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Revisiting the Thompson boundary conditions

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Abstract— the Thompson boundary conditions are used for situations where data are lacking, e.g. in tidal computations where only the free surface is known. They can also allow a free exit of waves through an open boundary. The technique, originally published by Thompson in [1] and described in [2], is based on the theory of characteristics, which was applied so far in a direction normal to the boundary. Adapting Thompson boundary conditions to domain decomposition parallelism revealed a weakness of this approach which requires specific advection fields for every boundary point. These advection fields should have been transmitted to every processor, and this was considered too cumbersome. A modified theory is presented, which consists of applying the theory of characteristics in a direction following the flow. The resulting advection fields do not depend on the original boundary point, thus the standard method for characteristics in parallel may be used.

I. INTRODUCTION

The Thompson boundary conditions are used in Telemac-2D, in cases where data are unknown on open boundaries. This is the case with tidal computations when only the free surface is known, or when a wave exits an open boundary. The original method uses the theory of characteristics, linearized in a direction normal to the boundary. In Thompson publication finite differences are employed to solve the 3 advection problems of the method, this was done mainly because at that time regular grids were common practice. Eric David, at Sogreah, then resorted to the method of characteristics itself to solve these problems on unstructured grids. At that time (1999) it precluded parallelism. Then Jacek Jankowski (BAW Karlsruhe) wrote an amazing parallel version of the method of characteristics (module "streamline" in library BIEF). More recently, module streamline was adapted by Christophe Denis (Sinetics, EDF R&D) for dealing with a list of points that are not necessarily linked to mesh nodes, to enable the treatment of particles on one hand, and Thompson boundary points on the other hand. This was not the end of the story. As a matter of fact, the advection fields requested by Thompson boundary points depend on the starting point, and these specific fields must be defined for the whole domain. In parallel this implies that every Thompson boundary point has to send its advection fields to all processors, in case its characteristic path-lines would go to another sub-domain. This was considered too cumbersome, a dead end. Moreover, the Thompson theory leads to the fact that two nearby boundary points may have their characteristics path-lines crossing, because linearization was done in two different directions. This is somewhat against the nature of characteristics that do not cross unless they carry the same invariant. For all these reasons it was considered that the theory had to be modified. It seems natural that the linearization direction should be the direction of the flow. It is what is attempted here. We shall first fully explain what was done in previous versions, and then we shall move to the new idea.

II. A DETAILED EXPLANATION OF THE ORIGINAL TECHNIQUE

We explain hereafter in more detail what is said in Reference [1] page 105 to 108. We neglect diffusion and start from the conservative form of Saint-Venant equations, put in the following form taken from [1] at page 31, using the fact that the free surface Z_s is the bottom topography plus the depth h.

$$\frac{\partial h}{\partial t} + div(h\vec{u}) = Sce \qquad (1)$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}(huu + g\frac{h^2}{2}) + \frac{\partial}{\partial y}(huv) = -gh\frac{\partial Z_f}{\partial x} + hF_x \qquad (2)$$

$$\frac{\partial (hv)}{\partial t} + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hvv + g\frac{h^2}{2}) = -gh\frac{\partial Z_f}{\partial y} + hF_y \qquad (3)$$

We write:

$$F = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \qquad (4)$$

$$G_x = \begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \\ huv \end{pmatrix} \text{ and } G_y = \begin{pmatrix} hv \\ huv \\ hv^2 + g\frac{h^2}{2} \end{pmatrix} \qquad (5)$$

and

$$S(F) = \begin{pmatrix} Sce \\ -gh\frac{\partial Z_f}{\partial x} + hF_x \\ -gh\frac{\partial Z_f}{\partial y} + hF_y \end{pmatrix} \quad (6)$$

so that the system of three equations can be written in the following form:

$$\frac{\partial F}{\partial t} + \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} = S(F) \quad (7)$$

The Thompson method as implemented so far in Telemac-2D consists of considering a local system of coordinates based on a local normal vector $\vec{n}$ (normal to the boundary) and a local tangent vector $\vec{t}$. If the new system of coordinates is denoted η and ζ, we have

$$\vec{n} = \begin{pmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \end{pmatrix} \text{ and } \vec{t} = \begin{pmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial x} \end{pmatrix} \quad (8)$$

We keep these notations here, but the directions $\vec{n}$ and $\vec{t}$ may not be linked to the boundary. The components of velocity in the new system will be denoted u_η and u_ζ. We have

$$\begin{pmatrix} u_\eta \\ u_\zeta \end{pmatrix} = \begin{pmatrix} \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \\ -u\frac{\partial \eta}{\partial y} + v\frac{\partial \eta}{\partial x} \end{pmatrix} \quad (9)$$

and

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial \eta}{\partial x} & -\frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial x} \end{pmatrix} \begin{pmatrix} u_\eta \\ u_\zeta \end{pmatrix} = \begin{pmatrix} u_\eta \frac{\partial \eta}{\partial x} - u_\zeta \frac{\partial \eta}{\partial y} \\ u_\eta \frac{\partial \eta}{\partial y} + u_\zeta \frac{\partial \eta}{\partial x} \end{pmatrix} \quad (10)$$

We first want to put the system in the form:

$$\frac{\partial F}{\partial t} + A_x \frac{\partial F}{\partial x} + B_y \frac{\partial F}{\partial x} = S(F) \quad (11)$$

where A_x and B_y are matrices. For this goal:

- In (2):

$\frac{\partial}{\partial x}(g\frac{h^2}{2})$ is written $c^2 \frac{\partial h}{\partial x}$

$\frac{\partial}{\partial x}(huu)$ is written: $u^2 \frac{\partial h}{\partial x} + h\frac{\partial u^2}{\partial x} = u^2 \frac{\partial h}{\partial x} + 2uh\frac{\partial u}{\partial x}$

which is also equal to $u^2 \frac{\partial h}{\partial x} + 2u\frac{\partial(hu)}{\partial x} - 2u^2 \frac{\partial h}{\partial x}$

$\frac{\partial}{\partial y}(huv)$ is written:

$v\frac{\partial}{\partial y}(hu) + hu\frac{\partial v}{\partial y} = v\frac{\partial}{\partial y}(hu) + u\frac{\partial(hv)}{\partial y} - uv\frac{\partial h}{\partial y}$

- In (3):

$\frac{\partial}{\partial y}(g\frac{h^2}{2})$ is written $c^2 \frac{\partial h}{\partial y}$

$\frac{\partial}{\partial x}(hvv)$ is written $-v^2 \frac{\partial h}{\partial y} + 2v\frac{\partial(hv)}{\partial y}$

$\frac{\partial}{\partial x}(huv)$ is written $v\frac{\partial}{\partial x}(hu) + u\frac{\partial(hv)}{\partial x} - uv\frac{\partial h}{\partial x}$

We effectively get to (11) with:

$$A_x = \begin{pmatrix} 0 & 1 & 0 \\ c^2 - u^2 & 2u & 0 \\ uv & v & u \end{pmatrix} \quad (12)$$

and

$$B_y = \begin{pmatrix} 0 & 0 & 1 \\ uv & v & u \\ c^2 - v^2 & 0 & 2v \end{pmatrix} \quad (13)$$

Now we change the coordinates by writing that for every function f we have:

$$\frac{\partial f}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial f}{\partial \eta} + \frac{\partial \zeta}{\partial x}\frac{\partial f}{\partial \zeta} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial \eta}{\partial y}\frac{\partial f}{\partial \eta} + \frac{\partial \zeta}{\partial y}\frac{\partial f}{\partial \zeta}$$

It gives us a system in the form:

$$\frac{\partial F}{\partial t} + A_\eta \frac{\partial F}{\partial \eta} + B_\zeta \frac{\partial F}{\partial \zeta} = S(F) \quad (14)$$

with

$$A_\eta = \frac{\partial \eta}{\partial x} A_x + \frac{\partial \eta}{\partial y} B_y \quad (15)$$

and

$$B_\zeta = \frac{\partial \zeta}{\partial x} A_x + \frac{\partial \zeta}{\partial y} B_y = -\frac{\partial \eta}{\partial y} A_x + \frac{\partial \eta}{\partial x} B_y \quad (16)$$

which gives

$$A_\eta = \begin{pmatrix} 0 & \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial x}(c^2 - u^2) - \frac{\partial \eta}{\partial y}uv & 2u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} & u\frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial y}(c^2 - v^2) - \frac{\partial \eta}{\partial x}uv & v\frac{\partial \eta}{\partial x} & u\frac{\partial \eta}{\partial x} + 2v\frac{\partial \eta}{\partial y} \end{pmatrix} \quad (17)$$

and

$$B_\zeta = \begin{pmatrix} 0 & -\frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial x} \\ -\frac{\partial \eta}{\partial y}(c^2 - u^2) - \frac{\partial \eta}{\partial x}uv & -2u\frac{\partial \eta}{\partial y} + v\frac{\partial \eta}{\partial x} & u\frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial x}(c^2 - v^2) + \frac{\partial \eta}{\partial y}uv & -v\frac{\partial \eta}{\partial y} & -u\frac{\partial \eta}{\partial y} + 2v\frac{\partial \eta}{\partial x} \end{pmatrix} \quad (18)$$

or even, still denoting u_η as the normal component of velocity and u_ζ the tangential component:

$$A_\eta = \begin{pmatrix} 0 & \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial x}c^2 - uu_\eta & u\frac{\partial \eta}{\partial x} + u_\eta & u\frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial y}c^2 - vu_\eta & v\frac{\partial \eta}{\partial x} & u_\eta + v\frac{\partial \eta}{\partial y} \end{pmatrix} \quad (19)$$

and

$$B_\zeta = \begin{pmatrix} 0 & -\frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial x} \\ -\frac{\partial \eta}{\partial y} c^2 - uu_\zeta & -u\frac{\partial \eta}{\partial y} + u_\zeta & u\frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial x} c^2 - uu_\zeta & -v\frac{\partial \eta}{\partial y} & u_\zeta + v\frac{\partial \eta}{\partial x} \end{pmatrix} \quad (20)$$

Subsequently, we ignore the variations along the direction ζ and try to solve the system:

$$\frac{\partial F}{\partial t} + A_\eta \frac{\partial F}{\partial \eta} = S(F) \quad (21)$$

An open question is: which part of $S(F)$ should be kept in this equation? We discard Sce, F_x and F_y, and keep only the variations of bottom along the direction η. It gives:

$$S_\eta(F) = \begin{pmatrix} 0 \\ -gh\frac{\partial \eta}{\partial x}\frac{\partial Z_f}{\partial \eta} \\ -gh\frac{\partial \eta}{\partial y}\frac{\partial Z_f}{\partial \eta} \end{pmatrix} \quad (22)$$

For the time being, we call it $S_\eta(F)$ whatever its value and go on with the diagonalization of A_η. Now A_η is diagonalized as $A_\eta = L^{-1}\Lambda L$ with:

$$L = \begin{pmatrix} u_\zeta & \frac{\partial \eta}{\partial y} & -\frac{\partial \eta}{\partial x} \\ c - u_\eta & \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \\ c + u_\eta & -\frac{\partial \eta}{\partial x} & -\frac{\partial \eta}{\partial y} \end{pmatrix} \quad (23)$$

and

$$\Lambda = \begin{pmatrix} u_\eta & 0 & 0 \\ 0 & u_\eta + c & 0 \\ 0 & 0 & u_\eta - c \end{pmatrix} \quad (24)$$

This can be controlled by checking that $LA_\eta = \Lambda L$. By stating that $dW = LdF$, we then get back to the diagonalized system:

$$\frac{\partial W}{\partial t} + \Lambda \frac{\partial W}{\partial \eta} = LS_\eta \quad (25)$$

Each of whose lines is a simple transport equation with source term. Thompson proposes to consider that L is constant in the vicinity of a boundary point, and to write $W = LF$, where

$$\overline{L} = \begin{pmatrix} \overline{u_\zeta} & \frac{\partial \eta}{\partial y} & -\frac{\partial \eta}{\partial x} \\ \overline{c} - \overline{u_\eta} & \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \\ \overline{c} + \overline{u_\eta} & -\frac{\partial \eta}{\partial x} & -\frac{\partial \eta}{\partial y} \end{pmatrix} \quad (26)$$

the over-bar values being considered as constant (these are the values deduced from the local conditions: h, u and v at the original starting point of the characteristics). The Riemann invariants of the vector W are thus:

- $W_1 = h(\overline{u_\zeta} - u_\zeta)$ (advection with u_η).

- $W_2 = h(\overline{c} + u_\eta - \overline{u_\eta})$ (advection with $u_\eta + c$).

- $W_3 = h(\overline{c} - u_\eta + \overline{u_\eta})$ (advection with $u_\eta - c$).

and to which can be added, if a tracer T also has to be considered:

- $W_4 = h(T - \overline{T})$ (advection with u_η).

Pure advection is treated with the method of characteristics. To be more precise, a first advection is done with velocity u_η. This is done backwards in time. For every boundary point of Thompson type, we compute the backward trajectory and find, at what is called the foot of the characteristic curve (starting point of the trajectory which will arrive at the boundary point after Δt), the values of depth and components of velocity which we call $\tilde{h}_1$, $\tilde{u}_1$, etc. If we neglect the source terms and take the invariants at this foot of characteristic path-line, we have:

- $W_1 = h(\overline{u_\zeta} - u_\zeta) = \tilde{W}_1 = \tilde{h}_1(\overline{u_\zeta} - \tilde{u}_{\zeta 1})$

  with $\overline{u_\zeta} = -u\frac{\partial \eta}{\partial y} + v\frac{\partial \eta}{\partial x}$

  and $\tilde{u}_{\zeta 1} = -\tilde{u}\frac{\partial \eta}{\partial y} + \tilde{v}\frac{\partial \eta}{\partial x}$

- $W_4 = h(T - \overline{T}) = \tilde{W}_4 = \tilde{h}(\tilde{T}_1 - \overline{T})$ with $\overline{T} = T$

Then, after an advection with velocity $u_\eta + c$, i.e. with results now called $\tilde{h}_2$, $\tilde{u}_2$ and $\tilde{v}_2$:

- $W_2 = h(\overline{c} + u_\eta - \overline{u_\eta}) = \tilde{W}_2 = \tilde{h}_2(\overline{c} + \tilde{u}_{\eta 2} - \overline{u_\eta})$

  with $\overline{c} = \sqrt{gh}$, $\overline{u_\eta} = u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}$

  and $\tilde{u}_{\eta 2} = \tilde{u}_2\frac{\partial \eta}{\partial x} + \tilde{v}_2\frac{\partial \eta}{\partial y}$

Then, after an advection with velocity $u_\eta - c$, i.e. with yet other values denoted $\tilde{h}_3$, $\tilde{u}_3$ and $\tilde{v}_3$:

- $W_3 = h(\overline{c} - u_\eta + \overline{u_\eta}) = \tilde{W}_3 = \tilde{h}_3(\overline{c} - \tilde{u}_{\eta 3} + \overline{u_\eta})$

  with $\tilde{u}_{\eta 3} = \tilde{u}_3\frac{\partial \eta}{\partial x} + \tilde{v}_3\frac{\partial \eta}{\partial y}$

All this is valid only if the backwards characteristic goes inside the domain. This can be checked by the fact that $\vec{u}_{conv}.\vec{n} > 0$, where $\vec{u}_{conv}$ is the advection velocity field (i.e. based on u_η, $u_\eta + c$ or $u_\eta - c$, respectively for W_1, W_2 and W_3). If $\vec{u}_{conv}.\vec{n} < 0$, all variables with a tilde will be based on the boundary conditions prescribed by the user. For example, $\tilde{u}_{\zeta 1}$ may be taken equal to:

$-u_{bor}\frac{\partial \eta}{\partial y} + v_{bor}\frac{\partial \eta}{\partial x}$

where u_{bor} and v_{bor} are the prescribed components of the velocity field. Source terms will be considered later. Once

the Riemann invariants are known, the primitive variables can be restored by the following formulae:

$$h = \frac{W_2 + W_3}{2\bar{c}} \tag{27}$$

$$h(u - \bar{u}) = \frac{\partial \eta}{\partial y} W_1 + \frac{\partial \eta}{\partial x}(W_2 - W_3) \tag{28}$$

$$h(v - \bar{v}) = \frac{\partial \eta}{\partial y}(W_2 - W_3) - \frac{\partial \eta}{\partial x} W_1 \tag{29}$$

$$h(T - \bar{T}) = -W_4 \tag{30}$$

Equation (14) can be used to eliminate h from the three others, yielding

$$h = \frac{W_2 + W_3}{2\bar{c}} \tag{31}$$

$$hu = \frac{W_2 + W_3}{2\bar{c}} \bar{u} + \frac{\partial \eta}{\partial y} W_1 + \frac{\partial \eta}{\partial x}(W_2 - W_3) \tag{32}$$

$$hv = \frac{W_2 + W_3}{2\bar{c}} \bar{v} + \frac{\partial \eta}{\partial y}(W_2 - W_3) - \frac{\partial \eta}{\partial x} W_1 \tag{33}$$

$$hT = \frac{W_2 + W_3}{2\bar{c}} \bar{T} - W_4 \tag{34}$$

This form is not the most practical but readily gives, if necessary or for checking:

$$\bar{L}^{-1} = \begin{pmatrix} 0 & \frac{1}{2\bar{c}} & \frac{1}{2\bar{c}} \\ \frac{\partial \eta}{\partial y} & \frac{1}{2}\frac{\partial \eta}{\partial x} + \frac{\bar{u}}{2\bar{c}} & -\frac{1}{2}\frac{\partial \eta}{\partial x} + \frac{\bar{u}}{2\bar{c}} \\ -\frac{\partial \eta}{\partial x} & \frac{1}{2}\frac{\partial \eta}{\partial y} + \frac{\bar{v}}{2\bar{c}} & -\frac{1}{2}\frac{\partial \eta}{\partial y} + \frac{\bar{v}}{2\bar{c}} \end{pmatrix} \tag{35}$$

We will favour the following formulas for the implementation:

$$h = \frac{W_2 + W_3}{2\bar{c}} \tag{36}$$

$$u = \frac{\frac{\partial \eta}{\partial y} W_1 + \frac{\partial \eta}{\partial x}(W_2 - W_3)}{h} + \bar{u} \tag{37}$$

$$v = \frac{-\frac{\partial \eta}{\partial x} W_1 + \frac{\partial \eta}{\partial y}(W_2 - W_3)}{h} + \bar{v} \tag{38}$$

$$T = -\frac{W_4}{h} + \bar{T} \tag{39}$$

If we do not neglect source terms, they have to be integrated along the characteristic curve. Assuming a constant $\bar{L}$ as done before we have

$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \tilde{W}_3 \end{pmatrix} + Sce\Delta t \begin{pmatrix} \overline{u_\zeta} \\ \bar{c} - \overline{u_\zeta} \\ \bar{c} + \overline{u_\zeta} \end{pmatrix}$$

$$+ h\left(-g \frac{\partial \eta}{\partial x}\frac{\partial Z_f}{\partial \eta} + F_x\right)\Delta t \begin{pmatrix} \frac{\partial \eta}{\partial y} \\ \frac{\partial \eta}{\partial x} \\ -\frac{\partial \eta}{\partial x} \end{pmatrix} + h\left(-g \frac{\partial \eta}{\partial y}\frac{\partial Z_f}{\partial \eta} + F_y\right)\Delta t \begin{pmatrix} -\frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \\ -\frac{\partial \eta}{\partial y} \end{pmatrix}$$

Neglecting again Sce, F_x and F_y, we are left with

$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \tilde{W}_3 \end{pmatrix} - gh\Delta t \begin{pmatrix} 0 \\ \frac{\partial Z_f}{\partial \eta} \\ -\frac{\partial Z_f}{\partial \eta} \end{pmatrix} \tag{40}$$

Though the source terms could be treated in an explicit way, we do the following approximation: $\partial Z_f / \partial \eta$ is approximated as:

$$\frac{Z_f - \tilde{Z}_f}{(\bar{u} + \bar{c})\Delta t}$$

i.e. the variation of Z_f along the (backwards) characteristic curve divided by the length of the curve, then u is neglected so that we have:

$$\frac{\partial Z_f}{\partial \eta} \approx \frac{Z_f - \tilde{Z}_f}{\bar{c}\Delta t} \tag{41}$$

and eventually $-gh\Delta t \partial Z_f / \partial \eta$ is simplified into:

$$\bar{c}(Z_f - \tilde{Z}_f) \tag{42}$$

It gives the following new formulas for W_2 and W_3:

$$W_2 = \bar{c}(\tilde{h}_2 + \tilde{Z}_{f2} - Z_f) + \tilde{h}_2(\tilde{u}_{\eta2} - \overline{u_\eta}) \tag{43}$$

$$W_3 = \bar{c}(\tilde{h}_3 + \tilde{Z}_{f3} - Z_f) - \tilde{h}_3(\tilde{u}_{\eta3} - \overline{u_\eta}) \tag{44}$$

III. NEW THEORY

A. Linearisation in the direction of the flow

All what has been said in previous section is valid up to version 6.0 if we choose for $\vec{n}$ the outward normal vector to the boundary. The problem is that in this case the three advections fields depend on the boundary point under treatment. This was heavy in scalar mode, where points with the same normal were grouped for optimization and shared the same advection field. It becomes even more heavy in parallel because these advection fields should be built for the whole domain, which implies that for every Thompson point, its normal vector must be exported to all sub-domains. It also appears very strange that characteristics of the same family

stemming from two different points may cross because they have a different original direction.

The new theory consists of choosing advection fields that would not depend on a given boundary point. It seems very natural to choose, instead of the outward normal vector $\vec{n}$, the direction of the velocity field itself. We have then:

$$\vec{n} = \begin{pmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{u}{\sqrt{u^2+v^2}} \\ \frac{v}{\sqrt{u^2+v^2}} \end{pmatrix}$$

An important consequence of this choice is that the velocity u_ζ is always 0 by definition, which would lead to $W_1 = 0$. This is true in fact only if we consider that the direction $\vec{n}$ changes along characteristics, it is false if we keep the original $\vec{n}$, which would be consistent with the linearisation leading to $\overline{L}$. Tests show that it is better to consider that u_ζ is indeed not 0, thus sticking to the linearisation. A possibility that remains to be tested would be considering that u_ζ is indeed 0, and taking the norm of velocity for the component u_η.

In any case there is an obvious problem when there is no velocity, the direction where to apply the celerity c is then undefined. A first idea is to cancel also the celerity c in this case, so that all variables will keep their original value. This is not possible, because a velocity equal to 0 for a given boundary point would then trigger that the depth and velocity at this point remain unchanged. This is valid only if there is no wave approaching the point, i.e. no velocity and no free surface slope. When there is a free surface slope, it seems then natural to choose the direction of the vector $-g\,\vec{grad}Z_s$, which is the driving term in momentum equation that will create velocity at the next time step. This happens to be very important in tests, especially the Gaussian hill test case.

B. Depth and velocity for interpolation

An unexpected problem occurred in the results, showing that the tests $\vec{u}_{conv}.\vec{n} > 0$, to decide whether we should take e.g. the depth h or the prescribed depth h_{bor} for computing h, could happen to be wrong. As a matter of fact the method of characteristics itself is able to check if the path-line goes out of the domain, and in this case it stops and interpolates at this exit point. In a corner the average $\vec{n}$ of the corner point may lead to a different decision, thus leading to wrongly choose for example h instead of h_{bor}. Any case where the value $\vec{u}_{conv}.\vec{n}$ is very close to 0 will lead to a random choice, and then to large differences if h and h_{bor} are very different. It was thus decided to discard the tests $\vec{u}_{conv}.\vec{n} > 0$ and to use interpolation fields of h, u and v that already contain the prescribed boundary conditions. A characteristic path-line that exits a Thompson boundary will thus find naturally that $h = h_{bor}$, without resorting to testing $\vec{u}_{conv}.\vec{n} > 0$. A drawback is that for small Courant numbers, when the characteristics path-lines will not go far from boundaries,

their interpolated values will be influenced by the prescribed values of the boundary. When prescribed values are correct, which is generally the case with box models and measurements, this could be also an advantage. With this new approach there can be no discontinuity of choice due to a truncation error.

C. Tests

The more convincing test is the Gaussian hill test, if we consider that all the boundaries of the square domain are open (test thompson in folder test.gb in telemac-2D release). In this case no information is given on the boundaries. The circular wave spreading in the square domain is supposed to exit freely the domain, without any reflection on the boundaries. The results are shown on Figure 1. The new method, on the right, gives slightly more circular iso-lines of depth.

Test-case number 2 checks a boundary with prescribed elevation and free velocity. In a channel 600 m long and 6 m large, a solitary wave is imposed at the left entrance ($x = 0$). The original depth is 10 m and the wave height 2 m. The original position of the wave at the beginning of the computation is $x_0 = -80$ m, so that velocity and depth in the channel are undisturbed. With A the amplitude and $c = \sqrt{gh}$, the celerity of the wave, depth and velocity are:

$$h(x,t) = h_0 + \frac{A}{\cosh\left[\sqrt{\frac{3A}{4h_0^3}}(x+x_0-ct)\right]^2}$$

$$u(x,t) = c\left(\frac{h}{h_0}-1\right)$$

Two cases are considered, both with elevation declared as imposed at the entrance and velocity free at both ends, but the first one (top of Figure 2) with Thompson boundary conditions and the second one (bottom of Figure 2) without. The arrival of the wave is equally well treated, but the exit is correct only with Thompson conditions, a totally free output yielding a spurious reflection. In this solitary wave case we have solved the Boussinesq equations, knowing that the solitary wave employed here is a first order solution of Navier-Stokes equations, which is rather badly treated by Saint-Venant equations. The drawback is that Boussinesq equations will perhaps not comply with the theory of characteristics underlying Thompson conditions… however the result clearly shows the improvement brought by Thompson.

REFERENCES

[1] J.-M. Hervouet, Hydrodynamics of free surface flows. Wiley & sons, 2007.

[2] K.W. Thompson, Time dependent boundary conditions for hyperbolic systems. Journal of Computational physics 68,1-24. 1987.
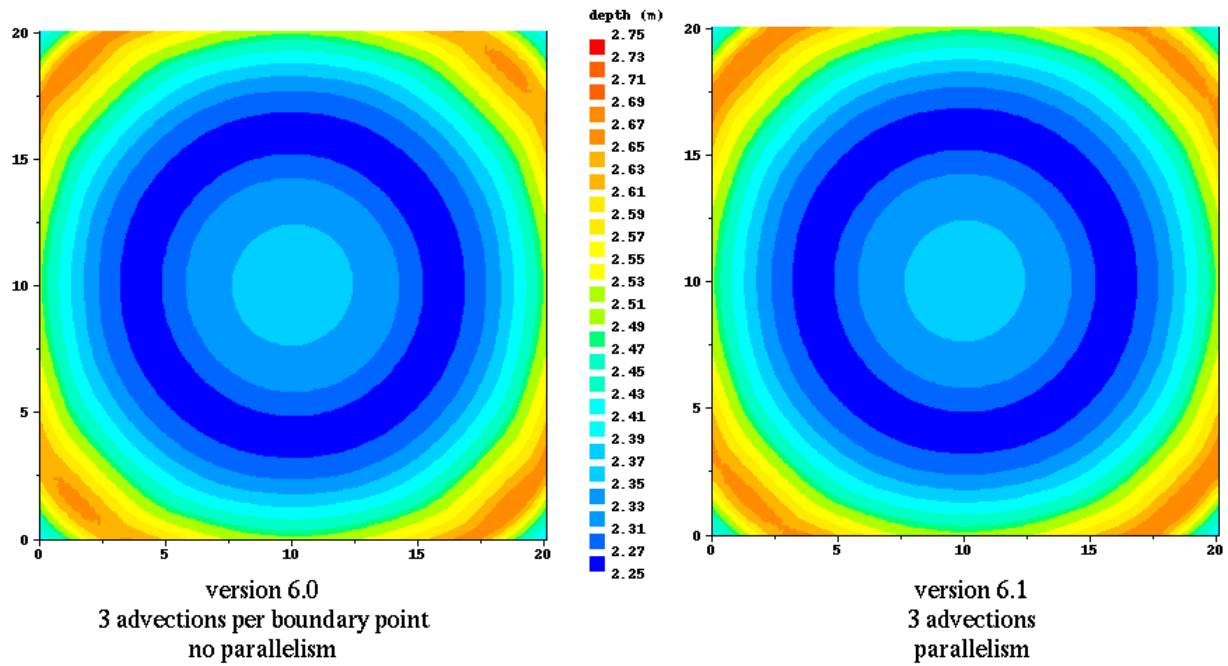
Figure 1 – A circular wave exiting through a square open boundary.
Comparison of version 6.0 and 6.1 of Telemac-2D.
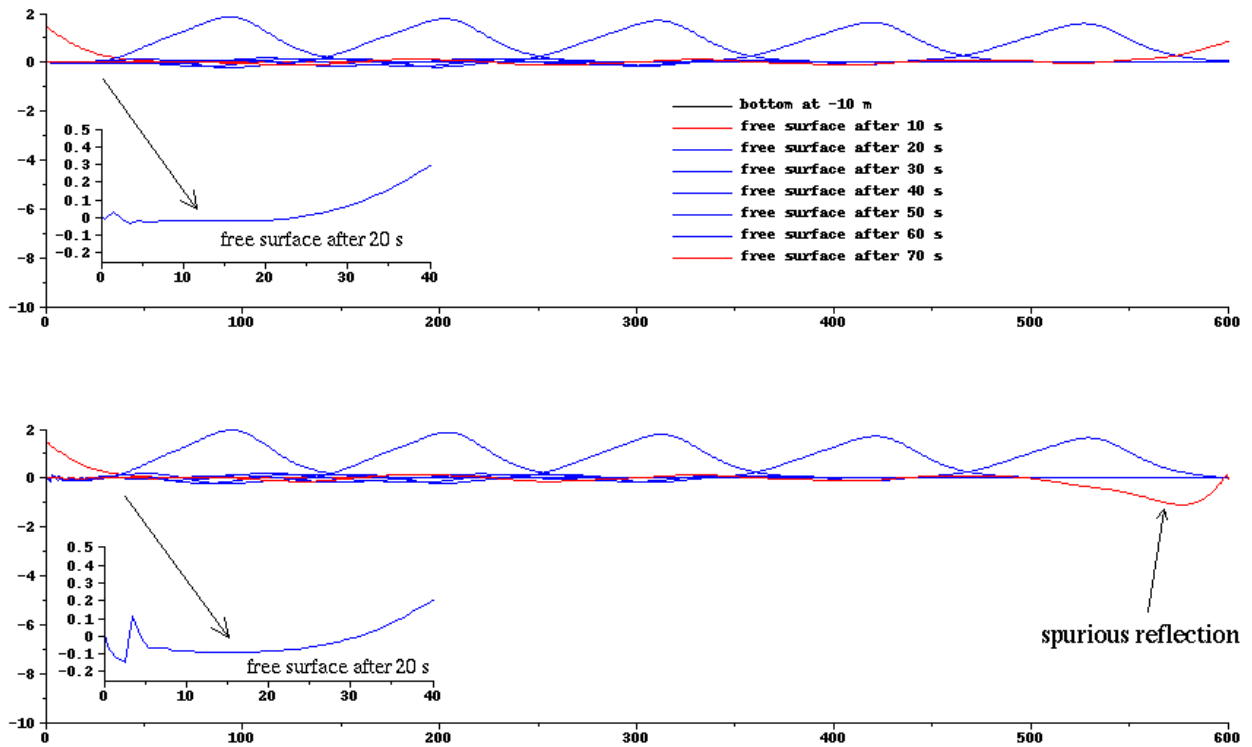
Figure 2 – A solitary wave with Boussinesq equations, with Thompson
boundary conditions (top) and without (bottom).