Optimal portfolio choice with path dependent labor income: the infinite horizon case

Enrico Biffis* Fausto Gozzi† Cecilia Prosdocimi†

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Abstract

We consider an infinite horizon portfolio problem with borrowing constraints, in which an agent receives labor income which adjusts to financial market shocks in a path dependent way. This path-dependency is the novelty of the model, and leads to an infinite dimensional stochastic optimal control problem. We solve the problem completely, and find explicitly the optimal controls in feedback form. This is possible because we are able to find an explicit solution to the associated infinite dimensional Hamilton-Jacobi-Bellman (HJB) equation, even if state constraints are present. To the best of our knowledge, this is the first infinite dimensional generalization of Merton's optimal portfolio problem for which explicit solutions can be found. The explicit solution allows us to study the properties of optimal strategies and discuss their financial implications.

Key words: Stochastic functional (delay) differential equations; Optimal control problems in infinite dimension with state constraints; Second order Hamilton-Jacobi-Bellman equations in infinite dimension; Verification theorems and optimal feedback controls; Life-cycle optimal portfolio with labor income; Wages with path dependent dynamics (sticky).

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*Biffis (e.biffis@imperial.ac.uk) is at the Department of Finance, Imperial College Business School, London SW7 2AZ, UK.

†Gozzi (f.gozzi@luiss.it) and Prosdocimi (c.prosdocimi@luiss.it) are at the Dipartimento di Economia e Finanza, Libera Università Internazionale degli Studi Sociali "Guido Carli", Rome, Italy.

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1 Introduction

We consider the life-cycle optimal portfolio choice problem faced by an agent receiving labor income and allocating her wealth to risky assets and a riskless bond subject to a borrowing constraint. The main novelty of the model is that the dynamics of labor income is path dependent, in line with the empirical literature showing that wages adjust slowly to financial market shocks, and income shocks have modest persistency when individuals can learn about their earning potential. The resulting optimal control problem is infinite dimensional, and can be seen as an infinite dimensional generalization of Merton's optimal portfolio problem.

The problem entails maximization of the expected power utility from lifetime consumption and bequest, subject to a linear state equation containing delay, as well as a state constraint, which is well known to make the problem considerably harder to solve. We are nonetheless able to exploit the structure of the model to solve it completely, and obtain the optimal controls in feedback form (Theorem 5.1), thus allowing us to fully understand the economic implications of the setting. To the best of our knowledge, the model presented here offers the first infinite dimensional generalization of the explicit solution to Merton's optimal portfolio problem.

Solving the problem is possible because we are able to find an explicit solution (which we call v) of the associated infinite dimensional HJB equation, even if state constraints are present (Proposition 4.7). Availability of the explicit solution, however, is not the end of the story, as proving that v is indeed the actual value function and finding the feedback map (Theorems 4.17 and 4.24) require considerable technical work. The solution strategy developed in this paper can be used to solve other types of problems with structure similar to the one considered here. As such structure arises naturally in finite dimensional economic and financial models, we think that our solution method could open the way to solving infinite dimensional generalizations of several interesting models.

Our interest in path-dependent labor income dynamics originates from at least three strands of literature addressing the empirical evidence on lifecycle consumption and portfolio decisions with stochastic labor income. First, a common approach used to model the stochastic component of the income process is to use auto-regressive moving average (ARMA) processes (e.g., [35], [1], [36]), and several authors have shown that a parsimonious AR(1) process provides a good description of wage dynamics (see, e.g., [27],[38], [46],[24]). As demonstrated by [42], [34], and [15], stochastic delay differential equations (SDDEs) can be understood, in some cases, as the weak limit of discrete time ARMA processes: it would therefore seem natural to extend the continuous time Merton's model to include a labor income process with a delayed dynamics which is simple enough to deliver closed form solutions. Second, as discussed in [24]-[25], shocks in labor income have modest persistence when heterogeneity in income growth rates is taken into account. In particular, [24] shows that allowing individuals to learn about their income growth rate in a Bayesian way can match several features of consumption data. As is well known, bounded rationality and rational inattention can support the use of moving averages instead of an optimal filter (e.g., [49]), which is exactly what our path dependent labor income dynamics can deliver, while retaining tractability and offering explicit solutions to the portfolio optimization problem. Finally, the empirical evidence on wage rigidity (e.g., [31], [18], and [33], among others) suggests that delayed dynamics may represent a very tractable way of modelling wages that adjust slowly to financial market shocks (e.g., [17], section 6).

Although the model solved here is infinite horizon, it is apparent that our findings would provide important insights in settings where an agent can retire, due to the growing relative importance of the past vs. future component of human capital as the retirement date approaches. This problem will be addressed in future work.

The structure of the paper is as follows. In the next Section 2, we outline the model and provide the economic motivation for the setting. In Section 3, we rewrite the state equation (Subsection 3.1) by exploiting the representation of human wealth provided in [6], as well as the relevant state constraints (Subsection 3.2), both in an infinite dimensional setting where the states are Markovian. In Section 4, the core of the paper, we solve the problem explicitly after recalling the infinite dimensional formulation (Subsection 4.1):

- In Subsection 4.2, we find the explicit solution of the associated HJB equation.
- In Subsection 4.3, we provide a lemma to understand what happens to admissible strategies when the boundary of the constraint set is reached, a key feature in dealing with state constraints problems.
- In Subsections 4.4-4.5, we prove the fundamental identity and the verification theorem, which allow us to find the optimal strategies in feedback form. Here, we pay special attention to the case of risk aversion coefficient $\gamma > 1$, which involves some technical complications relative to the more standard case of $\gamma \in (0,1)$.

Finally, Section 5 summarizes the main results of the paper, which are collected in Theorem 5.1, and discusses the implications for optimal portfolio choice, as well as possible extensions of the model.

2 Problem formulation

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where we define the $\mathbb{F}$-adapted vector valued process (S_0, S) representing the price evolution of a riskless asset, S_0, and n risky assets, $S = (S_1, \ldots, S_n)^\top$, with dynamics

$$\begin{cases} dS_0(t) = S_0(t) r dt \\ dS(t) = \operatorname{diag}(S(t)) \left(\mu dt + \sigma dZ(t)\right) \\ S_0(0) = 1 \\ S(0) \in \mathbb{R}^n_+, \end{cases} \tag{1}$$

where we assume the following.

Hypothesis 2.1.

(i) Z is a n-dimensional Brownian motion. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, is the one generated by Z, augmented with the $\mathbb{P}$-null sets.

(ii) $\mu \in \mathbb{R}^n$, and the matrix $\sigma \in \mathbb{R}^n \times \mathbb{R}^n$ is invertible.

An agent is endowed with initial wealth $w \geq 0$, and receives labor income y until the random time $\tau_\delta > 0$, which represents the agent's time of death (see the discussion in Section 5 for possible extensions). We assume the following.

Hypothesis 2.2.

(i) τ_δ *is independent of* Z*, and it has exponential law with parameter* $\delta > 0$.

(ii) The reference filtration is accordingly given by the enlarged filtration $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$*, where each sigma-field* $\mathcal{G}_t$ *is defined as*

$$\mathcal{G}_t := \cap_{u>t} \left(\mathcal{F}_u \vee \sigma_g\left(\tau_\delta \wedge u\right)\right),$$

and augmented with the $\mathbb{P}$*-null sets. Here by* $\sigma_g(U)$ *we denote the sigma-field generated by the random variable* U.

Note that, with the above choice, $\mathbb{G}$ is the minimal enlargement of the Brownian filtration satisfying the usual assumptions and making τ_δ a stopping time (see [41, Section VI.3, p.370] or [28, Section 7.3.3, p.420]). Moreover, see [2, Proposition 2.11-(b)], we have the following result. If a process A is $\mathbb{G}$-predictable then there exists a process a which is $\mathbb{F}$-predictable and such that

$$A(s,\omega) = a(s,\omega), \qquad \forall \omega \in \Omega,\ \forall s \in [0, \tau_\delta(\omega)] \tag{2}$$

We will therefore introduce the problem relative to the larger filtration $\mathbb{G}$, and then solve it by first working with pre-death processes (i.e. $\mathbb{F}$-predictable processes associated with $\mathbb{G}$-predictable processes as in (2)) and then finally expressing our results in terms of the original filtration $\mathbb{G}$ in Section 6.

During her lifetime, the agent can invest her resources in the riskless and risky assets, and can consume her wealth $W(t)$ at rate $c(t) \geq 0$. We denote by $\theta(t) \in \mathbb{R}^n$ the amounts allocated to the risky assets at each time $t \geq 0$. The agent can also purchase life insurance to reach a bequest target $B(\tau_\delta)$ at death, where $B(\cdot) \geq 0$ is also chosen by the agent. We let the agent pay an insurance premium of amount $\delta(B(t) - W(t))$ to purchase coverage of face value $B(t) - W(t)$ for $t < \tau_\delta$. As in [17], we interpret a negative face value $B(t) - W(t) < 0$ as a life annuity trading wealth at death for a positive income flow $\delta(W(t) - B(t))$ while living. We assume the pre-death controls (c, B, θ) to live in

$$\Pi^0 := \Big\{ \mathbb{F} - \text{predictable } c(\cdot), B(\cdot), \theta(\cdot), \text{ such that: } c(\cdot), B(\cdot) \in L^1(\Omega \times [0, +\infty); \mathbb{R}_+); \tag{3}$$

$$\theta(\cdot) \in L^2(\Omega \times \mathbb{R}; \mathbb{R}^n) \Big\}.$$

Again using [2, Proposition 2.11-(b)], we see that the processes

$$\overline{c}(t) = 1_{\tau_\delta \geq t} c(t), \quad \overline{\theta}(t) = 1_{\tau_\delta \geq t} \theta(t), \quad \overline{B}(t) = 1_{\tau_\delta \geq t} B(t), \quad \overline{W}(t) = 1_{\tau_\delta \geq t} W(t) \tag{4}$$

are all $\mathbb{G}$-predictable processes (we say that $c(\cdot), B(\cdot), \theta(\cdot), W(\cdot)$ are their pre-death counterparts), so that the controls $(\overline{c}, \overline{B}, \overline{\theta})$ live in

$$\overline{\Pi}^0 := \Big\{ \mathbb{G} - \text{predictable } \overline{c}(\cdot), \overline{B}(\cdot), \overline{\theta}(\cdot), \text{ such that: } \overline{c}(\cdot), \overline{B}(\cdot) \in L^1(\Omega \times [0, +\infty); \mathbb{R}_+); \tag{5}$$

$$\overline{\theta}(\cdot) \in L^2(\Omega \times \mathbb{R}; \mathbb{R}^n) \Big\}.$$

The agent's pre-death wealth W is assumed to obey the following dynamics,

$$\begin{cases} dW(t) = & \left[W(t)r + \theta(t)^\top(\mu - r\mathbf{1}) + y(t) - c(t) - \delta\left(B(t) - W(t)\right)\right] dt + \theta(t)^\top \sigma dZ(t), \qquad t \geq 0. \\ \overline{W}(0) = & w, \end{cases} \tag{6}$$

where the $y(\cdot)$ is the pre-death labour income process (similarly to what we did above in (4), we set $\overline{y}(t) := 1_{\tau_\delta \geq t} y(t)$) whose dynamics is described by the following SDDE:

$$\begin{cases} dy(t) = & \left[y(t)\mu_y + \int_{-d}^0 \phi(s)y(t+s)ds\right]dt + y(t)\sigma_y^\top dZ(t), \\ y(0) = & x_0, \quad y(s) = x_1(s) \text{ for } s \in [-d,0), \end{cases} \tag{7}$$

where $\mu_y \in \mathbb{R}$, $\sigma_y \in \mathbb{R}^n$, $\mathbf{1} = (1,\ldots,1)^\top$ is the unitary vector in $\mathbb{R}^n$, and the functions $\phi(\cdot), x_1(\cdot)$ live in $L^2(-d,0;\mathbb{R})$. Existence and uniqueness of a strong solution (with $\mathbb{P}$-a.s. continuous paths) to the SDDE for y are ensured by [39, Theorem I.1 and Remark I.3(iv)] (see also, for a more general result, [44, Section 3]). Existence and uniqueness of a strong solution to the SDE for W are ensured, e.g., by the results of [29, Chapter 5.6].

Remark 2.3. *We note that the function ϕ allows one to modulate the contribution of different subsets of the labor income path in shaping its dynamics going forward. For example, ϕ could give more weight to the most recent labor income realizations relative to wage level in the more distant past. One could similarly introduce an additional delay term in the volatility component of y, writing for example*

$$\begin{cases} dy(t) = & \left[y(t)\mu_y + \int_{-d}^0 y(t+s)\phi(s)\,ds\right]dt \\ & + \left[y(t)\sigma_y^\top + \begin{pmatrix} \int_{-d}^0 y(t+s)\varphi_1(s)\,ds \\ \vdots \\ \int_{-d}^0 y(t+s)\varphi_n(s)\,ds \end{pmatrix}^\top\right] dZ(t), \\ y(0) = & x_0, \quad y(s) = x_1(s) \textit{ for } s \in [-d,0), \end{cases}$$

with $\varphi_1,\ldots,\varphi_n$ belonging to $L^2(-d,0;\mathbb{R})$. This is the setup considered, for example, in [6] where no control problem is considered. In this paper, we consider path dependency in the drift of y only: the extension of our results to the above general case seems possible, although it would entail an increase in complexity of notation and technicalities. We leave it for future work.

We study the problem of maximizing the expected utility from lifetime consumption and bequest,

$$\mathbb{E}\left(\int_0^{+\infty} e^{-\rho t}\frac{\overline{c}(t)^{1-\gamma}}{1-\gamma}dt + e^{-\rho t}\frac{\left(k\overline{B}(t)\right)^{1-\gamma}}{1-\gamma}dN(t)\right), \tag{8}$$

over all triplets $\left(\overline{c},\overline{\theta},\overline{B}\right) \in \overline{\Pi}^0$ satisfying a suitable state constraint introduced further below in (14), where we denote by $N_t := 1_{\tau_\delta \leq t}$ the death indicator process and let parameteres k,γ,ρ satisfy

$$k > 0, \qquad \gamma \in (0,1) \cup (1,+\infty), \qquad \rho > 0, \tag{9}$$

an assumption that will stand throughout the paper.

As the death time is independent of Z and exponentially distributed, we can rewrite the objective functional in (8) as follows (e.g., [40, Section 3.6.2])

$$\mathbb{E}\left(\int_0^{+\infty} e^{-(\rho+\delta)t}\left(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta\frac{\left(kB(t)\right)^{1-\gamma}}{1-\gamma}\right)dt\right). \tag{10}$$

Here we work with the pre-death controls $(c,B,\theta) \in \Pi^0$ and with the pre-death state variables (W,y) whose dynamics is given by the state equation:

$$\begin{cases} dW(t) = & \left[W(t)r + \theta(t)^\top(\mu - r\mathbf{1}) + y(t) - c(t) - \delta\left(B(t) - W(t)\right)\right]dt + \theta(t)^\top\sigma dZ(t) \\ dy(t) = & \left[y(t)\mu_y + \int_{-d}^0 \phi(s)y(t+s)ds\right]dt + y(t)\sigma_y^\top dZ(t), \\ W(0) = & w, \\ y(0) = & x_0, \quad y(s) = x_1(s) \text{ for } s \in [-d,0). \end{cases} \tag{11}$$

Let us now introduce a state constraint which is natural in our context. We first observe that, given the financial market described by (1), the pre-death state-price density of the agent obeys the stochastic differential equation

$$\begin{cases} d\xi(t) &= -\xi(t)(r+\delta)dt - \xi(t)\kappa^\top dZ(t), \\ \xi(0) &= 1. \end{cases} \tag{12}$$

where κ is the market price of risk and is defined as follows (e.g., [30]):

$$\kappa := (\sigma)^{-1}(\mu - r\mathbf{1}). \tag{13}$$

We will then require the agent to satisfy the following constraint

$$W(t) + \xi^{-1}(t)\mathbb{E}\left(\int_t^{+\infty} \xi(u)y(u)du \middle| \mathcal{F}_t\right) \geq 0, \tag{14}$$

which is a no-borrowing-without-repayment constraint, as the second term in (14) represents the agent's market value of human capital at time t. In other words, human capital can be pledged as collateral, and represents the agent's maximum borrowing capacity. The agent cannot default on his/her debt upon death, as the bequest target, B, is nonnegative. We note that by ignoring the delay term (i.e., setting $\phi = 0$ a.e.), the constraint reduces to $W(t) \geq -\beta^{-1}y(t)$, with

$$\beta := r + \delta - \mu_y + \sigma_y^\top \kappa, \tag{15}$$

a parameter expressing the effective discount rate for labor income. We thus recover the borrowing constraints considered in the benchmark model of [17], for example.

Let us denote by $W^{w,x_0,x_1}(t; c, B, \theta)$ and $y^{x_0,x_1}(t)$ the solutions at time t of system (11), where we emphasize the dependence of the solutions on the initial conditions (w, x_0, x_1) and strategies (c, B, θ). We can then define the set of admissible controls as follows:

$$\Pi(w, x_0, x_1) := \Bigg\{ c(\cdot), B(\cdot), \theta(\cdot) \in \Pi^0, \text{ such that:} \\ W^{w,x_0,x_1}(t; c, B, \theta) + \xi^{-1}(t)\mathbb{E}\left(\int_t^{+\infty} \xi(u)y^{x_0,x_1}(u)du \middle| \mathcal{F}_t\right) \geq 0 \quad \forall t \geq 0 \Bigg\}. \tag{16}$$

Our problem is then to maximize the functional given in (10). over all controls in $\Pi(w, x_0, x_1)$.

We now introduce some standing assumptions. Let us first define the following quantities:

$$\beta_\infty := \int_{-d}^0 e^{(r+\delta)s}\phi(s)ds, \tag{17}$$

$$\overline{\beta}_\infty := \int_{-d}^0 e^{(r+\delta)s}|\phi(s)|ds. \tag{18}$$

We have that $\overline{\beta}_\infty \geq \beta_\infty$, with the equality holding if and only if $\phi \geq 0$ a.e.. We then introduce the following standing assumptions:

Hypothesis 2.4.

(i)

$$\beta - \overline{\beta}_\infty > 0. \tag{19}$$

(ii)

$$\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^\top\kappa}{2\gamma}) > 0. \tag{20}$$

Remark 2.5. *Hypothesis 2.4 is needed to rewrite in a convenient way constraint* (14)*. In particular, it implies that* $\beta > 0$*, and hence that the effective discount rate for labor income is positive (e.g., [17]). This allows us to apply Theorem 2.1 of [6], which is recalled, in the form we need here, in Proposition 3.3. When* $\phi \geq 0$ *a.e. (which also implies* $\beta_\infty > 0$*), strict positivity of the labor income process is ensured, as shown in Proposition 2.7 below, when the initial data are positive.*

Remark 2.6. *Hypothesis 2.4-(ii) is required to ensure that the value function is finite, as proved in Proposition 4.7. When* $\rho + \delta > 0$*, such hypothesis is always satisfied if the relative risk aversion* $\gamma > 1$*, but not in the case when* $\gamma \in (0, 1)$*. It can actually be proved that, when* $\gamma \in (0, 1)$ *and*

$$\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma}) < 0,$$

the value function is infinite; for example, see [22] for the deterministic case.

Even if it is not necessary to solve the problem, it is useful to provide conditions guaranteeing the positivity of the labor income process. One is given in the following proposition.

Proposition 2.7. *Let* $y(t) = y^{x_0,x_1}(t)$ *be the solution at time* t *of the second equation of system* (11) *with initial data* $x_0 \in \mathbb{R}$*,* $x_1 \in L^2(-d, 0; \mathbb{R})$*. Defining*

$$E(t) := e^{(\mu_y - \frac{1}{2}\sigma_y^\top \sigma_y)t + \sigma_y^\top Z(t)} \tag{21}$$

$$I(t) := \int_0^t E^{-1}(u) \left(\int_{-d}^0 \phi(s) y(u+s) ds \right) du, \tag{22}$$

we have

$$y(t) = E(t)\big(x_0 + I(t)\big). \tag{23}$$

Moreover, if $x_0 > 0$*,* $x_1 \geq 0$ *a.e. and* $\phi \geq 0$ *a.e., then* $y(t) > 0$ *must hold* $\mathbb{P}$*-a.s..*

Proof. Expression (23), immediately follows by the stochastic variation of constants formula (see, e.g., [8], Theorem 1.1). Concerning the next part of the proposition, let $x_0 > 0$, $x_1 \geq 0$ a.e., and $\phi \geq 0$ on $[-d, 0]$ a.e.. Let $\tau := \inf\{t \geq 0 : \ y(t) = 0\}$. Since $x_0 > 0$ and since y is continuous $\mathbb{P}$-a.s., then it must be $\tau > 0$, $\mathbb{P}$-a.s.. Moreover, assume that $\tau < +\infty$ in a set $\Omega_0 \subseteq \Omega$ of positive probability. Then, from (23) and the fact that $E(t) > 0$ for every $t \geq 0$, we immediately get $I(\tau) = -x_0$ in Ω_0. Since $x_1 \geq 0$ a.e., $\phi \geq 0$ a.e., and $y(t) \geq 0$ $\mathbb{P}$-a.s. where $t \leq \tau$, a contradiction follows. □

3 Reformulation of the problem

In this section we rewrite the problem in a form which is then solved in the subsequent Section 4. In Subsection 3.1 we show how to rewrite the stochastic delay equation for the labor income as a Markov SDE in a Hilbert space. In Subsection 3.2 we show how to rewrite the no borrowing constraint (14) in a way which is suitable for our needs.

3.1 Reformulating the SDE for the labor income

We aim to solve the stochastic optimal control problem introduced in the previous section by using the dynamic programming method. The state equation for the labor income y is a stochastic delay differential equation, and hence y is not Markovian, and the same applies to the state (W, y) of the control problem. Thus, the dynamic programming principle in its standard formulation does not apply. As usual, (see on this e.g. [47], [11] or the books [12, Section 0.2] [21, Section 2.6.8]), it is convenient to reformulate the problem in an infinite dimensional Hilbert space, where the Markov property of the state holds, and which takes into account both the present and the past values of the states. To be precise, let us introduce the Delfour-Mitter Hilbert space M_2 (see e.g. [4, Part II - Chapter 4]):

$$M_2 := \mathbb{R} \times L^2\big(-d, 0; \mathbb{R}\big),$$

with inner product, for $x = (x_0, x_1), y = (y_0, y_1) \in M_2$, defined as

$$\langle x, y \rangle_{M_2} := x_0 y_0 + \langle x_1, y_1 \rangle_{L^2},$$

where

$$\langle x_1, y_1 \rangle_{L^2} := \int_{-d}^{0} x_1(s) y_1(sd) ds.$$

(For ease of notation, we will drop below the subscript L^2 from the inner product of such space, writing simply $\langle x_1, y_1 \rangle$). To embed the state y of the original problem in the space M_2 we now introduce the linear operators A (unbounded) and C (bounded). Define the domain $\mathcal{D}(A)$ as follows

$$\mathcal{D}(A) := \left\{ (x_0, x_1) \in M_2 : x_1(\cdot) \in W^{1,2}\left([-d, 0]; \mathbb{R}\right), x_0 = x_1(0) \right\}.$$

The operator $A : \mathcal{D}(A) \subset M_2 \to M_2$ is then defined as

$$A(x_0, x_1) := \left(\mu_y x_0 + \langle \phi, x_1 \rangle, x_1'\right), \tag{24}$$

with μ_y, ϕ appearing in equation (11). The operator $C : M_2 \to \mathbb{R}^n \times L^2(-d, 0; \mathbb{R})$ is bounded and defined as

$$C(x_0, x_1) := (x_0 \sigma_y, 0),$$

where σ_y shows up in (11) and where 0 in the expression above stays for the null function in $L^2(-d, 0; \mathbb{R})$. Proposition A.27 in [12] shows that A generates a strongly continuous semigroup in M_2.

Consider the following stochastic differential equation in M_2

$$\begin{cases} dX(t) & = AX(t)dt + (CX(t))_0^\top dZ(t), \\ X_0(0) & = x_0, \\ X_1(0)(s) & = x_1(s) \text{ for } s \in [-d, 0), \end{cases} \tag{25}$$

where $(CX(t))_0^\top$ denotes the transpose of the $\mathbb{R}^n$-component of the operator C applied to $X(t)$. The equation above admits a (mild) solution in M_2, as ensured by Theorem 7.2 in [12], (see the same reference for the definition of mild solution). Denote the solution of (25) with $X(\cdot)$ (or $X^x(\cdot)$ if we want to underline its dependence on the initial condition). Note that X is Markovian. Theorem 3.9 and Remark 3.7 in [11] (see also [23], [20]) show that we can identify the solution $X(t) = (X_0(t), X_1(t))$ of (25) with the couple $\big(y(t), y(t+s)_{|s \in [-d,0)}\big)$, where $y(t)$ is the solution of the second equation in (11).

For the reader's convenience we rewrite the system (25) decoupling the two components of X.

$$\begin{cases} dX_0(t) & = \left[\mu_y X_0(t) + \langle \phi, X_1(t) \rangle\right] dt + X_0(t) \sigma_y^\top dZ(t), \\ dX_1(t) & = \frac{\partial}{\partial s} X_1(t) dt, \\ X_0(0) & = x_0, \\ X_1(0)(s) & = x_1(s) \text{ for } s \in [-d, 0), \end{cases} \tag{26}$$

We also clarify, for the reader's convenience, that the derivative of X_1 must be intended, in general, in distributional sense.

We will need the following standard result on the adjoint operator of A.

Proposition 3.1. *The adjoint of A is $A^* : \mathcal{D}(A^*) \subset M_2 \longrightarrow M_2$, defined as follows:*

$$\mathcal{D}(A^*) := \left\{ (y_0, y_1) \in M_2 : y_1(\cdot) \in W^{1,2}\left([-d, 0]; \mathbb{R}\right),\, y_1(-d) = 0 \right\}, \tag{27}$$

$$A^*(y_0, y_1) := \left(\mu_y y_0 + y_1(0), -y_1' + y_0 \phi\right). \tag{28}$$

Proof. We provide a sketch of the proof for the reader's convenience. We have to check that $\mathcal{D}(A^*)$ coincides with the set of points $(y_0, y_1) \in M_2$ such that

$$|\langle A(x_0, x_1), (y_0, y_1)\rangle_{M_2}| \le M|(x_0, x_1)|_{M_2} \quad \forall (x_0, x_1) \in \mathcal{D}(A). \tag{29}$$

Moreover we have to check that for any $(x_0, x_1) \in \mathcal{D}(A)$, $(y_0, y_1) \in \mathcal{D}(A^*)$,

$$\langle A(x_0, x_1), (y_0, y_1)\rangle_{M_2} = \langle (x_0, x_1), A^*(y_0, y_1)\rangle_{M_2}. \tag{30}$$

By (24) we have, for all $x \in \mathcal{D}(A)$,

$$\langle A(x_0, x_1), (y_0, y_1)\rangle_{M_2} = \mu_y x_0 y_0 + y_0 \langle \phi, x_1\rangle + \langle x_1', y_1\rangle. \tag{31}$$

It can be proved that (29) holds only if $y_1 \in W^{1,2}([-d, 0]; \mathbb{R})$. Then, using integration by parts we get

$$\langle x_1', y_1\rangle = x_1(0)y_1(0) - x_1(-d)y_1(-d) - \langle x_1, y_1'\rangle. \tag{32}$$

From this we get that (29) also implies $y_1(-d) = 0$, hence $y \in \mathcal{D}(A^*)$. When $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(A^*)$ we get from (31) and (32)

$$\langle A(x_0, x_1), (y_0, y_1)\rangle_{M_2} = \mu_y x_0 y_0 + y_0 \langle \phi, x_1\rangle + x_1(0)y_1(0) - \langle x_1, y_1'\rangle \tag{33}$$

and (29) is satisfied thanks to the boundary condition $x_0 = x_1(0)$. Finally (29) follows by straightforward computations. □

3.2 Rephrasing the no-borrowing constraint

In this section we express the no-borrowing constraint at each time $t \ge 0$ in terms of $X_0(t)$ and $X_1(t)$ given in the previous subsection. To do so, we introduce the constant $g_\infty > 0$ and the function $h_\infty : [-d, 0] \longrightarrow \mathbb{R}$ defined as follows:

$$\begin{cases} g_\infty & := \dfrac{1}{\beta - \beta_\infty}, \\ h_\infty(s) & := g_\infty \displaystyle\int_{-d}^{s} e^{-(r+\delta)(s-\tau)} \phi(\tau) d\tau, \end{cases} \tag{34}$$

with β and β_∞ defined in (15) and (17) respectively.

Lemma 3.2. *For a.e. $s \in [-d, 0]$, the function h_∞ defined in (34) is differentiable and it satisfies*

$$\begin{cases} h_\infty'(s) & = g_\infty \phi(s) - (r + \delta)h_\infty(s), \\ h_\infty(0) & = \beta g_\infty - 1. \end{cases} \tag{35}$$

Moreover $(g_\infty, h_\infty) \in \mathcal{D}(A^*)$.

Proof. By definition of β_∞ we have

$$h_\infty(0) = \beta_\infty g_\infty,$$

and therefore

$$\beta g_\infty - h_\infty(0) = \beta g_\infty - \beta_\infty g_\infty = 1,$$

thus h_∞ satisfies the terminal condition in (35). Differentiability a.e. of $h_\infty(s)$ follows by standard differentiability of integral functions. Differentiating we then get

$$h_\infty'(s) = -(r + \delta)h_\infty(s) + g_\infty \phi(s),$$

and (35) then follows. Let us now check that $h_\infty \in L^2(-d,0,\mathbb{R})$:

$$\int_{-d}^{0} h_\infty^2(s)ds = \frac{1}{(\beta-\beta_\infty)^2}\int_{-d}^{0}\Big(\int_{-d}^{s} e^{-(r+\delta)(s-\tau)}\phi(\tau)d\tau\Big)^2 ds$$
$$\leq \frac{1}{(\beta-\beta_\infty)^2}\int_{-d}^{0}\int_{-d}^{s} e^{-2(r+\delta)(s-\tau)}\phi^2(\tau)d\tau ds \leq \frac{1}{(\beta-\beta_\infty)^2}\int_{-d}^{0}\int_{-d}^{s}\phi^2(\tau)d\tau ds$$
$$\leq \frac{1}{(\beta-\beta_\infty)^2}\int_{-d}^{0}\int_{-d}^{0}\phi^2(\tau)d\tau ds = \frac{d}{(\beta-\beta_\infty)^2}\|\phi\|_2^2 < +\infty,$$

where the first inequality follows by Jensen's inequality. Using (35) we then immediately get that $h_\infty \in W^{1,2}([-d,0];\mathbb{R})$. Finally, since $h_\infty(-d)=0$ we get (g_∞,h_∞) is in $\mathcal{D}(A^*)$. □

The following Proposition, which is a direct consequence of Theorem 2.1 of [6], provides an explicit expression for the market value of human capital.

Proposition 3.3. *Let $X(t)$ solve (25) and W solve the first of (11) with $X_0(t)$ in place of $y(t)$. Let $\xi(t)$ solve (12). Then, the market value of human capital admits the following representation*

$$\xi(t)^{-1}\mathbb{E}\left(\int_t^{+\infty}\xi(u)X_0(u)du\,\middle|\,\mathcal{F}_t\right) = g_\infty X_0(t) + \langle h_\infty, X_1(t)\rangle \qquad \mathbb{P}\text{-}a.s.. \tag{36}$$

Proof. See [6, Theorem 2.1]. □

Expression (36) shows that the market value of human capital can be decomposed into two terms: one capturing the current market value of the past trajectory of labor income over $[t-d,t]$, and one capturing the current market value of the future labor income stream (see the discussion in [6]). This distinction will be important when interpreting the solution to our optimization problem.

Proposition 3.3 implies that the original constraint (14) can be reformulated as follows:

$$W(t) + g_\infty X_0(t) + \langle h_\infty, X_1(t)\rangle \geq 0 \text{ for all } t. \tag{37}$$

Notation 3.4. *We note that, when $t=0$, the above implies that the initial datum $(w,x) \in \mathcal{H} := \mathbb{R}\times M_2$ must belong to the half-space*

$$\mathcal{H}_+ = \{(w,x)\in\mathcal{H}:\ w + g_\infty x_0 + \langle h_\infty, x_1\rangle \geq 0\} \tag{38}$$

It is also convenient to introduce the open half-space

$$\mathcal{H}_{++} = \{(w,x)\in\mathcal{H}:\ w + g_\infty x_0 + \langle h_\infty, x_1\rangle > 0\} \tag{39}$$

Moreover, defining the linear map $\Gamma_\infty : \mathcal{H}\to\mathbb{R}$ as

$$\Gamma_\infty(w,x) := w + g_\infty x_0 + \langle h_\infty, x_1\rangle, \tag{40}$$

we have $\mathcal{H}_+ = \{\Gamma_\infty \geq 0\}$, $\mathcal{H}_{++} = \{\Gamma_\infty > 0\}$. We observe that $\mathcal{H}_+$ contains the cone of positive functions in $\mathcal{H}$ if and only if $\phi \geq 0$ a.e..

4 Solving the problem

4.1 Statement of the reformulated problem

Using the results of the previous Section 3, the problem exposed in Section 2 can be reformulated as follows.

Problem 4.1. *The state space is* $\mathcal{H} := \mathbb{R} \times M_2$. *The control space is* $U := \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$. *The state equation is*

$$\begin{cases} dW(t) = & \big[(r+\delta)W(t) + \theta^\top(t)(\mu - r\mathbf{1}) + X_0(t) - c(t) - \delta B(t)\big]\, dt \\ & +\theta^\top(t)\sigma dZ(t), \\ dX(t) = & AX(t)dt + \big(CX(t)\big)^\top dZ_t, \\ W(0) = & w, \\ X_0(0) = & x_0, \qquad X_1(s) = x_1(s) \text{ for } s \in [-d, 0). \end{cases} \tag{41}$$

Denote by $W^{w,x}(s; c, B, \theta)$ *the solution at time* s *of the first equation in (41) starting at time* 0 *in* (w, x) *and following the strategy* (c, B, θ), *and by* $X^x(s)$ *the solution at time* s *of the second equation in (41), starting at time* 0 *in* x. *The set of admissible controls is (see (16) and Proposition 3.3):*

$$\Pi(w, x_0, x_1) = \Pi(w, x) = \Big\{\mathbb{F} - \text{adapted } c(\cdot), B(\cdot), \theta(\cdot), \text{ such that: } c(\cdot), B(\cdot) \in L^1(\Omega \times [0, +\infty), \mathbb{R}_+);$$

$$\theta(\cdot) \in L^2(\Omega \times [0, +\infty), \mathbb{R}^n);$$

$$W^{w,x}(t; c, B, \theta) + g_\infty X_0^x(t) + \langle h_\infty, X_1^x(t)\rangle \geq 0 \quad \forall t \geq 0\Big\},$$

(the last constraint being equivalent to ask $(W(t), X(t)) \in \mathcal{H}_+$ *for every* $t \geq 0$*), find a strategy* $(c, B, \theta) \in \Pi(w, x_0, x_1)$ *maximizing the functional*

$$J\big(w, x; c, B, \theta\big) := \mathbb{E}\left(\int_0^{+\infty} e^{-(\rho+\delta)t}\left(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta\frac{\big(kB(t)\big)^{1-\gamma}}{1-\gamma}\right) dt\right), \tag{42}$$

assuming Hypotheses 2.1, 2.2 and 2.4.

Note that the functional J may possibly take value $-\infty$ (e.g. when $\gamma > 1$ and both $c(\cdot)$ and $B(\cdot)$ are both identically zero) or $+\infty$ (when $\gamma \in (0, 1)$ but, thanks to Hypothesis 2.4, this will be proved to be impossible, see Corollary 4.12 and Proposition 4.18).

We solve the problem by using the dynamic programming method. Define, for $(w, x) \in \mathcal{H}_+$, the value function $V(w, x)$ as

$$V(w, x) := \sup_{\big(c,B,\theta\big)\in\Pi(w,x)} J\big(w, x; c, B, \theta\big). \tag{43}$$

Similarly to what we noted above for the functional J we see that, up to now, V may possibly take the values $-\infty$ or $+\infty$ in $\mathcal{H}_+$.

Notation 4.2. *Sometimes, given an initial point* $(w, x) \in \mathcal{H}_+$ *and an admissible strategy* $(c, B, \theta) \in \Pi(w, x)$, *for readability, we will use the shorthand notations:*

$$\pi := (c, B, \theta)$$

and

$$W_\pi(s) := W^{w,x}(s; c, B, \theta), \qquad X(s) := X^x(s),$$

shrinking the dependence on the controls and omitting the dependence on the initial conditions.

4.2 The HJB equation and its explicit solution

Notation 4.3. *Let $p = (p_1, p_2)$ be a generic vector of $\mathcal{H} = \mathbb{R} \times M_2$, and let $S(2)$ denote the space of real symmetric matrices of dimension 2, and P an element of $S(2)$, with*

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

For any given function $u : \mathcal{H} \longrightarrow \mathbb{R}$, we denote by $Du = (u_w, u_x) = (u_w, (u_{x_0}, u_{x_1})) \in \mathcal{H}$ its gradient and by

$$D^2_{wx_0} u = \begin{pmatrix} u_{ww} & u_{wx_0} \\ u_{x_0 w} & u_{x_0 x_0} \end{pmatrix} \in S(2)$$

its second derivatives with respect to the first two components (w, x_0), whenever they exist.

The HJB equation associated with Problem 4.1 is

$$(\rho + \delta)v = \mathbb{H}\left(w, x, Dv, D^2_{wx_0} v\right), \tag{44}$$

where the Hamiltonian $\mathbb{H} : \mathbb{R} \times M_2 \times (\mathbb{R} \times \mathcal{D}(A^*)) \times S(2) \longrightarrow \overline{\mathbb{R}}$ [1] is, informally speaking, defined as follows

$$\mathbb{H}(w, x, p, P) := \sup_{(c,B,\theta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n} \langle \mathcal{B}(w, x, \theta, c, B), p \rangle_{\mathbb{R} \times M_2} + \frac{1}{2} Tr \Sigma(\theta, x_0) P \Sigma^*(\theta, x_0) + U(c, B) \tag{45}$$

where we call $\mathcal{B}(w, x, \theta, c, B)$ and $\Sigma(\theta, x_0)$,[2] the drift and the diffusion of (41), while U is the utility function in the integral (42). To compute the Hamiltonian we separate the part depending on the controls from the other one, which can be taken out of the supremum. Hence we write:

$$\mathbb{H}(w, x, p, P) := \mathbb{H}_1(w, x, p, P_{22}) + \mathbb{H}_{max}(x_0, p_1, P_{11}, P_{12}), \tag{46}$$

where

$$\mathbb{H}_1(w, x, p, P_{22}) := (r + \delta) w p_1 + x_0 p_1 + \langle x, A^* p_2 \rangle_{M_2} + \frac{1}{2} \sigma_y^\top \sigma_y x_0^2 P_{22}, \tag{47}$$

and

$$\mathbb{H}_{max}(x_0, p_1, P_{11}, P_{12}) := \sup_{(c,B,\theta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n} \mathbb{H}_{cv}(x_0, p_1, P_{11}, P_{12}; c, B, \theta) \tag{48}$$

with

$$\begin{aligned} \mathbb{H}_{cv}(x_0, p_1, P_{11}, P_{12}; c, B, \theta) :=& \frac{c^{1-\gamma}}{1-\gamma} + \frac{\delta \left(kB\right)^{1-\gamma}}{1-\gamma} + [\theta^\top (\mu - r\mathbf{1}) - c - \delta B] p_1 \\ &+ \frac{1}{2} \theta^\top \sigma \sigma^\top \theta P_{11} + \theta^\top \sigma \sigma_y x_0 P_{12} \\ :=& \frac{c^{1-\gamma}}{1-\gamma} - c p_1 + \frac{\delta \left(kB\right)^{1-\gamma}}{1-\gamma} - \delta B p_1 \\ &+ \theta^\top (\mu - r\mathbf{1}) p_1 + \frac{1}{2} \theta^\top \sigma \sigma^\top \theta P_{11} + \theta^\top \sigma \sigma_y x_0 P_{12}. \end{aligned} \tag{49}$$

Now note that, thanks to the last equality above, whenever $p_1 > 0$ and $P_{11} < 0$, the maximum in (48) is achieved at

$$\begin{cases} c^* := p_1^{-\frac{1}{\gamma}}, \\ B^* := k^{-b} p_1^{-\frac{1}{\gamma}}, \\ \theta^* := -(\sigma \sigma^\top)^{-1} \dfrac{(\mu - r\mathbf{1}) p_1 + \sigma \sigma_y x_0 P_{12}}{P_{11}}, \end{cases} \tag{50}$$

[1] Note that we allow the Hamiltonian to take values in $\overline{\mathbb{R}}$, hence to be possibly $\pm\infty$.

[2] Note that $\Sigma(\theta, x_0)$ is a 2×2 matrix since the component X_1 of the state in (41) has no diffusion coefficient.

where

$$b = 1 - \frac{1}{\gamma}. \tag{51}$$

Hence, for $p_1 > 0$ and $P_{11} < 0$ we have, by simple computations,

$$\begin{aligned}\mathbb{H}(w, x, p, P) =& (r + \delta)wp_1 + x_0 p_1 + \langle x, A^* p_2\rangle_{M_2} + \frac{\gamma}{1-\gamma} p_1^b \big(1 + \delta k^{-b}\big) \\ &+ \frac{1}{2}\sigma_y^\top \sigma_y x_0^2 P_{22} \\ &- \frac{1}{2P_{11}} \left[(\mu - r\mathbf{1})p_1 + \sigma\sigma_y x_0 P_{12}\right]^\top (\sigma\sigma^\top)^{-1} \left[(\mu - r\mathbf{1})p_1 + \sigma\sigma_y x_0 P_{12}\right].\end{aligned} \tag{52}$$

Therefore, if the unknown v satisfies $v_w > 0$ and $v_{ww} < 0$, the HJB equation in (44) reads

$$\begin{aligned}(\rho + \delta)v =& (r + \delta)wv_w + x_0 v_w + \langle x, A^* v_x\rangle_{M_2} + \frac{\gamma}{1-\gamma} v_w^b \big(1 + \delta k^{-b}\big) \\ &+ \frac{1}{2}\sigma_y^\top \sigma_y x_0^2 v_{x_0 x_0} \\ &- \frac{1}{2v_{ww}} \left[(\mu - r\mathbf{1})v_w + \sigma\sigma_y x_0 v_{wx_0}\right]^\top (\sigma\sigma^\top)^{-1} \left[(\mu - r\mathbf{1})v_w + \sigma\sigma_y x_0 v_{wx_0}\right].\end{aligned} \tag{53}$$

On the other hand we must also note that, when $p_1 < 0$ or $P_{11} > 0$, the Hamiltonian $\mathbb{H}$ is $+\infty$, while, when $p_1 P_{11} = 0$, different cases may arise depending on γ and on the sign of other terms.

Definition 4.4. *A function $u : \mathcal{H}_{++} \longrightarrow \mathbb{R}$ is a* classical solution *of the HJB equation (44) in $\mathcal{H}_{++}$ if the following holds:*

- *(i) u is continuously Fréchet differentiable in $\mathcal{H}_{++}$ and admits continuous second derivatives with respect to (w, x_0) in $\mathcal{H}_{++}$;*
- *(ii) $u_x(w, x) \in \mathcal{D}(A^*)$ for every $(w, x) \in \mathcal{H}_{++}$ and $A^* u_x$ is continuous in $\mathcal{H}_{++}$;*
- *(iii) for all $(w, x) \in \mathcal{H}_{++}$ we have*

$$(\rho + \delta)u - \mathbb{H}\big(w, x, Du, D^2_{wx_0} u\big) = 0. \tag{54}$$

Remark 4.5. *It is important to note that, in the Hamiltonian $\mathbb{H}_1$, the natural infinite dimensional term involving A should be written as $\langle Ax, p_2\rangle_{M_2}$, which would make sense only for $x \in \mathcal{D}(A) \cap \mathcal{H}_{++}$. Here we decided to write it in the different form $\langle x, A^* p_2\rangle_{M_2}$, which makes sense for all $x \in \mathcal{H}_{++}$ but only if p_2 (which stands for u_x) belongs to $\mathcal{D}(A^*)$. This choice substantially means that, on classical solutions, we assume a further regularity on the gradient. This will simplify the application of Ito's formula in next proposition and will be enough for our needs, as the explicit solution that we compute satisfies such a regularity.*

Remark 4.6. *In Definition 4.4 above (classical solution u of the HJB equation* (44)*) we did not include the requirement $u_w > 0$ and $u_{ww} < 0$ even if it seems natural in this context. A reason is that, in the case $\gamma > 1$, the function $u \equiv 0$ is indeed a solution (see Subsection 4.6).*

Proposition 4.7. *Define, for $(w, x) \in \mathcal{H}_{++}$,*

$$\bar{v}(w, x) := \frac{f_\infty^\gamma \Gamma_\infty^{1-\gamma}(w, x)}{1 - \gamma}, \tag{55}$$

with $f_\infty > 0$ defined as

$$f_\infty := (1 + \delta k^{-b})\nu, \tag{56}$$

where

$$\nu := \frac{\gamma}{\rho + \delta - (1 - \gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})} > 0, \tag{57}$$

Γ_∞ defined in Notation 3.4, and b as in (51). Then, $\bar{v}$ is a classical solution of the HJB equation (44) on $\mathcal{H}_{++}$.

Proof. Let $\bar{v}$ as in (55). Since Γ_∞ is linear, it is immediate to check that $\bar{v}$ satisfies the assumption required at point (i) of Definition 4.4. Moreover we have, in $\mathcal{H}_{++}$,

$$
\begin{aligned}
\bar{v}_w(w,x) &= f_\infty^\gamma \Gamma_\infty^{-\gamma}(w,x),\\
\bar{v}_x(w,x) &= f_\infty^\gamma \Gamma_\infty^{-\gamma}(w,x)(g_\infty, h_\infty),\\
\bar{v}_{ww}(w,x) &= -\gamma f_\infty^\gamma \Gamma_\infty^{-\gamma-1}(w,x),\\
\bar{v}_{wx_0}(w,x) &= -\gamma f_\infty^\gamma \Gamma_\infty^{-\gamma-1}(w,x) g_\infty,\\
\bar{v}_{x_0x_0}(w,x) &= -\gamma f_\infty^\gamma \Gamma_\infty^{-\gamma-1}(w,x) g_\infty^2.
\end{aligned}
\tag{58}
$$

Hence, since $(g_\infty, h_\infty) \in \mathcal{D}(A^*)$ by Lemma 3.2, then also point (ii) of Definition 4.4 holds for $\bar{v}$. Concerning point (iii) of Definition 4.4 we observe first that $\bar{v}_w > 0$ and $\bar{v}_{ww} < 0$ on $\mathcal{H}_{++}$ and then we can use the explicit expression of the HJB equation (53) which holds in such a case. Using the expression of A^* given in Proposition 3.1 we have, in $\mathcal{H}_{++}$,

$$
\begin{aligned}
\langle x, A^*\bar{v}_x(w,x)\rangle_{M_2} &= f_\infty^\gamma \Gamma_\infty^{-\gamma}(w,x)\langle x, A^*(g_\infty, h_\infty)\rangle_{M_2}\\
&= f_\infty^\gamma \Gamma_\infty^{-\gamma}(w,x)\left[x_0\mu_y g_\infty + x_0 h_\infty(0) + \langle x_1, -h'_\infty + g_\infty\phi\rangle\right].
\end{aligned}
$$

Moreover we have, omitting the variables (w,x) for simplicity of notation,

$$
\frac{\gamma}{1-\gamma}\bar{v}_w^b\left(1+\delta k^{-b}\right) = \frac{\gamma}{1-\gamma} f_\infty^{\gamma-1}\Gamma_\infty^{1-\gamma}(1+\delta k^{-b})
$$

$$
\frac{1}{2}\sigma_y^\top \sigma_y x_0^2 \bar{v}_{x_0x_0} = -\frac{\gamma}{2}\sigma_y^\top\sigma_y x_0^2 f_\infty^\gamma \Gamma_\infty^{-\gamma-1} g_\infty^2
$$

and, using, in the last line below, the definition of κ in (13),

$$
\begin{aligned}
&-\frac{1}{2\bar{v}_{ww}}\left[(\mu - r\mathbf{1})\bar{v}_w + \sigma\sigma_y x_0 \bar{v}_{wx_0}\right]^\top (\sigma\sigma^\top)^{-1}\left[(\mu - r\mathbf{1})\bar{v}_w + \sigma\sigma_y x_0\bar{v}_{wx_0}\right] =\\
&= \frac{1}{2\gamma} f_\infty^{-\gamma}\Gamma_\infty^{1+\gamma}\left[(\mu - r\mathbf{1}) f_\infty^\gamma \Gamma_\infty^{-\gamma} - \gamma x_0 f_\infty^\gamma \Gamma_\infty^{-1-\gamma} g_\infty \sigma\sigma_y\right]^\top (\sigma\sigma^\top)^{-1}\left[(\mu - r\mathbf{1}) f_\infty^\gamma\Gamma_\infty^{-\gamma} - \gamma x_0 f_\infty^\gamma \Gamma_\infty^{-1-\gamma} g_\infty\sigma\sigma_y\right] =\\
&= \frac{1}{2\gamma} f_\infty^\gamma \Gamma_\infty^{1-\gamma}\left[\mu - r\mathbf{1} - \gamma x_0 \Gamma_\infty^{-1} g_\infty \sigma\sigma_y\right]^\top (\sigma\sigma^\top)^{-1}\left[\mu - r\mathbf{1} - \gamma x_0\Gamma_\infty^{-1} g_\infty\sigma\sigma_y\right] =\\
&= \frac{1}{2\gamma} f_\infty^\gamma \Gamma_\infty^{1-\gamma}\left[\kappa^\top\kappa - \gamma x_0 \Gamma_\infty^{-1} g_\infty\left(\kappa^\top\sigma_y + \sigma_y^\top\kappa\right) + \gamma^2 x_0^2 \Gamma_\infty^{-2} g_\infty^2 \sigma_y^\top\sigma_y\right].
\end{aligned}
$$

Now we substitute $\bar{v}$ inside (53) using the last three equalities. Then we multiply by $f_\infty^{-\gamma}\Gamma_\infty^\gamma(w,x)$, obtaining, in $\mathcal{H}_{++}$,

$$
\begin{aligned}
\frac{(\rho+\delta)}{1-\gamma}\Gamma_\infty(w,x) &= (r+\delta)w + x_0 + x_0\mu_y g_\infty + x_0 h_\infty(0) + \langle x_1, -h'_\infty + g_\infty\phi\rangle\\
&\quad + \frac{\gamma}{1-\gamma} f_\infty^{-1}\Gamma_\infty(w,x)(1+\delta k^{-b}) + \frac{\kappa^\top\kappa}{2\gamma}\Gamma_\infty(w,x) - \kappa^\top\sigma_y x_0 g_\infty.
\end{aligned}
\tag{59}
$$

Recalling (35) the equality above can be rewritten as

$$
\begin{aligned}
\frac{(\rho+\delta)}{1-\gamma}\Gamma_\infty(w,x) &= (r+\delta)w + x_0 + x_0\mu_y g_\infty + x_0\big(\beta g_\infty - 1\big) + (r+\delta)\langle x_1, h_\infty\rangle\\
&\quad + \frac{\gamma}{1-\gamma} f_\infty^{-1}\Gamma_\infty(w,x)(1+\delta k^{-b}) + \frac{\kappa^\top\kappa}{2\gamma}\Gamma_\infty(w,x) - \kappa^\top\sigma_y x_0 g_\infty,
\end{aligned}
\tag{60}
$$

which, by the definitions of β in (15) and of Γ_∞ in (40), reads

$$
\frac{(\rho+\delta)}{1-\gamma}\Gamma_\infty(w,x) = (r+\delta)\Gamma_\infty(w,x) + \frac{\gamma}{1-\gamma} f_\infty^{-1}\Gamma_\infty(w,x)(1+\delta k^{-b}) + \frac{\kappa^\top\kappa}{2\gamma}\Gamma_\infty(w,x).
$$

We can now cancel $\Gamma_\infty(w,x)$, since it is strictly positive, and use the definition of f_∞ in (56), getting

$$\frac{(\rho+\delta)}{1-\gamma}-(r+\delta)-\frac{\gamma}{1-\gamma}\nu^{-1}-\frac{\kappa^\top\kappa}{2\gamma}=0.$$

This holds true thanks to the definition of ν given in (57), so the claim is proved. □

Remark 4.8. *The function $\bar{v}$ can be defined also in $\mathcal{H}_+$ by setting, on $\partial\mathcal{H}_+=\{\Gamma_\infty(w,x)=0\}$,*

$$\bar{v}(w,x)=0,\quad \text{when } \gamma\in(0,1),\qquad \text{and}\qquad \bar{v}(w,x)=-\infty,\quad \text{when } \gamma\in(1,+\infty).$$

From now on we will consider $\bar{v}$ defined on $\mathcal{H}_+$.

In next subsection we use the function $\bar{v}$ to prove the fundamental identity, the key step to get the Verification Theorem and the existence of optimal feedbacks, see Theorems 4.17 and 4.24.

4.3 A lemma on the evolution of Γ_∞ on the admissible paths

We need the following lemma.

Lemma 4.9. *Fix $(w,x)\in\mathcal{H}_+$ and let $\pi=(c,B,\theta)\in\Pi(w,x)$. Then we have the following:*

(i) The process

$$\bar{\Gamma}_\infty(t):=\Gamma_\infty(W^{w,x}(t;c,B,\theta),X^x(t))$$

satisfies the equation

$$\begin{aligned} d\bar{\Gamma}_\infty(t)=&\left[(r+\delta)\bar{\Gamma}_\infty(t)-c(t)-\delta B(t)+\left(\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right)\kappa\right]dt\\ &+\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]dZ_t, \end{aligned} \tag{61}$$

(ii) Assume that $\Gamma_\infty(w,x)=0$, i.e. that $\bar{\Gamma}_\infty(0)=0$. Then for every $t\geq 0$ it must be

$$\bar{\Gamma}_\infty(t)=0,\qquad \mathbb{P}-a.s.$$

and

$$c(t,\omega)=0,\quad B(t,\omega)=0,\quad \theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top=0,\qquad dt\otimes\mathbb{P}-a.e.\ in\ [0,+\infty)\times\Omega \tag{62}$$

Let now $\Gamma_\infty(w,x)>0$ and set τ be the first exit time of $(W_\pi(\cdot),X(\cdot))$ from $\mathcal{H}_{++}$. Then for every $t\geq 0$

$$\mathbf{1}_{\{\tau<t\}}\bar{\Gamma}_\infty(t)=0,\qquad \mathbb{P}-a.s.$$

and

$$\mathbf{1}_{\{\tau<t\}}(\omega)c(t,\omega)=0,\quad \mathbf{1}_{\{\tau<t\}}(\omega)B(t,\omega)=0,\quad \mathbf{1}_{\{\tau<t\}}\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]=0,$$

$dt\otimes\mathbb{P}-a.e.$ in $[0,+\infty)\times\Omega$.

Proof. Fix $(w,x)\in\mathcal{H}_+$ and $\pi=(c,B,\theta)\in\Pi(w,x)$. Recall that $W^{w,x}(s;c,B,\theta)$ is the solution at time s of the first equation in (41) starting at time 0 in (w,x) and following the strategy (c,B,θ); recall also that $X^x(s)$ is the solution at time s of the second equation in (41), starting at time 0 in x. For readability we will use the shorthand notation

$$\begin{aligned} W_\pi(s)&:=W^{w,x}(s;c,B,\theta),\\ X(s)&:=X^x(s), \end{aligned}$$

omitting the dependence on the controls and on the initial conditions.

Now we prove each step separately.

Proof of (i). We apply Ito's formula of [21, Proposition 1.165], to the process $\langle (g_\infty, h_\infty), (X_0(t), X_1(t))\rangle_{M_2}$ getting

$$\begin{aligned} d\langle (g_\infty, h_\infty), &(X_0(t), X_1(t))\rangle_{M_2} = \\ &= \langle A^*(g_\infty, h_\infty), (X_0(t), X_1(t))\rangle_{M_2} dt + \langle (g_\infty, h_\infty), \big(C(X_0(t), X_1(t))\big)_0^\top dZ_t\rangle_{M_2} \\ &= \big[\mu_y g_\infty X_0(t) + h_\infty(0) X_0(t) + \langle -h'_\infty + g_\infty \phi, X_1(t)\rangle\big] dt + g_\infty X_0(t) \sigma_y^\top dZ_t \\ &= \big[\big(\mu_y + \beta\big) g_\infty X_0(t) - X_0(t) + (r+\delta)\langle h_\infty, X_1(t)\rangle\big] dt + g_\infty X_0(t)\sigma_y^\top dZ_t, \end{aligned} \tag{63}$$

where the last equality follows from (35). Therefore, writing

$$\bar\Gamma_\infty(t) = \Gamma_\infty\big(W_\pi(t), X(t)\big) = W_\pi(t) + g_\infty X_0(t) + \langle h_\infty, X_1(t)\rangle,$$

we have, from (41) and (63),

$$\begin{aligned} d\bar\Gamma_\infty(t) = &\big[W(t)(r+\delta) + \theta^\top(t)(\mu - r\mathbf{1}) + X_0(t) - c(t) - \delta B(t) \\ &+ \big(\mu_y + \beta\big) g_\infty X_0(t) - X_0(t) + (r+\delta)\langle h_\infty, X_1(t)\rangle\big] dt \\ &+ \big[\theta^\top(t)\sigma + g_\infty X_0(t)\sigma_y^\top\big] dZ_t \\ = &\Big[(r+\delta)\bar\Gamma_\infty(t) - c(t) - \delta B(t) + \Big(\theta^\top(t)\sigma + g_\infty X_0(t)\sigma_y^\top\Big)\kappa\Big] dt \\ &+ \big[\theta^\top(t)\sigma + g_\infty X_0(t)\sigma_y^\top\big] dZ_t, \end{aligned} \tag{64}$$

where we have used $\mu_y + \beta = r + \delta + \sigma_y^\top \kappa$ (see (15)). This is the claim.

Proof of (ii). By Girsanov's Theorem (see e.g. [29, Theorem 3.5.1], for any $T > 0$, under the probability (depending on T, defined on $\mathcal{F}_T$ and, there, equivalent to $\mathbb{P}$)

$$\tilde{\mathbb{P}}_T = \exp\left(-\kappa^\top Z(T) - \frac{1}{2}\kappa^\top \kappa T\right)\mathbb{P}, \tag{65}$$

the process $t \mapsto \tilde Z(t) = \kappa t + Z(t)$ is a d-dimensional Brownian motion on $[0,T]$ and $\bar\Gamma_\infty$ satisfies, on $[0,T]$,

$$d\bar\Gamma_\infty(t) = \big[(r+\delta)\bar\Gamma_\infty(t) - c(t) - \delta B(t)\big] dt + \big[\theta^\top(t)\sigma + g_\infty X_0(t)\sigma_y^\top\big] d\tilde Z_t.$$

Hence we obtain, under $\tilde{\mathbb{P}}_T$, for every $0 \le t \le T$,

$$\bar\Gamma_\infty(t) = e^{(r+\delta)t}\left[\bar\Gamma_\infty(0) - \int_0^t e^{-(r+\delta)s}(c(s) + \delta B(s)) ds + \int_0^t e^{-(r+\delta)s}\big[\theta^\top(s)\sigma + g_\infty X_0(s)\sigma_y^\top\big] d\tilde Z_s\right]. \tag{66}$$

Setting $\Gamma^0_\infty(t) := e^{-(r+\delta)t}\bar\Gamma_\infty(t)$ the above (66) is rewritten as

$$\Gamma^0_\infty(t) = \Gamma^0_\infty(0) - \int_0^t e^{-(r+\delta)s}(c(s) + \delta B(s)) ds + \int_0^t e^{-(r+\delta)s}\big[\theta^\top(s)\sigma + g_\infty X_0(s)\sigma_y^\top\big] d\tilde Z_s, \tag{67}$$

which implies that the process $\Gamma^0_\infty(t)$ is a supermartingale under $\tilde{\mathbb{P}}_T$ on $[0,T]$. By the optional sampling theorem we then have, for every couple of stopping times $0 \le \tau_1 \le \tau_2 \le T$, and calling $\tilde{\mathbb{E}}_T$ the expectation under $\tilde{\mathbb{P}}_T$,

$$\tilde{\mathbb{E}}_T\left[\Gamma^0_\infty(\tau_2) | \mathcal{F}_{\tau_1}\right] \le \Gamma^0_\infty(\tau_1), \qquad \tilde{\mathbb{P}}_T - a.s. \tag{68}$$

The admissibility of the strategy π, and the fact that $\mathbb{P}$ and $\tilde{\mathbb{P}}_T$ are equivalent on $\mathcal{F}_T$, implies that $\Gamma^0_\infty(\tau_2) \ge 0$, $\tilde{\mathbb{P}}_T$-a.s., hence also

$$\tilde{\mathbb{E}}_T\left[\Gamma^0_\infty(\tau_2) | \mathcal{F}_{\tau_1}\right] \ge 0, \qquad \tilde{\mathbb{P}}_T - a.s.$$

Now let $\tau_1 := \tau \wedge T$ where τ is the first exit time of $(W_\pi(t), X(t))$ from $\mathcal{H}_{++}$ (which is taken to be identically 0 when $\Gamma_\infty(w,x) = 0$). Hence $\Gamma^0_\infty(\tau_1) = 0$ on $\{\tau < T\}$ and, from (68), we get

$$\mathbf{1}_{\{\tau<T\}}\tilde{\mathbb{E}}_T\left[\Gamma^0_\infty(\tau_2)|\mathcal{F}_{\tau_1}\right] = \tilde{\mathbb{E}}_T\left[\Gamma^0_\infty(\tau_2)\mathbf{1}_{\{\tau<T\}}|\mathcal{F}_{\tau_1}\right] = 0, \qquad \tilde{\mathbb{P}}_T - a.s.$$

and, consequently,

$$\Gamma^0_\infty(\tau_2)\mathbf{1}_{\{\tau<T\}} = 0, \qquad \tilde{\mathbb{P}}_T - a.s. \tag{69}$$

We now use (67) to compute $\Gamma^0_\infty(\tau_2) - \Gamma^0_\infty(\tau_1)$ getting

$$\Gamma^0_\infty(\tau_2) - \Gamma^0_\infty(\tau_1) = -\int_{\tau_1}^{\tau_2} e^{-(r+\delta)s}(c(s)+\delta B(s))ds + \int_{\tau_1}^{\tau_2} e^{-(r+\delta)s}\left[\theta^\top(s)\sigma + g_\infty X_0(s)\sigma_y^\top\right]d\tilde{Z}_s. \tag{70}$$

Again using the optional sampling theorem we get

$$\tilde{\mathbb{E}}_T\left[\Gamma^0_\infty(\tau_2)|\mathcal{F}_{\tau_1}\right] - \Gamma^0_\infty(\tau_1) = -\tilde{\mathbb{E}}_T\left[\int_{\tau_1}^{\tau_2} e^{-(r+\delta)s}(c(s)+\delta B(s))ds|\mathcal{F}_{\tau_1}\right], \qquad \tilde{\mathbb{P}}_T - a.s.$$

Hence, taking $\tau_2 \equiv T$

$$0 \le \mathbf{1}_{\{\tau<T\}}\tilde{\mathbb{E}}_T\left[\Gamma^0_\infty(T)|\mathcal{F}_{\tau_1}\right] = -\tilde{\mathbb{E}}_T\left[\int_0^T \mathbf{1}_{\{\tau<s\}}e^{-(r+\delta)s}(c(s)+\delta B(s))ds|\mathcal{F}_{\tau_1}\right], \qquad \tilde{\mathbb{P}}_T - a.s.$$

which implies

$$\mathbf{1}_{\{\tau<s\}}(\omega)c(s,\omega) = \mathbf{1}_{\{\tau<s\}}(\omega)B(s,\omega) = 0, \qquad ds\otimes\tilde{\mathbb{P}}_T - a.e.\ in\ [0,T]\times\Omega. \tag{71}$$

We now multiply (70) by $\mathbf{1}_{\{\tau<T\}}$ and we use (69) and (71) to get

$$0 = \int_0^{\tau_2} e^{-(r+\delta)s}\mathbf{1}_{\{\tau<s\}}\left[\theta^\top(s)\sigma + g_\infty X_0(s)\sigma_y^\top\right]d\tilde{Z}_s, \qquad \tilde{\mathbb{P}}_T - a.s. \tag{72}$$

Since the integral of the right hand side is a martingale the above implies that

$$\mathbf{1}_{\{\tau<s\}}\theta^\top(s)\sigma + g_\infty X_0(s)\sigma_y^\top = 0, \qquad ds\otimes\tilde{\mathbb{P}}_T - a.e.\ in\ [0,T]\times\Omega. \tag{73}$$

Using (69)-(71)-(73), that $\mathbb{P}$ and $\tilde{\mathbb{P}}_T$ are equivalent on $\mathcal{F}_T$, and the arbitrariness of T, we get the claim. □

4.4 The fundamental identity when $\gamma \in (0,1)$

We start with the following lemma.

Lemma 4.10. *Let $\gamma \in (0,1)$. For any initial condition $(w,x) \in \mathcal{H}_{++}$ and for any strategy $(c,B,\theta) \in \Pi(w,x)$, let τ be the first exit time of $(W_\pi(\cdot), X(\cdot))$ from $\mathcal{H}_{++}$. Then we have, for every $T > 0$,*

$$\mathbb{E}\left[e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)(T\wedge\tau)}\bar{v}\big(W_\pi((T\wedge\tau)), X((T\wedge\tau))\big)\right] \le \bar{v}(x,w). \tag{74}$$

Moreover,[3]

$$\lim_{T\to+\infty}\mathbb{E}\left[e^{-(\rho+\delta)(T\wedge\tau)}\bar{v}\big(W_\pi((T\wedge\tau)), X((T\wedge\tau))\big)\right] = 0. \tag{75}$$

[3]This is a sort of transversality condition, along the lines of what is usually done in the context of the maximum principle in infinite horizon problems.

Proof. Fix $(w,x)\in\mathcal{H}_+$ and $\pi=(c,B,\theta)\in\Pi(w,x)$. As in the previous proof, for readability, we will use the shorthand notation

$$W_\pi(s):=W^{w,x}(s;c,B,\theta),\qquad X(s):=X^x(s),$$

omitting the dependence on the controls and on the initial conditions.

Let then $\Gamma_\infty(w,x)>0$ and set τ be the first exit time of $(W_\pi(\cdot),X(\cdot))$ from $\mathcal{H}_{++}$. We apply Itô's formula to the process

$$e^{-(1-\gamma)\left((r+\delta)+\frac{\kappa^\top\kappa}{2\gamma}\right)t}\bar{v}\big(W_\pi(t),X(t)\big)$$

up to time τ. Since by definition

$$\bar{v}(w,x)=f_\infty^\gamma\frac{\Gamma_\infty^{1-\gamma}(w,x)}{1-\gamma}, \tag{76}$$

then, using, as in the previous proof, the shorthand notation

$$\bar{\Gamma}_\infty(t):=\Gamma_\infty\big(W_\pi(t),X(t)\big),$$

we have

$$e^{-(1-\gamma)\left((r+\delta)+\frac{\kappa^\top\kappa}{2\gamma}\right)t}\bar{v}\big(W_\pi(t),X(t)\big)=e^{-(1-\gamma)\left((r+\delta)+\frac{\kappa^\top\kappa}{2\gamma}\right)t}f_\infty^\gamma\frac{\bar{\Gamma}_\infty^{1-\gamma}(t)}{1-\gamma}.$$

We first apply the finite-dimensional Itô's formula (up to time τ) to the process $\frac{\bar{\Gamma}_\infty^{1-\gamma}(t)}{1-\gamma}$. Since the function $h(z)=\frac{z^{1-\gamma}}{1-\gamma}$ belongs to $C^2((0,+\infty),\mathbb{R})$ and not to $C^2(\mathbb{R},\mathbb{R})$, we use the Remark 2 to Theorem IV.3.3 of [43] which takes account of this case. We get, up to time τ,

$$\begin{aligned} d\frac{\bar{\Gamma}_\infty^{1-\gamma}(t)}{1-\gamma}=&\bar{\Gamma}_\infty^{-\gamma}(t)\left[(r+\delta)\bar{\Gamma}_\infty(t)-c(t)-\delta B(t)+\left(\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right)\kappa\right]dt\\ &-\frac{1}{2}\gamma\bar{\Gamma}_\infty^{-\gamma-1}\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]^\top\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]dt\\ &+\bar{\Gamma}_\infty^{-\gamma}(t)\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]dZ_t. \end{aligned}\tag{77}$$

Hence, up to time τ,

$$\begin{aligned} &d\left[e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)t}f_\infty^\gamma\frac{\bar{\Gamma}_\infty^{1-\gamma}(t)}{1-\gamma}\right]\\ =&e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)t}f_\infty^\gamma\Bigg\{-\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)\bar{\Gamma}_\infty^{1-\gamma}(t)\\ &+\bar{\Gamma}_\infty^{-\gamma}(t)\left[(r+\delta)\bar{\Gamma}_\infty(t)-\big(c(t)+\delta B(t)\big)+\left(\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right)\kappa\right]\\ &-\frac{1}{2}\gamma\bar{\Gamma}_\infty^{-\gamma-1}(t)\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]^\top\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]\Bigg\}dt\\ &+e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)t}f_\infty^\gamma\bar{\Gamma}_\infty^{-\gamma}(t)\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]dZ_t \end{aligned}\tag{78}$$

$$\begin{aligned} =&e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)t}f_\infty^\gamma\Bigg\{-\bar{\Gamma}_\infty^{-\gamma}(t)\big(c(t)+\delta B(t)\big)\\ &-\frac{1}{2\gamma}\bar{\Gamma}_\infty^{-\gamma-1}(t)\left[\bar{\Gamma}_\infty(t)\kappa-\gamma\Big(\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\Big)\right]^\top\left[\bar{\Gamma}_\infty(t)\kappa-\gamma\Big(\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\Big)\right]\Bigg\}dt\\ &+e^{-(1-\gamma)\left((r+\delta)+\frac{\kappa^\top\kappa}{2\gamma}\right)t}f_\infty^\gamma\bar{\Gamma}_\infty^{-\gamma}(t)\left[\theta^\top(t)\sigma+g_\infty X_0(t)\sigma_y^\top\right]dZ_t. \end{aligned}\tag{79}$$

The strategy $(c, B, \theta) \in \Pi(w, x)$, thus the drift in (79) is negative. It follows that, up to time τ, the process

$$e^{-(1-\gamma)\left((r+\delta)+\frac{\kappa^\top\kappa}{2\gamma}\right)t} f_\infty^\gamma \frac{\bar{\Gamma}_\infty^{1-\gamma}(t)}{1-\gamma} \tag{80}$$

is a local $\mathbb{F}$-supermartingale. Moreover, calling

$$\tau_N = \inf\left\{t \geq 0 : \bar{\Gamma}_\infty(t) \leq \frac{1}{N}\right\},$$

for N sufficiently big we have $\tau_N > 0$ and both the drift and the diffusion coefficient above are integrable. Hence $\bar{\Gamma}_\infty(t)$ is an integrable supermartingale up to τ_N. We then have, for every $T > 0$,

$$\mathbb{E}\left[e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)(T\wedge\tau_N)} f_\infty^\gamma \frac{\bar{\Gamma}_\infty^{1-\gamma}(T\wedge\tau_N)}{1-\gamma}\right] \leq f_\infty^\gamma \mathbb{E}\left(\frac{\bar{\Gamma}_\infty^{1-\gamma}(0)}{1-\gamma}\right). \tag{81}$$

Since $\tau_N \nearrow \tau$, sending N to $+\infty$ we get, using the definition of $\bar{v}$, the a.s. convergence:

$$e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)(T\wedge\tau_N)} f_\infty^\gamma \frac{\bar{\Gamma}_\infty^{1-\gamma}(T\wedge\tau_N)}{1-\gamma} \to e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)(T\wedge\tau)} \bar{v}(W_\pi(T\wedge\tau), X(T\wedge\tau)).$$

Since $\gamma \in (0,1)$, the integrand in the left hand side of (81) is always nonegative, hence we can apply Fatou's Lemma to get the claim (74).

Now we prove the claims (75). First, observe that, for every $T > 0$,

$$\mathbb{E}\left[e^{-(\rho+\delta)(T\wedge\tau)}\bar{v}\big(W_\pi(T\wedge\tau), X(T\wedge\tau)\big)\right] =$$
$$\mathbb{E}\left[\mathbf{1}_{\{\tau\leq T\}} e^{-(\rho+\delta)\tau}\bar{v}\big(W_\pi(\tau), X(\tau)\big)\right] + e^{-(\rho+\delta)T}\mathbb{E}\left[\mathbf{1}_{\{\tau>T\}}\bar{v}\big(W_\pi(T), X(T)\big)\right] =: I_1 + I_2. \tag{82}$$

Second, the term I_1 in (82) is zero since, by definition of τ and continuity of $\bar{\Gamma}$ we have $\bar{\Gamma}(\tau) = 0$.

Third, observe that the term I_2 converges to 0. Indeed

$$I_2 = e^{-\left[\rho+\delta-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)\right]T}\mathbb{E}\left[\mathbf{1}_{\{\tau>T\}} e^{-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)T}\bar{v}\big(W_\pi(T), X(T)\big)\right]$$
$$\leq e^{-\left[\rho+\delta-(1-\gamma)\left(r+\delta+\frac{\kappa^\top\kappa}{2\gamma}\right)\right]T}\bar{v}\big(w, x\big),$$

where in the last inequality we used (74). By Hypothesis 2.4-(ii) we immediately get that $I_2 \to 0$ as $T \to +\infty$. The claim (75) immediately follows. □

Proposition 4.11. *Let $\gamma \in (0,1)$. Take any initial condition $(w, x) \in \mathcal{H}_{++}$ and take any admissible strategy $\pi = (c, B, \theta) \in \Pi(w, x)$. Let τ be the first exit time of $(W_\pi(\cdot), X(\cdot))$ from $\mathcal{H}_{++}$. Then we have, for every $T > 0$,*

$$\bar{v}\big(w, x\big) = J(w, x; \pi)$$
$$+\mathbb{E}\int_0^\tau e^{-(\rho+\delta)s}\left\{\mathbb{H}_{max}\big(W_\pi(s), X(s), D\bar{v}(W_\pi(s), X(s)), D^2\bar{v}(W_\pi(s), X(s))\big)\right. \tag{83}$$
$$\left.-\mathbb{H}_{cv}\big(W_\pi(s), X(s), D\bar{v}(W_\pi(s), X(s)), D^2\bar{v}(W_\pi(s), X(s)); \pi(s)\big)\right\} ds.$$

Proof. Calling

$$\tau_N = \inf\left\{t \geq 0 : \bar{\Gamma}_\infty(t) \leq \frac{1}{N}\right\}$$

for N sufficiently big we have $\tau_N > 0$. Let us take such N from now on. We want to apply Itô's formula to the process

$$e^{-(\rho+\delta)s}\bar{v}\big(W_\pi(s), X(s)\big)$$

varying s between 0 and $T \wedge \tau_N$. We were not able to find, in the current literature, an Itô's formula which can be applied to this case. Indeed all results on Itô's formula in infinite dimension concern the case of functions defined over the whole space, which is not the case here. However it is not difficult to check that the proof of Proposition 1.164 in [21] can be easily generalized to this case as our process, on $[0, \tau_N]$, lives in the set $\left\{t \geq 0 : \bar{\Gamma}_\infty(t) \geq \frac{1}{N}\right\}$ and, on the same set the function $\bar{v}$ satisfies all required assumptions.[4] Hence we apply Proposition 1.164 in [21] obtaining

$$
\begin{aligned}
&e^{-(\rho+\delta)(T\wedge\tau_N)}\bar{v}\big(W_\pi(T\wedge\tau_N), X(T\wedge\tau_N)\big) - \bar{v}\big(w,x\big) = \\
&= \int_0^{T\wedge\tau_N} e^{-(\rho+\delta)s}\left\{-(\rho+\delta)\bar{v}\big(W_\pi(s), X(s)\big)\right. \\
&\quad + \bar{v}_w\big(W_\pi(s), X(s)\big)\big[(r+\delta)W_\pi(s) + \theta^\top(s)(\mu - r\mathbf{1}) + X_0(s) - c(s) - \delta B(s)\big] \\
&\quad + \langle AX(s), \bar{v}_x\big(W_\pi(s), X(s)\big)\rangle_{M_2} + \frac{1}{2}\bar{v}_{ww}\big(W_\pi(s), X(s)\big)\theta^\top\sigma\sigma^\top\theta \\
&\quad \left. + \frac{1}{2}\bar{v}_{x_0x_0}\big(W_\pi(s), X(s)\big)\sigma_y^\top\sigma_y X_0^2(s) + \bar{v}_{wx_0}\big(W_\pi(s), X(s)\big)\theta^\top(s)\sigma\sigma_y X_0(s)\ \right\} ds \\
&\quad + \int_0^{T\wedge\tau_N} e^{-(\rho+\delta)s}\left\{\bar{v}_w\big(W_\pi(s), X(s)\big)\theta^\top(s)\sigma + \bar{v}_{x_0}\big(W_\pi(s), X(s)\big)X_0(s)\sigma_y^\top\right\} dZ_s.
\end{aligned}
\tag{84}
$$

By Proposition 4.7, the function $\bar{v}$ solves the HJB equation (44) thus, for any $s \in [0, T\wedge\tau_N]$ we get

$$
(\rho+\delta)\bar{v}(W_\pi(s), X(s)) = \mathbb{H}\big(W_\pi(s), X(s), D\bar{v}(W_\pi(s), X(s)), D^2\bar{v}(W_\pi(s), X(s))\big).
$$

By this last observation, and recalling (47) and (48), the equality in (84) can be rewritten as

$$
\begin{aligned}
&e^{-(\rho+\delta)(T\wedge\tau_N)}\bar{v}\big(W_\pi(T\wedge\tau_N), X(T\wedge\tau_N)\big) - \bar{v}\big(w,x\big) = \\
&= -\int_0^{T\wedge\tau_N} e^{-(\rho+\delta)s}\left\{\mathbb{H}_{max}\big(W_\pi(s), X(s), D\bar{v}(W_\pi(s), X(s)), D^2\bar{v}(W_\pi(s), X(s))\big)\right. \\
&\qquad \left. -\mathbb{H}_{cv}\big(W_\pi(s), X(s), D\bar{v}(W_\pi(s), X(s)), D^2\bar{v}(W_\pi(s), X(s)); \pi(s)\big)\right\} ds \\
&\quad - \int_0^{T\wedge\tau_N} e^{-(\rho+\delta)s}\left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta\frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma}\right) ds \\
&\quad + \int_0^{T\wedge\tau_N} e^{-(\rho+\delta)s}\left\{\bar{v}_w\big(W_\pi(s), X(s)\big)\theta^\top(s)\sigma + \bar{v}_{x_0}\big(W_\pi(s), X(s)\big)X_0(s)\sigma_y^\top\right\} dZ_s.
\end{aligned}
\tag{85}
$$

By definition of $\mathbb{H}_{max}$ and $\mathbb{H}_{cv}$, we have

$$
\mathbb{H}_{max} - \mathbb{H}_{cv} \geq 0. \tag{86}
$$

Moreover the integral with respect to the Brownian motion in (85) is a martingale. Taking the

[4] To avoid using such a generalization of Itô's formula it is possible to argue differently: one can use first the equation (61) given in Lemma 4.9, for the process $\bar{\Gamma}$ and then apply the one-dimensional Itô's formula (see e.g. Remark 2 to Theorem IV.3.3 of [43]) to the process $e^{-(\rho+\delta)s}f_\infty\bar{\Gamma}(s)^{1-\gamma}$. Using then the HJB equation one would get the same result. We prefer not to use this path as the procedure would be less clear and intuitive for the reader.

expectation we then get

$$
\begin{aligned}
&\mathbb{E}\left[e^{-(\rho+\delta)(T\wedge\tau_N)}\bar{v}\big(W_\pi(T\wedge\tau_N),X(T\wedge\tau_N)\big)\right]-\bar{v}\big(w,x\big)=\\
&=-\,\mathbb{E}\int_0^{T\wedge\tau_N} e^{-(\rho+\delta)s}\left\{\mathbb{H}_{max}\big(W_\pi(s),X(s),D\bar{v}(W_\pi(s),X(s)),D^2\bar{v}(W_\pi(s),X(s))\big)\right.\\
&\qquad\left.-\mathbb{H}_{cv}\big(W_\pi(s),X(s),D\bar{v}(W_\pi(s),X(s)),D^2\bar{v}(W_\pi(s),X(s));\pi(s)\big)\right\}ds\\
&\quad-\,\mathbb{E}\int_0^{T\wedge\tau_N} e^{-(\rho+\delta)s}\left(\frac{c(s)^{1-\gamma}}{1-\gamma}+\delta\frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma}\right)ds.
\end{aligned}
\tag{87}
$$

Now we let $N\to+\infty$. The first expectation converges thanks to the dominated convergence theorem. Moreover the integrands on the right hand side are both nonnegative. Thus we can apply to them the monotone convergence theorem. Their limits must be finite since also the left hand side is finite. Hence

$$
\begin{aligned}
&\bar{v}\big(w,x\big)=\mathbb{E}\int_0^{T\wedge\tau} e^{-(\rho+\delta)s}\left(\frac{c(s)^{1-\gamma}}{1-\gamma}+\delta\frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma}\right)ds\\
&+\mathbb{E}\int_0^{T\wedge\tau} e^{-(\rho+\delta)s}\left\{\mathbb{H}_{max}\big(W_\pi(s),X(s),D\bar{v}(W_\pi(s),X(s)),D^2\bar{v}(W_\pi(s),X(s))\big)\right.\\
&\qquad\left.-\mathbb{H}_{cv}\big(W_\pi(s),X(s),D\bar{v}(W_\pi(s),X(s)),D^2\bar{v}(W_\pi(s),X(s));\pi(s)\big)\right\}ds\\
&+\,\mathbb{E}\left[e^{-(\rho+\delta)(T\wedge\tau)}\bar{v}\big(W_\pi(T\wedge\tau),X(T\wedge\tau)\big)\right].
\end{aligned}
\tag{88}
$$

Let now $T\to+\infty$. The last term above converges to 0 thanks to (75). The two integrals converge again thanks to monotone convergence and their limits are finite since the left hand side is finite. Hence we get

$$
\begin{aligned}
&\bar{v}\big(w,x\big)=\mathbb{E}\int_0^{\tau} e^{-(\rho+\delta)s}\left(\frac{c(s)^{1-\gamma}}{1-\gamma}+\delta\frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma}\right)ds\\
&+\mathbb{E}\int_0^{\tau} e^{-(\rho+\delta)s}\left\{\mathbb{H}_{max}\big(W_\pi(s),X(s),D\bar{v}(W_\pi(s),X(s)),D^2\bar{v}(W_\pi(s),X(s))\big)\right.\\
&\qquad\left.-\mathbb{H}_{cv}\big(W_\pi(s),X(s),D\bar{v}(W_\pi(s),X(s)),D^2\bar{v}(W_\pi(s),X(s));\pi(s)\big)\right\}ds
\end{aligned}
\tag{89}
$$

By the definition of τ and Lemma 4.9-(ii) it follows that

$$
J(w,x;\pi)=\mathbb{E}\int_0^{\tau} e^{-(\rho+\delta)s}\left(\frac{c(s)^{1-\gamma}}{1-\gamma}+\delta\frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma}\right)ds<+\infty
$$

and the claim follows. □

By the statement of Proposition 4.11 we get the following.

Corollary 4.12. *Let $\gamma\in(0,1)$. Then the value function V is finite on $\mathcal{H}_+$ and $V(w,x)\le\bar{v}(w,x)$ for every $(w,x)\in\mathcal{H}_+$.*

Proof. It is enough to observe that the integrand in (83) is positive, hence, we have, for every $(w,x)\in\mathcal{H}_+$ and $\pi=(c,B,\theta)\in\Pi(w,x)$,

$$
\bar{v}(w,x)\ge J(w,x;\pi).
$$

Since $V(w,x) = \sup_{\pi\in\Pi(w,x)} J(w,x;\pi)$, the claim immediately follows. □

Remark 4.13. *Observe that, from the proof the above Proposition 4.11, we easily obtain that the fundamental identity* (83) *holds when, in place of* $\bar{v}$, *we put any classical solution* v *of the HJB equation* (44) *which satisfies* (τ *being the first exit time from* $\mathcal{H}_{++}$),

$$\lim_{T\to+\infty} \mathbb{E}\left[e^{-(\rho+\delta)(T\wedge\tau)} v\big(W_\pi(T\wedge\tau), X(T\wedge\tau)\big)\right] = 0. \tag{90}$$

A sufficient condition for such limit to hold is the following: $v = 0$ *on* $\partial\mathcal{H}_{++}$ *and* $|v(w,x)| \le C(1+|w|^{1-\gamma}+|x|^{1-\gamma})$ *on* $\mathcal{H}_{++}$. *The latter indeed allows us to prove that* (75) *holds for* v *with essentially the same proof provided in Lemma 4.10 for* $\bar{v}$.
The above implies that Corollary 4.12 holds for v, *too. This proves a one-side comparison result: every classical solution of the HJB equation* (44), *satisfying* (90), *is larger than the value function* V.
One may then ask if classical solutions (satisfying the boundary condition $v = 0$ *on* $\partial\mathcal{H}_{++}$*) are unique. In state constraints problem like this one, the answer is, in general, negative (see e.g. [3, Remark 6.2] for a simple example in one dimension). The best one can expect in such cases is one-side comparison results of the type just mentioned (see e.g. [32, 45]).*
The reason of this fact lies exactly in the presence of state constraints. Indeed, if no constraints are present, we can take any classical solution v *of the HJB equation and use identity* (83) *for such* v. *It is clear that, if we find an admissible control such that the integrand in* (83) *is zero, then this control is optimal and we have* $v = V$. *Without state constraints this would be true possibly under further regularity assumptions (in this case, for example, by requiring* $v_w > 0$ *and* $v_{ww} < 0$*) to give sense to the feedback map and obtain a solution of the closed loop equation. In the presence of state constraints, however, it may happen that for any regular solution* v *we find a control that makes the integrand in* (83) *vanish but is not admissible.*
Clearly this leaves the possibility that, in this particular case, uniqueness hold in some form. However, this is not needed to solve our problem as we will see in the next subsections.

4.5 Verification Theorem and optimal feedbacks when $\gamma \in (0,1)$

Now we proceed to show that $\bar{v} = V$ and to find the optimal strategies in feedback form.

First, we provide the following definitions.

Definition 4.14. *Fix* $(w,x) \in \mathcal{H}_+$. *A strategy* $\bar{\pi} := \big(\bar{c}, \bar{B}, \bar{\theta}\big)$ *is called an optimal strategy if* $\big(\bar{c}, \bar{B}, \bar{\theta}\big) \in \Pi(w,x)$, *and if it achieves the supremum in (43), i.e.*

$$V(w,x) = \mathbb{E}\left(\int_0^{+\infty} e^{-(\rho+\delta)s}\left(\frac{\bar{c}(s)^{1-\gamma}}{1-\gamma} + \delta\frac{\big(k\bar{B}(s)\big)^{1-\gamma}}{1-\gamma}\right) ds\right). \tag{91}$$

Definition 4.15. *We say that a function* $(\boldsymbol{C}, \boldsymbol{B}, \Theta) : \mathcal{H}_+ \longrightarrow \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ *is an optimal feedback map if, for every* $(w,x) \in \mathcal{H}_+$ *the closed loop equation*

$$\begin{cases} dW(t) = & \big[(r+\delta)W(t) + \Theta^\top(W(t),X(t))(\mu - r) + X_0(t) - \boldsymbol{C}(W(t),X(t)) - \delta\boldsymbol{B}(W(t),X(t))\big]\,dt \\ & +\Theta^\top(W(t),X(t))\,\sigma dZ(t), \\ W(0) = & w, \end{cases} \tag{92}$$

coupled with the second of (41), i.e.

$$dX(t) = AX(t) + (CX(t))^\top dZ_t, \qquad X(0) = x, \tag{93}$$

has a unique solution (W^*, X), *and the associated control strategy* $\big(\bar{c}, \bar{B}, \bar{\theta}\big)$

$$\begin{cases} \bar{c}(t) & := \boldsymbol{C}(W^*(t), X(t)), \\ \bar{B}(t) & := \boldsymbol{B}(W^*(t), X(t)), \\ \bar{\theta}(t) & := \Theta(W^*(t), X(t)) \end{cases} \tag{94}$$

is an optimal strategy.

Recalling that

$$\Gamma_\infty(w,x) := w + g_\infty x_0 + \langle h_\infty, x_1\rangle, \tag{95}$$

we define the map

$$\begin{cases} \mathbf{C}_f(w,x) & := f_\infty^{-1}\Gamma_\infty(w,x), \\ \mathbf{B}_f(w,x) & := k^{-b} f_\infty^{-1}\Gamma_\infty(w,x), \\ \Theta_f(w,x) & := (\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1})\frac{\Gamma_\infty(w,x)}{\gamma} - (\sigma\sigma^\top)^{-1}\sigma\sigma_y g_\infty x_0 \\ & = \frac{1}{\gamma}\Gamma_\infty(w,x)(\sigma^\top)^{-1}\kappa - g_\infty x_0 (\sigma^\top)^{-1}\sigma_y. \end{cases} \tag{96}$$

Observe that this map is obtained taking the maximum points of the Hamiltonian given in (50) and substituting, in place of p_1, P_{11}, P_{12} the derivatives $\bar{v}_w, \bar{v}_{ww}, \bar{v}_{w,x_0}$, respectively.

We aim to prove that this is an optimal feedback map. For given $(w,x) \in \mathcal{H}_+$, denote with $W_f^*(t)$ the unique solution of the associated closed loop equation (coupled with (93)),

$$\begin{cases} dW(t) = & \Big[(r+\delta)W(t) + \Theta_f^\top\left(W(t),X(t)\right)(\mu - r\mathbf{1}) + X_0(t) - \mathbf{C}_f\left(W(t),X(t)\right) - \delta\mathbf{B}_f\left(W(t),X(t)\right)\Big]\,dt \\ & +\Theta_f^\top(t)\sigma dZ(t), \\ W(0) = & w, \end{cases} \tag{97}$$

and set

$$\Gamma_\infty^*(t) = \Gamma_\infty\big(W_f^*(t), X(t)\big) = W_f^*(t) + g_\infty X_0(t) + \langle h_\infty, X_1(t)\rangle. \tag{98}$$

The control strategy associated with (96) is

$$\begin{cases} \bar{c}_f(t) := \mathbf{C}_f\left(W_f^*(t), X(t)\right) = f_\infty^{-1}\Gamma_\infty^*(t), \\ \bar{B}_f(t) := \mathbf{B}_f\left(W_f^*(t), X(t)\right) = k^{-b} f_\infty^{-1}\Gamma_\infty^*(t), \\ \bar{\theta}_f(t) := \Theta_f\left(W_f^*(t), X(t)\right) = \dfrac{\Gamma_\infty^*(t)}{\gamma}(\sigma^\top)^{-1}\kappa - g_\infty X_0(t)(\sigma^\top)^{-1}\sigma_y. \end{cases} \tag{99}$$

Note that $\Gamma_\infty^*(t)$ is the total wealth associated to the strategy $(\bar{c}_f, \bar{B}_f, \bar{\theta}_f)$. We need the following lemma which is needed to show the admissibility of such strategy.

Lemma 4.16. *Let $(w,x) \in \mathcal{H}_+$. The process Γ_∞^* defined in (98) is a stochastic exponential, and it has dynamic*

$$d\Gamma_\infty^*(t) = \Gamma_\infty^*(t)\Big(r + \delta + \frac{1}{\gamma}\kappa^\top\kappa - f_\infty^{-1}\big(1 + \delta k^{-b}\big)\Big)dt + \frac{\Gamma_\infty^*(t)}{\gamma}\kappa^\top dZ(t). \tag{100}$$

Proof. Substituting (99) into the equation (97) for $W_f^*(t)$, we get

$$\begin{aligned} dW_f^*(t) = & \Big\{W_f^*(t)(r+\delta) + \Gamma_\infty^*(t)\big[\frac{\kappa^\top\kappa}{\gamma} - f_\infty^{-1}\big(1+\delta k^{-b}\big)\big] + X_0(t) - g_\infty X_0(t)\sigma_y^\top\kappa\Big\}dt \\ & + \left\{\frac{\Gamma_\infty^*(t)}{\gamma}\kappa^\top - g_\infty X_0(t)\sigma_y^\top\right\}dZ(t). \end{aligned} \tag{101}$$

Then, recalling (63) we have

$$\begin{aligned} & d\langle (g_\infty, h_\infty), (X_0(t), X_1(t))\rangle_{M_2} = \\ & \quad = \big[\big(\mu_y + \beta\big)g_\infty X_0(t) - X_0(t) + (r+\delta)\langle h_\infty, X_1(t)\rangle\big]dt + g_\infty X_0(t)\sigma_y^\top dZ_t, \end{aligned} \tag{102}$$

We thus obtain, similarly to (64),

$$
\begin{aligned}
d\Gamma^*_\infty(t) &= dW^*_f(t) + d\langle (g_\infty, h_\infty), \big(X_0(t), X_1(t)\big)\rangle_{M_2} \\
&= \bigg\{ W^*_f(t)(r+\delta) + \Gamma^*_\infty(t)\left[\frac{\kappa^\top\kappa}{\gamma} - f_\infty^{-1}\big(1+\delta k^{-b}\big)\right] + X_0(t) - g_\infty X_0(t)\sigma_y^\top\kappa \\
&\quad + (\mu_y+\beta)g_\infty X_0(t) - X_0(t) + (r+\delta)\langle h_\infty, X_1(t)\rangle \bigg\} dt \\
&\quad + \left\{ \frac{\Gamma^*_\infty(t)}{\gamma}\kappa^\top - g_\infty X_0(t)\sigma_y^\top + g_\infty X_0(t)\sigma_y^\top \right\} dZ(t) \\
&= \Gamma^*_\infty(t)\left[(r+\delta) + \frac{\kappa^\top\kappa}{\gamma} - f_\infty^{-1}\big(1+\delta k^{-b}\big)\right] dt + \frac{\Gamma^*_\infty(t)}{\gamma}\kappa^\top dZ(t),
\end{aligned}
\tag{103}
$$

where the last equality follows by noting that $\mu_y + \beta - \sigma_y^\top\kappa = r + \delta$. The claim is proved. □

Theorem 4.17 (Verification Theorem and Optimal feedback Map, $\gamma \in (0,1)$)**.** *Let $\gamma \in (0,1)$. We have $V = \bar{v}$ in $\mathcal{H}_+$. Moreover the function $(\boldsymbol{C}_f, \boldsymbol{B}_f, \Theta_f)$ defined in (96) is an optimal feedback map. Finally, for every $(w,x) \in \mathcal{H}_+$ the strategy $\bar{\pi}_f := (\bar{c}_f, \bar{B}_f, \bar{\theta}_f)$ is the unique optimal strategy.*

Proof. First take $(w,x) \in \partial\mathcal{H}_+ = \{\Gamma_\infty = 0\}$. In this case we have, by equation (100), for every $t \geq 0$, $\Gamma^*_\infty(t) = 0$, $\mathbb{P}$-a.s.. This implies, by (99), that it must be

$$
\bar{c}_f \equiv 0, \qquad \bar{B}_f \equiv 0, \qquad \bar{\theta}_f \equiv -g_\infty X_0(t)(\sigma^\top)^{-1}\sigma_y.
$$

Hence, by Lemma 4.9-(ii), we get that this is the only admissible strategy (see (62)), hence it is optimal.

Now take $(w,x) \in \mathcal{H}_{++} = \{\Gamma_\infty > 0\}$. First we observe that $(\bar{c}_f, \bar{B}_f, \bar{\theta}_f)$ is an admissible strategy. Indeed by Lemma 4.16 $\Gamma^*_\infty(\cdot)$ is a stochastic exponential, it is therefore $\mathbb{P}$-a.s. strictly positive for any strictly positive initial condition $\Gamma^*_\infty(0) = \Gamma_\infty(w,x)$. This implies, in particular, that the constraint in (37) is always satisfied and, consequently, that the couple $(\bar{c}_f, \bar{B}_f)$ is non negative, which is enough to prove admissibility.

Concerning optimality we observe that, as recalled above, the feedback map is obtained taking the maximum points of the Hamiltonian given in (96) and substituting, in place of p_1, P_{11}, P_{12} the derivatives $\bar{v}_w, \bar{v}_{ww}, \bar{v}_{w,x_0}$, respectively.

This implies that, substituting the strategy $\bar{\pi}_f := (\bar{c}_f, \bar{B}_f, \theta_f)$ in the fundamental identity (83) we obtain

$$
\bar{v}(w,x) = J\left(w,x;\bar{\pi}_f\right).
$$

Hence, using Corollary 4.12 and the definition of the value function, we get

$$
V(w,x) \leq \bar{v}(w,x) = J\left(w,x;\bar{\pi}_f\right) \leq V(w,x),
$$

which immediately gives $V(w,x) = J\left(w,x;\bar{\pi}_f\right)$, hence the required optimality.

We now prove uniqueness. When $(w,x) \in \partial\mathcal{H}_+$ the claim follows from Lemma 4.9-(ii). When $(w,x) \in \mathcal{H}_{++}$, the claim follows from the fundamental identity (83). Indeed, since $\bar{v} = V$ if a given strategy π is optimal at $(w,x) \in \mathcal{H}_{++}$ it must satisfy $\bar{v}(w,x) = J(w,x;\pi)$, which implies, substituting in (83), that the integral in (83) is zero. This implies that, on $[0,\tau]$ we have $\pi = \bar{\pi}_f$, $dt \otimes \mathbb{P}$-a.e. This gives uniqueness, as, for $t > \tau$, we still must have $\pi = \bar{\pi}_f$, $dt \otimes \mathbb{P}$-a.e., due to Lemma 4.9-(ii). □

4.6 The case $\gamma > 1$

We now treat the case when $\gamma > 1$. We cannot follow the same path as done in the case $\gamma \in (0,1)$. Indeed the proof of the crucial limiting condition (75) in Lemma 4.10 does not work as it is, since

Fatou's Lemma cannot be applied here. Indeed in such proof, we use Fatou's Lemma with the liminf inequality: this requires a uniform bound from below which we are not able to prove here.[5]

We have to follow a different path. We start by looking at the value function V. We already know that $-\infty \le V(w,x) \le 0$ for every $(w,x) \in \mathcal{H}_+$ and that, by Lemma 4.9-(ii), $V(w,x) = -\infty$ for every $(w,x) \in \partial\mathcal{H}_+$.

To prove that $V(w,x) > -\infty$ on $\mathcal{H}_{++}$ it is enough to find an admissible strategy π such that $J(w,x;\pi) > -\infty$. This is given in the following proposition.

Proposition 4.18. *Let $\gamma > 1$ and let $(w,x) \in \mathcal{H}_{++}$. The strategy $\bar{\pi}_f := (\bar{c}_f, \bar{B}_f, \bar{\theta}_f)$ defined in* (99) *is admissible at (w,x). Moreover*

$$V(w,x) \ge J(w,x;\bar{\pi}_f) = \bar{v}(w,x) > -\infty. \tag{104}$$

Proof. The admissibility follows from the proof of Theorem 4.17 since the constraint to be satisfied is the same as in the case $\gamma \in (0,1)$. The validity of (104) is achieved by direct computation, using (99) and the fact that $\Gamma^*_\infty(\cdot)$ is a stochastic exponential as from Lemma 4.16. □

The above Proposition 4.18 implies that

$$\bar{v}\big(w,x\big) = J(w,x;\bar{\pi}_f) \le V(w,x) \le 0.$$

We now want to prove that

$$\bar{v}\big(w,x\big) \ge V(w,x).$$

To do this we look closely at the value function.

Proposition 4.19 (Homogeneity of V)**.** *The value function in $\mathcal{H}_{++}$ satisfies the following*

$$V(w,x) = \eta \frac{\Gamma_\infty(w,x)^{1-\gamma}}{1-\gamma}, \qquad \text{for some } \eta \ge 0. \tag{105}$$

Proof. We divide the proof in two steps.

Step I. If, for $(w_1,x_1),(w_2,x_2) \in \mathcal{H}_{++}$, we have $\Gamma_\infty(w_1,x_1) = \Gamma_\infty(w_2,x_2)$, then it must be

$$V(w_1,x_1) = V(w_2,x_2).$$

Indeed let $\pi_1 := (c_1, B_1, \theta_1) \in \Pi(w_1,x_1)$. Consider the strategy $\pi_2 := (c_2, B_2, \theta_2)$ where $c_2 = c_1$, $B_2 = B_1$, and

$$\theta_2(t)^\top \sigma + g_\infty X_0^{x_2}(t)\sigma_y^\top = \theta_1(t)^\top \sigma + g_\infty X_0^{x_1}(t)\sigma_y^\top.$$

With this choice of π_1, π_2 it is clear that

$$\Gamma_\infty(W^{w_1,x_1}(t;\pi_1), X^{x_1}(t)) = \Gamma_\infty(W^{w_2,x_2}(t;\pi_2), X^{x_2}(t)), \quad t \text{ a.e. and } \mathbb{P}\text{-a.s.}$$

since both processes satisfy equation (61) with the same initial condition. Hence we also have $\pi_2 \in \Pi(w_2,x_2)$. Moreover we clearly have

$$J(w_1,x_1;\pi_1) = J(w_2,x_2;\pi_2).$$

Since this construction can be done for every $\pi_1 \in \Pi(w_1,x_1)$ we immediately get $V(w_1,x_1) \le V(w_2,x_2)$. The same argument also applies to show the opposite inequality, hence the claim follows.

Step II. Homogeneity. Since, by the previous Proposition 4.18, $0 \ge V > -\infty$ then we must have, for some $f : \mathbb{R}_+ \to \mathbb{R}_-$,

$$V(w,x) = f(\Gamma_\infty(w,x)), \qquad \forall (w,x) \in \mathcal{H}_{++}. \tag{106}$$

[5]In this respect it seems that the proof of Proposition 4.26 in [13] (case ($\gamma < 0$) is not completely correct as it uses Fatou's lemma in the wrong direction.

Now, from the homogeneity of the objective functional J and from the linearity of the state equation and of the constraints we get that f must be $(1-\gamma)$-homogeneous. Indeed, let $(w,x) \in \mathcal{H}_{++}$ and $\pi \in \Pi(w,x)$. For $a>0$ we have, by linearity,

$$W^{aw,ax}(t;a\pi) = aW^{w,x}(t;\pi), \quad and \quad X^{ax}(t) = aX^{x}(t).$$

Hence, by linearity of Γ_∞, it must be $a\pi \in \Pi(aw,ax)$, so $a\Pi(w,x) \subseteq \Pi(aw,ax)$. With the same argument we can prove that also $a\Pi(w,x) \supseteq \Pi(aw,ax)$, hence the two sets are equal. We then have

$$V(aw,ax) = \sup_{\pi\in\Pi(w,x)} J((aw,ax);a\pi) = a^{1-\gamma} \sup_{\pi\in\Pi(w,x)} J((w,x);\pi) = a^{1-\gamma}V(w,x).$$

From the above we immediately get that the function f in (106) is $(1-\gamma)$-homogeneous, hence a power. Since $V \le 0$ we must have $\eta \ge 0$ in (105). The claim is proved. □

Proposition 4.20 (Dynamic Programming Principle)**.** *For any stopping time τ with respect to $\mathbb{F}$, the value function V satisfies the dynamic programming principle*

$$V(w,x) = \sup_{(c,B,\theta)\in\Pi(w,x)} \mathbb{E}\Bigg\{ \int_0^\tau e^{-(\rho+\delta)s}\left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta\frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma}\right)ds + e^{-(\rho+\delta)\tau}V\left(W^{w,x}(\tau;c,B,\theta),X^{x}(\tau)\right)\Bigg\}. \tag{107}$$

Proof. See Theorem 3.70 in [21]. The only differences are that, in such theorem, one has:

- the horizon is finite;
- the current cost is assumed to be state dependent and uniformly bounded in the control.

The first difference is easily overcome by standard shift arguments as is done, e.g. in Section 2.4 of [21]. The second difference can be resolved observing that, in the proof of Theorem 3.70 in [21], such boundedness is used to apply dominated convergence inside the integral (see equation (3.166), p.242 of [21]. In our case, due to the specific form of the functional, we can apply monotone convergence to get the same result. □

Proposition 4.21. *The value function V is a classical solution of the HJB equation* (44) *in $\mathcal{H}_{++}$.*

Proof. The proofs uses exactly the same arguments of the proof of Theorem 2.41 in [21]. □

Now we substitute the explicit expression (105) into the HJB equation (44). It is immediate to get that we have equality only in two cases: either when $V \equiv \bar{v}$ or when $V \equiv 0$. We now show that we can exclude the second possibility. First we give a simple lemma.

Lemma 4.22. *Let $\gamma > 1$, let $(w,x) \in \mathcal{H}_{++}$ and let $\pi := (c,B,\theta) \in \Pi(w,x)$ be such that $J(w,x;\pi) > -\infty$. Let $\bar{\Gamma}_\infty(t) := \Gamma_\infty(W_\pi(t),X(t))$. Let τ be the first exit time of the process $\bar{\Gamma}_\infty(\cdot)$ from the open set $\mathcal{H}_{++}$. Then it must be $\mathbb{P}(\tau=+\infty)=1$.*

Proof. Assume by contradiction that $\mathbb{P}(\tau=+\infty)<1$. Then, for some $T_1>0$ we would have $\mathbb{P}(\tau \le T_1)>0$. By Lemma 4.9-(ii), on the set $\{\tau \le T_1\}$ we have

$$\int_{T_1}^{+\infty} e^{-(\rho+\delta)s}\left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta\frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma}\right)ds = -\infty.$$

Hence also $J(w,x;\pi) = -\infty$, a contradiction. □

Proposition 4.23. *The value function V is not always 0 in $\mathcal{H}_{++}$.*

Proof. Assume by contradiction that $V \equiv 0$ over $\mathcal{H}_{++}$. Then by dynamic programming principle (107) we get that, for every $(w,x) \in \mathcal{H}_{++}$ and $t \geq 0$

$$0 = \sup_{(c,B,\theta)\in\Pi(w,x)} \mathbb{E}\left\{ \int_0^t e^{-(\rho+\delta)s} \left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{\big(kB(s)\big)^{1-\gamma}}{1-\gamma} \right) ds \right\}. \tag{108}$$

In particular we fix $T > 0$ and we choose, for every $n \in \mathbb{N}$, (w_n, x_n) such that $\Gamma_\infty(w_n, x_n) < 1/n$, $t_n = T$, and $(c_n, B_n, \theta_n) \in \Pi(w_n, x_n)$ such that

$$-\frac{1}{n} < \mathbb{E}\left\{ \int_0^T e^{-(\rho+\delta)s} \left(\frac{c_n(s)^{1-\gamma}}{1-\gamma} + \delta \frac{\big(kB_n(s)\big)^{1-\gamma}}{1-\gamma} \right) ds \right\} < 0. \tag{109}$$

Now we use equation (67) for $t = T$, $(c,B,\theta) = (c_n, B_n, \theta_n)$, and take the expectation $\tilde{\mathbb{E}}_T$ under $\tilde{\mathbb{P}}_T$ (defined in (65)). In this way, using the previous lemma, the stochastic integral in (67) disappears and we get

$$\tilde{\mathbb{E}}_T \left\{ \int_0^T e^{-(r+\delta)s} \left(c_n(s) + \delta B_n(s) \right) ds \right\} \leq \Gamma_\infty(w_n, x_n) < 1/n. \tag{110}$$

This implies that the sequences c_n and B_n, since they are positive, converge to 0 in $L^1(\Omega \times [0,T], d\mathbb{P} \otimes dt)$. Hence there exist subsequences c_{n_k} and B_{n_k} which converge a.e. to 0 in $d\mathbb{P} \otimes dt$. Since $\gamma > 1$, this implies that the subsequences $c_{n_k}^{1-\gamma}/(1-\gamma)$ and $B_{n_k}^{1-\gamma}/(1-\gamma)$ converge a.e. to $-\infty$ in $d\mathbb{P} \otimes dt$. This contradicts (108), so the claim follows. □

The next, final, theorem, is then a straightforward consequence of the results of the present subsection.

Theorem 4.24 (Verification Theorem and Optimal feedback Map, $\gamma > 1$)**.** *Let $\gamma > 1$. We have $V = \bar{v}$ in $\mathcal{H}_+$. Moreover the function $(\boldsymbol{C}_f, \boldsymbol{B}_f, \Theta_f)$ defined in (96) is an optimal feedback map. Finally, for every $(w,x) \in \mathcal{H}_+$ the strategy $\bar{\pi}_f := (\bar{c}_f, \bar{B}_f, \bar{\theta}_f)$ is the unique optimal strategy.*

Remark 4.25. *Observe that, similarly to the case $\gamma \in (0,1)$ (see Remark 4.13), also in this case from the proof the above Proposition 4.11, we easily get that the fundamental identity* (83) *holds when, in place of $\bar{v}$, we put any classical solution v of the HJB equation* (44) *which satisfies (τ being the first exit time from $\mathcal{H}_{++}$),*

$$\lim_{T\to+\infty} \mathbb{E}\left[e^{-(\rho+\delta)(T\wedge\tau)} v\big(W_\pi(T\wedge\tau), X(T\wedge\tau)\big) \right] = 0.$$

This would provide a one-side comparison result also in this case. However, due to the asymmetry explained at the beginning of this subsection, this condition may not be easy to check, so such comparison result may be less interesting here. Moreover, as we said in Remark 4.13, we do not expect here uniqueness to hold, in general. This doe not affect the solution of our problem.

5 Discussion and extensions

Going back to the portfolio choice problem presented in Section 2, we recall that the pair $(X_0(t), X_1(t))$ can be identified with $(y(t), y(t+s)_{s\in[-d,0]})$, where $y(t)$ denotes labor income at time t, and $y(t+s)_{s\in[-d,0]}$ is the path of labor income between time t and d units of time in the past. We can then summarize the results of the previous sections in the following theorem, which also states the results for the original problem (8) by considering the original reference filtration $\mathbb{G}$ and the admissible control space $\overline{\Pi}$ which is defined as the set of all controls in $\overline{\Pi}_0$ whose pre-death counterpart satisfies the state constraint (14).

Theorem 5.1. *The value function V of Problem 4.1 is given by*

$$V(w, x_0, x_1) = \frac{f_\infty^\gamma \left(w + g_\infty x_0 + \int_{-d}^0 h_\infty(s) x_1(s)\, ds\right)^{1-\gamma}}{1-\gamma}, \tag{111}$$

where f_∞ is defined in (56) and (g_∞, h_∞) in (34). Moreover for every $(w, x) \in \mathbb{R} \times M_2$ there exists a unique optimal strategy $\pi^* = (c^*, B^*, \theta^*) \in \Pi_0$ starting at (w, x). Such strategy can be represented as follows. Denote total wealth by

$$\Gamma_\infty^*(t) := W^*(t) + g_\infty y(t) + \int_{-d}^0 h_\infty(s) y(t+s)\, ds, \tag{112}$$

where $W^*(\cdot)$ is the solution of equation (41) with initial datum w and control π^*, whereas $y(\cdot)$ is the solution of the second equation in (11) with datum $x = (x_0, x_1) \in M_2$. Then, Γ_∞^* has dynamics

$$d\Gamma_\infty^*(t) = \Gamma_\infty^*(t)\Big(r + \delta + \frac{\kappa^\top \kappa}{\gamma} - f_\infty^{-1}\big(1 + \delta k^{-b}\big)\Big) dt + \frac{\Gamma_\infty^*(t)}{\gamma} \kappa^\top dZ(t), \tag{113}$$

and the optimal strategy triplet $\pi^* = (c^*, B^*, \theta^*)$ for Problem 4.1 is given by

$$\begin{aligned} c^*(t) &:= f_\infty^{-1} \Gamma_\infty^*(t), \\ B^*(t) &:= k^{-b} f_\infty^{-1} \Gamma_\infty^*(t), \\ \theta^*(t) &:= \frac{\Gamma_\infty^*(t)}{\gamma} (\sigma^\top)^{-1} \kappa - g_\infty y(t) (\sigma^\top)^{-1} \sigma_y. \end{aligned} \tag{114}$$

Finally, there exists a unique $\mathbb{G}$-adapted optimal strategy $\overline{\pi}^* = (\overline{c}^*, \overline{B}^*, \overline{\theta}^*) \in \overline{\Pi}$ coinciding with $\pi^* = (c^*, B^*, \theta^*) \in \Pi$ on $\{\tau_\delta \geq t\}$. The optimal controls in the original filtration $\mathbb{G}$ are given by

$$\overline{c}^*(t) = 1_{\{\tau_\delta \geq t\}} c^*(t), \qquad \overline{B}^*(t) = 1_{\{\tau_\delta \geq t\}} B^*(t), \qquad \overline{\theta}^*(t) = 1_{\{\tau_\delta \geq t\}} \theta^*(t),$$

with associated total and financial wealth given by

$$\overline{\Gamma}^*(t) = 1_{\{\tau_\delta \geq t\}} \Gamma^*(t), \qquad \overline{W}^*(t) = 1_{\{\tau_\delta \geq t\}} W^*(t),$$

respectively.

To understand the optimal solution, we note that quantity (112) represents the agent's total wealth, given by the sum of financial wealth and human capital. In line with [7, 17], the agent considers the capitalized value of future wages as if they were a traded asset. The solution follows the logic of Merton [37], in that the agent chooses constant fractions of total wealth to consume and leave as bequest. The same would apply to the risky assets allocation if the agent's labor income were uncorrelated with the financial market. As it is instead instantaneously perfectly correlated with the risky assets, a negative income hedging demand arises (the term $-g_\infty y(t)(\sigma^\top)^{-1}\sigma_y$) reducing the allocation to the risky assets accordingly ([10]). The riskier the human capital, the less aggressive the agent's asset allocation.

We look now at the relation with the benchmark model with no delay, i.e. when $\phi = 0$. First, we observe that the dynamics of Γ_∞^* is not influenced by the path dependent component of the model in the sense that, *ceteris paribus*, changing ϕ (and hence y) leaves the dynamics of the total wealth unchanged at each time. Then, calling $\widetilde{\Gamma}_\infty$ the total wealth when $\phi = 0$, we have, for any initial point (w, x)

$$\Gamma_\infty^*(0) - \widetilde{\Gamma}_\infty(0) = x_0 \left(\frac{1}{\beta - \beta_\infty} - \frac{1}{\beta}\right) + \langle h_\infty, x_1 \rangle$$

so that we have

$$\Gamma_\infty^*(t) - \widetilde{\Gamma}_\infty(t) = \left[x_0 \left(\frac{1}{\beta - \beta_\infty} - \frac{1}{\beta}\right) + \langle h_\infty, x_1 \rangle\right] e^{\left(r + \delta + \frac{\kappa^\top \kappa}{\gamma}(1 - (2\gamma)^{-1}) - f_\infty^{-1}\left(1 + \delta k^{-b}\right)\right)t + \frac{\kappa^\top}{\gamma} Z(t)}.$$

This means that, when $\beta_\infty \geq 0$ and $\langle h_\infty, x_1 \rangle \geq 0$ (which are both true when $\phi(s) \geq 0$ for every $s \in [-d, 0]$), the total capital and hence the optimal consumption level c^* and bequest target B^* are larger than what they would be in the non path-dependent case (i.e. when $\phi \equiv 0$). This is a consequence of the predictable, past component of labor income shaping human capital and hence total wealth. The situation is less clear cut for the risky asset allocation θ^*, as there is a complex interplay between risk preferences and financial market parameters. Indeed, denoting by Θ_ϕ the feedback map associated with θ^* (see (96)), we have

$$\Theta_\phi(w, x) - \Theta_0(w, x) = (\sigma^\top)^{-1} \left[\left(\frac{1}{\beta - \beta_\infty} - \frac{1}{\beta} \right) x_0 \left(\frac{\kappa}{\gamma} - \sigma_y \right) + \langle h_\infty, x_1 \rangle \frac{\kappa}{\gamma} \right]. \tag{115}$$

The result suggests a very rich set of empirical predictions on risky asset allocations depending on risk preferences, financial market parameters, and the relative contribution of the past vs. future component of human wealth. In the special case of $\beta_\infty = 0$, for example, we have that the wedge between Θ_ϕ and Θ_0 is entirely driven by the past component of human capital, as the negative hedging demand appearing in both Θ_ϕ and Θ_0 only depends on the present component of human capital, and not on the capitalized market value of the labor income's past trajectory. The latter can tilt the asset allocation above or below the baseline optimum resulting in the case of no path dependency.

It is clear that our results could provide even richer empirical predictions in more realistic settings. The introduction of a fixed retirement date,[6] for example, would allow the relative importance of the past vs. future component of labor income to change as the retirement date approaches. This could generate a hump shaped pattern in the risky asset allocation, which would be consistent with empirical evidence often treated as a puzzle or more recently explained by assuming stock prices to be cointegrated with labor income ([5]). The model discussed here could then offer an interesting way to reconcile theory and empirical observation within a tractable setting. The solution of the finite horizon version of the model is the object of current further research.

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