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COUPLING OF BOUNDARY ELEMENT REGIONS WITH THE BOUNDARY ELEMENT TEARING AND INTERCONNECTING METHOD (BETI)

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Abstract. The boundary integral equation for elasticity is valid for a single domain consisting of homogeneous material properties. In the case of heterogeneity the consideration of different material properties is possible with a coupling of boundary element regions. Of course each region is again homogeneous. Another simulation application of multiple regions is the simulation of an industrial process, where different subdomains of a homogenous domain are treated differently due to a mechanical process. For instance, this is the case in tunnelling, where excavation is performed in a staged procedure. In the simulation of such an excavation process regions are deactivated step by step. As the material behaviour can be nonlinear an accurate simulation of such a staged process is a necessary requirement. Thus, the domain is decomposed into subregions which are coupled to neighbouring regions. There are different coupling strategies existing. In some of them stiffness matrices of subdomains are worked out which are the basis for the coupling and solution of the problem. A traditional method is the coupling of interface surfaces only [1]. In this method the stiffness matrix of a region is computed on the basis of the coupling surfaces (interfaces), whereas the coupling surface may be not identical to the complete surface of a subdomain and the size of the stiffness matrix is determined by the degrees of freedom of the coupling surface. In an application where the boundary conditions change (e.g. from interface to Neumann condition) from one calculation step to the other, the stiffness matrix has to be calculated new. A modern coupling technique is the Boundary Element Tearing and Interconnecting (BETI) method [2], similar to the method of Finite Element Tearing and Interconnecting (FETI) [3]. In

this method the region stiffness matrix is worked out for the entire boundary of the region. The stiffness matrices of all regions remain the same during the whole analysis, even if the boundary conditions change during the simulation process. In setting up the equation system each subdomain is treated completely separated and independent from the others. Thus, a parallelisation of the computational work is ideally suited and implemented in the present computer code. In this work the theory of both mentioned coupling techniques are introduced briefly. The differences of both methods are worked out and advantages/disadvantages are shown and will be demonstrated. The accuracy of the results as well as the computational performance will be shown and compared based on a realistic simulation example.

1 INTRODUCTION

Within a tunnel excavation according the New Austrian Tunneling Method (NATM) parts of the tunnel volume are excavated in a staged procedure. Due to this process the tunnel construction is dependent on spatial and temporal development. In order to provide certain predictions or in the case of verification of ongoing tunnel constructions a numerical simulation has to consider those requirements. A typical tunnel construction is shown in Figure 1. The tunnel cross section is divided into a top heading and a bench part. In longitudinal direction the excavation of the top heading is more advanced then the bench. The excavation is done in a way where volumes of rock with specified thickness are removed.

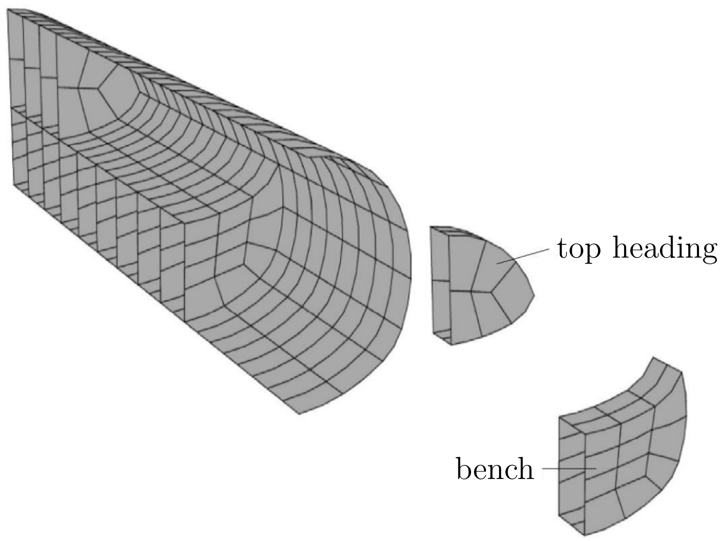

Figure 1: Sequential tunnel excavation

As a tunnel construction typicaly occurs in an infinite or semi-infinite domain the Boundary Element Method (BEM) is ideally suited for the numerical simulation of this type of problems [4]. The fundamental solution used in BEM [5] are special solutions for infinite domains, they allready fullfill the radiation conditons. There is no need to truncate a mesh and therefore no artificial boundary conditions have to be applied. With the BEM only the surface of the regions has to be discretised, thus the effort of mesh generation is drastically reduced for such simulation problems. As shown in Figure 1 the domain is subdivided into several regions. For the excavation volumes top heading and bench finite regions are used. These regions are embedded in an infinite region which represents the infinite extend of the domain. For the numerical solution of such problems there is a need to couple these regions. In chapter 2 the displacement boundary integral equation, its discretisation and solution is shown. In chapter 3 the development of two multiple region BEM (MRBEM) coupling methods will be addressed. In chapter 4 the coupling methods are demonstrated on a practical tunnel excavation example in 3D.

2 BOUNDARY INTEGRAL EQUATION

The displacement boundary integral equation [5] shown in Equation (1) is the basis of the BEM:

$$\mathbf{C}(\mathbf{y})\mathbf{u}(\mathbf{y}) + \oint_{\Gamma} \mathbf{T}(\mathbf{y},\mathbf{x})\mathbf{u}(\mathbf{x})\,\mathrm{d}\Gamma = \int_{\Gamma} \mathbf{U}(\mathbf{y},\mathbf{x})\mathbf{t}(\mathbf{x})\,\mathrm{d}\Gamma \tag{1}$$

$\mathbf{T}(\mathbf{y},\mathbf{x})$ and $\mathbf{U}(\mathbf{y},\mathbf{x})$ are the fundamental solutions and $\mathbf{u}(\mathbf{x})$ and $\mathbf{t}(\mathbf{x})$ are the boundary displacements and tractions, respectively. The boundary integral equation (1) is valid for a single region whose boundary is discretised by boundary elements. Due to the discretisation of the integral equation the boundary Γ is divided into a sum of elements E and nodes N. Introducing the discretisation the integral equation (1) is transformed into the following form:

$$\mathbf{C}\mathbf{u}_i + \sum_{e=1}^{E}\sum_{n=1}^{N} \boldsymbol{\Delta}\mathbf{T}_{ni}^{e}\mathbf{u}_n^{e} = \sum_{e=1}^{E}\sum_{n=1}^{N} \boldsymbol{\Delta}\mathbf{U}_{ni}^{e}\mathbf{t}_n^{e} \tag{2}$$

$\boldsymbol{\Delta}\mathbf{T}_{ni}^{e}$ and $\boldsymbol{\Delta}\mathbf{U}_{ni}^{e}$ are integrated kernel coefficients with respect to the collocation node i and element n. $\mathbf{C}$ is the integral free term which depends on the geometrical conditions at node i. Equation (2) is evaluated for all collocation points i and the coefficients $\boldsymbol{\Delta}\mathbf{T}_{ni}^{e}$ and $\boldsymbol{\Delta}\mathbf{U}_{ni}^{e}$ are assembled into matrices $[\Delta T]$ and $[\Delta U]$, whereas the following equation arise:

$$[\Delta T]\{u\} = [\Delta U]\{t\} \tag{3}$$

With equation (3) a single boundary element region can be solved. At the nodes of the boundary either displacements or tractions are known. The unknown boundary conditions

(BC's) are solved by rearranging equation (3). The unknown BC's with its corresponding columns of matrices $[\Delta T]$ and $[\Delta U]$ are shifted to the left side and the known BC's are multiplied with the columns of the matrices $[\Delta T]$ or $[\Delta U]$ and form the right hand side vector $\{f\}$ of the following equation:

$$[A]\{x\} = \{f\} \tag{4}$$

In case of a mixed boundary value problem the content of the solution vector $\{x\}$ are either displacements or tractions, matrix $[A]$ is filled up either with columns of matrix $[\Delta T]$ or $[\Delta U]$.

3 MULTIPLE REGION BEM (MRBEM)

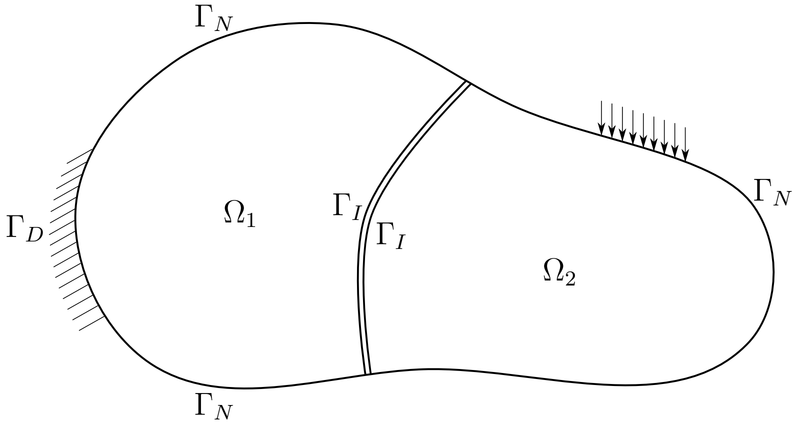

Figure 2: Multiple region system

If there is a system with more then one region present (as shown in Figure 2) a multiple region boundary element method (MRBEM) is a possibility to solve such a problem. In comparison to the single region BEM the boundary of a region is extended by the interface boundary. The whole boundary of a region may be devided into a Neumann (Γ_N), Dirichlet (Γ_D) and Interface (Γ_I) part. At the interface displacements and tractions are unknown. Thus additional conditions are necessary to solve this problem. These conditions are equilibrium and compatability at the interface of adjacent regions. For the solution of such a coupled system of regions different method formulations are available. Two of them are discussed next. For both methods stiffness matrices are worked out. In the first method (Interface Coupling (IC) method) the stiffness matrix of each region is based on the degrees of freedom (DOFs) at the interface, in the second (BETI method) the stiffness matrix is based on the DOFs of the whole region.

3.1 IC method

For the derivation of the equation system of a multiple region system Equation (3) is shown again as following:

$$[\Delta U]\{t\} = [\Delta T]\{u\} \tag{5}$$

Due to the three different boundary types (Neumann, Dirichlet and Interface) Equation (5) is expanded to:

$$\begin{bmatrix} [U_I] \; [U_N] \; [U_D] \end{bmatrix} \left\{ \begin{array}{c} \{t_I\} \\ \{t_N\} \\ \{t_D\} \end{array} \right\} = \begin{bmatrix} [T_I] \; [T_N] \; [T_D] \end{bmatrix} \left\{ \begin{array}{c} \{u_I\} \\ \{u_N\} \\ \{u_D\} \end{array} \right\} \tag{6}$$

The tractions $\{t_N\}$ are known BC's and they will be shifted to the right side of the equation system together with the associated kernel matrix $[U_N]$. $\{u_N\}$ are unknown displacements at the Neumann boundary and they will be moved to the left side together with matrix $[T_N]$ as shown next:

$$\begin{bmatrix} [U_I] \; -[T_N] \; [U_D] \end{bmatrix} \left\{ \begin{array}{c} \{t_I\} \\ \{u_N\} \\ \{t_D\} \end{array} \right\} = \begin{bmatrix} [T_I] \; -[U_N] \; [T_D] \end{bmatrix} \left\{ \begin{array}{c} \{u_I\} \\ \{t_N\} \\ \{u_D\} \end{array} \right\} \tag{7}$$

The matrix on the left side of Equation (7) is renamed by $[A]$ and on the right hand side the interface displacements $\{u_I\}$ are seperated as following:

$$\begin{bmatrix} \quad A \quad \end{bmatrix} \left\{ \begin{array}{c} \{t_I\} \\ \{u_N\} \\ \{t_D\} \end{array} \right\} = [T_I] \; \{u_I\} + \begin{bmatrix} -[U_N] \; [T_D] \end{bmatrix} \left\{ \begin{array}{c} \{t_N\} \\ \{u_D\} \end{array} \right\} \tag{8}$$

Solving Equation (8) for the vector on the left will result in:

$$\left\{ \begin{array}{c} \{t_I\} \\ \{u_N\} \\ \{t_D\} \end{array} \right\} = [A]^{-1}[T_I] \; \{u_I\} + [A]^{-1}\big[-[U_N] \; [T_D]\big] \left\{ \begin{array}{c} \{t_N\} \\ \{u_D\} \end{array} \right\} \tag{9}$$

This equation can be simplified to:

$$\left\{ \begin{array}{c} \{t_I\} \\ \{u_N\} \\ \{t_D\} \end{array} \right\} = \begin{bmatrix} [K]^* \\ [D] \end{bmatrix} \{u_I\} + \left\{ \begin{array}{c} \{t_{I0}\} \\ \{u_{N0}\} \\ \{t_{D0}\} \end{array} \right\} \tag{10}$$

Taking the first equation from (10), which is:

$$\{t_I\} = [K]^*\{u_I\} + \{t_{I0}\} \tag{11}$$

and multiplying it with the mass matrix $[M]$ the tractions of Equation (11) are transformed to work equivalent nodal point forces. Because of this a coupling to the Finite Element Method (FEM) is possible. As a result Equation (11) is transformed to:

$$\{f_I\}_r = [K]_r \{u_I\}_r + \{f_{I0}\}_r \tag{12}$$

whereas $\{f_I\}_r = [M]_r\{t_I\}_r$, $[K]_r = [M]_r[K]_r^*$ and $\{f_{I0}\}_r = [M]_r\{t_{I0}\}_r$. The final forces at the interface of region r are the forces due to the interface displacements plus the forces at the interface due to the loading (given tractions $\{t_N\}_r$ and applied displacements $\{u_D\}_r$). The final interface forces $\{f_I\}_r$ and the interface displacements $\{u_I\}_r$ are unknown at the present state. Thus Equation (12) has to be applied to every region of the coupled system and the final system of equation can be assembled under the following conditions:

- Equilibrium of forces at the interface:

$$\{f_I\}_1 + \{f_I\}_2 = 0 \tag{13}$$

  Equation (13) states that the forces at the interface of region 1 are in equilibrium with those of the neighbouring region 2.

- Compatability of displacements at the interface:

$$\{u_I\}_1 = \{u_I\}_2 \tag{14}$$

  Equation (14) states that the displacements at the interface of region 1 are equal with those of the neighbouring region 2.

Considering these conditions Equation (12) for every region is assembled into a global equation system which is shown as following:

$$\{f_I\}_r = [K]^{sys}\{u_I\} + \{f_{I0}\} = 0 \tag{15}$$

where $[K]^{sys}$ is the assembled stiffness matrix related to all coupling interfaces of the system. $\{f_{I0}\}$ is the right hand side vector related to the loading of the system and $\{u_I\}$ is the vector of interface displacements. This equation is solved for the interface displacements. Once $\{u_I\}$ is known all remaining unknowns (tractions at the interface $\{t_I\}$, displacements at the Neumann boundary $\{u_N\}$ and tractions at the Dirichlet boundary $\{t_D\}$) can be evaluated using Equation (10).

3.2 BETI method

The Boundary Element Tearing and Interconnecting Method (BETI) is a domain decomposition method for the Symmetric Galerkin BEM [2] similar to the Finite Element Tearing and Interconnecting Method (FETI) for the FEM introduced by [3]. Using similar concepts the BETI method will be applied in this work to the collocation BEM. For each region this method works out a stiffness matrix which is based on the DOFs of the whole region surface in contrast to the method applied in chapter 3.1 where the stiffness matrix is based on the coupled DOFs only. From Equation (3) the boundary tractions of a region can be calculated as following:

$$[\Delta U]^{-1}[\Delta T]\{u\} = \{t\} \tag{16}$$

To make possible a coupling to the FEM, Equation (16) is multiplied with the mass matrix $[M]$. Using this procedure the boundary tractions are transformed to work equivalent nodal point forces acting at the nodes of the boundary:

$$[M][\Delta U]^{-1}[\Delta T]\{u\} = [M]\{t\} = \{f\} \tag{17}$$

where the stiffness matrix $[K]$ is:

$$[K] = [M][\Delta U]^{-1}[\Delta T] \tag{18}$$

Inserting this into Equation (17) will result in the well known relation between displacements and forces:

$$[K]\{u\} = \{f\} \tag{19}$$

In order to formulate a coupled system of R boundary element regions two conditions have to be satisfied:

- Equilibrium
- Compatability

3.2.1 Equilibrium of a boundary element region

The equilibrium state of a region can be described by using Equation (19):

$$[K]\{u\} = \{f_N\} + [B]^T\{\lambda\} \tag{20}$$

whereas the force vector on the right hand side of Equation (19) is splitted into:

$$\{f\} = \{f_N\} + [B]^T\{\lambda\} \tag{21}$$

and inserted into Equation (20). $[K]\{u\}$ are the forces at the boundary of the region due to deformation, $\{f_N\}$ is the force vector of the given loading (Neumann boundary conditions) and $[B]^T\{\lambda\}$ are the coupling forces (Lagrange multipliers) to the neighbouring regions.

3.2.2 Compatability of interface displacements

The compatability of a system of R regions can be written in following form:

$$[B]_1\{u\}_1 + [B]_2\{u\}_2 + \cdots + [B]_R\{u\}_R = \{b\} \tag{22}$$

Equation (22) either guaranties that the displacements at the interface of adjacent regions are equal or that the displacements at the Dirichlet boundary are equal to the applied Dirichlet boundary conditions which are entries of vector $\{b\}$.

3.2.3 System of equation

The final system of equation of a coupled system of R boundary element regions is shown as following:

$$\begin{bmatrix} [K]_1 & & 0 & & -[B]_1^T \\ & [K]_2 & & & -[B]_2^T \\ 0 & & \ddots & & \vdots \\ & & & [K]_R & -[B]_R^T \\ [B]_1 & [B]_2 & \cdots & [B]_R & 0 \end{bmatrix} \cdot \begin{Bmatrix} \{u\}_1 \\ \{u\}_2 \\ \vdots \\ \{u\}_R \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{f_N\}_1 \\ \{f_N\}_2 \\ \vdots \\ \{f_N\}_R \\ \{b\} \end{Bmatrix} \tag{23}$$

Equations 1 to R of Equation (23) are representing the equilibrium of each region and the last equation of (23) guaranties compatability of displacements at every node at the interface of adjacent regions and at the nodes of the Dirichlet boundary.

In the implementation of the BETI method Equation (23) is not assembled to an equation system. The equation system (23) is condensed to the solution of the coupling forces λ (Lagrange multipliers). This is done by inserting equations 1 to R into the last equation of (23). From this equation λ is solved either directly or iteratively with a BiCGSTAB iterative solver. As the stiffness matrix $[K]_r$ of a finite region (floating region) is singular special treatment of rigid body motions have to be considered. The whole solution formulation is shown in detail by [2].

3.3 Comparison of coupling methods

The main advantage of the BETI method is that the stiffness matrix of each region has to be calculated only once and in the case of a sequential tunnel excavation these matrices can be used for each load step of excavation. Using the BETI method for this application type the stiffness matrices are independent on the changing boundary conditions. Changing boundary conditions due to sequential excavation are considered by the coupling matrix $[B]_r$ of Equation (23). The coupling matrices have to be computed again for each calculation step. As those matrices are sparsely populated they are implemented as sparse matrices. The effort to set up those matrices is small and it is insignificant compared to the overall computing time. The way how the equation system is formulated makes the treatment of operations at the region independently from the other regions. Thus, the BETI method is ideally suited for parallelisation.

The advantage of the IC method is that the size of the stiffness matrix is related to the number of DOFs at the interface of the coupled system of regions. In the case of a sequential excavation the coupling surfaces are reduced from one excavation step to the other. In each load step one or more regions are deactivated from the simulation model. Due to the deactivation the boundary condition of surfaces adjacent to the deactivated regions change from Interface condition to Neumann condition. Thus, the size of the assembled system stiffness matrix reduces from one load step to the other and the solution of the equation system gets faster. For regions for which a change of boundary conditions happens the stiffness matrix has to be calculated again. Compared to the BETI method stiffness matrices do not remain constant throughout the entire analysis of such an excavation simulation.

4 EXAMPLE - 3D TUNNEL EXCAVATION

In this example a 3D tunnel excavation is investigated. In Figure 3 the boundary discretisation is shown. The tunnel geometry is subdivided into top heading and bench regions (finite regions). These regions are embedded in an infinite region which represent the infinite extend of the domain. The mesh is discretised with quadratic boundary elements. At the starting of the tunnel quadratic plane strain infinite boundary elements are used to simulate the infinite extend of the tunnel behind the tunnel face. Two different meshes are investigated, a coarse mesh and a refined mesh as shown in Figure (3).

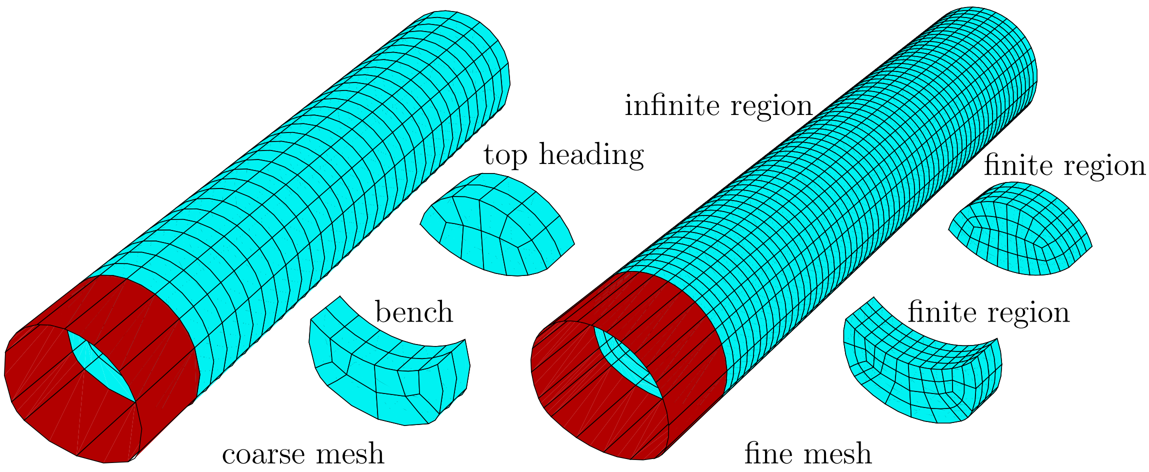

Figure 3: Discretisation

A constant primary stress field with $\sigma_{xx} = -1.375$, $\sigma_{yy} = -1.375$, $\sigma_{zz} = -2.75$ and $\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0.0$ is assumed. From this stress field the excavation tractions are computed by $t = \sigma \cdot n$, where n is the unit vector normal to the surface of the tunnel.

This traction is applied at the free surface, this is the surface of the tunnel just excavated in load step 1. The loading for every subsequent step is the traction applied at the parts of the tunnel surface which changed from coupled interface conditions to free surfaces, these are surfaces at adjacent regions to the deactivated regions. The tractions which are applied in the current load step are taken from the result tractions at the coupled interfaces from the previous laod step. From step 2 to 15 top heading and bench regions are excavated alternately. The sequence of excavation is shown in Figure 4, where regions are marked by the number of load step they are excavated. The excavation of the top heading part takes place further ahead then the bench, which can be observed in Figure 4, too. Within an excavation step one or several regions might be excavated. In the first step from the beginning top heading and bench regions are excavated to reach a typical staged excavation configuration in the longitudinal direction.

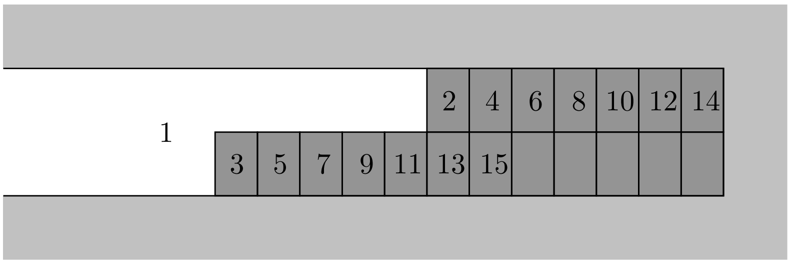

Figure 4: Sequence of excavation load steps

In Figure 5 results for the displacements u_z are shown at the infinite region and at finite regions (embedded in the infinite region) for the refined discretisation.

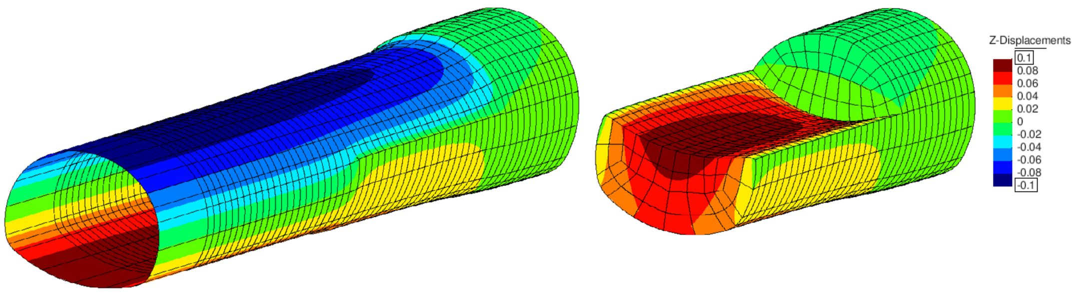

Figure 5: Displacements u_z for excavation load step 8

In the diagram of Figure 6 the deformation curves along a line (parallel to the longitudinal tunnel axis) at the crown of the tunnel for the displacements u_z are shown . The curves are shown for the coarse and fine discretisation and for all load cases. Results are shown only for the interpolation type Discontinuous. As can be seen the solutions for the coarse mesh are almost of the same quality as the solution of the fine mesh.

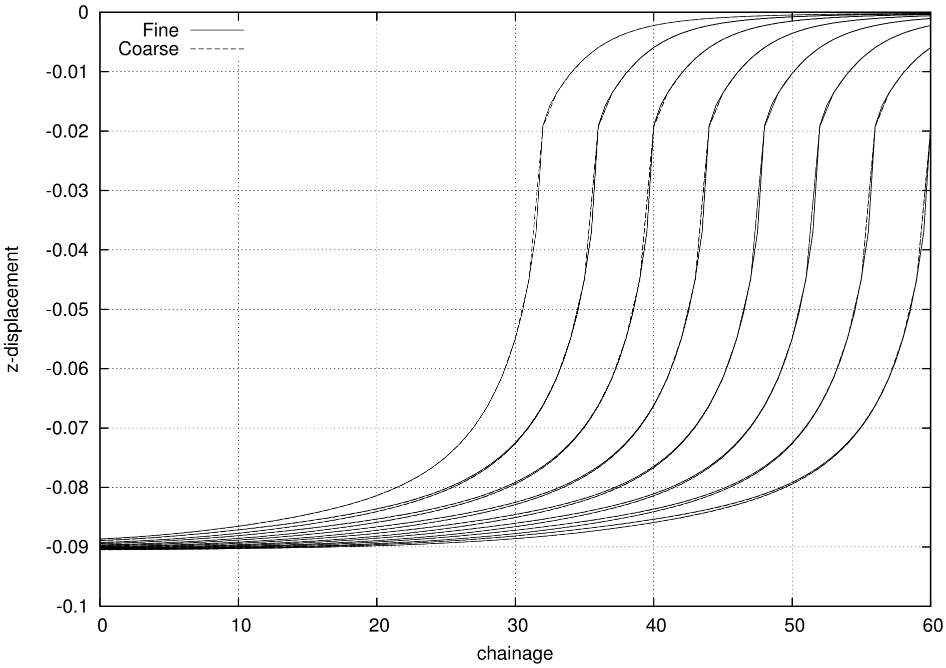

Figure 6: Displacements u_z at the crown of the tunnel

In Table 1 the number of DOFs for the infinite region and for the size of the global system of equation are shown for the coarse and fine discretisation. Three types of interpolation are choosen for the evaluation of this example - Continuous, Mixed and Discontinuous. For the Continuous type continuous interpolation is applied for the displacement and traction field, the Mixed type uses continuous interpolation for the displacement field and discontinuos for the traction field. Within the Discontinuous type both fields are interpolated discontinuous. The order of interpolation is choosen quadratic for all types of interpolation.

Table 1: Degrees of Freedom

	infinite Region				Global DOFs	
	Dirichlet DOFs		Neumann DOFs			
	Coarse	Fine	Coarse	Fine	Coarse	Fine
Continous	3459	13755	3459	13755	5403	19785
Mixed	3459	13755	9192	36624	5403	19785
Discontinous	9192	36624	9192	36624	12048	48192

In relation to the numbers of DOFs of Table 1 in Table 2 the calculation times are shown for the methods IC and BETI. The calculations are done with the parallised version of the code using 24 CPUs. As can be seen for all discretisations and interpolation types the BETI method is faster then the IC method. According to our experiences the difference in computing time of both methods is getting greater if the number of load cases is increased. The reason for this is the repeated calculation of region stiffness matrices due to changing boundary conditions for the method IC.

Table 2: Calculation Time

	Calculation Time			
	IC		BETI	
	Coarse	Fine	Coarse	Fine
Continous	140	2027	59	524
Mixed	241	3613	136	1116
Discontinous	394	8706	215	3034

5 CONCLUSIONS

In the present work two coupling methods are investigated, the IC method and the BETI method. Due to the formulation of stiffness matrices based on work equivalent nodal point forces both methods are able to couple BE regions (BEM/BEM coupling) and BE regions to FE regions (BEM/FEM coupling). The main differences of both methods are worked out and on a practical 3D tunnel excavation example results are shown. Particular attention has been paid to the computational performance of both coupling techniques.

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