 2a_{46}\epsilon_{23} + 2a_{56}\epsilon_{13} + 2a_{66}\epsilon_{12}.
\end{aligned} \tag{2.47}$$

Here the elastic stiffness constants a_{ik} are given in terms of Voigt notation [39], that is, $a_{1111} = a_{11}$, $a_{2222} = a_{22}$, $a_{3333} = a_{33}$, $a_{4444} = a_{44}$, etc., and

$$\begin{bmatrix} a_{1111} & a_{1122} & a_{1133} & a_{1123} & a_{1113} & a_{1112} \\ a_{2211} & a_{2222} & a_{2233} & a_{2223} & a_{2213} & a_{2212} \\ a_{3311} & a_{3322} & a_{3333} & a_{3323} & a_{3313} & a_{3312} \\ a_{2311} & a_{2322} & a_{2333} & a_{2323} & a_{2313} & a_{2312} \\ a_{1311} & a_{1322} & a_{1333} & a_{1323} & a_{1313} & a_{1312} \\ a_{1211} & a_{1222} & a_{1233} & a_{1223} & a_{1213} & a_{1212} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix}.$$

Now assume that for small deformations, the linear strain is given by (1.7). Therefore, the strain-displacement relations after applying (2.3) to (2.47) become

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial x}\right) = \frac{\partial u_1}{\partial x} = 0, \quad \epsilon_{22} = \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_1}{\partial y}\right) = \frac{\partial v_1}{\partial y} = 0, \\
\epsilon_{33} &= \frac{1}{2}\left(\frac{\partial w_1}{\partial z} + \frac{\partial w_1}{\partial z}\right) = \frac{\partial w_1}{\partial z} = 0, \quad \epsilon_{23} = \frac{1}{2}\left(\frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial v_1}{\partial z}\right), \\
\epsilon_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x}\right) = 0, \quad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial v_1}{\partial x}\right).
\end{aligned} \tag{2.48}$$

It is seen from (2.48) that only two components of the strain tensor, ϵ_{12} and ϵ_{23}, are non-zero, and thus the corresponding shear stresses σ_{12} and σ_{23} are non-zero. Thus substitution of (2.48) into (2.46) yields

$$\sigma_{11} = \sigma_{13} = \sigma_{22} = \sigma_{33} = 0, \ \sigma_{23} = a_{44}\frac{\partial v_1}{\partial z} + a_{46}\frac{\partial v_1}{\partial x}, \text{ and } \sigma_{12} = a_{46}\frac{\partial v_1}{\partial z} + a_{66}\frac{\partial v_1}{\partial x}. \tag{2.49}$$

Inserting equations (2.49) into (2.46) yields

$$\rho_0 \frac{\partial^2 v_2}{\partial t^2} = \frac{\partial}{\partial x}\Big[a_{46}\frac{\partial v_2}{\partial z} + a_{66}\frac{\partial v_1}{\partial x}\Big] + \frac{\partial}{\partial z}\Big[a_{44}\frac{\partial v_2}{\partial z} + a_{46}\frac{\partial v_2}{\partial x}\Big]. \tag{2.50}$$

Simplifying and dividing through (2.50) by a_{44} yields,

$$\frac{\rho_0}{a_{44}}\frac{\partial^2 v_1}{\partial t^2} = \frac{\partial^2 v_1}{\partial z^2} + 2\frac{a_{46}}{c_{44}}\frac{\partial^2 v_1}{\partial x \partial z} + \frac{a_{66}}{a_{44}}\frac{\partial^2 v_1}{\partial x^2}. \tag{2.51}$$

Putting

$$\beta = \frac{a_{66}}{a_{44}}, \quad \alpha = \frac{a_{46}}{a_{44}}, \quad c_1^2 = \frac{a_{66}}{\rho_2}, \tag{2.52}$$

one has

$$\frac{\rho_0}{a_{44}}\frac{\partial^2 v_1}{\partial t^2} = \frac{\partial^2 v_1}{\partial z^2} + 2\alpha\frac{\partial^2 v_1}{\partial z \partial x} + \beta\frac{\partial^2 v_1}{\partial x^2}. \tag{2.53}$$

The wave displacement in the layer is given by

$$v_1(x,z,t) = V_1(z)\exp\big(i(kx - \omega t)\big), \tag{2.54}$$

where the function $V_1(z)$ describes the amplitude factor of the Love wave as a function of the depth, $k = \omega/c$ denotes the wave number of the Love wave, $\omega = 2\pi f$ is its angular frequency, and c is the phase velocity of the Love wave. Putting (2.54) and its derivatives (with respect to x, z and t) into the PDE (2.53) transforms it into ODE given by

$$\frac{d^2 V_1}{dz^2} + 2\alpha i k\frac{dV_1}{dz} - \beta k^2\Big(1 - \frac{c^2}{c_1^2}\Big)V_1 = 0. \tag{2.55}$$

Here, note from (2.52) that, the ratio $\rho_0/a_{44} = \beta\rho_0/a_{66}$. Solving the equation (2.55) yields

$$V_1(z) = A_1\exp\left(-ik\xi_1 z\right) + A_2\exp\left(ik\xi_2 z\right), \tag{2.56}$$

where $\xi_1 = \alpha + \sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}$ and $\xi_2 = -\alpha + \sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}$. Hence the displacement and stress component for the anisotropic layer are given by

$$v_1(x,z,t) = \big[A_1\exp\left(-ik\xi_1 z\right) + A_2\exp\left(ik\xi_2 z\right)\big]\exp\big(ik(x - ct)\big), \tag{2.57}$$

$$\sigma_{23} = a_{44}\frac{\partial v_1}{\partial z} + a_{46}\frac{\partial v_1}{\partial x}. \tag{2.58}$$

The displacement (2.57) and stress component (2.58) in the anisotropic layer posses many elastic stiffness constants (a_{44}, a_{46}, etc.) as compared to only one elastic constant (shear modulus μ_1) in the displacement (2.21) and stress component (2.7) in the isotropic layer.

2.3.4 Boundary Conditions

To obtain the full solution to the problem (2.53) and (2.38), let us first assume that both the anisotropic elastic layer and the anisotropic elastic half-space are in welded contact and the required boundary conditions for the propagation of Love wave are given as follows

1. At the interface $z = -H$, $\sigma_{23} = 0$ where there is no stress due to free boundary surface (i.e., the normal component of the stress vanishes on the boundary).

2. At the interface $z = 0$, the displacement components are continuous.

3. At the interface $z = 0$, the stress is continuous, that is, $\sigma_{23} = \eta_{23}$.

2.3.5 Dispersion Relation for Love Waves in an Anisotropic Layer Overlying an Anisotropic Half-space

Using the boundary conditions provided in section (2.3.4) as numbered from 1 to 3 in (2.57), (2.58),(2.42), and (2.43), leads to three linear homogeneous equations with three unknown constants A_1, A_2, and A_3 [110]. From the first boundary condition ($\sigma_{23} = 0$ at the interface $z = -H$), we have

$$\sigma_{23}\Big|_{z=-H} = a_{44}\frac{\partial v_1}{\partial z} + a_{46}\frac{\partial v_1}{\partial x}\Big|_{z=-H} = 0. \tag{2.59}$$

Substituting (2.57) into (2.58) yields,

$$\begin{aligned} a_{44}\frac{\partial}{\partial z}\Big(\left(A_1\exp\left(-ik\xi_1 z\right) + A_2\exp\left(ik\xi_2 z\right)\right)\exp\big(ik(x-ct)\big)\Big)\Big|_{z=-H} \\ +a_{46}\frac{\partial}{\partial x}\Big(\left(A_1\exp\left(-ik\xi_1 z\right) + A_2\exp\left(ik\xi_2 z\right)\right)\exp\big(ik(x-ct)\big)\Big)\Big|_{z=-H} = 0. \end{aligned} \tag{2.60}$$

Now simplifying (2.60), we get the first linear homogeneous equation from the first boundary condition denoted by

$$\Big(\big(a_{46}\left(\exp\left(ik\xi_1 H\right)\right) - a_{44}\xi_1\left(\exp\left(ik\xi_1 H\right)\right)A_1 + \big(a_{46}\left(\exp\left(-ik\xi_2 H\right)\right) + a_{44}\xi_2\left(\exp\left(-ik\xi_2 H\right)\right)\big)A_2\Big). \tag{2.61}$$

Next, using the second boundary condition ($v_1 = v_2$ at the interface $z = 0$) gives

$$\Big(\left(A_1\exp\left(-ik\xi_1 z\right) + A_2\exp\left(ik\xi_2 z\right)\right)\exp\big(ik(x-ct)\big)\Big)\Big|_{z=0} = \Big(\left(A_3\exp\left(-ik\Upsilon_1 z\right)\right)\exp\big(ik(x-ct)\big)\Big)\Big|_{z=0}.$$

After evaluating at $z = 0$ gives the second linear homogeneous equation

$$A_1 + A_2 - A_3 = 0. \tag{2.62}$$

Using the last boundary condition ($\sigma_{23} = \eta_{23}$ at the interface $z = 0$) gives

$$\begin{aligned} a_{44}\frac{\partial}{\partial z}\Big(\left(A_1\exp\left(-ik\xi_1 z\right) + A_2\exp\left(ik\xi_2 z\right)\right)\exp\big(ik(x-ct)\big)\Big) \\ +a_{46}\frac{\partial}{\partial x}\Big(\left(A_1\exp\left(-ik\xi_1 z\right) + A_2\exp\left(ik\xi_2 z\right)\right)\exp\big(ik(x-ct)\big)\Big)\Bigg|_{z=0} \\ = c_{44}\frac{\partial}{\partial z}\Big(\left(A_3\exp\left(-ik\Upsilon_1 z\right)\right)\exp\big(ik(x-ct)\big)\Big) \\ +c_{46}\frac{\partial}{\partial x}\Big(\left(A_3\exp\left(-ik\Upsilon_1 z\right)\right)\exp\big(ik(x-ct)\big)\Big)\Bigg|_{z=0}. \end{aligned} \tag{2.63}$$

Simplifying and evaluating (2.63) at $z = 0$ yields

$$(a_{46} - a_{44}\xi_1)A_1 + (a_{46} + a_{44}\xi_2)A_2 - (c_{46} - c_{44}\Upsilon_1)A_3 = 0. \tag{2.64}$$

Now the three homogeneous equations (2.62), (2.64), and (2.59) are given by

$$\begin{gathered} A_1 + A_2 - A_3 = 0, \\ (a_{46} - a_{44}\xi_1)A_1 + (a_{46} + a_{44}\xi_2)A_2 - (c_{46} - c_{44}\Upsilon_1)A_3 = 0, \\ (a_{46}\left(\exp(ik\xi_1 H)\right) - a_{44}\xi_1\left(\exp(ik\xi_1 H)\right)A_1 + (a_{46}\left(\exp(-ik\xi_2 H)\right) + a_{44}\xi_2\left(\exp(-ik\xi_2 H)\right))A_2 = 0. \end{gathered} \tag{2.65}$$

The homogeneous system (2.65) is given in a matrix form by

$$\begin{bmatrix} 1 & 1 & -1 \\ a_{46} - a_{44}\xi_1 & a_{46} + a_{44}\xi_2 & -c_{46} + c_{44}\Upsilon_1 \\ a_{46}\exp(ik\xi_1 H) - a_{44}\xi_1\exp(ik\xi_1 H) & (a_{46} + a_{44}\xi_2)\exp(-ik\xi_2 H) & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0. \tag{2.66}$$

The system (2.65) has a non-trivial solution if the determinant of the coefficient matrix of the linear system (2.66) is zero. The unknown constants A_1, A_2 and A_3 are therefore eliminated from the three homogeneous equations as

$$\begin{vmatrix} 1 & 1 & -1 \\ a_{46} - a_{44}\xi_1 & a_{46} + a_{44}\xi_2 & -c_{46} + c_{44}\Upsilon_1 \\ a_{46}\exp(ik\xi_1 H) - a_{44}\xi_1\exp(ik\xi_1 H) & (a_{46} + a_{44}\xi_2)\exp(-ik\xi_2 H) & 0 \end{vmatrix} = 0. \tag{2.67}$$

In order to derive the dispersion relation, we take the determinant of the coefficient matrix in (2.66) and equate to zero as given by (2.67). Simplifying leads to the expression

$$\tanh\left[i\sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}kH\right] = \frac{\theta_1\sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}\sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_2^2} - 1\Big)}}{\beta\Big(\frac{c^2}{c_1^2} - 1\Big) + \left(\alpha + \sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)} - \left[\gamma + \sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_2^2} - 1\Big)}\right]\right)\Phi_1} \tag{2.68}$$

where $\theta_1 = c_{44}/a_{44}$ and $\Phi_1 = c_{46}/a_{44}$. Using the relation $\tanh(ix) = i\tan(x)$, (2.68) can be written as

$$i\tan\left[\frac{2\pi f H}{c}\sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}\right] = \frac{\theta_1\sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)}\sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_2^2} - 1\Big)}}{\beta\Big(\frac{c^2}{c_1^2} - 1\Big) + \left(\alpha + \sqrt{\alpha^2 + \beta\Big(\frac{c^2}{c_1^2} - 1\Big)} - \left[\gamma + \sqrt{\gamma^2 + \psi\Big(\frac{c^2}{c_2^2} - 1\Big)}\right]\right)\Phi_1}, \tag{2.69}$$

where $k = \omega/c$, the angular frequency $\omega = 2\pi f$, c is the velocity of Love wave, and f is the frequency. Here one can choose the term $\gamma^2 + \psi\Big(c^2/c_2^2 - 1\Big) < 0$. The equation (2.69) has no relevant solution if $\alpha^2 + \beta\Big(c^2/c_1^2 - 1\Big) < 0$. Equation (2.68) is called the dispersion equation for the Love waves. The unknown terms are the phase velocities (c/c_1), (c/c_2) and the wave number k. Here the presence of the transcendental relation (2.69) of the frequency under the sign of tangent shows the nonlinear dependence of Love wave phase velocity on frequency. The relation (2.69) is a generalized dispersion equation for Love wave in anisotropic media. Other forms of dispersion relations for orthotropic and transversely isotropic materials can be obtained from (2.69).

2.3.6 The isotropic case

For isotropic materials, one has $a_{44} = a_{66} = \mu_1$, $a_{46} = 0$, $c_{44} = c_{66} = \mu_2$, $c_{46} = 0$. Using these constants and the relation $\tanh(ix) = i\tan(x)$ (2.68) becomes

$$\tan\left[\sqrt{\left(\frac{c^2}{c_1^2} - 1\right)}kH\right] = \frac{\mu_2}{\mu_1}\frac{\sqrt{\left(1 - \frac{c^2}{c_2^2}\right)}}{\sqrt{\left(\frac{c^2}{c_1^2} - 1\right)}}. \tag{2.70}$$

The above equation (2.70) is the same as the classical Love wave dispersion equation derived in [3], and written above in equation (2.25). The dispersion relation (2.70) coincides with the dispersion relation obtained in (2.25), thus the phase velocity depends on frequency and that Love waves are dispersive waves. This serves as a validation check for the obtained dispersion relation. The dependence of phase velocity on the wave number was analyzed in [110].

2.4 Discussion

In this chapter, we considered linear Love waves in elastic media. We discussed a Love wave propagating in a homogeneous isotropic layer overlying a homogeneous elastic half-space. We also considered linear Love waves in anisotropic layer overlying an anisotropic half-space. Using the appropriate boundary conditions, we obtained the full solution to the linear wave equation in the isotropic layer, and the isotropic elastic half-space was presented. Also, the complete solution to the linear wave equation in the anisotropic layer and the anisotropic half-space were obtained. The dispersion relations for Love waves in both the isotropic and anisotropic media were presented. From the dispersion relation of the linear Love wave in anisotropic media, it was shown how to obtain the dispersion relation for a linear Love wave in isotropic media [3].

We now discuss work in the literature related to linear Love surface waves. Love surface waves in elastic media constitute a broad topic. Many authors, including Abd-Alla and Ahmed [1] and Farnell and Adler [50], have extensively researched it, primarily related to anisotropic and isotropic media. Significant work has been done on the solutions to both classical linear elastic Love wave and nonlinear elastic Love waves. Love [82] in his early work studied the dispersion of Rayleigh and Love waves, and talked extensively on solid elastic half-space being covered by a single solid layer. Abd-Alla and Ahmed [1] studied Love waves propagating in a non-homogeneous orthotropic elastic medium under changeable initial stress. They employed the Fourier transform method for finding the dispersion equation.

Farnell and Adler [50] studied wave traveling in thin layers. They discussed in detail both Love waves and Rayleigh waves in anisotropic and isotropic media. Bouden and Datta [25] investigated the effects of anisotropy of the transversely isotropic substrate on the dispersive wave propagation in an isotropic layer. Ohnabe and Nowinski [106] discussed Love waves in an elastic isotropic half-space with a superficial layer of differing incompressible material. Ahmed and Abo-Dahab [5] in their work studied the propagation of Love waves in an orthotropic granular layer under initial stress overlying a semi-infinite granular medium. They used the Fourier transform method in finding the dispersion relation.

Vaishnav *et al* [137] investigated the propagation of the Love-type wave in an initially stressed porous medium over a semi-infinite orthotropic medium with a rectangular irregular interface. They employed the method of separation of variables in finding the dispersion relation of the Love-type wave. Chattopadhyay and Kar [32] studied Love wave propagation in an isotropic homogeneous medium due to a point source under initial stress. They studied Love waves in a porous layer overlying an inhomogeneous half-space. The Green's function technique was applied in deriving the dispersion equation for Love waves in the porous layer.

The propagation of Love waves in a homogeneous medium overlying an inhomogeneous half-space was studied in [54]. The propagation of SH waves in a homogeneous viscoelastic isotropic layer lying over a semi-infinite heterogeneous viscoelastic isotropic half-space due to point source was presented in [30]. In this work, the authors assumed the inhomogeneity parameters associated with rigidity, internal friction, and density to be functions of depth. They employed the Green's function technique to obtain the dispersion relation of the SH wave. Chattopadhyay *et al* [31] investigated the propagation of SH wave due to a point source in a magnetoelastic self-reinforced layer overlying a heterogeneous self-reinforced half-space. They used the Green's function to derive the dispersion relation in the closed form.

The propagation of Love waves in a poroelastic layer resting over a poroelastic half-space was discussed by [43]. The authors' study revealed that such a medium transmits two types of Love waves, and they determined the velocity of poroelastic half-space and poroelastic layer. Ke, *et al.* [75], in their work, dealt with Biot's theory for transversely isotropic fluid-saturated porous media. They also derived dispersion Love waves equation in a transversely isotropic media with an inhomogeneous layer under consideration. They solved this equation by using numerical methods. Love waves in a heterogeneous orthotropic layer under changeable initial stress over a gravitating porous half-space was studied in [123]. Saha *et al.* [123] derived the dispersion equation of Love waves in closed-form using the variable separation approach. In the next chapter, we will present some shear wave propagation models in nonlinear hyperelasticity framework. In particular, nonlinear Love waves in isotropic homogeneous elastic media are discussed.

3 SOME WAVE PROPAGATION MODELS IN NONLINEAR HYPERELASTICITY FRAMEWORK

3.1 Introduction and Motivation

Materials such as polymers, rubbers, foams, and biological tissues exhibit essential nonlinear elastic behavior when they undergo large elastic deformations. This nonlinear elastic behavior under load or prescribed displacement can be modeled through phenomenological approach [86], physical description approach [134], or use of experimental data [105]. The strain energy density derived from a physical description approach is somewhat complex and material-specific. In a phenomenological approach, one normally treats the solid or material as a continuum and postulates a strain energy density, which is defined in terms of the deformation invariants. Phenomenological models are used to derive constitutive equations for a chosen stress tensor in general coordinates. A good phenomenological model provides stable results for a range of loadings and in application to various materials. As seen in Chapter 1, there are multiple proposed strain energy density functions in literature. These strain energy density functions are classified into those dealing with compressible and incompressible materials, those depending on the material being modeled, for example, polymer, foam, etc., [11], and those that do or do not satisfy the Valanis-Landel hypothesis [138]. The Valanis-Landel hypothesis states that a strain energy density function for rubbers can be written as a sum of independent functions of the principal stretches:

$$W = u(\lambda_1) + u(\lambda_2) + u(\lambda_3), \tag{3.1}$$

where λ_1, λ_2, and λ_3 are the principal stretches.

In nonlinear hyperelasticity modeling, it is also important to account for different types of shear deformation. Shear deformation (stress-strain response) has gained significant attention from researchers in recent times. The two main ways of describing shear deformation are simple shear and pure shear. These are well-known concepts in continuum mechanics. A pure shear may be defined in mechanics as a 3D-homogeneous flattening of a body. It is the deformation in the X-axis resulting in no change of area. Simple shear results when a body is subjected to a uniform shear, parallel to some direction, involving no change in area. For small deformation, pure shear and simple shear differ by only a rotation. That is, pure shear may be considered a simple shear followed by a rigid rotation. But in the case of large deformations, the relationship between these two notions is not well-defined [84]. Much research has been conducted on the comparison of simple and pure shears for incompressible isotropic hyperelastic materials. (See [94, 134] and references therein for details.)

In hyperelastic constitutive models, compressible and incompressible materials are of great importance. The study of the mathematical and computational modeling of incompressible materials was conducted by Ogden [100, 103, 104] and many authors. Several authors, including Horgan and Murphy [66, 68], have developed accurate, effective, and efficient ways of modeling rubber-like materials, polymers, and biological tissues. This aspect of nonlinear elasticity deals with nonlinear material laws (effects of large strain), geometric nonlinearity (large deformations), and accurate modeling of incompressible materials. Rivlin [118] studied a cuboidal block of either a compressible or incompressible material subject to simple strain. (See [4] and references therein for details.)

In this chapter, we present equations of motion for some specific incompressible and compressible hyperelastic models of shear waves. We discuss shear waves and present linear Love wave modeling in the hyperelasticity framework in Subsection 3.2. Next, we study in general the propagation of linear shear horizontal waves in Section 3.2.1. In Section 3.3, we first review nonlinear Love wave in literature and we formulate a nonlinear Love wave problem which from a geometrical point of view coincides with the formulation of linear Love waves in chapter 2, but from a mechanics point of view, it is different (That is, the layer and the half-space consist of nonlinear elastic materials with different properties, the materials follow the Murnaghan model, and the half-space and layer are in perfect mechanical contact). We also present equations for a nonlinear Love wave using the Murnaghan potential [97]. In Section 3.3.6, we present some compressible and incompressible shear wave one-dimensional models. Also, we present wave equations corresponding to the neo-Hookean, Mooney-Rivlin, and the Murnaghan material models. In Section 3.4, we present two-dimensional and one-dimensional vertical shear wave models in hyperelasticity framework. Some solutions to linear and nonlinear wave models are presented in Section 3.5. The equations of motion presented here are in terms of both actual and reference coordinates. We derive the nonlinear wave equations for hyperelastic models of shear waves. Finally, we give a general overview of the chapter in Section 3.7. In what follows, let us discuss shear waves.

3.2 Linear Love Wave Modeling in the Hyperelasticity Framework

Let us consider a general motion of an elastic medium, represented in the material framework by

$$\mathbf{x} = \boldsymbol{\xi}\left(\mathbf{X}, t\right) = \mathbf{X} + \mathbf{u}, \tag{3.2}$$

where $\mathbf{x}$ and $\mathbf{X}$ are respectively the Eulerian and the Lagrangian coordinates, and $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ represents the displacements of the corresponding material points. The displacements are assumed not to be small, and most of the equations are presented in terms of actual particle positions $\mathbf{x}$. The material coordinates may be chosen to coincide with the initial conditions: $\boldsymbol{\xi}(\mathbf{X}, 0) = \mathbf{X}$. The velocity and acceleration of the material point $\mathbf{X}$ are given by

$$\mathbf{u}(\mathbf{X},t) = \frac{\partial \mathbf{x}}{\partial t}, \quad \mathbf{a}(\mathbf{X},t) = \frac{\partial \mathbf{u}}{\partial t}.$$

The mapping (3.2) is assumed to be sufficiently smooth.

3.2.1 Linear Horizontal Shear Wave Equation

Let (x, y, z) and (X, Y, Z) be the Eulerian and Lagrangian Cartesian coordinates of a given material point in a three-dimensional domain. Let (u, v, w) represent the displacements along (X, Y, Z) axes, respectively. Let $v = v(X, Z, t)$ represent the displacement along the $Y-$direction, and let ρ_0 represent the mass density. The waves propagate in the $XZ-$plane since $v = v(X, Z, t)$; the displacement is transverse to the $XZ-$plane. Such *antiplane motion* [48, 49] is described by the equations

$$\begin{aligned} x &= X, \\ y &= Y + v(X, Z, t), \\ z &= Z. \end{aligned} \tag{3.3}$$

The matrices denoting the deformation gradient and its inverse are given by

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ v_X & 1 & v_Z \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -v_X & 1 & -v_Z \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.4}$$

where $v_X = \partial v/\partial X$, and $v_Z = \partial v/\partial Z$ represent the amount of shear. The right and left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ are given respectively by

$$\mathbf{C} = \begin{bmatrix} 1+(v_X)^2 & v_X & v_X v_Z \\ v_X & 1 & v_Z \\ v_X v_Z & v_Z & 1+(v_Z)^2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & v_X & 0 \\ v_X & (v_X)^2+(v_Z)^2+1 & v_Z \\ 0 & v_Z & 1 \end{bmatrix}. \tag{3.5}$$

The principal invariants (the first three invariants of $\mathbf{B}$) are given by

$$I_1 = I_2 = v_X^2 + v_Z^2 + 3 = 3 + |\nabla v|^2, \quad I_3 = 1. \tag{3.6}$$

Since det $\mathbf{F} = 1$, the deformation field defined by (3.3) is isochoric (incompressible), and the density ρ_0 in the motion remains unaltered. Now let $\mathbf{P}$ be the first Piola-Kirchhoff stress tensor associated with the deformation field (3.3) given in the classical relation (1.50). In the absence of body forces, the equation of motion (1.40a) becomes

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial P_{11}}{\partial X} + \frac{\partial P_{12}}{\partial Y} + \frac{\partial P_{13}}{\partial Z}, \tag{3.7a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \frac{\partial P_{21}}{\partial X} + \frac{\partial P_{22}}{\partial Y} + \frac{\partial P_{23}}{\partial Z}, \tag{3.7b}$$

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = \frac{\partial P_{31}}{\partial X} + \frac{\partial P_{32}}{\partial Y} + \frac{\partial P_{33}}{\partial Z}. \tag{3.7c}$$

Considering the situation of propagation of a transversely polarized harmonic wave where displacements $u = w = 0$, (3.7) becomes

$$0 = \frac{\partial P_{11}}{\partial X} + \frac{\partial P_{12}}{\partial Y} + \frac{\partial P_{13}}{\partial Z}, \tag{3.8a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \frac{\partial P_{21}}{\partial X} + \frac{\partial P_{22}}{\partial Y} + \frac{\partial P_{23}}{\partial Z}, \tag{3.8b}$$

$$0 = \frac{\partial P_{31}}{\partial X} + \frac{\partial P_{32}}{\partial Y} + \frac{\partial P_{33}}{\partial Z}. \tag{3.8c}$$

Now, we assume that the material is homogeneous, nonlinear, isotropic, incompressible and elastic, and strain energy density function is a function of the first principal invariant of the left Cauchy-Green tensor Consequently, the strain energy density, in this case, is a function of v_X and v_Z, given by

$$W = W_{\text{iso}}(I_1) = W\left(v_X^2 + v_Z^2 + 3\right), \tag{3.9}$$

depending only on I_1 (3.6). If we assume that we have a generalized neo-Hookean material [49], then stress constitutive equation for an incompressible neo-Hookean material can be expressed by $W = aI_1$ given in Table 1.1. Using the relation (1.66c), the transpose of the inverse of the deformation gradient (3.4), and the strain energy density for the incompressible neo-Hookean materials, one can write the first Piola-Kirchhoff stress tensor explicitly as

$$\mathbf{P} = \begin{bmatrix} 2a - p & pv_X & 0 \\ 2av_X & 2a - p & 2av_Z \\ 0 & pv_Z & 2a - p \end{bmatrix}. \tag{3.10}$$

where $p = p(X, Y, Z, t)$ is the hydrostatic pressure function which can be determined under certain restrictions as in [130]. The constant a associated with the neo-Hookean materials is related to the Lamé elastic constants by the relation $a = \mu/2$, where μ is the modulus of rigidity. The stress components (3.10) are written in terms of the modulus of rigidity as

$$P_{11} = P_{22} = P_{33} = \mu - p, \;\; P_{12} = pv_X, \quad P_{21} = \mu v_X, \;\; P_{23} = \mu v_Z, \quad P_{32} = pv_Z, \quad P_{13} = P_{31} = 0. \tag{3.11}$$

Substituting the components of the first Piola-Kirchhoff stress tensor in (3.11) into the equations of motion (3.8) yield the following system:

$$0 = \frac{\partial}{\partial X}\left(-p + \mu\right) + \frac{\partial}{\partial Y}\left(pv_X\right), \tag{3.12a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial X}\left(\mu v_X\right) + \frac{\partial}{\partial Y}\left(-p + \mu\right) + \frac{\partial}{\partial Z}\left(\mu\, v_Z\right), \tag{3.12b}$$

$$0 = \frac{\partial}{\partial Y}\left(p\, v_Z\right) + \frac{\partial}{\partial Z}\left(-p + \mu\right). \tag{3.12c}$$

The equations of motion (3.12) can be written after simplification as

$$0 = -\frac{\partial p}{\partial X} + \frac{\partial \mu}{\partial X} + \frac{\partial p}{\partial Y}\frac{\partial v}{\partial X}, \tag{3.13a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \mu\left(\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Z^2}\right) - \frac{\partial p}{\partial Y}, \tag{3.13b}$$

$$0 = \frac{\partial p}{\partial Y}\frac{\partial v}{\partial Z} - \frac{\partial p}{\partial Z} + \frac{\partial \mu}{\partial Z}. \tag{3.13c}$$

Now from (3.13b), we have

$$\frac{\partial p}{\partial Y} = -\rho_0 \frac{\partial^2 v}{\partial t^2} + \frac{\partial}{\partial X}(\mu v_X) + \frac{\partial}{\partial Z}(\mu v_Z), \tag{3.14}$$

and since the right hand side of (3.14) is independent of Y, it follows that the hydrostatic pressure p is linear in Y. Therefore, equations (3.12a) and (3.12c) require that $\partial p/\partial X$ and $\partial p/\partial Z$ be independent of Y and it is found that

$$\frac{\partial p}{\partial Y} = \alpha_0(X, Z, t). \tag{3.15}$$

Integrating (3.15) with respect to Y gives

$$p = \alpha_0(X, Z, t)Y + \alpha_1(X, Z, t), \tag{3.16}$$

where α_0 and α_1 are arbitrary functions. Using (3.16) in (3.12a) and (3.12c) gives

$$\frac{\partial \alpha_0}{\partial X} = 0, \quad \frac{\partial \alpha_0}{\partial Z} = 0. \tag{3.17}$$

Hence

$$\alpha_0 = q(t), \tag{3.18}$$

where q are arbitrary functions and the equations (3.12a) and (3.12c) become

$$0 = -\frac{\partial \alpha_1}{\partial X} + \frac{\partial \mu}{\partial X} + v_X q(t), \quad 0 = -\frac{\partial \alpha_1}{\partial Z} + \frac{\partial \mu}{\partial Z} + v_Z q(t). \tag{3.19}$$

By integrating (3.19) we find that

$$\alpha_1 = \mu + vq(t) - s(t), \tag{3.20}$$

where s is an arbitrary function, and hence

$$P_{11} = P_{22} = P_{33} = -vq(t) + s(t) - Yq(t),\ P_{2k} = \big(\mu + vq(t) - s(t) + Yq(t)\big)\, v_{\mathbf{X}},\ k = 1, 3,\ \mathbf{X} = (X, Z). \tag{3.21}$$

From the results in (3.21), it is observed that the stress components depend linearly on Y. This is a non-physical situation because all the stress components must remain bounded at $Y \to \pm\infty$. Hence we must have $q(t) = 0$. Now if we also assume that the natural state of the materials are stress free, then $s(t) = 0$. In the simplest case $q(t) = s(t) = 0$, one finds that the hydrostatic pressure is constant $p = \mu$. The stress tensor components in this case become

$$P_{11} = P_{22} = P_{33} = P_{13} = P_{31} = 0, \quad P_{12} = P_{21} = \mu v_X, \quad P_{23} = P_{32} = \mu v_Z, \tag{3.22}$$

and equations (3.12a) and (3.12c) are satisfied identically. This yields the linear wave equation for the displacement v

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \mu \left(\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Z^2} \right), \tag{3.23}$$

and the equations (3.13a) and (3.13c) are identically satisfied. The wave equation (3.23) is the common linear constant-coefficient two-dimensional wave equation (see, e.g., [55] for details).

3.2.2 Linear Love Waves in the Hyperelasticity Framework

Let us consider the propagation of small but finite amplitude Love wave-type displacement in an elastic half-space ($Z < 0$) covered by an elastic layer of uniform thickness H with different elastic properties. Let (u, v, w) represent the displacement along (X, Y, Z) axes respectively. Let the Y-displacements in the layer and the half-space be given by v^1 and v^2 and represent the displacements along $Y-$directions in the layer and half-space respectively, ρ_1, and ρ_2 represent the mass densities in the layer and half-space respectively, and μ_1 and μ_2 represent the modulus of rigidity in the layer and half-space respectively. The wave travels in the $XZ-$ plane since $v = v(X, Z, t)$. It is also assumed that the boundary $Z = -H$ is free of tractions, and the stress and displacements are continuous at the interface $Z = 0$. Hence, the condition of vanishing tractions (vanishing of the normal component of the stresses) on the surface $Z = -H$ imposes the boundary condition $P^1_{23} = 0$ on $Z = -H$, and the continuity of stresses, and the displacements at the interface $Z = 0$ are satisfied if $P^1_{23} = P^2_{23}$, and $v^{(1)} = v^{(2)}$ on $Z = 0$. The geometry of the problem is shown in Figure 2.1.

Let the Love wave model in the layered half-space with displacement in the $Y-$axis [48, 49] be described by the equations

$$\begin{aligned} x &= X, \\ y &= Y + v^j(X, Z, t), \quad j = 1, 2, \\ z &= Z, \end{aligned} \tag{3.24}$$

where v^j ($j = 1$ is for the layer and $j = 2$ for the half-space) is the antiplane displacement of the wave in the $Y-$ direction.

The simplest model for the Love waves can be obtained by using the incompressible hyperelasticity framework with the neo-Hookean constitutive equation as in Section 3.2.1. Then one can use the formulas from Section 3.2.1 for $j = 1,\ 2$ in two different media. The equations in the two media together with boundary conditions are given by

$$\rho_j \frac{\partial^2 v^{(j)}}{\partial t^2} = \mu_j \left(\frac{\partial^2 v^{(j)}}{\partial X^2} + \frac{\partial^2 v^{(j)}}{\partial Z^2} \right), \quad j = 1, 2, \tag{3.25a}$$

$$\frac{\partial v^{(1)}}{\partial Z} = 0 \ \text{ on the free boundary } Z = -H, \tag{3.25b}$$

$$v^{(1)} = v^{(2)} \text{ and } \mu_1 \frac{\partial v^{(1)}}{\partial Z} = \mu_2 \frac{\partial v^{(2)}}{\partial Z}, \text{ when } Z = 0. \tag{3.25c}$$

The equations (3.25) are equivalent to equations (2.9) in Chapter 2. The equations (3.25) admit solutions describing waves traveling along the X-direction and decaying in Z-direction given in (2.27) and (2.28).

3.3 Nonlinear Love Wave Modeling in the Hyperelasticity Framework

In the study of elastic waves in seismology, it is usually assumed that the displacements are so small that all nonlinear terms can be neglected from the relevant dynamical equations. Bataille and Lund [16] verified that the displacements and strains in earthquakes were indeed small. Typical earthquake displacement of the ground ranges from 10^{-10}m to 10^{-1}m and typical earthquake strain of the ground ranges from 10^{-6} to 10^{-2}. However, as discussed in [14], when there are physical situations, nonlinear effects can appear in a regime where one would have thought they were negligible. It is, therefore, possible one might miss important physical effects by retaining only linear terms in the equations for elastic waves. Seismic waves might be considered nonlinear when they are close to the focus of an earthquake. Bataille and Lund [16] argued that in the case of body waves, both linear and nonlinear terms are taken into consideration. In the case of surface waves, one is missing some terms since surface waves are dispersive, and nonlinearities may balance this dispersive effect [16]. The study of nonlinear surface wave propagation and the study of the impact of the nonlinearity on the propagation characteristics of elastic surface waves have been of research interest given its applications ranging from industry, and construction, seismology, and technological applications.

Significant research has been conducted on the nonlinear propagation of Love waves (see, e.g., [16, 72, 120, 129]). Kalyanasundaram [72] studied the propagation characteristics of finite amplitudes quasi-monochromatic Love waves on an isotropic half-space using the method of multiple scales. Rushchitsky [120] gave a nonlinear description of Love waves and solved the resulting wave equations by the successive approximation approach. Bataille and Lund [16] derived an equation of Love waves, taking into accounts the dispersive nature as well as the nonlinear effects. Teymur [129] studied the propagation of small but finite amplitude weakly nonlinear Love waves in a layered half-space with homogeneous, isotropic, compressible hyperelastic constitution using the derivative expansion method. In this section, we derive a nonlinear Love wave equation using a strain energy density function suggested in Murnaghan [97] and Bland [20].

3.3.1 Hyperelastic Two-Dimensional Shear Waves

Let us again consider a shear horizontal wave traveling along the $X-$axis in the layered half-space, or antiplane motion [48, 49] described by the equations (3.3), where v is the displacement of the wave in the $Y-$ direction, and t is the time. The matrices of the deformation gradient and its inverse are given in (3.4). The right and left Cauchy-Green tensors ($\mathbf{C}$ and $\mathbf{B}$) are given in (3.5). The principal invariants for the deformation field (3.3) are given by (3.6). The shear displacements (3.3) are naturally incompressible: det $\mathbf{F} \equiv 1$. Let $\mathbf{P}$ represent the first Piola-Kirchhoff stress tensor corresponding to the shear displacements (3.3) given in (1.50) [38]. The equations of motion (1.66a) neglecting body forces are given in (3.7). In the case of propagation of transversely polarized harmonic wave where displacements $u = w = 0$, the equation (3.7) becomes (3.8). One can assume that the material is nonlinear, isotropic, homogeneous, incompressible, and elastic, and the stored energy density function is a function of the first principal invariant of the Cauchy-Green tensor. Here without loss of generality, the strain energy density function is only a function of I_1 and is still given by

$$W(I_1, I_2, I_3) = W(I_1) = W(3 + v_X^2 + v_Z^2). \tag{3.26}$$

Using the relation (1.66c), where $\mathbf{F}^{-T}$ represent the transpose of the inverse of the deformation gradient, the hydrostatic pressure $p = p(X, Z, t)$, and the general strain energy density function (3.26), one can write the first Piola-Kirchhoff stress tensor explicitly as

$$\mathbf{P} = \begin{bmatrix} 2W_1 - p & pv_X & 0 \\ 2v_X W_1 & 2W_1 - p & 2v_Z W_1 \\ 0 & pv_Z & 2W_1 - p \end{bmatrix}, \tag{3.27}$$

where $W_1 \equiv \partial W / \partial I_1$. Substitution of the first Piola-Kirchhoff stress tensor (3.27) into the equation of motion (3.8) yields the following systems:

$$0 = \frac{\partial}{\partial X}(-p + 2W_1) + \frac{\partial}{\partial Y}(pv_X), \tag{3.28a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial X}(2v_X W_1) + \frac{\partial}{\partial Y}(-p + 2W_1) + \frac{\partial}{\partial Z}(2\, v_Z W_1), \tag{3.28b}$$

$$0 = \frac{\partial}{\partial Y}(p\, v_Z) + \frac{\partial}{\partial Z}(-p + 2W_1). \tag{3.28c}$$

The equations of motion (3.28) can be written as

$$\frac{\partial p}{\partial X} = 2\frac{\partial W_1}{\partial X}, \tag{3.29a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = 2\left[\frac{\partial}{\partial X}(v_X W_1) + \frac{\partial}{\partial Z}(v_Z W_1)\right], \tag{3.29b}$$

$$\frac{\partial p}{\partial Z} = 2\frac{\partial W_1}{\partial Z}. \tag{3.29c}$$

From (3.29a) and (3.29c), one consequently finds that the hydrostatic pressure is given by an exact explicit formula

$$p = 2W_1 + p_0(t). \tag{3.30}$$

Since the hydrostatic pressure is defined up to an arbitrary additive function of time which has no effect on the equations of motion, one can assume $p_0(t) = p_0 = \text{const}$. The $Y-$displacement v satisfies the nonlinear wave equation

$$\rho_0 v_{tt} = 2 \text{ div } (W_1 \text{grad v}) = 2\left[\frac{\partial}{\partial X}(v_X W_1) + \frac{\partial}{\partial Z}(v_Z W_1)\right], \tag{3.31}$$

where div and grad are taken with respect to (X, Z). We note that if the gravity force $\mathbf{e} = \rho_0 g \mathbf{e}_Z$ is taken into account, the PDE (3.29c) is replaced by

$$\frac{\partial p}{\partial Z} = \rho_0 g + 2\frac{\partial W_1}{\partial Z}, \tag{3.32}$$

and the pressure instead of (3.30) takes the form

$$p = 2W_1 + \rho_0 g Z, \tag{3.33}$$

Z being directed downward. We also observe that the wave equation (3.31) can be written as

$$\rho_0 v_{tt} = 2W_1 (v_{XX} + v_{ZZ}) + 4W_{11}\left(v_X^2 v_{XX} + 2v_X v_Z v_{XZ} + v_Z^2 v_{ZZ}\right), \tag{3.34}$$

thus the model is linear if and only if $W_{11} = 0$, i. e., the constitutive function $W(I_1, I_2)$ is linear in I_1, I_2. This is generally the case for Mooney-Rivlin materials.

3.3.2 The General Hyperelastic Love Wave Model

Let us consider the propagation of small but finite amplitude Love wave-type displacement in an elastic half-space ($Z < 0$) covered by an elastic layer of uniform thickness H with different elastic properties. Let (u, v, w) represent the displacement along (X, Y, Z) axes respectively. Let the Y-displacements in the layer and the half-space be given by v^1 and v^2 and represent the displacements along $Y-$directions in the layer and half-space respectively, ρ_1, and ρ_2 represent the mass densities in the layer and half-space respectively, and μ_1 and μ_2 represent the modulus of rigidity in the layer and half-space respectively. The wave travels in the $XZ-$ plane since $v = v(X, Z, t)$. It is also assumed that the boundary $Z = -H$ is free of tractions, and the stress and displacements are continuous at the interface $Z = 0$. Hence, the condition of vanishing tractions (vanishing of the normal component of the stresses) on the surface $Z = -H$ imposes the boundary condition $P^1_{23} = 0$ on $Z = -H$, and the continuity of stresses, and the displacements at the interface $Z = 0$ are satisfied if $P^1_{23} = P^2_{23}$, and $v^{(1)} = v^{(2)}$ on $Z = 0$. The geometry of the problem is shown in Figure 2.1.

Let the Love wave model in the layered half-space with displacement in the $Y-$axis [48, 49] is described by the equations

$$\begin{aligned} x &= X, \\ y &= Y + v^j(X, Z, t), \quad j = 1, 2, \\ z &= Z, \end{aligned} \tag{3.35}$$

where v^j ($j = 1$ is for the layer and $j = 2$ for the half-space) is the antiplane displacement of the wave in the $Y-$ direction. (Here "antiplane" means transverse to the XZ-plane.)

The general model for the Love waves can be obtained by using the incompressible hyperelasticity framework with the general constitutive equation $W = W(I_1)$. Then one can use the formulas from Section 3.2.1 for $j = 1, 2$ in two different media. The equations in the two media together with boundary conditions are given by

$$\rho_0 v^j_{tt} = 2W^j_1 \left(v^j_{XX} + v^j_{ZZ} \right) + 4W^j_{11} \left((v^j_X)^2 v^j_{XX} + 2v^j_X v^j_Z v^j_{XZ} + (v^j_Z)^2 v^j_{ZZ} \right), \quad j = 1, 2, \tag{3.36a}$$

$$\frac{\partial v^{(1)}}{\partial Z} = 0 \text{ on the free boundary } Z = -H, \tag{3.36b}$$

$$v^{(1)} = v^{(2)} \text{ and } \mu_1 \frac{\partial v^{(1)}}{\partial Z} = \mu_2 \frac{\partial v^{(2)}}{\partial Z} \text{ when } Z = 0. \tag{3.36c}$$

The equations (3.36) are the general Love wave equations from which both linear and nonlinear Love-type wave equations can be obtained, depending on the particular strain energy density form. Note that if W has terms linear in I_1 and I_2 then the first term on the right hand side of equation (3.36a) will involve a linear term $(v_{XX} + v_{ZZ})$.

3.3.3 A Mooney-Rivlin Model

In Section 3.3.1, it has been shown that for any constitutive relation $W = W(I_1, I_2)$, the Love wave-type incompressible $Y-$displacements $v(X, Z, t)$ (3.3) satisfy the wave equation (3.31), which can also be written as

$$\rho_0 v_{tt} = 2W_1 \left(v_{XX} + v_{ZZ}\right) + 4W_{11} \left(v_X^2 v_{XX} + 2v_X v_Z v_{XZ} + v_Z^2 v_{ZZ}\right), \tag{3.37}$$

and the hydrostatic pressure is given by (3.33). It is straightforward to observe that the PDE (3.37) is linear if and only if $W_{11} \equiv \partial^2 W(I_1)/\partial I_1^2 = 0$. Since for shear displacements (3.3), $I_1 = I_2$, the most general convex constitutive relation yielding the linear wave model (3.37) is the Mooney-Rivlin model $W = aI_1 + bI_2$. For this model, for each layer $j = 1, 2$, the displacement and hydrostatic pressure are governed by

$$\rho_0 v_{tt}^j = 2(a+b)\left(v_{XX}^j + v_{ZZ}^j\right), \quad j = 1,\ 2, \tag{3.38a}$$

$$p = 2(a+b) + \rho_0 g Z, \tag{3.38b}$$

and boundary conditions (3.36b) and (3.36c) hold.

3.3.4 The Murnaghan Model

We now use the general framework of Section 3.3.2 to construct a model of nonlinear surface-type waves (3.35), (3.36) following the so-called Murnaghan constitutive relation [97]. For displacements (3.35), the deformation gradient and the right and left Cauchy-Green tensors ($\mathbf{C}$ and $\mathbf{B}$) are given respectively by (3.4) and (3.5). The Lagrange strain tensor defined by

$$\epsilon_{lm} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) = \frac{1}{2}\left(v_{m,l} + v_{l,m} + v_{k,l} v_{k,m}\right) \tag{3.39}$$

is used to describe the deformation in nonlinear hyperelasticity framework [26], where $\mathbf{C}$ is the right Cauchy stress, $\mathbf{I}$ is the identity matrix, and $v_{m,l},\ v_{l,m},\ v_{k,l},$ and $v_{k,m}$ are components of the deformation gradient. For an isotropic solid, the strain density function $W = W_{iso}$ can be written as a function of the invariants of $\boldsymbol{\epsilon}$ defined by

$$I_I = \text{Tr}\ \boldsymbol{\epsilon} \equiv \epsilon_{ll}, \quad I_{II} = \text{Tr}\ \boldsymbol{\epsilon}^2 \equiv \epsilon_{lk}\epsilon_{kl}, \quad I_{III} = \text{Tr}\ \boldsymbol{\epsilon}^3 \equiv \epsilon_{lr}\epsilon_{rk}\epsilon_{kl}. \tag{3.40}$$

The invariants (3.40) are the first three algebraic invariants of ϵ. Algebraic invariants are invariants that remain unaltered under certain class of algebraic transformation. Principal invariants do not change with rotations of the coordinate system, i. e., they are objective, satisfying the principle of material frame indifference, and any function of the principal invariant is bijective. Algebraic invariants can also be defined as functions of the principal invariants (see equation (3.41) below). This idea of representation through invariants is often used in the nonlinear theory of elasticity. In particular, the principal invariants I_1, I_2, I_3 (1.56) do not change upon a transformation of coordinates in (X, Z). The second-order strain tensor in 3D posseses only three independent invariants. All other invariants are functions of the principal invariants. The principal invariants (1.57) are linked to the algebraic invariants (3.40) by the formulas

$$\begin{gathered} I_I = I_1, \quad I_{II} = I_1^2 - 2I_2, \quad I_{III} = I_1^3 - 3I_1 I_2 + 3I_3, \\ I_1 = I_I, \quad I_2 = \frac{1}{2}\left(I_I^2 - I_{II}\right), \quad I_2 = \frac{1}{3}\left(I_I^3 + 2I_{III} - 3I_I I_{II}\right). \end{gathered} \tag{3.41}$$

The algebraic invariants (3.41) can be written in terms of the first principal invariants I_1 as

$$I_I = I_1, \quad I_{II} = I_1^2 - 2I_1, \quad I_{III} = I_1^3 - 3I_1^2 + 3. \tag{3.42}$$

Thus

$$\begin{aligned} I_I = 3 + v_X^2 + v_Z^2, \quad I_{II} = \left(3 + v_X^2 + v_Z^2\right)^2 - 2\left(3 + v_X^2 + v_Z^2\right), \\ I_{III} = \left(3 + v_X^2 + v_Z^2\right)^3 - 3\left(3 + v_X^2 + v_Z^2\right)^2 + 3. \end{aligned} \tag{3.43}$$

Teymur [129] used the Murnaghan potential to describe the nonlinear modulation of Love waves. Kalyansaundaram [72] studied finite-amplitude Love waves on an elastic isotropic layer overlying over an isotropic elastic half-space. Rushchitsky [120, 121] gave a nonlinear description of Love waves in elastic materials. In these models,

$$W = \frac{1}{2}\lambda I_I^2 + \mu I_{II} + \frac{1}{3}AI_I^3 + BI_I I_{II} + \frac{1}{3}CI_{III} + DI_I^4 + EI_I^2 I_{II} + FI_I I_{III} + GI_{II}^2 + O\left(I_I^5, I_{II}^5, I_{III}^5\right), \tag{3.44}$$

where λ, μ are the usual second-order Lamé coefficients, A, B, C are the third-order Murnaghan constants, and D, E, F, G are fourth-order elastic constants. For the model of Love waves in two-layered medium following the Murnaghan model, we use the wave equation (3.31), or specifically, the problem setup (3.36) for the two-layers $j = 1,\ 2$, and the pressure is given by (3.33). For the Murnaghan potential

$$W = \frac{1}{2}\lambda I_I^2 + \mu I_{II} + \frac{1}{3}AI_I^3 + BI_I I_{II} + \frac{1}{3}CI_{III}, \tag{3.45}$$

and displacement (3.3), substituting the algebraic invariants (3.42), we obtain

$$W = W(I_1) = \frac{1}{2}\lambda I_1^2 + \mu(I_1^2 - 2I_1) + \frac{1}{3}AI_1^3 + BI_1(I_1^2 - 2I_1) + \frac{1}{3}C\left(I_1^3 - 3I_1^2 + 3\right) \tag{3.46}$$

Simplifying (3.46) gives

$$W = C - 2\mu I_1 + \left(\mu + \frac{\lambda}{2} - 2B - C\right)I_1^2 + \left(\frac{A}{3} + B + \frac{C}{3}\right)I_1^3. \tag{3.47}$$

Substitution of I_1 in equation (3.6) into (3.47) yields

$$W = W(I_1) = C - 2\mu\left(3 + v_X^2 + v_Z^2\right) + \left(\mu + \frac{\lambda}{2} - 2B + 2C\right)\left(3 + v_X^2 + v_Z^2\right)^2 + \left(\frac{A}{3} + B + \frac{C}{3}\right)\left(3 + v_X^2 + v_Z^2\right)^3. \tag{3.48}$$

The derivatives of W are given by

$$\begin{aligned} W_1 \equiv \frac{\partial W}{\partial I_1} &= -2\mu + (2\mu + \lambda - 4B - 2C)\,I_1 + (A + 2B + C)\,I_1^2 \\ &= -2\mu + (2\mu + \lambda - 4B - 2C)\left(3 + v_X^2 + v_Z^2\right) + (A + 2B + C)\left(3 + v_X^2 + v_Z^2\right)^2, \end{aligned} \tag{3.49a}$$

$$\begin{aligned} W_{11} \equiv \frac{\partial^2 W}{\partial I_1^2} &= (2\mu + \lambda - 4B - 2C) + (2A + 6B + 2C)\,(I_1) \\ &= (2\mu + \lambda - 4B - 2C) + (2A + 6B + 2C)\left(3 + v_X^2 + v_Z^2\right). \end{aligned} \tag{3.49b}$$

We note that equation (3.49a) dictates the form of the linear term. Substituting (3.49) into the wave equation (3.36) for the $Y-$displacement and the formula (3.33) for the hydrostatic pressure, we obtain

$$\begin{aligned}\rho_0 v_{tt} = {} & S_0(v_{XX}+v_{ZZ}) + S_1 v_X^2 v_{XX} + S_2 v_Z^2 v_{XX} + S_1 v_Z^2 v_{ZZ} + S_2 v_X^2 v_{ZZ} + 4S_2 v_X v_Z v_{XZ} + Q_1 v_X^4 v_{XX} \\ & + Q_1 v_Z^4 v_{ZZ} + Q_2 v_Z^4 v_{XX} + Q_2 v_X^4 v_{ZZ} + Q_3 v_X^3 v_Z v_{XZ} + Q_3 v_Z^3 v_X v_{XZ} + Q_4 v_X^2 v_Z^2 v_{XX} + Q_4 v_X^2 v_Z^2 v_{ZZ},\end{aligned} \tag{3.50a}$$

$$p = 2\left(-2\mu + (2\mu+\lambda-4B-2C)\left(3+v_X^2+v_Z^2\right) + (A+2B+C)\left(3+v_X^2+v_Z^2\right)^2\right) + \rho_0 g Z. \tag{3.50b}$$

where

$$\begin{aligned} S_0 = 18A+30B+6C+6\lambda+8\mu, \quad S_1 = 3\left[2\lambda+4\mu+12A+28B+8C\right] = 3S_2, \\ S_2 = 2\lambda+4\mu+12A+28B+8C, \quad Q_1 = 5\left(2A+6B+2C\right) = 5Q_2, \quad Q_2 = 2A+6B+2C, \\ Q_3 = 8\left[2A+6B+2C\right] = 8Q_2, \quad Q_4 = 4\left(2A+6B+2C\right) = 4Q_2. \end{aligned} \tag{3.51}$$

Equation (3.50a) contains the nonlinear polynomial terms of the third and fifth orders in terms of v. The wave equation (3.50a) may be called a nonlinear surface wave equation. With the appropriate boundary conditions, (3.50a) can be solved up to the third approximation using the method of successive approximations by considering only cubic nonlinearity in (3.50a) [121].

The Murnghan potential (3.45) which is an extension of the Saint Venant-Kirchhoff potential given in terms of the Lagrangian strain tensor is not polyconvex (see Section 1.5.6 for details). In order to obtain a polyconvex potential, one can replace the algebraic invariants in (3.45) by the principal invariants:

$$W = \frac{1}{2}\lambda(I_1-3)^2 + \mu(I_2-3) + \frac{1}{3}A(I_1-3)^3 + B(I_1-3)(I_2-3) + \frac{1}{3}CI_3. \tag{3.52}$$

Now since $I_1 = I_2$ and $I_3 = 1$, then without loss of generality (3.52) can be written only in terms of the first principal invariants as

$$W = \frac{1}{2}\lambda(I_1-3)^2 + \mu(I_1-3) + \frac{1}{3}A(I_1-3)^3 + B(I_1-3)^2 + \frac{1}{3}C. \tag{3.53}$$

Substituting the first principal invariant (3.6) into (3.53), one obtains

$$W = \frac{1}{2}\lambda(v_X^2+v_Z^2)^2 + \mu(v_X^2+v_Z^2) + \frac{1}{3}A(v_X^2+v_Z^2)^3 + B(v_X^2+v_Z^2)^2 + \frac{1}{3}C. \tag{3.54}$$

The equation (3.54) may be written as

$$W = \frac{1}{3}C + \mu\left(v_X^2+v_Z^2\right) + \left(\frac{1}{2}\lambda+B\right)\left(v_X^2+v_Z^2\right)^2 + \frac{1}{3}A\left(v_X^2+v_Z^2\right)^3. \tag{3.55}$$

Differentiating W with respect to I_1 twice and plugging in $I_1 = 3+v_X^2+v_Z^2$ yields

$$\begin{aligned} W_1 \equiv \frac{\partial W}{\partial I_1} &= \mu + (\lambda+2B)\left(I_1-3\right) + A\left(I_1-3\right)^2 \\ &= \mu + (\lambda+2B)\left(v_X^2+v_Z^2\right) + A\left(v_X^2+v_Z^2\right)^2, \end{aligned} \tag{3.56a}$$

$$W_{11} \equiv \frac{\partial^2 W}{\partial I_1^2} = (\lambda + 2B) + 2A\,(I_1 - 3) \\ = (\lambda + 2B) + 2A\left(v_X^2 + v_Z^2\right). \tag{3.56b}$$

Substituting the derivatives (3.56) into the equation (3.36) for the $Y-$displacement and the formula (3.33) for the hydrostatic pressure, give

$$\rho_0 v_{tt} = 2\left(\mu + (\lambda + 2B)\left(v_X^2 + v_Z^2\right) + A\left(v_X^2 + v_Z^2\right)^2\right)(v_{XX} + v_{ZZ}) \\ +4\left((\lambda + 2B) + 2A\left(v_X^2 + v_Z^2\right)\right)\left(v_{XX}^2 + 2v_X v_Z v_{XZ} + v_Z^2 v_{ZZ}\right), \tag{3.57a}$$

$$p = 2\left(\mu + (\lambda + 2B)\left(v_X^2 + v_Z^2\right) + A\left(v_X^2 + v_Z^2\right)^2\right) + \rho_0 g Z. \tag{3.57b}$$

The Yeoh Model

To avoid the idea of transforming from the non-polyconvexity of the Murnaghan potential to a polyconvex Murnaghan potential, one can use the incompressible Yeoh model, which is a good fit in modeling rubber-like materials and its strain energy density function is cubic in terms of the first principal invariant I_1. Since we assume the strain energy density function $W(I_1, I_2) = W(I_1)$ without loss of generality, then one requires a nonlinear model, which is a power series of strain invariants I_1. This cubic strain energy function is also efficient in nonlinear elasticity. We use the general framework of Section 3.3.2 to construct a model of nonlinear Love-type waves (3.35) following the Yeoh constitutive relation [142] (See Section 1.5.6). Its strain energy potential is written in terms of the first principal invariant can be re-written as

$$W = a(I_1 - 3) + b(I_1 - 3)^2 + c(I_1 - 3)^3, \tag{3.58}$$

where a, b, and c are elastic constants. Now the first principal invariant for the Cauchy-Green tensor can be written as

$$I_1 = 3 + v_X^2 + v_Z^2 = 3 + Q, \tag{3.59}$$

where $Q = |\text{grad } v|^2$. So that $I_1 - 3 = Q$. Now substitution of (3.59) into (3.58) yields

$$W = aQ + bQ^2 + cQ^3 = U(Q).$$

The derivatives of W are given by

$$W_1 \equiv \frac{\partial W}{\partial I_1} = \frac{\partial U}{\partial Q} = a + 2bQ + 3cQ^2 = a + 2b\left(v_X^2 + v_Z^2\right) + 3c\left(v_X^2 + v_Z^2\right)^2, \tag{3.60a}$$

$$W_{11} \equiv \frac{\partial^2 W}{\partial I_1^2} = \frac{\partial^2 U}{\partial Q^2} = 2b + 6CQ = 2b + 6c\left(v_X^2 + v_Z^2\right). \tag{3.60b}$$

Plugging in the derivatives (3.60a) into the equation (3.36) for the $Y-$displacement and the formula (3.33) for the hydrostatic pressure, give

$$\rho_0 v_{tt} = 2a(v_{XX} + v_{ZZ}) + 4b\left(3v_X^2 v_{XX} + v_Z^2 v_{XX} + 3v_Z^2 v_{ZZ} + v_X^2 v_{ZZ}\right) + 4c\Big(4v_X v_Z v_{XZ} + 7v_X^4 v_{XX} \\ +7v_Z^4 v_{ZZ} + v_Z^4 v_{XX} + v_X^4 v_{ZZ} + 12v_X^3 v_Z v_{XZ} + 12v_Z^3 v_X v_{XZ} + 8v_X^2 v_Z^2 v_{XX} + 8v_X^2 v_Z^2 v_{ZZ}\Big), \tag{3.61a}$$

$$p = 2\left(a + 2b\left(v_X^2 + v_Z^2\right) + 3c\left(v_X^2 + v_Z^2\right)^2\right) + \rho_0 g Z, \tag{3.61b}$$

where $\mu = 2a$.

3.3.5 Discussion

Equations (3.50a), (3.57a), and (3.61a) contain the nonlinear polynomial terms of the third and fifth orders in terms of v. Assuming $|v| \ll 1$ in (3.61a), one obtains linear Love wave equations coinciding with the PDEs (2.9) presented in Section 2.2. The model of nonlinear Love waves based on displacements (3.3) and the Murnaghan potential (3.48) was considered in Kalyanasundaram [72], Teymur [129], and Rushchitsky [120, 121]. Kalyanasundaram [72], Teymur [129], and Rushchitsky [120, 121] used the same deformation field (3.3). They assumed the motion takes place in a homogeneous, isotropic, compressible hyperelastic material. They employed the first Piola-Kirchhoff stress tensor $P_{ik} = \partial W/\partial F_{ik}$, and the Lagrangian strain tensor ε in their description of nonlinear Love wave equations. They all obtained nonlinear Love wave equations which differed by elastic coefficients. Also, they all used the method of successive approximation to find approximate solutions to their respective nonlinear wave equations. In our approach, since the deformation field (3.3) is naturally isochoric, new nonlinear wave equations are obtained involving pressure fields. The nonlinear wave equations (3.31) and (3.33) are general; they hold for any form of incompressible constitutive relation $W = W(I_1, I_2)$.

3.3.6 Simplified One-Dimensional Models

Let us consider a shear horizontal wave traveling along the $X-$axis in the layered half-space, or antiplane motion [48, 49] described by the equations.

$$x = X, \quad y = Y + v(X,t), \quad z = Z, \tag{3.62}$$

where v is the displacement of the wave in the $Y-$ direction, and t is the time. The matrices representing the deformation gradient $\mathbf{F}$ and its inverse matrix are given by

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ v_X & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -v_X & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.63}$$

where $v_X = \partial v/\partial X$ denotes the amount of shear. We note that $v_X = |\nabla v| = |\partial v/\partial X|$, where ∇ is the gradient operator. The right and left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ are as well given respectively by

$$\mathbf{C} = \begin{bmatrix} 1+v_X^2 & v_X & 0 \\ v_X & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & v_X \\ 0 & 1+v_X^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.64}$$

The principal invariants (the first three invariants of $\mathbf{B}$) are computed and are given by

$$I_1 = I_2 = 3 + v_X^2, \quad I_3 = 1. \tag{3.65}$$

Consequently, the strain energy density in this case is a function of v_X and it is given by

$$W = W(I_1, I_2) = W(I_1) = W(3 + v_X^2). \tag{3.66}$$

Assuming the hydrostatic pressure $p = p(X, Z, t)$, the equations of motion (3.28) simplify to

$$0 = -\frac{\partial p}{\partial X} + 2\frac{\partial W_1}{\partial X}, \tag{3.67a}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = 2\left(W_1 \frac{\partial^2 v}{\partial X^2} + \frac{\partial v}{\partial X}\frac{\partial W_1}{\partial X}\right), \tag{3.67b}$$

$$0 = -\frac{\partial p}{\partial Z} + \rho_0 g. \tag{3.67c}$$

Without loss of generality, this yields the simplified form of the wave equation (3.31) on $v(X, t)$:

$$\rho_0 v_{tt} = 2\frac{\partial}{\partial X}\left(v_X W_1\right), \tag{3.68}$$

and the pressure expression

$$p = p_0 + 2W_1 + \rho_0 g Z, \tag{3.69}$$

where $p_0 = \text{const}$ or is a function of time, and $p = p(X, t)$ in the absence of gravity. In particular, for the neo-Hookean or the Mooney-Rivlin constitutive law $W = aI_1 + bI_2$, the wave equation (3.68) linearizes to

$$\rho_0 v_{tt} = 2(a + b)v_{XX}. \tag{3.70}$$

For Murnaghan potential (3.45) one has the derivatives of W as

$$\begin{aligned} W_1 \equiv \frac{\partial W}{\partial I_1} &= -2\mu + (2\mu + \lambda - 4B - 2C)(I_1) + (A + 2B + C)(I_1)^2 \\ &= -2\mu + (2\mu + \lambda - 4B - 2C)\left(3 + v_X^2\right) + (A + 2B + C)\left(3 + v_X^2\right)^2, \end{aligned} \tag{3.71a}$$

$$\begin{aligned} W_{11} \equiv \frac{\partial^2 W}{\partial I_1^2} &= (2\mu + \lambda - 4B - 2C) + (2A + 6B + 2C)(I_1) \\ &= (2\mu + \lambda - 4B - 2C) + (2A + 6B + 2C)\left(3 + v_X^2\right), \end{aligned} \tag{3.71b}$$

and the wave equation of (3.68) takes the form

$$\rho_0 v_{tt} = S_0 v_{XX} + S_1 v_X^2 v_{XX} + Q_1 v_X^4 v_{XX}, \tag{3.72}$$

where S_0, S_1, and Q_1 are given by (3.51), with the right hand side of (3.72) containing the usual linear term involving v_{XX} and a nonlinear term, which for small displacements $v \approx \varepsilon$, has the order $O(\varepsilon^1)$. The equation (3.72) contains the nonlinear terms of orders $O(\varepsilon^3)$ and $O(\varepsilon^5)$. The hydrostatic pressure in this case is given by

$$p = 2\left(-2\mu + (2\mu + \lambda - 4B - 2C)\left(3 + v_X^2\right) + (A + 2B + C)\left(3 + v_X^2\right)^2\right) + \rho_0 g Z. \tag{3.73}$$

Using the Yeoh strain energy potential (3.58), one has the derivatives of W as

$$W_1 \equiv \frac{\partial W}{\partial I_1} = \frac{\partial U}{\partial Q} = a + 2bQ + 3cQ^2 = a + 2bv_X^2 + 3cv_X^4, \tag{3.74a}$$

$$W_{11} \equiv \frac{\partial^2 W}{\partial I_1^2} = \frac{\partial^2 U}{\partial Q^2} = 2b + 6CQ = 2b + 6cv_X^2, \tag{3.74b}$$

where $Q = v_X^2$. Substitution of the derivatives (3.74) into the equation (3.36) for the $Y-$displacement and the formula (3.33) for the hydrostatic pressure, give

$$\rho_0 v_{tt} = \mu v_{XX} + 12bv_X^2 v_{XX} + 30cv_X^4 v_{XX}, \tag{3.75a}$$

$$p = 2\left(a + 2bv_X^2 + 3cv_X^4\right) + \rho_0 gZ. \tag{3.75b}$$

From the Murnaghan potential (3.58) which depends only on the first principal invariant, one has the derivatives of W as

$$W_1 \equiv \frac{\partial W}{\partial I_1} = \mu + (\lambda + 2B)(I_1 - 3) + A(I_1 - 3)^2 = \mu + (\lambda + 2B)v_X^2 + Av_X^4, \tag{3.76a}$$

$$W_{11} \equiv \frac{\partial^2 W}{\partial I_1^2} = (\lambda + 2B) + 2A(I_1 - 3) = (\lambda + 2B) + 2Av_X^2, \tag{3.76b}$$

where $Q = v_X^2$. Inserting the derivatives (3.76) into the equation (3.36) for the $Y-$displacement and the formula (3.33) for the hydrostatic pressure, give

$$\rho_0 v_{tt} = 2\mu v_{XX} + 3(2\lambda + 4B)v_X^2 v_{XX} + 10Av_X^4 v_{XX}, \tag{3.77a}$$

$$p = 2\left(\mu + (\lambda + 2B)v_X^2 + Av_X^4\right) + \rho_0 gZ. \tag{3.77b}$$

One can observe that, equation (3.77a) is equivalent up to renaming of the constants. That is,

$$b = \frac{(\lambda + 2B)}{2}, \quad C = \frac{A}{3}, \quad a = \mu. \tag{3.78}$$

3.4 Vertical Shear Wave Models in the Hyperelasticity Framework

Hyperelastic materials are usually considered to be incompressible. However, compressible hyperelastic models are indispensable in considering generalized shear wave motion in elastodynamic problems. Boyce and Arruda [27] highlighted some approaches to extend an incompressible hyperelastic model to a compressible form. Shear wave models in compressible and incompressible hyperelastic solids have been considered by various authors, including Jeffery and Teymur [56, 71, 129] and many others. In incompressible elastic material, there is no change of volume, and dilation stresses are transmitted instantaneously. Jeffery and Teymur [71] studied the formation of shock waves from the generalized simple waves in an isotropic incompressible hyperelastic material. Chu [37] discussed finite-amplitude shear waves generated in incompressible Cauchy elastic materials. Anti-plane shear in the context of elasticity is of interest in the study of compressible and incompressible hyperelastic models. Knowles [77] studied the finite anti-plane shear for incompressible elastic materials. Destrade and Saccomandi [42] studied a particular case of anti-plane shear called the *rectilinear* shear models of compressible and incompressible elastic slabs. In what follows, we present two-and one-dimensional vertical shear waves and give the general wave equations which depend on $W = W(I_1)$ for both compressible and incompressible materials.

3.4.1 Two-Dimensional Vertical Shear Waves

In this section, we discuss the propagation of small but finite amplitude vertical shear motion in a plate of uniform thickness, composed of two homogeneous isotropic incompressible hyperelastic layers with different elastic properties. Firstly, we consider a class of motion where the displacement in medium occurs in the $Z-$direction. We also consider the two-layered model in the later part of this Section. Let the $Z-$displacement be given by $v = v\,(X, Z, t)$ in the $Z-$direction. The shear displacements for Rayleigh waves are given by

$$x = X + u(X, Z, t), \quad y = Y, \quad z = Z + v(X, Z, t), \tag{3.79}$$

where $u = u(X, Z, t)$ and $v = v(X, Z, t)$ are the displacements in the $X-$ and $Z-$directions respectively.

Shear motion for surface waves

Let $\mathbf{X} = (X, Y, Z)$ be the material (reference) coordinates in the orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, $\mathbf{X} = X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$. The vertical shear motion, which can be considered a simplified version of shear displacements (3.79) (actual positions of material points in the body (Eulerian coordinates)) are described by equations

$$x = X, \quad y = Y, \quad z = Z + v(X, Z, t), \tag{3.80}$$

where (x, y, z) are the Eulerian coordinates, and $v = v(X, Z, t)$ represents the finite displacements of the corresponding material points. The Jacobian matrices representing the deformation gradient as well as its inverse matrix are given by

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_X & 0 & 1+v_Z \end{bmatrix}, \quad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{v_X}{1+v_Z} & 0 & \frac{1}{1+v_Z} \end{bmatrix}, \quad J = \det \mathbf{F} = 1 + v_Z \tag{3.81}$$

The right and left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ are given respectively as

$$\mathbf{C} = \begin{bmatrix} 1+v_X^2 & 0 & v_X v_Z + v_X \\ 0 & 1 & 0 \\ v_X v_Z + v_X & 0 & \left(1+v_Z^2\right)^2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & v_X \\ 0 & 1 & 0 \\ v_X & 0 & \left(1+v_Z\right)^2 \end{bmatrix}. \tag{3.82}$$

Since the $J \neq 1$, the motion is not incompressible. Consequently, the alternative set of invariants for finite deformation models of nearly incompressible materials (1.64) maybe used:

$$\bar{I}_1 = \frac{I_1}{(1+v_Z)^{2/3}}, \quad \bar{I}_2 = \frac{I_2}{(1+v_Z)^{4/3}}, \quad \bar{I}_3 = \left(1+v_Z\right)^{1/2}. \tag{3.83}$$

Here

$$I_1 = 3 + 2v_Z + v_X^2 + v_Z^2, \quad I_2 = 3 + 4v_Z + 2v_Z^2 + v_X^2 \tag{3.84}$$

are the first and second principal invariants respectively, and $I_3 = J = 1 + v_Z$ is the determinant of the deformation gradient $\mathbf{F}$.

3.4.2 Mooney-Rivlin and neo-Hookean model of acoustic shear waves in a compressible medium

Due to the complexity of the governing equation for a general strain energy density function $W = W(\bar{I}_1, \bar{I}_2)$, one can consider the special case where the stored energy function for Mooney-Rivlin constitutive model (compressible) $W = a\bar{I}_1 + b\bar{I}_2$ provided in Table 1.1 is used. For compressible models the components of the first Piola-Kirchhoff stress tensor are given using (1.41) [38] by

$$
\begin{aligned}
P_{11} &= -\frac{2a\left(v_X^2 + 2v_Z + v_Z^2\right)(1+v_Z)^{2/3} + 2b\left(v_X^2 + 2v_X + (1/2)v_Z^2\right)}{3\,(1+v_Z)^{4/3}},\\
P_{22} &= -\frac{2a\left(v_X^2 + 2v_Z + v_Z^2\right)(1+v_Z)^{2/3} + 2b\left(v_X^2 - 2v_Z - v_Z^2\right)}{3\,(1+v_Z)^{4/3}},\\
P_{31} &= \frac{2a\left(v_X^2 + 2v_X + v_Z^2 + 3\right)(1+v_X)^{2/3} + 2b\left(v_X^2 + 2v_X + v_Z^2 + 3\right)}{3\,(1+v_X)^{7/3}},\\
P_{12} &= P_{21} = P_{23} = P_{32} = 0, \quad P_{31} = \frac{2v_X\left(a\,(1+v_X)^{2/3} + b\right)}{(1+v_Z)^{4/3}},\\
P_{13} &= \frac{2av_X\left(v_X^2 + 2v_Z + v_Z^2 + 3\right)(1+v_Z)^{2/3} + 2v_X b\left(v_X^2 + v_Z + (1/2)v_Z^2 + 3/2\right)}{3\,(1+v_Z)^{4/3}},\\
P_{33} &= \frac{2a\left(v_X^2 - 4v_Z - 2v_Z^2\right)(1+v_Z)^{2/3} + 2b\left(v_X^2 - 2v_Z - v_Z^2\right)}{3\,(1+v_Z)^{7/3}}
\end{aligned}
\tag{3.85}
$$

From (1.40a), when we assume there are no body forces and ρ_0 is also assumed to be constant mass density then, the $X-$ and $Z-$ equations of motion is given by

$$
\begin{aligned}
0 = \frac{2a}{9\Big(1+v_Z\Big)^{10/3}}\Bigg\{\Bigg[&-12v_Xv_{XX}v_Z^2 + 26v_Zv_{XZ}v_X^2 - 24v_Xv_{XX}v_Z - 2v_Xv_{ZZ}v_Z^2 - 4v_Xv_{ZZ}v_Z + v_{XZ}v_Z^3\\
&+3v_{XZ}v_Z^2 + 26v_{XZ}v_X^2 + 5v_{XZ}v_Z - 12v_Xv_{XX} - 14v_{ZZ}v_X^3 - 10v_Xv_{ZZ} + 3v_{XZ}\Bigg]\Bigg\}\\
+\frac{2b}{9\Big(1+v_Z\Big)^{8/3}}\Bigg\{\Bigg[&6v_Xv_{XX}v_Z^2 + 11v_Zv_{XZ}v_X^2 - 12v_Xv_{XX}v_Z + v_Xv_{ZZ}v_Z^2 - 2v_Xv_{ZZ}v_Z - v_{XZ}v_Z^3\\
&-3v_{XZ}v_Z^2 + 11v_{XZ}v_X^2 - v_{XZ}v_Z - 12v_Xv_{XX} - 5v_{ZZ}v_X^3 - 9v_Xv_{ZZ} + 3v_{XZ}\Bigg]\Bigg\},
\end{aligned}
\tag{3.86a}
$$

$$
\begin{aligned}
\rho_0 v_{tt} = \frac{a}{(1+v_Z)^{8/3}}\Bigg[&\frac{10}{9}v_{ZZ}v_X^2 + \frac{4}{9}v_{ZZ}v_Z^2 + \frac{8}{9}v_{ZZ}v_Z + \frac{8}{3}v_{ZZ} - \frac{8}{3}v_Xv_{XZ} + 2v_{XX} + 4v_{XX}v_Z - \frac{8}{3}v_Xv_Zv_{XZ}\\
&+2v_{XX}v_Z^2\Bigg] + \frac{b}{(1+v_Z)^{10/3}}\Bigg[\frac{28}{9}v_{ZZ}v_X^2 - \frac{4}{9}v_{ZZ}v_Z^2 - \frac{8}{9}v_{ZZ}v_Z + \frac{8}{3}v_{ZZ} - \frac{16}{3}v_Xv_{XZ} + 2v_{XX} + 4v_{XX}v_Z\\
&-\frac{16}{3}v_Zv_{XZ}v_X + 2v_{ZZ}v_X^2\Bigg],
\end{aligned}
\tag{3.86b}
$$

where (3.86b) and (3.86a) are the nonlinear wave equation and the constraint equation respectively. The equation in the Y-direction is identically satisfied. When we assume that the mass density ρ_0 is a function of X, and Z, then in the absence of body forces, the full system of equations of motion is similar to (3.86), only that a variable mass density replaces the constant density in the system. For small displacements v, the equation (3.86a) is identically satisfied, and (3.86b) becomes the linear wave equation

$$\rho_0 v_{tt} = \frac{8}{3}(a+b)v_{ZZ} + 2(a+b)v_{XX}. \tag{3.87}$$

Now one may set $b = 0$ in all the formulas in this subsection for neo-Hookean models. From (3.87) after rescaling $Z = \sqrt{\frac{4}{3}}\tilde{Z}$ and $X = \tilde{X}$, one finds

$$\begin{aligned}
\rho_0 v_{tt} &= 2(a+b)\frac{4}{3}v_{ZZ} + 2(a+b)v_{XX} \\
&= 2(a+b)\frac{4}{3}\frac{\partial^2 v}{\partial\left(\sqrt{\frac{4}{3}}\tilde{Z}\right)^2} + 2(a+b)\frac{\partial^2 v}{\partial \tilde{X}^2}, \\
&= 2(a+b)\frac{4}{3}\frac{\partial^2 v}{\partial\left(\sqrt{\frac{4}{3}}\tilde{Z}\right)^2} + 2(a+b)\frac{\partial^2 v}{\partial \tilde{X}^2} \\
&= 2(a+b)\left(\frac{\partial^2 v}{\partial \tilde{Z}^2} + \frac{\partial^2 v}{\partial \tilde{X}^2}\right).
\end{aligned} \tag{3.88}$$

Ignoring tildes, (3.88) coincides with the linear wave equation (3.38) in Section 3.3.3.

3.4.3 One-dimensional Vertical Shear Wave Models

In this Section, we discuss a special class of vertical shear motion where the displacement in the medium occurs in the $Z-$direction, and the displacement of the medium $v = v(X,t)$ is in the $Z-$direction. The model introduced is used to investigate a special class of isochoric motions known as anti-plane shear motions. Such motions arise in a wide variety of applications, ranging from mechanics to biological tissues [34, 35]. We consider the shear motions

$$x = X, \quad y = Y, \quad z = Z + v(X,t), \tag{3.89}$$

which is a reduced ($Z-$independent) version of (3.80). For such motions,

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_X & 0 & 1 \end{bmatrix}, \quad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -v_X & 0 & 1 \end{bmatrix}, \quad \det \mathbf{F} = 1, \tag{3.90}$$

hence the nonlinear motions (3.89) are naturally incompressible. We note that $v_X = |\nabla v| = |\partial v/\partial X|$ and the right and left Cauchy-Green tensors $\mathbf{C}$ and $\mathbf{B}$ are given respectively by

$$\mathbf{C} = \begin{bmatrix} 1+v_X^2 & 0 & v_X \\ 0 & 1 & 0 \\ v_X & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & v_X \\ 0 & 1 & 0 \\ v_X & 0 & 1+v_X^2 \end{bmatrix}. \tag{3.91}$$

The principal invariants (the first three invariants of $\mathbf{B}$) are computed and are given by

$$I_1 = I_2 = 3 + v_X^2, \quad I_3 = 1. \tag{3.92}$$

Consequently, the strain energy density in this case is a function of v_X and it is given by

$$W = W(I_1, I_2) = W(I_1). \tag{3.93}$$

Shear Waves in Compressible Media

Let us consider the shear wave model (3.89) in the compressible framework first. Then the hydrostatic pressure $p = \text{const}$ and $\mathbf{P} = \partial W/\partial \mathbf{F}$:

$$\mathbf{P} = \begin{bmatrix} 2W_1 & 0 & 0 \\ 0 & 2W_1 & 0 \\ 2W_1 v_X & 0 & 2W_1 \end{bmatrix}. \tag{3.94}$$

The $X-$ and $Z-$ components of equations of motion (1.37) become

$$0 = 4W_{11} v_X v_{XX}, \tag{3.95a}$$

$$\rho_0 v_{tt} = 2\frac{\partial}{\partial X}\left(v_X W_1\right) - \rho_0 g, \tag{3.95b}$$

and the equation in the $Y-$direction is identically satisfied. The $X-$projection (3.95a) is a differential constraint.

Shear Waves in Incompressible Media

Let us consider shear wave models in the more natural incompressible media with hydrostatic pressure $p = p(X, Z, t) \neq 0$. From the shear motion given in equation (3.89), suppose the strain density function is chosen as a function of the first two principal invariants I_1, and I_2 denoted by $W = W(I_1,\ I_2)$, where $J = 1$, and $I_1 = I_2 = 3 + (v_X)^2$. Without loss of generality, the strain energy density can be written as $W = W(I_1) = W(3 + v_X^2)$. The components of the first Piola-kirchhoff stress tensor $\mathbf{P} = -p\mathbf{F}^{-T} + \partial W/\partial \mathbf{F}$ as in (1.41) are written explicitly by

$$\mathbf{P} = \begin{bmatrix} 2W_1 - p & 0 & p v_X \\ 0 & 2W_1 - p & 0 \\ 2W_1 v_X & 0 & 2W_1 - p \end{bmatrix}. \tag{3.96}$$

Now in the presence of body forces (gravity) pointing downwards and a constant mass density ρ_0, the $X-$ and $Z-$ projections of equations of motion (1.40a) for an incompressible hyperelastic material become

$$p_X = 4W_{11} v_X v_{XX} + p_Z v_X, \tag{3.97a}$$

$$\rho_0 v_{tt} = 2\frac{\partial}{\partial X}\left(v_X W_1\right) - p_Z + \rho_0 g, \tag{3.97b}$$

where $p = p(X, Z, t)$ is the hydrostatic pressure and g is the free fall acceleration due to gravity. The equation in the $Y-$direction is satisfied identically. Making p_Z the subject in equation (3.97b) and substituting into equation (3.97a) yields

$$p(X, Z, t) = p_0(X, t) + \rho_0 g Z, \quad p_0(X, t) = p_0 + 2W_1, \quad p_0 = \text{const : reference pressure.}$$

Consequently, the vertical displacement $v(X, t)$ satisfies the wave equation

$$\rho_0 v_{tt} = 2\frac{\partial}{\partial X}\left(v_X W_1\right), \tag{3.98}$$

which is the same as (3.68): the equation of one-dimensional shear waves in Subsection 3.3.6.

3.5 Solutions to Linear and Nonlinear Wave Equations

3.5.1 Solutions to One-Dimensional Linear Wave Equation

Let us consider the linear wave equation (3.70) derived from the simplified one-dimensional model $x = X$, $y = Y + v(X, t)$, and $z = Z$ in Section 3.3.6. Let us consider

$$v_{tt} = c^2 v_{XX}, \tag{3.99}$$

$X \in [0, L]$, where $c^2 = 2(a + b)/\rho_0$. Firstly, let us solve for the general solution to (3.70) subject to Dirichlet boundary conditions and appropriate initial conditions. One can write the initial boundary value problem as

$$v_{tt} = c^2 v_{XX}, \quad x \in [0, L], \tag{3.100a}$$

$$v(0, t) = 0, \quad v(L, t) = 0, \tag{3.100b}$$

$$v(X, 0) = \alpha(X), \quad v_t(X, 0) = \beta(X). \tag{3.100c}$$

Using the method of separation of variables. Let us seek for the product solution to the wave equation (3.100a) of the form

$$v(X, t) = T(t)\Phi(X). \tag{3.101}$$

Substituting (3.101) into (3.100a) yields

$$T''(t)\Phi(X) = c^2 T(t)\Phi''(X). \tag{3.102}$$

Dividing equation (3.102) by $T(t)\Phi(X)$ gives

$$\frac{T''(t)}{c^2 T(t)} = \frac{\Phi''(X)}{\Phi(X)}. \tag{3.103}$$

Thus

$$\frac{T''(t)}{c^2 T(t)} = \frac{\Phi''(X)}{\Phi(X)} = -\lambda, \tag{3.104}$$

where λ is the separation constant. From (3.104) one finds two ordinary differential equation as

$$T''(t) + c^2\lambda T(t) = 0, \quad \Phi''(X) + \lambda\Phi = 0. \tag{3.105}$$

The two homogeneous boundary conditions (3.100b) show that

$$\Phi(0) = \Phi(L) = 0. \tag{3.106}$$

Equation (3.105) form a boundary value problem. Let us analyze the solution of (3.105) as follows. From the time-dependent equation in (3.105), one can deal with three cases, $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. If $\lambda = 0$, the $T(t) = a_1 + a_2 t$, and if $\lambda < 0$, $T(t) = a_1 \exp(c\sqrt{\lambda}t) + a_2 \exp(-c\sqrt{\lambda}t)$. The time-dependent solution needs to oscillate and so one considers the case where $\lambda > 0$. Thus,

$$T(t) = a_1 \cos c\sqrt{\lambda}t + a_2 \sin c\sqrt{\lambda}t. \tag{3.107}$$

For the boundary value problem

$$\Phi'' + \lambda\Phi = 0, \quad \Phi(0) = \Phi(L) = 0. \tag{3.108}$$

One can solve for three cases, $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. When $\lambda = 0$, $\Phi'' = 0$, which implies $\Phi(X) = c_1 + c_2 X$. Applying the boundary conditions give $\Phi(X) = 0$. Thus there are no eigenfunctions corresponding to $\lambda = 0$ and hence $\lambda = 0$ is not an eigenvalue. Similarly, $\lambda < 0$ there are no eigenvalues and eigenfunctions. For $\lambda > 0$, one obtains eigenvalues and corresponding eigenfunctions as

$$\lambda_m = \left(\frac{m\pi}{L}\right)^2, \quad \Phi_m(X) = \sin\frac{m\pi X}{L}, \quad m = 1, 2, \ldots. \tag{3.109}$$

For $m = 1$, one has the first harmonics or the fundamental. For $m = 2$, one has the second harmonics, etc. To solve the IBVP (3.100), one considers a general solution given by a linear combination of modes (3.101), (3.107), and (3.109):

$$v_{qm}(X, t) = \sum_{m=1}^{\infty} \left(C_m \cos\frac{m\pi ct}{L} + D_m \sin\frac{m\pi ct}{L}\right) \sin\frac{m\pi X}{L}. \tag{3.110}$$

Using the initial conditions (3.100c), one finds that

$$\alpha(X) = \sum_{m=1}^{\infty} C_m \sin\frac{m\pi X}{L}, \quad \beta(X) = \sum_{m=1}^{\infty} D_m \frac{m\pi c}{L} \sin\frac{m\pi X}{L}. \tag{3.111}$$

The coefficient C_m and D_m are obtained from the Fourier sine series as

$$C_m = \frac{2}{L}\int_0^L \alpha(X)\sin\frac{m\pi X}{L}\mathrm{d}X, \quad D_m = \frac{2}{m\pi c}\int_0^L \beta(X)\sin\frac{m\pi X}{L}\mathrm{d}X. \tag{3.112}$$

Let us solve for the general solution to (3.70) subject to Neumann boundary conditions and appropriate initial conditions. Here, equation (3.100b) in (3.100) is replaced by

$$v_X(0, t) = 0, \quad v_X(L, t) = 0, \tag{3.113}$$

and the initial boundary value problem becomes

$$v_{tt} = c^2 v_{XX}, \quad x \in [0, L], \quad v_X(0,t) = 0, \quad v_X(L,t) = 0, \quad v(X,0) = \alpha(X), \quad v_t(X,0) = \beta(X). \tag{3.114}$$

Again using the method of separation of variables, one obtains (3.105). But subjected to the boundary conditions

$$\Phi'(0) = \Phi'(L) = 0. \tag{3.115}$$

The time-dependent solution in this case coincides with equation (3.107). Now let us consider the boundary value problem

$$\Phi'' + \lambda\Phi = 0, \quad \Phi'(0) = 0, \quad \Phi'(L) = 0. \tag{3.116}$$

To find all the product solutions, one needs to find all the eigenvalues and eigenfunctions satisfying the boundary condition (3.115). One needs to consider three cases, $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$ separately. For $\lambda = 0$, one finds $\Phi'' = 0$, which implies $\Phi(X) = c + dX$. The derivative $\Phi'(X) = d$, and the boundary conditions (3.115) implies $d = 0$. So every constant function $\Phi(X) = C$, is an eigenfunction for $\lambda_0 = 0$. Next, for $\lambda = -m^2 < 0$, in which case the equation for Φ becomes

$$\Phi''(X) - m^2\Phi(X) = 0. \tag{3.117}$$

The solution to equation (3.117) is

$$\Phi(X) = c\exp(mX) + d\exp(-mX). \tag{3.118}$$

The derivative of (3.118) is

$$\Phi'(X) = cm\exp(mX) + dm\exp(-mX). \tag{3.119}$$

Using the boundary conditions (3.115) in equations (3.117) and (3.118) and for $m \neq 0$ give $c = d = 0$. This leads to the identical zero solution $\Phi(X) = 0$, which means there are no eigenvalues. For $\lambda = m^2 > 0$, the equation for Φ becomes

$$\Phi'' + m^2\Phi = 0$$

with solution

$$\Phi(X) = c\sin mX + d\cos mX. \tag{3.120}$$

The derivative

$$\Phi'(X) = c\,m\cos mX - d\,m\sin mX. \tag{3.121}$$

Using the boundary conditions in (3.120) and (3.121), the eigenvalues and corresponding eigenfunctions are respectively given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \Phi_n(X) = \cos\left(\frac{n\pi X}{L}\right), \quad n = 0, 1, 2, \cdots. \tag{3.122}$$

Note that for $n = 0$, gives zero eigenvalue $\lambda_0 = 0$ with eigenfunction $\Phi_0 = 1$. The set of all eigenvalues, called spectrum, for Neumann boundary conditions differs from the Dirichlet boundary conditions, by additional eigenvalue. The time-dependent solution part is given by (3.107). For $\lambda_0 = 0$, $T''(t) = 0$ implies $T(t) = a+bt$,

$$T(t) = \frac{1}{2}a_0 + \frac{1}{2}b_0 t. \tag{3.123}$$

The factors of 1/2 included in (3.123) helps to have a single formula for Fourier coefficients. The solutions $T_n(t)$ corresponding to $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, for $n = 1, 2, \cdots$ is given by

$$T_n(t) = a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}. \tag{3.124}$$

Now putting $T_n(t)$ and $\Phi_n(X)$ together gives the general series expansion for solution of (3.114)

$$v(X,t) = \frac{1}{2}a_0 + \frac{1}{2}b_0 t + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}\right)\cos\frac{n\pi X}{L} \tag{3.125}$$

as long as the initial data can be expressed into Fourier series

$$\alpha(X) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi X}{L}, \quad \beta(X) = \frac{1}{2}b_0 + \sum_{n}^{\infty}\frac{n\pi c}{L} b_n \cos\frac{n\pi X}{L}. \tag{3.126}$$

The series (3.126) for the data comes from substituting in $t = 0$ into (3.125) and its derivatives with respect to t. The coefficient a_n and b_n are obtained from the Fourier cosine series as

$$a_n = \frac{2}{L}\int_0^L \alpha(X)\cos\frac{n\pi X}{L}\mathrm{d}X, \quad b_n = \frac{2}{n\pi c}\int_0^L \beta(X)\cos\frac{n\pi X}{L}\mathrm{d}X. \tag{3.127}$$

D'Alembert Solution to One-Dimensional Wave Equation

On $X \in \mathbb{R}$, the linear wave equation (3.99) has solution which is given by a sum of left- and right- traveling waves: the D'Alembert solution. In order to derive it, we first obtain a second order canonical form of (3.99), using the light-cone variables

$$\alpha = X - ct, \quad \beta = X + ct. \tag{3.128}$$

Let $w(\alpha,\beta) = v(X,t)$. By chain rule

$$\frac{\partial w}{\partial X} = \frac{\partial w}{\partial \alpha}\frac{\partial \alpha}{\partial X} + \frac{\partial w}{\partial \beta}\frac{\partial \beta}{\partial X} = \frac{\partial w}{\partial \alpha} + \frac{\partial w}{\partial \beta}, \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial w}{\partial \beta}\frac{\partial \beta}{\partial t} = -c\frac{\partial w}{\partial \alpha} + c\frac{\partial w}{\partial \beta}.$$

Also

$$\frac{\partial^2 w}{\partial X^2} = \frac{\partial^2 w}{\partial \alpha^2} + 2\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{\partial^2 w}{\partial \beta^2}, \quad \frac{\partial^2 w}{\partial t^2} = c^2\left(\frac{\partial^2 w}{\partial \alpha^2} - 2\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{\partial^2 w}{\partial \beta^2}\right).$$

Plugging these derivatives into the linear wave PDE $v_{tt} = c^2 v_{XX}$ yields

$$c^2\left(\frac{\partial^2 w}{\partial \alpha^2} - 2\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{\partial^2 w}{\partial \beta^2}\right) = c^2\left(\frac{\partial^2 w}{\partial \alpha^2} + 2\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{\partial^2 w}{\partial \beta^2}\right).$$

Simplifying leads to a new form of the linear wave equation

$$\frac{\partial^2 w}{\partial\alpha\partial\beta}(\alpha,\beta) = 0. \tag{3.129}$$

The equation (3.129) can be written as

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial w}{\partial \beta}(\alpha,\beta)\right) = 0, \quad \frac{\partial}{\partial \beta}\left(\frac{\partial w}{\partial \alpha}(\alpha,\beta)\right) = 0. \tag{3.130}$$

Integrating the first equation in (3.130) yields

$$\frac{\partial w}{\partial \beta}(\alpha,\beta) = \xi(\beta), \tag{3.131}$$

where $\xi(\beta)$ is some function of β. One can define the antiderivative of $\xi(\beta)$ as $G(\beta)$, that is, such that $G'(\beta) = \xi(\beta)$. Substitution of this into (3.131) leads to

$$\frac{\partial w}{\partial \beta}(\alpha,\beta) = G'(\beta). \tag{3.132}$$

Equation (3.132) can be written as

$$\frac{\partial}{\partial \beta}\big(w(\alpha,\beta) - G(\beta)\big) = 0. \tag{3.133}$$

Integrating (3.133) with respect to β gives

$$w(\alpha,\beta) - G(\beta) = F(\alpha), \tag{3.134}$$

where $F(\alpha)$ is some function of α. Therefore (3.134) can be written as

$$w(\alpha,\beta) = F(\alpha) + G(\beta). \tag{3.135}$$

Plugging in α and β from (3.128) and keeping in mind $v(X,t) = w(\alpha,\beta)$ yield

$$v(X,t) = w(\alpha,\beta) = F(X-ct) + G(X+ct). \tag{3.136}$$

Now let us consider the initial conditions

$$v(X,0) = f(X), \quad v_t(X,0) = g(X). \tag{3.137}$$

Applying the initial conditions (3.137) to (3.136), one finds

$$v(X,0) = F(X) + G(X) = f(X), \quad v_t(X,0) = -F'(X) + G'(X) = \frac{g(X)}{c}. \tag{3.138}$$

Now integrating the second equation (3.138) with respect to X from 0 to X leads to

$$G(X) - F(X) = G(0) - F(0) + \frac{1}{c}\int_0^X g(S)\mathrm{d}S. \tag{3.139}$$

Solving for $F(X)$ and $G(X)$ in equations (3.138) and (3.139), one finds

$$F(X) = \frac{1}{2}\left(f(X) + F(0) - G(0) - \frac{1}{c}\int_0^X g(S)\mathrm{d}S\right), \tag{3.140a}$$

$$G(X) = \frac{1}{2}\left(f(X) - F(0) + G(0) + \frac{1}{c}\int_0^X g(S)\mathrm{d}S\right). \tag{3.140b}$$

Therefore an initial-value problem given on $X \in \mathbb{R}$

$$v_{tt} = c^2 v_{XX}, \quad v(X,0) = f(X), \quad v_t(X,0) = g(X) \tag{3.141}$$

has the solution

$$v(X,t) = F(X-ct) + G(X-ct) = \frac{1}{2}\left[f(X-ct) + f(X+ct)\right] + \frac{1}{2c}\int_{X-ct}^{X+ct} g(S)\mathrm{d}S, \tag{3.142}$$

which is also called the D'Alembert's solution to the linear wave equation (3.99).

3.5.2 The Linear Wave Equation in 2+1 Dimensions: Separated Solutions in Cartesian Coordinates

Let us consider the linear wave equation (3.23) that arises from the shear displacements (3.3). The problem for (3.23) with zero Dirichlet boundary conditions

$$v(0,Z,t) = 0, \quad v(X,0,t) = 0, \quad v(L,Z,t) = 0, \quad v(X,H,t) = 0 \tag{3.143}$$

is equivalent to that for the vibrations of a rectangular membrane. The displacement $v = v(X,Z,t)$ of the membrane satisfies the two dimensional wave equation (3.23). Now suppose that the boundary is given such that all four sides are fixed with zero displacement given by (3.143). One can find the displacement of such rectangular membrane at time t uniquely if the initial position and velocity are given by

$$v(X,Z,0) = \alpha(X,Z), \quad \frac{\partial v}{\partial t}(X,Z,0) = \beta(X,Z).$$

In summary, one can write the initial boundary value problem as

$$\frac{\partial^2 v}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Z^2}\right), \quad 0 < X < L, \quad 0 < Z < H, \tag{3.144a}$$

$$v(0,Z,t) = 0, \quad v(X,0,t) = 0, \quad v(L,Z,t) = 0, \quad v(X,H,t) = 0, \tag{3.144b}$$

$$v(X,Z,0) = \alpha(X,Z), \quad \frac{\partial v}{\partial t}(X,Z,0) = \beta(X,Z), \tag{3.144c}$$

where $c^2 = \text{const}$. Since the PDE (3.144a) and boundary conditions (3.144b) are linear homogeneous, one can apply the method of separation of variables. Now using the method of separation of variables, one can seek product solutions to (3.144) in the form

$$v(X,Z,t) = T(t)\Phi(X,Z). \tag{3.145}$$

Substitution of the second derivatives of (3.145) with respect to X, Z, and t into (3.144) yield

$$\frac{T''(t)}{c^2 T(t)} = \frac{\Delta\Phi(X,Z)}{\Phi(X,Z)} = -\lambda. \tag{3.146}$$

Now

$$\frac{T''(t)}{T(t)} = -\lambda, \quad \frac{\Delta\Phi(X,Z)}{\Phi(X,Z)} = -\lambda. \tag{3.147}$$

From the first equation in (3.147), when $\lambda > 0$, one finds

$$T(t) = A_1 \cos\left(\sqrt{\lambda} ct\right) + A_2 \sin\left(\sqrt{\lambda} ct\right), \tag{3.148}$$

where A_1 and A_2 are arbitrary constants. The homogeneous boundary conditions imply that the eigenvalue problem is

$$\Delta\Phi = -\lambda\Phi, \tag{3.149a}$$

$$\begin{aligned} \Phi(0, Z) = 0, \quad \Phi(X, 0) = 0, \\ \Phi(L, Z) = 0, \quad \Phi(X, H) = 0. \end{aligned} \tag{3.149b}$$

Thus $\Phi = 0$ along the entire boundary. Then (3.149) is a two-dimensional eigenvalue problem. Again since (3.149) is a linear homogeneous PDE in two-dimensional variables with homogeneous boundary conditions, (3.149) can be solved by separations of variables in Cartesian coordinates. From the second equation of (3.148), one can search for product solutions in the form

$$\Phi(X, Z) = A(X)B(Z). \tag{3.150}$$

Now substitution of the second derivatives of equation (3.150) with respect to X and Z into the second equation of (3.148) give

$$\frac{A''(X)}{A(X)} + \lambda = -\frac{B''(Z)}{B(Z)} = q. \tag{3.151}$$

Thus,

$$\frac{A''(X)}{A(X)} + \lambda = q, \quad -\frac{B''(Z)}{B(Z)} = q. \tag{3.152}$$

Equation (3.152) contains two separation constants λ and q, both of which must be determined. One needs to use the homogeneous boundary conditions (3.149b). Then equation (3.150) implies that

$$A(0) = 0, \quad A(L) = 0, \quad B(0) = 0, \quad B(L) = 0. \tag{3.153}$$

In general from equation (3.152),

$$B''(Z) + B(Z)q = 0, \quad A''(X) + (\lambda - q)A(Z) = 0. \tag{3.154}$$

For $q > 0$,

$$B(Z) = A_3 \cos\left(\sqrt{q} Z\right) + A_4 \sin\left(\sqrt{q} Z\right), \tag{3.155}$$

where A_3 and A_4 are arbitrary constants. Also for $q < 0$,

$$B(Z) = A_3 \exp\left(\sqrt{q} Z\right) + A_4 \exp\left(-\sqrt{q} Z\right). \tag{3.156}$$

From the second equation of (3.154), let $m = \lambda - q > 0$, then

$$A(X) = B_1 \cos\left(\sqrt{m} X\right) + B_2 \sin\left(\sqrt{m} X\right) \tag{3.157}$$

where B_1 and B_2 are constants. If $m = \lambda - q < 0$, then

$$A(X) = B_1 \exp\left(\sqrt{m}X\right) + B_2 \exp\left(-\sqrt{3}X\right). \tag{3.158}$$

Therefore the displacement can be written for $\lambda > 0$, $q > 0$, and $q \neq \lambda$ as

$$\begin{aligned} v(X,Z,t) = &\left(B_1\cos\left(\sqrt{m}X\right) + B_2\sin\left(\sqrt{m}X\right)\right)\left(A_3\cos\left(\sqrt{q}Z\right) + A_4\sin\left(\sqrt{q}Z\right)\right) \\ &\times\left(A_1\cos\left(\sqrt{\lambda}ct\right) + A_2\sin\left(\sqrt{\lambda}ct\right)\right). \end{aligned} \tag{3.159}$$

Similarly, for $\lambda > 0$, $q < 0$, and $\lambda \neq q$, the displacement is given by

$$\begin{aligned} v(X,Z,t) = &\left(B_1\cos\left(\sqrt{m}X\right) + B_2\sin\left(\sqrt{m}X\right)\right)\left(A_3\exp\left(\sqrt{q}Z\right) + A_4\exp\left(-\sqrt{q}Z\right)\right) \\ &\times\left(A_1\cos\left(\sqrt{\lambda}ct\right) + A_2\sin\left(\sqrt{\lambda}ct\right)\right). \end{aligned} \tag{3.160}$$

Now there are homogeneous boundary conditions in X and Z. Thus

$$B''(Z) + B(Z)q = 0, \text{ with } B(0) = 0 \text{ and } B(L) = 0 \tag{3.161}$$

is a Sturm-Liouville eigenvalue problem in the $Z-$variable, where q is the eigenvalue and $B(Z)$ is the eigenfunction. Similarly, the $Z-$dependent problem is a regular Sturm-Liouville problem

$$A''(X) + (\lambda - q)A(X) = 0, \text{ with } A(0) = 0, \quad A(L) = 0. \tag{3.162}$$

Here λ is the eigenvalue and $A(X)$ is the corresponding eigenfunction. Now from (3.161), for $q > 0$, one can compute eigenvalues as

$$q_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \cdots \tag{3.163}$$

and the corresponding eigenfunctions as

$$B_n(Z) = \sin\left(\frac{n\pi Z}{L}\right), \quad n = 1, 2, 3, \cdots. \tag{3.164}$$

For each value of q_n, (3.162) is still an eigenvalue problem. There are infinite number of eigenvalues λ for each n. Thus, λ should be double subscripted, λ_{nm}. From equation (3.162) the eigenvalues are obtained as

$$\lambda_{nm} - q_n = \left(\frac{m\pi}{H}\right)^2, \quad n, m = 1, 2, 3, \cdots. \tag{3.165}$$

The eigenfunction is

$$A_{nm}(X) = \sin\left(\frac{m\pi X}{H}\right). \tag{3.166}$$

The separated constant λ_{nm} can now be determined by

$$\lambda_{nm} = q_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n, m = 1, 2, 3, \cdots. \tag{3.167}$$

Thus the two-dimensional eigenvalue problem (3.149b) has eigenvalues (3.167) and eigenfunctions

$$\Phi(X,Z) = \sin\left(\frac{n\pi Z}{L}\right)\sin\left(\frac{m\pi X}{H}\right), \quad n, m = 1, 2, 3, \cdots. \tag{3.168}$$

The time-dependent part of the product solutions are $\sin\sqrt{\lambda_{nm}}t$ and $\cos\sqrt{\lambda_{nm}}t$, oscillating with natural frequencies,

$$c\sqrt{\lambda_{nm}} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}, \quad n, m = 1, 2, 3, \cdots .$$

The general solution of the problem (3.144) is

$$\begin{aligned} v(X, Z, t) = &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\sin\left(\frac{n\pi Z}{L}\right)\sin\left(\frac{m\pi X}{H}\right)\cos\left(c\sqrt{\lambda_{nm}}t\right) \\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm}\sin\left(\frac{n\pi Z}{L}\right)\sin\left(\frac{m\pi X}{H}\right)\sin\left(c\sqrt{\lambda_{nm}}t\right). \end{aligned} \tag{3.169}$$

The coefficients D_{nm} and E_{nm} can be determined form the two initial conditions. From the first initial condition in equation (3.144c),

$$\alpha(X, Z) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} D_{nm}\sin\left(\frac{m\pi X}{L}\right)\right)\sin\left(\frac{n\pi Z}{H}\right). \tag{3.170}$$

Equation (3.170) is called the double Fourier series and that

$$\sum_{m=1}^{\infty} D_{nm}\sin\left(\frac{m\pi X}{L}\right) = \frac{2}{H}\int_0^H \alpha(X, Z)\sin\left(\frac{n\pi Z}{H}\right)\mathrm{d}Z, \tag{3.171}$$

for each n. The equation (3.171) is valid for all X. The coefficients of this Fourier sine series in X are given by

$$D_{nm} = \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H \alpha(X, Z)\sin\left(\frac{n\pi Z}{H}\right)\mathrm{d}Z\right]\sin\left(\frac{m\pi X}{L}\right)\mathrm{d}X. \tag{3.172}$$

The other coefficient E_{nm} is computed in similar fashion. Using the second initial condition in (3.144c), one finds

$$\beta(X, Z) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}E_{nm}\sin\left(\frac{m\pi X}{L}\right)\right)\sin\left(\frac{n\pi Z}{H}\right). \tag{3.173}$$

Thus, again using a Fourier sine series in Z and a Fourier sine series in X, one obtains

$$E_{nm} = \frac{1}{c\sqrt{\lambda_{nm}}}\frac{4}{LH}\int_0^L\int_0^H \beta(X, Z)\sin\left(\frac{n\pi Z}{H}\right)\sin\left(\frac{m\pi X}{L}\right)\mathrm{d}Z\mathrm{d}X. \tag{3.174}$$

Thus the solution to the IBVP (3.144) is given by (3.169) where the coefficients are given by (3.172) and (3.174).

3.5.3 Separated Solutions in Polar Coordinates

Suppose one has the linear wave equation (3.23) and a polar domain in the $XZ-$plane: $r = \sqrt{X^2 + Z^2}$, and $0 < r < a$. An interesting application of two-dimensional eigenvalue problem occurs when considering the vibrations of a circular membrane. The displacement v satisfies the two-dimensional wave equation (3.23). The geometry suggest that one uses polar coordinates, in which case $v = v(r, \theta, t)$. One assumes that the membrane has zero displacement at the circular boundary $r = a$. That is, $v(a, \theta, t) = 0$. The initial position and velocity are given by

$$v(r, \theta, 0) = \alpha(r, \theta), \quad \frac{\partial v}{\partial t}(r, \theta, 0) = \beta(r, \theta). \tag{3.175}$$

In summary, the initial boundary value problem (IBVP) can be written as

$$\frac{\partial^2 v}{\partial t^2} = c^2 \left(\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Z^2} \right), \quad 0 < r < a, \tag{3.176a}$$

$$v(a, Z, t) = 0, \tag{3.176b}$$

$$v(r, \theta, 0) = \alpha(r, \theta), \quad \frac{\partial v}{\partial t}(r, \theta, 0) = \beta(r, \theta). \tag{3.176c}$$

One can first separate out the time variable by seeking

$$v(r, \theta, 0) = T(t)\Phi(r, \theta). \tag{3.177}$$

Then as in Section 3.5.2, $T(t)$ satisfies

$$T'' = -\lambda c^2 T(t), \tag{3.178}$$

where λ is a separation constant. From (3.178), the natural frequencies of vibration are $c\sqrt{\lambda}$, if $\lambda > 0$. In addition, $\Phi(r, \theta)$ satisfies the two-dimensional eigenvalue problem

$$\Delta\Phi + \lambda\Phi = 0 \tag{3.179}$$

with $\Phi = 0$ on the entire boundary, $r = a$:

$$\Phi(a, \theta) = 0. \tag{3.180}$$

One may seek product solutions of (3.179) in the form appropriate for polar coordinates.

$$\Phi(r, \theta) = A(r)B(\theta), \tag{3.181}$$

since for the circular membrane $0 < r < a$, $-\pi < \theta < \pi$. Now substitution of (3.181) into (3.179) yields

$$\frac{B(\theta)}{r} \frac{d}{dr} \left(r \frac{dA}{dr} \right) + \frac{A(r)}{r^2} \frac{d^2 B}{d\theta^2} = 0. \tag{3.182}$$

Dividing (3.182) by $A(r)B(\theta)$ and multiplying through by r^2 give

$$\frac{-1}{B} \frac{d^2 B}{d\theta^2} = \frac{r}{A} \frac{d}{dr} \left(r \frac{dA}{dr} \right) + \lambda r^2 = q. \tag{3.183}$$

Since $B(\theta)$ must oscillate in order to satisfy the periodic conditions in θ, one introduces a second separation constant q. One can write out the three ordinary differential equations with two separation constants as

$$T''(t) + \lambda c^2 T(t) = 0, \tag{3.184a}$$

$$B''(\theta) + qB(\theta) = 0, \tag{3.184b}$$

$$r^2 A''(r) + rA'(r) + (\lambda r^2 - q)A(r) = 0. \tag{3.184c}$$

Two of the three equations (3.184) must be eigenvalue problems. Now in the absence of initial conditions, the only boundary condition is $A(a) = 0$, which follows from $v(a, \theta, t) = 0$ or $\Phi(a, \theta) = 0$. Both r and θ are defined over finite intervals, $0 < r < a$ and $-\pi < \theta < \pi$ respectively. One needs boundary conditions at both ends. The periodic nature of the solution in θ implies that

$$B(-\pi) = B(\pi), \quad \frac{dB}{d\theta}(-\pi) = \frac{dB}{d\theta}(\pi). \tag{3.185}$$

So if $\lambda < 0$, the general solution of (3.184a) becomes

$$T(t) = E \exp\left(c\sqrt{\lambda}\right) t + F \exp\left(-c\sqrt{\lambda}\right) t, \tag{3.186}$$

where E and F are arbitrary constants. If $\lambda = m^2 > 0$ with $m > 0$ then the general solution of (3.184a) is

$$T(t) = E\cos\left(c\sqrt{\lambda}\right) t + F\sin\left(c\sqrt{\lambda}\right) t = E\cos(cmt) + F\sin(cmt), \tag{3.187}$$

The equation (3.184b) follows from the periodicity condition that

$$B(\theta) = a\cos\ (n\theta) + b\sin(n\theta), \tag{3.188}$$

where n is an integer. Equation (3.184c) which denotes the radial component $A(r)$ can be re-written as

$$r^2 A'' + rA' + \left(w^2 - n^2\right) A = 0, \tag{3.189}$$

where $w^2 = \lambda r^2$. Equation (3.189) is the Bessel differential equation. Thus the radial component $A(r)$ has the form

$$A(r) = \gamma J_n(w), \quad n = 0, 1, 2, \cdots \tag{3.190}$$

and

$$A(r) = J_n\left(\sqrt{\lambda_{nl}} r\right), \quad n = 0, 1, \cdots, l = 1, 2, \cdots,$$

where $q = n^2$, $\sqrt{\lambda_{nl}} = w_{nl}/a$ with w_{nl} representing the $n - th$ positive root of J_n. The Bessel function $J_n(w)$ satisfies the Bessel equation

$$w^2 J_n'' + wJ_n' + \left(w^2 - n^2\right) J_n = 0, \tag{3.191}$$

The radial function J_n has infinitely many roots for each n, denoted by w_{ln}. The boundary condition that V vanishes where $r = a$ will be satisfied if the corresponding wave numbers are given by

$$\sqrt{\lambda_{nl}} = w_{nl}/a, \quad l = 1, 2, \cdots, \quad n = 0, 1, \cdots$$

The general solution of V will take the form involving products of $\sin n\theta$, $\cos n\theta$, and $J_n(w_{nl} r)$. Thus

$$\Phi(r, \theta) = A(r)B(\theta) = (a\cos n\theta + b\sin n\theta) J_n\left(\sqrt{\lambda_{n,l}} r\right). \tag{3.192}$$

Therefore the general solution of the displacement is given by

$$v_{kl}(r, \theta, t) = \left(E\cos(cm_{kl}t) + F\sin(cm_{kl}t)\right)(A\cos k\theta + B\sin k\theta) J_k\left(\sqrt{\lambda_{kl}} r\right), \quad l = 1, 2, \cdots, \quad k = 0, 1, \cdots. \tag{3.193}$$

Now supposing one uses the boundary conditions (3.185), then from equation (3.184b), for $q > 0$, one computes eigenvalues as

$$q_n = n^2, \quad n = 0, 1, 2, \cdots . \tag{3.194}$$

The corresponding eigenfunctions are both

$$B(\theta) = \sin n\theta, \quad B(\theta) = \cos n\theta. \tag{3.195}$$

For $n = 0$, equation (3.195) reduces to one eigenfunction. The eigenvalue problem generates a full Fourier series in θ and n is the number of crests in $\theta-$direction. Now since the displacement must be finite, one can conclude that $\left|A(0)\right| < \infty$. For each integral value of n. One can define an eigenvalue problem for λ as

$$r^2 A'' + rA' + \left(w^2 - n^2\right) A = 0, \quad A(a) = 0, \quad \left|A(0)\right| < \infty. \tag{3.196}$$

The general solution to the Bessel differential equation (3.196) is given by

$$A(r) = a_1 J_n(\sqrt{\lambda} r) + a_2 Y_n(\sqrt{\lambda} r). \tag{3.197}$$

Applying the homogeneous boundary conditions in (3.196), one can compute the eigenvalues. Since $A(0)$ must be finite and $Y_n(0)$ is infinite. Thus $a_2 = 0$, implying that

$$A(r) = a_1 J_n(\sqrt{\lambda} r).$$

Using the condition $A(a) = 0$, one can compute the eigenvalues $J_n(\sqrt{\lambda} a) = 0$. It is observed that $\sqrt{\lambda} a$ must be a zero of the Bessel function $J_n(w)$ and there are in fact an infinite number of zeros of each Bessel function $J_n(w)$. Let w_{nm} denote the $n-$th zero of $J_n(w)$. Then

$$\sqrt{\lambda} a = w_{nm}, \quad \lambda_{nm} = \left(\frac{w_{nm}}{a}\right)^2. \tag{3.198}$$

The vibrations $v(r, \theta, t)$ of a circular membrane are usually described by the two-dimensional wave equation (3.176a), with v being fixed on the boundary subject to initial conditions. The general solution of (3.176) is given by

$$v(r, \theta, t) = \sum_{m=1}^{\infty} \frac{a_{0m}(t)}{2} J_0(\sqrt{\lambda_{0m} r}) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(a_{nm} J_n\left(\sqrt{\lambda_{nm} r}\right) \cos n\theta + b_{nm} J_n\left(\sqrt{\lambda_{nm} r}\right) \sin n\theta \right). \tag{3.199}$$

Now we can plug the expansion (3.199) into the wave equation (3.176a). One finds that the equation is satisfied if the time-dependent coefficients have the form

$$a_{0m}(t) = D_{nm} \cos\left(\frac{c\sqrt{\lambda_{nm}} t}{L}\right) + F_{nm} \sin\left(\frac{c\sqrt{\lambda_{nm}} t}{L}\right), \quad b_{0m}(t) = E_{nm} \cos\left(\frac{c\sqrt{\lambda_{nm}} t}{L}\right) + G_{nm} \sin\left(\frac{c\sqrt{\lambda_{nm}} t}{L}\right). \tag{3.200}$$

Using the first initial condition, the initial position $v(r, \theta, 0) = \alpha(r, \theta)$, one finds

$$\alpha(r, \theta) = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} D_{nm} J_n\left(\sqrt{\lambda_{nm} r}\right) \right) \cos n\theta + \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} E_{nm} J_n\left(\sqrt{\lambda_{nm} r}\right) \right) \sin n\theta, \tag{3.201}$$

where

$$D_{nm} = \frac{\int_{-\pi}^{\pi} \int_0^a J_n(\sqrt{\lambda_{nm}}r) \cos\ n\theta \alpha(r,\theta) r \mathrm{d}r \mathrm{d}\theta}{\pi \int_0^a J_n^2(\sqrt{\lambda_{nm}}r) r \mathrm{d}r} \tag{3.202}$$

and

$$E_{nm} = \frac{\int_{-\pi}^{\pi} \int_0^a J_n(\sqrt{\lambda_{nm}}r) \sin\ n\theta \alpha(r,\theta) r \mathrm{d}r \mathrm{d}\theta}{\pi \int_0^a J_n^2(\sqrt{\lambda_{nm}}r) r \mathrm{d}r}. \tag{3.203}$$

Now from the second initial condition,

$$\begin{aligned} \frac{\partial v}{\partial t}(r,\theta,0) = \sum_{m=1}^{\infty} \frac{c\sqrt{\lambda_{nm}}}{L} F_{nm} J_0(w_{nm}r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{c\sqrt{\lambda_{nm}}}{L} F_{nm} J_n\left(\sqrt{\lambda_{nm}}r\right) \cos n\theta \\ + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{c\sqrt{\lambda_{nm}}}{L} G_{nm} J_n\left(\sqrt{\lambda_{nm}}r\right) \sin n\theta = \beta(r,\theta), \end{aligned} \tag{3.204}$$

so that

$$F_{nm} = \frac{L}{c\sqrt{\lambda_{nm}}} \frac{\int_{-\pi}^{\pi} \int_0^a J_n(\sqrt{\lambda_{nm}}r) \cos\ n\theta \alpha(r,\theta) r \mathrm{d}r \mathrm{d}\theta}{\pi \int_0^a J_n^2(\sqrt{\lambda_{nm}}r) r \mathrm{d}r} \tag{3.205}$$

and

$$G_{nm} = \frac{L}{c\sqrt{\lambda_{nm}}} \frac{\int_{-\pi}^{\pi} \int_0^a J_n(\sqrt{\lambda_{nm}}r) \sin\ n\theta \alpha(r,\theta) r \mathrm{d}r \mathrm{d}\theta}{\pi \int_0^a J_n^2(\sqrt{\lambda_{nm}}r) r \mathrm{d}r}. \tag{3.206}$$

When one assumes that the membrane is initially at rest, then from the second initial condition $\frac{\partial v}{\partial t}(r,\theta,0) = \beta(r,\theta) = 0$. Then using the principle of superposition, one can satisfy the initial value problem by considering the infinite linear combination of the remaining product solutions

$$v(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} D_{nm} J_n\left(\sqrt{\lambda_{nm}}r\right) \cos(n\theta)\ \cos(c\sqrt{\lambda_{nm}}t) + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} E_{nm} J_n\left(\sqrt{\lambda_{nm}}r\right) \sin(n\theta) \cos(c\sqrt{\lambda_{nm}}t). \tag{3.207}$$

To compute the coefficients one can use the two-dimensional orthogonality. Let us consider the two-dimensional eigenvalue problem (3.179) with $\Phi = 0$ on a circular membrane with radius a, the two-dimensional eigenfunctions are the doubly infinite families

$$\Phi_\lambda(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} D_{nm} J_n\left(\sqrt{\lambda_{nm}}r\right) \cos\ (n\theta) + E_{nm} J_n\left(\sqrt{\lambda_{nm}}r\right) \sin\ (n\theta)\,. \tag{3.208}$$

Here

$$\alpha(r,\theta) = \sum_{\lambda} D_\lambda \Phi_\lambda(r,\theta),$$

where $\sum_\lambda$ represents summation over all eigenfunctions. The eigenfunctions are orthogonal with weight 1. The coefficient D_λ is computed as

$$D_\lambda = \frac{\int\int \alpha(r,\theta)\Phi_\lambda(r,\theta) \mathrm{d}A}{\int\int \Phi_\lambda^2(r,\theta) \mathrm{d}A}, \tag{3.209}$$

where D_λ represents the two coefficients E_{nm} and D_{nm} and dA $= r dr d\theta$. In two-dimensional problems the weighting function is a constant.

3.6 Exact Solutions of the Linear Wave Equation in Unbounded Domain in 2+1 Dimensions

3.6.1 The Cauchy Problem for the Two-Dimensional Wave Equation

Let us consider the linear wave equation (3.23) that comes from the deformation field (3.3). Suppose the equation (3.23) is the two-dimensional Cauchy problem with initial conditions

$$v(X,Z,0)=0, \quad v_t(X,Z,0)=g(X,Z),$$

then the initial value problem for the linear wave equation is given by

$$v_{tt}=c^2\left(v_{XX}+v_{ZZ}\right), \quad -\infty<X<\infty, \quad -\infty<Z<\infty, \quad t>0, \tag{3.210a}$$

$$v(X,Z,0)=0, \quad v_t(X,Z,0)=g(X,Z), \tag{3.210b}$$

$$v(X,Z,t)\to 0 \text{ at } X,Z\to\infty, \tag{3.210c}$$

$$v_t(X,Z,t)\to 0 \text{ at } X,Z\to\infty, \tag{3.210d}$$

where c is constant. Now one has to make use of the double Fourier transform in X and Z is given by

$$\mathcal{F}[v(X,Z)]=V(k,\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)v(X,Z)\mathrm{d}X\ \mathrm{d}Z, \tag{3.211}$$

where $\boldsymbol{\eta}=(k,\alpha)$, $\mathbf{r}=(X,Z)$, and $\boldsymbol{\eta}\cdot\mathbf{r}=kX+\alpha Z$. The inverse Fourier transform is given by

$$\mathcal{F}^{-1}[V(k,\alpha)]=v(X,Z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)V(k,\alpha)\mathrm{d}k\ \mathrm{d}\alpha, \tag{3.212}$$

Applying the two-dimensional Fourier transform in X and Z given by (3.211) to the equations (3.210) and (3.210a), one finds

$$\frac{d^2V}{dt^2}+c^2\tau^2V=0, \quad \tau^2=k^2+\alpha^2, \quad V(k,\alpha,0)=0, \quad \left.\frac{dV}{dt}\right|_{t=0}=G(k,\alpha). \tag{3.213}$$

Thus the solution of the transformed system becomes

$$V(k,\alpha,t)=A(k,\alpha)\cos(c\tau t)+B(k,\alpha)\sin(c\tau t).$$

Using the transformed initial data, one has

$$V(k,\alpha,t)=G(k,\alpha)\frac{\sin(c\tau t)}{c\tau}. \tag{3.214}$$

The inverse Fourier transform gives the solution

$$\begin{aligned}
v(X,Z,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)G(k,\alpha)\frac{\sin(c\tau t)}{c\tau}\mathrm{d}k\ \mathrm{d}\alpha,\\
&=\frac{1}{2\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)G(k,\alpha)\left(\frac{\exp(ic\tau t)-\exp(-ic\tau t)}{2i\tau}\right)\mathrm{d}k\ \mathrm{d}\alpha,\\
&=\frac{1}{4i\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)G(k,\alpha)\left(\frac{\exp(ic\tau t)-\exp(-ic\tau t)}{\tau}\right)\mathrm{d}k\ \mathrm{d}\alpha,\\
&=\frac{1}{4i\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(k,\alpha)}{\tau}\left[\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}+ct\right)\right)-\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}-ct\right)\right)\right]\mathrm{d}k\ \mathrm{d}\alpha.
\end{aligned} \tag{3.215}$$

The equation (3.215) shows an important characteristics of the wave equation (3.210a). The exponential terms $\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}\pm ct\right)\right)$ appearing under the integral solution (3.215) denotes a plane wave solution of equation (3.210a). Also, the integral solution (3.215) denotes a superposition of plane wave solutions traveling in all directions. Now when one replaces the initial condition (3.210b) in (3.210) with

$$v(X,Z,0)=f(X,Z),\quad v_t(X,Z,0)=0, \tag{3.216}$$

then following all the procedures above in this Section 3.6.1, one has

$$\frac{d^2V}{dt^2}+c^2\tau^2V=0,\quad \tau^2=k^2+\alpha^2,\quad V(k,\alpha,0)=F(k,\alpha),\quad \left.\frac{dV}{dt}\right|_{t=0}=0, \tag{3.217}$$

and the formal solution is given by the Fourier inverse

$$\begin{aligned}
v(X,Z,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)F(k,\alpha)\cos(c\tau t)\mathrm{d}k\,\mathrm{d}\alpha,\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)F(k,\alpha)\left(\frac{\exp(ic\tau t)+\exp(-ic\tau t)}{2}\right)\mathrm{d}k\,\mathrm{d}\alpha,\\
&=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)F(k,\alpha)\left(\exp(ic\tau t)+\exp(-ic\tau t)\right)\mathrm{d}k\,\mathrm{d}\alpha,\\
&=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(k,\alpha)\left[\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}+ct\right)\right)+\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}-ct\right)\right)\right]\mathrm{d}k\,\mathrm{d}\alpha.
\end{aligned} \tag{3.218}$$

Also when one replaces the initial condition (3.210b) in (3.210) with

$$v(X,Z,0)=f(X,Z),\quad v_t(X,Z,0)=G(X,Z), \tag{3.219}$$

then from the procedures above in this current Section 3.6.1, one has

$$\frac{d^2V}{dt^2}+c^2\tau^2V=0,\quad \tau^2=k^2+\alpha^2,\quad V(k,\alpha,0)=F(k,\alpha),\quad \left.\frac{dV}{dt}\right|_{t=0}=G(k,\alpha), \tag{3.220}$$

and the formal solution is just addition of (3.218) and (3.215) given by the Fourier inverse

$$\begin{aligned}
v(X,Z,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)F(k,\alpha)\cos(c\tau t)\mathrm{d}k\,\mathrm{d}\alpha\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(i\left(\boldsymbol{\eta}\cdot\mathbf{r}\right)\right)G(k,\alpha)\frac{\sin(c\tau t)}{c\tau}\mathrm{d}k\,\mathrm{d}\alpha,\\
&=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(k,\alpha)\left[\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}+ct\right)\right)+\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}-ct\right)\right)\right]\mathrm{d}k\,\mathrm{d}\alpha\\
&+\frac{1}{4i\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(k,\alpha)}{\tau}\left[\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}+ct\right)\right)-\exp\left(i\tau\left(\frac{\boldsymbol{\eta}\cdot\mathbf{r}}{\tau}-ct\right)\right)\right]\mathrm{d}k\,\mathrm{d}\alpha.
\end{aligned} \tag{3.221}$$

3.6.2 Depth-Decaying Solution in Half-Space

Let us consider a linear Love wave problem (3.23) that satisfies the Helmholtz equation in an infinite elastic half-space given by

$$v_{tt}=c_2^2\left(v_{XX}+v_{ZZ}\right),\quad -\infty<X<\infty,\quad 0<Z<\infty. \tag{3.222}$$

At the surface of the half-space ($Z = 0$), the boundary condition relating to the surface stress is given by

$$\frac{\partial v}{\partial Z} = 0 \text{ at } Z = 0, \tag{3.223}$$

and $v(X, Z, t) \to 0$ as $Z \to \infty$ for $-\infty < X < \infty$. Now assuming one restricts our consideration to a time harmonic dependence with time factor $\exp(-i\omega t)$, then the displacement can be written as

$$v(X, Z, t) = V(X, Z) \exp(-i\omega t). \tag{3.224}$$

plugging (3.224) into (3.222) gives the Helmholtz equation

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Z^2} + \frac{\omega^2}{c_2^2} V = 0, \tag{3.225}$$

where ω is the frequency and c_2 is the shear wave speed. The equation (3.225) can be written as

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Z^2} + q^2 V = 0, \quad q^2 = \frac{k^2 c^2}{c_2^2}, \tag{3.226}$$

where $k^2 = \omega^2/c^2$ is the wave number. Now applying Fourier transform in X gives

$$\frac{d^2 \hat{V}}{dZ^2} - \gamma^2 \hat{V} = 0, \quad \gamma^2 = \alpha^2 - q^2. \tag{3.227}$$

Now for $\gamma^2 > 0$, then

$$\hat{V}(\alpha) = A(\alpha) \exp(\gamma Z) + B(\alpha) \exp(-\gamma Z). \tag{3.228}$$

Thus

$$\hat{V}(\alpha) = B(\alpha) \exp(-\gamma Z). \tag{3.229}$$

Taking the Fourier inverse transform gives

$$V(X, Z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(\alpha) \exp(-i\alpha X) \exp(-\gamma Z) \mathrm{d}\alpha. \tag{3.230}$$

Hence the formal solution is given by

$$v(X, Z, t) = \frac{1}{2\pi} \exp(i\omega t) \int_{-\infty}^{\infty} B(\alpha) \exp(i\alpha X) \exp(-\gamma Z) \mathrm{d}\alpha. \tag{3.231}$$

Now suppose one has the initial boundary value problem for an elastic half-space

$$v_{tt} = c_2^2 \left(v_{XX} + v_{ZZ} \right), \quad -\infty < X < \infty, \quad 0 < Z < \infty, \tag{3.232a}$$

$$v(X, Z, 0) = f(X, Z), \quad v_t(X, Z, 0) = g(X, Z), \tag{3.232b}$$

$$v(X, Z, t) \to 0 \text{ at } X, Z \to \infty, \tag{3.232c}$$

$$v_t(X, Z, t) \to 0 \text{ at } X, Z \to \infty, \tag{3.232d}$$

Taking the double Fourier transform in X and t to the system (3.232) gives

$$\frac{d^2 V}{dZ^2} - k^2 V + \frac{\omega^2}{c_2^2} V = 0, \quad V(k, \omega) = F(k, \omega), \quad \left. \frac{dV}{dt} \right|_{t=0} = G(k, \omega). \tag{3.233}$$

Since $k = \omega/c$, one can re-write (3.233) as

$$\frac{d^2V}{dZ^2} - k^2\left(1 - \frac{c^2}{c_2^2}\right)V = 0. \tag{3.234}$$

Solving the ordinary differential equation (3.234) yields

$$V(k,\omega) = A(k,\omega)\exp(mZ) + B(k,\omega)\exp(-mZ)\,, \quad m^2 = k^2\left(1 - \frac{c^2}{c_2^2}\right). \tag{3.235}$$

The equation (3.235) can be written by putting $A(k,\omega) = 0$ as

$$V(k,\omega) = B(k,\omega)\exp(-mZ)\,, \quad m^2 = k^2\left(1 - \frac{c^2}{c_2^2}\right). \tag{3.236}$$

From equation (3.236), one needs $m \geq 0$ for decaying solutions and $1 - c^2/c_2^2 \leq 0$. This implies that $c^2 \leq c_2^2$. Thus

$$|c| = \left|\frac{\omega}{k}\right| \leq c_2 \implies |\omega| \leq c_2|k|\,, \quad \omega, k > 0.$$

The inverse Fourier transform gives the formal solution

$$v(X,Z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-c_2k}^{c_2k}\exp\big(i\,(kX + \omega t)\big)\,B(k,\omega)\exp(-mZ)\,\mathrm{d}\omega\,\mathrm{d}k. \tag{3.237}$$

Equation (3.237) may be written as

$$v(X,Z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-c_2k}^{c_2k}\exp\big(ik\,(X + ct)\big)\,B(k,\omega)\exp(-mZ)\,\mathrm{d}\omega\,\mathrm{d}k. \tag{3.238}$$

3.7 Discussion

Hyperelasticity provides a general framework to derive linear and nonlinear wave models, isotropic and anisotropic, homogeneous and inhomogeneous, compressible and incompressible models. We started from considering Love-wave type displacements (3.3), we showed that for these displacements, the strain energy function $W = W(I_1, I_2, I_3)$ has the form (3.9) depending only on $v_X^2 + v_Z^2$, i. e., the magnitude of displacement gradient. Importantly, it was shown that all motions (3.3) are naturally incompressible.

In Section 3.2.1, we showed that the neo-Hookean and Mooney-Rivlin constitutive equations (1.74) lead to linear wave equation for the unknown displacement v, (3.23), which coincides with (2.9) derived in the linear elasticity framework in Section 2.2 of Chapter 2. In Section 3.2.2, we showed that the simplest model for linear Love wave can be obtained in the incompressible setting by using the neo-Hookean constitutive relation in Section 3.2.1. The equations in the elastic Layer and the elastic half-space (3.25a) together with boundary conditions (3.25b) and (3.25c) are equivalent to (2.9).

We considered the nonlinear Love-type model with shear displacements (3.3) in the framework of hyperelasticity in Section 3.3. We showed that for every strain density function or constitutive equation W of the form (3.26), one can obtain the wave equation (3.34) and the hydrostatic pressure (3.33). We established that

(3.34) is the most general two-dimensional wave equation which constitute both linear and nonlinear parts for all strain energy functions or constitutive equation $W = W(I_1, I_2, I_3)$. And equation (3.33) is the most general hydrostatic pressure with gravity forcing term. Thus for every constitutive relation W, one will find its corresponding wave equation. Constitutive relations such as Mooney-Rivlin and neo-Hookean give linear wave equations whiles cubic strain energy functions such as the Yeoh model give nonlinear wave equations. Following (3.33) and (3.34) in Section 3.3 and Section 3.3.2, the general hyperelastic nonlinear Love wave equation together with boundary conditions were provided by equations (3.36a)-(3.36c).

The linear Love wave equation together with hydrostatic pressure given by (3.38a) and (3.38b) respectively were presented in Section 3.3.3. Nonlinear surface wave equations were derived using the Murnaghan potential (3.46) in Section 3.3.4. Due to the non-polyconvexity of the Murnaghan model, we adapted a new polyconvex Murnaghan potential (3.52) based on only the first principal invariant I_1 and derived a nonlinear wave equation (3.57a). We also showed that the Yeoh constitutive equation (1.79) used in (3.33) and (3.34) gave the nonlinear Love-wave type equation (3.61a) obtained by Teymur [129], Kalyanasundaram [72], and Rushchitsky [120]. In Section 3.3.6, a simplified one-dimensional version of the shear displacements (3.3) of the form (3.62) was provided. Also the one-dimensional analogue of all the formulas and wave equations in Section 3.3.1 to Section 3.3.4 were given in Section 3.3.6.

In Section 3.4, we considered a shear displacement (3.79) corresponding to Rayleigh waves. We showed a simplified version of the deformation field (3.79) in the form (3.80). We also showed that the Mooney-Rivlin and the neo-Hookean constitutive equations (1.74) lead to the nonlinear wave equation (3.86b) together with nonlinear constraint equation (3.86a). We showed that after linearization and rescaling of (3.86b) yield the linear wave equation (3.87). The one-dimensional analogue of shear displacement (3.79) was provided in Section 3.4.3. Again we derived the general wave equation (3.95b) for compressible materials together with constraint (3.95a) and for incompressible materials we derived equation (3.97b) for all strain energy density function $W = W(I_1)$.

Moreover, in Section 3.5, we presented solutions to some selected linear wave equation problems, using the separation of variables and related classical techniques. For the two-dimensional wave equation, we employed the separation of variables method to obtain the general solution. The solutions to the one-dimensional linear wave equation subject to both Dirichlet and Neumann boundary conditions and appropriate initial conditions are obtained using the method of separation of variables in Section 3.5. The D'Alembert solution to any initial value problem of the linear wave equation was determined. Also, for two-dimensional wave problems, we used the method of separation of variables to obtain a general solution for a boundary value problem on rectangular and circular domains.

Finally, in Section 3.6, we presented an exact solution of the two dimensional wave in an unbounded domain and depth-decaying solution in half-space. The double Fourier transform and its inverse were employed to obtain integral solutions to the linear wave equations. The solution (3.238) is the general solution to the initial boundary value problem (3.232).

4 HAMILTONIAN AND LAGRANGIAN FORMALISM IN NONLINEAR ELASTODYNAMICS

4.1 Introduction

Hamiltonian and Lagrangian systems play a significant role in multiple applications. Quantum mechanics, classical mechanics, general relativity, classical field theory, and many other areas contain examples of where Lagrangian and Hamiltonian structures are of great importance. A Hamiltonian system is a mathematical structure used to describe dynamical systems given by differential equations. It can be defined as a physical function used in interpreting the motion of particles in classical mechanics when they are subjected to potentials corresponding to different force fields. William Rowan Hamilton [57] first formulated Hamiltonian mechanics approximately 45 years after Joseph Lagrange has formulated the Lagrangian mechanics. Hamiltonian mechanics has two main sources, namely Lagrangian mechanics, which constitutes the derivation from the principle of least action and the Euler-Lagrange variational equations, and geometrical optics as the second source, with Fermats and Huyghens principles. These days, these two sources have been merged under a common theory in mathematics called the *symplectic geometry* which takes place on arbitrary manifolds, endowed with a symplectic structure [9].

In classical mechanics, the Hamiltonian function obeys Hamilton's equation, which is termed the canonical equation relating to the rate of change of the coordinate and momentum with time, and with variation in H. Hamiltonian formalism reveals analytical properties of a model even if the initial value problem cannot be solved analytically; for example, it is helpful to analyze solution stability. Hamiltonian formalism is essential in solving problems that arise in areas such as quantum mechanics, classical dynamics, and elastodynamics. It highlights the structure of a physical problem. The Hamiltonian system enables one to use geometric methods for investigating the properties of dynamical systems. It is helpful when changing variables and for writing equations of motion in new variables. In quantum systems, the Hamiltonians are differential operators in Hilbert space of wave functions.

Lagrangian formulation of classical mechanics on the other hand, was formulated by Lagrange about 100 years after Newton has formulated the Newtonian mechanics. Since the Lagrangian method is simpler than the Newtonian methods, it provides a computational advantage in terms of learning the Lagrangian formulation of classical mechanics. The Lagrangian formulation is the natural framework for quantum field theory and general relativity.

Lagrangian mechanics is suitable for systems with conservative forces and for circumventing constraint forces in any coordinates. It is widely used to analyze mechanics in physics when Newton's formulation of classical mechanics is not convenient. It applies to the dynamics of particles, while fields are described employing a Lagrangian density, Lagrangian's equations are also used in optimization problems of dynamical systems. Lagrangian mechanics further shows conserved quantities and their symmetries directly, as a particular case of Noether's theorem. It is also crucial in advancing broad understanding in physics .

For simple mechanical systems, classical Hamiltonian of a physical non-dissipative system is defined as the sum of the potential and kinetic energies and Lagrangian is the potential energy minus kinetic energy. They are written as functions of the position and their conjugate momenta [2, 9]. Mathematically, the Hamiltonian (H) and the Lagrangian (L) are represented by the formulas

$$H = W + K, \quad L = W - K,$$

where W denotes the potential energy, and K is the kinetic energy. Hamiltonian formalism is closely related to the Lagrangian formulation. The main motivation to use Hamiltonian mechanics instead of Lagrangian mechanics comes from a more apparent geometric structure present in the Hamiltonian systems. Active research has been, and is being conducted on Hamiltonian methods (see, for example, [2, 9, 33, 87, 88, 126] and references therein.)

4.2 Variational Principle and Lagrangian Formulation for ODE and PDE Models

The weak form of the system of equations in elasticity is relevant because it is often convenient for numerical computations; it helps in the study of uniqueness and existence of solutions. It also holds in cases when shock waves are present, and the strong form of the equations does not make sense [51, 87, 140]. One can say that not every differential equation model is coming from a Lagrangian, so not all models are variational. Let us consider the situation with m independent variables $y = \left(y^1, \cdots, y^m\right)$ and n arbitrary functions (dependent variables) $u = \left(u_1(y), \cdots, u_n(y)\right)$ and their partial derivatives to order j defined on a domain $\bar{\Omega}$

$$I[u] = \int_{\bar{\Omega}} L[u]\mathrm{d}y = \int_{\bar{\Omega}} L\left(y, u, \partial u, \cdots, \partial^j u\right)\, \mathrm{d}y. \tag{4.1}$$

The differential function $L[u] = L\left(y, u, \partial u, \cdots, \partial^j u\right)$ and the functional $I[u]$ in equation (4.1) are called the Lagrangian and action integral respectively. The variation in the Lagrangian $L[u]$ is given by

$$\begin{aligned} \delta L &= L\left(y, u + \epsilon w, \partial u + \epsilon \partial w, \cdots, \partial^j u + \epsilon \partial^j w\right) - L\left(y, u, \partial u, \cdots, \partial^j u\right), \\ &= \epsilon \left(\frac{\partial L[u]}{\partial u^r} w^r + \frac{\partial L[u]}{\partial u_i^r} w_i^r + \cdots + \frac{\partial L[u]}{\partial u_{i_1 \cdots i_j}^r} w_{i_1 \cdots i_j}^r\right) + O\left(\epsilon^2\right), \end{aligned} \tag{4.2}$$

where $u(y) \to u(y) + \epsilon w(y)$ is an infinitesimal change of u and $w(y)$ is any function such that $w(y)$ and its derivatives to order $j-1$ vanish on the boundary $\partial\bar{\Omega}$ of the domain $\bar{\Omega}$. Using integration by parts repeatedly in equation (4.2), one can show that

$$\delta L = \epsilon \left(w^r E_{u^r} \left(L[u] \right) + D_j v^j[u,w] \right) + O\left(\epsilon^2\right), \tag{4.3}$$

where E_{u^r} is the Euler operator with respect to u^r and

$$\begin{aligned} v^l[u,w] = w^r & \left(\frac{\partial L[u]}{\partial u^r_l} + \cdots + (-1)^{j-1} D_{i_1} \cdots D_{i_{j-1}} \frac{\partial L[u]}{\partial u^r_{l i_1 \cdots i_{j-1}}} \right) \\ + w^r_{i\,1} & \left(\frac{\partial L[u]}{\partial u^r_{l i 1}} + \cdots + (-1)^{j-2} D_{i_2} \cdots D_{i_{j-1}} \frac{\partial L[u]}{\partial u^r_{l i_1 i_2 \cdots i_{j-1}}} \right) \\ & + \cdots + w^r_{i_1 \cdots i_{j-1}} \frac{\partial L[u]}{\partial u^r_{l i_1 i_2 \cdots i_{j-1}}}. \end{aligned} \tag{4.4}$$

Thus from equation (4.3) and the divergence theorem, one finds the corresponding variation in the action integral $I[u]$ is given by

$$\begin{aligned} \delta I = I[u+\epsilon w] - I[u] &= \int_{\bar{\Omega}} \delta L \mathrm{d}y \\ &= \epsilon \int_{\bar{\Omega}} \left(w^r E_{u^r}(L[u]) + D_s v^s[u,w] \right) \mathrm{d}y + O(\epsilon^2) \\ &= \epsilon \left(\int_{\bar{\Omega}} w^r E_{u^r}(L[u]) \mathrm{d}y + \int_{\partial\bar{\Omega}} v^s[u,w] n_r \mathrm{d}S \right) + O(\epsilon^2), \end{aligned} \tag{4.5}$$

where $\int_{\bar{\Omega}}$ denotes the surface integral over the boundary $\partial\bar{\Omega}$ of the domain $\bar{\Omega}$ with $n = (n_1, \cdots, n_m)$ is the unit outward normal vector to $\bar{\Omega}$. It follows that if $u = u(y)$ extremizes the action integral $I[u]$, then the $O(\epsilon)$ term of δI must vanish and hence

$$\int_{\bar{\Omega}} w^r E_{u^r}(L[u]) \mathrm{d}y = 0. \tag{4.6}$$

for an arbitrary $w(y)$ defined on the domain $\bar{\Omega}$. Hence, it follows that if $u = U(y)$ extremizes the action integral $I[u]$ given by (4.1), then $U(y)$ must satisfy the PDE system

$$E_{U^r}(L[U]) = \frac{\partial L[U]}{\partial U^r} + \cdots + (-1)^j D_{i_1} \cdots D_{i_j} \frac{\partial L[U]}{\partial U^r_{i_1 \cdots i_k}} = 0. \quad r = 1, 2, \ldots, n. \tag{4.7}$$

Equations (4.7) are called the Euler-Lagrange equations satisfied by an extremum $u = u(y)$ of the action integral $I[u]$.

For ODE systems, the action integral associates a number to each curve or path $q^j = q^j(t)$ satisfying the given boundary conditions. The variation of action from the variational principle is defined as

$$\delta \int L(q^i, \dot{q}^i) dt = 0. \tag{4.8}$$

Here one can choose variations among the paths $q^i(t)$ with endpoints fixed. If we denote the variation by δq^i, then equation (4.8) becomes

$$\int \Big(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \Big) \mathrm{d}t = 0, \quad \forall \text{ variations } \delta q^i. \tag{4.9}$$

After making use of the mixed partial derivatives, one can see that

$$\delta \dot{q}^i = \frac{d}{dt}\delta q^i.$$

If we take the above variation (4.8) and employ the chain rule we have

$$\int \Big(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\delta q^i\Big)\mathrm{d}t = 0. \tag{4.10}$$

Integrating the second term by parts and using a boundary conditions $\delta q^i = 0$ at the end points of the time interval yield

$$\int \Big(\frac{\partial L}{\partial q^i}\delta q^i - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\delta q^i\Big)\mathrm{d}t = 0. \tag{4.11}$$

Assume this holds for all variations δq^i. Since δq^i is arbitrary (except vanishing at the end points), then we have the Euler-Lagrange equations as

$$E_q L = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0. \tag{4.12}$$

Equation (4.12) is the Lagrangian equations of motion for a conserved system, which is also called Newton's second law of motion. They are second-order differential equations which requires $2n$ initial conditions. The term $(\partial L/\partial \dot{q}^i)$ is called the canonical momentum.

Let us consider an example of the harmonic oscillator in one dimension. The kinetic energy and potential energy are given respectively by

$$K = \frac{mx_t^2(t)}{2}, \quad W(x) = \frac{kx^2(t)}{2}. \tag{4.13}$$

The Lagrangian density is given by

$$L[x] = \frac{mx_t^2(t)}{2} - \frac{kx^2(t)}{2}. \tag{4.14}$$

The Euler operator is given by

$$E_x = \frac{\partial}{\partial x} - D_t\frac{\partial}{\partial x_t} \tag{4.15}$$

Applying (4.15) to (4.14) yields

$$E_x L = \left(\frac{\partial}{\partial x} - D_t\frac{\partial}{\partial x_t}\right)\left(\frac{mx_t^2(t)}{2} - \frac{kx^2(t)}{2}\right) = 0 \implies kx + mx_{tt} = 0. \tag{4.16}$$

As a second example, consider a multi-variable problem with the dependent variable $v(X,t)$. The action integral is given by

$$I[x] = \int_0^\infty \int_{\mathbb{R}} L[x]\,\mathrm{d}x\mathrm{d}t. \tag{4.17}$$

As a PDE example, consider $v = v(x,t)$ and the Lagrangian density

$$L[v] = \frac{v_t^2}{2} - \frac{c^2v_x^2}{2}. \tag{4.18}$$

The corresponding Euler operator is given by

$$E_v = \left(\frac{\partial}{\partial v} - D_x\frac{\partial}{\partial x_t} - D_t\frac{\partial}{\partial x_t}\right). \tag{4.19}$$

The Euler-Lagrange equations for this Lagrangian (4.18) is hence

$$E_v L = \left(\frac{\partial}{\partial v} - D_x\frac{\partial}{\partial x_t} - D_t\frac{\partial}{\partial x_t}\right)\left(\frac{v_t^2}{2} - \frac{c^2v_x^2}{2}\right) = 0 \implies v_{tt} - c^2v_{xx} = 0. \tag{4.20}$$

which is the linear wave equation.

4.3 Finite-Dimensional Hamiltonian Systems

In this section, we give some concepts of the Hamiltonian system of differential equations. We discuss systems of ordinary differential equations and provide examples of finite-dimensional Hamiltonians. We look at the generalization to systems of partial differential equations in subsequent sections. Finite-dimensional Hamiltonian methods are ordinary differential equations that may be written as dynamical systems.

4.3.1 Finite-Dimensional Poisson Bracket and Hamiltonian Vector Field

The notion of the Poisson bracket is essential in the study of finite-dimensional Hamiltonians. The Poisson bracket on a smooth manifold N is defined as an operation that assigns a smooth real-valued function $\{G, H\}$ on N to each pair G, H of smooth, real-valued function with the following four properties.

1. Leibniz's rule, $\{G, HK\} = \{G, H\}K + H\{G, K\}$.
2. Bilinearity, $\{\alpha G + \beta K, H\} = \alpha\{G, H\} + \beta\{K, H\}, \quad \{G, \alpha H + \beta K\} = \alpha\{G, H\} + \beta\{G, K\}$.
3. Jacobi identity, $\{\{G, H\}, K\} + \{\{K, G\}, H\} + \{\{H, K\}, G\} = 0$.
4. Skew-symmetry, $\{G, H\} = -\{H, G\}$,

where G, H, and K are arbitrary smooth real-valued functions of the manifold N. The manifold N together with a Poisson bracket is called a *Poisson manifold*. In canonical coordinates, let N represents an even-dimensional Euclidean space $\mathbb{R}^{2n}$ with coordinates $\left(p_i, q^i\right) = \left(p_1, \cdots p_n, q^1, \cdots, q^n\right)$, where in mechanics p_i and q_j are the momenta and positions of mechanical objects respectively. Then if $G(p_i, q^i)$ and $H(p_i, q^i)$ are smooth functions, we obtain the Poisson bracket to be the function

$$\{G, H\} = \sum_{i=1}^{n} \left\{ \frac{\partial G}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i} \frac{\partial H}{\partial q^i} \right\}. \tag{4.21}$$

The Poisson bracket (4.21) acting on the canonical coordinates themselves yields

$$\{q^i, q^j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q^i, p_j\} = \delta^i_j, \tag{4.22}$$

where δ^i_j is the Kronecker delta which is 1 when $i = j$ and 0 when $i \neq 0$, and i and j run from 1 to n.

Finite-Dimensional Hamiltonian Vector Field

Let N be a Poisson manifold, so the Poisson bracket satisfies the fundamental requirements (4.21) and $H : N \to \mathbb{R}$ be a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field X_H on N satisfying

$$X_H(G) = \{G, H\} = -\{H, G\} \tag{4.23}$$

for every smooth function $G : N \to \mathbb{R}$. The equations governing the flow of X_H are referred to as *Hamilton's equations* for the Hamiltonian function H. In Hamiltonian systems, there is always a link between the Poisson bracket of two smooth functions and the Lie bracket of their corresponding Hamiltonian vector field [108]. In particular,

$$X_{\{G,H\}} = [X_H, X_G] = -[X_G, X_H]$$

holds. Here $[\cdot,\cdot]$ is the Lie bracket. The Lie bracket of a vector field is an operator that assigns two vector fields G and H on a smooth manifold N a third vector field denoted $[G,H]$. The Lie bracket or the commutator of G and H is defined as

$$[G,H] = GH - HG.$$

Let us now look at a particular class of functions called *structure functions*. Let N be a manifold, let $y = \left(y^1, \cdots, y^n\right)$ be local coordinates of N, and $H(y)$ be a real-valued function. The corresponding Hamiltonian vector field has a general form

$$X_H = \sum_{i=1}^{n} \alpha^i\left(y\right)\frac{\partial}{\partial y^i},$$

where $\alpha^i\left(y\right)$ are the coefficients of $\partial/\partial y^i$ and the unknown functions to be determined. Also, let us consider another smooth function $G(y)$. Applying the definition of Hamiltonian vector field (4.23), we obtain

$$\{G,H\} = X_H\left(G\right) = \sum_{i=1}^{n} \alpha^i\left(y\right)\frac{\partial G}{\partial y^i}. \tag{4.24}$$

From (4.23), the unknown coefficient functions $\alpha^i(y)$ to be determined are given by

$$\alpha^i\left(y\right) = \{y^i, H\} = X_H\left(G\right). \tag{4.25}$$

Therefore, the equation (4.24) becomes

$$\{G,H\} = \sum_{i=1}^{n}\{y^i,H\}\frac{\partial G}{\partial y^i}. \tag{4.26}$$

Similarly, using the anti-commutativity property of the Poisson bracket, one can re-write (4.24) as

$$\{y^i,H\} = -\{H,y^i\} = -X_{y^i}\left(H\right) = -\sum_{k=1}^{n}\{y^k,y^i\}\frac{\partial H}{\partial y^k}. \tag{4.27}$$

From (4.26), we have the formula for Poisson bracket given by

$$\{G,H\} = \sum_{i=1}^{n}\sum_{k=1}^{n}\{y^i,y^k\}\frac{\partial G}{\partial y^i}\frac{\partial H}{\partial y^k}. \tag{4.28}$$

It is imperative for one to know the Poisson bracket between the local coordinates in order to compute the Poisson bracket of any pair of functions in some given set of local coordinates. These basis Poisson brackets are called the *structure functions* of manifold N with respect to a given local coordinates and they are given by

$$J^{ik}\left(y\right) = \{y^i,y^k\}, \quad i,\ k = 1,\cdots,n. \tag{4.29}$$

The structure function is usually expressed as a skew symmetric $n \times n$ matrix $J(y)$ termed as *structure matrix* of N. Therefore the local coordinate form (4.28) for the Poisson bracket may be written as

$$\{G, H\} = \nabla G \cdot J \nabla H, \tag{4.30}$$

where ∇H represent the column gradient vector for H. In matrix notation, the Hamiltonian vector field associated with $H(y)$ is given by

$$X_H = (J \nabla H) \cdot \partial_y, \tag{4.31}$$

where ∂_y denote the vector with entries $\partial / \partial y^i$. The equation (4.31) may be written as

$$X_H = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} J^{ik}(y) \frac{\partial H}{\partial y^i} \frac{\partial}{\partial y^i} \right). \tag{4.32}$$

The Hamilton's equation in the given coordinate chart is of the form

$$\frac{dy}{dt} = J(y) \nabla H(y). \tag{4.33}$$

Equation (4.33) may be written in the form of Poisson bracket by using (4.26) as

$$\frac{dy}{dt} = \{y, H\}, \tag{4.34}$$

A finite-dimensional *Hamiltonian system* is defined as a first-order ordinary differential equation with a Hamiltonian function $H(y)$ and a structure matrix of functions $J(y)$ determining a Poisson bracket (4.30) through which the system can be written as equation (4.33). Indeed, the matrices $J(y)$ are the structure matrices for the Poisson bracket, which satisfies the skew-symmetric and Jacobi identity properties [108]. One may allow $H(y, t)$ to depend on time t, which leads to a time-dependent Hamiltonian vector field (see [108] for details).

4.4 The Legendre Transformation and Hamiltonian Equations of Motion

The Legendre transformation is a self-inverse transformation, and it is of significant interest to study what happens if it is applied to the Hamiltonian. As is to be expected, if a Hamiltonian is generated by a Legendre transform from a Lagrangian, then this Lagrangian can be recovered. A Legendre transform is used in classical mechanics to derive the Hamiltonian formulation from the Lagrangian formulation and conversely. It is an essential tool in mathematics which transforms function on a vector space to functions on the dual space. In the finite-dimensional setting, suppose we have a Lagrangian $L : TQ_c \to \mathbb{R}$, then a *fiber derivative* is a map $FL : TQ_c \to T^*Q_c$ given by

$$FL(u) \cdot v = \frac{d}{d\varepsilon} L(u + v\varepsilon) \Big|_{\varepsilon=0}, \tag{4.35}$$

where $u, v \in T_qQ_c$. Thus, the fiber derivative $FL(u)\cdot v$ is the derivative of L at u along the fiber T_qQ_c in the direction v with FL being fiber-preserving. However, we can define the fiber derivative in a local coordinate chart $V \times X$ for TQ_c where V is an open set in the model space X for Q_c. The fiber derivative is represented by

$$FL(u, v) = (u, D_2L(u, v)), \tag{4.36}$$

where D_2L denotes the partial derivative of L with respect to its second argument. We note that for finite-dimensional manifolds, the fiber derivative is given by $FL(q^i, \dot{q}^i) = (q^i, \partial L/\partial \dot{q}^i)$ with FL given by $p_i = \partial L/\partial \dot{q}^i$, where q^i are the coordinates on Q_c and $(q^i, \dot{q}^i)$ represent the coordinates on TQ_c. We therefore define the energy function as

$$H = E(u) = FL(u) \cdot u - L(u). \tag{4.37}$$

One may call FL the *Legendre transform.* The Lagrangian can also be computed from the Legendre transformation through the relation

$$L(u) = FL \cdot u - H(u).$$

Now one can put $p_i = \partial L/\partial \dot{q}^i$ and define the Hamiltonian by

$$H(q^i, p_j) = p_i\dot{q}^i - L(q^i, \dot{q}^i). \tag{4.38}$$

Here we can show that the Euler-Lagrange equations are equivalent to the Hamilton's equations. To do so, we notice that $\dot{q}^i = \partial H/\partial p_i$ by taking the derivative of the Hamiltonian $H(q^i, p_j)$ with respect to p_i and noting that $p_j = \partial L/\partial \dot{q}^j$ as

$$\frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i}\{p_i\dot{q}^i - L(q^i, \dot{q}^i)\} = \dot{q}^i + p_j\frac{\partial \dot{q}^j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial p_j} = \dot{q}^i. \tag{4.39}$$

Again from equation (4.38) and employing the chain rule, we have $\dot{p}_i = -\partial H/\partial q^i$ by taking the derivative of the Hamiltonian $H(q^i, p_j)$ with respect to q^j and noting that $p_j = \partial L/\partial \dot{q}^j$ give

$$\frac{\partial H}{\partial q^j} = \frac{\partial}{\partial q^j}\{p_i\dot{q}^i - L(q^i, \dot{q}^i)\} = -\frac{\partial L}{\partial q^j} + p_i\frac{\partial \dot{q}^j}{\partial q_j} - \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial p_j} = -\frac{\partial L}{\partial q^j}. \tag{4.40}$$

That is,

$$-\frac{\partial H}{\partial q^i} = \frac{dp_i}{dt}, \quad \frac{\partial H}{\partial p_i} = \frac{dq^i}{dt}. \tag{4.41}$$

Now we outline some steps needed to construct finite-dimensional Hamiltonian as follows

1. Calculate the kinetic energy, K and potential energy W, and write down the Lagrangian $L = K - W$, in terms of the coordinates in use, say q_j and their derivatives ${q_j}_t$.

2. Calculate $p_j \equiv \partial L/\partial {q_j}_t$ for each n coordinates.

3. Invert the expressions for n p_i to solve for the n ${q_j}_t$ in terms of the p_j and q_j

4. Write the Hamiltonian, $H = \sum p_j {q_j}_t - L$, and do away with all the ${q_j}_t$ in favor of the p_j and q_J.

5. Write down the Hamilton's equations.

6. Solve Hamilton's equations. Where possible, this can be done by transforming the first-order Hamilton's equations for coordinates and momenta to second-order differential equations for the coordinates.

Example 4.4.1 (The Simple Harmonic Oscillator as a Hamiltonian System) Let us consider a particle of mass m subjected to a linear restoring force $F = -kx$, which is associated to the quadratic potential

$$W(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2,$$

where $k = m\omega^2$. In the Lagrangian and Hamiltonian description of classical mechanics, the system is described by the dynamical variables $\{x, p\}$, and the evolution is governed by the Lagrangian and Hamiltonian written as

$$L = K - W, \quad H = K + W,$$

where K is the kinetic energy in the system. The kinetic energy $K = (1/2)mv^2 = (1/2m)p^2$. The Lagrangian and the Hamiltonian in the system are written as

$$L = \frac{p^2}{2m} - \frac{1}{2}m\omega^2x^2, \quad H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2.$$

The Hamiltonian can also be computed from the Legendre transformation using the formula

$$H(x, p) = px_t - L(x, p),$$

where $x_t = \partial H/\partial p = \partial L/\partial p = p/m$. Thus

$$H(x, p) = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2.$$

The Hamilton's equations of motion are given by

$$x_t = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad p_t = -\frac{\partial H}{\partial x} = -m\omega^2x.$$

Now taking the second derivative, one obtains the second-order, linear, autonomous ODEs

$$x_{tt} + \omega^2x = 0, \quad p_{tt} + \omega^2p = 0, \tag{4.42}$$

where ω is a positive constant. The solutions to equation (4.42) give the usual oscillatory behaviour

$$x(t) = B\sin(\omega t + \theta), \quad p(t) = Bm\omega\cos(\omega t + \theta). \tag{4.43}$$

Example 4.4.2 (A finite-Dimensional Hamiltonian System) Let us consider an n-particle system in $\mathbb{R}^3$ subject to a potential force field modeled on a Coulomb interaction. The kinetic energy in the system is given by

$$K(\mathbf{x}_t) = \frac{1}{2}\sum_{i=1}^{n} m_i|\mathbf{x}_t|^2, \tag{4.44}$$

where m_i denotes the mass in the system and $\mathbf{x}^i = \left(x_1^i, x_2^i, x_3^i\right)$ be the position of the $i-th$ particle. The potential energy $W(t,\mathbf{x})$ in the system may vary depending on the physical problem. One of such potential energy is given by the formula

$$W(t,\mathbf{x}) = \sum \alpha_{ij}|\mathbf{x}^i - \mathbf{x}^j|^{-1} = \sum \frac{\alpha_{ij}}{|\mathbf{x}^i - \mathbf{x}^j|}, \tag{4.45}$$

which may depend on the pairwise gravitational interaction between masses, or for a simple case where $n = 1$, we might have the central gravitational force of Kepler's problem. The Lagrangian is written as

$$L = K - W = \frac{1}{2}\sum_{i=1}^{n} m_i|\mathbf{x}_t|^2 - \sum \alpha_{ij}|\mathbf{x}^i - \mathbf{x}^j|^{-1}. \tag{4.46}$$

From the Legendre transformation,

$$\begin{aligned} H(\mathbf{x},\mathbf{x}_t) &= \sum_{i=1}^{n} \mathbf{x}_t \frac{\partial L}{\partial \mathbf{x}_t} - L(\mathbf{x},\mathbf{x}_t), \\ &= \sum_{i=1}^{n} m_i|\mathbf{x}_t|^2 - \frac{1}{2}\sum_{i=1}^{n} m_i|\mathbf{x}_t|^2 + \sum \alpha_{ij}|\mathbf{x}^i - \mathbf{x}^j|^{-1}, \\ &= \frac{1}{2}\sum_{i=1}^{n} m_i|\mathbf{x}_t|^2 + \sum \alpha_{ij}|\mathbf{x}^i - \mathbf{x}^j|^{-1} \end{aligned}$$

where $\mathbf{x}_t = \partial L/\partial \mathbf{x}_t$. The total energy or the Hamiltonian of the system is thus given by

$$H = K + W = \frac{1}{2}\sum_{i=1}^{n} m_i|\mathbf{x}_t|^2 + \sum \alpha_{ij}|\mathbf{x}^i - \mathbf{x}^j|^{-1} \tag{4.47}$$

The Newton's equations of motion given by

$$m_i\mathbf{x}_{tt}^i = -\nabla_i W \equiv \left(W_{x_1^i}, W_{x_2^i}, W_{x_3^i}\right), \quad i = 1, \cdots, n, \tag{4.48}$$

admits variational form which are the Euler-Lagrange equations for the action integral $\int_{-\infty}^{\infty} (K - W)\, d\mathbf{x}$. The example above can be written in a canonical Hamiltonian form. In local canonical coordinates, let $\mathbf{q}_i = \left(q_1^i, q_2^i, q_3^i\right)$ be the positions of particles and $\mathbf{p}_i = \left(p_1^i, p_2^i, p_3^i\right) = m_i\dot{\mathbf{q}}_i,\ i = 1, \cdots, n$ be the momenta. The Lagrangian is given by the formula

$$L(p,q) = K(p) - W(q) = \sum_{j=1}^{n} \frac{|\mathbf{p}_i|^2}{2m_j} - \sum_{j<k} m_j m_k W\left(|\mathbf{q}_j - \mathbf{q}_k|\right). \tag{4.49}$$

Using the Legendre transformation,

$$\begin{aligned} H(p,q) &= \sum_{j=1}^{n} \mathbf{p}_i\mathbf{q}_{it} - L(p,q), \\ &= \sum_{j=1}^{n} \frac{|\mathbf{p}_i|^2}{m_j} - \sum_{j=1}^{n} \frac{|\mathbf{p}_i|^2}{2m_j} + \sum_{j<k} m_j m_k W\left(|\mathbf{q}_j - \mathbf{q}_k|\right), \end{aligned}$$

where $\mathbf{q}_{it} = \partial L/\partial \mathbf{p}_i$. The total energy or the Hamiltonian function is given by

$$H(p,q) = K(p) + W(q) = \sum_{j=1}^{n} \frac{|\mathbf{p}_i|^2}{2m_j} + \sum_{j<k} m_j m_k W\left(|\mathbf{q}_j - \mathbf{q}_k|\right), \tag{4.50}$$

which coincides with (4.47). The equations of motion are thus given by

$$\frac{d\mathbf{p}_i}{dt} = -\frac{\partial H}{\partial \mathbf{q}_i} = -\frac{\partial W}{\partial \mathbf{q}_i}, \quad \frac{d\mathbf{q}_i}{dt} = \frac{\partial H}{\partial \mathbf{p}_i} = \frac{\mathbf{p}_i}{m_i}, \quad i = 1, \cdots, n, \tag{4.51}$$

which are equivalent to (4.51).

4.5 Linear Hamiltonian Systems

In this section, we are concerned with some basic ideas of the theory of Hamiltonian systems which are linear differential equations (linear Hamiltonian systems) and their applications to elasticity. This includes several theorems that involve the existence of solutions. We begin this section by providing some basic definitions in Banach spaces as well as providing proof to some relevant theorems and propositions. We shall give examples (linear wave equation, etc.) illustrating the idea of linear Hamiltonian systems. We deal with this in the finite-dimensional setup and later extend it to the infinite-dimensional case. We also note that most of the definitions, theorems, and propositions are given in the framework of symplectic algebra since the symplectic manifold is a backbone to the study of Hamiltonian mechanics [2]. A linear Hamiltonian system is defined as a system of $2n$ ordinary differential equations

$$\frac{dy}{dt} = J\frac{\partial H}{\partial y} = J\nabla H = JS(t)y, \tag{4.52}$$

where

$$H = H(t,y) = \frac{1}{2} y^T S(t) y,$$

y is a coordinate vector in $\mathbb{R}^{2n}$, J is the skew symmetric structure matrix (4.29), $\mathbb{I}$ is an interval $\mathbb{R}$, $S : \mathbb{I} \to gl\,(2n, \mathbb{R})$ is continuous and symmetric, and H is the Hamiltonian [92]. We note that, the Hamiltonian above is the quadratic form in the $y's$ with coefficients that are continuous in $t \in \mathbb{I} \in \mathbb{R}$ [92].

For PDE one can write the Hamiltonian system (4.52) as

$$\frac{\partial u}{\partial t} = \mathcal{D} \cdot \delta\mathcal{H}[u], \tag{4.53}$$

where $\mathcal{H}[u]$ is the Hamiltonian functional, $\delta\mathcal{H}$ is the variational derivative, and $\mathcal{D}$ is the skew-adjoint differential operator, i. e., $\mathcal{D}^* = -\mathcal{D}$. Let us consider the following definitions that will be helpful in subsequent sections. If you

Definition 4.5.1 *Let X be a Banach space. A weak symplectic form on Banach space X is a continuous bilinear map $\Omega : X \times X \to \mathbb{R}$ that is*

1. *$\Omega(v,w) = -\Omega(w,v)$, that is, Ω is skew-symmetric and*

2. $\Omega(v,w)=0$ for all $w\in X$ implies $v=0$, that is, Ω is weakly nondegenerate.

A Banach space X with a symplectic structure Ω, (X,Ω) is called a phase space.

Definition 4.5.2 *A linear operator $\mathbf{F}:\tilde{D}(\mathbf{F})\to X$ with domain $\tilde{D}(\mathbf{F})$ which is a linear space of X is called Hamiltonian if*

$$\Omega(\mathbf{F}x,y)=-\Omega(x,\mathbf{F}y).$$

(That is $\Omega-$ skew for all x, $y\in\tilde{D}(\mathbf{F})$.)

Definition 4.5.3 *A Hamiltonian or energy function of F is defined by*

$$H(v)=\tfrac{1}{2}\Omega(\mathbf{F}v,v),\quad v\in\tilde{D}(\mathbf{F}).$$

In finite-dimensional settings, if we choose the Banach space $X=\mathbb{R}^{2s}$ with coordinates $(q_1,\cdots,q_s,p^1,\cdots,p^s)$ and $\mathbf{\Omega}=\sum_i \mathbf{d}q_i\wedge\mathbf{d}p^i$. As a bilinear map Ω is given by

$$\mathbf{\Omega}\left((\mathbf{p},\mathbf{q}),(\tilde{\mathbf{p}},\tilde{\mathbf{q}})\right)=\textstyle\sum_i \tilde{p}^i q_i-\tilde{q}_i p^i.$$

The form $\mathbf{\Omega}$ has the matrix

$$J=\begin{bmatrix}0 & \mathbf{I}\\ -\mathbf{I} & 0\end{bmatrix}$$

where $\mathbf{I}$ is an $s\times s$ identity matrix. Thus, $\mathbf{\Omega}(v,w)=v^T\cdot J\cdot w$. The linear operator $\mathbf{F}$ in this case can be denoted in a block form as

$$\mathbf{F}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}$$

where $\mathbf{F}$ is Ω-skew, when $\mathbf{B}$ and $\mathbf{C}$ are symmetric and $\mathbf{A}^T=-\mathbf{D}$. The linear operator $\mathbf{F}$ and the energy function H are related by the Hamilton's equations:

If $\mathbf{F}\left(\mathbf{q},\mathbf{p}\right)=(\dot{\mathbf{q}},\dot{\mathbf{p}})$, then

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix}=\begin{bmatrix}\dot{\mathbf{q}}\\ \dot{\mathbf{p}}\end{bmatrix}$$

Therefore,

$$\begin{aligned}\dot{\mathbf{q}}&=\mathbf{A}\mathbf{q}+\mathbf{B}\mathbf{p}=\frac{\partial H}{\partial \mathbf{p}},\\ \dot{\mathbf{p}}&=\mathbf{C}\mathbf{q}+\mathbf{D}\mathbf{p}=-\frac{\partial H}{\partial \mathbf{q}}.\end{aligned}\tag{4.54}$$

The relationship between the operator $\mathbf{F}$ and H may be written as

$$\mathbf{F}v=JDH(v),\tag{4.55}$$

where $DH(v)$ is the derivative of H.

4.6 Infinite-Dimensional Hamiltonian Systems

In this section, let us consider how the ideas from the finite-dimensional Hamiltonian setting are generalized to the infinite-dimensional Hamiltonian framework. Infinite-dimensional Hamiltonian systems are given by partial differential equations. We give examples of Hamiltonian PDEs, including the one-dimensional linear wave equation, the nonlinear Korteweg -de Vries equation, and the vorticity system.

4.6.1 Some Examples of Hamiltonian PDEs

We start by first defining and presenting some examples of functional derivative since it is crucial in this section. Let M be a manifold denoting smooth and continuous functions v satisfying certain boundary conditions and a function $H : M \to \mathbb{R}$, then the functional derivative of $H[v]$ given by $\delta H/\delta v$ is defined by

$$\begin{aligned} \mathrm{d}\mathcal{H} = \int \frac{\delta H}{\delta v}(x)u(x)\mathrm{d}x &= \lim_{\epsilon \to 0} \frac{H[v+\epsilon u] - H[v]}{\epsilon}, \\ &= \frac{d}{d\epsilon} H\left[v + \epsilon u\right]\Big|_{\epsilon=0}, \end{aligned} \tag{4.56}$$

where u is an arbitrary function and $\delta H/\delta v$ is the functional derivative. For example let us find the functional derivative of

$$H[v] = \int_0^1 v^2(x)\,\mathrm{d}x.$$

To calculate the functional derivative, we first calculate the change $\mathrm{d}H$ that is due to infinitesimal change $\epsilon u(x)$ in the dependent variables:

$$\begin{aligned} \frac{d}{d\epsilon} H[v+\epsilon u]\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_0^1 \left(v(x) + \epsilon u\right)^2 \mathrm{d}x, \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_0^1 \Big(v^2(x) + 2v(x)\epsilon u(x) + \epsilon^2 u^2(x)\Big)\,\mathrm{d}x, \\ &= \int_0^1 \Big(2v(x)u(x + 2\epsilon u^2(x)\Big)\Big|_{\epsilon=0} \mathrm{d}x, \\ &= \int_0^1 2u(x)v(x)\,\mathrm{d}x. \end{aligned}$$

Hence

$$\int_0^1 \frac{\delta H}{\delta v}(x)u(x)\mathrm{d}x = \int_0^1 2u(x)v(x)\mathrm{d}x \implies \frac{\delta H}{\delta v}(x) = 2v(x).$$

Let us again find the function derivative of

$$H[v_t] = \frac{\rho_0}{2} \int v_t^2\,\mathrm{d}x.$$

Using the definition (4.56), one computes the functional derivative as

$$\begin{aligned}
\frac{d}{d\epsilon}H[v_t+\epsilon u_t]\Big|_{\epsilon=0} &= \frac{\rho_0}{2}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int (v+\epsilon u)_t^2 \ \mathrm{d}x, \\
&= \frac{\rho_0}{2}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int \Big((v(x))^2+2v(x)\epsilon u(x)+\epsilon^2u^2\Big)_t \ \mathrm{d}x, \\
&= \frac{\rho_0}{2}\int_0^1 \Big(2v(x)u(x)+2\epsilon u(x)^2\Big)_t\Big|_{\epsilon=0}\mathrm{d}x, \\
&= \frac{\rho_0}{2}\int_0^1 2u_tv_t\mathrm{d}x.
\end{aligned}$$

Hence

$$\int \frac{\delta H}{\delta v_t}u_t\mathrm{d}x=\rho_0\int v_t\mathrm{d}x \implies \frac{\delta H}{\delta v_t}=\rho_0 v_t.$$

Let us consider the Hamiltonian of some linear and nonlinear PDEs as below.

Example 4.6.1 (The linear wave equation) Let us consider the wave equation governing the motion of a homogeneous elastic material subjected to small displacement from equilibrium (3.23)

$$u_{tt}=c^2u_{xx}, \tag{4.57}$$

where $u=u(x,t)$ denotes the displacement at time t, at $x\in\mathbb{R}$. The corresponding kinetic and potential energies are given by

$$\mathcal{K}=\tfrac{1}{2}\int_{\mathbb{R}^s}u_t^2\ \mathrm{d}\mathbf{x},\quad \mathcal{W}=\tfrac{1}{2}c^2\int_{\mathbb{R}^s}u_x^2\ \mathrm{d}\mathbf{x}.$$

The Lagrangian

$$\mathcal{L}=\mathcal{K}-\mathcal{W}=\frac{1}{2}\int_{\mathbb{R}^s}(u_t)^2\mathrm{d}\mathbf{x}-\frac{1}{2}c^2\int_{\mathbb{R}^s}u_x^2\mathrm{d}\mathbf{x}. \tag{4.58}$$

In equation (4.58), the Lagrangian density is

$$L[u]=c^2\frac{u_x^2}{2}-\frac{(u_t)^2}{2}. \tag{4.59}$$

The Hamiltonian density is given by

$$H=\frac{u_t^2}{2}+c^2\frac{u_x^2}{2} \tag{4.60}$$

and $\mathcal{H}=\int_{\mathbb{R}}H\ \mathrm{d}x$. Let $u_t=v$ which implies that

$$H=\frac{v^2}{2}+c^2\frac{u_x^2}{2}. \tag{4.61}$$

Applying the Euler operator to H in (4.60) leads to

$$E_uH=-c^2u_{xx},\quad E_vH=v. \tag{4.62}$$

Hence

$$\delta\mathcal{H}=\begin{bmatrix}-c^2u_{xx}\\ v\end{bmatrix}$$

It is easy to show that $\mathcal{D}$ is skew-adjoint as matrix in terms of inner product in $\mathbb{R}^2$. To form the Hamiltonian equations (4.53), consider the linear operator

$$\mathcal{D} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

then the Hamiltonian equations (4.53) $u_t = \mathcal{D} \cdot \delta\mathcal{H}$ yield

$$u_t = v = E_v H, \quad v_t = c^2 u_{xx} = -E_u H. \tag{4.63}$$

After excluding v, one gets the wave equation

$$u_{tt} = c^2 u_{xx}.$$

Example 4.6.2 (Two Hamiltonian Forms of Korteweg -de Vries (KdV) Equation)
The KdV equation

$$u_t = uu_x + u_{xxx} \tag{4.64}$$

can be written in Hamiltonian form in two different ways. First, one can observe that

$$u_t = \mathrm{D}_x \left(u_{xx} + \frac{1}{2}u^2 \right) = \mathcal{D}\delta\mathcal{H}_1,$$

where $\mathcal{D} = \mathrm{D}_x$ and

$$\mathcal{H}_1[x] = \int \left(-\frac{1}{2}u_x^2 + \frac{1}{6}u^3 \right) \mathrm{d}x$$

is one of the classical conservation quantity. One can write that

$$\begin{aligned} \delta\mathcal{H}_1 = E_u\mathcal{H}_1 &= \left(\frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - \right) \left(-\frac{1}{2}u_x^2 + \frac{1}{6}u^3 \right), \\ &= \frac{1}{2}u^2 + D_x(u_x) = \frac{1}{2}u^2 + u_{xx}. \end{aligned}$$

Thus the Hamiltonian equation (4.53) is given by

$$u_t = \mathcal{D}\delta\mathcal{H}_1 = \mathrm{D}_x \left(\frac{1}{2}u^2 + u_{xx} \right) = u_{xxx} + uu_x.$$

Here $\mathcal{D} = D_x$ is skew-adjoint hence $\mathcal{H}_1$ is automatically Hamiltonian (see Olver [108] Corollary 7.5.) Remarkably, KdV (4.64) has a second Hamiltonian form, with

$$\mathcal{H}_0[u] = \int \frac{1}{2}u^2 \mathrm{d}x.$$

The corresponding skew-adjoint operator is given by

$$\xi = \mathrm{D}_x^3 + \frac{2}{3}u\mathrm{D}_x + \frac{1}{3}u_x.$$

and (4.64) is obtained from

$$u_t = \left(\mathrm{D}_x^3 + \frac{2}{3}u\mathrm{D}_x + \frac{1}{3}u_x \right) u = \xi\delta\mathcal{H}_0 = uu_x + u_{xxx}.$$

Example 4.6.3 (The Vorticity System) Let us consider the Euler equations of inviscid ideal fluid flow

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p, \quad \nabla\cdot\mathbf{u} = 0. \tag{4.65}$$

Equations (4.65) as they stand, cannot take the form of Hamiltonian system since one cannot obtain an equation governing the temporal evolution of the pressure p. One simple way to circumvent this challenge is to re-write the equations in terms of the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{u}$. Taking curl of the first set of equations, one finds the vorticity equation

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \mathbf{\Omega}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{\Omega} \tag{4.66}$$

will possess Hamiltonian of the form

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \mathcal{D}\frac{\delta\mathcal{H}}{\delta\mathbf{\Omega}}$$

for a suitable Hamiltonian which is the kinetic energy

$$\mathcal{H} = \int_R \frac{1}{2}|\mathbf{u}|^2\,\mathrm{d}x,$$

and $\mathcal{D}$ is a skew-adjoint operator. For example, in the two-dimensional planer flow $\mathbf{\Omega} = \vec{k}\Omega$, and $\mathcal{D} = \Omega_x D_y - \Omega_y D_x$. (See Olver [108], example 7.10 for details.)

4.6.2 Hamiltonians for General Nonlinear Elastodynamics

We now consider some examples that are very important to the study of nonlinear elasticity. In [21, 34], when we assume there are no body forces, or when external forces are potential forces, the equations of motion for a three-dimensional incompressible or compressible model (1.40) admit a variational formulation. The partial differential equations are derived from the variation of the action functional

$$\mathrm{S} = \int_0^\infty \int_{\mathbb{R}^3} L\,\mathrm{d}x\,\mathrm{d}t, \tag{4.67}$$

where L is the volumetric Lagrangian density for isochoric materials given by

$$L = p\,(1-J) + (W-K)\,. \tag{4.68}$$

For compressible materials, the Lagrangian density is written as

$$L = (W-K)\,. \tag{4.69}$$

When the external body forces D_f in (1.40) vanish, we have the strain energy density function per unit mass as $W = W^h(\mathbf{X},\mathbf{F})$ where $W^h = W^h_{iso} + W^h_{aniso}$, and the kinetic energy per unit mass is given by

$$K = \frac{\rho_0}{2}|\mathbf{x}_t|^2,$$

where ρ_0 is constant density in material coordinates. To derive the PDEs (1.40), we denote the specific scalar dependent variable by v, the derivatives with respect to the $j-th$ independent variable by v_j, the total derivative operator by D_j. The Euler operators are written as

$$\mathrm{E}_v = \frac{\partial}{\partial v} - D_j\frac{\partial}{\partial v_j} + \cdots + (-1)^k D_{j_1}\dots D_{j_k}\frac{\partial}{\partial v_{j_1\dots j_k}} + \cdots\,, \tag{4.70}$$

where the total derivative operator $\mathbf{D}_j$ [22] is given by

$$D_j = D_x = \frac{D}{D_x} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial^2}{\partial y^2} + \cdots + y^{n+1} \frac{\partial^n}{\partial y_n} + \cdots . \tag{4.71}$$

One can say that, the extremals of action (4.67) satisfy the Euler-Lagrange equations $\mathrm{E}_v L = 0$, where $v = x^1, x^2, x^3, p$. The PDEs are therefore given by

$$\frac{\delta L}{\delta p} \equiv \mathrm{E}_p L = 1 - J = 0, \tag{4.72a}$$

$$\frac{\delta L}{\delta \mathbf{x}} \equiv \mathrm{E}_{\mathbf{x}} L = \rho_0 \mathbf{x}_{tt} - \frac{\partial \mathbf{P}}{\partial \mathbf{X}} = 0. \tag{4.72b}$$

The equations of motion (4.72b) can be obtained also from the Hamiltonian formulation. The Hamiltonian density

$$H[x] = K + W = \frac{\rho_0}{2} |\mathbf{x}_t|^2 + W(\mathbf{F}). \tag{4.73}$$

The kinetic energy $K = (\rho_0/2)|\mathbf{x}|^2 = (1/2\rho_0)|\mathbf{p}|^2$ and (4.73) can be written as

$$H[\mathbf{p}, \mathbf{x}] = K + W = \frac{1}{2\rho_0} |\mathbf{p}|^2 + W(\mathbf{F}), \quad \mathbf{F} = \mathrm{grad}_{\mathbf{X}} \mathbf{x}. \tag{4.74}$$

and $\mathcal{H} = \int_{\mathbb{R}^3} H \, \mathrm{d}x$. Now applying the Euler operator to H in (4.74) yields

$$E_{\mathbf{x}} H = -D_j \frac{\partial W}{\partial \mathbf{F}}, \quad E_{\mathbf{p}} H = \frac{\mathbf{p}}{\rho_0}. \tag{4.75}$$

In component form, (4.75) becomes

$$E_{x^j} H = -D_j \frac{\partial W}{\partial F_{ij}}, \quad E_{p^j} H = \frac{p^i}{\rho_0}. \tag{4.76}$$

Hence

$$\delta \mathcal{H} = \begin{bmatrix} -D_j \dfrac{\partial W}{\partial F_{ij}} \\ \dfrac{p^i}{\rho_0} \end{bmatrix}, \quad \text{i.e., } 6 \times 1 \text{ vector.}$$

Again one can show that the operator $\mathcal{D}$ is skew-adjoint as matrix in terms of inner products in $\mathbb{R}^6$. To form the Hamiltonian equations (4.53) one can consider the operator

$$\mathcal{D} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix}.$$

where $\mathbf{I}$ is 3×3 identity matrix. Then from the Hamiltonian equations (4.53) $u_t = \mathcal{D} \cdot \delta\mathcal{H}$, one finds

$$\begin{bmatrix} \mathbf{x}_t \\ \mathbf{p}_t \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} -D_j \dfrac{\partial W}{\partial F_{ij}} \\ \dfrac{\mathbf{p}}{\rho_0} \end{bmatrix}$$

which implies

$$\begin{bmatrix} \mathbf{x}_t \\ \mathbf{p}_t \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{p}}{\rho_0} \\ D_j(P_{ij}) \end{bmatrix}. \tag{4.77}$$

where $P_{ij} = \partial W / \partial F_{ij}$ is the first Piola-Kirchhoff stress tensor. Therefore

$$\mathbf{x}_t = \frac{\mathbf{p}}{\rho_0}, \quad \mathbf{p}_t = D_j P_{ij} = \text{div } \mathbf{P}. \tag{4.78}$$

Hence since $\rho_0 \mathbf{x}_{tt} = \mathbf{p}_t$ then one gets the general wave equation

$$\rho_0 \mathbf{x}_{tt} = \text{div } \mathbf{P}. \tag{4.79}$$

which coincides with (4.72b).

4.7 Hamiltonian and Lagrangian Structure for Nonlinear Wave Equations

In this section, the Lagrangian and Hamiltonian structures of the nonlinear and linear Love-type wave equations obtained in chapter 3 are derived systematically.

4.7.1 Hamiltonian and Lagrangian Structures for Nonlinear Wave Equations

Example 4.7.1 We construct the infinite-dimensional Hamiltonian for the nonlinear wave equation obtained from the Love-type shear displacements $x = X,\ y = Y + v(X, Z, t),\ z = Z$ (3.3). The Y- direction equation of motion (3.34) is re-written as

$$\rho_0 v_{tt} = (2W_1 v_X)_X + (2W_1 v_Z)_Z .$$

The Lagrangian density is given by

$$L = K - W = \frac{\rho_0}{2} v_t^2 - W(I_1), \tag{4.80}$$

where $W(I_1)$ is the potential energy, $I_1 = v_X^2 + v_Z^2 + 3$, the kinetic energy is $K = (\rho_0/2)v_t^2$, and $\mathcal{L} = \int_{\mathbb{R}} L \mathrm{dx}$. Applying the Euler operator to the Lagrangian density leads to

$$\begin{aligned} E_v L &= \left(\frac{\partial}{\partial v} - D_j \frac{\partial}{\partial v_j} - D_t \frac{\partial}{\partial v_t} \right) \left(\frac{p^2}{2\rho_0} - W(I_1) \right) = 0, \\ &= -D_t (\rho_0 v_t) + D_j \left(\frac{\partial W(I_1)}{\partial v_j} \right) = 0, \\ &= -\rho_0 v_{tt} + D_j \left(\frac{\partial W(I_1)}{\partial I_1} \frac{\partial (I_1)}{\partial v_j} \right) = 0. \end{aligned} \tag{4.81}$$

Simplifying (4.81) and noting ρ_0 is a constant yields

$$\rho_0 v_{tt} = D_X \left(\frac{\partial W(I_1)}{\partial I_1} \frac{\partial (I_1)}{\partial v_X} \right) + D_Z \left(\frac{\partial W(I_1)}{\partial I_1} \frac{\partial (I_1)}{\partial v_Z} \right) = (2W_1 v_X)_X + (2W_1 v_Z)_Z , \tag{4.82}$$

which coincides with (3.34).

For the Hamiltonian formulation, let us define the Hamiltonian density as

$$H = H[p, v] = \frac{p^2}{2\rho_0} + W(I_1) \tag{4.83}$$

and $\mathcal{H} = \int_{\mathbb{R}} H \, \mathrm{d}x$. From (4.83), the kinetic energy is given by $K = \big(\rho_0/2\big)\, v_t^2 = \frac{p^2}{2\rho_0}$. Now applying the Euler operator to H in (4.83) yields

$$E_v H = -D_j \frac{\partial W}{\partial v_j}, \quad E_p H = \frac{p}{\rho_0}. \tag{4.84}$$

Hence

$$\delta\mathcal{H} = \begin{bmatrix} -D_j \dfrac{\partial W}{\partial v_j} \\ \dfrac{p}{\rho_0} \end{bmatrix}.$$

Here one can show again that the operator $\mathcal{D}$ is skew-adjoint as matrix in terms of inner products in $\mathbb{R}^2$. From the Hamiltonian equations (4.53) one can consider the operator

$$\mathcal{D} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then from the Hamiltonian equations (4.53) $u_t = \mathcal{D} \cdot \delta\mathcal{H}$, one finds

$$\begin{bmatrix} v_t \\ p_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -D_j \dfrac{\partial W}{\partial v_j} \\ \dfrac{p}{\rho_0} \end{bmatrix}$$

which implies

$$\begin{bmatrix} v_t \\ p_t \end{bmatrix} = \begin{bmatrix} \dfrac{p}{\rho_0} \\ \\ D_j \dfrac{\partial W}{\partial v_j} \end{bmatrix}. \tag{4.85}$$

Therefore

$$v_t = \frac{p}{\rho_0}, \quad p_t = D_j \frac{\partial W}{\partial v_j} = D_X \left(\frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial v_X} \right) + D_Z \left(\frac{\partial W}{\partial I_1} \frac{\partial I_1}{\partial v_Z} \right) = (2W_1 v_X)_X + (2W_1 v_Z)_Z \,. \tag{4.86}$$

Hence since $\rho_0 \mathbf{v}_{tt} = p_t$, one gets the general two-dimensional wave equation

$$\rho_0 v_{tt} = (2W_1 v_X)_X + (2W_1 v_Z)_Z \,, \tag{4.87}$$

which again coincides with (3.34).

4.8 Discussion

Hamiltonian formalism is important in analyzing problems that arise in areas such as quantum mechanics and elastodynamics. In quantum systems, the Hamiltonians are differential operators in Hilbert space of wave functions. Physical systems such as the simple harmonic oscillator and the pendulum are examples of time-independent Hamiltonian systems. In a situation where the Hamiltonian

$$H\left(t, \mathbf{q}, \mathbf{p}\right) = H\left(\mathbf{q}, \mathbf{p}\right), \tag{4.88}$$

that is, the Hamiltonian does not change with time, it is constant of motion, often interpreted as the total energy in the system, $H = E$. In this chapter, we presented briefly finite- and infinite-Hamiltonian systems.

In Section 4.2, we presented the variational and Lagrangian formulation for ODE and PDE models. We stated that not every differential equation model is coming from a Lagrangian, i. e., not all differential equations are variational. We defined the action functional, Euler operator, and the Euler-Lagrange equations. We also provided the potential energy, kinetic energy, and Lagrangian for basic examples such as the Harmonic oscillator in one-dimension and derived the 1+1-dimensional linear wave equation from a variational principle.

We presented finite- and infinite- dimensional Hamiltonian systems in Section 4.3. We showed that the Hamilton's equation can be written in terms of the Poisson bracket in equation (4.34) which satisfies all the properties of Poisson bracket listed in Section 4.3.1.

In Section 4.4, we defined the Legendre transformation and showed how one can write the Hamiltonian of a system using the Legendre transformation (4.38) when the Lagrangian is known and vice versa. We outlined some basic steps in constructing the finite-dimensional Hamiltonian system in Section 4.4 and showed how the Legendre transformation is used in computing the Hamiltonian for the simple harmonic oscillator.

In Section 4.5, we gave some general statements about linear Hamiltonian systems. The Hamiltonian for the wave equation (4.57) was computed and we derived the Hamilton's equation. For the Love-type shear displacement (3.3), $x = X$, $y = Y + v(X, Z, t)$, $z = Z$ with $Y-$ equation of motion (3.98) has the general Hamiltonian (4.83) for all forms of the stored energy function W.

In Section 4.6, we presented infinite-dimensional Hamiltonian. We started by defining functional derivatives and provided examples of how to use the functional derivative. Hamiltonian formulations of some PDEs such as one-dimensional linear wave equation, KdV, vorticity equation were presented. In Section 4.6.2, we formulated the Lagrangian and Hamiltonian for the general nonlinear elastodynamics.

Finally, in Section 4.7, we showed that the Lagrangian and the Hamiltonian for the general two-dimensional wave equation (3.34) arising as nonlinear Love wave models (Chapter 3) for all strain energy functions $W = W(I_1)$ were given by (3.136).

While the presence of Hamiltonian and Lagrangian structure provides significant analytical insights and computational tools, the actual existence of Hamiltonian and Lagrangian remains a priori unknown. The general rule is that a system must be in some sense non-dissipative [108]. Some models such as integrable PDEs, admit multiple Hamiltonians (e. g, Olver [108] example 7.6). Models can admit a Lagrangian formulation after a change of variables or a writing of equations [21]. In general, it remains an open problem to characterize ODE and PDE models admitting a Lagrangian and/or Hamiltonian formulation.

5 CONCLUSION AND DISCUSSION

We now overview the content of this thesis. In Chapter 1, we started by describing the theory of linear and nonlinear elasticity. We presented that, in linear elasticity, one assumes small deformations and the Hooke's law holds. The elastic stiffness or compliance tensor from the generalized Hooke's law (1.4) shows that material is isotropic or anisotropic. Earthquakes and seismic waves were discussed. Earthquake vibrations depend on the intensity, nature of materials on the ground, and many other factors. We discussed the types of seismic waves, and in particular Love surface waves were of greater interest. We introduced the equations of motion in hyperelasticity and presented the basic ideas of nonlinear elasticity. The balance laws in both the Eulerian and Lagrangian configuration were discussed. Also, equations of motion which depend on the first Piola-Kirchhoff stress tensor and the Cauchy stress tensors in both Lagrangian and Eulerian configuration were discussed. Afterwards, constitutive laws for elastic solids in isotropic hyperelasticity were presented. Lastly, constitutive models for anisotropic hyperelasticity were briefly discussed.

The second chapter of the current thesis is dedicated to discussing Love surface waves in isotropic and anisotropic elastic media. Firstly, the problem formulation of linear Love waves propagating in a homogeneous isotropic elastic layer overlying a homogeneous isotropic elastic half-space with dissimilar elastic properties was provided. Using the appropriate boundary conditions, the full solution to both the two-dimensional linear wave equation for the layer and half-space was obtained and the dispersion relation was computed. Secondly, the problem definition of Love waves traveling in a homogeneous anisotropic elastic layer lying over a homogeneous anisotropic elastic half-space was studied. The solution to the problem was presented, satisfying the appropriate boundary conditions, and the dispersion equation for Love waves was derived. We observed that the dispersion equation obtained in an anisotropic model can reduce to the corresponding dispersion relation for the isotropic model.

In Chapter 3, we presented some wave propagation models in linear and nonlinear hyperelasticity framework. We dwelt mainly on horizontal and vertical shear wave displacements that produce Love wave-type and Rayleigh wave-type models. First, we provided a general shear motion of an elastic material and in particular discussed a linear horizontal shear wave model. The shear displacement model considered was naturally incompressible with strain energy function written in terms of only the first principal invariant $W = W(I_1)$. For Mooney-Rivlin's constitutive relation, the governing equation obtained is the usual two-dimensional linear wave equation. Next, we derived new general one- and two-dimensional wave equations from a one- and two-dimensional shear wave set up from the general strain energy function $W = W(I_1)$ together with the general pressure equation. Thus, all strain energy function $W = W(I_1)$, satisfy the derived general equation

(3.34) for two-dimensional and (3.68) for one-dimensional. Linear models such as the Mooney-Rivlin give linear wave equations and nonlinear models such as Saint Venant-Kirchhoff, Murnaghan, Yeoh give nonlinear wave equations. Afterward, we studied vertical shear displacement coming from the shear displacements for Rayleigh wave-type models. Lastly, using the method of separation of variables and the Fourier theory, we presented some exact solutions of linear wave models in one- and two-dimensional settings.

Chapter 4 was dedicated to a brief overview of Hamiltonian and Lagrangian mechanics for ODE and PDE models. We started by presenting the variational principles and Lagrangian formulation for PDE and ODE models in general. We defined the action integral, Euler operator, and Euler-Lagrange equations and presented some basic examples. Next, finite- and infinite-Lagrangian and Hamiltonian was studied. The Legendre transformation was introduced and used to derive Hamiltonian from Lagrangian formulation and vice versa. We outlined some steps in formulating finite-dimensional Hamiltonian. Lastly, the Hamiltonian and Lagrangian formulations of general one- and two-dimensional wave equations modeling nonlinear Love waves with strain energy function $W = W(I_1)$ were presented.

Possible Further Work

There are some potential directions that need to be researched further based on the material in this thesis which could be of interest.

Firstly, the study of Love wave in the nonlinear elastic model needs much attention and requires further investigations. Due to the complexity of the nonlinear Love wave equation (involving cubic and fifth-order nonlinearities), an appropriate numerical scheme may be developed and used for approximate solution. Also, in further studies, one may consider Love wave in an inhomogeneous anisotropic elastic layer overlying an inhomogeneous anisotropic elastic half-space. Specifically, particular classes of anisotropic materials (transversely isotropic and orthotropic materials), which are also significant in seismology, may be of interest. With regards to shear wave models derived in Chapter 3, the mathematical aspects of the constraint equations may be discussed and analyzed further. The Hamiltonian formulation of the derived nonlinear shear wave equations, in particular, the possibility of existence of a second Hamiltonian for some models, and its mathematical implications, with constraints may be analyzed.

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