View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by University of Strathclyde Institutional Repository

Proceedings of the 6th World Congress on Electrical Engineering and Computer Systems and Science (EECSS'20)
Prague, Czech Republic - August 2020
Paper No. MHCI 104
DOI: 10.11159/mhci20.104

Mathematical Tools for Processing Broadband Multi-Sensor Signals

Stephan Weiss
Centre for Signal & Image Processing
Department of Electronic & Electrical Engineering
University of Strathclyde
Glasgow G1 XW, Scotland
stephan.weiss@strath.ac.uk

Abstract - Spatial information in broadband array signals is embedded in the relative delay with which sources illuminate different sensors. Therefore, second order statistics, on which cost functions such as the mean square rest, must include such delays. Typically, a space-time covariance matrix therefore arises, which can be represented as a Laurent polynomial matrix. The optimisation of a cost function then requires extending the utility of the eigenvalue decomposition from narrowband covariance matrices to the broadband case of operating in a space-time covariance matrix. This overview paper summarises efforts in performing such factorisations, and demonstrated via the exemplar application of a broadband beamformer how thus well-known narrowband solutions can be extended to the broadband case using polynomial matrices and their factorisations.

Keywords: array processing, broadband beamforming, sensor processing, polynomial matrices, matrix factorisation.

1. Introduction

Signals acquired by M sensors may be expressed in a data vector $\mathbf{x}[n] \in \mathbb{C}^M$, where each vector element represents one time series in the discrete time index n. If we assume that these sensors pick up emissions by some sources, then this data contains both temporal clues about the source signals and spatial information on the sensor position, since e.g. a source signal will arrive at different sensor element with different delays. In the narrowband case, such delays reduce to phase shifts and are e.g. contained in the covariance matrix $\mathbf{R} = \mathscr{E}\left\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]\right\} \in \mathbb{C}^{M\times M}$, where $\{\cdot\}^{\mathrm{H}}$ denotes Hermitian transposition and $\mathscr{E}\{\cdot\}$ is the expectation operator. In the case of linearly combining sensor signals to general an output with enhanced signal to noise ratio over a single sensor signal, the output power can be formulated based on $\mathbf{R}$. Optimisation of the weighting coefficients in the mean square or least squares error sense then generally rests centrally on the eigenvalue decomposition (EVD) of $\mathbf{R}$ [1, 2] or the singular value decomposition of the data matrix [3].

For broadband signals, the frequency-dependent phase shift of a signal is insufficient to preserve spatial information, and we have to work with explicit delays. Therefore, the second order statistics are captured by the space-time covariance matrix $\mathbf{R}[\tau] = \mathscr{E}\left\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\right\} \in \mathbb{C}^{M\times M}$, with the discrete lag parameter τ embedding the required delay. With $\mathbf{x}[n] = [x_1[n], \ldots x_M[n]]^{\mathrm{T}}$,

$$\mathbf{R}[\tau] = \mathscr{E}\left\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\right\} = \begin{bmatrix} \mathscr{E}\{x_1[n]x_1^*[n-\tau]\} & \ldots & \mathscr{E}\{x_1[n]x_M^*[n-\tau]\} \\ \vdots & \ddots & \vdots \\ \mathscr{E}\{x_M[n]x_1^*[n-\tau]\} & \ldots & \mathscr{E}\{x_M[n]x_M^*[n-\tau]\} \end{bmatrix} = \begin{bmatrix} r_{x_1x_1}[\tau] & \ldots & r_{x_1x_M}[\tau] \\ \vdots & \ddots & \vdots \\ r_{x_Mx_1}[\tau] & \ldots & r_{x_Mx_M}[\tau] \end{bmatrix}, \tag{1}$$

and the cross-correlation

$$r_{x_ix_j}[\tau] = \mathscr{E}\left\{x_i[n]x_j^*[n-\tau]\right\} = r^*_{x_jx_i}[-\tau], \qquad i,j \in \{1,\ldots,M\}, \tag{2}$$

the space-time covariance satisfies the symmetry $\mathbf{R}[\tau] = \mathbf{R}^{\mathrm{H}}[-\tau]$. The z-transform of the space-time covariance matrix, $\boldsymbol{R}(z) = \sum_\tau \mathbf{R}[\tau]z^{-\tau}$, is the cross-spectral density (CSD) matrix of $\mathbf{x}[n]$, and now depends on z and is a matrix of Laurent polynomials if of finite order, or Laurent series generally. The symmetry of $\mathbf{R}[\tau]$ translates into $\boldsymbol{R}(z)$ satisfying the para-hermitian property $\boldsymbol{R}(z) = \boldsymbol{R}^{\mathrm{P}}(z)$, where parahermitian conjugate $\boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^*)$ implies a Hermitian transposition and a time reversal [4]. This represents a generalisation of the Hermitian (or in the real-valued case symmetric) property of the narrowband covariance matrix $\mathbf{R} = \mathbf{R}[0]$ from above, and is also sometimes referred to as a palindromic matrix [5], or as a self-adjoint matrix [6] when evaluated on the unit circle, $z = \mathrm{e}^{\mathrm{j}\Omega}$, where $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{R}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})$.

Using polynomial matrix notation, broadband problems can be formulated analogously to narrowband ones, including applications such as optimum source coding [7], channel coding [8], optimum precoding and equalisation for multiple-input multiple-output systems [9, 10, 11, 12], angle of arrival estimation [13, 14, 15], broadband beamforming [16, 17, 18, 19], or blind source separation [20]. These formulations then include the CSD matrix $\boldsymbol{R}(z)$; the utility of the EVD to diagonalise narrowband covariance matrices and identify signal and noise subspaces does however not directly extend to the broadband case. If the EVD is, say, applied to $\mathbf{R}[0]$, such that $\mathbf{R}[0] = \mathbf{Q}\Lambda\mathbf{Q}^{\mathrm{H}}$, then the unitary matrix $\mathbf{Q}$ will diagonalise $\mathbf{R}[\tau]$ for $\tau = 0$ but not simultaneously for $\tau \neq 0$. Hence an extension of the EVD to the broadband case is required that diagonalises $\mathbf{R}[\tau]$ for all lags of τ, or that analogously diagonalises the CSD matrix $\boldsymbol{R}(z)$ for all z within its region of convergence.

If the CSD matrix $\boldsymbol{R}(z)$ is analytic, i.e. produced by a causal, stable scenario, then with the exception of multiplexed data [21], it is possible to find a parahermitian matrix EVD (PhEVD)

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z);, \tag{3}$$

where the analytic, diagonal $\boldsymbol{\Lambda}(z)$ contains the eigenvalues and the columns of the analytic, paraunitary $\boldsymbol{Q}(z)$ hold the eigenvectors of $\boldsymbol{R}(z)$ [22]. Previously, the McWhirter decomposition [23, 24] defines a polynomial EVD (PEVD) that approximates (3) as

$$\boldsymbol{R}(z) \approx \boldsymbol{U}(z)\boldsymbol{\Gamma}(z)\boldsymbol{U}^{\mathrm{P}}(z) , \tag{4}$$

where the factors $\boldsymbol{U}(z)$ and $\boldsymbol{\Gamma}(z)$ are Laurent polynomial matrices, with $\boldsymbol{U}(z)$ paraunitary and $\boldsymbol{\Gamma}(z)$ diagonal. However, $\boldsymbol{\Gamma}(z)$ will not necessarily approximate the analytic eigenvalues in $\boldsymbol{\Lambda}(z)$ but aims to provide a spectrally majorised solution, such that for $\boldsymbol{G}(z) = \mathrm{diag}\{\gamma_1(z), \ldots, \gamma_M(z)\}$ evaluated on the unit circle,

$$\gamma_1(\mathrm{e}^{\mathrm{j}\Omega}) \geq \gamma_2(\mathrm{e}^{\mathrm{j}\Omega}) \geq \ldots \geq \gamma_M(\mathrm{e}^{\mathrm{j}\Omega}) \tag{5}$$

holds, i.e. $\boldsymbol{\Gamma}(z)$ is spectrally majorised [25]. Both PhEVD and PEVD can at least approximately diagonalise $\boldsymbol{R}(z)$, and hence offer a route to solutions to the earlier stated broadband extensions to classical narrowband optimum solutions such as angle of arrival estimation or beamforming.

The past decade has seen the development of a number of algorithms that generate the McWhirter decomposition, which includes the second order sequential best rotation algorithm (SBR2 [23]) and the sequential matrix diagonalisation (SMD) families of algorithms [26]. Various versions of SBR2 [7, 27] and SMD [28, 29, 30] have since emerged that enhance convergence in one aspect or another. Additionally, the computational complexity of these algorithms has been addressed by various means, including linear algebraic approximations of the EVD [31, 32], the truncation of large polynomials [33, 34, 35, 36], reduction of the optimisation parameter space [37, 38, 39, 40] as well as the exploitation of the symmetry of $\boldsymbol{R}(z)$ [41], and the parallelisation of algorithms in [41, 42, 43, 44]. For the PhEVD in (3), and initial approximation has been provided in [45, 46], with further developments that exploit the analyticity of the extracted solution for eigenvalues [47] and eigenvectors [48], based in smoothness criteria in [49, 50].

The purpose of this paper is to provide an update on developments in the area of polynomial matrix decompositions beyond [51] and provide an example application where a broadband solution by a simple extension of narrowband results. Thus, Sec. 2 defines the signal model, the space-time covariance matrix, its parahermitian matrix factorisation and polynomial approximation. Sec. 3 discusses the state of the art in PhEVD and PEVD algorithms. These are then applied to a Capon beamforming problem, which is first introduced for the narrowband case and then is extended to the broadband scenario in Sec. 4. Conclusions and an outlook to emerging applications are given in Sec. 5.

2. Broadband Signal Model,Cross Spectral Density, and Parahermitian Matrix EVD

2.1. Signal Model and Cross Spectral Density

We assume that the measurements in $\mathbf{x}[n]$ are generated by L sources $s_\ell[n]$, which are convolutively mixed by an $M \times L$ mixing matrix $\mathbf{H}[n]$, as shown in Fig. 1. The power spectral densities $R_{s_\ell}(z)$ of these sources can be explain via an innovation filter [52] $g_\ell[n]$ with z transform $G_\ell(z) = \sum_n g_\ell[n]z^{-n}$, or short $G_\ell(z) \bullet\!\!-\!\!\circ\, g_\ell[n]$, such that $R_{s_\ell}(z) = G_\ell(z)G_\ell^{\mathrm{P}}(z)$. Thus, we have for the CSD matrix

$$\boldsymbol{R}(z) = \boldsymbol{H}(z)\mathrm{diag}\{R_{s_1}(z), \ldots, R_{s_L}(z)\}\,\mathbf{H}^{\mathrm{P}}(z) = \boldsymbol{H}(z)\boldsymbol{G}(z)\boldsymbol{G}^{\mathrm{P}}(z)\boldsymbol{H}^{\mathrm{P}}(z) , \tag{6}$$

where $\mathbf{H}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{H}(z) : \mathbb{C} \longrightarrow \mathbb{C}^{M\times L}$ and $\boldsymbol{G}(z) = \mathrm{diag}\{G_1(z), \ldots, G_L(z)\} : \mathbb{C} \longrightarrow \mathbb{C}^{L\times L}$. The product on the r.h.s. of (6) only formally looks like (3); note however that the factor $\boldsymbol{H}(z)$ in (6) is unconstrained, while $\boldsymbol{Q}(z)$ is parahermitian.

2.2. Parahermitian Matrix EVD

Since in a real measurement system, all components of $\boldsymbol{H}(z)$ and $\boldsymbol{G}(z)$ must be causal and stable, $\boldsymbol{R}(z)$ will be analytic at least on a disc including the unit circle. Therefore, the parahermitian matrix EVD in (3) exists for almost all cases [22, 21].

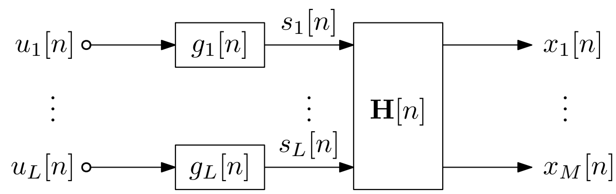

Fig. 1: Source model of measurements $x_m[n]$, $m = 1,\ldots,M$, with convolutive mixing matrix $\mathbf{H}[n]$ and source signals $s_\ell[n]$, $\ell = 1,\ldots,L$, which are generated by innovation filters $g_\ell[n]$ from uncorrelated zero-mean unit-variance Gaussian signals $u_\ell[n]$.

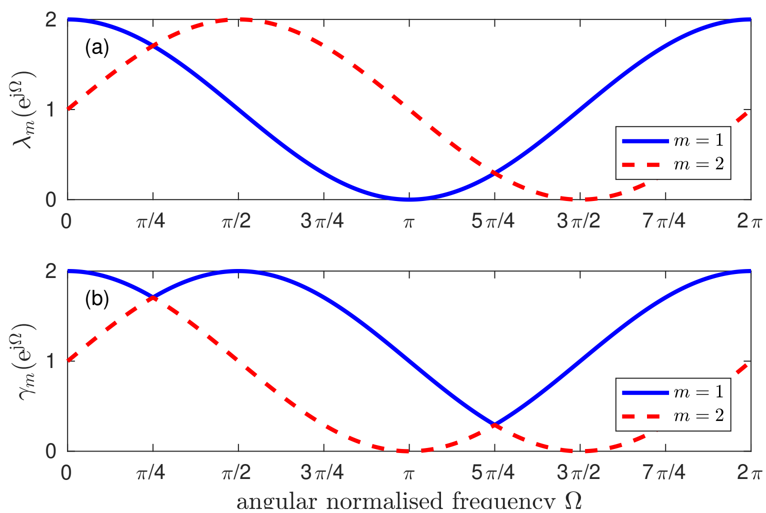

Fig. 2: (a) Analytic vs (b) spectrally majorised selection of eigenvalues.

A simple proof rests on the generalisation of Rellich's EVD [53], where analytic eigenvalues and -vectors are guaranteed to exist for some analytic self-adjoint matrix $\boldsymbol{A}(t)$ for $t \in \mathbb{R}$.

One specific class of parahermitian matrices $\boldsymbol{R}(z)$ does not a factorisation into analytic components. When evaluated on the unit circle $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$ is 2π-periodic in Ω. However, the same cannot be guaranteed for the EVD factors. If $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$ can be traced back to pseudo-circulant components, which is the case for multiplexed data [4], then the periodicity of these factors will be $2\pi N$ with some $N \in \mathbb{N}$ [21]. As a result, the factors are not analytic, and do not exist as realisable components. However, a spectrally majorised approximation can still be undertaken [21].

To highlight the difference between the analytic eigenvalues in $\boldsymbol{\Lambda}(z)$ of (3) and the spectrally majorised ones in $\boldsymbol{\Gamma}(z)$ of (4), consider the analytic eigenvalues $\lambda_1(\mathrm{e}^{\mathrm{j}\Omega}) = 1 + \cos\Omega$ and $\lambda_2(\mathrm{e}^{\mathrm{j}\Omega}) = 1 + \sin\Omega$. Both functions could be permuted at any frequency Ω and still be valid eigenvalues, provided that they retain a 2π-periodicity. Apart from the analytic selection $\lambda_1(\mathrm{e}^{\mathrm{j}\Omega})$ and $\lambda_2(\mathrm{e}^{\mathrm{j}\Omega})$ depicted in Fig. 2(a), an important alternative are the spectrally majorised eigenvalues $\gamma_1(\mathrm{e}^{\mathrm{j}\Omega})$ and $\gamma_2(\mathrm{e}^{\mathrm{j}\Omega})$ in Fig. 2(b), where spectral majorisation implies that $\lambda_1'(\mathrm{e}^{\mathrm{j}\Omega}) \geq \lambda_2'(\mathrm{e}^{\mathrm{j}\Omega})\ \forall\Omega$ [25] as formulated in (5).

The solutions for the PhEVD in (3) and the PEVD in (4) will be equivalent, i.e. analytic eigenvalues will be spectrally majorised, if the analytic eigenvalues to no intersect on the unit circle. This means that the $\lambda_m(\mathrm{e}^{\mathrm{j}\Omega})$, $m = 1,\ldots,M$ do not possess any algebraic multiplicities greater than one at any frequency Ω.

If on the unit circle eigenvalues have algebraic multiplicities greater than one, as in Fig. 2 for $\Omega = \frac{\pi}{4}$ and $\Omega = \frac{5\pi}{4}$, then the analytic selection in $\boldsymbol{\Lambda}(z)$ will not be spectrally majorised. Since only analytic functions are absolutely convergence, the spectrally majorised solution in (4) will need to approximate non-differentiable functions. This requires approximations with polynomials of significantly higher order than for the analytic eigenvalues, which impact on the computational cost of any application of this function.

Note that a spectrally majorised selection requires a permutation of the analytic eigenvalues, and therefore also implies a permutation of the analytic eigenvectors. Since the latter are orthogonal, a permutations leads to a discontinuity. Hence for a non-analytic selection, the approximation of discontinuous functions for the eigenvectors may require an even higher order than the approximation of non-differentiable functions in case of the eigenvalues. Hence it is hope that a shift from SBR2 and SMD algorithms, which encourage or can even be shown to converge to spectrally majorised solutions in case of SBR2 [54] towards the extraction of analytic solutions will significantly reduce computational complexity. Some features of

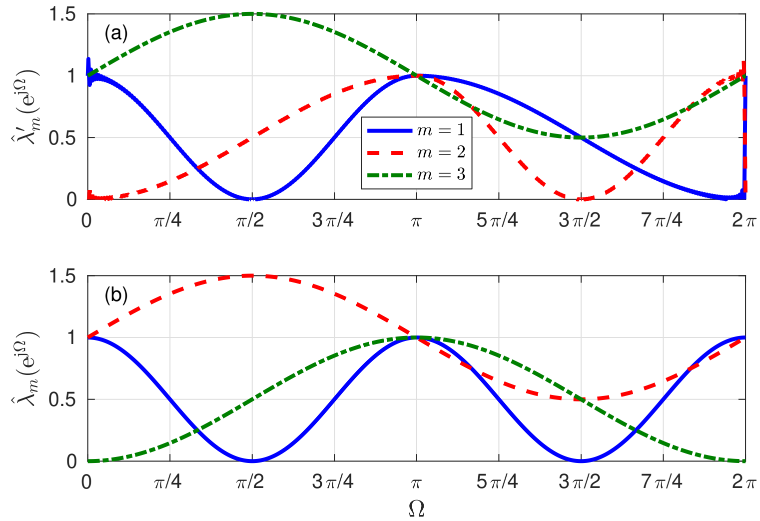

Fig. 3: Comparison of the extracted eigenvalues using (a) [45] and (b) [47].

these algorithms will be illuminated in the following section.

3. Algorithms for the Approximation of PEVD and PhEVD

Several algorithms targetting the McWhirter decomposition in (4) have emerged over the past decade [7, 23, 24, 26, 27, 28, 30, 45, 55, 56]. Most of these operate in the time domain, and iteratively shift energy from the off-diagonal components of $\boldsymbol{R}(z)$ onto the diagonal through a combination of delay operations and Givens rotations (in case of the SBR2 family [7, 23, 24, 27]) or through EVD [26, 28] or Householder transformations [30] in case of the SMD family. These algorithms iteratively reduce off-diagonal energy and can be proven to converge, even though it had not been clear to what solution. In case of SBR2, algorithms have also been shown to converge to a spectrally majorised solution [54], while for SMD algorithms spectral majorisation is generally achieved but yet unproven for any of its algorithms. In contrast to these iterative time-domain methods, in [55, 56] a fixed-order approximation is attempted by elementary paraunitary operations [4], while [45] has proposed a DFT-domain approach which also operates with a fixed order for its solution to (4).

The approach by Tohidian *et al.* [45] has been the first to extract maximally smooth — and therefore ideally the analytic — eigenvalues and -vectors. Their algorithm operates in the DFT domain, where an EVD is calculated in each bin. Working in independent frequency bins, though, the coherence is lost, and an association across successive frequency bins has to be established in order to extract M eigenvalues and -vectors from $\boldsymbol{R}(z) : \mathbb{C} \longrightarrow \mathbb{C}^{M\times M}$. This association is driven by the bin-wise eigenvectors. However, in case if a Q-fold multiplicity of eigenvalues, the associated eigenvectors can form an arbitrary basis in a Q-fold subspace; similarly, for closely-spaced eigenvalues, the calculation of individual eigenvectors may therefore be ill-conditioned [57, 46, 48]. The approach in [47] therefore establishes associations across frequency bins solely based on the eigenvalues, driven by a smoothness criterion of the interpolated functions [49, 50]. Additionally, the DFT-length is iterative adjusted until an approximation error is minimised.

As an example, Fig. 3 provides an comparison of [45] and [47] for a matrix $\boldsymbol{R}(z) : \mathbb{C} \longrightarrow \mathbb{C}^{3\times 3}$ with the following known ground-truth analytic eigenvalues:

$$\lambda_1(z) = \frac{1}{4}z^2 + \frac{1}{2} + \frac{1}{4}z^{-2} \qquad \lambda_2(z) = -\frac{1}{4}z^1 + \frac{1}{2} - \frac{1}{4}z^{-1} \qquad \lambda_3(z) = -\mathrm{j}\frac{1}{4}z^1 + 1 + \mathrm{j}\frac{1}{4}z^{-1}\,. \tag{7}$$

The CSD matrix possesses an algebraic multiplicity of $Q=3$ at $\Omega=\pi$, where the algorithm in [45] may be misled, since only adjacent bins of the ill-conditioned eigenvectors are inspected, resulting in the extracted eigenvalues $\lambda'(z)$ shown in Fig. 3(a). Picking an incorrect continuation at $\Omega=\pi$ ultimately leads to two of the eigenvalues approximating discontinuous functions at $\Omega = 2\pi$. In contrast, the result of [47] in Fig. 3(b) matches the analytic solution.

Enforcing spectral majorisation in the case of an algebraic multiplicity greater than one as shown in Fig. 2 leads to eigenvalues that are not infinitely differentiable and to eigenvectors with discontinuities [22, 51]. Current PEVD approaches based on the SBR2 and SMD algorithm families either empirically encourage or, in case of SBR2, can be shown to yield spectrally majorised eigenvalues [54]. Thus, they result generally can result in matrix factors of high polynomial orders to approximate the McWhirter decomposition in (4). Even though some mechanisms to curb the polynomial order of these factors have been suggested [33, 34, 23, 35, 36], these are generally truncation-based and therefore potentially impact on

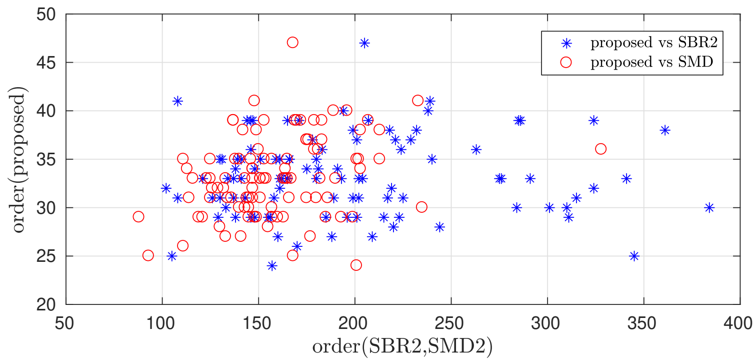

Fig. 4: Comparison of the order of extracted eigenvalues using SBR2 [23], SMD [26], and the method for analytic eigenvalues in [47].

the approximation error in (4). This matters, because he polynomial order of any extracted factors directly related to computational cost. The paraunitary matrices $\boldsymbol{Q}(z)$ and $\boldsymbol{U}(z)$ in (3) and (4) represent lossless filter bank [4], whose order is the order of filter bank required to e.g. provide a required subspace projection afforded by (3) and (4). Hence the extraction of factorisation with fundamentally lower order is important.

Since analytic solutions produce factors that in the time domain are absolutely convergent, they can generally be easily approximated by Laurent polynomials. As a result, approximations of the PhEVD in (3) yields factors of considerably lower order compared to algorithms that target (4) with its enforced spectral majorisation. For a simulation using 50 parahermitian matrices $\boldsymbol{R}(z) : \mathbb{C} \longrightarrow \mathbb{C}^{4\times 4}$ are generated from a randomised matrix $\boldsymbol{A}(z) : \mathbb{C} \longrightarrow \mathbb{C}^{4\times 4}$ with normally distributed entries of order 10 via $\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$. The order of factorisations to comparable approximation orders using SBR2 [23], SMD [26], and the approach in [47] are shown in Fig. 4, and highlight the order reduction by approximately one order of magnitude by the extraction of analytic eigenvalues.

4. Capon Beamformer

To demonstrate the polynomial approach for a sample broadband multi-sensor problem, we first introduce a standard narrowband problem, which is then generalised together with its classical narrowband solution. As an exemplar application, we target the Capon beamformer [58, 2].

4.1. Narrowband Problem and Solution

Given is an array of sensors generating measurements $\mathbf{x}[n] \in \mathbb{C}^M$, and a set of weights $\mathbf{w} \in \mathbb{C}^M$ to perform a linear combination. The result of this linear combination, $e[n] = \mathbf{w}^{\mathrm{H}}\mathbf{x}[n]$ should be minimised in the mean square error sense, subject to constraints that preserve the gain in look direction and null interferers whose angle of arrival is known. If a source signal s_ℓ arrives at the mth sensor with a delay $\tau_{m,\ell}$, then the direction of this source is characterised by a narrowband steering vector

$$\mathbf{a}_\ell = \frac{1}{\sqrt{M}} \left[\mathrm{e}^{\mathrm{j}\Omega\tau_{1,\ell}},\, \mathrm{e}^{\mathrm{j}\Omega\tau_{2,\ell}}, \ldots,\, \mathrm{e}^{\mathrm{j}\Omega\tau_{M,\ell}} \right]^{\mathrm{T}} , \tag{8}$$

such that at a particular narrowband frequency Ω, the delays simplify to phase shifts. Thus, for L known sources, whereby the first source, $\ell = 1$, is assumed to be the signal of interest while the remaining sources are seen as interferers, the problem can be formulated as a minimum variance distortionless response (MVDR) problem

$$\mathbf{w}_{\mathrm{opt}} = \arg\min_{\mathbf{w}} \mathscr{E}\left\{|e[n]|^2\right\} = \arg\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{w} \quad , \qquad \text{s.t.} \qquad \mathbf{w}^{\mathrm{H}} \underbrace{[\mathbf{a}_1,\, \mathbf{a}_2,\, \ldots,\, \mathbf{a}_L]}_{\mathbf{C}} = [0,\, \underbrace{1, \ldots, 0}_{L-1}] = \mathbf{f}^{\mathrm{H}} . \tag{9}$$

With the MVDR setup, up to $L = M-1$ interfering sources can be suppressed. Any remaining degrees of freedom are invested such that unknown structured interferers and omnidirectional noise are minimised in the mean square error sense.

The Capon solution to (9) can be obtained by constrained optimisation e.g. via Lagrange multipliers, such that [58, 59]

$$\mathbf{w}_{\mathrm{opt}} = \boldsymbol{R}^{-1}(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{C}\left(\mathbf{C}^{\mathrm{H}}\boldsymbol{R}^{-1}(\mathrm{e}^{\mathrm{j}\Omega})\mathbf{C}\right)^{-1}\mathbf{f} , \tag{10}$$

and is well-known for the narrowband case. Because the error $e[n] = \mathbf{w}^{\mathrm{H}}\mathbf{x}[n]$ is formulated via the Hermitian transposition, $\mathbf{w}$ holds the complex conjugate record of the actual weights.

4.2. Broadband Problem and Solution

In the broadband case, tap-delay line processing of the sensor signals is required in order to formulate explicit delays, and the linear combination is now performed by filters $w_m[n]$, $m = 1, \ldots, M$, the operate on each sensor signal $x_m[n]$. We first demonstrate how the problem formulation in (9) generalises to the broadband case. For this, the complex conjugate record of weights generalises to a parahermitian record, such that $\boldsymbol{w}^{\mathrm{P}}(z) = [w_1[n], \ \ldots, \ w_m[n]]$. The mean square error is obtained in the frequency domain by integrating over the PSD, such that [18]

$$\boldsymbol{w}_{\mathrm{opt}}(z) = \arg\min_{\boldsymbol{w}(z)} \oint_{|z|=1} \boldsymbol{w}^{\mathrm{P}}(z)\boldsymbol{R}(z)\boldsymbol{w}(z)\frac{\mathrm{d}z}{z} \quad , \qquad \text{s.t.} \quad \boldsymbol{w}_{\mathrm{opt}}(z)\boldsymbol{C}(z) = \boldsymbol{f}^{\mathrm{P}}(z) \; , \tag{11}$$

with the constraint equation now made up of broadband steering vectors

$$\boldsymbol{a}(z) = \frac{1}{\sqrt{M}} \left[\; d[n-\tau_1] \quad d[n-\tau_2] \quad \ldots \quad d[n-\tau_M] \; \right] \; , \tag{12}$$

where $d[n-\tau]$ is a fractional delay filter with $\tau \in \mathbb{R}$ [60, 61]. In case of $L-1$ interferers, we have

$$\boldsymbol{C}(z) = [\boldsymbol{a}_1(z), \boldsymbol{a}_2(z), \ldots \boldsymbol{a}_L(z)] \qquad \text{and} \qquad \boldsymbol{f}^{\mathrm{P}}(z) = \big[F_1(z), \underbrace{0, \ \ldots, \ 0}_{L-1}\big] \; , \tag{13}$$

where $F_1(z)$ is the desired transfer function in look direction. Note that when evaluated at a specific frequency Ω and for $F_1(z) = 1$, the broadband steering vector in (12) collapses to the narrowband one defined in (8). The same applies to the entire problem formulation. For the broadband Capon beamformer, the narrowband solution simply generalises to [19]

$$\boldsymbol{w}_{\mathrm{opt}}(z) = \boldsymbol{R}^{-1}(z)\boldsymbol{C}(z)\left[\boldsymbol{C}^{\mathrm{P}}(z)\boldsymbol{R}^{-1}(z)\boldsymbol{C}(z)\right]^{-1}\boldsymbol{f}(z) \; . \tag{14}$$

This requires the inversion of polynomial matrices, which can be computed via the PEVD or PhEVD [62].

5. Conclusions

This paper has summarised efforts in the area of polynomial matrices to formulate broadband multi-sensor problems, and find solutions to these formulations by means of generalising narrowband solutions. This typically requires the application of polynomial matrix decompositions, particularly the parahermitian (PhEVD) and polynomial matrix EVDs (PEVD). For these, algorithmic efforts have been reviewed, with the main focus currently directed to extracting analytic eigenvalues and -vectors that afford lower-order polynomial approximations than the state-of-the-art in PEVD approaches with proven convergence. Important future developments also target the estimation of space-time covariance [63, 64, 65], since the statistics typically have to be estimated from finite data sets, and interesting new applications where the polynomial approach permits solutions that were previously unobtainable, such as in the area of impulse response modelling [66] or speech enhancement [67].

Acknowledgements

I would like to very grateful acknowledge the immense help and input from a number of collaborators, in particular John McWhirter, who initiated this field of research, as well as Ian Proudler, Jennifer Pestana, Malcolm Macleod, Soydan Redif, Fraser Coutts, Jamie Corr, Chi Hieu Ta, Mohamed Alrmah, Connor Delaosa, Samir Bendoukha, Ahmed Alzin, Amr Nagy, and Zeliang Wang.

References

[1] B. Widrow and S. D. Stearns. *Adaptive Signal Processing*. Prentice Hall, Englewood Cliffs, New York, 1985.

[2] S. Haykin. *Adaptive Filter Theory*. Prentice Hall, Englewood Cliffs, 3rd edition, 1996.

[3] M. Moonen and B. de Moor. *SVD and Signal Processing, III: Algorithms, Architectures & Applications*. Elsevier, 1995.

[4] P. P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall, Englewood Cliffs, 1993.

[5] T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, and F. Tisseur. Nlevp: A collection of nonlinear eigenvalue problems. *ACM Trans. Math. Softw.*, 39(2), Feb. 2013.

[6] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982.

[7] S. Redif, J. McWhirter, and S. Weiss. Design of FIR paraunitary filter banks for subband coding using a polynomial eigenvalue decomposition. *IEEE Transactions on Signal Processing*, 59(11):5253–5264, Nov. 2011.

[8] S. Weiss, S. Redif, T. Cooper, C. Liu, P. Baxter, and J. McWhirter. Paraunitary oversampled filter bank design for channel coding. *EURASIP J. Advances in Signal Processing*, 2006:1–10, 2006.

[9] C. H. Ta and S. Weiss. A Design of Precoding and Equalisation for Broadband MIMO Systems. In *15th International Conference on Digital Signal Processing*, pp. 571–574, Cardiff, UK, July 2007.
[10] N. Moret, A. Tonello, and S. Weiss. Mimo precoding for filter bank modulation systems based on PSVD. In *IEEE 73rd Vehicular Technology Conference*, pp. 1–5, May 2011.
[11] A. Sandmann, A. Ahrens, and S. Lochmann. Resource allocation in SVD-assisted optical mimo systems using polynomial matrix factorization. In *Proc. 16. ITG Symp. Photonic Networks*, pp. 1–7, May 2015.
[12] A. Ahrens, A. Sandmann, E. Auer, and S. Lochmann. Optimal power allocation in zero-forcing assisted PMSVD-based optical MIMO systems. In *Sensor Signal Processing for Defence Conference*, pages 1–5, Dec. 2017.
[13] M. Alrmah, S. Weiss, and S. Lambotharan. An extension of the MUSIC algorithm to broadband scenarios using polynomial eigenvalue decomposition. In *19th European Signal Processing Conf.*, pp. 629–633, Barcelona, Spain, Aug. 2011.
[14] S. Weiss, M. Alrmah, S. Lambotharan, J. McWhirter, and M. Kaveh. Broadband angle of arrival estimation methods in a polynomial matrix decomposition framework. In *IEEE CAMSAP*, pp. 109–112, Dec. 2013.
[15] F. Coutts, K. Thompson, S. Weiss, and I. Proudler. Impact of fast-converging PEVD algorithms on broadband AoA estimation. In *Sensor Signal Processing for Defence Conference*, pages 1–5, London, UK, December 2017.
[16] M. Davies, S. Lambotharan, and J. McWhirter. Broadband MIMO beamforming using spatial-temporal filters and polynomial matrix decomposition. In *15th Int. Conf. Digital Signal Processing*, pp. 579–582, Cardiff, UK, July 2007.
[17] P. Vouras and T. Tran. Robust transmit nulling in wideband arrays. *IEEE Transactions on Signal Processing*, 62(14):3706–3719, July 2014.
[18] S. Weiss, S. Bendoukha, A. Alzin, F. Coutts, I. Proudler, and J. Chambers. MVDR broadband beamforming using polynomial matrix techniques. In *23rd European Signal Processing Conf.*, pp. 839–843, Nice, France, Sep. 2015.
[19] A. Alzin, F. Coutts, J. Corr, S. Weiss, I. Proudler, and J. Chambers. Polynomial matrix formulation-based Capon beamformer. In *IMA Int. Conf. Signal Processing in Mathematics*, Birmingham, UK, Dec. 2016.
[20] S. Redif, S. Weiss, and J. McWhirter. Relevance of polynomial matrix decompositions to broadband blind signal separation. *Signal Processing*, 134:76–86, May 2017.
[21] S. Weiss, J. Pestana, I. Proudler, and F. Coutts. Corrections to on the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix. *IEEE Transactions on Signal Processing*, 66(23):6325–6327, Dec 2018.
[22] S. Weiss, J. Pestana, and I. K. Proudler. On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix. *IEEE Transactions on Signal Processing*, 66(10):2659–2672, May 2018.
[23] J. G. McWhirter, P. D. Baxter, T. Cooper, S. Redif, and J. Foster. An EVD Algorithm for Para-Hermitian Polynomial Matrices. *IEEE Transactions on Signal Processing*, 55(5):2158–2169, May 2007.
[24] J. G. McWhirter and P. D. Baxter. A Novel Technqiue for Broadband SVD. In *12th Annual Workshop on Adaptive Sensor Array Processing*, MIT Lincoln Labs, Cambridge, MA, 2004.
[25] P. Vaidyanathan. Theory of optimal orthonormal subband coders. *IEEE Transactions on Signal Processing*, 46(6):1528–1543, June 1998.
[26] S. Redif, S. Weiss, and J. McWhirter. Sequential matrix diagonalization algorithms for polynomial EVD of parahermitian matrices. *IEEE Transactions on Signal Processing*, 63(1):81–89, January 2015.
[27] Z. Wang, J. G. McWhirter, J. Corr, and S. Weiss. Multiple shift second order sequential best rotation algorithm for polynomial matrix EVD. In *23rd European Signal Processing Conference*, pp. 844–848, Nice, France, Sep. 2015.
[28] J. Corr, K. Thompson, S. Weiss, J. McWhirter, S. Redif, and I. Proudler. Multiple shift maximum element sequential matrix diagonalisation for parahermitian matrices. In *IEEE SSP*, pp. 312–315, Gold Coast, Australia, June 2014.
[29] J. Corr, K. Thompson, S. Weiss, J. McWhirter, and I. Proudler. Maximum energy sequential matrix diagonalisation for parahermitian matrices. In *Asilomar Conf. Signals, Systems & Comp.*, pp. 470–474, Pacific Grove, CA, USA, Nov. 2014.
[30] V. W. Neo and P. A. Naylor. Second order sequential best rotation algorithm with householder reduction for polynomial matrix eigenvalue decomposition. In *IEEE ICASSP*, pp. 8043–8047, Brighton, UK, May 2019.
[31] J. Corr, K. Thompson, S. Weiss, J. McWhirter, and I. Proudler. Cyclic-by-row approximation of iterative polynomial EVD algorithms. In *Sensor Signal Processing for Defence*, pp. 1–5, Edinburgh, Scotland, Sep.2014.
[32] F. Coutts, J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter. Complexity and search space reduction in cyclic-by-row PEVD algorithms. In *Asilomar Conf. Signals, Systems &Comp.*, Pacific Grove, CA, Nov. 2016.
[33] J. Foster, J. G. McWhirter, and J. Chambers. Limiting the order of polynomial matrices within the SBR2 algorithm. In *IMA International Conference on Mathematics in Signal Processing*, Cirencester, UK, December 2006.
[34] C. H. Ta and S. Weiss. Shortening the order of paraunitary matrices in SBR2 algorithm. In *6th International Conference on Information, Communications & Signal Processing*, pages 1–5, Singapore, Dec. 2007.
[35] J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter. Row-shift corrected truncation of paraunitary matrices for PEVD algorithms. In *23rd European Signal Processing Conference*, pp. 849–853, Nice, France, Aug. 2015.
[36] J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter. Shortening of paraunitary matrices obtained by polynomial eigenvalue decomposition algorithms. In *Sensor Signal Processing for Defence*, Edinburgh, Scotland, Sep. 2015.
[37] J. Corr, K. Thompson, S. Weiss, J. G. McWhirter, and I. K. Proudler. Causality-Constrained multiple shift sequential matrix diagonalisation for parahermitian matrices. In *EUSIPCO*, pp. 1277–1281, Lisbon, Portugal, Sep. 2014.

[38] J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. McWhirter. Reduced search space multiple shift maximum element sequential matrix diagonalisation algorithm. In *IET/EURASIP Intelligent Signal Processing*, London, UK, Dec. 2015.
[39] J. Corr, K. Thompson, S. Weiss, J. McWhirter, and I. Proudler. Performance trade-offs in sequential matrix diagonalisation search strategies. In *IEEE CAMSAP*, pp. 25–28, Cancun, Mexico, Dec. 2015.
[40] F. Coutts, K. Thompson, I. Proudler, and S. Weiss. Restricted update sequential matrix diagonalisation for parahermitian matrices. In *IEEE CAMSAP*, Curacao, Dec. 2017.
[41] F. K. Coutts, I. K. Proudler, and S. Weiss. Efficient implementation of iterative polynomial matrix EVD algorithms exploiting structural redundancy and parallelisation. *IEEE TCAS I*, 66(12):4753–4766, Dec. 2019.
[42] F. K. Coutts, J. Corr, K. Thompson, S. Weiss, I. Proudler, and J. G. McWhirter. Memory and complexity reduction in parahermitian matrix manipulations of PEVD algorithms. In *EUSIPCO*, Budapest, Hungary, Aug. 2016.
[43] F. K. Coutts, K. Thompson, S. Weiss, and I. K. Proudler. Analysing the performance of divide-and-conquer sequential matrix diagonalisation for large broadband sensor arrays. In *IEEE SIPS*, pp. 1–6, Lorient, France, Oct. 2017.
[44] F. Coutts, J. Corr, K. Thompson, I. Proudler, and S. Weiss. Divide-and-conquer sequential matrix diagonalisation for parahermitian matrices. In *Sensor Signal Processing for Defence Conference*, pp. 1–5, London, UK, Dec. 2017.
[45] M. Tohidian, H. Amindavar, and A. M. Reza. A DFT-based approximate eigenvalue and singular value decomposition of polynomial matrices. *EURASIP Journal on Advances in Signal Processing*, 2013(1):1–16, 2013.
[46] F. Coutts, K. Thompson, S. Weiss, and I. Proudler. A comparison of iterative and DFT-based polynomial matrix eigenvalue decompositions. In *IEEE CAMSAP*, Curacao, Dec. 2017.
[47] S. Weiss, I. K. Proudler, F. K. Coutts, and J. Pestana. Iterative approximation of analytic eigenvalues of a parahermitian matrix EVD. In *IEEE International Conference on Acoustics, Speech and Signal Processing*, Brighton, UK, May 2019.
[48] F. K. Coutts, K. Thompson, J. Pestana, I. Proudler, and S. Weiss. Enforcing eigenvector smoothness for a compact DFT-based polynomial eigenvalue decomposition. In *10th IEEE SAM*, pp. 1–5, Sheffield, UK, July 2018.
[49] S. Weiss and M. D. Macleod. Maximally smooth dirichlet interpolation from complete and incomplete sample points on the unit circle. In *IEEE ICASSP*, Brighton, UK, May 2019.
[50] S. Weiss, I. K. Proudler, and M. D. Macleod. Measuring smoothness of real-valued functions defined by sample points on the unit circle. In *Sensor Signal Processing in Defence Conference*, Brighton, UK, May 2019.
[51] S. Weiss. Extending narrowband descriptions and optimal solutions to broadband sensor arrays. In *Int. Joint Conf. Pervasive and Embedded Computing and Communication Systems*, Porto, Portugal, July 2018.
[52] A. Papoulis. *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, New York, 3rd edition, 1991.
[53] F. Rellich. Störungstheorie der Spektralzerlegung. I. Mitteilung. Analytische Störung der isolierten Punkteigenwerte eines beschränkten Operators. *Mathematische Annalen*, 113:DC–DCXIX, 1937.
[54] J. G. McWhirter and Z. Wang. A novel insight to the SBR2 algorithm for diagonalising para-hermitian matrices. In *11th IMA Conference on Mathematics in Signal Processing*, Birmingham, UK, Dec. 2016.
[55] A. Tkacenko and P. Vaidyanathan. On the spectral factor ambiguity of fir energy compaction filter banks. *IEEE Transactions on Signal Processing*, 54(1):380–385, Jan. 2006.
[56] A. Tkacenko. Approximate eigenvalue decomposition of para-hermitian systems through successive FIR paraunitary transformations. In *IEEE ICASSP*, pp. 4074–4077, Dallas, TX, Mar. 2010.
[57] G. H. Golub and C. F. Van Loan. *Matrix Computations*. John Hopkins University Press, Baltimore, MD, 3rd ed., 1996.
[58] J. Capon. High-resolution frequency-wavenumber spectrum analysis. *Proc. IEEE*, 57(8):1408–1418, Aug. 1969.
[59] R. G. Lorenz and S. P. Boyd. Robust minimum variance beamforming. *IEEE Transactions on Signal Processing*, 53(5):1684–1696, May 2005.
[60] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine. Splitting the Unit Delay. *IEEE Signal Processing Magazine*, 13(1):30–60, Jan. 1996.
[61] J. Selva. An efficient structure for the design of variable fractional delay filters based on the windowing method. *IEEE Transactions on Signal Processing*, 56(8):3770–3775, Aug. 2008.
[62] S. Weiss, A. Millar, and R. W. Stewart. Inversion of parahermitian matrices. In *EUSIPCO*, pp. 447–451, Aalborg, Denmark, Aug. 2010.
[63] C. Delaosa, F. K. Coutts, J. Pestana, and S. Weiss. Impact of space-time covariance estimation errors on a parahermitian matrix EVD. In *IEEE SAM*, pp. 1–5, Sheffield, UK, July 2018.
[64] C. Delaosa, J. Pestana, N. J. Goddard, S. Somasundaram, and S. Weiss. Sample space-time covariance matrix estimation. In *IEEE ICASSP*, pp. 8033–8037, Brighton, UK, May 2019.
[65] C. Delaosa, J. Pestana, N. J. Goddard, S. D. Somasundaram, and S. Weiss. Support estimation of a sample space-time covariance matrix. In *SSPD*, Brighton, UK, Mar. 2019.
[66] S. Weiss, N. J. Goddard, S. Somasundaram, I. K. Proudler, and P. A. Naylor. Identification of broadband source-array responses from sensor second order statistics. In *SSPD*, pp. 1–5, London, UK, Dec. 2017.
[67] V. W. Neo, C. Evers, and P. A. Naylor. Speech enhancement using polynomial eigenvalue decomposition. In *IEEE WASPAA*, pp. 125–129, New Paltz, NY, Oct. 2019.