

# **Robust Economic Model Predictive Control: recursive feasibility, stability and average performance**

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This dissertation is submitted for the degree of  
*Doctor of Philosophy*

November 2019



I would like to dedicate this thesis to my loving families  
for their encouragement and support.



## **Declaration of Originality**

I hereby confirm that this thesis is the result of my own research work. Any ideas and results of other people have been properly referenced.

Zihang Dong  
November 2019



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## **Acknowledgements**

Firstly, I would like to express my genuine gratitude to my supervisor Dr. David Angeli for the continuous support of my PhD research, for his motivation, patience, and immense knowledge. His guidance helped me in all the time of the research and writing of publications. I could not have imagined having a better supervisor for my project.

I would also like to thank Dr. Matthias Müller and Dr. Eric Kerrigan for kindly accepting to be examiners for my VIVA. In particular, my sincere thanks goes to Dr. Eric Kerrigan, who taught us Model Predictive Control in MSc at Imperial College. From his lectures, I not only learned many concepts and methods, but I also developed a strong interest in Model Predictive Control.

I also want to thank my colleagues at Control and Power Group (CAP) for the stimulating discussions, for the days and nights we were working together, and for all the fun we have had in the past three years. You create an enjoyable and unique working atmosphere and I am looking forward to having future interaction.

I gratefully acknowledge the Department of Electrical Engineering for the financial support of the scholarship that made my PhD study a reality.

Finally, I would like to express my sincere gratitude to my families for their love, patience and support. Without them, this work would never have come into existence.



# Abstract

This thesis is mainly concerned with designing algorithms for Economic Model Predictive Control (EMPC), and analysis of its resulting recursive feasibility, stability and asymptotic average performance.

In particular, firstly, in order to extend and unify the formulation and analysis of economic model predictive control for general optimal operation regimes, including steady-state or periodic operation, we propose the novel concept of a “control storage function” and introduce upper and lower bounds to the best asymptotic average performance for nonlinear control systems based on suitable notions of dissipativity and controlled dissipativity. As a special case, when the optimal operation is periodic, we present a new approach to formulate terminal cost functions.

Secondly, aiming at designing a robust EMPC controller for nonlinear systems with general optimal regimes of operation, we present a tube-based robust EMPC algorithm for discrete-time nonlinear systems that are perturbed by disturbance inputs. The proposed algorithm minimizes a modified economic objective function, which considers the worst cost within a tube around the solution of the associated nominal system. Recursive feasibility and an a-priori upper bound to the closed-loop asymptotic average performance are ensured. Thanks to the use of dissipativity of the nominal system with a suitable supply rate, the closed-loop system under the proposed controller is shown to be asymptotically stable, in the sense that it is driven to an optimal robust invariant set.

Thirdly, for the purpose of combining robust EMPC design with a state observer in a single pure economic optimization problem, we consider homothetic tube-based EMPC synthesis for constrained linear discrete time systems. Since, in practical systems, full state measurement is seldom available, the proposed method integrates a moving horizon estimator to achieve closed-loop stability and constraint satisfaction despite system disturbances and output measurement noise. In contrast to existing approaches, the worst cost within a single

homothetic tube around the solution of the associated nominal system is minimized, which at the same time tightens the bound on the set of potential states compatible with past output and input data. We show that the designed optimization problem is recursively feasible and adoption of homothetic tubes leads to less conservative economic performance bounds. In addition, the use of strict dissipativity of the nominal system guarantees asymptotic stability of the resulting closed-loop system.

Finally, to deal with the unknown nonzero mean disturbance and the presence of plant-model error, we propose a novel economic MPC algorithm aiming at achieving optimal steady-state performance despite the presence of plant-model mismatch or unmeasured nonzero mean disturbances. According to the offset-free formulation, the system's state is augmented with disturbances and transformed into a new coordinate framework. Based on the new variables, the proposed controller integrates a moving horizon estimator to determine a solution of the nominal system surrounded by a set of potential states compatible with past input and output measurements. The worst cost within a single homothetic tube around the nominal solution is chosen as the economic objective function which is minimized to provide a tightened upper bound for the accumulated real cost within the prediction horizon window. Thanks to the combined use of the nominal system and homothetic tube, the designed optimization problem is recursively feasible and less conservative economic performance bounds are achieved.

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# Notation

## Symbols

$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^n$	The set of all real valued $n$ -dimensional (column) vectors
$\mathbb{R}^{n \times m}$	The set of all $m \times n$ matrices with real entries
$\mathbb{I}$	The set of all integer numbers
$\mathbb{R}_{\geq a} (\mathbb{I}_{\geq 0})$	The real numbers (integers) greater or equal to $a$
$\mathbb{I}_{[a, b]}$	The integers $\{a, a + 1, \dots, b\}$ for some $a \in \mathbb{I}$ and $b \in \mathbb{I}_{\geq a}$
$I_n$	The $n \times n$ identity matrix
$O_{n \times m}$	The $n \times m$ null matrix, with the dimensions omitted when defined by the context

## Operators

$\oplus$	Minkowski sum of two sets, that is $X \oplus Y := \{x + y \in \mathbb{R}^n : x \in X, y \in Y\}$
$\ominus$	Minkowski difference of two sets, that is $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$
$ x $	The Euclidean norm of $x$
$ x _{\Pi}$	The distance of a point $x \in \mathbb{R}^n$ to a set $\Pi$ , that is $ x _{\Pi} := \min_{y \in \Pi}  x - y $

## Functions

$\mathcal{K}$	A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is of <i>class</i> $\mathcal{K}$ , if it is strictly increasing and satisfies $\alpha(0) = 0$
$\mathcal{K}_\infty$	A continuous function $\alpha(s)$ is of <i>class</i> $\mathcal{K}_\infty$ , if it belongs to <i>class</i> $\mathcal{K}$ , defined for all $s \geq 0$ and satisfies $\alpha(s) \rightarrow +\infty$ as $s \rightarrow +\infty$
$\mathcal{KL}$	A continuous function $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$ is a <i>class</i> $\mathcal{KL}$ function if $\beta(\cdot, t)$ is a <i>class</i> $\mathcal{K}$ function for every fixed $t$ , and $\beta(s, \cdot)$ is strictly decreasing with $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$ for every fixed $s$

### Acronyms / Abbreviations

MPC	Model Predictive Control
EMPC	Economic Model Predictive Control
MHE	Moving Horizon Estimation
RPI	Robust Positively Invariant
mRPI	minimal Robust Positively Invariant
FNS	Fixed Nominal State
VNS-AC	Variable Nominal State with Additional Constraint
VNS-IWF	Variable Nominal State with Initial Weighting Function
HT	Homothetic Tube
RT	Rigid Tube
LO	Luenberger Observer

# Chapter 1

## Introduction

### 1.1 Stabilizing MPC

The development of Model Predictive Control (MPC) or Receding Horizon Control (RHC) can be traced back to the work of Kalman in the early 1960s when a linear quadratic regulator (LQR) was designed to minimise an unconstrained quadratic cost function of states and inputs. Although LQR control has powerful stabilising properties, it has some limitations in real-world implementations such as the infinite horizon in the objective function, the absence of constraints in the formulation and the ability to handling nonlinear property of the real system. After many years of revolution, the applications of MPC, whose strengths are the ability to handle constraints directly in its framework and satisfaction of some optimal performance criteria by solving online optimization problems, become successfully popular in industry [1].

In the MPC dynamic optimization problem, the future state trajectory of the system is predicted using the process model, and based on this prediction, an accumulated cost which is usually the deviation between the predicted values and the reference signals is minimized. This optimization is solved online by taking into account the current conditions, process operation and safety constraints. Then the first move of this optimal input sequence is injected into the system, and based on the measurements and information about the process model and the disturbances, the next state of the system is estimated. This estimate is used to call the dynamic optimizer again and the feedback loop repeats.

## 1.2 Economic MPC

In many practical applications, the main goal of designing advanced controllers is to operate the system at its most profitable regime of operation, while guaranteeing stability [1]. Typically, in process industries, this objective is achieved based on a hierarchical architecture which is composed of two optimization-based layers. The upper layer, usually referred to as real-time optimization (RTO), is dedicated to computing the economically optimal steady-state. Then, this optimal operation point is sent to the MPC layer that is used to determine a control action aiming at stabilizing the closed-loop system to the desired set-point as closely as possible. Although the system can be regulated to achieve the optimal equilibrium, the hierarchical separation brings two main disadvantages. The first one is that the objective of the MPC controllers is designed in the form of positive definite tracking error without considering the economic costs incurred during transients. On the other hand, model uncertainties and unknown disturbances can cause offset problems and unsuccessful determination of the actual economically optimal equilibrium.

In recent years, however, an alternative approach, Economic MPC (EMPC), has looked into the issue of directly addressing economic optimization in real-time, and to this end, adopts cost functionals which are not required to be positive definite with respect to an equilibrium point (refer to [2–11] and references therein on EMPC for deterministic systems). In spite of this, the implementation mechanism of EMPC is the same as that of traditional MPC, that is, both adopt the receding horizon approach to achieve the system's closed-loop state (or output) feedback control.

However, the closed-loop system of directly optimizing the economic performance shows potentially complex behaviour because the cost is chosen as dictated by economics and may take an arbitrary form. To analyse the closed-loop behaviour and economic performance of Economic MPC controllers, various tools have been proposed [2–6, 12–15]. An EMPC controller without terminal constraints [4, 6, 12] is presented and analyzed based on a turnpike property and suitable controllability assumptions. Practical stability and approximate optimality in the transient phase of this controller are analyzed. The benefits of omitting terminal constraints are: (i) a simpler optimization problem and (ii) a larger feasible set. However, recursive feasibility and asymptotic average performance bounds are not straightforward. In particular, explicit controllability assumptions (asymptotic controllability concerning stage cost  $\ell$  [4], local and finite time controllability [7]) are required compared with a setting with terminal constraints where such properties follow a posteriori within the feasible set. In the

context of traditional MPC [16], three ingredients are elaborated for enforcement of recursive feasibility and stability, consisting of terminal cost, terminal constraint and local controller, respectively. In analogy, similar tools have been proposed for Economic MPC and have allowed feasibility, stability and performance analysis of the closed-loop system. These approaches require the introduction of strong duality or dissipativity assumptions to convert the economic cost function to positive definite functions, in combination with the terminal equality or inequality constraints so that recursive feasibility and closed-loop stability of the economically optimal steady-state are established by using a rotated stage cost in an auxiliary optimization problem [2, 3, 5].

Notice the crucial role that the turnpike property and dissipativity have in Economic MPC. In particular, the relations between these two properties and the optimality of steady-state operation have been discussed in several publications: the equivalence between dissipativity and the turnpike property under suitable assumptions is elaborated in [17]. The sufficiency of dissipativity for optimal steady-state operation is obtained in [3], and the necessity, under a mild additional controllability assumption, is proved in [9].

Apart from the stabilization of the optimal feasible setpoint, a priori bounds on the closed-loop economic performance are also remarkable features of EMPC. When considering long time horizons, the infinite horizon averaged performance is a useful criterion. In the work [5], EMPC algorithms with terminal constraints result in a closed-loop asymptotic average performance which is no worse than the optimal steady-state cost, whereas, in the framework without terminal constraints, In [4], the author shows that closed-loop averaged cost approximates the optimal steady-state cost with an error term which vanishes as the prediction horizon grows to infinity. Nevertheless, the infinite horizon averaged performance does not necessarily enhance profitability during the transient phase. Hence, non-averaged performance criteria have also been proposed, see references [12, 18] in the context of EMPC, with and without terminal constraints, respectively.

Although dissipativity provides an elegant method to design EMPC controllers with terminal constraints, alternative approaches have been proposed. A new generalized terminal constraint where the terminal state can be a free steady-state instead of the optimal steady-state in the optimization process is studied [19]. The main advantage of this approach, compared to a fixed terminal constraint, is that the feasible region is enlarged. Based on the generalized terminal equality constraint, several update rules for the self-tuning terminal weight are illustrated in [8]. Furthermore, the closed-loop asymptotic average performance bounds can

be improved if the generalized terminal equality is relaxed by adopting regional constraints as in [20].

In general, however, optimal regimes of operation (from an economic point of view) are not limited to steady-state behaviours but may have complex nature, periodic operation and even non-periodic optimal regimes could sometimes arise. The work [21] proposes two economically oriented EMPC formulations, a periodic terminal equality constraint and an infinite horizon method with discounting factor. To achieve lower computational burden and larger feasibility region, EMPC using a multi-step scheme without terminal constraints are analyzed and near optimal performance of the closed-loop system is established [22]. For time-varying discrete-time linear systems, in [23], an artificial periodic trajectory is introduced which is considered as the optimization variable to minimize the average economic cost without knowing the optimal periodic regime a priori. In addition, the authors of [10] suggest three solutions of the resulting terminal cost functions under different assumptions to induce convergence for the time-varying discrete-time closed-loop system. For the continuous-time system, the time-varying dissipativity property implies the convergence to the time-dependent steady-states [11]. Accordingly, the notions of dissipativity that ensure practical asymptotic stability of the optimal periodic orbit are extended [24, 25]. For a further non-stationary mode of optimal operation that a system is optimally operated in a certain subset of the state space, an economic MPC scheme with optimized terminal equality constraint is proposed and a dissipativity condition with a parametric storage function guarantees the convergence to the set of optimal operation [26].

In summary, the above discussion shows some existing literature on economic MPC methods for different optimal regimes of operation. However, there is no literature on designing EMPC controllers with time-invariant terminal regional constraints for discrete-time nonlinear systems. Motivated by this issue, in this thesis, we develop a novel economic MPC controllers and will discuss in details in Chapter 2.

### 1.3 Robust Economic MPC

While traditional EMPC approaches require exact knowledge of the plants' dynamics, many practical application scenarios are inherently affected by uncertainties. If disturbances are affecting the real system, predicted states will be different from actual states and the determined optimal control may even violate constraints and cause the optimization problem

at the next time step to become infeasible. To deal with uncertainties, various approaches have been proposed in the literature. In [13], a min-max EMPC approach for linear systems is discussed which recasts the underlying optimization problem as a second-order cone program to improve the computational efficiency of its solution. In [14, 27], the authors use the concept of robust invariant set and adopt the standard tube-based MPC scheme to an EMPC setting by averaging the economic cost function within the invariant tube, but only the bound for expectation of the close-loop performance is evaluated instead of a robust guarantee. Another stream of research, which explicitly considers the stochastic information of the disturbance, includes chance-constrained stochastic EMPC [28] and multi-stage EMPC [29].

Tube-based EMPC formulates a state-feedback control policy of the form  $u = v + K(x - z)$ , where  $v$  is the nominal control input,  $u$  is the applied input to the true plant,  $x$  and  $z$  are the true and nominal states of the system. The feedback gain  $K$  is a pre-defined parameter and used to compute the rigid tube which is a robust positively invariant set under the feedback affine policy. In contrast to the tube-based tracking MPC, the economic framework employs a modified economic stage cost on which the optimal equilibrium is defined in [14, 27].

Unless the full state information is directly available at every time instant, an observer should be used to estimate the states. Besides, for constrained systems, a direct application of the separation principle, which obtains the control inputs based on current estimates from observers, such as high gain observer, Kalman filtering, or Moving Horizon Estimation (MHE), is not possible as the estimation error may cause constraint violation and thus no recursive feasibility is ensured. For instance, in the framework of tracking MPC, state estimation and optimal control problems are usually formulated separately, as in [30, 31], where a robust output feedback control policy is adopted. Moreover, when the estimation problem is separated from the controller design, the resulting estimate is normally optimal in the sense that it maximizes a criterion that captures the likelihood of the estimate for the given measurements. Thus, this optimal state estimate may not be beneficial from the economic point of view.

In [32–34], the approach proposed is to solve both the MPC and MHE problems simultaneously in a single min-max optimization problem under the assumption of existence of a saddle-point without requiring recursive feasibility. The authors additionally derive that existence of saddle-point equilibrium for a linear system with quadratic cost is satisfied if the system is observable and weights in the cost function are appropriately chosen. To satisfy recursive feasibility, output-feedback MPC scheme based on set-valued estimation from finite past measurements is established and its complexity and conservatism are addressed [35].

Furthermore, within the above-mentioned literature, the mathematical models used for prediction are assumed to be able to match the true plant dynamics and system disturbances are zero-mean and bounded. Therefore, under mild assumptions, the feedback control policy obtained using economic MPC controllers can drive the actual or nominal system's state (respectively in a deterministic or robust setup) to the economically optimal equilibrium. However, when there is an unknown constant disturbance entering the process or if a plant-model error is present, the closed-loop system may converge to a suboptimal steady-state. To overcome such difficulties, some authors propose approaches using multi-model linear offset-free formulations [36, 37], by adopting a finite family of linear models describing the behaviour of the plant in different operation points, where recursive feasibility, convergence to the economic set-point and stability are ensured. Another remedy is to augment the system's state with offset-free disturbances and adopt an economic steady-state modifier-adaptation strategy [38, 39], but the state estimation and optimal control problems are formulated separately and use the current estimated state and disturbance in prediction, which may cause constraint violation.

In summary, although the above discussed literature develop many economic MPC methods to deal with uncertainties, there are still many open problems to address in the field of robust economic MPC. These include (i) the development and convergence analysis of suitable economic MPC schemes; (ii) a thorough investigation of robust bound for the closed-loop asymptotic performance; (iii) development of optimization-based controller with pure economic cost and desired closed-loop performance which integrates both state estimation and controller synthesis problems. Chapter 3 - 5 will answer these questions and more details in these chapters will be specified in the next section.

## 1.4 Outlines and contributions

In the following, we specify in more detail the contributions and the outline of this thesis.

### Chapter 2

Chapter 2 designs EMPC controllers with time-invariant terminal constraints for discrete-time nonlinear systems other than those which are optimally operated at steady-state. In particular, the contribution of this chapter are: (i) propose a novel notion of “control storage function”

(CSF) suitable as a terminal cost within a control invariant set regardless of the nature of the underlying optimal regime of operation (ii) closed-loop performance and stability analysis under an arbitrary optimal regime of operation (iii) construction of terminal penalty functions for generalized optimal regime of operation (iv) relate the CSF to a periodic CSF and employ it as the terminal cost function when the optimal regime of operation is periodic.

## Chapter 3

Motivated by the approaches in [14, 27], which present an asymptotic average performance bound and stability results for systems with optimal steady-state operation under some additional constraints on the storage function that could restrict the feasibility region, Chapter 3 designs a robust EMPC controller which is able to achieve closed-loop asymptotic stability for nonlinear systems with general optimal regimes of operation and a robust bound of the long run average cost. In particular, the main contributions include: (i) it is in fact possible to design a tube-based robust EMPC algorithm by constructing the optimization problem using the initial nominal sequences and a weighting function on nominal initial state. There is no need to have additional constraints based on the storage function limiting the selection of the nominal initial state. (ii) Recursive feasibility and closed-loop asymptotic stability hold not only for steady-state operation but also for optimal periodic operation, provided that dissipativity with respect to nominal dynamics is fulfilled with a suitable supply rate. (iii) The bound on asymptotic average performance, in the case of optimal periodic operation, depends on whether the components of the optimal robust invariant set intersect.

## Chapter 4

Chapter 4 shows several contributions in the design of robust economic MPC together with state estimation using MHE: (i) a homothetic tube-based robust EMPC algorithm combined with a MHE state observer is proposed and the resulting recursive feasibility is guaranteed; (ii) closed-loop asymptotic stability holds both for optimal steady-state and periodic operation, provided that dissipativity with respect to nominal dynamics is fulfilled with a suitable supply rate; (iii) the use of the homothetic tube provides a less conservative robust economic performance bound.

## Chapter 5

Chapter 5 proposes the formulation of an optimization based controller with pure economic cost achieving offset-free for the optimal equilibrium, while ensuring recursive feasibility and providing an asymptotic average performance evaluation. In particular, the main contributions include: (i) the closed-loop operation using our proposed controller can achieve an offset-free optimal steady-state; (ii) we can design a homothetic tube-based robust Economic MPC algorithm integrated with moving horizon estimation state observer to guarantee recursive feasibility; (iii) the use of the homothetic tube provides a less conservative robust economic performance bound.

## 1.5 Publications

### Conference papers

- Dong, Z. and Angeli, D., 2017. A generalized approach to economic model predictive control with terminal penalty functions. *IFAC-PapersOnLine*, 50(1), pp.518-523.
- Dong, Z. and Angeli, D., 2018, December. Tube-Based Robust Economic Model Predictive Control on Dissipative Systems with Generalized Optimal Regimes of Operation. In *2018 IEEE Conference on Decision and Control (CDC)* (pp. 4309-4314). IEEE.

### Journal papers

- Dong, Z. and Angeli, D., 2018. Analysis of economic model predictive control with terminal penalty functions on generalized optimal regimes of operation. *International Journal of Robust and Nonlinear Control*, 28(16), pp.4790-4815.
- Dong, Z. and Angeli, D. Homothetic tube-based robust Economic MPC with integrated Moving Horizon Estimation. Submitted to *IEEE Transactions on Automatic Control* (Under the second-round review)
- Dong, Z. and Angeli, D. Homothetic tube-based robust offset-free Economic Model Predictive Control. (Prepared to submit)

## Chapter 2

# Economic MPC with generalized optimal regimes of operation

This chapter is organized as follows. Problem formulations of the deterministic nonlinear system are described in Section 2.1. Section 2.2 provides an estimate to some upper and lower bounds for the system's asymptotic average performance. The extension of EMPC formulation to general optimal operation regimes by adopting these new concepts and the resulting closed-loop analysis are addressed in Section 2.3.1-2.3.3. Section 2.3.4 elaborates the design of control storage function and Section 2.3.5 discusses a special optimal regime of operation which is periodic. In Section 2.3.6, the MATLAB toolbox SOSTOOLS is introduced to approximate the storage function. Several examples and counterexamples illustrating the features of the proposed tools are included in Section 2.4. Section 2.5 summarizes this chapter.

The results presented in this chapter are based on [40, 41].

### 2.1 Problem setup

Throughout this chapter, we consider finite dimensional discrete-time nonlinear control systems described by difference equations

$$x_{t+1} = f(x_t, u_t) \tag{2.1}$$

with state  $x_t \in \mathbb{X} \subset \mathbb{R}^n$ , input  $u_t \in \mathbb{U} \subset \mathbb{R}^m$ , at time  $t \in \mathbb{I}_{\geq 0}$ , and a continuous state transition map  $f : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{X}$ . Together with system (2.1), let us consider a time-invariant, nonlinear and continuous stage cost given as

$$\ell(x, u) : \mathbb{Z} \rightarrow \mathbb{R} \quad (2.2)$$

where  $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$  is a compact set capturing the pointwise-in-time state and input constraints which our system is subject to:

$$(x_t, u_t) \in \mathbb{Z}, \quad \forall t \in \mathbb{I}_{\geq 0}. \quad (2.3)$$

Our goal is to enhance profitability by minimizing the economic costs incurred in the long term system operation:

$$\sum_t \ell(x_t, u_t). \quad (2.4)$$

To this end, we need to identify a viable subset of the state space and corresponding control actions. As is well known, the notion of a control invariant set is crucial in this respect.

**Definition 2.1.** A control invariant set is any non-empty set  $\bar{\mathbb{X}} \subseteq \mathbb{X}$ , such that

$$\forall x \in \bar{\mathbb{X}}, \exists u : f(x, u) \in \bar{\mathbb{X}} \text{ and } (x, u) \in \mathbb{Z}. \quad (2.5)$$

The non-empty and continuous set-valued map of all admissible control inputs which keeps the system state inside  $\bar{\mathbb{X}}$  is denoted as:

$$\bar{\mathbb{U}}(x) := \{u \mid (x, u) \in \mathbb{Z} \text{ and } f(x, u) \in \bar{\mathbb{X}}\}. \quad (2.6)$$

The set of state and corresponding admissible input pairs is given as:

$$\bar{\mathbb{Z}} := \bigcup_{x \in \bar{\mathbb{X}}} [\{x\} \times \bar{\mathbb{U}}(x)]. \quad (2.7)$$

**Remark 2.1.** We consider a closed control invariant set  $\bar{\mathbb{X}} \subseteq \mathbb{X}$ . Accordingly, constraints (2.3) can be strengthened as follows:

$$(x_t, u_t) \in \bar{\mathbb{Z}}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (2.8)$$

and viability is guaranteed whenever the system is initialized in  $\bar{\mathbb{X}}$ . Notice that investigating state trajectories within a control invariant set is also a fundamental step in standard MPC Lyapunov stability analysis.

For later use, it will also be convenient to define an additional control invariant set as in the assumption below.

**Assumption 2.1.** *There exists a closed control invariant set  $\mathbb{X}_f \subseteq \bar{\mathbb{X}}$ .*

The set of all admissible control input which keeps the system's state inside  $\mathbb{X}_f$  is defined for all  $x \in \mathbb{X}_f$  as:

$$\mathbb{U}_f(x) := \{u \in \bar{\mathbb{U}}(x) \mid f(x, u) \in \mathbb{X}_f\}. \quad (2.9)$$

The set of state and corresponding admissible input pairs is given as:

$$\mathbb{Z}_f := \bigcup_{x \in \mathbb{X}_f} [\{x\} \times \mathbb{U}_f(x)]. \quad (2.10)$$

## 2.2 Dissipativity and control storage functions

Design of Economic MPC control schemes is intimately related to the identification of the “long run” optimal regime of operation of the system as in references [42–44]. In order to have a grasp of the system long-run optimal average performance, three quantities  $\ell_{av}^*$ ,  $\underline{\ell}$  and  $\bar{\ell}$  are explicitly defined below and will be discussed throughout this section.

**Definition 2.2.** *Let  $x \in \bar{\mathbb{X}}$  be a given initial state, then the best average asymptotic cost is defined as:*

$$\begin{aligned} \ell_{av}^*(x) := & \inf_{u(\cdot)} \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T}. \end{aligned} \quad (2.11)$$

$$\begin{aligned} & x_0 = x \\ & x_{t+1} = f(x_t, u_t) \\ & (x_t, u_t) \in \bar{\mathbb{Z}} \\ & \forall t \in \mathbb{I}_{\geq 0} \end{aligned}$$

Moreover, we denote by  $\ell_{av}^* = \inf_{x \in \bar{\mathbb{X}}} \ell_{av}^*(x)$ .

Notice that it is challenging to provide sufficient conditions for the infimums in the infinite horizon optimal control problems to be obtained, and, in particular, to determine the properties of  $\ell_{av}^*(\cdot)$ . Throughout this paper, we tentatively assume the infimums are attainable and denote the limit set of the optimal solution, achieving  $\ell_{av}^*$ , as  $(x_{s,i}, u_{s,i})$  such that  $x_{s,i+1} = f(x_{s,i}, u_{s,i})$  for all  $i \in \mathbb{I}_{\geq 0}$  and  $f(\cdot, \cdot)$  is as in (2.1). If this set is a singleton or has finite number  $P \in \mathbb{I}_{\geq 2}$  of elements, the system has optimal steady-state operation or optimal  $P$  periodic operation. In any case, the optimality of the optimal solution implies

$$\liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} \geq \lim_{T \rightarrow +\infty} \frac{\sum_{i=0}^{T-1} \ell(x_{s,i}, u_{s,i})}{T}, \quad (2.12)$$

for any feasible solution  $(x_{0:+\infty}, u_{0:+\infty})$ .

For convenience, the solution set is explicitly denoted as the closure of the countably infinite set

$$\Pi := \text{cl} \{(x_{s,0}, u_{s,0}), (x_{s,1}, u_{s,1}), \dots, (x_{s,i}, u_{s,i}), \dots\}, \quad (2.13)$$

and the projection of  $\Pi$  on  $\bar{\mathbb{X}}$  is

$$\Pi_{\bar{\mathbb{X}}} := \text{cl} \{x_{s,0}, x_{s,1}, \dots, x_{s,i}, \dots\}, \quad i \in \mathbb{I}_{\geq 0}, \quad (2.14)$$

where  $x_{s,i}$  is an element in the complete separable metric space  $\Pi_{\bar{\mathbb{X}}}$ .

Recall the notion of dissipativity as given in [3, Definition 4.1]

**Definition 2.3.** A discrete time system is dissipative with respect to a supply rate  $s : \bar{\mathbb{Z}} \rightarrow \mathbb{R}$  if there is a continuous storage function  $\lambda : \bar{\mathbb{X}} \rightarrow \mathbb{R}$  such that:

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u) \quad (2.15)$$

for all  $(x, u) \in \bar{\mathbb{Z}}$ . If, in addition, a positive definite function  $\rho : \bar{\mathbb{X}} \rightarrow \mathbb{R}_{\geq 0}$  exists such that:

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(x) + s(x, u), \quad (2.16)$$

then the system is said to be strictly dissipative.

Alternatively, given the role of dissipativity in providing lower bounds to the best asymptotic performance, one may consider the following quantity.

**Definition 2.4.** The quantity  $\underline{\ell}$  (later referred to as the “tightest lower bound to the optimal asymptotic cost”) is defined as follows:

$$\underline{\ell} := \sup \{ c \mid \exists \lambda(\cdot) : \bar{\mathbb{X}} \rightarrow \mathbb{R}, \text{continuous} : \lambda(f(x, u)) \leq \lambda(x) + \ell(x, u) - c, \forall (x, u) \in \bar{\mathbb{Z}} \}. \quad (2.17)$$

**Remark 2.2.** The dissipativity property is used to estimate a lower bound on the asymptotic average performance. This bound is independent of the storage function  $\lambda(\cdot)$  and only depends on the specific form of the supply rate. The same applies to Definition 2.6 below.

Note that the quantity  $\underline{\ell}$  is a finite real number whose upper bound will be proved in Theorem 2.1. For the lower bound, by taking  $\lambda(\cdot) = 0, \forall x \in \bar{\mathbb{X}}$ , we can have  $\underline{\ell} \geq \min_{(x,u) \in \bar{\mathbb{Z}}} \ell(x, u) > -\infty$  which is finite because the function  $\ell$  is continuous and the set  $\bar{\mathbb{Z}}$  is compact.

Next, along the lines of the widely adopted tool of Control Lyapunov Function (CLF) (see definition [45]) and its role in stabilizability theory, we propose similar concepts referred to as Control Storage Function (CSF) and controlled dissipativity.

**Definition 2.5.** Let  $\mathbb{X}_f$  be a control invariant set as in Assumption 2.1. A control storage function is a function  $V_f : \mathbb{X}_f \rightarrow \mathbb{R}$  that is continuous and such that for all  $x \in \mathbb{X}_f$

$$\min_{u \in \mathbb{U}_f(x)} V_f(f(x, u)) - s(x, u) \leq V_f(x), \quad (2.18)$$

where  $s : \mathbb{Z}_f \rightarrow \mathbb{R}$  is the supply rate. Moreover, the system (2.1), admitting a control storage function, is said to fulfill controlled dissipativity with respect to the supply rate  $s(x, u)$ .

**Remark 2.3.** Notice that inequality (2.18) appears in [5, Assumption 6] in the form

$$V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x)) - \ell(x_s, u_s) \leq V_f(x), \quad (2.19)$$

where  $\kappa_f : \mathbb{X}_f \rightarrow \mathbb{U}$  is the terminal control policy. This can be seen as a special case of (2.18), where  $s(x, u) = \ell(x_s, u_s) - \ell(x, u)$  and  $\kappa_f(x)$  can be chosen to be any selection in  $\operatorname{argmin}_{u \in \mathbb{U}_f(x)} V_f(f(x, u)) - s(x, u)$ .

From this definition, we can see that the CLF is a special case of the CSF, in which  $s(x, u)$  is any negative definite function of  $|x|$ . Since the CLF is frequently used to approximate the tail of the infinite horizon cost of tracking MPC (see [46] for instance), our CSF is meant to

be an appropriate choice of terminal cost in an economic setup. This will be discussed later in Section 2.3.

In order to estimate upper bounds for the best asymptotic performance, the quantity below can be specified,

**Definition 2.6.** *The quantity  $\bar{\ell}$  (later referred to as the “tightest upper bound to the long run average performance”) is defined as:*

$$\bar{\ell} := \inf \{ c \mid \exists V_f : \mathbb{X}_f \rightarrow \mathbb{R} : \forall x \in \mathbb{X}_f, \inf_{u \in \mathbb{U}_f(x)} V_f(f(x, u)) + \ell(x, u) \leq V_f(x) + c \}. \quad (2.20)$$

**Remark 2.4.** Note that  $\bar{\ell}$  is a finite real number. Its lower bound will be proved in Theorem 2.1, and the upper bound can be seen by taking  $V_f(x) = 0, \forall x \in \mathbb{X}_f$ , which yields  $\bar{\ell} \leq \max_{x \in \mathbb{X}_f, u \in \mathbb{U}_f(x)} \ell(x, u) < +\infty$ .

As mentioned in Remark 2.3, inequality (2.18) appears in [5] with  $s(x, u) = \ell(x_s, u_s) - \ell(x, u)$  and  $\bar{\ell}$  is only allowed to be  $\ell(x_s, u_s)$ . In particular, the formal definition of penalty function is crucially linked with the identification of a candidate equilibrium control input pair  $(x_s, u_s)$ . The existence of functions  $V_f(\cdot)$  and  $\kappa_f(\cdot)$  as in (2.19) ensures

$$\inf_{u \in \mathbb{U}_f(x)} V_f(f(x, u)) + \ell(x, u) \leq V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x)) \leq V_f(x) + \ell(x_s, u_s),$$

which implies  $\bar{\ell} \leq \ell(x_s, u_s)$ . Notice that strict inequality may, in general, occur (see Example 2.4.1).

Under the conditions that the terminal control policy  $\kappa_f(\cdot)$  is chosen such that the optimal steady-state  $x_s \in \mathbb{X}_f$  is exponentially stabilizable and the resulting stage cost  $\ell(x, \kappa_f(x))$  is Lipschitz continuous, the terminal penalty function  $V_f(\cdot)$  exists and the inequality (2.19) is an equality by designing the function as  $V_f(x) = \sum_{t=0}^{\infty} [\ell(x_t, \kappa_f(x_t)) - \ell(x_s, u_s)]$ , where  $x_0 = x$  and  $x_{t+1} = f(x_t, \kappa_f(x_t))$ .

Moreover, the above CSF inequality in Definition 2.5 follows the same form of the Hamilton-Jacobi-Bellman (HJB) inequality [47], so any CSF can also be regarded as a solution of the HJB inequality, which is a value function of an infinite horizon optimal control problem.

**Remark 2.5.** We emphasize that Definition 2.5 is given without reference to any underlying regime of operation and it only depends on the specific form of the supply rate. Throughout this paper, two different forms of supply rates are adopted, one is  $s(x, u) = \ell(x, u) - \underline{\ell}$  for

dissipativity and the other is  $s(x, u) = \bar{\ell} - \ell(x, u)$  for controlled dissipativity. Sometimes, shifts of supply rate are also needed as we might not be able to find the tightest value.

We are now ready to state the main result of this section, that is the relation between dissipativity, controlled dissipativity and optimal averaged performance.

**Theorem 2.1.** *Consider system (2.1) subject to constraints (2.8). Then, the following inequality holds:*

$$\underline{\ell} \leq \ell_{av}^*(x), \quad \forall x \in \bar{\mathbb{X}}. \quad (2.21)$$

*In addition, if Assumption 2.1 is fulfilled, we have the following upper bound for  $\ell_{av}^*(x)$ :*

$$\ell_{av}^*(x) \leq \bar{\ell}, \quad \forall x \in \mathbb{X}_f. \quad (2.22)$$

*Proof.* i) We first prove inequality  $\underline{\ell} \leq \ell_{av}^*(x), \forall x \in \bar{\mathbb{X}}$ .

By assumption, let system (2.1) be, for all  $\epsilon > 0$ , dissipative with supply rate  $s(x, u) = \ell(x, u) - \ell_a$  where  $\ell_a = \underline{\ell} - \epsilon$ . Then, there exists a continuous storage function  $\lambda^\epsilon$  such that for all  $(x, u) \in \bar{\mathbb{Z}}$

$$\lambda^\epsilon(f(x, u)) \leq \lambda^\epsilon(x) + \ell(x, u) - \ell_a.$$

Next, for any time  $T$ , and any given feasible solution, it holds

$$\sum_{t=0}^{T-1} \lambda^\epsilon(x_{t+1}) - \lambda^\epsilon(x_t) \leq \sum_{t=0}^{T-1} (\ell(x_t, u_t) - \ell_a).$$

By applying liminf on both sides, we obtain

$$\liminf_{T \rightarrow +\infty} \frac{\lambda^\epsilon(x_T) - \lambda^\epsilon(x_0)}{T} \leq \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} - \ell_a.$$

Moreover, exploiting boundedness of solutions, we see that

$$\liminf_{T \rightarrow +\infty} \frac{\lambda^\epsilon(x_T) - \lambda^\epsilon(x_0)}{T} = 0,$$

and therefore,

$$\ell_a \leq \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T}.$$

Then, taking infimums with respect to  $u$  for any fixed  $x_0 = x \in \bar{\mathbb{X}}$ , we can see that

$$\ell_a \leq \ell_{av}^*(x).$$

Since  $\epsilon > 0$  was taken arbitrary to start with,

$$\underline{\ell} \leq \ell_{av}^*(x).$$

ii) Next, we prove the inequality  $\ell_{av}^*(x) \leq \bar{\ell}, \forall x \in \mathbb{X}_f$ .

Assume system (2.1) admits CSFs with supply rate  $s(x, u) = \ell_b - \ell(x, u)$  where  $\ell_b = \bar{\ell} + \epsilon$  for all  $\epsilon > 0$ . The corresponding inequality is:

$$\inf_{u \in \mathbb{U}_f(x)} V_f^\epsilon(f(x, u)) + \ell(x, u) \leq V_f^\epsilon(x) + \ell_b, \quad \forall x \in \mathbb{X}_f.$$

Next, let us consider any state trajectory starting from arbitrary initial state  $x_0 = x \in \mathbb{X}_f$  with corresponding control input sequence defined as:

$$\begin{aligned} u_t &\in \underset{u \in \mathbb{U}_f(x_t)}{\operatorname{argmin}} V_f^\epsilon(f(x_t, u)) + \ell(x_t, u), \\ x_{t+1} &= f(x_t, u_t), \quad t \in \mathbb{I}_{\geq 0}. \end{aligned}$$

The state-input pair at any time instant is denoted as  $(x_t, u_t) \in \mathbb{X}_f \times \mathbb{U}_f(x_t), \forall t \in \mathbb{I}_{\geq 0}$ ; then, it holds

$$\sum_{t=0}^{T-1} V_f^\epsilon(x_{t+1}) - V_f^\epsilon(x_t) \leq \sum_{t=0}^{T-1} (\ell_b - \ell(x_t, u_t)).$$

Dividing by  $T$  and applying limsup on both sides, we see that

$$\limsup_{T \rightarrow +\infty} \frac{V_f^\epsilon(x_T) - V_f^\epsilon(x_0)}{T} \leq \ell_b - \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T},$$

and exploiting boundedness of solutions,

$$\liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} \leq \ell_b.$$

Then, by definition of  $\ell_{av}^*(x)$ , it holds

$$\ell_{av}^*(x) \leq \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} \leq \ell_b,$$

and, since  $\epsilon > 0$  was taken arbitrary to start with,

$$\ell_{av}^*(x) \leq \bar{\ell}.$$

□

**Remark 2.6.** If a system is optimally operated at an equilibrium point  $x_s$  with corresponding input  $u_s$ , then  $\ell_{av}^* = \ell(x_s, u_s)$  and one may consider  $s(x, u) = \ell(x, u) - \ell(x_s, u_s)$ . This supply rate and associated storage function are used [3] as a sufficient condition to prove Lyapunov stability of the equilibrium point. In addition, the closed-loop asymptotic average performance is bounded by  $\ell(x_s, u_s)$  from above as shown by [3, Theorem 18].

If a system is controllable within finite time to the best optimal operation, every initial condition gives the same best asymptotic average performance, that is  $\ell_{av}^*(x) = \ell_{av}^*, \forall x \in \bar{\mathbb{X}}$ . However, if this is not the case, the strict inequality  $\ell_{av}^*(x) > \ell_{av}^*$  may hold. This gap arising from uncontrollable systems will be illustrated in the Example 2.4.2.

## 2.3 Economic MPC algorithm and analysis

This section is dedicated to the formulation of EMPC with generalized terminal penalty functions and the analysis of asymptotic average performance and stability of its associated closed-loop system. To this end, we start by introducing the basic setup in EMPC.

### 2.3.1 Economic MPC formulation

Now, we consider the EMPC problem for a given finite prediction horizon  $N \in \mathbb{I}_{\geq 1}$ . The open loop EMPC problem at time instant  $t \in \mathbb{I}_{\geq 0}$  can be formulated as

$$V_N(x_t) = \min_{x_{t+N|t}, u_{t+N-1|t}} J_N(x_{t+N|t}, u_{t+N-1|t}) \quad (2.23)$$

$$s.t. \quad x_{k+1|t} = f(x_{k|t}, u_{k|t}), \quad x_{t|t} = x_t \quad (2.23a)$$

$$(x_{k|t}, u_{k|t}) \in \mathbb{Z}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]} \quad (2.23b)$$

$$x_{t+N|t} \in \mathbb{X}_f \quad (2.23c)$$

where  $J_N(x_{t:t+N|t}, u_{t:t+N-1|t}) := \sum_{k=t}^{t+N-1} \ell(x_{k|t}, u_{k|t}) + V_f(x_{t+N|t})$  is the economic objective function, and  $V_f(\cdot)$  is the terminal penalty function and a CSF according to Definition 2.5 with respect to the supply function

$$s(x, u) = \bar{\ell} - \ell(x, u).$$

The decision variables in this optimization problem are

$$u_{t:t+N-1|t} := (u_{t|t}, u_{t+1|t}, \dots, u_{t+N-1|t}), \quad x_{t:t+N|t} := (x_{t|t}, x_{t+1|t}, \dots, x_{t+N|t}),$$

which are the input and state sequences over the prediction horizon  $N$ . For the ease of feasibility analysis, we may use  $\bar{\mathbb{Z}}$  instead of  $\mathbb{Z}$ , and it will not affect the EMPC controller with the original  $\mathbb{Z}$ . Then, the admissible set  $\mathcal{Z}_N \subseteq \bar{\mathbb{Z}}$  for  $(x, u_{0:N-1})$  pairs is

$$\begin{aligned} \mathcal{Z}_N := \{ & (x, u_{0:N-1}) \mid \exists x_1, \dots, x_N : x_{t+1} = f(x_t, u_t), x_0 = x, \\ & x_N \in \mathbb{X}_f, (x_t, u_t) \in \bar{\mathbb{Z}}, \forall t \in \mathbb{I}_{[0, N-1]} \}, \end{aligned} \quad (2.24)$$

and the projection of  $\mathcal{Z}_N$  onto  $\bar{\mathbb{X}}$  is

$$\mathcal{X}_N := \{x \in \bar{\mathbb{X}} \mid \exists u_{0:N-1} \text{ such that } (x, u_{0:N-1}) \in \mathcal{Z}_N\}. \quad (2.25)$$

Notice that the terminal set fulfills  $\mathbb{X}_f \subseteq \mathcal{X}_N \subseteq \bar{\mathbb{X}}$ , the set  $\mathcal{X}_N$  is non-empty and hence the optimization problem  $\mathbb{P}_N(x)$  has at least one feasible solution for any  $x \in \mathbb{X}_f$ .

As customary in MPC, the optimal input from EMPC controller (2.23) is denoted as:

$$u_{t:t+N-1|t}^* := (u_{t|t}^*, u_{t+1|t}^*, \dots, u_{t+N-1|t}^*), \quad (2.26)$$

with the corresponding state trajectory:

$$x_{t:t+N|t}^* := (x_{t|t}^*, x_{t+1|t}^*, \dots, x_{t+N-1|t}^*, x_{t+N|t}^*), \quad (2.27)$$

and the optimal value function

$$V_N(x_t) := J_N(x_{t:t+N|t}^*, u_{t:t+N-1|t}^*). \quad (2.28)$$

Since only the first element of the optimal control sequence (2.26), which is

$$\kappa_N(x_t) := u_{t|t}^*, \quad (2.29)$$

will be implemented into the system, the closed-loop system dynamic by using EMPC controller is

$$x^+ = f(x, \kappa_N(x)). \quad (2.30)$$

Notice that  $u_{t|t}^*$  might be non-unique, in which case, we take any element that achieves the optimum.

### 2.3.2 Performance analysis

In order to address the asymptotic average performance of the closed-loop system (2.30), it is useful to have an explicit notation for the terminal control policy.

**Definition 2.7.** *The terminal control policy is defined as any function  $\kappa_f : \mathbb{X}_f \rightarrow \mathbb{U}$  fulfilling:*

$$\kappa_f(x) \in \operatorname{argmin}_{u \in \mathbb{U}_f(x)} V_f(f(x, u)) + \ell(x, u). \quad (2.31)$$

One of the main concerns of EMPC problem is the closed-loop performance which is analyzed as below:

**Proposition 2.1.** *The asymptotic average performance of system (2.30) is no worse than  $\bar{\ell}$ .*

*Proof.* At any time instant  $t \in \mathbb{I}_{\geq 0}$ , let  $x_t \in \mathcal{X}_N$ , a sub-optimal control input sequence and resulting state trajectory in next time instance is

$$\begin{aligned} u_{t+1:t+N|t+1}^{sub} &= (u_{t+1|t}^*, u_{t+2|t}^*, \dots, u_{t+N-1|t}^*, \kappa_f(x_{t+N|t}^*)) \\ x_{t+1:t+N+1|t+1}^{sub} &= (x_{t+1|t}^*, x_{t+2|t}^*, \dots, x_{t+N-1|t}^*, x_{t+N|t}^*, x_{t+N+1|t}), \end{aligned} \quad (2.32)$$

where  $x_{t+N+1|t} = f(x_{t+N|t}^*, \kappa_f(x_{t+N|t}^*))$ .

Therefore, the optimal cost-to-go functions fulfill

$$V_N(x_{t+1}) \leq J_N(x_{t+1:t+N+1|t+1}^{sub}, u_{t+1:t+N|t+1}^{sub}),$$

which is

$$\begin{aligned} V_N(x_{t+1}) &\leq V_N(x_t) + V_f(x_{t+N+1|t}) + \ell(x_{t+N|t}^*, \kappa_f(x_{t+N|t}^*)) \\ &\quad - \ell(x_t, \kappa_N(x_t)) - V_f(x_{t+N|t}^*) \\ &\leq V_N(x_t) - \ell(x_t, \kappa_N(x_t)) + \bar{\ell}. \end{aligned}$$

Then, for any time  $T$ , we have,

$$\sum_{t=0}^{T-1} V_N(x_{t+1}) - V_N(x_t) \leq \sum_{t=0}^{T-1} (\bar{\ell} - \ell(x_t, \kappa_N(x_t))).$$

By applying  $\liminf$  on both sides and exploiting boundedness of solutions, we finally obtain the asymptotic average cost using proposed economic MPC controller satisfies

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, \kappa_N(x_t))}{T} \leq \bar{\ell}.$$

which proves the claim. □

### 2.3.3 Stability analysis

This sub-section explores the asymptotic stability of the closed-loop system under EMPC control actions. In this context, we only consider the no gap case, viz:

**Assumption 2.2.** *There is no gap between the tightest upper and lower bounds of  $\ell_{av}^*$ :*

$$\underline{\ell} = \ell_{av}^* = \ell_{av}^*(x) = \bar{\ell}, \quad \forall x \in \bar{\mathbb{X}}. \quad (2.33)$$

**Remark 2.7.** This assumption entails that, although  $\underline{\ell}$  and  $\bar{\ell}$  are defined without any a priori reference to a specific optimal regime of operation, the optimal average performance has been exactly computed both by means of dissipativity and controlled dissipativity inequalities.

Notice that the duality relation between  $\underline{\ell}$  and  $\ell_{av}^*$  for continuous time systems has been studied in [43]. For discrete time systems, the authors of [44] have proved  $\underline{\ell} = \ell_{av}^*$  using the concept of occupational measure.

Before the analysis of closed-loop stability, it is necessary to define strict dissipativity with respect to a set of points

**Definition 2.8.** *A discrete time system is strictly dissipative at a set  $\mathbb{S} \in \bar{\mathbb{X}}$  if there is a continuous storage function  $\lambda : \bar{\mathbb{X}} \rightarrow \mathbb{R}$  such that:*

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u) + \rho(|x|_{\mathbb{S}}) \quad (2.34)$$

for all  $(x, u) \in \bar{\mathbb{Z}}$ , where  $s(x, u)$  is the supply rate and  $\rho(\cdot|x|_{\mathbb{S}})$  is a positive definite function with respect to the set  $\mathbb{S}$ .<sup>1</sup>

As pioneered in [48] and dictated in [5], we introduce the notion of rotated stage cost and terminal cost as defined below:

$$L(x, u) := \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - \ell_{av}^*, \quad (2.35)$$

$$\bar{V}_f(x) := V_f(x) + \lambda(x), \quad \forall x \in \mathbb{X}_f. \quad (2.36)$$

Then, the new optimization problem is

$$\bar{V}_N(x_t) = \min_{x_{t:t+N}|t, u_{t:t+N-1}|t} \bar{J}_N(x_{t:t+N}|t, u_{t:t+N-1}|t) \quad (2.37)$$

$$s.t. \quad x_{k+1}|t = f(x_k|t, u_k|t), \quad x_t|t = x_t \quad (2.37a)$$

$$(x_k|t, u_k|t) \in \bar{\mathbb{Z}}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]} \quad (2.37b)$$

$$x_{t+N}|t \in \mathbb{X}_f \quad (2.37c)$$

where  $\bar{J}_N(x_{t:t+N}|t, u_{t:t+N-1}|t) := \sum_{k=t}^{t+N-1} L(x_k|t, u_k|t) + \bar{V}_f(x_{t+N}|t)$  is the rotated total economic cost,

$$u_{t:t+N-1}|t := (u_t|t, u_{t+1}|t, \dots, u_{t+N-1}|t), \quad x_{t:t+N}|t := (x_t|t, x_{t+1}|t, \dots, x_{t+N}|t)$$

are the input and corresponding state sequences.

With the notion of rotated stage cost and terminal cost functions, let us define two sets:

$$\hat{\Pi}_{\bar{\mathbb{X}}} := \operatorname{argmin}_{x \in \mathbb{X}_f} \bar{V}_f(x), \quad \tilde{\Pi}_{\bar{\mathbb{X}}} := \{x \mid \min_{u \in \bar{\mathbb{U}}(x)} L(x, u) = 0\}. \quad (2.38)$$

**Remark 2.8.** Without loss of generality, shifting  $\bar{V}_f(\cdot)$  by some constant real number, we may assume  $\bar{V}_f(x) = 0$  for all  $x \in \hat{\Pi}_{\bar{\mathbb{X}}}$  and  $\bar{V}_f(x) > 0$  for all  $x \in \mathbb{X}_f \setminus \hat{\Pi}_{\bar{\mathbb{X}}}$ .

**Assumption 2.3.** We assume there exists a continuous storage function  $\lambda(\cdot)$  such that the system (2.1) is strictly dissipative at  $\tilde{\Pi}_{\bar{\mathbb{X}}}$  with supply rate  $s(x, u) = \ell(x, u) - \ell_{av}^*$ , and a control storage function  $V_f(\cdot)$  for which inequalities (2.18) is achieved with supply rate  $s(x, u) = \ell_{av}^* - \ell(x, u)$ .

<sup>1</sup>A continuous function  $\rho(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is positive definite with respect to a set  $\mathbb{S}$  if  $\rho(\cdot|x|_{\mathbb{S}}) = 0$  for all  $x$  such that  $|x|_{\mathbb{S}} = 0$ , and  $\rho(\cdot|x|_{\mathbb{S}}) > 0$  otherwise.

The use of rotated stage costs in the stability analysis of EMPC starts from a powerful analogy to the case of tracking MPC. In fact, under Assumption 2.3, the rotated stage cost  $L(x, u)$  is positive definite with respect to  $\tilde{\Pi}_{\bar{\mathbb{X}}}$ . Together with the compactness of the set  $\mathbb{Z}$ , it implies the function  $L(x, u)$  is bounded from below by a *class*  $\mathcal{K}$  function  $\alpha_1(\cdot)$  of the distance to  $\tilde{\Pi}_{\bar{\mathbb{X}}}$ :

$$L(x, u) \geq \alpha_1(|x|_{\tilde{\Pi}_{\bar{\mathbb{X}}}}) \geq 0, \forall (x, u) \in \bar{\mathbb{Z}}. \quad (2.39)$$

Therefore, a rotated stage cost is at least non-negative definite. Moreover, the solution to an optimization problem is unaffected by “rotation”, as defined in the following Lemma. Similarly, the rotated terminal functions fulfill (2.40).

**Lemma 2.1.** [5, Lemma 14] *For every function  $\lambda(\cdot)$  and constant  $\ell_{av}^* \in \mathbb{R}$ , the set of solutions of optimization problem (2.23) is identical to the set of solutions to problem (2.37).*

**Lemma 2.2.** [5, Lemma 9] *If Assumption 2.3 holds, then for all  $x \in \mathbb{X}_f$  and the terminal control  $\kappa_f(x)$  in (2.31), the CSF inequality in (2.18) with  $s(x, u) = \ell_{av}^* - \ell(x, u)$  holds if and only if the following inequality is satisfied*

$$\bar{V}_f(f(x, \kappa_f(x))) - \bar{V}_f(x) \leq -L(x, \kappa_f(x)). \quad (2.40)$$

**Lemma 2.3.** *In addition to Lemma 2.2, under Assumption 2.3, the following inequalities are satisfied:*

$$\alpha_1(|x|_{\tilde{\Pi}_{\bar{\mathbb{X}}}}) \leq \min_{u \in \bar{\mathbb{U}}(x)} L(x, u) \leq \bar{V}_N(x), \forall x \in \bar{\mathbb{X}} \text{ and } \bar{V}_N(x) \leq \bar{V}_f(x), \forall x \in \mathbb{X}_f. \quad (2.41)$$

*Proof.* Consider the compound inequality, the first half is trivially satisfied given (2.39), and the other side is true given the formulation in (2.37) and the non-negativity of functions  $L(\cdot, \cdot)$  and  $\bar{V}_f(\cdot)$  as in Remark 2.8,

$$\bar{V}_N(x) = L(x, \kappa_N(x)) + \sum_{k=1}^{N-1} L(x_k^*, u_k^*) + \bar{V}_f(x_N^*) \geq L(x, \kappa_N(x)) \geq \min_{u \in \bar{\mathbb{U}}(x)} L(x, u).$$

Now, let us show the inequality  $\bar{V}_N(x) \leq \bar{V}_f(x), \forall x \in \mathbb{X}_f$ . Based on Lemma 2.2, it holds

$$L(x, \kappa_f(x)) + \bar{V}_f(f(x, \kappa_f(x))) \leq \bar{V}_f(x),$$

which means

$$\bar{V}_1(x) \leq L(x, \kappa_f(x)) + \bar{V}_f(f(x, \kappa_f(x))) \leq \bar{V}_f(x).$$

By induction, for any  $N \in \mathbb{I}_{\geq 2}$ , we have

$$\bar{V}_N(x) \leq L(x, \kappa_f(x)) + \bar{V}_{N-1}(f(x, \kappa_f(x))) \leq L(x, \kappa_f(x)) + \bar{V}_f(f(x, \kappa_f(x))) \leq \bar{V}_f(x).$$

Therefore, we proved the last inequality in (2.41).  $\square$

**Lemma 2.4.** *Let Assumption 2.3 hold, the sets  $\hat{\Pi}_{\bar{\mathbb{X}}}$  and  $\tilde{\Pi}_{\bar{\mathbb{X}}}$  in (2.38) fulfill the following relation:*

$$\hat{\Pi}_{\bar{\mathbb{X}}} \subseteq \tilde{\Pi}_{\bar{\mathbb{X}}}. \quad (2.42)$$

*Proof.* The definition of  $\hat{\Pi}_{\bar{\mathbb{X}}}$  implies  $\bar{V}_f(x) = 0$  for all  $x \in \hat{\Pi}_{\bar{\mathbb{X}}}$ . Together with (2.41), it can be seen that, for all  $x \in \hat{\Pi}_{\bar{\mathbb{X}}} \subseteq \mathbb{X}_f$ ,  $0 \leq \min_{u \in \bar{\mathbb{U}}(x)} L(x, u) \leq \bar{V}_f(x) = 0$ . Thus,  $x \in \tilde{\Pi}_{\bar{\mathbb{X}}}$  and  $\hat{\Pi}_{\bar{\mathbb{X}}} \subseteq \tilde{\Pi}_{\bar{\mathbb{X}}}$ .  $\square$

**Lemma 2.5.** *The set  $\hat{\Pi}_{\bar{\mathbb{X}}}$  is positively invariant with respect to system  $x^+ = f(x, \kappa_f(x))$ .*

*Proof.* From Assumption 2.3, taken any  $x \in \hat{\Pi}_{\bar{\mathbb{X}}}$  and selecting control  $u = \kappa_f(x)$  as in (2.31),

$$\begin{aligned} \bar{V}_f(f(x, \kappa_f(x))) &= \lambda(f(x, \kappa_f(x))) + V_f(f(x, \kappa_f(x))) \\ &\leq \lambda(x) + \ell(x, \kappa_f(x)) - \ell_{av}^* + V_f(f(x, \kappa_f(x))) \\ &\leq \lambda(x) + V_f(x) = \bar{V}_f(x) = 0. \end{aligned}$$

Meanwhile,  $\bar{V}_f(f(x, \kappa_f(x)))$  is non-negative from (2.41), hence  $\bar{V}_f(f(x, \kappa_f(x))) = 0$  and  $f(x, \kappa_f(x)) \in \hat{\Pi}_{\bar{\mathbb{X}}}$ . Then,  $\hat{\Pi}_{\bar{\mathbb{X}}}$  is shown to be positively invariant as  $x$  was taken arbitrary in  $\hat{\Pi}_{\bar{\mathbb{X}}}$ .  $\square$

While the two sets  $\tilde{\Pi}_{\bar{\mathbb{X}}}$  and  $\hat{\Pi}_{\bar{\mathbb{X}}}$  are always nested as in (2.4) for some stability results to hold, the following stronger assumption is needed,

**Assumption 2.4.** *The sets  $\Pi_{\bar{\mathbb{X}}}$ ,  $\tilde{\Pi}_{\bar{\mathbb{X}}}$  and  $\hat{\Pi}_{\bar{\mathbb{X}}}$  fulfill*

$$\Pi_{\bar{\mathbb{X}}} = \tilde{\Pi}_{\bar{\mathbb{X}}} = \hat{\Pi}_{\bar{\mathbb{X}}}. \quad (2.43)$$

where  $\Pi_{\bar{\mathbb{X}}}$  is the best regime of operation defined in (2.14).

**Remark 2.9.** The relation between sets  $\Pi_{\bar{\mathbb{X}}}$  and  $\tilde{\Pi}_{\bar{\mathbb{X}}}$  will be further discussed in Section 2.3.4. In case of steady-states and periodic orbits, it is easy to find suitable storage function  $\lambda(\cdot)$  and terminal penalty function  $V_f(\cdot)$  in order to verify  $\tilde{\Pi}_{\bar{\mathbb{X}}} = \hat{\Pi}_{\bar{\mathbb{X}}} = \Pi_{\bar{\mathbb{X}}}$ . In Example 2.4.3, even if the optimal regime is non-periodic,  $\tilde{\Pi}_{\bar{\mathbb{X}}}$ ,  $\hat{\Pi}_{\bar{\mathbb{X}}}$  and  $\Pi_{\bar{\mathbb{X}}}$  are still the same, which is the unit circle.

To the end, we recall the definition of asymptotic stability of a set [49], and then state our main result on asymptotic stability of closed-loop Economic Model Predictive Control with generalized terminal penalty function.

**Definition 2.9.** Let  $\mathcal{D}_0$  be a compact positively invariant set for a nonlinear dynamic system.  $\mathcal{D}_0$  is Lyapunov stable if, for all  $\epsilon > 0$ , there exists  $\delta = \delta(\epsilon) > 0$  such that if  $|x_0|_{\mathcal{D}_0} < \delta$ , then the state, at any time  $t \in \mathbb{I}_{\geq 0}$ , fulfills  $|\varphi(t, x_0)|_{\mathcal{D}_0} < \epsilon$ , where  $\varphi(t, x_0)$  is the system state at time  $t$  with initial condition  $x_0$ .  $\mathcal{D}_0$  is attractive if there exists an open neighborhood  $\mathcal{O}$  of  $\mathcal{D}_0$  such that the  $\omega$ -limit set  $\omega(x_0) \subseteq \mathcal{D}_0$  for all  $x_0 \in \mathcal{O}$ .  $\mathcal{D}_0$  is asymptotically stable if it is Lyapunov stable and attractive.

**Theorem 2.2.** Let Assumptions 2.1, 2.2, 2.3 and 2.4 hold, the set  $\Pi_{\bar{\mathbb{X}}}$  by using control policy (2.29), is asymptotically stable within a region of attraction  $\mathcal{X}_N$ .

*Proof.* A candidate Lyapunov function is the optimal value function

$$\bar{V}_N(x_t) = \bar{J}_N(x_{t:t+N|t}^*, u_{t:t+N-1|t}^*),$$

where  $u_{t:t+N-1|t}^*$  and  $x_{t:t+N|t}^*$  are the optimal control and corresponding state sequences in (2.26) and (2.27).

Suppose a terminal region constraint enforces  $x_{t+N|t}^* \in \mathbb{X}_f$ , a suboptimal input and state sequences as in (2.32) guarantees  $x_{t+N+1|t} \in \mathbb{X}_f$ . Then, the cost of this sub-optimal sequence is given as:

$$\begin{aligned} & \bar{J}_N(x_{t+1:t+N+1|t}^{sub}, u_{t+1:t+N|t+1}^{sub}) \\ &= \sum_{k=1}^{N-1} L(x_k^*, u_k^*) + L(x_{t+N|t}^*, \kappa_f(x_{t+N|t}^*)) + \bar{V}_f(x_{t+N+1|t}) \\ &= \bar{V}_N(x_t) - L(x_t, \kappa_N(x_t)) - \bar{V}_f(x_{t+N|t}^*) + L(x_{t+N|t}^*, \kappa_f(x_{t+N|t}^*)) + \bar{V}_f(x_{t+N+1|t}) \\ &\leq \bar{V}_N(x_t) - L(x_t, \kappa_N(x_t)) \quad (\text{from (2.40)}). \end{aligned}$$

Because of  $\bar{V}_N(x_{t+1}) \leq \bar{J}_N(x_{t+1:t+N+1|t}^{sub}, u_{t+1:t+N|t+1}^{sub})$ , it yields

$$\bar{V}_N(x_{t+1}) - \bar{V}_N(x_t) \leq -L(x_t, \kappa_N(x_t)) \leq -\alpha_1(|x_t|_{\tilde{\Pi}_{\bar{\mathbb{X}}}}) \quad (\text{from (2.39)}),$$

which shows the attractivity of  $\tilde{\Pi}_{\bar{\mathbb{X}}}$ .

Notice that the function  $\bar{V}_N(x)$  is lower bounded by  $\alpha_1(|x|_{\tilde{\Pi}_{\bar{\mathbb{X}}}})$  as in (2.41). We need to provide an upper bound for this function. By definition,  $\bar{V}_f(x) = 0, \forall x \in \hat{\Pi}_{\bar{\mathbb{X}}}$  and  $\bar{V}_f(\cdot)$  is continuous due to continuity of the storage function  $\lambda(\cdot)$  and the penalty function  $V_f(\cdot)$ . Hence, there exists  $\tilde{\alpha}_2(\cdot) \in \mathcal{K}$  such that  $\bar{V}_f(x) \leq \tilde{\alpha}_2(|x|_{\hat{\Pi}_{\bar{\mathbb{X}}}})$ . Then, by (2.41), we obtain  $\bar{V}_N(x) \leq \tilde{\alpha}_2(|x|_{\hat{\Pi}_{\bar{\mathbb{X}}}}), \forall x \in \mathbb{X}_f$ , which further yields  $\bar{V}_N(x) \leq \alpha_2(|x|_{\hat{\Pi}_{\bar{\mathbb{X}}}}), \forall x \in \mathcal{X}_N$  [45, Propositions 2.17 & 2.18], where  $\alpha_2(\cdot)$  is some *class*  $\mathcal{K}$  function. Therefore, the following inequalities hold

$$\alpha_1(|x|_{\tilde{\Pi}_{\bar{\mathbb{X}}}}) \leq \bar{V}_N(x) \leq \alpha_2(|x|_{\hat{\Pi}_{\bar{\mathbb{X}}}}), \forall x \in \mathcal{X}_N.$$

With Assumption 2.4, which is  $\hat{\Pi}_{\bar{\mathbb{X}}} = \tilde{\Pi}_{\bar{\mathbb{X}}} = \Pi_{\bar{\mathbb{X}}}$ , we finally obtain

$$\alpha_1(|x|_{\Pi_{\bar{\mathbb{X}}}}) \leq \bar{V}_N(x) \leq \alpha_2(|x|_{\Pi_{\bar{\mathbb{X}}}})$$

$$\bar{V}_N(f(x, \kappa_N(x))) - \bar{V}_N(x) \leq -\alpha_1(|x|_{\Pi_{\bar{\mathbb{X}}}})$$

for all  $x \in \mathcal{X}_N$  which implies the closed-loop system is Lyapunov stable, and any trajectory that starts in  $\mathcal{X}_N$  satisfies  $|\varphi_{\kappa_N}(t, x_0)|_{\Pi_{\bar{\mathbb{X}}}} \leq \beta(|x_0|_{\Pi_{\bar{\mathbb{X}}}}, t), \forall t \geq 0$  for some class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$ . Therefore, the set  $\Pi_{\bar{\mathbb{X}}}$  is asymptotically stable with a region of attraction  $\mathcal{X}_N$ .  $\square$

**Remark 2.10.** The asymptotic stability of the general optimal regimes of operation in this subsection and the averaged performance of closed-loop system in previous subsection are not necessarily assuming any relation between terminal region and the optimal regimes. However, the next section which provides a candidate of terminal cost function is based on the condition that terminal states are restricted by the optimal operation regime.

### 2.3.4 Design of terminal control storage functions

Designing suitable terminal penalty functions is, in general, a non-trivial task. In particular one would normally hope to find a terminal region  $\mathbb{X}_f$  that strictly includes the support of the optimal regime of operation  $\Pi_{\bar{\mathbb{X}}}$ . This is desirable, among other things, to increase the set of feasible initial states  $\mathcal{X}_N$ , as well as possibly avoiding terminal equality constraints. Before showing the main result in this subsection, we define the limit occupational measure, also called sojourn time, of a set  $S$  generated by the process as

$$\gamma(S) = \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \chi_S(x_t, u_t)}{T} \quad (2.44)$$

where  $\chi_S(\cdot)$  is the indicator function of  $S$

$$\chi_S(x, u) := \begin{cases} 1 & (x, u) \in S \\ 0 & \text{otherwise.} \end{cases} \quad (2.45)$$

**Assumption 2.5.** *The limit occupational measure  $\gamma(\cdot)$  generated by  $(x_{s,i}, u_{s,i})$ , has support  $\Pi$ , i.e.,  $\forall (x, u) \in \Pi$  and  $\forall \epsilon > 0$  it holds  $\gamma(B_\epsilon(x, u)) > 0$ , where  $B_\epsilon(x, u) := \{(\tilde{x}, \tilde{u}) \mid |(\tilde{x}, \tilde{u})|_{(x, u)} < \epsilon\}$ .*

**Remark 2.11.** This assumption is trivially fulfilled for steady-state or periodic optimal regimes of operation. It also holds in Example 2.4.3 and can be straightforwardly verified. However, in general, this might be difficult to verify as one might need explicit knowledge of the optimal solution.

**Assumption 2.6.** *There exists a continuous storage function  $\lambda(\cdot)$  such that dissipativity holds on  $\bar{\mathbb{X}}$  with supply rate  $\ell(x, u) - \ell_{av}^*$  as follows:*

$$\lambda(f(x, u)) - \lambda(x) \leq \ell(x, u) - \ell_{av}^*. \quad (2.46)$$

**Lemma 2.6.** *If Assumption 2.5 and 2.6 hold, then for any  $(x, u) \in \Pi$ , the following equality is fulfilled:*

$$\lambda(f(x, u)) - \lambda(x) = \ell(x, u) - \ell_{av}^*. \quad (2.47)$$

*Proof.* Suppose Assumption 2.6 holds, viz.

$$\lambda(f(x, u)) - \lambda(x) \leq \ell(x, u) - \ell_{av}^*, \quad \forall (x, u) \in \Pi.$$

The rotated stage cost function  $L(x, u)$

$$L(x, u) = \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - \ell_{av}^* \geq 0.$$

Because of the continuity of  $f(\cdot, \cdot)$ ,  $\ell(\cdot, \cdot)$  and  $\lambda(\cdot)$ ,  $L(x, u)$  is also continuous.

Next, we consider the asymptotic average value of the function  $L(x, u)$  along the optimal trajectory in  $\Pi_{\bar{x}}$ ,

$$\begin{aligned}
& \liminf_{T \rightarrow +\infty} \frac{\sum_{i=0}^{T-1} L(x_{s,i}, u_{s,i})}{T} \\
&= \liminf_{T \rightarrow +\infty} \frac{\sum_{i=0}^{T-1} \ell(x_{s,i}, u_{s,i})}{T} - \ell_{av}^* + \lim_{T \rightarrow +\infty} \frac{\sum_{i=0}^{T-1} (\lambda(x_{s,i}) - \lambda(x_{s,i+1}))}{T} \\
&= \lim_{T \rightarrow +\infty} \frac{\sum_{i=0}^{T-1} (\lambda(x_{s,i}) - \lambda(x_{s,i+1}))}{T} \\
&= \lim_{T \rightarrow +\infty} \frac{\lambda(x_{s,0}) - \lambda(x_{s,T})}{T} = 0.
\end{aligned} \tag{2.48}$$

Notice that this equality is not sufficient to indicate  $L(x_{s,i}, u_{s,i}) \equiv 0$ . One can use Assumption 2.5 to prove this result by contradiction.

Assume there exist  $(\bar{x}, \bar{u}) \in \Pi$  and  $\bar{\epsilon} > 0$  such that  $L(\bar{x}, \bar{u}) > 0$  and  $L(x, u) \geq \frac{L(\bar{x}, \bar{u})}{2}$  for all  $(x, u) \in B_{\bar{\epsilon}}(\bar{x}, \bar{u})$ . Considering the asymptotic average value of  $L(x_{s,i}, u_{s,i})$  again,

$$\begin{aligned}
\liminf_{T \rightarrow +\infty} \frac{\sum_{i=0}^{T-1} L(x_{s,i}, u_{s,i})}{T} &\geq \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} L(x_{s,i}, u_{s,i}) \chi_{B_{\bar{\epsilon}}(\bar{x}, \bar{u})}(x_{s,i}, u_{s,i})}{T} \\
&\geq \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \frac{L(\bar{x}, \bar{u})}{2} \chi_{B_{\bar{\epsilon}}(\bar{x}, \bar{u})}(x_{s,i}, u_{s,i})}{T} \\
&= \frac{L(\bar{x}, \bar{u})}{2} \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \chi_{B_{\bar{\epsilon}}(\bar{x}, \bar{u})}(x_{s,i}, u_{s,i})}{T} \\
&= \frac{L(\bar{x}, \bar{u})}{2} \gamma(B_{\bar{\epsilon}}(\bar{x}, \bar{u})) > 0.
\end{aligned}$$

The last inequality holds due to Assumption 2.5, and contradicts (2.48) previously established. Therefore,  $L(x, u) = 0, \forall (x, u) \in \Pi$  and this in turn implies the equality (2.47).  $\square$

From Lemma 2.6, it can be seen that the rotated stage cost  $L(x, u) = 0, \forall (x, u) \in \Pi$ , which in turn implies:

**Corollary 2.1.** *Under Assumption 2.5 and 2.6, the two sets  $\Pi_{\bar{x}}$  and  $\tilde{\Pi}_{\bar{x}}$  fulfill the following relation:*

$$\Pi_{\bar{x}} \subseteq \tilde{\Pi}_{\bar{x}}. \tag{2.49}$$

**Assumption 2.7.** The terminal region  $\mathbb{X}_f$ , the optimal regime of operation  $\Pi_{\bar{\mathbb{X}}}$  and the set  $\tilde{\Pi}_{\bar{\mathbb{X}}}$  fulfill

$$\tilde{\Pi}_{\bar{\mathbb{X}}} = \Pi_{\bar{\mathbb{X}}} \subseteq \mathbb{X}_f \subseteq \bar{\mathbb{X}}. \quad (2.50)$$

**Remark 2.12.** This assumption indicates the dissipation inequality (2.46) is strict on  $\bar{\mathbb{X}} \setminus \Pi_{\bar{\mathbb{X}}}$ . Moreover, the terminal region must be designed large enough to include states of the optimal regime of operation.

**Proposition 2.2.** Let Assumptions 2.5, 2.6 and 2.7 hold, then, we can select the terminal control storage function as

$$V_f(x) = -\lambda(x), \quad \forall x \in \mathbb{X}_f. \quad (2.51)$$

Then, the resulting EMPC problem is

$$V_N(x_t) = \min_{u_{t:t+N-1|t}} \sum_{k=t}^{t+N-1} \ell(x_{k|t}, u_{k|t}) - \lambda(x_{t+N|t}) \quad (2.52)$$

$$s.t. \quad x_{k+1|t} = f(x_{k|t}, u_{k|t}), \quad x_{t|t} = x_t \quad (2.52a)$$

$$(x_{k|t}, u_{k|t}) \in \mathbb{Z}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]} \quad (2.52b)$$

$$x_{t+N|t} \in \Pi_{\bar{\mathbb{X}}}. \quad (2.52c)$$

*Proof.* By selecting the terminal penalty function as (2.51), the rotated terminal cost function fulfills  $\bar{V}_f(x) = V_f(x) + \lambda(x) = 0, \forall x \in \mathbb{X}_f$ , and hence  $\mathbb{X}_f \subseteq \hat{\Pi}_{\bar{\mathbb{X}}}$  from the definition of  $\hat{\Pi}_{\bar{\mathbb{X}}}$ . Together with Lemma 2.4, it implies  $\tilde{\Pi}_{\bar{\mathbb{X}}} = \hat{\Pi}_{\bar{\mathbb{X}}} = \Pi_{\bar{\mathbb{X}}} = \mathbb{X}_f$  which means the terminal region  $\mathbb{X}_f$  coincides with the support of the optimal trajectory. For any optimal state-input pair  $(x_{s,i}, u_{s,i}) \in \Pi, \forall i \in \mathbb{I}_{\geq 0}$ , given the supply rate of the CSF in form of  $s(x, u) = \ell_{av}^* - \ell(x, u)$ , the CSF inequality follows from the equality:

$$(-\lambda(x_{s,i+1})) - (-\lambda(x_{s,i})) = \ell_{av}^* - \ell(x_{s,i}, u_{s,i}). \quad (2.53)$$

Therefore, function  $-\lambda(x)$  is a suitable candidate of terminal penalty function and the EMPC problem is shown as (2.52).  $\square$

### 2.3.5 Periodic optimal regimes of operation

This section introduces suitable terminal ingredients to specialize the stability analysis of EMPC to the case of systems with a periodic optimal regime of operation.

Denote the optimal period- $P$  solution and associated control of the optimal control problem (2.52) as  $\Pi^P := \{(x_{s,i}, u_{s,i}), i \in \mathbb{I}_{[0,P-1]}\}$  for some  $P \in \mathbb{I}_{\geq 1}$ . For convenience, we denote the projection of  $\Pi^P$  on  $\bar{\mathbb{X}}$  as  $\Pi_{\bar{\mathbb{X}}}^P$ , and we simply consider  $i \in \mathbb{I}_{[0,P-1]}$  and use  $i + j$  to represent  $(i + j) \bmod P, \forall j \in \mathbb{I}$ .

The system cost over a time period  $P$  achieves its minimum at  $\Pi^P$ , viz.

$$\begin{aligned} \sum_{i=0}^{P-1} \ell(x_{s,i}, u_{s,i}) &\leq \sum_{i=0}^{P-1} \ell(x_i, u_i), \\ \text{s.t. } x_{i+1} &= f(x_i, u_i) \\ (x_i, u_i) &\in \mathbb{Z}, i \in \mathbb{I}_{[0,P-1]}. \end{aligned} \quad (2.54)$$

For every state  $x_{s,i} \in \Pi_{\bar{\mathbb{X}}}^P$ , three important terminal ingredients, as discussed on optimal steady-state operation, [5] are adopted. A terminal region, containing  $x_{s,i}$ , is denoted by  $\mathbb{X}_f^i$ , with the corresponding terminal control policy and penalty functions denoted by  $\kappa_f^i(x)$  and  $V_f^i(x)$ , respectively.

Let us first make an assumption on the relation between  $\mathbb{X}_f^i$  and  $\kappa_f^i(x)$ :

**Assumption 2.8.** *There exists a family of compact sets  $\mathbb{X}_f^i, i \in \mathbb{I}_{[0,P-1]}$ , each containing  $x_{s,i}$ , such that for all  $x \in \mathbb{X}_f^i$ , the solutions of  $x_{t+1} = f(x_t, \kappa_f^{i+t}(x_t))$  converge exponentially to  $\Pi_{\bar{\mathbb{X}}}^P$ .*

Accordingly, the feedback law  $\kappa_f^i(x)$  which drives any state  $x \in \mathbb{X}_f^i$  to  $\mathbb{X}_f^{i+1}$  takes values in:

$$\mathbb{U}_f^i(x) := \{u \in \bar{\mathbb{U}}(x) \mid f(x, u) \in \mathbb{X}_f^{i+1}\}. \quad (2.55)$$

Then, the  $i^{\text{th}}$  set of feasible state-input pairs is given as:

$$\mathbb{Z}_f^i := \bigcup_{x \in \mathbb{X}_f^i} [\{x\} \times \mathbb{U}_f^i(x)], \quad (2.56)$$

and we let

$$\mathbb{X}_f = \bigcup_i \mathbb{X}_f^i, \quad \mathbb{U}_f(x) = \bigcup_i \mathbb{U}_f^i(x). \quad (2.57)$$

With regards to the terminal penalty functions  $V_f^i(x)$  for period- $P$  optimal operation, the concept of CSF in Definition 2.5 is adapted to give us a  $P$ -CSF as follows:

**Definition 2.10.** A  $P$ -CSF is a family of continuous functions  $V_f^i : \mathbb{X}_f^i \rightarrow \mathbb{R}$ ,  $i \in \mathbb{I}_{[0, P-1]}$  and the following inequalities hold

$$\min_{u \in \mathbb{U}_f^i(x)} V_f^{i+1}(f(x, u)) - s(x, u) \leq V_f^i(x), \quad \forall x \in \mathbb{X}_f^i, \quad (2.58)$$

where  $s : \bigcup_{i \in \mathbb{I}_{[0, P-1]}} \mathbb{Z}_f^i \rightarrow \mathbb{R}$  is the supply rate.

Notice that  $V_f^i(x)$  is the  $i^{th}$  component of a  $P$ -CSF defined for all  $x \in \mathbb{X}_f^i$ . Then, for any point  $x \in \mathbb{X}_f$  a single terminal penalty function can be defined as:

$$V_f(x) := \min_i V_f^i(x). \quad (2.59)$$

**Lemma 2.7.** The minimum of a  $P$ -CSF as in (2.58) and (2.59) is a suitable CSF.

*Proof.* Denote the  $P$ -CSF by  $\{V_f^0, V_f^1, \dots, V_f^{P-1}\}$ , and let  $x \in \mathbb{X}_f$  be arbitrary. Then for any  $i^* \in \argmin_i V_f^i(x)$ ,  $\forall x \in \mathbb{X}_f^{i^*}$ , the following inequality is satisfied,

$$V_f^{i^*+1}(f(x, \kappa_f^{i^*}(x))) \leq V_f^{i^*}(x) + s(x, \kappa_f^{i^*}(x))$$

hence, from (2.59) and the above inequality,

$$\begin{aligned} V_f(f(x, \kappa_f^{i^*}(x))) &\leq V_f^{i^*+1}(f(x, \kappa_f^{i^*}(x))) \\ &\leq V_f^{i^*}(x) + s(x, \kappa_f^{i^*}(x)) \\ &= V_f(x) + s(x, \kappa_f^{i^*}(x)). \end{aligned}$$

Therefore, there exists an admissible control policy  $\kappa_f^{i^*}(x) \in \mathbb{U}_f^{i^*}(x) \subseteq \mathbb{U}_f(x)$  such that

$$\inf_{u \in \mathbb{U}_f^{i^*}(x) \subseteq \mathbb{U}_f(x)} V_f(f(x, u)) - s(x, u) \leq V_f(x),$$

and  $V_f(x)$  defined in (2.59) is a CSF. □

Let us recall the definition of orbital stability:

**Definition 2.11.** [49, Definition 4.12] A periodic orbit  $\mathcal{O}$  is orbital stable if for all  $\epsilon > 0$ , there exists  $\delta = \delta(\epsilon) > 0$  such that if  $|x_0|_{\mathcal{O}} < \delta$ , then the state, at time  $t \in \mathbb{I}_{\geq 0}$ , fulfills

$|\varphi(t, x_0)|_{\mathcal{O}} < \epsilon$ . A periodic orbit is orbitally asymptotically stable if it is orbital stable and there exists  $\delta > 0$  such that if  $|x_0|_{\mathcal{O}} < \delta$ , then  $|\varphi(t, x_0)|_{\mathcal{O}} \rightarrow 0$  as  $t \rightarrow 0$ .

Now, it is sufficient to state the stability property of the periodic orbit by using EMPC controller with terminal region (2.57) and terminal cost function (2.59),

**Theorem 2.3.** *Let Assumptions 2.1, 2.2, 2.3, 2.4 and 2.8 hold and there exists a P-CSF with supply rate  $s(x, u) = \ell_{av}^* - \ell(x, u)$ , according to Theorem 2.2, the optimal periodic solution is orbitally asymptotically stable.*

**Remark 2.13.** In [7], periodic strict dissipativity and local controllability imply the near optimal performance of the closed-loop EMPC without terminal constraints, whereas, in our work, optimal performance is enforced by employing terminal constraints.

Notice that our notion of non-periodic dissipativity, which is motivated by the continuous-time results proposed in [43], is different from the three extended notions of dissipativity for periodic systems in [50]. Exploring the link of our concept with respect to those extensions and analyzing the case in which Assumption 2.2 does not hold is an interesting question for future investigations.

**Lemma 2.8.** *Suppose a continuous autonomous system  $x^+ = \tilde{f}(x)$  has a periodic solution. Then this periodic solution is Lyapunov asymptotically stable if and only if it is orbitally asymptotically stable.*

*Proof.* Let us denote  $\bar{\Pi}^P := \{x_{s,0}, \dots, x_{s,P-1}\}$  as the optimal  $P$ -period solution fulfilling  $x_{s,i+1} = \tilde{f}(x_{s,i})$ ,  $\forall i \in \mathbb{I}_{[0,P-2]}$  and  $x_{s,0} = \tilde{f}(x_{s,P-1})$ . The optimal state at  $t = 0$  is assigned as  $x_{s,0}$  without loss of generality.

Since the periodic solution is orbitally stable, according to [51, Theorem 1], there exists a Lyapunov function  $V(x)$  fulfilling

$$\hat{\alpha}_1(|x|_{\bar{\Pi}^P}) \leq V(x) \leq \hat{\alpha}_2(|x|_{\bar{\Pi}^P}), \quad V(\tilde{f}(x)) \leq V(x) - \hat{\alpha}_3(|x|_{\bar{\Pi}^P})$$

where  $\hat{\alpha}_1(\cdot)$ ,  $\hat{\alpha}_2(\cdot)$  and  $\hat{\alpha}_3(\cdot)$  are class  $\mathcal{K}$  functions.

We select  $\varepsilon > 0$  and  $\bar{\mathbb{X}}_f^i$  for every  $i \in \mathbb{I}_{[0,P-1]}$  as the connected components of  $\bar{\mathbb{X}}_f := \{x \mid V(x) \leq \varepsilon\}$  which contains  $x_{s,i}$ . Then,  $\bar{\mathbb{X}}_f = \bigcup_{i=\{0,\dots,P-1\}} \bar{\mathbb{X}}_f^i$  is forward invariant.

Moreover, for  $\varepsilon > 0$  sufficiently small,  $\bar{\mathbb{X}}_f^i \cap \bar{\mathbb{X}}_f^j = \emptyset$  if  $i \neq j$ , and in addition we have

$$\operatorname{argmin}_k |x - x_{s,k}| = \{i\}, \quad \forall x \in \bar{\mathbb{X}}_f^i.$$

Notice that, by [52, Proposition 4.6],  $\tilde{f}(\bar{\mathbb{X}}_f^i)$  is connected. Hence, there exist  $j$  such that  $\tilde{f}(\bar{\mathbb{X}}_f^i) \subseteq \bar{\mathbb{X}}_f^j$ . Since  $\tilde{f}(x_{s,i}) = x_{s,i+1}$ , we concluded that  $\tilde{f}(\bar{\mathbb{X}}_f^i) \subseteq \bar{\mathbb{X}}_f^{i+1}$ .

Therefore, arguing by induction, we see that solutions initiated from  $x \in \bar{\mathbb{X}}_f^0$  will evolve in phase with the optimal periodic solution, that is, the state at any time  $t \in \mathbb{I}_{\geq 0}$  which is denoted by  $\varphi(t, x)$ , fulfills  $\varphi(t, x) \in \bar{\mathbb{X}}_f^{t \bmod P}, \forall x \in \bar{\mathbb{X}}_f^0$ .

Thus, for any  $x \in \bar{\mathbb{X}}_f^0$ ,

$$\operatorname{argmin}_k |\varphi(t, x) - x_{s,k}| = \{t \bmod P\},$$

or equivalently,

$$\min_k |\varphi(t, x) - x_{s,k}| = |\varphi(t, x) - \varphi(t, x_{s,0})|,$$

since

$$\varphi(t, x_{s,0}) = x_{s,t \bmod P}.$$

By substituting the above equality into the following definition of orbital asymptotic stability:

$$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \text{ such that } \min_{k \in \mathbb{I}_{[0, P-1]}} |x - x_{s,k}| \leq \delta_1 \Rightarrow \min_{k \in \mathbb{I}_{[0, P-1]}} |\varphi(t, x) - x_{s,k}| \leq \epsilon_1, \forall t \in \mathbb{I}_{\geq 0},$$

and

$$\exists \delta_2 > 0, \forall x, \text{ such that } \min_{k \in \mathbb{I}_{[0, P-1]}} |x - x_{s,k}| \leq \delta_2, \text{ it holds } \lim_{t \rightarrow +\infty} \min_{k \in \mathbb{I}_{[0, P-1]}} |\varphi(t, x) - x_{s,k}| = 0,$$

we obtain the conditions for Lyapunov asymptotic stability as follows

$$\forall \epsilon_1 > 0, \exists \delta_1 > 0, \text{ such that } |x - x_{s,0}| \leq \delta_1 \Rightarrow |\varphi(t, x) - \varphi(t, x_{s,0})| \leq \epsilon_1, \forall t \in \mathbb{I}_{\geq 0}.$$

and

$$\exists \delta_2 > 0, \forall x, \text{ such that } |x - x_{s,0}| \leq \delta_2, \text{ it holds } \lim_{t \rightarrow +\infty} |\varphi(t, x) - \varphi(t, x_{s,0})| = 0.$$

Hence, the conclusions of the lemma follow.  $\square$

**Assumption 2.9.** *The value function  $V_N(\cdot)$  and the optimal control policy  $\kappa_N(\cdot)$  : as in (2.23) and (2.29) are continuous functions of  $x$  in a neighborhood of the optimal periodic trajectory.*

**Remark 2.14.** The continuity properties of the value function and optimal control law in constrained optimal control problems have been extensively studied in the literature. For instance, the conditions of [53, Corollary 5.4.2, Theorem 5.4.3] that functions  $f(\cdot, \cdot)$ ,  $\ell(\cdot, \cdot)$  being continuous and the admissible control input  $\bar{\mathbb{U}}(\cdot)$  being continuous and compact valued, imply  $V_N(\cdot)$  is continuous and  $u^*(\cdot)$  is outer semicontinuous. If, in addition, the minimizer is unique, then  $u^*(\cdot)$  is continuous at  $x$ . However, in general, the continuity of the set map  $\bar{\mathbb{U}}(\cdot)$  and thus the optimal value function  $V_N(\cdot)$  is difficult to be satisfied.

Now, it is sufficient to have a statement on Lyapunov stability of the optimal periodic solution.

**Corollary 2.2.** *If Assumption 2.9 holds, the optimal periodic solution is orbitally asymptotically stable and asymptotically stable in the sense of Lyapunov.*

**Remark 2.15.** Orbital stability only ensures that state trajectories converge to the orbit and there is no information on the phase. However, asymptotic stability in the sense of Lyapunov is a stronger version of stability which tracks the phase of the optimal solution.

We propose next a constructive formula for computing  $P$ -CSF. In particular,

$$\bar{V}_f^i(x) = \sum_{k=0}^{\infty} L(x_k^i, \kappa_f^{i+k}(x_k^i)). \quad (2.60)$$

where  $x_k^i$  is the solution of  $x_{k+1}^i = f(x_k^i, \kappa_f^{i+k}(x_k^i))$  initiated at  $x_0^i = x \in \mathbb{X}_f^i$ .

To analyze the convergence of (2.60), we first consider Lemma 2.6 in the context of  $P$ -periodic optimal regimes of operation. Notice that Assumption 2.5 is trivially satisfied and this implies the rotated stage cost at the periodic solution fulfills

$$L(x_{s,i}, u_{s,i}) = 0 \quad \forall i \in \mathbb{I}_{[0, P-1]} \quad (2.61)$$

**Assumption 2.10.** *There exists a  $\delta_L > 0$  such that the functions  $\ell(\cdot)$  and  $\lambda(\cdot)$ , and hence the rotated stage cost  $L(\cdot)$  are Lipschitz continuous for all  $(x, u)$  fulfilling  $|(x, u)|_{\Pi^P} < \delta_L$ . Then, the corresponding Lipschitz constants are denoted by  $L_\ell$ ,  $L_\lambda$  and  $L_L$ . Moreover, for all  $i \in \mathbb{I}_{[0, P-1]}$ ,  $\kappa_f^i(x)$  is Lipschitz continuous with respect to  $x$ .*

Since Assumption 2.8 implies that there exist constant real numbers  $A$  and  $|a| < 1$  such that  $|(x_k, u_k)|_{\Pi^P} \leq A \cdot |a|^k$ . Together with Assumption 2.10 and (2.61), we conclude that  $|L(x_k, u_k)| \leq L_L \cdot A \cdot |a|^k$ , and therefore the  $P$ -CSF in (2.60) is upper bounded by  $\frac{L_L \cdot A}{1-a}$ .

Notice that, provided the series converges for arbitrary  $i \in \mathbb{I}_{[0, P-1]}$ , it is straightforward that

$$\begin{aligned} \bar{V}_f^i(x) &= L(x, \kappa_f^i(x)) + \sum_{k=1}^{\infty} L(x_k^i, \kappa_f^{i+k}(x_k^i)) \\ &= L(x, \kappa_f^i(x)) + \sum_{k=0}^{\infty} L(x_k^{i+1}, \kappa_f^{i+k}(x_k^{i+1})) \\ &= L(x, \kappa_f^i(x)) + \bar{V}_f^{i+1}(f(x, \kappa_f^i(x))). \end{aligned} \tag{2.62}$$

Thus, the proposed  $P$ -CSF follows Lemma 2.2 and the terminal cost function is selected as

$$\bar{V}_f(x) = \min_i \bar{V}_f^i(x). \tag{2.63}$$

Moreover, knowledge of the storage function  $\lambda(\cdot)$  used in the definition of rotated stage and terminal costs is not needed in designing the EMPC controller. To see this, we consider the terminal cost

$$V_f(x) = \min_i V_f^i(x), \tag{2.64}$$

where

$$\begin{aligned}
V_f^i(x) &= \bar{V}_f^i(x) - \lambda(x) \\
&= \left[ \sum_{k=0}^{\infty} L(x_k^i, \kappa_f^{i+k}(x_k^i)) \right] - \lambda(x) \\
&= \left[ \sum_{k=0}^{\infty} \left[ \ell(x_k^i, \kappa_f^{i+k}(x_k^i)) + \lambda(x_k^i) - \ell_{av}^* - \lambda(f(x_k^i, \kappa_f^{i+k}(x_k^i))) \right] \right] - \lambda(x) \\
&= \left[ \sum_{m=0}^{\infty} \left[ \sum_{k=0}^{P-1} \left[ \ell(x_{mP+k}^i, \kappa_f^{i+k}(x_{mP+k}^i)) - \ell_{av}^* \right] - \lambda(x_{(m+1)P}^i) + \lambda(x_{mP}^i) \right] \right] - \lambda(x) \\
&= \lim_{M \rightarrow +\infty} \left[ \sum_{m=0}^M \left[ \sum_{k=0}^{P-1} \left[ \ell(x_{mP+k}^i, \kappa_f^{i+k}(x_{mP+k}^i)) - \ell_{av}^* \right] - \lambda(x_{(M+1)P}^i) \right] \right] \\
&= \left[ \sum_{m=0}^{\infty} \sum_{k=0}^{P-1} \left[ \ell(x_{mP+k}^i, \kappa_f^{i+k}(x_{mP+k}^i)) - \ell_{av}^* \right] \right] - \lim_{M \rightarrow +\infty} \lambda(x_{(M+1)P}^i) \\
&= \left[ \sum_{m=0}^{\infty} \sum_{k=0}^{P-1} \left[ \ell(x_{mP+k}^i, \kappa_f^{i+k}(x_{mP+k}^i)) - \ell_{av}^* \right] \right] - \lambda(x_{s,i}).
\end{aligned} \tag{2.65}$$

Notice that this  $V_f^i(x)$  is entirely dictated by the stage cost, provided the value of the storage function  $\lambda(x_{s,i})$ ,  $i \in \mathbb{I}_{[0,P-1]}$  is known.

**Remark 2.16.** Thanks to independence of  $i$  in the terminal constraint  $x \in \mathbb{X}_f$ , our setup shows a larger region where the EMPC problem is feasible without the need for a long prediction horizon. However, a disadvantage of this terminal constraint is the storage function must be known a priori. In the paper written by Zanon et al,[24] a periodic ( $i$ -dependent) terminal set  $\mathbb{X}_f^{N+i}$  is used in the EMPC problem formulation, where the dissipativity is periodic, the phase is fixed and thus the terminal cost function (2.65) can reduce to the following without the storage function:

$$V_f^i(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{P-1} (\ell(x_{mP+k}^i, \kappa_f^{i+k}(x_{mP+k}^i)) - \ell_{av}^*). \tag{2.66}$$

### 2.3.6 SOSTOOLs for storage functions

The MATLAB toolbox SOSTOOLs proposed in [54] can be used to determine a candidate for  $\lambda(\cdot)$  in polynomial form. SOSTOOLs is a free MATLAB toolbox which solves two types of sum of squares programs: the feasibility and optimization problems. This toolbox has been applied to many control problems, such as construction of Lyapunov functions, state feedback control synthesis, and nonlinear optimal control [55, 56].

In this paper, in order to address the construction of storage functions, we consider the optimization problem as follows:

$$\min_{\underline{\ell}, c_0, c_1, \dots, c_P} \underline{\ell} \quad (2.67)$$

such that

$$\lambda(x) = \sum_{i=0}^P c_i x^i \quad (2.68a)$$

$$\begin{aligned} \lambda(x) - \lambda(f(x, u)) + \ell(x, u) - \underline{\ell} \text{ is sum of squares } (\geq 0), \\ \forall (x, u) \in \overline{\mathbb{Z}}, \end{aligned} \quad (2.68b)$$

where it requires that the system dynamic function  $f$  to be polynomial. In this formulation,  $x$  and  $u$  are independent scalar variables which are created as symbolic variables in MATLAB, whereas  $\underline{\ell}$ ,  $c_i$  are decision variables.

The objective in (2.67) is to minimize the lower bound of asymptotic average. (2.68a) shows the construction of the storage function that is formulated as a polynomial of order  $P$ . In addition, (2.68b) comes from the dissipation inequality or the rotated stage cost, this inequality must hold for all admissible state-input pairs.

By using the returned values of coefficients, the storage function can be formulated as a polynomial function of the system state.

## 2.4 Numerical examples

### 2.4.1 No gap: $\bar{\ell} = \underline{\ell}$ - optimal period-2 operation

Consider the nonlinear system, known as a logistic map or demographic model, described by the following difference equation

$$x^+ = ux(1 - x) \quad (2.69)$$

in which  $x$  is the ratio between current population and the maximum possible population with value in  $[0, 1]$  and a parameter  $u$  in the interval  $(0, 4]$ . To avoid the trivial case  $x = 0$ , we only consider the control invariant set  $\bar{\mathbb{X}} := [\epsilon, 1 - \epsilon]$ ,  $\bar{\mathbb{U}} := [\frac{1}{1-\epsilon}, 4(1 - \epsilon)]$  and  $\bar{\mathbb{Z}} = \bar{\mathbb{X}} \times \bar{\mathbb{U}}$ , where  $\epsilon$  is a small positive value. Throughout this example, we select  $\epsilon = 0.01$ .

In EMPC, the cost can be any function, so we use the following stage cost which is not meant to be a physically or economically motivated example:

$$\ell(x, u) = -x^4.$$

The optimal steady-state operation based on this stage cost is  $(x_s, u_s) = (0.7475, 3.96)$  with an economic cost  $\ell(x_s, u_s) = -0.3122$ . However, a periodic operation regime can be achieved and it can be expressed as

$$x_{0,1}^* = \frac{(5 - 4\epsilon) \mp \sqrt{(1 - 4\epsilon)(5 - 4\epsilon)}}{8(1 - \epsilon)}, \quad u_{0,1}^* = 4(1 - \epsilon).$$

Numerically, the optimal periodic operation is  $x_{s,0} \approx 0.3507$  and  $x_{s,1} \approx 0.9018$  with optimal control  $u_0^* = u_1^* = 3.96$ , so that the best asymptotic average cost for all initial states  $x \in \bar{\mathbb{X}}$  is  $\underline{\ell} = \ell_{av}^* = \ell_{av}^*(x) = \bar{\ell} = -\frac{(x_{s,0})^4 + (x_{s,1})^4}{2} \approx -0.3382 < \ell(x_s, u_s)$ .

To construct the terminal cost function, a terminal state feedback control law is needed  $\kappa_f^i(x) = u_{s,i} + K_i(x - x_{s,i})$ ,  $i \in \mathbb{I}_{[0,1]}$ , where  $K_i$  is determined from the linearized system of (2.69) along the optimal solution that is

$$\begin{aligned} \delta x_0^+ &= -3.1821\delta x_1 + 0.0886\delta u_1, \\ \delta x_1^+ &= 1.1821\delta x_0 + 0.2277\delta u_0. \end{aligned} \quad (2.70)$$

Correspondingly, constraints of this linearized system are

$$\begin{aligned} -0.3407 \leq \delta x_0 \leq 0.6393, \quad -2.9499 \leq \delta u_0 \leq 0, \\ -0.8918 \leq \delta x_1 \leq 0.0882, \quad -2.9499 \leq \delta u_1 \leq 0. \end{aligned} \quad (2.71)$$

Since there is no  $[K_0 \ K_1]^T$  able to stabilize system (2.70) within the whole region in (2.71), we consider maximizing the region of attraction of the linearized system by solving a bilinear matrix inequality problem based on the method.[57] The resulting feedback gain is  $[K_0 \ K_1]^T = [-5.2 \ 10.75]^T$  which is able to stabilize the linear system for  $0 \leq \delta x_0 \leq 0.5681$  and  $-0.2606 \leq \delta x_1 \leq 0$ .

Next, we determine a potentially smaller set where the above control policy works for the nonlinear system (2.69). If the terminal state feedback control law is implemented into (2.69), that is,

$$\begin{aligned} x_{s,0} + \delta x_0^+ &= (u_1^* + K_1 \cdot \delta x_1)(x_{s,1} + \delta x_1)(1 - x_{s,1} - \delta x_1) \\ x_{s,1} + \delta x_1^+ &= (u_0^* + K_0 \cdot \delta x_0)(x_{s,0} + \delta x_0)(1 - x_{s,0} - \delta x_0), \end{aligned}$$

the nonlinear dynamic of state deviation by using state feedback control, after substituting the values of optimal periodic solutions and state feedback gains, becomes

$$\begin{aligned} \delta x_0^+ &= -10.75 \delta x_1^3 - 12.6 \delta x_1^2 - 2.23 \delta x_1, \\ \delta x_1^+ &= 5.2 \delta x_0^3 - 5.513 \delta x_0^2 - 0.0016 \delta x_0. \end{aligned}$$

Trajectories described by these two equations admit a small region  $0 \leq \delta x_0 \leq 0.22$  and  $-0.21 \leq \delta x_1 \leq 0$  such that any interior point can be attracted to the origin without constraints violation. Therefore, the terminal region is  $\mathbb{X}_f = \mathbb{X}_f^0 \cup \mathbb{X}_f^1 = [0.3507, 0.5707] \cup [0.6918, 0.9018]$ .

Then, according to the equations in (2.64) and (2.65), it is sufficient to construct the terminal cost function as follows

$$V_f(x) = \min_i \left[ \sum_{m=0}^{\infty} \sum_{k=0}^{P-1} \left[ \ell(x^i(mP+k), \kappa_f^{i+k}(x^i(mP+k))) - \ell_{av}^* \right] \right] - \lambda(x_{s,i}),$$

where  $\kappa_f^i(x) = u_{s,i} + K_i(x - x_{s,i})$ .

Moreover, by using SOSTOOLS, a candidate polynomial storage function of 3rd order which fulfills dissipativity approximately is

$$\lambda(x) = 0.30471x^3 - 0.81183x^2 + 1.2215x.$$

Fig.2.1 shows the closed loop state transition and input at initial condition  $x = 0.5$  and prediction horizon  $N = 6$ . It can be seen that the state trajectory converges to the optimal periodic solution in several steps with the control input remaining at its upper bound. It also shows the convergence of asymptotic average performance to  $\bar{\ell} = \underline{\ell}$  and the decreasing of the optimal cost-to-go which is therefore a Lyapunov function. Moreover, the average computational time of *fmincon* is 0.0101s approximately, and the average number of calls for objective functions and nonlinear constraints are around 94 times.

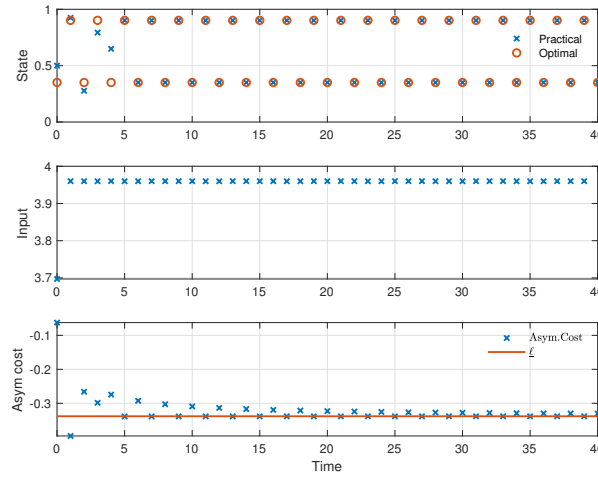


Fig. 2.1 State trajectory, optimal input and asymptotic average performance of the logistic map system (2.69).

### 2.4.2 With gap: $\bar{\ell} > \underline{\ell}$ - optimal period-4 operation

Next, we consider a bi-dimensional nonlinear system of equations:

$$x^+ = \begin{bmatrix} \frac{1-u^2}{1+u^2} & \frac{2u}{1+u^2} \\ \frac{-2u}{1+u^2} & \frac{1-u^2}{1+u^2} \end{bmatrix} x \quad (2.72)$$

where  $\mathbf{x}$  is the state variable and  $u$  is the input. This nonlinear system gives solutions which rotate on a circle with radius that is equal to the norm of the initial state. Thus, only states on this circle are reachable.

Consider the state space  $\mathbb{X} := \{|\mathbf{x}| \leq 1\}$  and input constraint  $u \in \mathbb{U} := (-\infty, 0]$ , along with the stage cost:

$$\ell(\mathbf{x}, u) = -(\mathbf{x}^T Q \mathbf{x})^2 + (u + 1)^2, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This results into the globally optimal periodic solution  $\bar{\mathbf{x}}_0^* = [1, 0]^T$ ,  $\bar{\mathbf{x}}_1^* = [0, 1]^T$ ,  $\bar{\mathbf{x}}_2^* = [-1, 0]^T$  and  $\bar{\mathbf{x}}_3^* = [0, -1]^T$ .

However, due to the structure of reachable sets, any given initial condition with distance to the origin  $|\mathbf{x}| = r$ ,  $0 < r < 1$  moves within a robustly invariant set  $\bar{\mathbb{X}}_r := \{\mathbf{x} \mid |\mathbf{x}| = r\}$ . Then, a sub-optimal periodic operation is  $\mathbf{x}_0^* = [r, 0]^T$ ,  $\mathbf{x}_1^* = [0, r]^T$ ,  $\mathbf{x}_2^* = [-r, 0]^T$  and  $\mathbf{x}_3^* = [0, -r]^T$  with optimal control  $u_0^* = u_1^* = u_2^* = u_3^* = -1$ . Thus, there is a gap between  $\ell_{av}^*(\mathbf{x})$  and  $\underline{\ell}$ ,

$$\underline{\ell} = \ell_{av}^* = -1 \leq \ell_{av}^*(\mathbf{x}) = -|\mathbf{x}|^4.$$

The terminal control policy is expected to be a continuous function of state  $\mathbf{x}$ . Therefore, we consider a terminal feedback  $\kappa_f(\mathbf{x})$ ,  $\mathbf{x} \in \mathbb{X}_f = \bar{\mathbb{X}}_r \setminus \bar{\mathbb{X}}_s$  where  $\bar{\mathbb{X}}_s := \{\mathbf{x} \mid \frac{(2k+1)\pi}{4} - \epsilon \leq \arg(\mathbf{x}) \leq \frac{(2k+1)\pi}{4} + \epsilon, k \in \mathbb{I}\}$ , for any arbitrarily small number  $\epsilon > 0$ , which forces the system to evolve according to the following dynamic in polar coordinate,

$$\theta^+ = \theta + \frac{\pi}{2} - K \sin(4\theta),$$

where  $\theta$  is the angle of the state vector with respect to the positive horizontal axis and  $K$  is the state feedback gain. Since the optimal solution evolves  $\frac{\pi}{2}$  radius counterclockwise every step,  $K \sin(4\theta)$  is used to reduce the gap between the current state to the optimal ones. To keep in phase with the rotation of periodic solution,  $K$  should be selected from  $(0, \frac{1}{4}]$ . In this example,  $K = \frac{1}{4}$  for faster convergence.

Then, with  $\kappa_f(\cdot)$  defined above, the terminal cost function for  $\mathbf{x} \in \mathbb{X}_f$  is

$$V_f(x) = \sum_{m=0}^{\infty} \sum_{k=0}^{P-1} (\ell(\mathbf{x}(mP + k), \kappa_f(\mathbf{x}(mP + k))) - \ell_{av}^*).$$

Notice that, this terminal cost function will only be finite for  $x$  in  $\{x \mid |x| = 1\}$  and Lyapunov asymptotic stability discussed in Section 2.3 applies for  $\{x \mid |x| = 1\}$ . In other words, we have restricted our area of interest to the unit circle and therefore artificially enforced the Assumption 2.2. Furthermore, a candidate of storage function is

$$\lambda(x) = (x^T Q x)^2.$$

The state trajectory initialized at  $x = [0.48 \ 0.64]^T$  and with a prediction horizon  $N = 6$  is shown in Fig.2.2 in which we see that closed-loop system catches the periodic solution after several control moves. The system average performance for this cost objectives convergences to  $\ell_{av}^*(x) = -0.4096$  instead of  $\underline{\ell} = \ell_{av}^* = -1$ . Moreover, the average computational time of  $fmincon$  is  $0.2805s$  approximately, and the average number of calls for objective functions and nonlinear constraints are around 160 times.

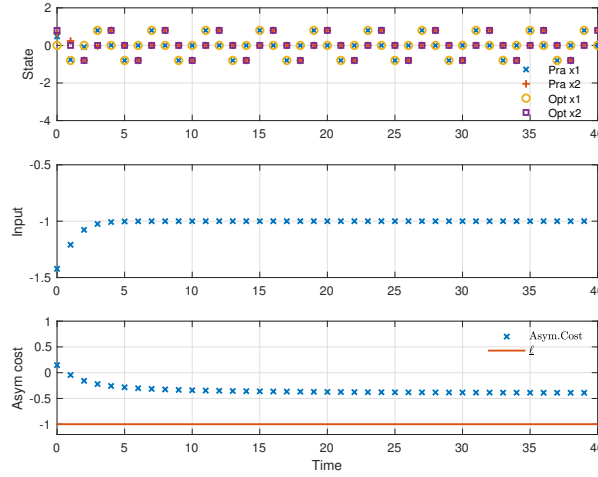


Fig. 2.2 State trajectory, optimal input and asymptotic average performance of the bi-dimensional nonlinear system (2.72) with optimal period-4 operation.

### 2.4.3 Controllable Non-periodic optimal operation

We consider a bi-dimensional nonlinear control system with two input variables:

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = u_1 \cdot \begin{bmatrix} \frac{1-u_2^2}{1+u_2^2} & \frac{2u_2}{1+u_2^2} \\ \frac{-2u_2}{1+u_2^2} & \frac{1-u_2^2}{1+u_2^2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2.73)$$

where  $\mathbf{x} = [x_1 \ x_2]^T$  is the state variable and  $\mathbf{u} = [u_1 \ u_2]^T$  is the input. This nonlinear system gives state trajectories whose rotation speed and distance to the origin are controlled by  $u_2$  and  $u_1$ , respectively.

Consider  $\mathbb{X} := \{\mathbf{x} \mid \|\mathbf{x}\| \leq 8\}$  with input admissible set  $\mathbb{U} := [\frac{1}{2}, 2] \times [0, 1]$ , and  $\mathbb{Z} := \mathbb{X} \times \mathbb{U}$ .

The stage cost

$$\ell(\mathbf{x}, \mathbf{u}) = \frac{1}{3}x_1 + \left|u_2 - \frac{1}{2}\right| + \left|\mathbf{x}^T \mathbf{x} - 1\right|$$

results into the global optimal trajectory that rotates on the unit circle at  $-\arccos(\frac{3}{5}) \text{ rad/step}$  corresponding to  $u_2 = \frac{1}{2}$ . Since  $2\pi$  is not commensurable to this speed, the optimal solution is not periodic and asymptotically approaches the whole unit circle which is denoted by  $\Pi_{\mathbb{X}}$ . As a consequence, the optimal asymptotic average performance is expected to be  $\underline{\ell} = \ell_{av}^* = 0$ .

We choose the following candidate storage function:

$$\lambda(\mathbf{x}) = -\frac{x_1}{6\sqrt{x_1^2 + x_2^2}} - \frac{x_2}{3\sqrt{x_1^2 + x_2^2}}. \quad (2.74)$$

To see this storage function fulfills the strict dissipativity, we transform the system (2.73) into polar coordinate as follows:

$$\begin{bmatrix} u_1 \cdot r \cdot \cos(\theta - \varphi) \\ u_1 \cdot r \cdot \sin(\theta - \varphi) \end{bmatrix} = u_1 \cdot \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix} \cdot \begin{bmatrix} r \cdot \cos(\theta) \\ r \cdot \sin(\theta) \end{bmatrix}$$

where  $[x_1, x_2]^T = [r \cdot \cos(\theta), r \cdot \sin(\theta)]^T$  and  $u_2 = \sqrt{\frac{1-\cos(\varphi)}{1+\cos(\varphi)}}$ . Particularly, as stated above, when  $u_2 = \frac{1}{2}$ ,  $\varphi = -\arccos(\frac{3}{5}) \text{ rad/step}$ . Based on this equivalent representation, the candidate storage function becomes

$$\lambda(r, \theta) = -\frac{\cos(\theta)}{6} - \frac{\sin(\theta)}{3}.$$

Then, let us consider the following function from the definition of dissipation inequality

$$S(\mathbf{x}, \mathbf{u}) = \lambda(\mathbf{x}) - \lambda(f(\mathbf{x}, \mathbf{u})) + \ell(\mathbf{x}, \mathbf{u}) - \ell_{av}^* - \frac{1}{2} \left| \mathbf{x}^T \mathbf{x} - 1 \right|.$$

Converting it to polar coordinates, we have

$$S(r, \theta, \varphi) = -\frac{\cos(\theta)}{6} - \frac{\sin(\theta)}{3} + \frac{\cos(\theta - \varphi)}{6} + \frac{\sin(\theta - \varphi)}{3} \\ + \frac{r \cdot \cos(\theta)}{3} + \left| \sqrt{\frac{1 - \cos(\varphi)}{1 + \cos(\varphi)}} - \frac{1}{2} \right| - \ell_{av}^* + \frac{1}{2} |r^2 - 1|.$$

Notice that for  $u_2 = \frac{1}{2}$  and  $r = 1$ , we have

$$\begin{cases} \frac{2u_2^2}{1+u_2^2} - \frac{1-u_2^2}{1+u_2^2} - (2r-1) = 0 \\ \frac{2(1-u_2^2)}{1+u_2^2} + \frac{2u_2^2}{1+u_2^2} - 2 = 0. \end{cases}$$

and in polar coordinates

$$\begin{cases} 2\sin(\varphi) - \cos(\varphi) - (2r-1) = 0 \\ 2\cos(\varphi) + \sin(\varphi) - 2 = 0 \end{cases}$$

which in turn implies that

$$\frac{\partial S}{\partial \theta} = 0, \quad \forall \theta \in \mathbb{R}.$$

It is seen that for all  $z \in \{z \mid |z| = 1\}$ ,  $v_1 \in \mathbb{R}$  and all  $(x, u) \in \mathbb{Z}$ , it holds

$$0 = S(z, [v_1, \frac{1}{2}]^T) \leq S(x, u),$$

explicitly,

$$\lambda(x) - \lambda(f(x, u)) + \ell(x, u) - \ell_{av}^* \geq \frac{1}{2} |x^T x - 1|.$$

Since the following inequality holds

$$\frac{1}{2} |x^T x - 1| = \frac{1}{2} |(|x| - 1)(|x| + 1)| = \frac{1}{2} ||x| + 1| \cdot ||x| - 1| = \frac{1}{2} ||x| + 1| \cdot |x|_{\Pi_{\mathbb{X}}} = \rho(|x|_{\Pi_{\mathbb{X}}}),$$

the storage function (2.74) fulfills strict dissipativity, viz.

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(|x|_{\Pi_{\mathbb{X}}}) + \ell(x, u) - \ell_{av}^*, \quad \forall (x, u) \in \mathbb{Z},$$

where  $\rho(|x|_{\Pi_{\mathbb{X}}})$  is a positive definite function with respect to  $\Pi_{\mathbb{X}}$ .

Given this candidate storage function, the rotated stage cost is

$$L(\mathbf{x}, \mathbf{u}) = \ell(\mathbf{x}, \mathbf{u}) + \lambda(\mathbf{x}) - \lambda(f(\mathbf{x}, \mathbf{u})) - \ell_{av}^* \geq \rho(|\mathbf{x}|_{\Pi_{\mathbb{X}}}), \quad (2.75)$$

and particularly  $L(\mathbf{x}, \mathbf{u}) = 0$  for state trajectories on unit circle with associated inputs  $\mathbf{u} = [1, \frac{1}{2}]^T$ .

Furthermore, the terminal cost function is constructed as

$$\bar{V}_f(\mathbf{x}) = \sum_{k=0}^{\infty} L(\mathbf{x}(k), \mathbf{u}_f(\mathbf{x}(k))).$$

where

$$\mathbf{u}_f(\mathbf{x}) = \begin{cases} [\max(\frac{1}{2}, \frac{1}{|\mathbf{x}|}), \frac{1}{2}]^T, & |\mathbf{x}| > 1 \\ [1, \frac{1}{2}]^T, & |\mathbf{x}| = 1 \\ [\min(2, \frac{1}{|\mathbf{x}|}), \frac{1}{2}]^T, & |\mathbf{x}| < 1 \end{cases}$$

such that  $\mathbf{u}_f(\mathbf{x}) \in \mathbb{U}, \forall \mathbf{x} \in \mathbb{X}$  and terminal region  $\mathbb{X}_f = \mathbb{X}$ . Since the first component of  $\mathbf{u}_f(\mathbf{x})$  controls the radius of state trajectory to 1 and the second component is  $u_2 = \frac{1}{2}$ , the rotated stage cost  $L(\mathbf{x}, \mathbf{u}_f(\mathbf{x}))$  reaches the value 0 in a finite number of steps; therefore, the series in (2.75) converges.

Moreover, the terminal cost definition implies the following equality:

$$\bar{V}_f(f(\mathbf{x}, \mathbf{u}_f(\mathbf{x}))) - \bar{V}_f(\mathbf{x}) = -L(\mathbf{x}, \mathbf{u}_f(\mathbf{x}))$$

which in turn from Lemma 2.2 implies the non-rotated terminal cost function  $V_f(\mathbf{x})$  is a CSF.

The resulting state trajectory initialized at  $\mathbf{x} = [-3, 4]^T$  and with a prediction horizon  $N = 6$  is shown in Fig.2.3 in which we see that the unit circle and solutions are stabilized and rotate with the speed corresponding to  $u_2 = \frac{1}{2}$ . Notice that there is a phase shift between the practical and optimal state angles which is formed during the transition period of radius. However, after the radius achieves 0.8, angular velocity keeps at the optimal value, in other words, the phase lag of the practical angle with respect to the optimal one is constant. In addition, the system average performance for this cost objective converges to  $\underline{\ell} = \ell_{av}^*(x) = 0$ . The average computational time of *fmincon* is 0.0362s approximately, and

the average number of calls for objective functions and nonlinear constraints are around 100 times.

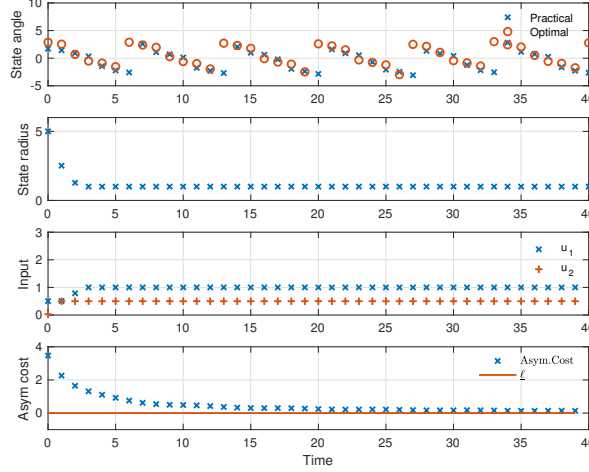


Fig. 2.3 State trajectory, optimal input and asymptotic average performance of the bi-dimensional nonlinear system (2.73) with controllable optimal non-periodic operation.

#### 2.4.4 Uncontrollable Non-periodic optimal operation

We consider a bi-dimensional nonlinear control system with single input variable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{(1 + (1 - (x_1^2 + x_2^2))^2)}{4 + (1 - x_1^2 + u)^2} \cdot \begin{bmatrix} 4 - (1 - x_1^2 + u)^2 & -4(1 - x_1^2 + u) \\ 4(1 - x_1^2 + u) & 4 - (1 - x_1^2 + u)^2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2.76)$$

where  $\mathbf{x} = [x_1 \ x_2]^T$  is the state variable and  $u$  is the input. Suppose the state space is  $\mathbb{X} := \{\mathbf{x} \mid |\mathbf{x}| \leq 1\}$ , input admissible set is  $\mathbb{U} := [0, \frac{1}{2}]$  and  $\mathbb{Z} := \mathbb{X} \times \mathbb{U}$ . Any initial state within the unit disk results into a spiral approaching to the unit circle and the rotation speed is dependent on the current location. Let us consider the stage cost

$$\ell(\mathbf{x}, u) = 1 - x_1^2. \quad (2.77)$$

which enforces the trajectory to evolve mostly around  $\Pi_{\mathbb{X}} = \{[1, 0]^T, [-1, 0]^T\}$ , and the number of states close to these two points is increasing since the growth of  $x_1$  slows down the rotation speed. As a consequence, the asymptotic average performance is expected to be  $\underline{\ell} = 0$ .

We choose the following candidate storage function:

$$\lambda(\mathbf{x}) = -(x_1^2 + x_2^2). \quad (2.78)$$

To see this storage function fulfills the strict dissipativity, it is noticed that

$$\begin{aligned} S(\mathbf{x}, u) &= \lambda(\mathbf{x}) - \lambda(\mathbf{x}^+) + \ell(\mathbf{x}, u) - \ell_{av}^* - \rho(\|\mathbf{x}\|_{\Pi_{\mathbb{X}}}) \\ &= -(x_1^2 + x_2^2) + ((x_1^+)^2 + (x_2^+)^2) + (1 - x_1^2) - \frac{1}{2}((1 - x_1^2)^2 + x_2^4) \\ &= -(x_1^2 + x_2^2) + ((x_1^+)^2 + (x_2^+)^2) + (1 - x_1^2) - \frac{1}{2}(1 - 2x_1^2 + x_1^4 + x_2^4) \\ &= -(x_1^2 + x_2^2) + ((x_1^+)^2 + (x_2^+)^2) + \frac{1}{2}(1 - (x_1^4 + x_2^4)) \geq 0 \end{aligned}$$

where  $\rho(\|\mathbf{x}\|_{\Pi_{\mathbb{X}}}) = \frac{1}{2}((1 - x_1^2)^2 + x_2^4)$  is strict positive definite with respect to  $\Pi_{\mathbb{X}}$ , hence the strict dissipativity holds

$$\lambda(\mathbf{x}^+) - \lambda(\mathbf{x}) \leq -\rho(\|\mathbf{x}\|_{\Pi_{\mathbb{X}}}) + \ell(\mathbf{x}, u) - \ell_{av}^*.$$

However, for the stage cost in (2.77), CSF does not exist, since it should fulfill

$$V_f(\mathbf{x}^+) - V_f(\mathbf{x}) \leq \ell_{av}^* - \ell(\mathbf{x}, u) = x_1^2 - 1.$$

By analyzing the state equation (2.76) and expressing  $\mathbf{x}$  in polar coordinates, the rotation angle achieved in a single step is between  $0^\circ$  and  $73.74^\circ$  counter-clockwise, which are obtained at  $x_1 = 1, u = 0$  and  $x_1 = 0, u = \frac{1}{2}$ , respectively. Therefore,  $x_1^2 - 1$  does not converge to zero, which implies  $V_f(\mathbf{x})$  is not bounded from below along the trajectory.

Therefore, we choose another stage cost

$$\ell(\mathbf{x}, u) = 1 - x_1^2 - x_2^2 + e^{-\frac{4x_2^2}{x_1^2}} \cdot u.$$

and expand the optimal set to the unit circle  $\Pi'_{\mathbb{X}} = \{\mathbf{x} \mid |\mathbf{x}| = 1\}$ . In this case, storage function in (2.78) is still applicable, and the strict dissipativity holds as

$$\begin{aligned} S(\mathbf{x}, u) &= \lambda(\mathbf{x}) - \lambda(\mathbf{x}^+) + \ell(\mathbf{x}, u) - \ell_{av}^* - \rho(\mathbf{x}|_{\Pi'_{\mathbb{X}}}) \\ &= -(x_1^2 + x_2^2) + ((x_1^+)^2 + (x_2^+)^2) + (1 - x_1^2 - x_2^2) + e^{-\frac{4x_2^2}{x_1^2}} \cdot u - \frac{1}{2}(1 - x_1^2 - x_2^2) \\ &= -(x_1^2 + x_2^2) + ((x_1^+)^2 + (x_2^+)^2) + \frac{1}{2}(1 - x_1^2 - x_2^2) + e^{-\frac{4x_2^2}{x_1^2}} \cdot u \geq 0 \end{aligned}$$

where  $\rho(\mathbf{x}|_{\Pi'_{\mathbb{X}}}) = \frac{1}{2}(1 - x_1^2 - x_2^2)$  is strict positive definite with respect to  $\Pi'_{\mathbb{X}}$ .

Next, the terminal cost function is defined as

$$V_f(\mathbf{x}) = \sum_{k=0}^{\infty} \ell(\mathbf{x}(k), 0) = \sum_{k=0}^{\infty} 1 - x_1^2(k) - x_2^2(k),$$

and thus the CSF inequality is satisfied

$$V_f(\mathbf{x}^+) - V_f(\mathbf{x}) = -\ell(\mathbf{x}, 0) \leq \ell_{av}^* - \ell(\mathbf{x}, 0) = x_1^2 + x_2^2 - 1.$$

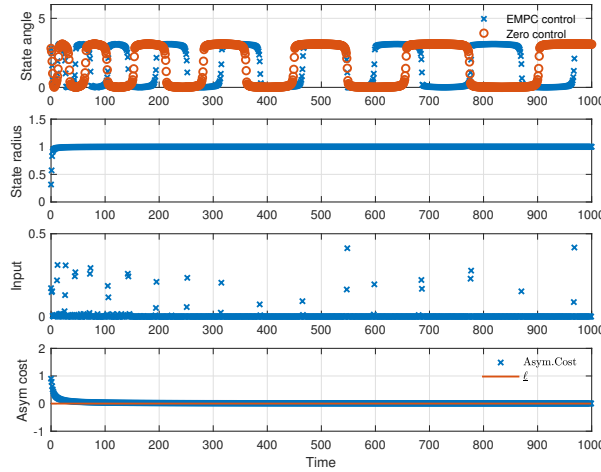


Fig. 2.4 State trajectory, optimal input and asymptotic average performance of the bi-dimensional nonlinear system (2.76) with uncontrollable optimal non-periodic operation.

Finally, the system performance starting from  $x = [-0.3, 0.1]^T$  is shown in Fig.2.4. It can be seen that the trajectory is approaching to the unit circle and the system has shorter transition period by using the EMPC controller. Moreover, there are no inputs when states are close to  $[\pm 1, 0]$ , whereas non-zero inputs are applied if the state is far from these two points. The average computational time of *fmincon* is 0.2589s approximately, and the average number of calls for objective functions and nonlinear constraints are around 879 times.

## 2.5 Summary

In conclusion, this chapter discusses a generalized approach for estimation of system asymptotic average performance from above and below by means of the CSF and dissipation inequalities. Such tools are adapted to formulate EMPC control schemes and eventually analyze their performance and stability. In the case of “no-gap”, as previously defined, if the assumption on continuity of optimal control policy from the economic MPC controller holds, the optimal periodic operation is Lyapunov asymptotically stable when the CSF is used as the terminal penalty function.

## Chapter 3

# Tube-based robust economic MPC on dissipative nonlinear systems

This chapter is organized as follows. Section 3.1 reviews the problem setup on EMPC and demonstrates the motivation of our proposed algorithm. Section 3.1 addresses the robust EMPC algorithm with its formulation in Section 3.2.1 and recursive feasibility analysis in Section 3.2.2. The asymptotic stability and average performance of the closed-loop system by using this EMPC controller are analyzed in Section 3.3. Finally, simulative examples using the proposed EMPC algorithm are presented in Section 3.4 and Section 3.5 summarizes this chapter.

The results presented in this chapter are based on [58].

### 3.1 Problem setup

Throughout this chapter, we consider finite dimensional discrete-time nonlinear control systems described by difference equations

$$x_{t+1} = f(x_t, u_t, \omega_t), \quad (3.1)$$

with  $t \in \mathbb{I}_{\geq 0}$ , state variable  $x_t \in \mathbb{X} \subseteq \mathbb{R}^n$ , control input  $u_t \in \mathbb{U} \subseteq \mathbb{R}^m$ , unknown but bounded disturbance  $\omega_t \in \mathbb{W} \subseteq \mathbb{R}^q$  and continuous state-transition map  $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \rightarrow \mathbb{X}$ . We assume the set  $\mathbb{W}$  is compact, with the origin as an interior.

The state and input of the system must fulfill the pointwise-in-time state and input constraints

$$(x_t, u_t) \in \mathbb{Z}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (3.2)$$

for some compact set  $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$ .

Our objective is to enhance the profitability by minimizing the economic costs accumulated in the long term system operation

$$\sum_t \ell(x_t, u_t), \quad (3.3)$$

where  $\ell(\cdot, \cdot) : \mathbb{Z} \rightarrow \mathbb{R}$  is the economic stage cost, which may take arbitrary form coherently with the EMPC setup (see [3]), and need not be positive definite with respect to any equilibrium state.

### 3.1.1 Invariant error sets

As customary in robust MPC, we introduce the nominal system, associated with system (3.1), that is

$$z_{t+1} = f(z_t, v_t, 0), \quad (3.4)$$

where  $z_t \in \mathbb{X}$  and  $v_t \in \mathbb{U}$  are the nominal state and input at time  $t$ , respectively. Then, the resulting state error is  $e_t = x_t - z_t$ , which tracks the deviation between the real system (3.1) and the nominal system (3.4). To derive bounds on the error, we apply the control policy

$$u_t = \mu(v_t, x_t, z_t) \quad (3.5)$$

to the real system (3.1), such that the state error at subsequent time instant  $t + 1$  fulfills

$$e_{t+1} = f(x_t, \mu(v_t, x_t, z_t), \omega_t) - f(z_t, v_t, 0). \quad (3.6)$$

Before proceeding, we recall the definition of robust invariant set from [14, 59, 60].

**Definition 3.1.** A set  $\Omega \subseteq \mathbb{R}^n$  is *robust positively invariant (RPI)* for the error system (3.6) if and only if for all  $x_t, z_t \in \mathbb{X}$  with  $e_t = x_t - z_t \in \Omega$ , all  $v_t \in \mathbb{U}$ , and all  $\omega_t \in \mathbb{W}$ , it holds  $e_{t+1} \in \Omega$  and  $(x_t, \mu(v_t, x_t, z_t)) \in \mathbb{Z}$ .

**Remark 3.1.** The feedback control law  $u_t = \mu(v_t, x_t, z_t)$  ensures that state trajectories of the real system are always inside the robust invariant set  $\Omega$  around the nominal state trajectory generated by system (3.4), regardless of the disturbance realization.

**Assumption 3.1.** There exists a robust positively invariant set  $\Omega$  as in Definition 3.1 for error system (3.6) under state feedback control policy (3.5).

For linear systems with additive disturbance, the error system (3.6) is able to be simplified and there is a broad literature in order to compute some robust invariant sets, e.g., [59, 61]. However, determining a robust invariant set for general nonlinear systems is a difficult task. For special types of nonlinear systems, there are some systematic techniques, see, e.g., [60, 62, 63].

In order to guarantee feasibility of the real system, we should consider tightened constraints for the nominal system

$$(z_t, v_t) \in \bar{\mathbb{Z}}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (3.7)$$

where  $\bar{\mathbb{Z}} = \{(z, v) \mid (x, \mu(v, x, z)) \in \mathbb{Z}, \forall x \in \{z\} \oplus \Omega\}$ , and the projection of set  $\bar{\mathbb{Z}}$  on the state space is denoted by  $\bar{\mathbb{X}}$ .

### 3.1.2 Optimal regimes of operation

As indicated by the motivating example in [14], simply transferring the design procedure of tube-based tracking MPC to an EMPC setup may be sub-optimal, we define a modified robust cost function as

$$\tilde{\ell}(z, v) = \max_{x \in \{z\} \oplus \Omega} \ell(x, \mu(v, x, z)), \quad (3.8)$$

where  $\Omega$  is the associated robust invariant set for the error system (3.6). This modification takes the worst cost into account and it provides an upper bound of the real system cost regardless of the uncertainty realizations. The new robust optimal steady-state operation is

$$(z_s, v_s) = \arg \min_{(z, v) \in \bar{\mathbb{Z}}, z=f(z, v, 0)} \tilde{\ell}(z, v). \quad (3.9)$$

While optimal steady-state operation is often plausible, in general one might need to identify more complex optimal operation regimes, for nominal system (3.4) with respect to the modified robust cost function (3.8). For instance, any fixed integer  $P$ , one might define the best feasible  $P$ -periodic solution fulfilling

$$\begin{aligned} z_{s,i+1} &= f(z_{s,i}, v_{s,i}, 0) \quad \forall i \in \mathbb{I}_{[0,P-2]} \\ z_{s,0} &= f(z_{s,P-1}, v_{s,P-1}, 0) \\ (z_{s,i}, v_{s,i}) &\in \bar{\mathbb{Z}} \quad \forall i \in \mathbb{I}_{[0,P-1]} \end{aligned} \quad (3.10)$$

where  $f(\cdot, \cdot, 0)$  is the dynamic function in (3.4), and the optimal average performance is

$$\tilde{\ell}_{av}^* = \frac{\sum_{i=0}^{P-1} \tilde{\ell}(z_{s,i}, v_{s,i})}{P}. \quad (3.11)$$

For convenience, the set of states visited along the solution is explicitly denoted as

$$\Pi = \{z_{s,0}, z_{s,1}, \dots, z_{s,i}, \dots, z_{s,P-1}\}, \quad \forall i \in \mathbb{I}_{[0,P-1]}, \quad (3.12)$$

and the corresponding tube is denoted by  $\bigcup_{z \in \Pi} \{z\} \oplus \Omega$  which is referred to as the optimal robust invariant set in later discussions.

Next, we define the optimal operation at a set  $\Pi$  for the nominal dynamics.

**Definition 3.2.** *The nominal system (3.4) is optimally operated at a set  $\Pi$  with respect to the modified cost function  $\tilde{\ell}$ , if for any solution of (3.4) such that  $(z_t, v_t) \in \bar{\mathbb{Z}}, \forall t \in \mathbb{I}_{\geq 0}$ , it holds*

$$\liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_t, v_t)}{T} \geq \tilde{\ell}_{av}^*, \quad (3.13)$$

where  $\tilde{\ell}_{av}^*$  is the average modified cost of the optimal robust invariant set as in (3.11).

According to this definition, the nominal system, operating at  $\Pi$ , provides the best economic cost compared to the long-run average modified cost along any other feasible nominal dynamics. If this set is a singleton ( $P = 1$ ), system (3.4) is optimally operated at steady-state (similar to [27, Definition 4] in terms of average integral cost). Otherwise, the system is optimally operated at some  $P$ -periodic solution.

The sufficiency of dissipativity for a nominal system to be optimally operated at steady-state is obtained in [3], and the necessity, under a mild additional controllability assumption, is

proved in [9]. We recall the notion of dissipativity as given in [3, Definition 4.1], and also introduce a definition of strict dissipativity with respect to a set.

**Definition 3.3.** *A discrete time system is dissipative with a supply rate  $s : \overline{\mathbb{Z}} \rightarrow \mathbb{R}$  if there is a continuous storage function  $\lambda : \overline{\mathbb{X}} \rightarrow \mathbb{R}$  such that:*

$$\lambda(f(z, v, 0)) - \lambda(z) \leq s(z, v) \quad (3.14)$$

*for all  $(z, v) \in \overline{\mathbb{Z}}$ . If, in addition, a positive definite function  $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  exists such that:*

$$\lambda(f(z, v, 0)) - \lambda(z) \leq -\rho(|z|_{\Pi}) + s(z, v), \quad (3.15)$$

*then the system is strictly dissipative with respect to a set  $\Pi$ .*

The generalized notions of dissipativity and controlled dissipativity proposed in Chapter 2 are used to derive upper and lower bounds on the optimal asymptotic average performance without reference to any underlying regime of operation and only depending upon the specific form of the supply. For stability analysis, the supply rate  $s(z, v) = \tilde{\ell}(z, v) - \tilde{\ell}_{av}^*$  will be adopted for the dissipation inequality.

In [27], a more restrictive dissipativity statement called  $\Omega$ -robust dissipativity, which takes the supremum over all possible nominal initial states at the next sampling time, is adopted to establish  $\Omega_{\infty}$ -robustly optimal steady-state operation, in order to cope with the lack of connection between the nominal initial states at two consecutive time instants. However, the nominal dissipativity as in Definition 3.3, which provides a sufficient condition for system (3.4) to be optimally operated at the set  $\Pi$ , is enough for the purpose of closed-loop stability analysis. Moreover, as shown in the analysis in Section 3.3.2, the closed-loop nominal sequence will typically follow the nominal dynamics after finite number of steps because of its convergence (as a result of strict dissipativity (3.15)).

### 3.1.3 Motivation

The authors of [14] have proposed an algorithm which controls both the disturbed and nominal closed-loop system by using a specifically defined integral stage cost. The resulting closed-loop asymptotic average performance, stability and optimal steady-state operation are analyzed in detail. Following the same approaches in the context, it can be found that these properties, regarding both optimal steady-state operation and periodic operation, still

hold if we change the stage cost to the form of (3.8). Moreover, in order to possibly afford some performance improvement, they introduced the nominal initial state as an optimization variable. While this provides an additional degree of freedom, this technique alone prevents convergence of nominal sequences. To overcome this issue, the authors in [14] state that the nominal initial state can be used as an optimization variable provided that asymptotic convergence is ensured by imposing an additional constraint, viz.

$$\lambda(z_{t|t}) \leq \lambda(z_{t|t-1}). \quad (3.16)$$

However, this constraint may restrict the selection of the nominal initial state. Therefore, we propose an alternative approach which considers the nominal initial state as an optimization variable at every sampling time, without the need of the additional constraint for stability purpose, so that the closed-loop economic cost in the transient phase might be improved.

## 3.2 The robust economic MPC algorithm

This section is to propose a new recursively feasible EMPC algorithm which employs the modified economic cost function together with a terminal region and a penalty function.

### 3.2.1 Robust Economic MPC formulation

Notice that the nominal (optimal) initial states at two consecutive time instants, in general, do not fulfill the nominal dynamic (3.4). Given the measured system state  $x$ , let us denote the set of possible nominal states by

$$\mathcal{A}(x) := \{z \in \bar{\mathbb{X}} \mid x \in \{z\} \oplus \Omega\}. \quad (3.17)$$

Then, for a fixed prediction horizon  $N \in \mathbb{I}_{\geq 0}$ , the finite horizon economic optimization problem is formulated as

$$\mathbb{P}_N(x_t) : \min_{z_{t:t+N|t}, v_{t:t+N-1|t}} J_N(z_{t:t+N|t}, v_{t:t+N-1|t}) \quad (3.18)$$

$$s.t. \quad z_{t|t} \in \mathcal{A}(x_t), \quad (3.18a)$$

$$z_{k+1|t} = f(z_{k|t}, v_{k|t}, 0), \quad (3.18b)$$

$$(z_{k|t}, v_{k|t}) \in \overline{\mathbb{Z}} \quad \forall k \in \mathbb{I}_{[t, t+N-1]}, \quad (3.18c)$$

$$z_{t+N|t} \in \overline{\mathbb{X}}_f, \quad (3.18d)$$

where

$$J_N(z_{t:t+N|t}, v_{t:t+N-1|t}) = \lambda(z_{t|t}) + \sum_{k=t}^{t+N-1} \tilde{\ell}(z_{k|t}, v_{k|t}) + V_f(z_{t+N|t}) \quad (3.19)$$

is the economic objective function and the set  $\overline{\mathbb{X}}_f$  is the tightened terminal region.

The storage function  $\lambda(\cdot)$  appears in the objective function and it can be seen as a weighting function for the nominal initial state. Compared with constraint (3.16), this initial weighting function is more natural and will not reduce the feasible region. When the inequality  $\tilde{\ell}(z, v) \geq \tilde{\ell}(z_s, v_s), \forall (z, v) \in \overline{\mathbb{Z}}$  holds, the supply rate  $s(z, v) = \tilde{\ell}(z, v) - \tilde{\ell}(z_s, v_s)$  is positive semidefinite and any constant function may be regarded as a storage function fulfilling the dissipativity inequality. In this situation, the weighting function is unnecessary and the constraint (3.16) is automatically satisfied, so that these two methods boil down to the same optimization problem.

Then, we make the following assumptions on feasibility, robust invariance and terminal stability.

**Assumption 3.2.** *There exists a terminal set  $\mathbb{X}_f \subseteq \mathbb{X}$  with associated set  $\overline{\mathbb{X}}_f := \{x \in \overline{\mathbb{X}} \mid \{x\} \oplus \Omega \subseteq \mathbb{X}_f\}$  that contains the support of nominal optimal operation  $\Pi$  in its interior, and a control policy  $\kappa_f : \overline{\mathbb{X}}_f \rightarrow \overline{\mathbb{U}}$  such that for all  $z \in \overline{\mathbb{X}}_f$*

- (i)  $(z, \kappa_f(z)) \in \overline{\mathbb{Z}}$
- (ii)  $f(z, \kappa_f(z), 0) \in \overline{\mathbb{X}}_f$
- (iii)  $V_f(f(z, \kappa_f(z), 0)) - V_f(z) \leq \tilde{\ell}_{av}^* - \tilde{\ell}(z, \kappa_f(z))$ .

**Remark 3.2.** Conditions (i) and (ii) are standard requirements in economic MPC framework. Condition (iii) on terminal cost can be regarded as the controlled dissipativity with supply rate  $s(z, v) = \tilde{\ell}_{av}^* - \tilde{\ell}(z, v)$ .

The decision variables of this optimization problem are

$$z_{t:t+N|t} = (z_{t|t}, z_{t+1|t}, \dots, z_{t+N|t}), \quad v_{t:t+N-1|t} = (v_{t|t}, v_{t+1|t}, \dots, v_{t+N-1|t}). \quad (3.20)$$

Once the optimization problem  $\mathbb{P}_N(x_t)$  admits a feasible solution, it is denoted by

$$z_{t:t+N|t}^* = (z_{t|t}^*, z_{t+1|t}^*, \dots, z_{t+N|t}^*), \quad v_{t:t+N-1|t}^* = (v_{t|t}^*, \dots, v_{t+N-1|t}^*), \quad (3.21)$$

and the corresponding optimal value function is

$$V_N(x_t) = J_N(z_{t:t+N|t}^*, v_{t:t+N-1|t}^*). \quad (3.22)$$

Let us define the admissible set  $\mathcal{Z}_N$  for a fixed state  $x$  as

$$\begin{aligned} \mathcal{Z}_N(x) := \{ & (z_{0:N}, v_{0:N-1}) \mid z_0 \in \mathcal{A}(x), z_{k+1} = f(z_k, v_k, 0) \\ & z_N \in \bar{\mathbb{X}}_f, (z_k, v_k) \in \bar{\mathbb{Z}}, \forall k \in \mathbb{I}_{[0, N-1]} \}. \end{aligned} \quad (3.23)$$

and the collection of admissible states  $x$ ,

$$\mathcal{X}_N = \{x \in \mathbb{R}^n \mid \exists (z_{0:N}, v_{0:N-1}) \in \mathcal{Z}_N(x)\}. \quad (3.24)$$

Notice that the terminal set fulfills  $\mathbb{X}_f \subseteq \mathcal{X}_N$ , the set  $\mathcal{X}_N$  is non-empty and hence the optimization problem  $\mathbb{P}_N(x)$  has at least one feasible solution for any  $x \in \mathbb{X}_f$ .

Following the manner of feedback algorithm in EMPC, the optimal control implemented to the system (3.1) is

$$\kappa_N(x_t) := \mu(v_{t|t}^*, x_t, z_{t|t}^*), \quad (3.25)$$

and the resulting closed-loop dynamic is

$$x_{t+1} = f(x_t, \kappa_N(x_t), \omega_t). \quad (3.26)$$

Notice that,  $v_{t|t}^*$  and  $z_{t|t}^*$  might be non-unique, in which case, we take any pair that achieves the optimum.

### 3.2.2 Recursive feasibility

Note that, provided that the optimization problem  $\mathbb{P}_N(\cdot)$  is feasible, the optimal nominal state-input pair  $(z_{t|t}^*, v_{t|t}^*)$  is restricted by the tightened constraint (3.7). The rationale is that true state and input solutions, in the usual spirit of tube MPC, will then fulfill

$$(x_t, \mu(v_{t|t}^*, x_t, z_{t|t}^*)) \in \mathbb{Z}, \forall t \in \mathbb{I}_{\geq 0}, \quad (3.27)$$

whatever the disturbance signal is until time  $t$ . Therefore, the real system's solutions fulfill pointwise-in-time constraints.

To this end, we are ready to claim the recursive feasibility of the optimization problem proposed in Section 3.2.1.

**Proposition 3.1.** *Let Assumptions 3.1 and 3.2 hold. Then, for any initial state  $x \in \mathcal{X}_N$ , the EMPC optimization problem  $\mathbb{P}_N(\cdot)$  is recursively feasible.*

*Proof.* The proof for recursive feasibility follows the standard argumentation in robust MPC (see i.e. [14]). It is worth to be noted that by constraint (3.18a), we have  $x_t - z_{t|t}^* \in \Omega$ . Then, due to Assumption 3.1, for all  $\omega_t \in \mathbb{W}$ , it holds  $x_{t+1} - z_{t+1|t}^* = f(x_t, \kappa_N(x_t), \omega_t) - f(z_{t|t}^*, v_{t|t}^*, 0) \in \Omega$ , which implies  $z_{t+1|t}^* \in \mathcal{A}(x_{t+1})$ .  $\square$

## 3.3 Asymptotic performance and stability analysis

This section is dedicated to the analysis of closed-loop behaviors of the system using the proposed EMPC controller. We start by proving the stability of the optimal robust invariant set, which is inferred from the asymptotic convergence of the nominal state sequence. Then, an upper bound of the asymptotic average performance, regardless of the disturbance realizations is derived.

### 3.3.1 Stability analysis

This section explores the asymptotic stability of the closed-loop system under EMPC control actions. Before the statement of asymptotic stability, we define an explicit notation for the terminal control policy and make some assumptions.

**Definition 3.4.** *The terminal control policy is defined as the set valued function  $\kappa_f : \bar{\mathbb{X}}_f \rightarrow \bar{\mathbb{U}}$  which is the solution of the following optimization problem*

$$\min_{v \in \bar{\mathbb{U}}, f(z, v, 0) \in \bar{\mathbb{X}}_f} V_f(f(z, v, 0)) + \tilde{\ell}(z, v). \quad (3.28)$$

Note that if the  $\kappa_f(\cdot)$  exists as in Assumption 3.2, this Definition 3.4 provides an appropriate approach to compute the terminal control policy.

**Assumption 3.3.** *There exists a continuous storage function  $\lambda(\cdot)$  such that the nominal system (3.4) is strictly dissipative with supply rate  $\tilde{\ell}(z, v) - \tilde{\ell}_{av}^*$ , i.e.,*

$$\lambda(f(z, v, 0)) - \lambda(z) \leq \tilde{\ell}(z, v) - \tilde{\ell}_{av}^* - \alpha_1(|z|_\Pi), \quad (3.29)$$

where  $\alpha_1(\cdot)$  is a class  $\mathcal{K}$  function.

**Assumption 3.4.** *The storage function and the terminal cost function satisfy  $\lambda(z) + V_f(z) = 0$  for all  $z \in \Pi$  and  $\lambda(z) + V_f(z) \geq 0$  for all  $z \in \bar{\mathbb{X}}_f \setminus \Pi$ .*

Now, we are ready to state the first main result in this note

**Theorem 3.1.** *Let Assumptions 3.1, 3.2, 3.3 and 3.4 hold, under the application of economic MPC feedback control policy (3.25), the set  $\bigcup_{z \in \Pi} [\{z\} \oplus \Omega]$  is asymptotically stable for the closed-loop system (3.26) with region of attraction  $\mathcal{X}_N$ .*

*Proof.* The idea of this proof is to follow the usual way to prove asymptotic stability of a closed-loop system under an EMPC controller with the aid of dissipativity assumption. With the notion of rotated cost functions, the optimal rotated objective function is shown to be a Lyapunov function. The details are omitted for brevity, but readers may refer to [5, Theorem 15].  $\square$

### 3.3.2 Closed-loop asymptotic performance

Another main goal of EMPC is the guaranteed closed-loop performance. Before stating the main result, we make the following assumptions.

**Assumption 3.5.** *For any  $z_{s,i}, z_{s,j} \in \Pi$  where  $i \neq j \in \mathbb{I}_{[0, P-1]}$ , the corresponding components in optimal robust invariant sets  $\{z_{s,i}\} \oplus \Omega$  and  $\{z_{s,j}\} \oplus \Omega$  are disjoint.*

**Assumption 3.6.** For any disturbance realization, there exists a finite time  $t$  such that the closed-loop system (3.26) enters the optimal robust invariant set, i.e.,  $x_t \in \bigcup_{z \in \Pi} [\{z\} \oplus \Omega]$ .

Now, we state the second main result in this note.

**Theorem 3.2.** Suppose Assumptions 3.1, 3.2, 3.5 and 3.6 are satisfied, the asymptotic average performance of the closed-loop system (3.26) is no worse than  $\tilde{\ell}_{av}^*$ , regardless of  $\omega(\cdot) \in \mathbb{W}^\infty$ , i.e.,

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, \kappa_N(x_t))}{T} \leq \tilde{\ell}_{av}^*. \quad (3.30)$$

*Proof.* The proof of Theorem 3.2 is similar to that of [5, Theorem 18]. One should note that, at the first time  $x_t \in \bigcup_{z \in \Pi} [\{z\} \oplus \Omega]$ , we denote the phase of the optimal solution by  $i(t) \in \mathbb{I}_{[0, P-1]}$ . Then, the one step ahead predicted nominal state is  $z_{t+1|t}^* = z_{s, i(t)+1}$ . Because of Assumption 3.5, the real system state at time  $t+1$  satisfies  $x_{t+1} \in \{z_{s, i(t)+1}\} \oplus \Omega$ . Then, the optimal nominal initial state is  $z_{t+1|t+1}^* = z_{s, i(t)+1} = z_{t+1|t}^*$  which in turn yields  $\lambda(z_{t+1|t+1}^*) = \lambda(z_{t+1|t}^*)$ .  $\square$

**Remark 3.3.** This upper bound of the asymptotic performance is not guaranteed if the components of the optimal robust invariant sets intersect. In this case, the phase of the optimal nominal solution might “reset” at each sampling time. Therefore, the closed-loop nominal sequence may not be synchronized with the optimal periodic operation. The asymptotic performance bound is more conservative than the average cost and it is still bounded by  $\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, \kappa_N(x_t))}{T} \leq \max_{i \in \mathbb{I}_{[0, P-1]}} \tilde{\ell}(z_{s, i}, v_{s, i})$ .

## 3.4 Numerical examples

Throughout the examples, we compare the economic performance by adopting three EMPC methods: fixed nominal initial state with dynamic update according to nominal system model (FNS), variable nominal initial state with additional constraint (VNS-AC) using (3.16), and variable nominal initial state with initial weighting function (VNS-IWF).

### 3.4.1 Linear System: Robust optimal steady-state operation

We revisit the scalar example in [14],

$$x_{t+1} = 0.9x_t + u_t + \omega_t, \quad (3.31)$$

constrained by  $\mathbb{Z} = \{(x, u) \in \mathbb{R}^2 \mid |x| \leq 10, |u| \leq 1\}$  and  $\mathbb{W} = \{\omega \in \mathbb{R} \mid |\omega| \leq 0.1\}$ . We also choose control policy  $u_t = \mu(v_t, x_t, z_t) = v_t + K(x_t - z_t)$  with  $K = -0.9$  and the resulting RPI set is  $\Omega = \mathbb{W}$ . Then, the tightened constraints for nominal system is  $\bar{\mathbb{Z}} = \mathbb{Z} \ominus (\Omega \times K\Omega) = \{(z, v) \in \mathbb{R}^2 \mid |z| \leq 9.9, |v| \leq 0.91\}$ , that is,  $\bar{\mathbb{X}} = \{z \in \mathbb{R} \mid |z| \leq 9.9\}$ . The economic cost function is defined as  $\ell(x, u) = \ell_1(x) + \ell_2(u)$ , where

$$\ell_1(x) = \begin{cases} x^2 + 0.1x & x < -0.05 \\ 0.225x - 0.0363 & -0.05 \leq x < 0.15 \\ x^2 - 0.3x + 0.02 & x \geq 0.15 \end{cases}$$

$$\ell_2(u) = \begin{cases} u^2 + 1.18u + 0.0981 & u < -0.59 \\ -0.25 & -0.59 \leq u < -0.41 \\ u^2 + 0.82u - 0.0819 & u \geq -0.41 \end{cases}$$

and the corresponding modified cost function is

$$\tilde{\ell}(z, v) = z^2 + v^2 - z + f(z, v, 0) = z^2 + v^2 - 0.1z + v.$$

One can find a possible storage function  $\lambda(z) = z$  fulfilling the dissipativity condition. According to (3.9), the robust optimal equilibrium is  $(z_s, v_s) = (0, 0)$ , with the best modified robust cost  $\tilde{\ell}_{av}^* = \tilde{\ell}(z_s, v_s) = 0$ . By using terminal control policy  $\kappa_f(z) = -Kv$ , the terminal region can be defined as  $\bar{\mathbb{X}}_f = \bar{\mathbb{X}}$  and the terminal penalty function is  $V_f(z) = \tilde{\ell}(z, \kappa_f(z))$  which fulfills controlled dissipativity.

Table 3.1 compares the accumulated transient economic stage cost performance under various initial conditions and 1000 randomly generated disturbance realizations by using the three approaches. Generally, the two methods changing the nominal initial state as a variable lead to less accumulated cost during the transient steps compared with the approach using the nominal closed-loop system. Moreover, it indicates that our proposed controller VNS-IWF performs better in more than 70% of cases. In particular, the FNS and VNS-IWF have the same results at  $x_0 = 0$ , which cost less in 723 out of 1000 simulations.

	FNS(%)	VNS-AC(%)	VNS-IWF(%)
$x_0 = -9$	0	0	100
$x_0 = -5$	2	0	99.8
$x_0 = 0$	72.3	27.7	72.3
$x_0 = 5$	0	27.4	72.6
$x_0 = 9$	0	27.2	72.8

Table 3.1 Percentage of best performance achieved in 1000 simulations with random disturbances in the case of robust optimal steady-state operation for the scalar system (3.31).

### 3.4.2 Linear System: Robust optimal periodic operation

We reconsider the linear system (3.31) again in Section 3.4.1. Now, the point-wise-in-time constraint is  $\mathbb{Z} = \{(x, u) \in \mathbb{R}^2 \mid |x| \leq 10, 0.1 \leq |u| \leq 1\}$  and cost function is

$$\ell(x, u) = \begin{cases} (x + 0.1)^2 & x < 0 \\ (x - 0.1)^2 & x \geq 0. \end{cases}$$

By using the same control policy with  $K = -0.9$ , the RPI set is  $\Omega = \mathbb{W}$  and the tightened constraint is  $\bar{\mathbb{Z}} = \mathbb{Z} \ominus (\Omega \times K\Omega) = \{(z, v) \in \mathbb{R}^2 \mid |z| \leq 9.9, 0.19 \leq |v| \leq 0.91\}$ . The modified cost function is

$$\tilde{\ell}(z, v) = \begin{cases} 0.01 & -0.1 \leq z < 0.1 \\ z^2 & \text{otherwise,} \end{cases}$$

and the resulting robust optimal periodic solution is

$$(z_{s,0}, v_{s,0}) = (-0.1, 0.19), (z_{s,1}, v_{s,1}) = (0.1, -0.19).$$

with average economic cost  $\tilde{\ell}_{av}^* = \frac{\tilde{\ell}(z_{s,0}, v_{s,0}) + \tilde{\ell}(z_{s,1}, v_{s,1})}{2} = 0.01$ . Using the same terminal control policy and terminal region as in the previous example, the terminal cost function in this case is

$$V_f(z) = \begin{cases} 0.01 & -0.1 \leq z < 0.1 \\ z^2 & \text{otherwise.} \end{cases}$$

To fulfill the dissipativity inequality with supply rate  $s(z, v) = \tilde{\ell}(z, v) - \tilde{\ell}_{av}^*$ , the storage function can be chosen as  $\lambda(z) = c$  where  $c \in \mathbb{R}$  is a constant real number or  $\lambda(z) = -\tilde{\ell}(z, v)$ .

In the first case, constant storage function causes two methods VNS-IWF and VNS-AC are equivalent, the simulation results with initial condition  $x_0 = 0.3$  and prediction horizon  $N = 20$  are shown in Fig.3.1 and Fig.3.2, in which the real system switches between the intervals  $[-0.2, 0]$  and  $[0, 0.2]$  which are essentially non-overlapping if the single point 0 is removed. The asymptotic average performance is lower than and approaching to the cost  $\tilde{\ell}_{av}^*$  asymptotically for the random disturbance realization and the worst scenario, respectively. The average computational time of *fmincon* is 0.1512s approximately, and the average number of calls for objective functions and nonlinear constraints are around 813 times.

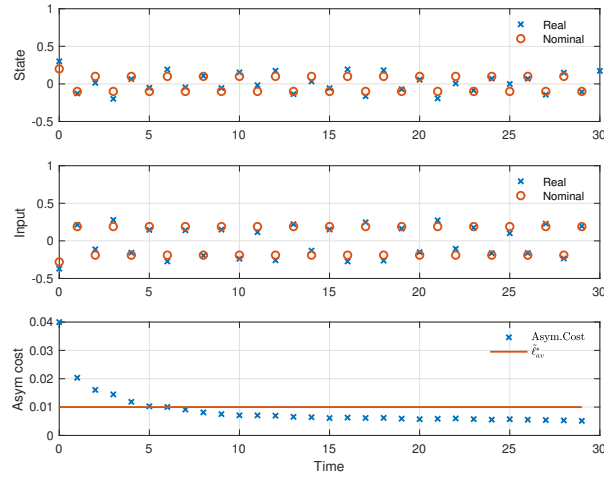


Fig. 3.1 State trajectory, optimal input and asymptotic average cost of system (3.31) with optimal periodic operation under a randomly generated disturbance sequence.

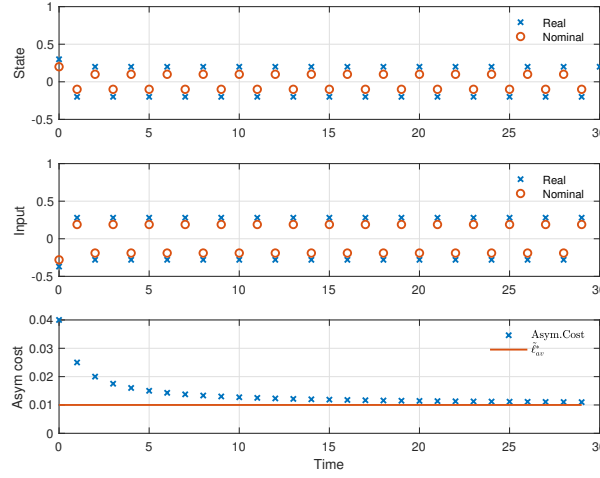


Fig. 3.2 State trajectory, optimal input and asymptotic average cost of system (3.31) with optimal periodic operation under the worst disturbance realization.

When the storage function is  $\lambda(z) = -\tilde{\ell}(z, v)$ , the best performing controller depends on the initial condition and the disturbance. Simulated results are in Table 3.2. It can be seen that the advantage of using VNS-IWF is obvious if the initial condition is far away from the origin.

	FNS(%)	VNS-AC(%)	VNS-IWF(%)
$x_0 = -9$	0	0	100
$x_0 = -5$	0	0	100
$x_0 = -1$	28.5	31.6	39.9
$x_0 = 0$	25.6	36.9	37.5
$x_0 = 1$	30.8	31.3	37.9
$x_0 = 5$	0	0	100
$x_0 = 9$	0	0	100

Table 3.2 Percentage of best performance achieved in 1000 simulations with random disturbances in the case of robust optimal periodic operation for the system 3.31.

It can be seen from the robust optimal periodic solution, the optimal robust invariant sets are intervals  $[-0.2, 0]$  and  $[0, 0.2]$  which are essentially non-overlapping if the single point 0 is removed. In order to demonstrate the effect of substantial overlaps, we modify the point-wise-in-time constraint to  $\mathbb{Z} = \{(x, u) \in \mathbb{R}^2 \mid |x| \leq 10, 0.05 \leq |u| \leq 1\}$ , and the stage

cost

$$\ell(x, u) = \begin{cases} (x + 0.05)^2 & -10 \leq x < -0.05 \\ 4(x + 0.05)^2 & -0.05 \leq x < 0 \\ 4(x - 0.05)^2 & 0 \leq x < 0.05 \\ (x - 0.05)^2 & 0.05 \leq x < 10 \end{cases}$$

which results into the robust optimal periodic solution

$$(z_{s,0}, v_{s,0}) = (-0.05, 0.095), (z_{s,1}, v_{s,1}) = (0.05, -0.095).$$

with corresponding invariant intervals  $[-0.15, 0.05]$  and  $[-0.05, 0.15]$ . Fig.3.3 shows the simulated results, in which there are some phase resets under the random disturbance. While the worst disturbances, in Fig.3.4, can always force the system states outside the overlapping area  $[-0.05, 0.05]$ , so that the phase is synchronized. For both cases, the asymptotic performance is bounded by 0.01 which is the average and the worst cost at the robust optimal periodic solution. To demonstrate the effect from the phase reset, the simulation with terminal equality constraint which is used to enforce the phase of the state is compared under the same disturbance realization. In Fig.3.5, as expected, the asymptotic average cost is lower than that when phase reset occurs, since the optimal periodic-2 solution should perform better than any other solutions. Moreover, the computational time when phase resets exist is longer than the average computational time of *fmincon* is 0.2726s approximately whereas it is around 0.1534s when there is no phase reset.

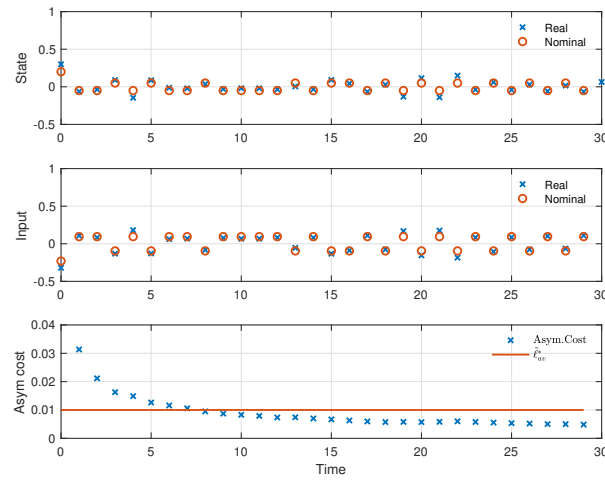


Fig. 3.3 State trajectory, optimal input and asymptotic average cost of system (3.31) with optimal periodic operation under a randomly generated disturbance sequence, when the optimal invariant sets overlap.

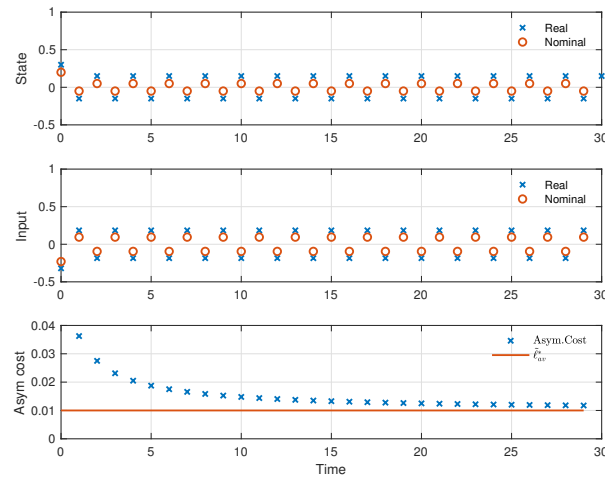


Fig. 3.4 State trajectory, optimal input and asymptotic average cost of system (3.31) with optimal periodic operation under the worst disturbance realization, when the optimal invariant sets overlap.

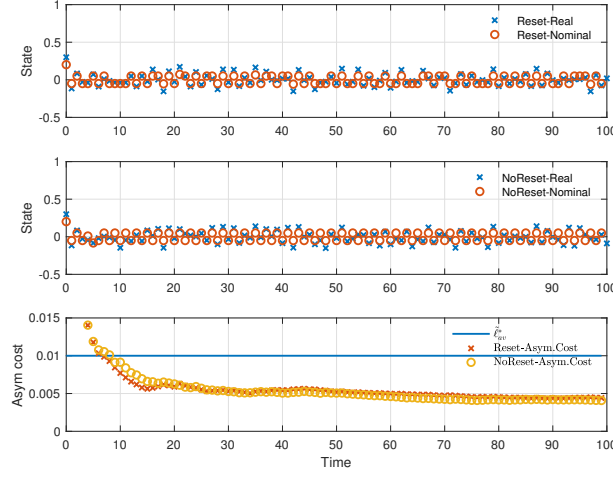


Fig. 3.5 Real and nominal state trajectories of the systems with and without phase reset as well as the corresponding asymptotic average cost under a randomly generated disturbance sequence, when the optimal invariant sets overlap.

### 3.5 Summary

In conclusion, we presented a tube-based robust economic MPC algorithm for nonlinear systems subject to disturbances. By considering the worst cost of the tube along the artificial nominal system and a weighting function on nominal initial state, the robustness against the disturbances is guaranteed in our EMPC design. Within this chapter, constraint tightening is used to prove recursive feasibility and a bound of the closed-loop asymptotic average performance is derived. Moreover, the stability of the optimal robust invariant set is inferred from the asymptotic stability of the nominal state sequence.

## Chapter 4

# Homothetic tube-based robust economic MPC with integrated MHE

This chapter is organized as follows. Section 4.1 introduces the basic formulations and setup. Section 4.2 addresses the homothetic tube-based robust EMPC algorithm with its formulation in Section 4.2.1 and recursive feasibility analysis in Section 4.2.2. The asymptotic stability and average performance of the closed-loop system adopting this EMPC controller are analyzed in Section 4.3. Finally, illustrative examples using the proposed EMPC algorithm are presented in Section 4.4 and Section 4.5 summarizes this chapter.

The results presented in this chapter are based on [41, 64].

### 4.1 Problem setup

Throughout this chapter, we consider finite dimensional discrete-time linear systems described by difference equations

$$\begin{aligned}x_{t+1} &= Ax_t + Bu_t + \omega_t \\ y_t &= Cx_t + \eta_t\end{aligned}\tag{4.1}$$

where  $x_t \in \mathbb{R}^{n_x}$  and  $u_t \in \mathbb{R}^{n_u}$  are the system state and the control input, respectively, at time  $t \in \mathbb{I}_{\geq 0}$ . The system is subject to pointwise-in-time constraints

$$(x_t, u_t) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (4.2)$$

where the set  $\mathbb{Z}$  is compact. Moreover, the variables  $\omega_t \in \mathbb{W} \subseteq \mathbb{R}^{n_x}$  and  $\eta_t \in \mathbb{H} \subseteq \mathbb{R}^{n_y}$ , where  $\mathbb{W}$  and  $\mathbb{H}$  are compact sets containing zero, denote the state disturbance and the measurement noise, respectively. The signal  $y_t \in \mathbb{R}^{n_y}$  is known as the measured output that is available for feedback. Throughout this chapter, we adopt the following assumptions about the system (4.1):

**Assumption 4.1.** *The pairs  $(A, B)$  and  $(C, A)$  are stabilizable and detectable, respectively.*

Our objective is to enhance profitability by minimizing the economic costs accumulated in the long term system operation

$$\sum_t \ell(x_t, u_t), \quad (4.3)$$

where  $\ell(x, u) : \mathbb{Z} \rightarrow \mathbb{R}$  is the economic stage cost, which may take arbitrary form coherently with the economic MPC setup (see [3]), and need not be positive definite with respect to any equilibrium state.

#### 4.1.1 Feedback control and observer

One natural approach to control system (4.1), while ensuring the satisfaction of the constraints (4.2), is to apply an optimization-based feedback control policy. However, in most literature, Economic MPC is formulated assuming full-state feedback, whereas, in practical cases, the full state measurement is often not available. This motivates the use of a pre-defined state observer

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + L(y_t - C\hat{x}_t) \quad (4.4)$$

where  $\hat{x}_t \in \mathbb{R}^{n_x}$  is the observer state and  $L$  satisfies  $\rho(A - LC) < 1$ , which will be used in the computation of the input  $u_t$ . Therefore, as in [31, 65, 66], the state  $x_t$  can be decomposed into different components:

$$x_t = \hat{x}_t + e_t = z_t + \xi_t + e_t, \quad (4.5)$$

where these three terms are later referred to as nominal state, observer state error, and estimation error, respectively. The nominal state  $z_t$  depends on the nominal input  $v_t$  and fulfills the dynamic

$$z_{t+1} = Az_t + Bv_t. \quad (4.6)$$

In order to determine the optimal control input  $u_t$  at each sampling time instant, an optimization problem is solved with the nominal state and input sequences as decision variables. As customary in tube-based EMPC, the optimal nominal state and input pair  $(z_t, v_t)$  are used to calculate the control policy

$$u_t = v_t + K(\hat{x}_t - z_t), \quad (4.7)$$

which is implemented to the systems (4.1) and (4.4), where  $K$  is a pre-defined feedback gain such that  $\rho(A + BK) < 1$ .

The state estimation error  $e \in \mathbb{R}^{n_x}$  is defined as  $e_t := x_t - \hat{x}_t$  and governed by

$$e_{t+1} = (A - LC)e_t + \omega_t - L\eta_t, \quad (4.8)$$

Moreover,  $\xi_t$  the difference between the observer state  $\hat{x}_t$  and its nominal counterpart  $z_t$  follows the difference equation

$$\xi_{t+1} = (A + BK)\xi_t + L(Ce_t + \eta_t). \quad (4.9)$$

At this point, we recall the standard definition of robust positively invariant set from [65]:

**Definition 4.1.** A set  $S \subseteq \mathbb{R}^n$  is robust positively invariant for the system  $x^+ = f(x, \omega)$  and the constraint set  $(\mathbb{X}, \mathbb{W})$  if  $S \subseteq \mathbb{X}$  and  $f(x, \omega) \in S, \forall \omega \in \mathbb{W}, \forall x \in S$ .

Since  $\rho(A - LC) < 1$ , there exists a  $C$  set that is finite time computable and robustly positively invariant [67] for the estimation error system (4.8). Then, following the ideas in [65], we can bound  $e_t$  and  $\xi_t$  using sets  $E$  and  $\Xi$ , respectively, provided that the system state  $x_0$ , observer state  $\hat{x}_0$  and nominal system state  $z_0$  satisfy  $e_0 = x_0 - \hat{x}_0 \in E$  and  $\xi_0 = \hat{x}_0 - z_0 \in \Xi$ . Furthermore, it can be indicated that the system state  $x_t$  fulfills  $x_t \subseteq z_t \oplus \Omega = z_t \oplus (E \oplus \Xi)$ .

**Remark 4.1.** Notice that sets  $E$ ,  $\Xi$  and  $\Omega$  for a linear system (3.1) and given convex polyhedrons  $\mathbb{W}$  and  $\mathbb{H}$  are, in turn, convex polyhedrons and can be computed based on MPT3 toolbox. Moreover, the paper [31] analyzed the conservatism of using separate sets for  $e_t$  and  $\xi_t$ , and proposed a single set on an augmented system  $\delta_t = [\xi_t \ e_t]^T$  to obtain tighter approximations on the worst case effect of the uncertainties. However, the computation of

RPI sets is performed on a space with double dimension, which reduces the applicability of the approach for high dimensional systems.

In order to ensure that the unknown state satisfies the state constraint, we need to introduce the tightened constraint for the nominal system (4.6) using the sets  $E$  and  $\Xi$

$$\mathcal{Z} = \{(z, v) \mid (z + \xi + e, v \oplus K\xi) \subseteq \mathbb{Z}, \forall \xi \in \Xi, \forall e \in E\}, \quad (4.10)$$

and, accordingly, this tightened constraint guarantees that

$$(z_t, v_t) \in \mathcal{Z} \Rightarrow (x_t, u_t) \in \mathbb{Z}. \quad (4.11)$$

For later use, the projection of  $\mathcal{Z}$  on  $\mathbb{X}$  is denoted by  $\mathcal{X}$ .

Since the set  $\Omega$  may not provide a tight estimate of the real system state value, it may be possible to find a less conservative bound for the economic cost incurred in operation. Indeed, one might take advantage of the past measured outputs in order to further reduce the region where the real system state belongs. For instance, it is possible to find the best scalar  $\gamma \in [0, 1]$  such that the state is guaranteed to fulfill  $x \in z \oplus \gamma\Omega$ . Notice that this scaling parameter is only used to derive a tighter performance bound and it will not affect the robust optimal regime of operation which will be discussed in the next subsection. Based on the available information at time  $t$ , we are able to recursively define the reachable sets

$$\begin{aligned} \mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) = \{x_t \mid \exists x_{t-1} \in \mathcal{R}_{t-1}(y_{0:t-1}, \hat{x}_0, u_{0:t-2}), \exists \omega_{t-1} \in \mathbb{W}, y_t - Cx_t \in \mathbb{H}, \\ x_t = Ax_{t-1} + Bu_{t-1} + \omega_{t-1}\}. \end{aligned} \quad (4.12)$$

where  $\mathcal{R}_0(y_0, \hat{x}_0) = \{x_0 \mid x_0 \in \hat{x}_0 \oplus E, y_0 - Cx_0 \in \mathbb{H}\}$ .

### 4.1.2 Robust optimal operation regimes

Clearly, the presence of an additive persistent disturbance  $\omega$  and measurement noise  $\eta$  acting on system (4.1) means it is not possible to guarantee asymptotic convergence to the generally defined optimal steady-state, i.e.

$$(x^*, u^*) = \arg \min_{(x, u) \in \mathbb{Z}, x = Ax + Bu} \ell(x, u). \quad (4.13)$$

Hence, an achievable goal is to steer the system state to a neighborhood of a certain steady-state which may differ from (4.13). When the deviation of the true state  $x$  from the nominal state  $z$  is bounded by  $\gamma\Omega$ , the worst economic cost, over the homothetic set centered at  $z$ , is defined as

$$\tilde{\ell}(z, v, \gamma) = \max_{\delta \in \gamma\Omega, \xi \in \Xi} \ell(z + \delta, v + K\xi). \quad (4.14)$$

This modification takes the worst cost into account and it provides an upper bound of the real system cost regardless of the state disturbance and measurement uncertainty realizations. Readers may refer to [14] for the motivations to consider a modified economic stage cost. We should also notice that  $\tilde{\ell}(z, v, \gamma)$  is monotonically non-decreasing with respect to  $\gamma$ , so that the tightest estimates of the state region (corresponding to small  $\gamma$ ) provide a lower upper-bound to the economic cost.

At this point, we are able to define the robust optimal steady-state operation by

$$(z_s, v_s) = \arg \min_{(z, v) \in \mathcal{Z}, z = Az + Bv} \tilde{\ell}(z, v, 1). \quad (4.15)$$

Given this robust optimal state and input pair, the worst cost within  $z \oplus \gamma\Omega$  is  $\tilde{\ell}_{av}^*(\gamma) = \tilde{\ell}(z_s, v_s, \gamma)$ , in particular, the robust optimal average economic cost  $\tilde{\ell}_{av}^*(1) = \tilde{\ell}(z_s, v_s, 1)$ . Notice that  $\tilde{\ell}_{av}^*(1)$  is independent of the parameter  $\gamma$ , so it denotes an a-priori upper bound to the closed-loop average cost.

Although a steady-state often provides the minimal long run average economic cost, sometimes more complex operations analyzed in Chapter 2 and [22], such as a periodic solution, can outperform the steady-state, for the nominal system (4.6) with respect to the modified robust cost function (4.14).

Let us consider the more general case where system (4.1) is optimally operated at some periodic orbit with period  $P \in \mathbb{I}_{\geq 1}$ . Solving the following optimization problem,

$$\begin{aligned} & \min_{\substack{z_{i+1} = Az_i + Bv_i \ \forall i \in \mathbb{I}_{[0, P-2]} \\ z_0 = Az_{P-1} + Bv_{P-1} \\ (z_i, v_i) \in \mathcal{Z} \ \forall i \in \mathbb{I}_{[0, P-1]}}} \frac{\sum_{i=0}^{P-1} \tilde{\ell}(z_i, v_i, 1)}{P}, \end{aligned}$$

we obtain the best  $P$ -periodic solution

$$\Pi = \{(z_{s,0}, v_{s,0}), (z_{s,1}, v_{s,1}), \dots, (z_{s,P-1}, v_{s,P-1})\}, \quad (4.16)$$

with corresponding optimal periodic orbit  $\Pi_{\mathbb{X}} = \{z_{s,0}, z_{s,1}, \dots, z_{s,P-1}\}$ , and its robust optimal average performance can be denoted by

$$\tilde{\ell}_{av}^*(\gamma) = \frac{\sum_{i=0}^{P-1} \tilde{\ell}(z_{s,i}, v_{s,i}, \gamma)}{P}. \quad (4.17)$$

Next, we define the notion of optimal operation at a set  $\Pi$  for the nominal dynamic:

**Definition 4.2.** *The nominal system (4.6) is optimally operated at a set  $\Pi$  with respect to the modified cost function  $\tilde{\ell}(\cdot, \cdot, \cdot)$ , if for any solution such that  $(z_t, v_t) \in \mathcal{Z}, \forall t \in \mathbb{I}_{\geq 0}$ , it holds*

$$\liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_t, v_t, 1)}{T} \geq \tilde{\ell}_{av}^*(1), \quad (4.18)$$

where  $\tilde{\ell}_{av}^*(1)$  is the average modified cost of the optimal robust invariant set for  $\gamma = 1$  as in (4.17).

Definition 4.2 means that each feasible solution will result in an asymptotic average performance which is as good as or worse than the average performance of the periodic orbit  $\Pi$ . Particularly, for  $P = 1$  the notion of optimal steady-state operation in [3] is recovered.

After clarifying the robust optimal operation of the set  $\Pi$ , we make an assumption based on the definition of strict dissipativity for disturbance-free systems as given in [24, 25]:

**Assumption 4.2.** *There exists a continuous storage function  $\lambda(\cdot) : \mathcal{X} \rightarrow \mathbb{R}$  bounded from below on  $\mathcal{X}$ , and a class  $\mathcal{K}$  function  $\alpha_1(\cdot)$  such that*

$$\lambda(Az + Bv) - \lambda(z) \leq \tilde{\ell}(z, v, \gamma) - \tilde{\ell}_{av}^*(\gamma) - \alpha_1(\|z\|_{\Pi_{\mathbb{X}}}), \quad (4.19)$$

for all  $(z, v) \in \mathcal{Z}$  and  $\gamma \in [0, 1]$ .

**Remark 4.2.** The authors in [68] propose a computational method for automatic verification of dissipativity for discrete time systems with polynomial dynamics and stage cost. Assumption 4.2 requires that inequality (4.19) holds for all  $\gamma \in [0, 1]$ , whereas in the standard definition of dissipativity  $\gamma = 0$  and the supply rate is based on the original cost function  $\ell$ , rather than  $\tilde{\ell}$ . This rather strong condition, ensures an optimal operation regime that is

independent of our confidence level  $\gamma$  on the location of the true system's state. In particular, provided that the stage cost  $\ell$  is convex, the modified function  $\tilde{\ell}$  is convex as well, and under special symmetry conditions, this may imply the optimal operation is independent of the factor  $\gamma$  (see Example 4.4.3).

## 4.2 The Homothetic tube-based economic MPC algorithm

### 4.2.1 Problem formulation

In this section, a robust economic MPC controller is designed. To this end, further notation has to be introduced:  $z_{0:t+N|t}$  is the sequence of nominal states from sampling instants 0 to  $t + N$ , considered at time  $t$ ;  $v_{t:t+N-1|t}$  is the sequence of future nominal control moves;  $u_{0:t-1}$  (or  $v_{0:t-1}$  equivalently),  $y_{0:t}$  and  $\gamma_{0:t-1}$  are the (known) past system inputs, measured outputs and scaling factors, respectively, at time  $t$ . Because the observer states are auxiliary variables processed within the controller, the sequence  $\hat{x}_{0:t}$  is known at time  $t$ .

Notice that the Luenberger observer only comes with an a-priori (constant) bound  $E$  of the estimation error, which, in some sense, corresponds to the worst case disturbance realization. The moving horizon estimation, instead, is a set membership estimator that, depending on disturbance realization, may allow a much tighter bound on the estimation error. This is beneficial for the economic performance as tighter characterizations of the state whereabouts allow sharper optimization of the economic criterion.

Before presenting the EMPC algorithm, we first make the following assumption to restrict the error between the real system and the estimates:

**Assumption 4.3.** *The initial observer state  $\hat{x}_0$  is provided along with the minimal RPI set  $E$ , such that the real system state  $x_0$  is initially known to satisfy  $x_0 \in \hat{x}_0 \oplus E$ .*

Next we propose a feedback control law by solving online the following optimization problem with a prediction horizon  $N \in \mathbb{I}_{\geq 1}$ ,

$$\min_{\mathbf{d}_t} J_N(z_{t:t+N}|t, v_{t:t+N-1}|t, \gamma_{t:t+N}|t) \quad (4.20)$$

$$s.t. \quad z_t|t \in \hat{x}_t \oplus \Xi, \quad (4.20a)$$

$$z_{j+1}|t = Az_j|t + Bv_j, \quad (4.20b)$$

$$(z_j|t, v_j) \in \mathcal{Z}, \quad \left. \begin{array}{l} \\ \end{array} \right\} \forall j \in \mathbb{I}_{[0, t-1]} \quad (4.20c)$$

$$y_j - Cz_j|t \in \gamma_j C\Omega \oplus \mathbb{H}, \quad (4.20d)$$

$$\mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) \subseteq z_t|t \oplus \gamma_t|t\Omega, \quad (4.20e)$$

$$As + B(v_k|t + K\Xi) \oplus \mathbb{W} \subseteq z_{k+1}|t \oplus \gamma_{k+1}|t\Omega,$$

$$\forall s \in z_k|t \oplus \gamma_k|t\Omega, \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (4.20f)$$

$$z_{k+1}|t = Az_k|t + Bv_k|t, \quad \left. \begin{array}{l} \\ \end{array} \right\} \forall k \in \mathbb{I}_{[t, t+N-1]} \quad (4.20g)$$

$$(z_k|t, v_k|t) \in \mathcal{Z}, \quad (4.20h)$$

$$0 \leq \gamma_k|t \leq 1, \quad \forall k \in \mathbb{I}_{[t, t+N]}, \quad (4.20i)$$

$$z_{t+N}|t \in \mathcal{X}_f, \quad (4.20j)$$

where  $\mathbf{d}_t = [z_{0:t+N}|t, v_{t:t+N-1}|t, \gamma_{t:t+N}|t]$  is the vector of decision variables,

$$J_N(z_{t:t+N}|t, v_{t:t+N-1}|t, \gamma_{t:t+N}|t) = \lambda(z_t|t) + \sum_{k=t}^{t+N-1} \tilde{\ell}(z_k|t, v_k|t, \gamma_k|t) + V_f(z_{t+N}|t) \quad (4.21)$$

is the economic objective function with initial and terminal penalty costs  $\lambda(\cdot)$  and  $V_f(\cdot)$ , and  $\mathcal{X}_f$  is a terminal constraint set. Note that the initial weighting function is the storage function defined in Assumption 4.2, and it has been verified that this function is crucial in the proof of stability as shown in Chapter 3.

This optimization problem utilizes the full past information, which is not practically realizable as the number of decision variables increases linearly with time. To overcome this issue, one method is keeping the estimation window as small as possible with guaranteed recursive feasibility that is only discarding the redundant output measurements. The other approach is adopting moving horizon estimation which only considers the past data within a finite window horizon  $N_E \in \mathbb{I}_{\geq 1}$  after the time instant  $t = N_E$ . In particular, the decision variables  $z_{0:t-N_E-1}|t$  are removed and the index of (4.20b)-(4.20d) becomes  $j \in \mathbb{I}_{[t-N_E, t-1]}$ . In addition, the reachable set  $\mathcal{R}_t$  might often be trimmed by the new output measurements,

so that discarding past measurements could avoid its number of vertices growing unbounded. However, there might be a violation of recursive feasibility when the reachable set  $\mathcal{R}_t$  is discarding the previous outputs. To explain this result, let us denote the reachable set at time  $t$  after discarding the output measurement  $y_{0:t-N_E}$  by  $\mathcal{R}_t^{N_E}$ , and accordingly it should be defined as

$$\begin{aligned} \mathcal{R}_t^{N_E} = \{x_t \mid \exists x_{t-1} \in \mathcal{R}_{t-1}^{N_E} \cup Y_{t-N_E-1}, \exists \omega_{t-1} \in \mathbb{W}, \\ y_t - Cx_t \in \mathbb{H}, x_t = Ax_{t-1} + Bu_{t-1} + \omega_{t-1}\}, \end{aligned} \quad (4.22)$$

where  $Y_t = \{x \mid y_t - Cx \in \mathbb{H}\}$ . Based on this definition, it is concluded that  $\mathcal{R}_{t-1} \subseteq \mathcal{R}_{t-1}^{N_E}$ , and hence there is no guarantee that  $\mathcal{R}_{t-1}^{N_E} \subseteq z_{t-1|t-1}^* \oplus \gamma_{t-1|t-1}^* \Omega$ , which is necessary in the proof of recursive feasibility. In particular, when the output  $y_{t-N_E-1}$  is important, this set inclusion may be violated and thus breaks the constraint (4.20e) at time  $t$ . Therefore, one may resort to methods in [35] to treat this problem.

**Remark 4.3.** In particular, rather than tightening the nominal state-input pairs by the homothetic tube, the constraint (4.20c) is enforced because it is not always satisfied that  $\hat{x} - z \in \gamma \Xi$  in which case the input constraint is violated. The constraint (4.20d) is to select the nominal dynamic compatible with the output measurements, and the restriction of scalar factors along the prediction horizon is described by the constraint (4.20f) as in [69].

**Remark 4.4.** The constraint (4.20e) employs a homothetic tube with scalar factor  $\gamma_{t|t}$  to circumscribe the set of potential states  $\mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1})$ . This ensures that  $\gamma_{t|t} \in [0, 1]$  is selected so that the real system state fulfills  $x_t \in z_{t|t} \oplus \gamma_{t|t} \Omega$ . Moreover, it is less conservative compared to  $x_t \in z_{t|t} \oplus \Omega$ , and may result in better economic performance due to the monotonicity of  $\tilde{\ell}$  with respect to  $\gamma$ . To mitigate the computational difficulty of this set constraint, only its vertices are considered as the computed RPI sets are convex polyhedrons.

**Remark 4.5.** Notice that we are not applying certainty equivalence or any kind of separation principle in order to compute state-estimates, and control policies. On the contrary, a unique “coupled” optimization problem of “pure” economic nature is solved in order to simultaneously compute them. By integrating the MHE with EMPC optimization problems and adopting a pure economic cost, there is no need to maximize the likelihood but only to select the nominal state sequence compatible with the known measurements, so that the economic criterion is optimized within the considered prediction horizon. The proposed approach seeks to optimize performance without probabilistic assumptions on the disturbances and provide a guaranteed bound on performance. Modelling the probability distribution of the disturbances and using the expected stage cost could reduce unwanted conservatism, but it will not afford a performance bound and beyond the scope of this thesis.

For the terminal periodic sets, we make the following assumptions:

**Assumption 4.4.** *There exist terminal sets  $\mathcal{X}_f \subseteq \mathbb{X} \ominus \Omega$ , containing the nominal optimal state  $z_{s,i}$  in their interior, a terminal control policy  $\kappa_f(\cdot)$ , and a continuous terminal penalty function  $V_f : \mathcal{X}_f \rightarrow \mathbb{R}$ , such that for all  $z \in \mathcal{X}_f$  and  $\gamma \in [0, 1]$ , it holds*

- (i)  $(z, \kappa_f(z)) \in \mathcal{Z}$
- (ii)  $Az + B\kappa_f(z) \in \mathcal{X}_f$
- (iii)  $V_f(Az + B\kappa_f(z)) - V_f(z) \leq \tilde{\ell}_{av}^*(\gamma) - \tilde{\ell}(z, \kappa_f(z), \gamma)$ .

When the system enters the optimization routine at time  $t$ , the following sub-optimal and feasible solution which is generated from the previous optimal solution can be adopted as an initial guess

$$\begin{aligned} z_{0:t+N|t}^{sub} &= (z_{0|t-1}^*, z_{1|t-1}^*, \dots, z_{t+N-1|t-1}^*, Az_{t+N-1|t-1}^* + B\kappa_f(z_{t+N-1|t-1}^*)), \\ v_{t:t+N-1|t}^{sub} &= (v_{t|t-1}^*, v_{t+1|t-1}^*, \dots, v_{t+N-2|t-1}^*, \kappa_f(z_{t+N-1|t-1}^*)), \\ \gamma_{t:t+N-1|t}^{sub} &= (\gamma_{t|t-1}^*, \gamma_{t+1|t-1}^*, \dots, \gamma_{t-1+N|t-1}^*, 1). \end{aligned} \quad (4.23)$$

Finally, the optimization solver converges to the optimal solution  $d_t^*$  which is

$$\begin{aligned} z_{0:t+N|t}^* &= (z_{0|t}^*, z_{1|t}^*, \dots, z_{t|t}^*, \dots, z_{t+N|t}^*), \\ v_{t:t+N-1|t}^* &= (v_{t|t}^*, v_{t+1|t}^*, \dots, v_{t+N-1|t}^*), \\ \gamma_{t:t+N|t}^* &= (\gamma_{t|t}^*, \gamma_{t+1|t}^*, \dots, \gamma_{t+N|t}^*), \end{aligned} \quad (4.24)$$

and the corresponding optimal value function is

$$V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) = J_N(z_{t:t+N|t}^*, v_{t:t+N-1|t}^*, \gamma_{t:t+N|t}^*). \quad (4.25)$$

For an initial real system state  $x$ , let us denote the admissible set fulfilling constraints (4.20a)-(4.20j) by  $\mathcal{Z}_N(x)$  and the collection of admissible states  $x$ ,

$$\mathcal{X}_N = \{x \in \mathbb{R}^{n_x} \mid \exists(z, v, \gamma) \in \mathcal{Z}_N(x)\}. \quad (4.26)$$

Next, following the manner of feedback algorithm in EMPC, the optimal control implemented to the system (4.1) is

$$u_t = v_{t|t}^* + K(\hat{x}_t - z_{t|t}^*), \quad (4.27)$$

and the resulting closed-loop dynamic is

$$x_{t+1} = Ax_t + Bu_t + \omega_t. \quad (4.28)$$

In addition, the optimal value  $\gamma_{t|t}^*$  is recorded as  $\gamma_t$  and will be used in future optimization problems. If the optimal solution  $d_{t|t}^*$  is non-unique, we may take any sequence that achieves the optimum.

### 4.2.2 Recursive feasibility

Note that, provided the optimization problem is feasible, the optimal nominal state-input pair  $(z_{t|t}^*, v_{t|t}^*)$  is restricted in the set  $\mathcal{Z}$  defined in (4.10). The rationale is that true state and input solutions, in the usual spirit of tube MPC, will then fulfill

$$(x_t, v_{t|t}^* + K(\hat{x}_t - z_{t|t}^*)) \in \mathbb{Z}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (4.29)$$

whatever the disturbance and noise signals are until time  $t$ . Therefore, the real system's solutions fulfill pointwise-in-time constraints.

**Proposition 4.1.** *Let Assumption 4.1, 4.3, 4.4.(i), and 4.4.(ii) hold, then the economic MPC optimization problem (4.20) is recursively feasible.*

*Proof.* The main idea to prove recursive feasibility is to explicitly construct feasible solutions of problems (4.20), at the current sampling time, given the previous feasible and optimal solutions. Specifically, the sequences (4.23) will be shown as a feasible solution.

Notice that the sequences (4.23) are generated from the optimization problem at time  $t - 1$ , they fulfill the initial nominal state constraint (4.20a), the nominal state dynamic equality constraints (4.20b) and (4.20g), the pointwise-in-time constraints (4.20c) and (4.20h), and the scalar factor constraints (4.20f) and (4.20i).

Because the set  $\Omega$  is invariant with respect to system disturbance and measurement noise, and the homothetic sets circumscribe the reachable states, it holds

$$x_{t-1} \in z_{t-1|t-1}^* \oplus \gamma_{t-1|t-1}^* \Omega, \quad \text{and} \quad x_t \in z_{t|t-1}^* \oplus \gamma_{t|t-1}^* \Omega$$

which further imply

$$\begin{aligned} y_{t-1} - Cz_{t-1|t-1}^* &\in \gamma_{t-1|t-1}^* C\Omega \oplus \mathbb{H}, \\ \mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) &\subseteq z_{t|t-1}^* \oplus \gamma_{t|t-1}^* \Omega. \end{aligned}$$

In addition, satisfaction of the terminal constraint follows by Assumption 4.4, viz.

$$z_{t+N-1|t-1}^* \in \mathcal{X}_f \Rightarrow Az_{t+N-1|t-1}^* + B\kappa_f(z_{t+N-1|t-1}^*) \in \mathcal{X}_f.$$

Therefore, the optimization problem at current time  $t$  has at least one solution, and recursive feasibility of this optimization problem is ensured.  $\square$

### 4.3 Stability and asymptotic performance analysis

This section is dedicated to the analysis of closed-loop behaviors of the system using the proposed EMPC controller. The first part is to prove the stability of the optimal robust invariant set, which is inferred from the asymptotic convergence of the nominal state sequence. Then, an upper bound of the asymptotic average performance, regardless of the disturbance realizations is derived. Before discussing these two closed-loop properties, we need to make the following assumption:

**Assumption 4.5.** *The solution in (4.15) and (4.16) are the unique optimal steady-state and the optimal period- $P$  operation, respectively, for any  $\gamma \in [0, 1]$ .*

**Remark 4.6.** This assumption ensures that the optimal regime of operation is independent of the choice of homothetic tube. It is worth to notice that the stability result discussed later is not valid if this condition does not hold. Nevertheless, the use of homothetic tube may provide economic profits, which will be indicated by examples in Section 4.4.

#### 4.3.1 Stability analysis

Asymptotic stability of economic MPC employing Lyapunov function was first proved in [48] under the assumption of strong duality. Such condition is subsequently relaxed by dissipativity in [3] and [5], where it is shown how to rotate the economic cost to standard positive definite function and choosing the rotated cost-to-go as a candidate Lyapunov

function. In analogy, this section explores the asymptotic stability of the closed-loop nominal sequence under EMPC control actions.

**Assumption 4.6.** *The terminal regions  $\mathcal{X}_f$  are compact. In addition, the storage function and the terminal cost function satisfy  $\lambda(z_{s,i}) + V_f(z_{s,i}) = 0, \forall i \in \mathbb{I}_{[0,P-1]}$  and  $\lambda(z) + V_f(z) \geq 0$  for all  $z \in \mathcal{X}_f \setminus \Pi_{\mathbb{X}}$ .*

We are now in position to state the first main result in this note:

**Theorem 4.1.** *Let Assumptions 4.1, 4.2, 4.3, 4.4, 4.5, and 4.6 hold, under the application of economic MPC feedback control policy (4.27), the set  $\bigcup_{i=0}^{P-1} [z_{s,i} \oplus \Omega]$  is asymptotically stable for the closed-loop system (4.28) with region of attraction  $\mathcal{X}_N$ .*

*Proof.* Following the usual way to prove asymptotic stability of a closed-loop system under an EMPC controller with the aid of dissipativity assumptions, we first introduce the rotated stage cost function, rotated terminal cost function as

$$\begin{aligned} L(z, v, \gamma) &= \tilde{\ell}(z, v, \gamma) - \tilde{\ell}_{av}^*(\gamma) + \lambda(z) - \lambda(Az + Bv) \geq 0, \\ \overline{V}_f(z) &= V_f(z) + \lambda(z) \geq 0. \end{aligned}$$

Accordingly, the rotated objective function at time  $t$  is

$$\overline{J}_N(z_{t:t+N|t}, v_{t:t+N-1|t}, \gamma_{t:t+N|t}) = \sum_{k=0}^{N-1} L(z_{t+k|t}, v_{t+k|t}, \gamma_{t+k|t}) + \overline{V}_f(z_{t+N|t}).$$

With the notion of rotated cost functions, readers can refer to [5, Theorem 15] and prove

$$\begin{aligned} \overline{V}_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) - \overline{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \\ \leq -L(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) \leq -\alpha_1(|z_{t|t}^*|_{\Pi_{\mathbb{X}}}), \end{aligned}$$

where

$$\overline{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) = \overline{J}_N(z_{t:t+N|t}^*, v_{t:t+N-1|t}^*, \gamma_{t:t+N|t}^*).$$

Thus, the optimal value of the rotated value function  $\overline{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})$  is non-increasing along the trajectory of closed-loop system (4.28) which implies  $|z_{t|t}^*|_{\Pi_{\mathbb{X}}} \rightarrow 0$  as  $t \rightarrow +\infty$ .

Next, we provide lower and upper bounds for the function  $\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})$ . Because of the non-negativity of rotated cost functions, by construction, the optimal rotated objective function fulfills

$$\begin{aligned}\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) &= L(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + \sum_{k=1}^{N-1} L(z_{t+k|t}^*, v_{t+k|t}^*, \gamma_{t+k|t}^*) + \bar{V}_f(z_{t+N|t}) \\ &\geq L(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) \geq \alpha_1(|z_{t|t}^*|_{\Pi_{\mathbb{X}}}).\end{aligned}$$

On the other hand, it holds

$$\bar{V}_1(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \leq L(z_{t|t}^*, \kappa_f(z_{t|t}^*)) + \bar{V}_f(Az_{t|t}^* + B\kappa_f(z_{t|t}^*)) \leq \bar{V}_f(z_{t|t}^*).$$

By induction, for any  $N \in \mathbb{I}_{\geq 2}$  and  $z_{t|t}^* \in \mathcal{X}_f$ , we have

$$\begin{aligned}\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) &\leq L(z_{t|t}^*, \kappa_f(z_{t|t}^*)) + \bar{V}_{N-1}(Az_{t|t}^* + B\kappa_f(z_{t|t}^*)) \\ &\leq L(z_{t|t}^*, \kappa_f(z_{t|t}^*)) + \bar{V}_f(Az_{t|t}^* + B\kappa_f(z_{t|t}^*)) \\ &\leq \bar{V}_f(z_{t|t}^*)\end{aligned}$$

Suppose that Assumption 4.6 is satisfied, the continuous function  $\bar{V}_f(z_{t|t}^*)$  fulfills  $\bar{V}_f(z_{t|t}^*) \leq \alpha_2(|z_{t|t}^*|_{\Pi_{\mathbb{X}}})$ , where  $\alpha_2(\cdot)$  is a *class*  $\mathcal{K}$  function. Hence, we obtain the optimal rotated objective function is bounded from above and below

$$\alpha_1(|z_{t|t}^*|_{\Pi_{\mathbb{X}}}) \leq \bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \leq \alpha_2(|z_{t|t}^*|_{\Pi_{\mathbb{X}}}).$$

Therefore, the function  $\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})$  is a Lyapunov function and the optimal set  $\Pi_{\mathbb{X}}$  is locally Lyapunov stable. Together with the global attractivity, the set  $\Pi_{\mathbb{X}}$  is asymptotic stable in the sense that there exists a *class*  $\mathcal{KL}$  function  $\beta$  such that  $|z_{t|t}^*|_{\Pi_{\mathbb{X}}} \leq \beta(|z_{0|0}^*|_{\Pi_{\mathbb{X}}}, t)$  for all  $t \in \mathbb{I}_{\geq 0}$ , which in turn implies the set  $\bigcup_{i=0}^{P-1} [z_{s,i} \oplus \Omega]$  is asymptotically stable for the real system.  $\square$

### 4.3.2 Closed-loop asymptotic performance

Due to convergence to the optimal regime of operation, the average cost of asymptotically stable closed-loop systems does not deserve special attention. Therefore, in the following we

analyze asymptotic average cost of more general systems in which the stability conditions, such as Assumption 4.2 and Assumption 4.4.(iii) for  $\gamma \in [0, 1)$ , are not required.

Next, we state our second main result in this note:

**Theorem 4.2.** *Suppose Assumptions 4.1, 4.3, 4.4.(i), 4.4.(ii) and 4.4.(iii) only for  $\gamma = 1$  are satisfied. Let us choose the initial penalty function  $\lambda(\cdot) = 0$ , then the asymptotic average performance of the closed-loop system (4.28), regardless of the system disturbance and measurement noise, fulfills,*

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} \leq \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \tilde{\ell}_{av}^*(1). \quad (4.30)$$

*Proof.* Given the sub-optimal control input sequence and its resulting state trajectory at time instant  $t + 1$  in the same structure as (4.23), for all disturbance  $\omega_t \in \mathbb{W}$  and measurement noise  $\eta_t \in \mathbb{H}$ , the optimal cost-to-go functions fulfill  $V_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) \leq J_N(z_{0:t+N+1|t+1}^{sub}, v_{0:t+N|t+1}^{sub}, \gamma_{0:t+N|t+1}^{sub})$ , which is

$$\begin{aligned} V_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) &\leq V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) - \lambda(z_{t|t}^*) + \lambda(z_{t+1|t}^*) \\ &\quad + V_f(Az_{t+N|t}^* + B\kappa_f(z_{t+N|t}^*)) - V_f(z_{t+N|t}^*) \\ &\quad + \tilde{\ell}(z_{t+N|t}^*, \kappa_f(z_{t+N|t}^*), 1) - \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) \\ &\leq V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) - \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + \tilde{\ell}_{av}^*(1), \end{aligned}$$

where the last inequality follows considering  $\lambda(\cdot) = 0$  and Assumption 4.4.(iii) for  $\gamma = 1$ . Then, integrating the previous inequalities, for any time  $T \in \mathbb{I}_{\geq 0}$ , we have,

$$\begin{aligned} &V_N(u_0, \hat{x}_0, y_{0:1}, \gamma_0) - V_N(\hat{x}_0, y_0) \\ &+ V_N(u_{0:1}, \hat{x}_0, y_{0:2}, \gamma_{0:1}) - V_N(u_0, \hat{x}_0, y_{0:1}, \gamma_0) \\ &+ \cdots + V_N(u_{0:T}, \hat{x}_0, y_{0:T+1}, \gamma_{0:T}) \\ &- V_N(u_{0:T-1}, \hat{x}_0, y_{0:T}, \gamma_{0:T-1}) \\ &= \sum_{t=0}^{T-1} [V_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) - V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})] \\ &\leq \sum_{t=0}^{T-1} [\tilde{\ell}_{av}^*(1) - \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)]. \end{aligned} \quad (4.31)$$

Finally, by applying  $\liminf$  on both sides of (4.31) and exploiting boundness of solutions, we obtain

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \tilde{\ell}_{av}^*(1).$$

To this end, based on the definition of modified cost function, for any  $x_t \in z_{t|t}^* \oplus \gamma_{t|t}^* \Omega$ , it holds  $\ell(x_t, u_t) \leq \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)$ ,  $\forall t \in \mathbb{I}_{\geq 0}$ . Therefore, the following inequality is fulfilled

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} \leq \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \tilde{\ell}_{av}^*(1).$$

which proves the claim.  $\square$

### 4.3.3 Rigid tube-based economic MPC algorithm

As mentioned in Remark 4.6, in case of Assumption 4.5 failing, asymptotic stability is not in general guaranteed which might be a dangerous disadvantage of homothetic tube-based economic MPC for some specific applications. However, the asymptotic average performance of the real system can be preserved, and to illustrate this point, a rigid tube-based economic MPC algorithm, in which  $\gamma = 1$ , defined as following will be employed for comparison,

$$\min_{\mathbf{d}_t} \lambda(z_{t|t}) + \sum_{k=t}^{t+N-1} \tilde{\ell}(z_{k|t}, v_{k|t}, 1) + V_f(z_{N|t}) \quad (4.32)$$

$$s.t. \quad z_{t|t} \in \hat{x}_t \oplus \Xi, \quad (4.32a)$$

$$z_{j+1|t} = Az_{j|t} + Bv_j, \quad (4.32b)$$

$$(z_{j|t}, v_j) \in \mathcal{Z}, \quad \left. \vphantom{\begin{matrix} z_{j+1|t} = Az_{j|t} + Bv_j \\ (z_{j|t}, v_j) \in \mathcal{Z} \end{matrix}} \right\} \forall j \in \mathbb{I}_{[0, t-1]} \quad (4.32c)$$

$$y_j - Cz_{j|t} \in C\Omega \oplus \mathbb{H}, \quad (4.32d)$$

$$z_{k+1|t} = Az_{k|t} + Bv_{k|t}, \quad (4.32e)$$

$$(z_{k|t}, v_{k|t}) \in \mathcal{Z}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]}, \quad (4.32f)$$

$$z_{t+N|t} \in \mathcal{X}_f, \quad (4.32g)$$

where  $\mathbf{d}_t = [z_{0:t+N|t}, v_{t:t+N-1|t}]$  is the vector of decision variables. The main advantage of this formulation is reduced computational complexity. Similarly to the previous formulation, recursive feasibility, asymptotic stability and asymptotic performance bounds for this rigid

tube-based algorithm can be proved. However, the modified cost function over a rigid set might reach the sub-optimal solution in some situations which will result into conservatism.

Furthermore, it is also possible to achieve less computational complexity by abandoning moving horizon estimation without losing desired recursive feasibility and stability. However, this may sacrifice the economic performance as the single Luenberger observer for current state estimation is not able to provide a tighter bound on the estimation error. The optimization problem of using Luenberger observer is formulated as:

$$\min_{\mathbf{d}_t} \lambda(z_{t|t}) + \sum_{k=t}^{t+N-1} \tilde{\ell}(z_{k|t}, v_{k|t}, 1) + V_f(z_{N|t}) \quad (4.32)$$

$$s.t. \quad z_{t|t} \in \hat{x}_t \oplus \Xi, \quad (4.33a)$$

$$z_{k+1|t} = Az_{k|t} + Bv_{k|t}, \quad (4.33b)$$

$$(z_{k|t}, v_{k|t}) \in \mathcal{Z}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]}, \quad (4.33c)$$

$$z_{t+N|t} \in \mathcal{X}_f, \quad (4.33d)$$

which has less number of decision variables  $\mathbf{d}_t = [z_{t:t+N|t}, v_{t:t+N-1|t}]$  compared to that in (4.32).

## 4.4 Numerical examples

### 4.4.1 Unstable scalar system - Assumption 4.5 fails

Let us consider a simple scalar example

$$\begin{aligned} x_{t+1} &= 1.1x_t + u_t + \omega_t \\ y_t &= x_t + \eta_t \end{aligned} \quad (4.34)$$

subjected to state and input constraints  $\mathbb{X} = [-10, 10]$  and  $\mathbb{U} = [-1, 1]$ . The process disturbance and measurement noise are restricted to  $\mathbb{W} = [-0.2, 0.2]$  and  $\mathbb{H} = [-0.1, 0.1]$ .

We select feedback gains  $K = -0.9$  and  $L = 0.9$ , such that  $A + BK = A - LC = 0.2$ , then the following equation is satisfied

$$\begin{bmatrix} \xi_{t+1} \\ e_{t+1} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.9 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} \xi_t \\ e_t \end{bmatrix} + \begin{bmatrix} 0 & 0.9 \\ 1 & -0.9 \end{bmatrix} \begin{bmatrix} \omega_t \\ \eta_t \end{bmatrix},$$

which provides the invariant sets  $\Xi \times E$  and  $\Delta$  in Fig.4.1 by using approaches in [66] and [31]. The error  $\xi + e$ , under these two methods, are positively invariant in intervals  $[-0.8828, 0.8828]$  and  $[-0.6578, 0.6578]$ , respectively. Hence, we use the tighter invariant set  $\Delta$  and the resulting tightened constraints on state and input are  $[-9.342, 9.342]$  and  $[-0.5722, 0.5722]$ .

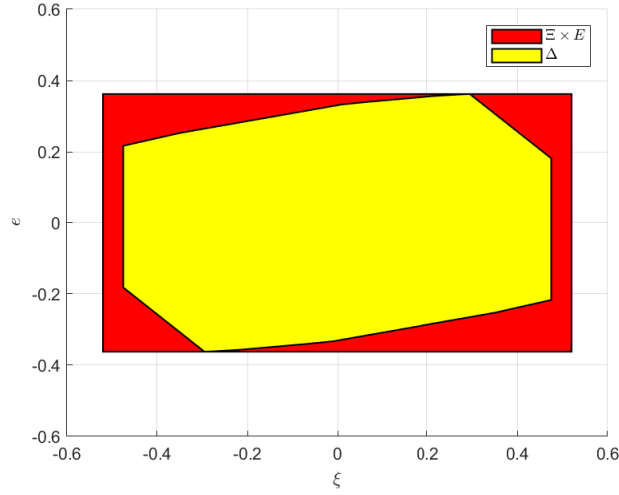


Fig. 4.1 Robust positively invariant sets of errors using two different approaches for the unstable scalar system (4.34).

In order to explore the effect of losing Assumption 4.5, we allow existence of  $\gamma \in [0, 1]$  whose corresponding robust optimal regime of operation differs from that of  $\gamma = 1$ , we define a stage cost

$$\ell(x, u) = \begin{cases} |x + \frac{0.6578}{2}| & -10 \leq x \leq 0 \\ |x - \frac{0.6578}{2}| & 0 < x \leq 10. \end{cases}$$

Accordingly, the modified economic cost function with respect to any  $\gamma \in [0, 1]$  is

$$\begin{aligned} \tilde{\ell}(z, v, \gamma) &= \begin{cases} \max\{|z - 0.6578\gamma + \frac{0.6578}{2}|, |z + 0.6578\gamma + \frac{0.6578}{2}|\} & -9.342 \leq z \leq -0.6578\gamma \\ \max\{|z - 0.6578\gamma + \frac{0.6578}{2}|, |z + 0.6578\gamma - \frac{0.6578}{2}|, \frac{0.6578}{2}\} & -0.6578\gamma < z \leq 0.6578\gamma \\ \max\{|z - 0.6578\gamma - \frac{0.6578}{2}|, |z + 0.6578\gamma - \frac{0.6578}{2}|\} & 0.6578\gamma < z \leq 9.342. \end{cases} \end{aligned}$$

which has the robust optimal steady-state  $(z_s, v_s) = (0, 0)$  and the optimal economic cost  $\tilde{\ell}_{av}^*(1) = \tilde{\ell}(z_s, v_s, 1) = \frac{0.6578}{2}$  for  $\gamma \in [0.5, 1]$ . Meanwhile, when  $\gamma \in [0, 0.5]$ , the robust optimal equilibrium points are  $(z, v) = (\frac{0.6578}{2}, -\frac{0.6578}{20})$  and  $(z, v) = (-\frac{0.6578}{2}, \frac{0.6578}{20})$ .

Next, for  $\gamma = 1$ , the worst cost over the rigid tube is

$$\begin{aligned} \tilde{\ell}(z, v, 1) &= \begin{cases} |z - 0.6578 + \frac{0.6578}{2}| & -9.342 \leq z \leq 0 \\ |z + 0.6578 - \frac{0.6578}{2}| & 0 \leq z \leq 9.342 \end{cases} \\ &= |z| + \frac{0.6578}{2}, \end{aligned}$$

hence the storage function is designed to be  $\lambda(z) = c$  where  $c$  is a finite constant real number, so that the following dissipativity inequality holds

$$\lambda(Az + Bv) - \lambda(z) \leq |z| + \frac{0.6578}{2} - (|0| + \frac{0.6578}{2}) = \tilde{\ell}(z, v, \gamma) - \tilde{\ell}_{av}^*(1).$$

For simplicity, in both homothetic tube-based EMPC (MHE-HT) and rigid tube-based EMPC (MHE-RT) methods, let us use the terminal equality constraint in which case  $V_f(z) = 0$ ,  $\mathcal{X}_f = z_s$ , and  $\kappa_f(z_s) = v_s$  such that Assumption 3.2 is satisfied. Under different disturbances and noise, Figures 4.2, 4.3, 4.4, and 4.5 illustrate the closed-loop states of the real system (3.1), observer system (4.4), and the optimal nominal sequences at initial states  $x = \hat{x} = 0.3289$  and prediction horizon  $N = 10$ . It can be seen that the nominal sequences converge to the optimal steady state  $z_s = 0$  in finite steps by using MHE-RT, whereas the sequence of initial nominal states, in the case of MHE-HT, stays at the  $z_s$  only in Fig.4.3. Moreover, as expected, for any control sequence, the total cost is monotone with respect to the initial scaling factor, which is indicated by every figure with the minimal scaling factor  $\underline{\gamma}$  defined as

$$\underline{\gamma}_t = \min_{\gamma \in [0, 1]} \{\gamma : \mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) \subseteq z_t^* \oplus \gamma\Omega\}.$$

Finally, the closed-loop asymptotic average cost of MHE-HT, in Fig.4.6, is no worse than that of MHE-RT, regardless of disturbances and noise. The average computational time using *fmincon* to solve the MHE-HT optimization problem is 1.3208s approximately, and the average number of calls for objective functions and nonlinear constraints are around  $4.1646 \times 10^3$  times.

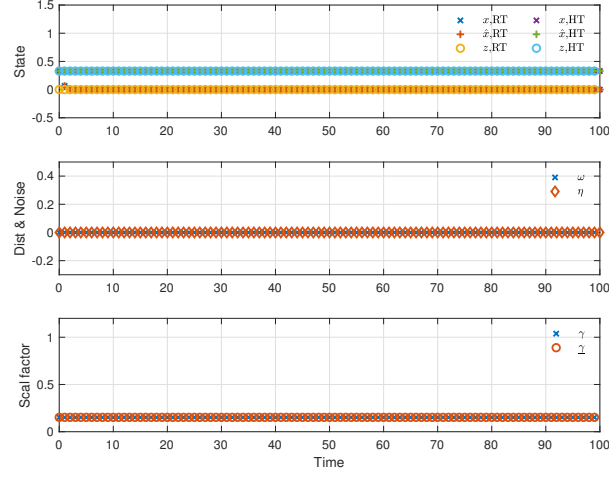


Fig. 4.2 Closed-loop simulation results of system (4.34) under zero disturbances.

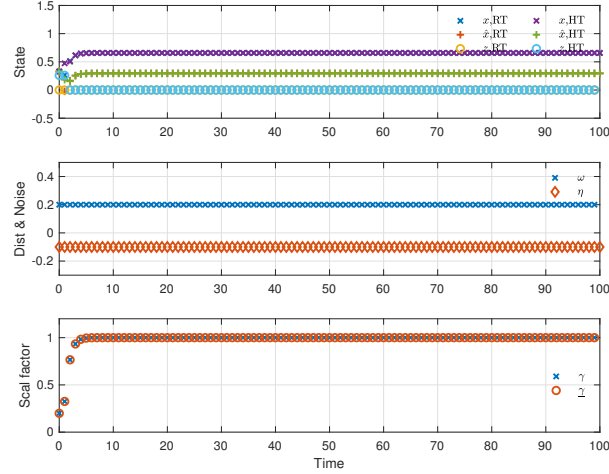


Fig. 4.3 Closed-loop simulation results of system (4.34) under non-zero constant disturbances.

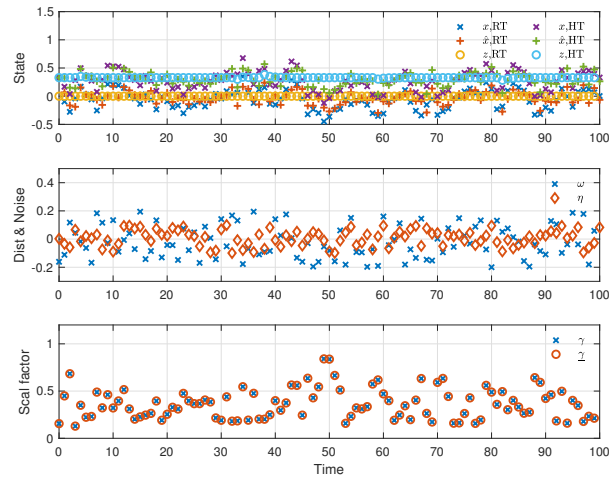


Fig. 4.4 Closed-loop simulation results of system (4.34) under uniformly distributed random disturbances.

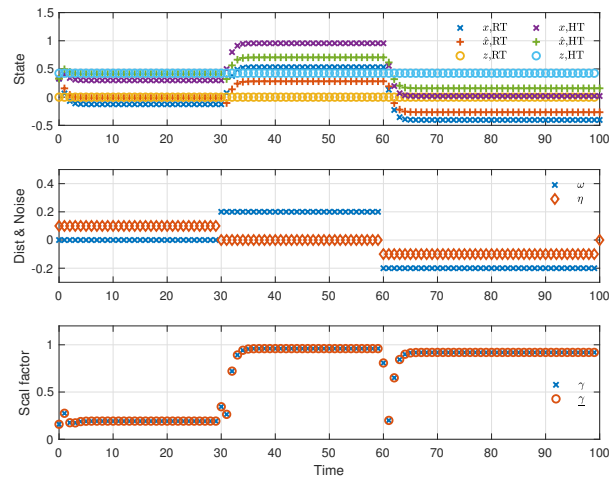


Fig. 4.5 Closed-loop simulation results of system (4.34) under piecewise constant disturbances.

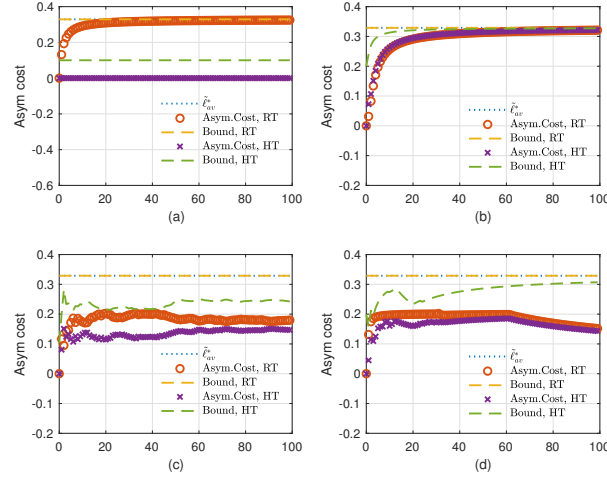


Fig. 4.6 Closed-loop performance comparison of system (4.34) using MHE-RT and HTEPMC under different disturbances, (a) nominal; (b) non-zero constant; (c) uniformly distributed random; (d) piecewise constant.

#### 4.4.2 Double integrator system - Assumption 4.5 holds

##### Optimal steady-state operation

Considering a constrained double integrator system defined as

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + B\mathbf{u}_t + \boldsymbol{\omega}_t \\ y_t &= C\mathbf{x}_t + \eta_t \end{aligned} \quad (4.35)$$

where  $\mathbf{x} = [x_1 \ x_2]^T$  and the system parameters are

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

This system is subject to constraints  $(\mathbf{x}, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}$  where  $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid [-10 \ -10]^T \leq \mathbf{x} \leq [10 \ 10]^T\}$  and  $\mathbb{U} = \{u \in \mathbb{R} \mid -5 \leq u \leq 5\}$ . The sets of state disturbance and measurement noise are  $\mathbb{W} = \{\boldsymbol{\omega} \in \mathbb{R}^2 \mid [-0.2 \ -0.2]^T \leq \boldsymbol{\omega} \leq [0.2 \ 0.2]^T\}$  and  $\mathbb{H} = \{\eta \in \mathbb{R} \mid -0.1 \leq \eta \leq 0.1\}$ , respectively. By selecting  $K = [-1 \ -1]$  and  $L = [1 \ 1]^T$ , the closed-loop

system including the observer is described as

$$\begin{bmatrix} \xi_{t+1} \\ e_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \xi_t \\ e_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \omega_t \\ \nu_t \end{bmatrix}$$

and the resulting tightened constraints on state and input are  $\mathcal{Z} = \{(z, v) \mid [-8.9 \ -8.2]^T \leq z \leq [8.9 \ 8.2]^T, -2 \leq v \leq 2\}$ .

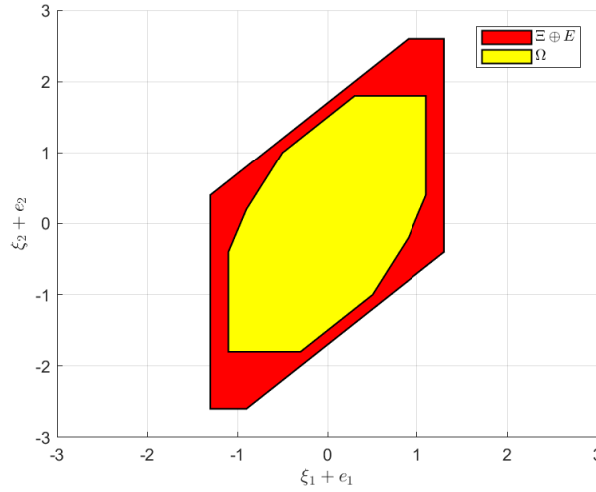


Fig. 4.7 Robust positively invariant sets of errors for the double integrator system (4.35).

Moreover, we define the economic stage cost as  $\ell(x, u) = x_1 + x_2$ , which implies  $\tilde{\ell}(z, v, \gamma) = z_1 + z_2 + 2.9\gamma$ ,  $(z_s, v_s) = ([-8.9 \ 0]^T, 0)$ , and  $\ell_{av}^* = -6$ . The storage function is  $\lambda(z) = z_1 - z_2$  satisfying the following dissipativity inequality

$$\begin{aligned} \lambda(Az + Bv) - \lambda(z) &= z_1 - (z_1 - z_2) \\ &\leq [1 \ 1](z + \gamma[1.1 \ 1.8]^T) - [1 \ 1](z_s + \gamma[1.1 \ 1.8]^T) \\ &= \tilde{\ell}(z, v, \gamma) - \tilde{\ell}(z_s, v_s, \gamma). \end{aligned}$$

Then, selecting the terminal ingredients by  $V_f(z) = 0$ ,  $\mathcal{X}_f = z_s$ , and  $\kappa_f(z_s) = v_s$ , we can obtain the simulated results in Fig. 4.8 - 4.11, under different disturbances and noise, with initial states  $x = [-2.6 \ -2.3]^T$ ,  $\hat{x} = [-2.4 \ -2]^T$  and prediction horizon  $N = 10$ . The average computational time using *fmincon* to solve the MHE-HT optimization problem is

0.0753s approximately, and the average number of calls for objective functions and nonlinear constraints are around 263 times.

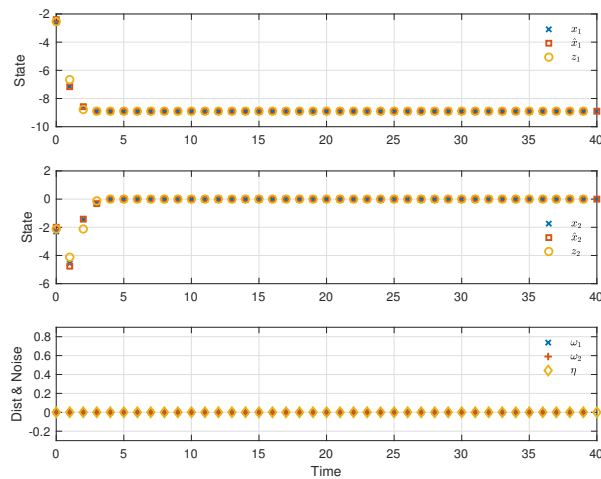


Fig. 4.8 Closed-loop simulation results of the double integrator system (4.35) with optimal steady-state operation under zero uncertainties.

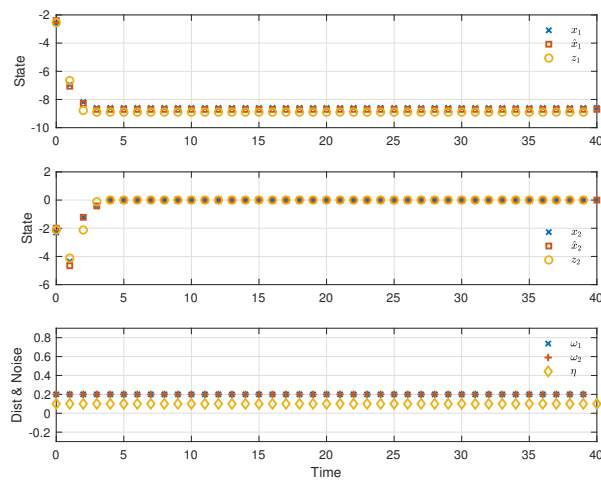


Fig. 4.9 Closed-loop simulation results of the double integrator system (4.35) with optimal steady-state operation under non-zero constant uncertainties.

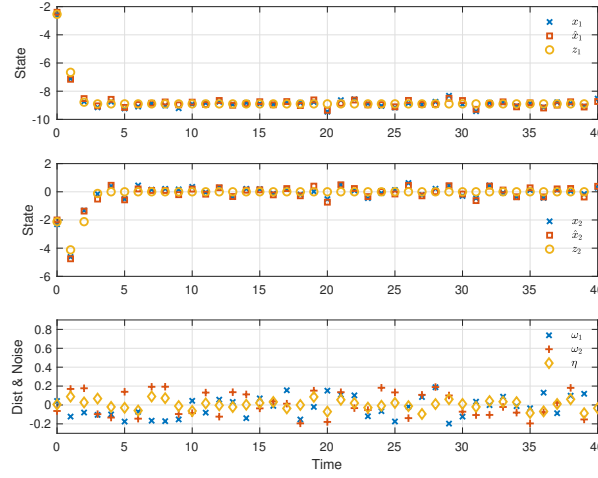


Fig. 4.10 Closed-loop simulation results of the double integrator system (4.35) with optimal steady-state operation under uniformly distributed uncertainties.

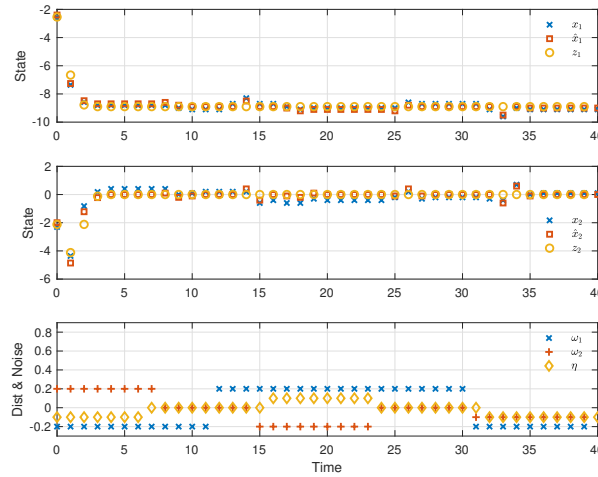


Fig. 4.11 Closed-loop simulation results of the double integrator system (4.35) with optimal steady-state operation under piecewise constant uncertainties.

### Optimal period-2 operation

Now, the input set is  $\mathbb{U} = \{u \in \mathbb{R} \mid 1 \leq |u| \leq 10\}$  and the stage cost function is defined as  $\ell(x, u) = 2|x_2|$ . In this case, the tightened constraints are  $\mathcal{Z} = \{(z, v) \mid [-8.9 \ -8.2]^T \leq z \leq [8.9 \ 8.2]^T, 4 \leq |v| \leq 7\}$ , and the modified cost function is  $\tilde{\ell}(z, v, \gamma) = 2|z_2| + 3.6\gamma$  which implies an infinite number of optimal periodic solutions because of translation

invariance with respect to the variable  $z_1$ . We select the robust optimal periodic pairs as  $(z_{s,0}, v_{s,0}) = ([-1 \ -2]^T, 4)$  and  $(z_{s,1}, v_{s,1}) = ([1 \ 2]^T, -4)$  whose robust optimal cost is  $\ell_{av}^* = \frac{2|-2-1.8|+2|2+1.8|}{2} = 7.2$ .

The storage function is chosen as  $\lambda(z) = -|[0 \ 1]z|$  that satisfies dissipativity inequality

$$\begin{aligned}
 \lambda(Az + Bv) - \lambda(z) &= -|z_2 + v| + |z_2| \\
 &\leq -|v| + |z_2| + |z_2| \\
 &\leq 2|z_2| - \frac{2|[0 \ 1]z_{s,0}| + 2|[0 \ 1]z_{s,1}|}{2} \\
 &= (2|z| + 3.6\gamma) - \frac{(2|[0 \ 1]z_{s,0}| + 3.6\gamma) + (2|[0 \ 1]z_{s,1}| + 3.6\gamma)}{2} \\
 &= \tilde{\ell}(z, v, \gamma) - \frac{\tilde{\ell}(z_{s,1}, v_{s,1}, \gamma) + \tilde{\ell}(z_{s,2}, v_{s,2}, \gamma)}{2}.
 \end{aligned}$$

Finally, the simulation results, with initial states  $x = -5.2$ ,  $\hat{x} = -5$  and prediction horizon  $N = 13$ , are in Fig.4.12 - 4.15, under different disturbances and noise. The average computational time using *fmincon* to solve the MHE-HT optimization problem is 0.3252s approximately, and the average number of calls for objective functions and nonlinear constraints are around 389 times.

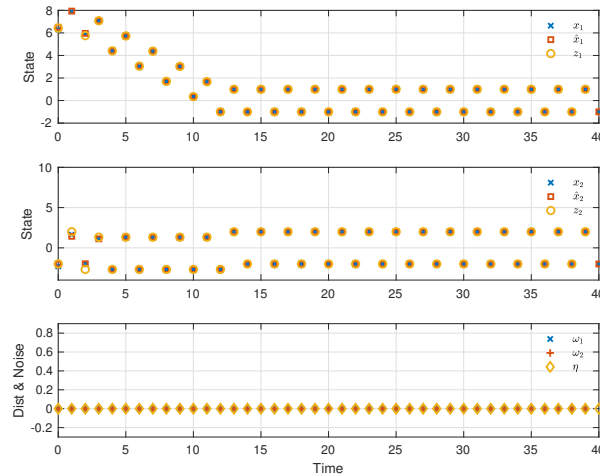


Fig. 4.12 Closed-loop simulation results of the double integrator system (4.35) with optimal period-2 solution under zero uncertainties.

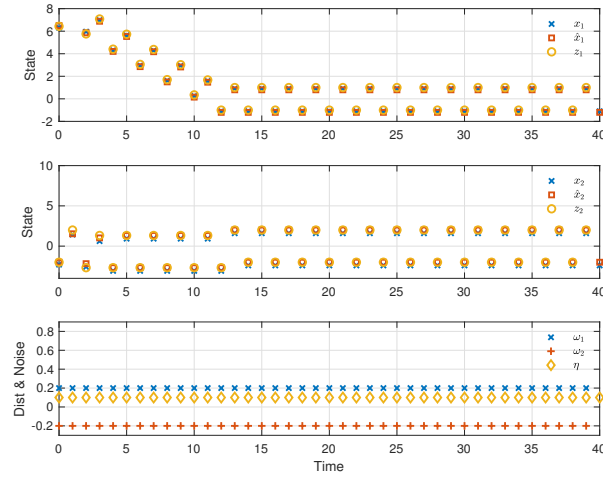


Fig. 4.13 Closed-loop simulation results of the double integrator system (4.35) with optimal period-2 solution under non-zero constant uncertainties.

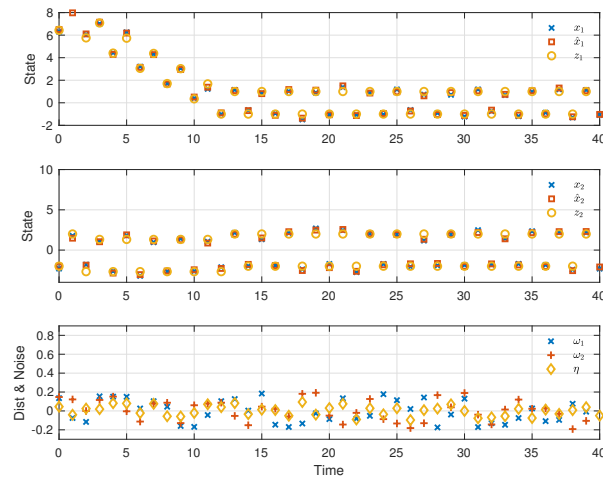


Fig. 4.14 Closed-loop simulation results of the double integrator system (4.35) with optimal period-2 solution under uniformly distributed uncertainties.

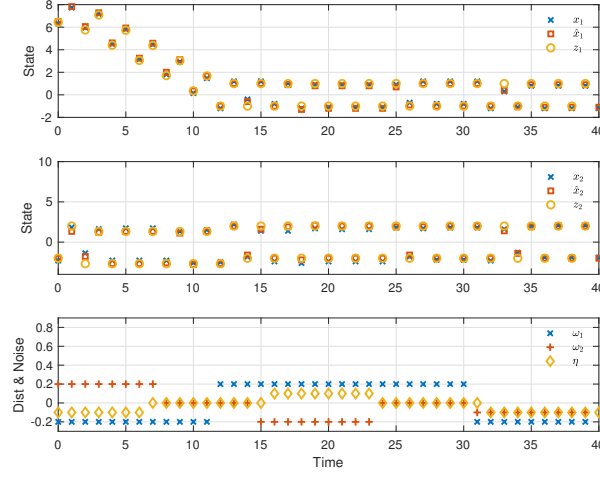


Fig. 4.15 Closed-loop simulation results of the double integrator system (4.35) with optimal period-2 solution under piecewise constant uncertainties.

#### 4.4.3 CSTR system - Assumption 4.5 fails

We consider a constant level continuous stirred tank reactor (CSTR) with a single exothermal irreversible first order reaction  $A \xrightarrow{k} B$ , which has been studied in [14] and [70]. The homothetic tube-based robust EMPC integrated with MHE (MHE-HT) algorithm, the rigid tube-based robust EMPC integrated with MHE (MHE-RT) algorithm and the rigid tube-based robust EMPC integrated with Luenberger observer (LO-RT) are applied to this simplified discrete time linearized CSTR model with sampling time  $T_s = 0.1 \text{ min}$ ,

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x} + Bu_t + B_\omega\boldsymbol{\omega}_t \\ y_t &= C\mathbf{x}_t + \eta_t \end{aligned} \quad (4.36)$$

where  $\mathbf{x} = [x_1 \ x_2]^T$  and system parameters

$$\begin{aligned} A &= \begin{bmatrix} 0.7776 & -0.0045 \\ 26.6185 & 1.8555 \end{bmatrix} & B &= \begin{bmatrix} -0.0004 \\ 0.2907 \end{bmatrix} \\ B_\omega &= \begin{bmatrix} -0.0002 & 0.0893 \\ 0.1390 & 1.2267 \end{bmatrix} & C &= \begin{bmatrix} 0 & 1 \end{bmatrix}. \end{aligned}$$

This model consists of two states, the concentration  $x_1$  of the reactant and the temperature  $x_2$  in the reactor. The control input  $u$  is the cooling temperature, while the feed flow concentration and the feed flow temperature are regarded as disturbances.

In terms of constraints, the system is subject to

$$\begin{aligned}\mathbb{X} &= \{x \in \mathbb{R}^2 \mid |x_1| \leq 0.5, |x_2| \leq 5\} \\ \mathbb{U} &= \{u \in \mathbb{R} \mid |u| \leq 15\}.\end{aligned}$$

The disturbance is bounded within the set

$$\mathbb{W} = \{\omega \in \mathbb{R}^2 \mid |\omega_1| \leq 0.2, |\omega_2| \leq 0.05\}$$

and the measurement noise is bounded by

$$\mathbb{H} = \{\eta \in \mathbb{R} \mid |\eta| \leq 0.01\}.$$

By selecting  $K = [-84.5145 \ -6.3997]$  and  $L = [0.0182 \ 2.6331]^T$ , the bounds for errors are shown in Fig. 4.16, which result into the tightened constraints

$$\mathcal{Z} = \{(z, v) \mid [-0.4811 \ -4.139]^T \leq z \leq [0.4811 \ 4.139]^T, -9.997 \leq v \leq 9.997\}.$$

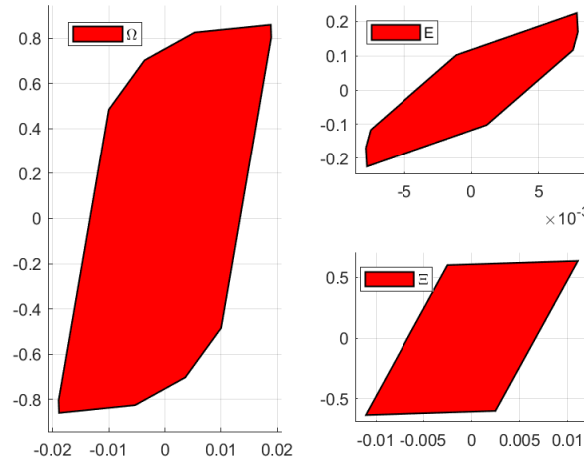


Fig. 4.16 Forward invariant sets  $\Omega$ ,  $E$  and  $\Xi$  of state errors for system (4.36).

The stage cost function is written as

$$\ell(\mathbf{x}, u) = x_1 + 0.01x_2^2,$$

where the first term is to boost the reaction from  $A$  to  $B$ , and the second term can be interpreted as a soft constraint on the temperature, which penalizes the deviation of  $x_2$  from 0. Hence, the modified economic stage cost with a scaling factor  $\gamma$  is

$$\tilde{\ell}(\mathbf{z}, v, \gamma) = \max_{\delta \in \Omega} z_1 + \gamma\delta_1 + 0.01(z_2 + \gamma\delta_2)^2.$$

Due to convexity, we know the maximum of the modified cost can be achieved on the vertex of the set  $\Omega$ , so it is reduced to

$$\tilde{\ell}(\mathbf{z}, v, \gamma) = \max_{\delta \in \mathcal{V}} z_1 + \gamma\delta_1 + 0.01(z_2 + \gamma\delta_2)^2.$$

where  $\mathcal{V}$  is the collection of vertices of  $\Omega$ .

According to [14, Theorem 3], the robust optimal regime of operation for the nominal system with  $\gamma = 1$  is a steady-state that is

$$(\mathbf{z}_s, v_s) = ([-0.0006 \ 0.0337]^T, -0.043954).$$

Note that, for this modified economic cost, different values of  $\gamma$  correspond to different optimal steady-state so that Assumption 4.5 does not hold. With  $\gamma = 1$ , a candidate of storage function obtained by using SOSTOOLS is

$$V(\mathbf{z}) = 2.67889z_1^2 - 0.04058z_1z_2 - 0.00387z_2^2 - 5.37833z_1 - 0.00717z_2,$$

so that the dissipativity inequality holds.

In the simulation, we assume the prediction horizon  $N = 12$ , initial states  $\mathbf{x}_0 = [0.2 \ -3]^T$  and  $\hat{\mathbf{x}}_0 = [0.205 \ -2.9]^T$ , and use the terminal equality constraint such that Assumption 4.4 is satisfied and the terminal state is driven to the optimal steady-state in  $N$  steps. The optimization is performed using the MATLAB routine *fmincon* with the *sqp* solver. In particular, the computation of sets are based on the Multi-Parametric Toolbox 3 (MPT3). The following Table.4.1 summarizes the computational usage and the asymptotic average economic costs (averaged over 200 simulations with  $\omega$  and  $\eta$  uniformly distributed over  $\mathbb{W}$  and  $\mathbb{H}$ , respectively) of the three algorithms.

	MHE-HT	MHE-RT	LO-RT
$\tau_f(\text{seconds})$	4.94332	0.68121	0.186665
$\ell_{av}(T = 20)$	0.06133	0.06138	0.06144
$\bar{\ell}_{av}(T = 20)$	0.07167	0.08383	0.07874
$\ell_{av}(T = 50)$	0.02383	0.02385	0.02387
$\bar{\ell}_{av}(T = 50)$	0.03339	0.04946	0.04725
$\ell_{av}(T = 100)$	0.01057	0.01059	0.01059
$\bar{\ell}_{av}(T = 100)$	0.0205	0.03786	0.03675

Table 4.1 Average computational time for  $fmincon(\tau_f)$ , corresponding averaged economic costs ( $\ell_{av}(T) = \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T}$ ), and the upper bounds of asymptotic average performance ( $\bar{\ell}_{av}(T) = \frac{\sum_{t=0}^{T-1} \bar{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T}$ , where  $\gamma_{t|t}^* = 1$  in MHE-RT) by using three proposed EMPC algorithms for the CSTR plant (4.36), averaged over 200 randomly generated disturbance and noise sequences, when Assumption 4.5 fails.

As expected, while Assumption 4.5 does not hold, the approach considering the homothetic tube in the optimization provides a better average performance and bounds at the expense of computational complexity. Moreover, it can be seen that the upper bound of MHE-RT is more conservative than that of LO-RT because more constraints on past information are enforced in the optimization problems.

We also run simulations consisting malicious disturbance and noise sequences,  $\omega_t = [0.2 \ 0.05]^T$  and  $\eta_t = 0.01$ ,  $\forall t \in \mathbb{I}_{\geq 0}$ , which intend to maximize the economic cost in prediction. The simulated results are shown in Table.4.2. For these non-zero constant disturbance and noise, the optimal value of the scaling factor  $\gamma_{t|t}^*$  converges to 0.8946. Comparing these two tables, it can be observed that the method MHE-HT provides a larger gap in average economic performance relative to other two methods when the disturbances and noises are malicious to the system. The reason is that the modified online optimization is considering the worst cost within the tube, so the results in Table.4.2 takes more advantages compared to Table.4.1 which corresponds to the random noise realizations.

	MHE-HT	MHE-RT	LO-RT
$\tau_f(\text{seconds})$	5.03089	0.799921	0.165637
$\ell_{av}(T = 20)$	0.0718	0.07539	0.0754
$\bar{\ell}_{av}(T = 20)$	0.07727	0.08413	0.08186
$\ell_{av}(T = 50)$	0.04031	0.04113	0.04114
$\bar{\ell}_{av}(T = 50)$	0.04469	0.04971	0.04875
$\ell_{av}(T = 100)$	0.02914	0.02944	0.02944
$\bar{\ell}_{av}(T = 100)$	0.03354	0.03798	0.0375

Table 4.2 Computational time for  $fmincon$  ( $\tau_f$ ), corresponding asymptotic economic costs ( $\ell_{av}(T)$ ), and the upper bounds of asymptotic performance ( $\bar{\ell}_{av}(T)$ ) by using three proposed EMPC algorithms for the CSTR plant (4.36), under the malicious disturbance and noise sequences, when Assumption 4.5 fails.

Note that, without Assumption 4.5, the algorithm does not guarantee convergence to the robust optimal equilibrium  $z_s$ . For instance, given arbitrary uniformly randomly generated disturbance and noise sequences of  $\omega$  and  $\eta$ , comparing the sequences of the nominal states shown in Fig.4.17, Fig.4.19 and Fig.4.20, the algorithms MHE-RT and LO-RT always enforce initial nominal states converging to the robust optimal steady state in finite time, whereas they do not have convergence behavior in MHE-HT as the value of  $\gamma$  varies.

Therefore, we can conclude that our proposed algorithm MHE-HT can bring some advantages in economic performance while stability convergence is not a necessary requirement. Moreover, as the total cost is monotone with respect to the initial scaling factor, it is observed that in Fig.4.18 the optimal current scaling factor  $\gamma_{t|t}^*$  coincides with the minimal scaling factor  $\underline{\gamma}_t$  defined as

$$\underline{\gamma}_t = \min_{\gamma \in [0,1]} \{ \gamma : \mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) \subseteq z_{t|t}^* \oplus \gamma \Omega \}.$$

The average computational time using  $fmincon$  to solve the MHE-HT optimization problem is 1.9585s approximately, and the average number of calls for objective functions and nonlinear constraints are around 3699 times.

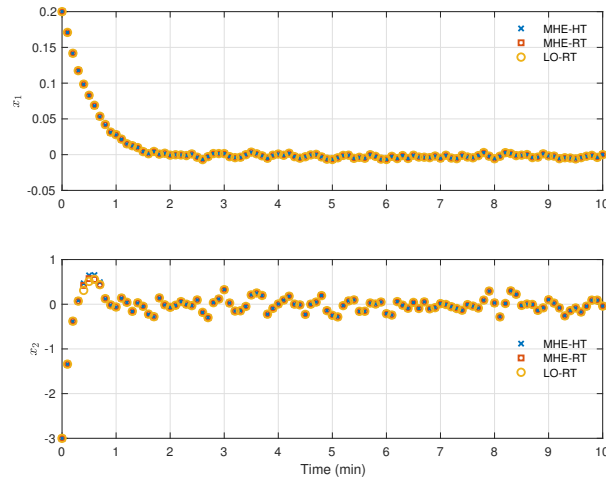


Fig. 4.17 The real state trajectory in closed-loop when Assumption 4.5 does not hold.

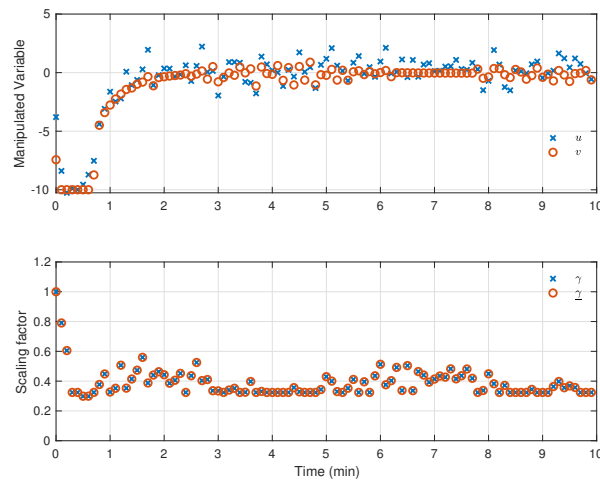


Fig. 4.18 Sequences optimal control inputs and scaling factors using MHE-HT EMPC algorithm, under randomly generated disturbances and noise sequence, when Assumption 4.5 does not hold and the optimal operation is a steady-state.

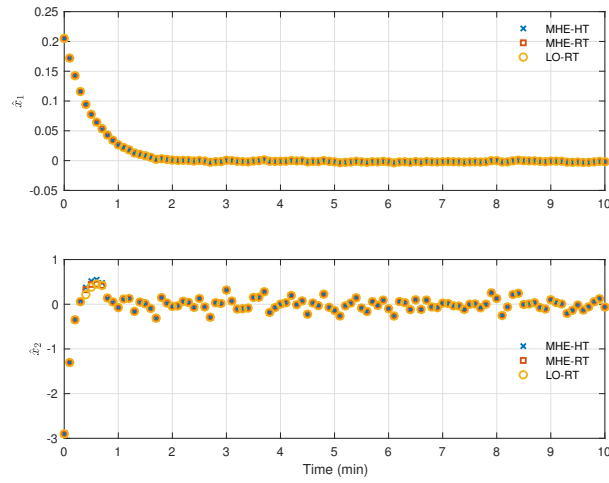


Fig. 4.19 The estimated state trajectory in closed-loop when Assumption 4.5 does not hold and the optimal operation is a steady-state.

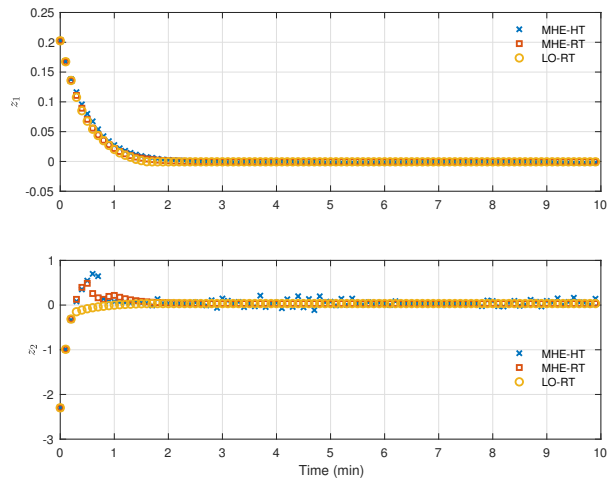


Fig. 4.20 The nominal state sequence in closed-loop when Assumption 4.5 does not hold and the optimal operation is a steady-state.

#### 4.4.4 CSTR system - Assumption 4.5 satisfied

##### Optimal steady-state operation

If different values of the scaling factor  $\gamma$  lead to the same optimal equilibrium, which means Assumption 4.5 holds, we compare the economic performance of using three algorithms

and show that the closed-loop nominal sequence of MHE-HT also converges to the robust optimal equilibrium  $z_s$ .

Considering the same system in previous section but with different stage cost function

$$\ell(x, u) = 50x_1$$

which is to accelerate the reaction from  $A$  to  $B$  without any penalty on the temperature. Thus, the modified stage cost is represented as

$$\tilde{\ell}(z, v, \gamma) = \max_{\delta \in \mathcal{V}} 50(z_1 + \gamma\delta_1).$$

In this case, for any  $\gamma \in [0, 1]$ , the robust optimal regime of operation is

$$(z_s, v_s) = ([-0.0740 \ 4.1390]^T, -5.4017).$$

Then, a candidate of storage function with respect to this new stage cost is

$$\begin{aligned} V(z) = & 2734.42z_1^2 - 1.9195z_1z_2 - 0.1929z_2^2 \\ & + 144.29z_1 + 1.0853z_2, \end{aligned}$$

For the initial states  $x_0 = [-0.2 \ 2.5]^T$  and  $\hat{x}_0 = [-0.195 \ 2.6]^T$ , and prediction horizon  $N = 12$ , the simulated results are shown in the following TABLE.4.3 and Fig.4.21 - Fig.4.24. Although the algorithm MHE-HT consumes more time in computation, the resulting economic performance is slightly better than other two cases as tighter characterizations of the state whereabouts allow sharper optimization of the economic criterion. In particular, there is an obvious improvement on performance bound.

	MHE-HT	MHE-RT	LO-RT
$\tau_f(\text{seconds})$	2.61487	0.253959	0.0224035
$\ell_{av}(T = 20)$	-5.269	-5.268	-5.268
$\bar{\ell}_{av}(T = 20)$	-4.341	-3.99	-3.982
$\ell_{av}(T = 50)$	-4.378	-4.377	-4.377
$\bar{\ell}_{av}(T = 50)$	-3.722	-3.25	-3.246
$\ell_{av}(T = 100)$	-4.097	-4.096	-4.096
$\bar{\ell}_{av}(T = 100)$	-3.506	-3.002	-3

Table 4.3 Average computational time for  $fmincon$  ( $\tau_f$ ), corresponding averaged economic costs ( $\ell_{av}(T)$ ), and the upper bounds of asymptotic average performance ( $\bar{\ell}_{av}(T)$ ) by using three proposed EMPC algorithms for the CSTR plant (4.36), averaged over 200 randomly generated disturbance and noise sequences, when Assumption 4.5 is satisfied.

An important observation of adopting these three algorithms is that the closed-loop nominal sequences converge to the common robust optimal state and input pair  $(z_s, v_s)$  regardless of the disturbances and noises or the optimal scaling factors. This fulfills our expectation because satisfaction of Assumption 4.5 ensures convergence, and hence the improvement in average economic cost by using MHE-HT might be less evident. The average computational time using  $fmincon$  to solve the MHE-HT optimization problem is 0.5591s approximately, and the average number of calls for objective functions and nonlinear constraints are around 781 times.

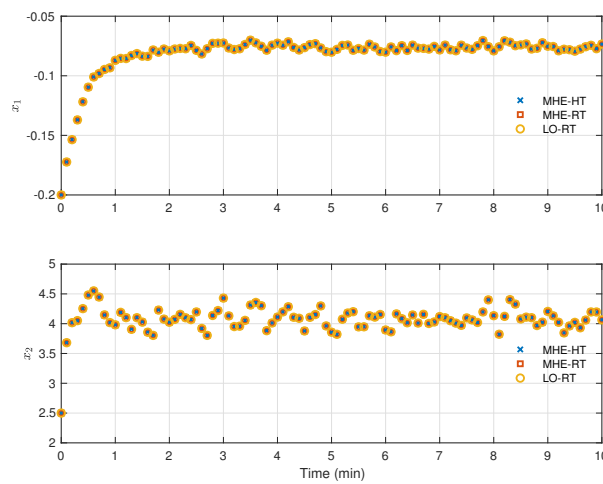


Fig. 4.21 The real state trajectory in closed-loop when Assumption 4.5 holds and the optimal operation is a steady-state.

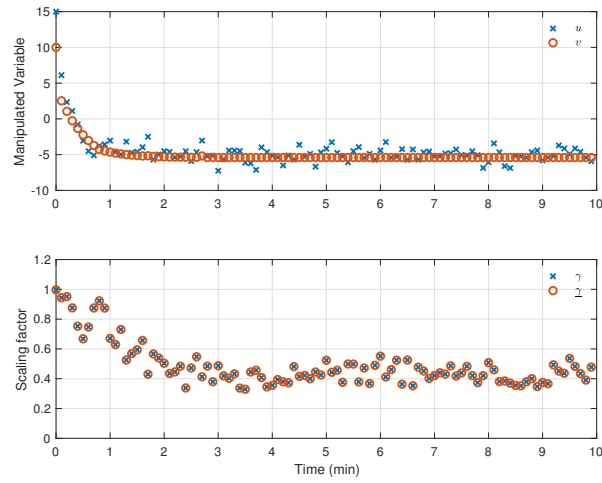


Fig. 4.22 Sequences optimal control inputs and scaling factors using MHE-HT EMPC algorithm, when Assumption 4.5 holds and the optimal operation is a steady-state.

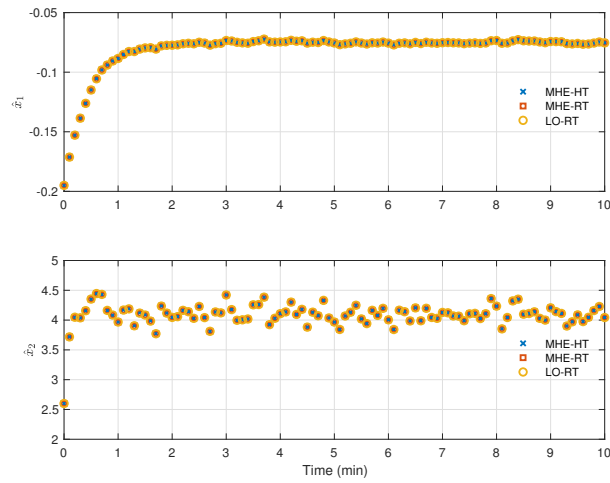


Fig. 4.23 The estimated state trajectory in closed-loop when Assumption 4.5 holds and the optimal operation is a steady-state.

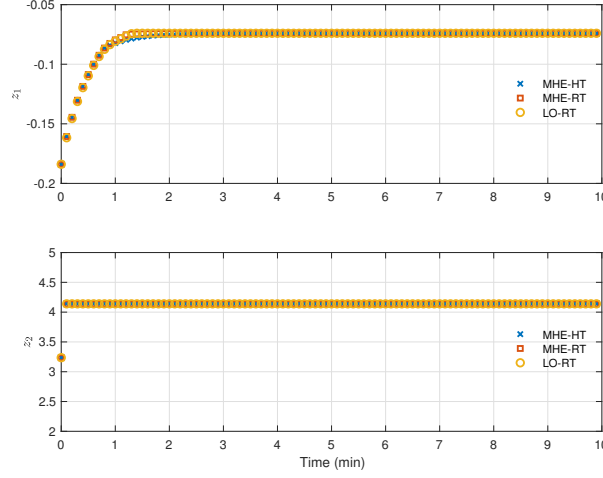


Fig. 4.24 The nominal state sequence in closed-loop when Assumption 4.5 holds and the optimal operation is a steady-state.

### Optimal period-2 operation

To illustrate the performance of the proposed controller when the periodic operation is optimal, we consider the CSTR system and its stage cost function same as that of the optimal steady-state operation. In this case, we introduce an auxiliary state  $p_{t+1} = v_t$  to capture the previous step input, hence the new nominal system state equation can be described as

$$\tilde{z}_{t+1} = \begin{bmatrix} A & O \\ O & O \end{bmatrix} \tilde{z}_t + \begin{bmatrix} B \\ I \end{bmatrix} v_t$$

where  $\tilde{z} = [z_1 \ z_2 \ p]^T$ . In addition to the constraints to define the state and input space, a coupled constraint is also considered so that the resulting tightened constraints for the augmented system is

$$\begin{aligned} \tilde{\mathcal{Z}} = \{(\tilde{z}, v) | & [-0.4811 \ -4.139]^T \leq \tilde{z} \leq [0.4811 \ 4.139]^T, \\ & -9.997 \leq p \leq 9.997, -9.997 \leq v \leq 9.997, \\ & |p - v| \geq 5\}. \end{aligned}$$

The last inequality in  $\tilde{\mathcal{Z}}$  is equivalent to enforcing the variation of input signal at two successive sampling time no less than 5. Since this constraint will not affect the stage cost function and the dissipation inequality, the modified cost function and the storage function

are unchanged. However, the resulting optimal regime of operation becomes a period-2 solution, that is,

$$\begin{aligned}(z_{s,1}, v_{s,1}) &= ([-0.0694 \ 4.1390]^T, -7.5705), \\ (z_{s,2}, v_{s,2}) &= ([-0.0696 \ 3.6315]^T, -2.5705).\end{aligned}$$

It has been verified that the optimal nominal sequence can always converge to this periodic operation by using the three algorithms. The simulated state trajectories are shown in the following Fig.4.25 - Fig.4.28. Notice, in Fig 4.28 (upper figure was zoomed in), the nominal sequences switch between the corresponding values of  $z_{s,1}$  and  $z_{s,2}$  in the optimal phase, because it is determined by the switching terminal equality constraints. The average computational time using *fmincon* to solve the MHE-HT optimization problem is 0.2964s approximately, and the average number of calls for objective functions and nonlinear constraints are around 498 times.

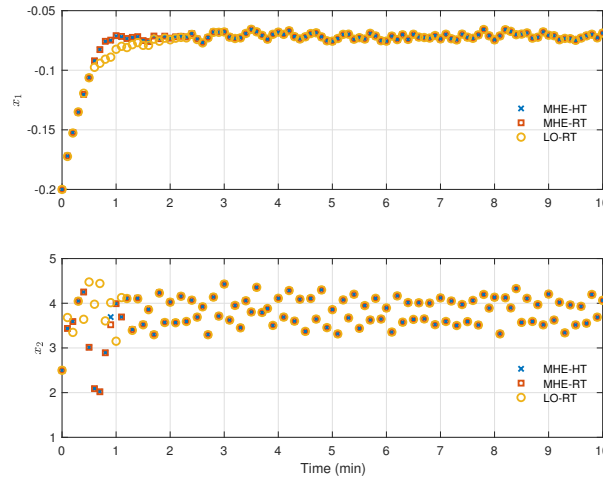


Fig. 4.25 The real state trajectory in closed-loop when Assumption 4.5 holds and the optimal operation is a period-2 solution.

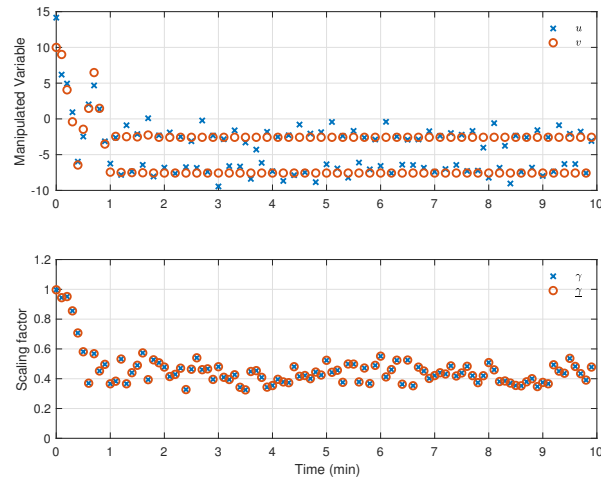


Fig. 4.26 Sequences optimal control inputs and scaling factors using MHE-HT EMPC algorithm, when Assumption 4.5 holds and the optimal operation is a period-2 solution.

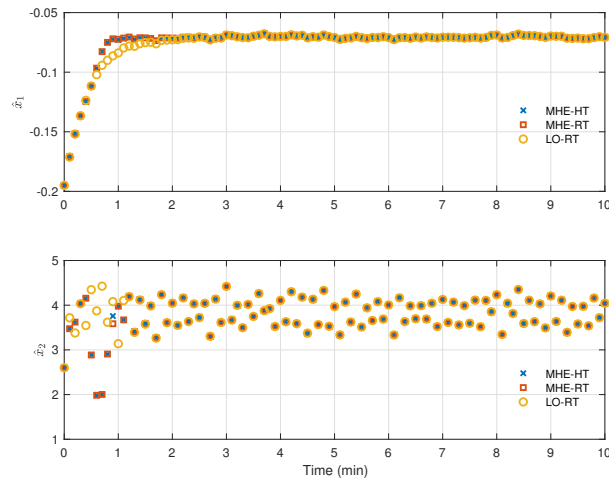


Fig. 4.27 The estimated state trajectory in closed-loop when Assumption 4.5 holds and the optimal operation is a period-2 solution.

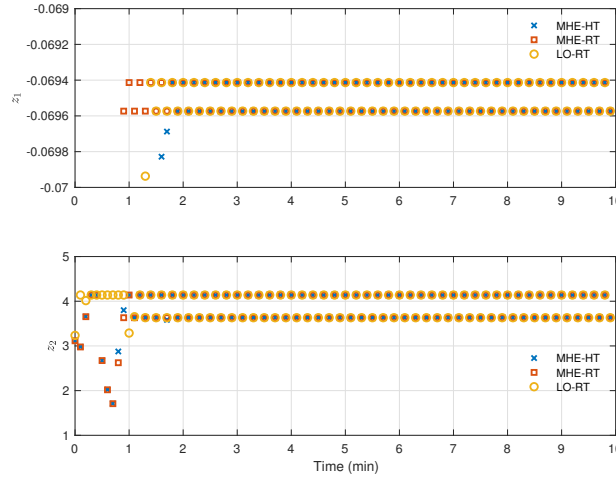


Fig. 4.28 The nominal state sequence in closed-loop when Assumption 4.5 holds and the optimal operation is a period-2 solution.

## 4.5 Summary

In conclusion, we presented a homothetic tube-based robust economic MPC algorithm integrated with moving horizon estimation for constrained linear discrete time systems. By considering the worst cost within the homothetic tube along the artificial nominal system and a weighting function on nominal initial state, the robustness against the disturbances and measurement noise is guaranteed in our EMPC design. Within this chapter, constraint tightening is used to prove recursive feasibility and a less conservative bound of the closed-loop asymptotic average performance is derived. Moreover, the stability of the optimal robust invariant set is inferred from the asymptotic stability of the nominal state sequence.

One of the major drawbacks of the algorithm is its computational burden. We highlight below possible future research directions to mitigate this aspect of the proposed solution. In particular, one may use ellipsoid or polyhedral sets with fixed number of vertices to approximate the robust invariant sets and the reachable sets. Another possibility is discarding past measurements and adoption of finite time-windows. All of these solutions, however, could lead to more conservative results as we are considering the worst cost within an enlarged tube. In this respect, there seems to be, as expected, a trade-off between computational burden and closed-loop system's performance.



## Chapter 5

# Homothetic tube-based robust offset-free Economic Model Predictive Control

This chapter is organized as follows. Section 5.1 introduces the basic notions and setup. Section 5.2 elaborates the homothetic tube-based robust economic algorithm with its formulation in Section 5.2.1 and recursive feasibility analysis in Section 5.2.2. The associated closed-loop asymptotic average performance by adopting this controller is discussed in Section 5.3. Finally, two illustrative practical examples are presented in Section 5.4 and Section 5.5 summarizes this chapter.

The results in this chapter are based on [71].

### 5.1 Problem setup

Throughout this chapter, we consider discrete-time linear systems of the form

$$\begin{aligned}x_{t+1} &= Ax_t + Bu_t + B_d d_t + B_\omega \omega_t \\d_{t+1} &= d_t + \theta_t \\y_t &= Cx_t + C_d d_t + \eta_t\end{aligned}\tag{5.1}$$

where  $x_t \in \mathbb{X} \subseteq \mathbb{R}^n$ ,  $u_t \in \mathbb{U} \subseteq \mathbb{R}^m$ ,  $y_t \in \mathbb{R}^p$ ,  $\forall t \in \mathbb{I}_{\geq 0}$  are, respectively, the state, input and output vectors, while  $\omega_t \in \mathbb{W} \subseteq \mathbb{R}^{n_\omega}$  and  $\eta_t \in \mathbb{E} \subseteq \mathbb{R}^y$  are process noise and measurement noise. The unknown but bounded variable  $d \in \mathbb{D} \subseteq \mathbb{R}^{n_d}$  is an additional integrating input, which models a constant or slowly varying *disturbance*, and its rate of variation  $\theta_t \in \Theta$  for any time  $t \in \mathbb{I}_{\geq 0}$ .

System's state and input are required to satisfy the following pointwise-in-time constraints

$$(x_t, u_t) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}, \forall t \in \mathbb{I}_{\geq 0} \quad (5.2)$$

where  $\mathbb{Z}$  is a compact set.

Our objective is to enhance profitability by minimizing the economic costs accumulated in the long term system operation

$$\sum_t \ell(x_t, u_t, d_t), \quad (5.3)$$

where  $\ell(x, u, d) : \mathbb{Z} \times \mathbb{R}^{n_d} \rightarrow \mathbb{R}$  is the economic stage cost, which may take arbitrary form coherently with the economic MPC setup (see [3]), and need not be positive definite with respect to any equilibrium state.

System (5.1) can also be written in state-space form with the augmented state  $x_t^a = [x_t^T, d_t^T]^T$ ,

$$\begin{aligned} \begin{bmatrix} x_{t+1} \\ d_{t+1} \end{bmatrix} &= \begin{bmatrix} A & B_d \\ O & I \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + \begin{bmatrix} B \\ O \end{bmatrix} u_t + \begin{bmatrix} B_\omega \\ O \end{bmatrix} \omega_t + \begin{bmatrix} O \\ I \end{bmatrix} \theta_t \\ y_t &= \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + \eta_t \end{aligned} \quad (5.4)$$

which is

$$\begin{aligned} x_{t+1}^a &= \bar{A}x_t^a + \bar{B}u_t + \bar{B}_\omega\omega_t + \Psi\theta_t \\ y_t &= \bar{C}x_t^a + \eta_t \end{aligned} \quad (5.5)$$

where

$$\bar{A} = \begin{bmatrix} A & B_d \\ O & I \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ O \end{bmatrix}, \quad \bar{B}_\omega = \begin{bmatrix} B_\omega \\ O \end{bmatrix}, \quad \Psi = \begin{bmatrix} O \\ I \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & C_d \end{bmatrix}. \quad (5.6)$$

It is useful, in the following, to transform system (5.4) to new coordinates:

$$z_t^a = \begin{bmatrix} z_t \\ d_t \end{bmatrix} = Mx_t^a = \begin{bmatrix} I & -(I - A)^{-1}B_d \\ O & I \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix}, \quad (5.7)$$

which yields the algebraically equivalent representation

$$\begin{aligned} z_{t+1}^a &= \tilde{A}z_t^a + \tilde{B}u_t + \tilde{B}_\omega\omega_t + \tilde{\Psi}\theta_t \\ y_t &= \tilde{C}z_t^a + \eta_t \end{aligned} \quad (5.8)$$

where

$$\begin{aligned} \tilde{A} &= M\bar{A}M^{-1} = \begin{bmatrix} A & O \\ O & I \end{bmatrix}, \quad \tilde{B} = M\bar{B} = \begin{bmatrix} B \\ O \end{bmatrix}, \quad \tilde{B}_\omega = M\bar{B}_\omega = \begin{bmatrix} B_\omega \\ O \end{bmatrix}, \\ \tilde{\Psi} &= M\Psi = \begin{bmatrix} -(I - A)^{-1}B_d \\ I \end{bmatrix}, \quad \tilde{C} = \bar{C}M^{-1} = \begin{bmatrix} C & C(I - A)^{-1}B_d + C_d \end{bmatrix}. \end{aligned} \quad (5.9)$$

Accordingly, the stage cost function  $\ell(x, u, d)$  is expressed as

$$\ell(x, u, d) = \ell(z + (I - A)^{-1}B_d d, u, d) =: \tilde{\ell}(z, u, d) \quad (5.10)$$

in terms of the new coordinates.

Throughout this chapter, we adopt the following assumption:

**Assumption 5.1.** *The pair  $(A, B)$  is stabilizable, the pair  $(\tilde{C}, \tilde{A})$  is detectable and the matrix  $I - A$  has full rank.*

**Remark 5.1.** Note that the transformation matrix  $M$  is non-singular, detectability of the pair  $(\tilde{C}, \tilde{A})$  implies that the pair  $(\bar{C}, \bar{A})$  is detectable. Necessary and sufficient conditions for detectability of  $(\bar{C}, \bar{A})$  have been addressed by [72, Theorem 3].

**Remark 5.2.** The rather strong condition that matrix  $I - A$  has full rank rules out some practical applications such as the double integrator dynamic which models a wide range of second order systems. To overcome this limitation, one can define a pre-stabilizing controller  $u = \bar{u} + Kx$  with the selected state feedback gain  $K$  such that the matrix  $I - (A + BK)$  has full rank. Then, accordingly, it is rational to transform the system state coordinate with equation  $z = x - (I - A - BK)^{-1}B_d d$ .

Given system (5.8), a pre-designed observer is introduced to track the new augmented system variable and the estimate is denoted by  $\hat{z}_t^a = [\hat{z}_t^T, \hat{d}_t^T]^T$ ,

$$\begin{aligned}\hat{z}_{t+1}^a &= \tilde{A}\hat{z}_t^a + \tilde{B}u_t + \tilde{L}(y_t - \hat{y}_t) \\ \hat{y}_t &= \tilde{C}\hat{z}_t^a,\end{aligned}\tag{5.11}$$

where the matrix  $\tilde{L} = [L^T, L_d^T]^T$ . Then, the estimation error is defined as  $e_t^a = z_t^a - \hat{z}_t^a$ , and governed by

$$e_{t+1}^a = (\tilde{A} - \tilde{L}\tilde{C})e_t^a + \tilde{B}\omega_t + \tilde{\Psi}\theta_t - \tilde{L}\eta_t.\tag{5.12}$$

This estimation error is the stack of state estimation error and disturbance estimation error components. Since  $\tilde{A} - \tilde{L}\tilde{C}$  is chosen as a Hurwitz matrix, a robust invariant set  $\mathcal{E}$  can be determined by the method in [67] such that  $e_0^a \in \mathcal{E}$  implies  $e_t^a \in \mathcal{E}, \forall t \in \mathbb{I}_{\geq 0}$  regardless of uncertainty realizations. For the ease of notation, we denote the projections of  $\mathcal{E}$  on system state space and integrating disturbance space by  $\mathcal{E}_z = F\mathcal{E} \subseteq \mathbb{R}^n$  and  $\mathcal{E}_d = G\mathcal{E} \subseteq \mathbb{R}^{n_d}$ , respectively, where  $F = [I, O]$  and  $G = [O, I]$ .

Next, in order to determine the optimal input, based on current and past information, we consider nominal system dynamics:

$$\bar{z}_{t+1} = A\bar{z}_t + Bv_t.\tag{5.13}$$

The mismatch between the observer state and the nominal state is  $\xi_t = \hat{z}_t - \bar{z}_t$ , and, by applying a state feedback control policy  $u_t = v_t + K(\hat{z}_t - \bar{z}_t)$  where  $K$  is a pre-defined matrix gain such that  $A + BK$  is Hurwitz, the observer mismatch fulfills the following recursion

$$\xi_{t+1} = (A + BK)\xi_t + F\tilde{L}\tilde{C}e_t^a + L\eta_t.\tag{5.14}$$

Thus, it is possible to construct a robust invariant set  $\Xi$  for the error system (5.14).

For an initial state  $\bar{z}_0$  and a nominal control sequence  $v_{0:t-1}$ , we can obtain a nominal state sequence  $\bar{z}_{0:t}$ . By the definitions of estimated state error  $Fe_t^a = z_t - \hat{z}_t$  and observer mismatch  $\xi_t = \hat{z}_t - \bar{z}_t$ , the actual transformed system's state differs from the nominal state is  $z_t - \bar{z}_t = Fe_t^a + \xi_t$ . Then, according to [65, Proposition 2], we can bound the difference  $z_t - \bar{z}_t$  by  $\Omega = F\mathcal{E} \oplus \Xi = \mathcal{E}_z \oplus \Xi$ , provided that  $z_0 - \hat{z}_0 \in \mathcal{E}_z$  and  $\hat{z}_0 - \bar{z}_0 \in \Xi$ .

Since the set  $\Omega$  may not provide a tight estimate of the real system state value, it may be possible to find a less conservative bound for the economic cost incurred. Indeed, one might take advantage of the past measured outputs in order to further reduce the region where

the real system state belongs. For instance, by solving the economic MPC optimization problem, it is possible to find the smallest scalar  $\gamma_t \in [0, 1]$  such that at any time  $t$  the state is guaranteed to fulfill

$$z_t \in \bar{z}_t \oplus \gamma_t \Omega. \quad (5.15)$$

In order to provide a robust feasibility region within the prediction horizon window, we may adopt some notion of “outer reachable sets” based on currently available information. We denote an a-priori estimate of the reachable set at time instant  $t$  of the integrating disturbance by  $\mathcal{D}_{t|t-1}$  updated as follows

$$\mathcal{D}_{t|t-1} = (\mathcal{D}_{t-1} \oplus \Theta) \cap \mathbb{D}, \quad (5.16)$$

where  $\mathcal{D}_{0|-1} = (\hat{d}_0 \oplus \mathcal{E}_d) \cap \mathbb{D}$ . In addition to the information from the previous time step, the current measured output  $y_t$  is also available to further tighten the disturbance set. To take outputs into account, we first recursively define the reachable sets for the augmented state  $z^a$  as

$$\begin{aligned} \mathcal{R}_t(y_{0:t}, \hat{z}_0^a, u_{0:t-1}) \\ = \{z_t^a \in \mathbb{R}^{n+n_d} \mid \exists z_{t-1}^a \in \mathcal{R}_{t-1}(y_{0:t-1}, \hat{z}_0^a, u_{0:t-2}), \exists \omega_{t-1} \in \mathbb{W}, \exists \theta_{t-1} \in \Theta, \\ z_t^a = \tilde{A}z_{t-1}^a + \tilde{B}u_{t-1} + \tilde{B}\omega_{t-1} + \tilde{\Psi}\theta_{t-1}, \\ M^{-1}z_t^a \in \mathbb{X} \times \mathbb{D}, y_t - \tilde{C}z_t^a \in \mathbb{H}, Gz_t^a \in \mathcal{D}_{t|t-1}\}. \end{aligned} \quad (5.17)$$

where  $\mathcal{R}_0(y_0, \hat{z}_0) = \{z_0^a \in \mathbb{R}^{n+n_d} \mid z_0^a \in \hat{z}_0^a \oplus \mathcal{E}, M^{-1}z_0^a \in \mathbb{X} \times \mathbb{D}, y_0 - \tilde{C}z_0^a \in \mathbb{H}, Gz_0^a \in \mathcal{D}_{0|-1}\}$ . Then, for all  $t \in \mathbb{I}_{\geq 0}$ , the reachable sets of state  $z_t$  and disturbance  $d_t$ , by making the best use of available information, are the projections of  $\mathcal{R}_t$  on the state space and the disturbance space,

$$\mathcal{Z}_t = F\mathcal{R}_t, \quad \mathcal{D}_t = G\mathcal{R}_t. \quad (5.18)$$

Given the a-posteriori estimation set  $\mathcal{D}_t \subseteq \mathbb{D}$ , as customary in standard robust MPC, the tightened state and input constraint for the nominal system is defined as

$$\bar{\mathcal{Z}}_{\mathcal{D}} = \{(z, v) \mid (z + \delta + (I - A)^{-1}B_d d, v + K\xi) \in \mathbb{Z}, \forall \xi \in \Xi, \forall \delta \in \xi \oplus \mathcal{E}_z, \forall d \in \mathcal{D}\}, \quad (5.19)$$

which, accordingly, guarantees that the true system state and input are feasible provided that the nominal ones are restricted by the tightened set, viz.,

$$(z_t, v_t) \in \bar{\mathcal{Z}}_{\mathcal{D}_t} \Rightarrow (x_t, u_t) \in \mathbb{Z}. \quad (5.20)$$

Although this set  $\mathcal{D}_t$  can provide bounds for the true integrating disturbance  $d_t$  and robustly tightened constraint (5.19), it does not take advantage of the fact that the variation of  $d_t$  is infrequent. To reduce the conservatism of using  $\mathcal{D}_t$ , we may consider the situation arising if the integrating disturbance is constant, i.e.,  $\Theta = \{0\}$ , and hence we consider a tighter modified definition together with (5.16)

$$\mathcal{D}_{t|t-1}^o = \mathcal{D}_{t-1}^o \cap \mathbb{D}, \quad (5.21)$$

where  $\mathcal{D}_{0|-1}^o = (\hat{d}_0 \oplus \mathcal{E}_d) \cap \mathbb{D}$ . Similar to (5.17), the available output sequence is also used to select those constant  $d$  values in  $\mathcal{D}_{t|t-1}^o$  which are compatible with measured outputs

$$\begin{aligned} \mathcal{R}_t^o(y_{0:t}, \hat{z}_0^a, u_{0:t-1}) &= \{z_t^a \in \mathbb{R}^{n+n_d} \mid \exists z_{t-1}^a \in \mathcal{R}_{t-1}^o(y_{0:t-1}, \hat{z}_0^a, u_{0:t-2}), \exists \omega_{t-1} \in \mathbb{W}, \\ &\quad z_t^a = \tilde{A}z_{t-1}^a + \tilde{B}u_{t-1} + \tilde{B}\omega_{t-1}, \\ &\quad M^{-1}z_t^a \in \mathbb{X} \times \mathbb{D}, y_t - \tilde{C}z_t^a \in \mathbb{H}, Gz_t^a \in \mathcal{D}_{t|t-1}^o\}. \end{aligned} \quad (5.22)$$

where  $\mathcal{R}_0^o(y_0, \hat{z}_0) = \{z_0^a \in \mathbb{R}^{n+n_d} \mid z_0^a \in \hat{z}_0^a \oplus \mathcal{E}, M^{-1}z_0^a \in \mathbb{X} \times \mathbb{D}, y_0 - \tilde{C}z_0^a \in \mathbb{H}, Gz_0^a \in \mathcal{D}_{0|-1}^o\}$ . As a result, the a-posteriori estimated disturbance set in this case is  $\mathcal{D}_t^o = G\mathcal{R}_t^o$ .

When the integrating disturbance is known to be constant, or equivalently  $\Theta = \{0\}$ ,  $d_t \in \mathcal{D}_t^o$  for all time  $t$  and one can expect that the increasing number of output measurements keeps shrinking the size of compatible disturbances  $\mathcal{D}_t^o$ , and hence even offset-free predictions may be achieved, i.e.,  $\lim_{t \rightarrow \infty} \mathcal{D}_t^o = \{\bar{d}\}$ , in the Hausdorff metric. However, if infrequent or slow disturbance variation occurs, there might exist a time instant  $\tau$  at which the set  $\mathcal{R}_\tau^o$  is empty, due to the incompatibility of current output measurement, and we would then reset  $\mathcal{D}_\tau^o$  according to  $\mathcal{D}_\tau^o = \mathcal{D}_\tau$  which certainly contains the true disturbance  $d_\tau$ . Therefore, by the updating rules of  $\mathcal{D}_t$  and  $\mathcal{D}_t^o$ , one can see that for all time  $t \in \mathbb{I}_{\geq 0}$  it holds

$$\mathcal{D}_t^o \subseteq \mathcal{D}_t. \quad (5.23)$$

**Remark 5.3.** It may happen that the set  $\mathcal{D}_t^o$  is non-empty even if  $d_t \notin \mathcal{D}_t^o$ . However, detecting whether  $d_t$  belongs to  $\mathcal{D}_t^o$  instantaneously is difficult, and we have to compromise using emptiness of  $\mathcal{D}_t^o$  as a proxy for  $d_t \notin \mathcal{D}_t^o$ .

In fact, due to the existence of uncertainties, it is not realistic to enforce the system to any optimal equilibrium. An achievable goal is to steer the system to a neighbourhood of a certain steady-state. Thanks to the restricted bound  $\gamma\Omega$ , the deviation of the true transformed state  $z$  from the nominal state  $\bar{z}$  is bounded. At this point, for any set  $\mathcal{S}$ , we define a modified

economic cost

$$\bar{\ell}_{\mathcal{S}}(z, v, \gamma) = \max_{\xi \in \Xi, \delta \in \gamma\Omega, d \in \mathcal{S}} \tilde{\ell}(z + \delta, v + K\xi, d), \quad (5.24)$$

which is monotonically non-decreasing with respect to  $\gamma$  and the set  $\mathcal{S}$  (according to set-inclusion). In particular, when  $\mathcal{S} = \mathcal{D}$ , this modified cost provides an upper bound of the real system cost regardless of the process and measurement noise. However, according to the inclusion in (5.23), the estimated cost by adopting  $\mathcal{D}$  is higher and possibly conservative with respect to that obtained by using  $\mathcal{D}^o$ . In addition, due to the desirable decreasing property of  $\mathcal{D}^o$  as long as no reset occurs the latter is preferred in the formulation of the proposed economic MPC optimization. Therefore, in the next section, we will only employ  $\mathcal{D}$  to guarantee robust feasibility while the sharper bound  $\mathcal{D}^o$  is used to approximate the upper bound of the economic stage cost in agreement with our modelling presumption that  $d_t$  is a piecewise constant or slowly-varying disturbance.

The rational having that in our considered scenario disturbances will mostly be constant, except for transient intervals of small variation where they might adjust to a different level. Based on the modified stage cost function (5.24) and the case  $\mathcal{S} = \mathcal{D} = \{d\}$ , the optimal equilibrium can be defined as

$$(z_s, v_s) = \underset{z = Az + Bv, (z, v) \in \bar{\mathbb{Z}}_{\mathcal{D}}}{\operatorname{argmin}} \bar{\ell}_d(z, v, \gamma). \quad (5.25)$$

Notice that the optimal equilibrium is a function of  $\gamma$  and  $\bar{\ell}_d(z_s, v_s, \gamma)$  provides the worst cost for states within  $z_s \oplus \gamma\Omega$  regardless of the effect from noises. In particular, when  $\gamma = 1$ , it gives the robust optimal cost  $\bar{\ell}_d(z_s, v_s, 1)$  which is relatively conservative compared with any other case with  $\gamma < 1$ .

## 5.2 Homothetic tube-based offset-free economic MPC algorithm

### 5.2.1 Problem formulation

In this section, a robust economic MPC controller is designed. Following the fashion of previously mentioned notation along the prediction:  $\bar{z}_{0:t+N|t}$  is the sequence of nominal states from sampling instants 0 to  $t + N$ , considered at time  $t$ ;  $v_{t:t+N|t}$  is the sequence of

future nominal control moves;  $u_{0:t-1}$  (or  $v_{0:t-1}$  equivalently) and  $y_{0:t}$  are the (known) past system inputs and measured outputs, respectively, at time  $t$ . Because the observer states are auxiliary variables processed within the controller, the sequence  $\hat{z}_{0:t}^a$  is known at time  $t$ . Moreover, we make the following assumption to restrict the error between the real system and the estimates:

**Assumption 5.2.** *The initial augmented observer state  $\hat{z}_0^a$  is provided, such that the true transformed system state  $z_0$  and disturbance  $d_0$  fulfill  $z_0^a - \hat{z}_0^a \in \mathcal{E}$ .*

Now, we propose a feedback control law by solving the following finite horizon economic MPC optimization problem with a prediction horizon  $N \in \mathbb{I}_{\geq 1}$ ,

$$\min_{\mathbf{s}_t} J_{N, \mathcal{D}_t^o}(\bar{z}_{t:t+N|t}, v_{t:t+N|t}, \gamma_{t:t+N-1|t}, \beta) \quad (5.26)$$

$$s.t. \quad \hat{z}_t - \bar{z}_{t|t} \in \Xi, \quad (5.26a)$$

$$\bar{z}_{j+1|t} = A\bar{z}_{j|t} + Bv_j, \quad \forall j \in \mathbb{I}_{[0, t-1]}, \quad (5.26b)$$

$$(\bar{z}_{j|t}, v_j) \in \bar{\mathbb{Z}}_{\mathcal{D}_j}, \quad \forall j \in \mathbb{I}_{[0, t-1]}, \quad (5.26c)$$

$$y_j - \tilde{C}[\bar{z}_{j|t}^T, \hat{d}_j^T]^T \in \tilde{C}\mathcal{E} \oplus C\Xi, \quad \forall j \in \mathbb{I}_{[0, t]}, \quad (5.26d)$$

$$\mathcal{Z}_t \subseteq \bar{z}_{t|t} \oplus \gamma_{t|t}\Omega, \quad (5.26e)$$

$$\begin{aligned} & A(\bar{z}_{t+k|t} \oplus \gamma_{t+k|t}\Omega) \oplus B(v_{t+k|t} \oplus K\Xi) \oplus B_\omega \mathbb{W} \oplus -(I - A)^{-1}B_d\Theta \\ & \subseteq \bar{z}_{t+k+1|t} \oplus \gamma_{t+k+1|t}\Omega, \quad \forall k \in \mathbb{I}_{[0, N-2]}, \end{aligned} \quad (5.26f)$$

$$\bar{z}_{t+k+1|t} = A\bar{z}_{t+k|t} + Bv_{t+k|t}, \quad (5.26g)$$

$$(\bar{z}_{t+k|t}, v_{t+k|t}) \in \bar{\mathbb{Z}}_{\mathcal{D}_{t+k|t}}, \quad \left. \vphantom{\begin{aligned} \bar{z}_{t+k+1|t} = A\bar{z}_{t+k|t} + Bv_{t+k|t}, \\ (\bar{z}_{t+k|t}, v_{t+k|t}) \in \bar{\mathbb{Z}}_{\mathcal{D}_{t+k|t}}, \end{aligned}} \right\} \forall k \in \mathbb{I}_{[0, N-1]}, \quad (5.26h)$$

$$0 \leq \gamma_{t+k|t} \leq 1, \quad (5.26i)$$

$$\bar{z}_{t+N|t} = A\bar{z}_{t+N|t} + Bv_{t+N|t}, \quad (5.26j)$$

$$\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}, v_{t+N|t}, 1) \leq \kappa(t), \quad (5.26k)$$

where  $\mathbf{s}_t = [\bar{z}_{0:t+N|t}, v_{t:t+N|t}, \gamma_{t:t+N-1|t}]$  is the vector of decision variables, the function

$$\begin{aligned} & J_{N, \mathcal{D}_t^o}(\bar{z}_{t:t+N|t}, v_{t:t+N|t}, \gamma_{t:t+N|t}, \beta) \\ & = \sum_{k=0}^{N-1} \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+k|t}, v_{t+k|t}, \gamma_{t+k|t}) + \beta(t) \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}, v_{t+N|t}, 1), \end{aligned} \quad (5.27)$$

is the economic objective with terminal constraint  $\kappa$  specified as

$$\kappa(t) = \begin{cases} \bar{\ell}_{\mathcal{D}_t}(\bar{z}_{t+N-1|t-1}^*, v_{t+N-1|t-1}^*, 1), & \text{if } \mathcal{R}_t^o \text{ is empty,} \\ \bar{\ell}_{\mathcal{D}_{t-1}^o}(\bar{z}_{t+N-1|t-1}^*, v_{t+N-1|t-1}^*, 1), & \text{otherwise.} \end{cases} \quad (5.28)$$

The motivation of adopting a time-varying terminal weight  $\beta$  and several self-tuning update rules are detailed in [8] and it satisfied the following assumption:

**Assumption 5.3.** [8, Assumption 1] *The function  $\beta(t)$  satisfies  $\beta(t+1) - \beta(t) \leq c$  and  $\beta(t) \geq \underline{\beta}$  for all  $t \in \mathbb{I}_{\geq 0}$  and some constant  $c, \underline{\beta} \in \mathbb{R}$ , and  $\limsup_{t \rightarrow +\infty} (\beta(t+1) - \beta(t)) \leq 0$ .*

**Remark 5.4.** In the optimization problem (5.26), we use the iteration  $\mathcal{D}_{t+k+1|t} = (\mathcal{D}_{t+k|t} \oplus \Theta) \cap \mathbb{D}$ , where  $\mathcal{D}_{t|t} = \mathcal{D}_t$ , to generate a prediction, at time  $t$ , of the sequence of sets  $\mathcal{D}_{t+k}$ , reachable by disturbances  $\forall k \in \mathbb{I}_{\geq 0}$ . Since new output measurements are available at time  $t+k$ , the predicted and the a-posteriori disturbance sets satisfy  $\mathcal{D}_{t+k} \subseteq \mathcal{D}_{t+k|t}$  and this implies that correspondingly tightened constraints fulfill  $\bar{\mathbb{Z}}_{\mathcal{D}_{t+k|t}} \subseteq \bar{\mathbb{Z}}_{\mathcal{D}_{t+k}}$ . In particular, for  $k=1$ , that is  $\mathcal{D}_{t+1} \subseteq \mathcal{D}_{t+1|t}$ , we can obtain the relation between the two sequentially generated tubes,  $\mathcal{D}_{t+k|t+1} \subseteq \mathcal{D}_{t+k|t}$ , and hence  $\bar{\mathbb{Z}}_{\mathcal{D}_{t+k|t}} \subseteq \bar{\mathbb{Z}}_{\mathcal{D}_{t+k|t+1}}$ , which is crucial to guarantee recursive feasibility. While, for the predicted stage cost, it is adopted that  $\mathcal{D}_{t+k|t}^o = \mathcal{D}_t^o, \forall k \in \mathbb{I}_{[0,N]}$ , because no future output measurements are available to further tighten  $\mathcal{D}_t^o$ .

**Remark 5.5.** The constraint (5.26d) is to guarantee that the nominal dynamics are compatible with output measurements. Because the set  $\mathcal{E}$  is robustly invariant with respect to the augmented estimation error dynamics, the difference between output measurement  $y_j$  and estimated output  $\hat{y}_j$  satisfies  $y_j - \hat{y}_j \in \tilde{C}\mathcal{E}$ . Meanwhile, it holds  $\hat{y}_j - \tilde{C}[\bar{z}_j^T, \hat{d}_j^T]^T = C(\hat{z}_j^T - \bar{z}_j^T) \in C\Xi$ . Therefore, if we consider  $y_j - \tilde{C}[\bar{z}_j^T, \hat{d}_j^T]^T$ , the constraint (5.26d) is obtained.

**Remark 5.6.** The constraint (5.26e) employs a homothetic tube with scalar factor  $\gamma_{t|t}$  to circumscribe the set of potential states  $\mathcal{Z}_t$ . This ensures that  $\gamma_{t|t} \in [0, 1]$  is selected so that the real system state fulfills  $z_t \in \bar{z}_{t|t} \oplus \gamma_{t|t}\Omega$ . Moreover, it is less conservative compared to  $z_t \in \bar{z}_{t|t} \oplus \Omega$ , and may result in better economic performance due to the monotonicity of function  $\bar{\ell}_{\mathcal{S}}(z, v, \gamma)$  with respect to  $\gamma$ .

**Remark 5.7.** Providing that the integrating disturbance is constant, no reset will happen and the time-varying terminal constraint enforces the sequence  $\kappa(t)$  to be non-increasing.

Moreover, this is bounded from below because of continuity of  $\ell$  and compactness of  $\mathbb{Z}$ . Hence, it converges to a limit value denoted by  $\kappa_\infty = \lim_{t \rightarrow +\infty} \kappa(t)$ .

When the system enters the optimization routine at time  $t$ , the following suboptimal and feasible solution which is generated from the previous optimal solution can be adopted as an initial guess

$$\begin{aligned}\bar{z}_{0:t+N|t}^{sub} &= (\bar{z}_{0|t-1}^*, \dots, \bar{z}_{t|t-1}^*, \dots, \bar{z}_{t+N-1|t-1}^*, \bar{z}_{t+N-1|t-1}^*), \\ v_{t:t+N|t}^{sub} &= (v_{t|t-1}^*, \dots, v_{t+N-1|t-1}^*, v_{t+N-1|t-1}^*), \\ \gamma_{t:t+N-1|t}^{sub} &= (\gamma_{t|t-1}^*, \dots, \gamma_{t+N-2|t-1}^*, 1).\end{aligned}\tag{5.29}$$

Finally, the solver converges to the optimal solution  $s_t^*$ , which is

$$\begin{aligned}\bar{z}_{0:t+N|t}^* &= (\bar{z}_{0|t}^*, \dots, \bar{z}_{t|t}^*, \dots, \bar{z}_{t+N-1|t}^*, \bar{z}_{t+N|t}^*), \\ v_{t:t+N|t}^* &= (v_{t|t}^*, \dots, v_{t+N-1|t}^*, v_{t+N|t}^*), \\ \gamma_{t:t+N-1|t}^* &= (\gamma_{t|t}^*, \dots, \gamma_{t+N-2|t}^*, \gamma_{t+N-1|t}^*).\end{aligned}\tag{5.30}$$

and the optimal value function is denoted by

$$V_{N, \mathcal{D}_t^o}(u_{0:t-1}, \hat{z}_{0:t}^a, y_{0:t}, \beta) = J_{N, \mathcal{D}_t^o}(\bar{z}_{t:t+N|t}^*, v_{t:t+N|t}^*, \gamma_{t:t+N-1|t}^*, \beta).\tag{5.31}$$

As usual in feedback algorithm in economic MPC, the optimal control input implemented to system (5.1) is

$$u_t = v_{t|t}^* + K(\hat{z}_t - \bar{z}_{t|t}^*),\tag{5.32}$$

and the resulting closed-loop dynamic is

$$x_{t+1} = Ax_t + Bu_t + \omega_t.\tag{5.33}$$

In addition, the optimal values  $v_{t|t}^*$  are recorded as  $v_t$  and will be used in future optimization problems. If the optimal solution is non-unique, we may take any sequence that achieves the optimum.

### 5.2.2 Recursive feasibility

Notice that, provided the optimization problem is feasible, the optimal nominal state-input pair  $(\bar{z}_{t|t}^*, v_{t|t}^*)$  is restricted in the set  $\bar{\mathbb{Z}}_t$  defined in (5.19). The rationale is that true state and input solutions, in the usual spirit of tube MPC, will then fulfill

$$(x_t, v_{t|t}^* + K(\hat{z}_t - \bar{z}_{t|t}^*)) \in \mathbb{Z}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (5.34)$$

whatever the disturbance and noise signals are until time  $t$ . Therefore, the real system's solutions fulfill pointwise-in-time constraints.

**Proposition 5.1.** *Let Assumption 5.1 and 5.2 hold, then the economic MPC optimization problem (5.26) is recursively feasible.*

*Proof.* The main idea to prove recursive feasibility is to explicitly construct feasible solutions of problems (5.26), at the current sampling time, given the previous feasible and optimal solutions. In particular, we will show the initial guess (5.29) is a feasible solution at time  $t$ .

Notice that the sequences in (5.29) are generated from the optimization problem at time  $t - 1$ , they satisfy the nominal state dynamic equality constraints (5.26b) and (5.26g), the pointwise-in-time constraints (5.26c), the output constraints (5.26d) until  $t-1$ , the scalar factor constraints (5.26f) and (5.26i), and the terminal equilibrium constraint (5.26j), automatically.

In terms of the constraint (5.26e), since the homothetic sets circumscribe the reachable states, it holds

$$z_t \in \mathcal{Z}_t \subseteq \bar{z}_{t|t-1}^* \oplus \gamma_{t|t-1}^* \Omega.$$

For the pointwise-in-time constraints (5.26h) within the prediction horizon  $k \in \mathbb{I}_{[1, N]}$ , it is indicated by

$$(\bar{z}_{t-1+k|t-1}^*, v_{t-1+k|t-1}^*) \in \bar{\mathbb{Z}}_{\mathcal{D}_{t-1+k|t-1}} \subseteq \bar{\mathbb{Z}}_{\mathcal{D}_{t-1+k|t}}.$$

Then, because the observer mismatch (5.14) is invariant with respect to the set  $\Xi$  and it is able to be tightened by the scalar factor, the initial constraint (5.26a),

$$\hat{z}_t \in \bar{z}_{t|t-1}^* \oplus \Xi,$$

$$\hat{y}_t - \tilde{C}[\bar{z}_{t|t-1}^{*T}, \hat{d}_t^T]^T \in C\Xi.$$

Together with the result that the estimation error at time  $t$  fulfills  $e_t^a \in \mathcal{E}$  from Assumption 5.2, we can obtain  $y_t - \hat{y}_t \in \tilde{C}\mathcal{E}$ , and hence

$$y_t - \tilde{C}[\bar{z}_{t|t-1}^{*T}, \hat{d}_t^T]^T \in \tilde{C}\mathcal{E} \oplus C\Xi,$$

which ensures the output constraint (5.26d) at time instant  $t$  for the sub-optimal solution.

Moreover, when there is no reset on  $\mathcal{D}_t^o$ , it fulfills  $\mathcal{D}_t^o \subseteq \mathcal{D}_{t-1}^o$ , hence

$$\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N-1|t-1}^*, v_{t+N-1|t-1}^*, 1) \leq \bar{\ell}_{\mathcal{D}_{t-1}^o}(\bar{z}_{t+N-1|t-1}^*, v_{t+N-1|t-1}^*, 1) = \kappa(t).$$

Once reset happens, we have  $\mathcal{D}_t^o = \mathcal{D}_t$  and then

$$\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N-1|t-1}^*, v_{t+N-1|t-1}^*, 1) = \bar{\ell}_{\mathcal{D}_t}(\bar{z}_{t+N-1|t-1}^*, v_{t+N-1|t-1}^*, 1) = \kappa(t).$$

Thus, the terminal inequality constraint (5.26k) is always satisfied. Therefore, the optimization problem (5.26) at current time  $t$  has at least one solution, and recursive feasibility of this optimization problem is ensured.  $\square$

### 5.3 Asymptotic average performance

For tube-based robust economic MPC problems analyzed in Chapter 3, 4 and reference [14], the nominal state sequence is stabilized to the robust optimal steady-state, and the true system state is invariant with respect to the generated tube. Thus, the worst cost evaluated within this tube guarantees an upper bound for the system long-run average operation. Similarly, having formulated the economic MPC optimization problem with self-tuning terminal cost, in this section, we study the resulting closed-loop average performance in the case of constant integrating disturbance. This result is stated in the following theorem.

**Theorem 5.1.** *Suppose Assumption 5.1, 5.2 and 5.3 are satisfied and the integrating disturbance is constant. The asymptotic average performance of the closed-loop system using optimal input (5.32), regardless of the noises, fulfills,*

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t, d_t)}{T} \leq \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \kappa_\infty. \quad (5.35)$$

*Proof.* Given the suboptimal control input sequence and its resulting state trajectory at time  $t + 1$  in the same structure as (5.29), for all noises  $\omega_t \in \mathbb{W}$ ,  $\theta_t \in \Theta$  and  $\eta_t \in \mathbb{E}$ , the optimal cost-to-go functions fulfill

$$\begin{aligned} V_{N, \mathcal{D}_{t+1}^o}(u_{0:t}, \hat{z}_{0:t+1}^a, y_{0:t+1}, \beta(t+1)) \\ \leq J_{N, \mathcal{D}_{t+1}^o}(\bar{z}_{t+1:t+N+1|t+1}^{sub}, v_{t+1:t+N|t+1}^{sub}, \gamma_{t+1:t+N|t+1}^{sub}, \beta(t+1)) \\ \leq J_{N, \mathcal{D}_t^o}(\bar{z}_{t+1:t+N+1|t+1}^{sub}, v_{t+1:t+N|t+1}^{sub}, \gamma_{t+1:t+N|t+1}^{sub}, \beta(t+1)), \end{aligned}$$

with the last inequality satisfied by  $\mathcal{D}_{t+1}^o \subseteq \mathcal{D}_t^o$ , which is

$$\begin{aligned} V_{N, \mathcal{D}_{t+1}^o}(u_{0:t}, \hat{z}_{0:t+1}^a, y_{0:t+1}, \beta(t+1)) - V_{N, \mathcal{D}_t^o}(u_{0:t-1}, \hat{z}_{0:t}^a, y_{0:t}, \beta(t)) \\ \leq J_{N, \mathcal{D}_t^o}(\bar{z}_{t+1:t+N+1|t}^*, v_{t+1:t+N|t}^*, \gamma_{t+1:t+N|t}^*, \beta(t+1)) \\ - J_{N, \mathcal{D}_t^o}(\bar{z}_{t:t+N|t}^*, v_{t:t+N-1|t}^*, \gamma_{t:t+N-1|t}^*, \beta(t)) \\ = -\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}^*, v_{t+N|t}^*, 1) \\ - \beta(t)\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}^*, v_{t+N|t}^*, 1) + \beta(t+1)\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}^*, v_{t+N|t}^*, 1) \\ = -\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + (1 - \beta(t) + \beta(t+1))\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}^*, v_{t+N|t}^*, 1). \end{aligned} \quad (5.36)$$

As mentioned in Remark (5.7), the sequence  $\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}^*, v_{t+N|t}^*, 1)$  is non-increasing in  $t$  and converge to  $\kappa_\infty$  as  $t \rightarrow +\infty$ . This implies that  $\varepsilon(t) = \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t+N|t}^*, v_{t+N|t}^*, 1) - \kappa_\infty$  is non-increasing and converges to zero for  $t \rightarrow +\infty$ .

Integrating the inequality (5.36), for any time  $T - 1 \in \mathbb{I}_{\geq 0}$ , we have,

$$\begin{aligned} V_{N, \mathcal{D}_1^o}(u_0, \hat{z}_{0:1}^a, y_{0:1}, \beta(1)) - V_{N, \mathcal{D}_0^o}(\hat{z}_0^a, y_0, \beta(0)) \\ + V_{N, \mathcal{D}_2^o}(u_{0:1}, \hat{z}_{0:2}^a, y_{0:2}, \beta(2)) - V_{N, \mathcal{D}_1^o}(u_0, \hat{z}_{0:1}^a, y_{0:1}, \beta(1)) \\ + \cdots + V_{N, \mathcal{D}_T^o}(u_{0:T-1}, \hat{z}_{0:T}^a, y_{0:T}, \beta(T)) - V_{N, \mathcal{D}_{T-1}^o}(u_{0:T-2}, \hat{z}_{0:T-1}^a, y_{0:T-1}, \beta(T-1)) \\ = \sum_{t=0}^{T-1} [V_{N, \mathcal{D}_{t+1}^o}(u_{0:t}, \hat{z}_{0:t+1}^a, y_{0:t+1}, \beta(t+1)) - V_{N, \mathcal{D}_t^o}(u_{0:t-1}, \hat{z}_{0:t}^a, y_{0:t}, \beta(t))] \\ \leq \sum_{t=0}^{T-1} [-\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + (1 + \beta(t+1) - \beta(t))(\kappa_\infty + \varepsilon(t))] \end{aligned} \quad (5.37)$$

Then, by taking arithmetic average, applying  $\liminf$  on both sides of (5.37) exploiting boundedness of solutions, we obtain for the right hand side

$$\begin{aligned}
& \liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} [-\bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + (1 + \beta(t+1) - \beta(t))(\kappa_\infty + \varepsilon(t))]}{T} \\
& \leq \kappa_\infty - \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} + \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} (\beta(t+1) - \beta(t) + 1)\varepsilon(t)}{T} \\
& \quad + \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} (\beta(t+1) - \beta(t))\kappa_\infty}{T} \\
& = \kappa_\infty - \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} + \limsup_{T \rightarrow +\infty} \frac{(\beta(T) - \beta(0))\kappa_\infty}{T} \\
& = \kappa_\infty - \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T}
\end{aligned}$$

where the last two equalities hold true since signals converging to 0 have zero average and because of Assumption 5.3 and the fact that  $\bar{\ell}_{\mathcal{D}_t^o}$  can be shifted to be positive without loss of generality.

On the other hand, as the stage cost function  $\bar{\ell}_{\mathcal{D}_t^o} > -\infty$  and  $\beta(t) \geq 0$  for all  $t \in \mathbb{I}_{\geq 0}$ , the optimal value function  $V_{N, \mathcal{D}_t^o}$  is bounded from below, then the left hand side becomes

$$\begin{aligned}
& \liminf_{T \rightarrow +\infty} \frac{1}{T} [V_{N, \mathcal{D}_1^o}(u_0, \hat{z}_{0:1}^a, y_{0:1}, \beta(1)) - V_{N, \mathcal{D}_0^o}(\hat{z}_0^a, y_0, \beta(0)) \\
& \quad + V_{N, \mathcal{D}_2^o}(u_{0:1}, \hat{z}_{0:2}^a, y_{0:2}, \beta(2)) - V_{N, \mathcal{D}_1^o}(u_0, \hat{z}_{0:1}^a, y_{0:1}, \beta(1)) \\
& \quad + \cdots + V_{N, \mathcal{D}_T^o}(u_{0:T-1}, \hat{z}_{0:T}^a, y_{0:T}, \beta(T)) - V_{N, \mathcal{D}_{T-1}^o}(u_{0:T-2}, \hat{z}_{0:T-1}^a, y_{0:T-1}, \beta(T-1))] \\
& \geq \liminf_{T \rightarrow +\infty} \frac{V_{N, \mathcal{D}_T^o}(u_{0:T-1}, \hat{z}_{0:T}^a, y_{0:T}, \beta(T))}{T} + \liminf_{T \rightarrow +\infty} \frac{-V_{N, \mathcal{D}_0^o}(\hat{z}_0^a, y_0, \beta(0))}{T} \\
& \geq 0.
\end{aligned}$$

Finally, combining both sides, we have

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^T \bar{\ell}_{\mathcal{D}_t}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \kappa_\infty$$

To this end, based on the definition of modified cost function (5.24), for any  $x_t \in \bar{z}_{t|t}^* \oplus \gamma_{t|t}^* \Omega \oplus (I - A)^{-1} B_d \mathcal{D}_t^o$ , it holds  $\ell(x_t, u_t, d_t) \leq \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*), \forall t \in \mathbb{I}_{\geq 0}$ . Therefore, the following inequality is fulfilled

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^T \ell(x_t, u_t, d_t)}{T} \leq \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^T \bar{\ell}_{\mathcal{D}_t^o}(\bar{z}_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \kappa_\infty.$$

which proves the claim.  $\square$

## 5.4 Numerical examples

We consider a two-tanks system addressed in [46, 73] and the structure of this system is presented in Fig.5.1. Let variables  $\mathbf{x} = [x_1 \ x_2]^T \in \mathbb{R}^2$ ,  $u \in \mathbb{R}$  and  $y \in \mathbb{R}$  denote the heights of two tanks, the total water inflow through the pump, and the total water outflow, respectively. There is an additional constant flow into tank 2 described as an integrating disturbance  $d$ .

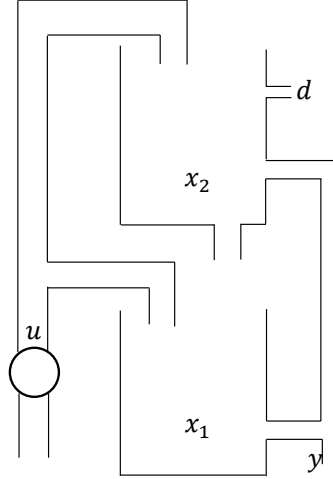


Fig. 5.1 Two tanks system.

The mathematical linear model can be written as

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ d^+ \end{bmatrix} = \begin{bmatrix} -0.0159 & 0.0419 & 0 \\ 0 & -0.0419 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ d \end{bmatrix} + \begin{bmatrix} 0.0521 \\ 0.0479 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.0718 \\ 0 \\ 0 \end{bmatrix} \omega, \quad (5.38)$$

$$y = \begin{bmatrix} 0.5 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \eta.$$

This system is subject to constraints  $(\mathbf{x}, u) \in \mathbb{X} \times \mathbb{U}$  where

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid [0 \ 0]^T \leq \mathbf{x} \leq [2 \ 1.5]^T\}$$

and

$$\mathbb{U} = \{u \in \mathbb{R} \mid 0 \leq u \leq 5\}.$$

The sets of system and measurement noise are

$$\mathbb{W} = \{\omega \in \mathbb{R} \mid -0.1 \leq \omega \leq 0.1\}$$

and

$$\mathbb{H} = \{\eta \in \mathbb{R} \mid -0.05 \leq \eta \leq 0.05\},$$

respectively. Moreover, it is known that the integrating disturbance satisfies

$$d \in \mathbb{D} = \{d \in \mathbb{R} \mid 0 \leq d \leq 1\}.$$

By selecting state feedback gain  $K = [0.5 \ 0.1]$  and observer gains  $L = [0.1 \ 0.2]^T$ ,  $L_d = 5$ , the computed sets to bound the estimation errors and the observer mismatch are shown in Fig.5.2 and Fig.5.3.

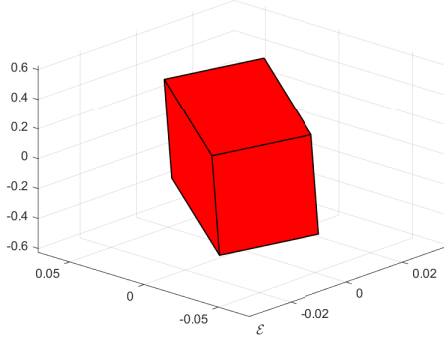


Fig. 5.2 Robust positively invariant set  $\mathcal{E}$  for system (5.38).

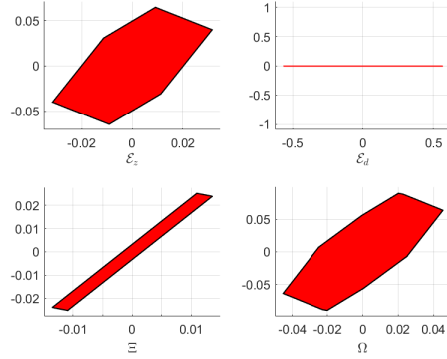


Fig. 5.3 Associated sets in system (5.38):  $\mathcal{E}_z$ ,  $\mathcal{E}_d$ ,  $\Xi$  and  $\Omega$ .

### 5.4.1 Linear cost function

The aim of this controlled system is to maximize the total height of the water in these two tanks, so the stage cost to be minimized is

$$\ell(\mathbf{x}, u, d) = -x_1 - x_2, \quad (5.39)$$

which leads to the modified stage cost

$$\bar{\ell}_{\mathcal{S}}(\bar{\mathbf{z}}, v, \gamma) = \max_{d \in \mathcal{S}} -\bar{z}_1 - \bar{z}_2 - 0.9994d + 0.09217\gamma. \quad (5.40)$$

- *Zero rate of variation*

Next, when the integrating disturbance is a constant that is  $\theta = 0$ , the size of the estimated disturbance set  $\mathcal{D}^o$  is expected to be monotonically non-increasing. Under the sequences of noise in Fig.5.6, the following Fig.5.4 shows the closed-loop results at the initial state  $\mathbf{x} = [1 \ 1]^T$ , initial transformed state estimate  $\hat{\mathbf{z}} = [0.9802 \ 0.5201]^T$ , constant disturbance  $d = 0.5$ , and prediction horizon  $N = 20$ .

As a result, the nominal state sequence converges to the steady state  $\bar{\mathbf{z}}_s = [0.2654 \ 0.2294]^T$  which corresponds to  $\mathbf{x}_s = [0.2852 \ 0.7093]^T$  in the original coordinate. We can verify that the obtained value of  $\bar{\mathbf{z}}_s$  indeed is the optimal equilibrium for any  $\gamma \in [0 \ 1]$ . As the function  $\bar{\ell}_{\mathcal{S}}(\bar{\mathbf{z}}, v, \gamma)$  is linear with respect to  $\bar{\mathbf{z}}$ , the optimal solution in (5.25) is independent of  $\mathcal{S}$  and

$\gamma$ . Therefore, the proposed economic MPC controller is able to achieve offset free behaviour for the nominal state without a priori knowledge of constant disturbances.

Regarding disturbance estimation, the noise realizations and a priori bounds determine the size of the disturbance sets,  $\mathcal{D}_t$  and  $\mathcal{D}_t^o$ . Our simulations indicate that there exist some scenarios which can ensure the set  $\mathcal{D}^o$  to estimate the actual disturbance accurately. This happens, in particular, when the process noise  $\omega$  and the measurement noise  $\eta$  are random signals and their actual supports are adopted as a priori estimates  $\mathbb{W}$  and  $\mathbb{H}$ . Specifically, convergence might be faster when the noise take values on the boundary of the respective supporting sets. Notice in Fig.5.5 how  $\mathcal{D}_t^o$  shrinks to the true value  $\{0.5\}$  (purple line). For comparison  $\mathcal{D}_t$  has a non-monotone behaviour and never is able to tightly characterize the constant disturbance acting on the system (yellow line).

Moreover, the sequence of optimal scaling factors and the average performance are shown in Fig.5.6. The use of the homothetic tube provide a less conservative upper bound for the actual asymptotic average economic cost. The average computational time using *fmincon* to solve the optimization problem is 1.2628s approximately, and the average number of calls for objective functions and nonlinear constraints are around 889 times.

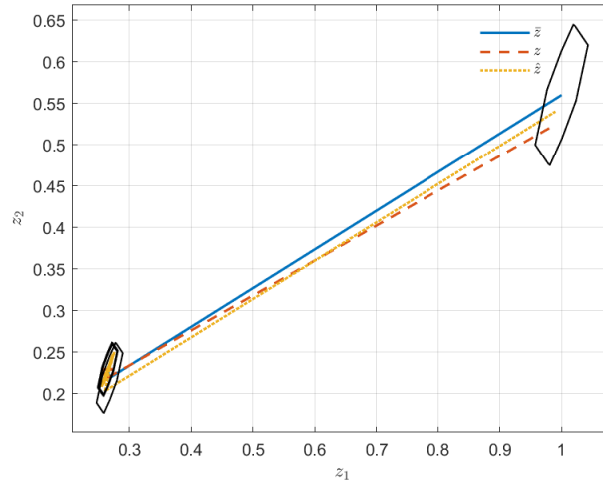


Fig. 5.4 Closed-loop transformed state trajectories with scaled bounded set  $\gamma\Omega$  for system (5.38), when the disturbance has zero rate of variation.

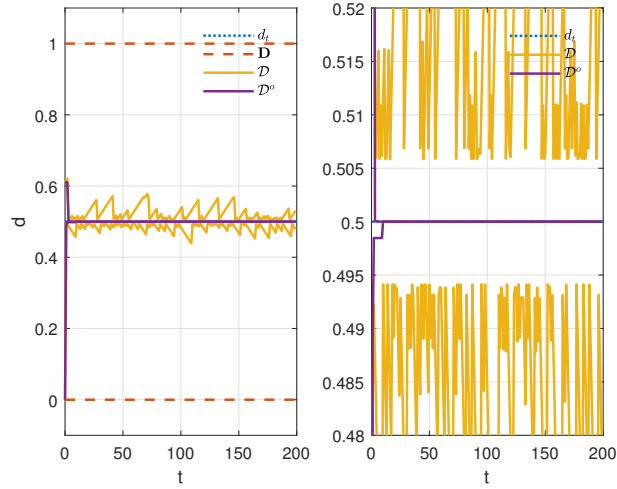


Fig. 5.5 True disturbance  $d$ , disturbance bound  $\mathbb{D}$ , robust estimated set  $\mathcal{D}$ , estimated set  $\mathcal{D}^o$  and its zoomed-in for system (5.38), when the disturbance has zero rate of variation and noises are randomly generated from the boundaries  $\mathbb{W}$  and  $\mathbb{H}$ .

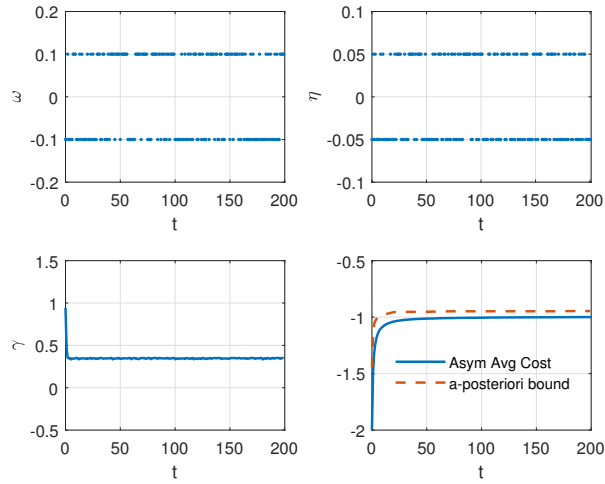


Fig. 5.6 Distribution of the noises, the optimal scaling factor, and asymptotic average performance for system (5.38), when the disturbance has zero rate of variation.

For a different randomly generated scenario of noises from sets  $\mathbb{W}$  and  $\mathbb{H}$ , the resulting disturbance sets are shown in Fig.5.7. We can see that  $\mathcal{D}_t^o$  is decreasing in time, however, it does not appear to converge asymptotically to a singleton or, alternatively, convergence might be very slow.

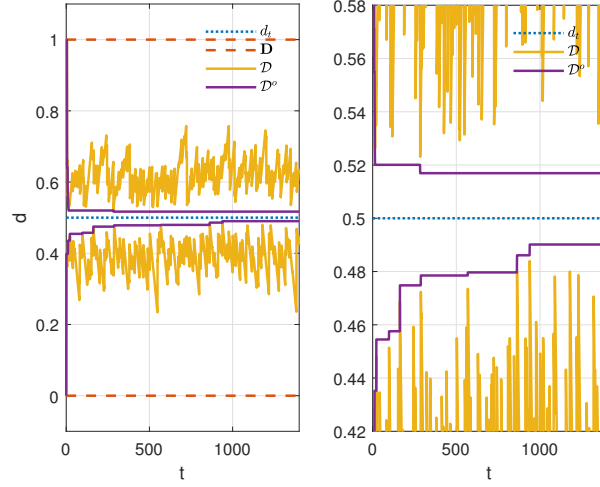


Fig. 5.7 Actual disturbance  $d$ , disturbance bound  $\mathbb{D}$ , robust estimated set  $\mathcal{D}$ , estimated set  $\mathcal{D}^o$  under uniformly generated noises, when the disturbance has zero rate of variation and noises are randomly generated from sets  $\mathbb{W}$  and  $\mathbb{H}$ .

- *Nonzero rate of variation*

Suppose the rate of variation is nonzero, i.e.,

$$\theta \in \Theta = \{\theta \in \mathbb{R} \mid -0.005 \leq \theta \leq 0.005\},$$

the simulation results of two noise scenarios are shown in Fig.5.8 and Fig.5.9. Particularly, in Fig.5.8, at  $t = 50, 100, 150$  where disturbance variation occurs and  $\mathcal{R}_t^o$  becomes empty,  $\mathcal{D}_t^o$  is reset to  $\mathcal{D}_t$  accordingly. To this end, we can conclude that the set  $\mathcal{D}^o$  can estimate the actual value of  $d$  exactly provided that the past output measurements are sufficiently rich. In addition, in both cases, it is observed that the sequences of the nominal state converge to  $\bar{z}_s = [0.2654 \ 0.2294]^T$ . Based on this  $\bar{z}_s$ , it has been verified that the nominal system of (5.38) can always be driven to its economically optimal equilibrium for different values of disturbance  $d$ . The average computational time using *fmincon* to solve the optimization problem is 1.5528s approximately, and the average number of calls for objective functions and nonlinear constraints are around 1014 times.

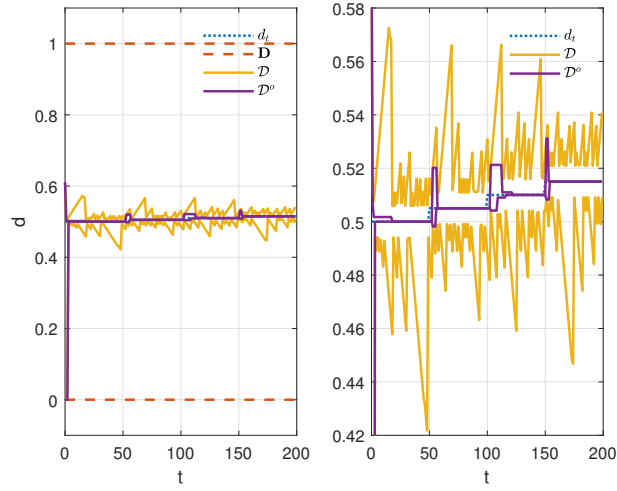


Fig. 5.8 True disturbance  $d$ , disturbance bound  $\mathbb{D}$ , robust estimated set  $\mathcal{D}$ , estimated set  $\mathcal{D}^o$  and its zoomed-in for system (5.38), when the disturbance has nonzero rate of variation and noises are randomly distributed on the boundaries of  $\mathbb{W}$  and  $\mathbb{H}$ .

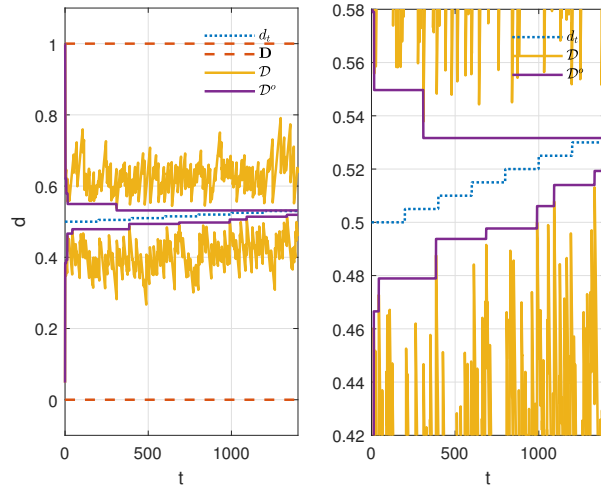


Fig. 5.9 True disturbance  $d$ , disturbance bound  $\mathbb{D}$ , robust estimated set  $\mathcal{D}$ , estimated set  $\mathcal{D}^o$  and its zoomed-in for system (5.38), when the disturbance has nonzero rate of variation and noises are randomly distributed on sets  $\mathbb{W}$  and  $\mathbb{H}$ .

### 5.4.2 Quadratic cost function

The optimal equilibrium of problem (5.25) with the linear cost function shows its independence to the scaling factor  $\gamma$ , and hence the closed-loop initial nominal sequence converges

to the same point regardless of the optimal solution  $\gamma_{t|t}^*$ . However, there are cases that the results are affected by the scaling factor which can be illustrated by considering the following quadratic stage cost function

$$\ell(\mathbf{x}, u, d) = -x_1 + (x_2 - 0.5)^2.$$

Then, the modified stage cost is represented as

$$\bar{\ell}_{\mathcal{S}}(\bar{\mathbf{z}}, v, \gamma) = \max_{d \in \mathcal{S}, \delta \in \gamma\Omega} (0.9598d + \bar{z}_2 + \delta_1\gamma)^2 - (\bar{z}_1 + \delta_1\gamma) - (\bar{z}_2 + \delta_2\gamma) - 0.9994d + 0.25.$$

We consider the initial state  $\mathbf{x} = [1 \ 1]^T$ , prediction horizon  $N = 12$ , and a slowly varying disturbance  $d$  increasing from 0.5 to 0.58 and subsequently decreasing to its initial value. The simulated results are shown in the following Fig.5.10 - Fig.5.12. Notice, in Fig.5.11, that the set  $\mathcal{D}_t^o$  can estimate the true constant values  $\{0.5\}$  and  $\{0.58\}$  (purple line) when the noise takes values on the boundary of the respective supporting sets as shown in Fig.5.12. More importantly, the closed-loop initial nominal sequence  $\bar{\mathbf{z}}_{t|t}^*$  converges to  $[0.7155 \ 0.6186]$  and  $[0.6267 \ 0.5418]$  for  $d = 0.5$  and  $d = 0.58$ , respectively, which are approximately identical to the optimal equilibria from problem (5.25) in the case of  $\gamma = 0.345$  in Fig.5.12. The average computational time using *fmincon* to solve the optimization problem is 2.6402s approximately, and the average number of calls for objective functions and nonlinear constraints are around 3525 times.

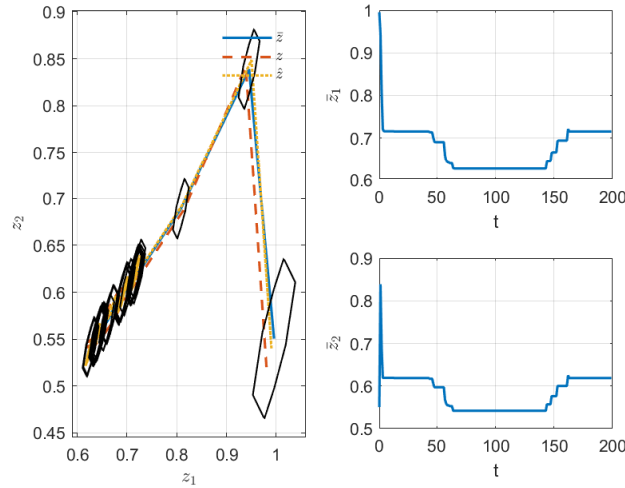


Fig. 5.10 Closed-loop transformed state trajectories with scaled bounded set  $\gamma\Omega$  for system (5.38), when the disturbance is as shown in Fig.5.11.

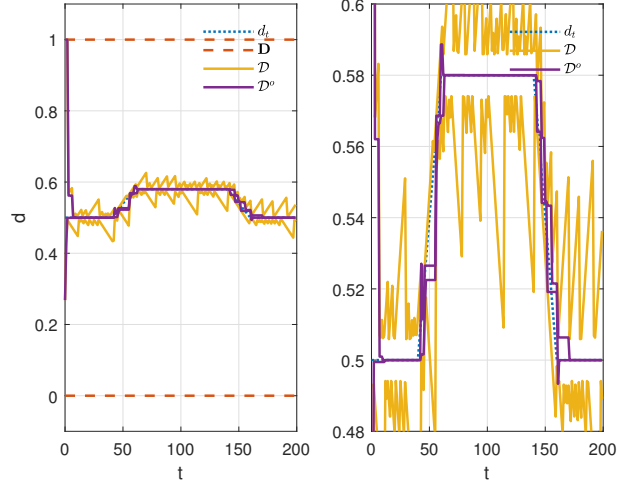


Fig. 5.11 True disturbance  $d$ , disturbance bound  $\mathbb{D}$ , robust estimated set  $\mathcal{D}$ , estimated set  $\mathcal{D}^o$  and the zoomed-in for system (5.38), when noises are randomly generated on the boundaries of sets  $\mathbb{W}$  and  $\mathbb{H}$ .

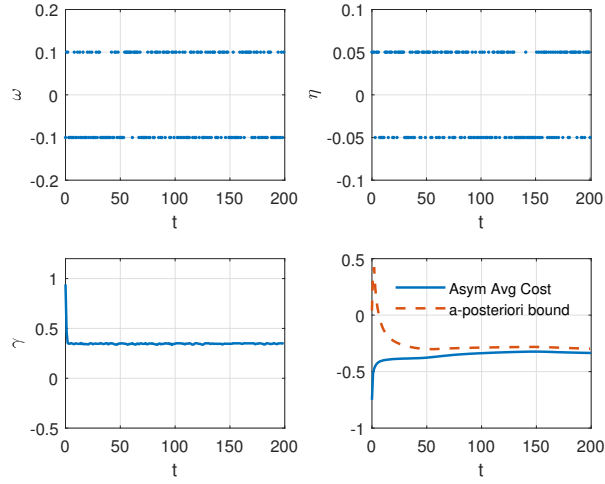


Fig. 5.12 Distribution of the noises, the optimal scaling factor, and asymptotic average performance for system (5.38), when the disturbance is as shown in Fig.5.11.

## 5.5 Summary

In summary, we presented a homothetic tube-based robust offset-free Economic MPC algorithm combined with moving horizon estimation for constrained linear discrete time

systems. By adopting separate iterative updating rules for the set of current disturbances compatible with past data, allows to differentiate the tightest estimate based on constant disturbances, from the worst case approach based on the assumption of slowly-varying disturbances. These are respectively used for cost estimation and to both guarantee robustness against process/measurement noises and recursive feasibility. Moreover, a tight robust economic performance bound of the closed-loop asymptotic average performance is derived.

## Chapter 6

# Conclusion and Future Work

### 6.1 Conclusion

In this thesis, we investigated recursive feasibility, asymptotic stability and average performance of robust economic MPC problems.

In particular, we introduced the new concept control storage function and adopt it as the terminal cost function. A generalized approach for estimation of system asymptotic average performance from above and below employing the CSF and dissipation inequalities has been elaborated. Such tools are adapted to formulate economic MPC control schemes and eventually analyze their performance and stability. In the case of “no-gap”, as previously defined, if the assumption on continuity of optimal control policy from the economic MPC controller holds, the optimal regime of operation is Lyapunov asymptotically stable when the CSF is used as the terminal penalty function.

When the system is subject to disturbances, a tube-based robust economic MPC algorithm was presented. By considering the worst cost of the tube along with the artificial nominal system, the robustness against the disturbances is guaranteed in our economic MPC design. Within this thesis, constraint tightening is used to prove recursive feasibility and the bound of the closed-loop asymptotic average performance is derived. The asymptotic stability of the robust optimal invariant set is inferred from the asymptotic convergence of the nominal state sequence.

Furthermore, MHE is integrated with the economic MPC in a single optimization problem and the homothetic tube is employed if the system state is not measurable. In this setup, a purely economic cost is formulated as the objective function, so that the controller can estimate the current state using the past output measurements and optimize the predicted cost at the same time. More importantly, thanks to the use of the artificial nominal system, recursive feasibility and asymptotic stability are ensured. This method is then extended to the offset-free framework, in which the optimal equilibrium is unknown a-priori and a generalized terminal cost function is used in the objective function.

In conclusion, the results provide a possibility of economic MPC for systems optimally operated at general solutions beyond steady-state and contribute to some design methods in the area of robust economic MPC.

## 6.2 Future work

In this thesis, we obtained some novel results of economic MPC on general optimal regimes of operation and explicitly considering disturbances, so this can be regarded as a starting point for various interesting future research topics.

For all of the presented ideas, the dissipativity condition is required to classifying the optimal operation regime and establish closed-loop stability. However, it is not always easy to find suitable storage functions to fulfill the dissipativity. Hence it would be an interesting direction to explore the analytically or approximate solutions. Meanwhile, as the counterpart of the dissipativity, controlled dissipation inequality, which is on terminal cost functions, is an open area. Therefore, the relation between the storage function and the control storage function is of interest.

We derived some bounds of asymptotic performance for systems operating in a long time, while the performance of the transient phase has not been considered. To find similar bounds on transient average cost is also a possible research direction. In addition, throughout this thesis, the disturbance is assumed to be uniformly distributed in a bounded set and the worst cost is utilized in problem formulations, thus this could lead to conservatism in closed-loop operation and performance bound estimation. Taking into account more statistical information of the disturbance and deriving a tighter robust performance bound for system realizations without losing recursive feasibility and stability is an interesting topic.

Adopting the tube-based controller requires the computation of control invariant set which is a complex procedure for general nonlinear systems. One possibility is to investigate other system properties to reduce the computation complexity such as monotonicity in which case only the extreme disturbance realization is needed.

Finally, in our proposed methods, non-zero disturbances are always treated as quantities which degrade economic performance. In fact, this is not the case. For example, we usually consider wind as a disturbance in aircraft control problems. However, in the forced landing problem where there is no thrust to control the aircraft, upward wind could help the flight have less sink rate and longer glide distance. Furthermore, the robust optimal steady-state is defined based on the nominal system, but there might be other steady-state of the system with non-zero disturbance, which is capable of providing a lower economic cost. Then, it could be very interesting to replace the nominal system consideration in tube-based economic MPC framework by other system scenarios, which takes the influence of the disturbance into consideration explicitly.



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