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NUMERICAL STABILITY OF A FIXED POINT ITERATIVE METHOD TO DETERMINE PATTERNS OF TURBULENT FLOW IN A RECTANGULAR CAVITY WITH DIFFERENT ASPECT RATIOS

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Abstract. 2D isothermal viscous incompressible flows are presented from the Navier-Stokes equations in the Stream function-vorticity formulation and in the velocity-vorticity formulation. The simulation is made using a numerical method based on a fixed point iterative process to solve the nonlinear elliptic system that results after time discretization. The iterative process leads us to the solution of uncoupled, well-conditioned, symmetric linear elliptic problems from which efficient solvers exist regardless of the space discretization. The experiments take place on the lid driven cavity problem for Reynolds numbers up to $Re = 10000$ and different aspect ratios A (A=ratio of the height to the width) $A = 1$ and $A \neq 1$ such as $A = 1/2$, till $A = 3$. It appears that with velocity and vorticity variables is more difficult to solve this kind of flows, at least with a numerical procedure similar to the one applied in stream function and vorticity variables to solve an analogous nonlinear elliptic system. To obtain such flows is not an easy task, especially with the velocity-vorticity formulation. We report here results for moderate Reynolds numbers ($Re \leq 10000$), although with them enough effectiveness is achieved to be able to vary the aspect ratio of the cavity A, which causes the flow to be more unstable. Contribution in this work is to consider rectangular cavities of drag, which can impact on isothermal turbulent flow patterns. Another contribution is to include a wide region of the Reynolds number as well as different aspect ratios where we tested stability of the numerical scheme.

1 INTRODUCTION

In this work, we are dealing with the Navier-Stokes in two different formulations: The Stream Function-vorticity and the Velocity-vorticity formulation. The problem we are going to solve is the well known lid driven cavity problem, with Reynolds numbers $Re \leq 10000$ and different aspect ratios A (A=ratio of the height to the width) of the cavity.

Results, in both formulations, are obtained using a simple numerical scheme based on a fixed point iterative process [1], applied to a nonlinear elliptic system resulting after time discretization. The scheme has shown to be robust enough to handle such Reynolds numbers, ([2] , and [3]) and different aspect ratios of the cavity [5].

Since we are working with Reynolds number up to 10000, as this number increases the mesh has to be refined and a smaller time step has to be used, numerically, by stability matters and physically, to capture the fast dynamics of the flow, as pointed out in ([2], [3]). With the Velocity-vorticity formulation ([6], [7], [4]), a finer mesh and a smaller time step has to be used, and because of this, computing time is in general very large.

2 Mathematical Models

Let $\Omega \subset R^N$ $(N = 2, 3)$ the region of a non-steady, viscous, incompressible flow, and Γ its boundary.

$$\left\{ \begin{array}{lll} \frac{\partial u}{\partial t} - \nabla^2 \mathbf{u} + \nabla p + (\mathbf{u} \cdot \nabla)\mathbf{u} & = f, & (a) \\ \nabla \cdot \mathbf{u} & = 0 & (b) \end{array} \right. \tag{1}$$

which are the Navier-Stokes equations in the primitive variables formulation. The system has to be supplemented with appropriate boundary and initial conditions.

2.1 Stream function-vorticity Formulation

First we are going to speak about the Stream function-vorticity formulation.

In this case, we will restrict ourselves to a bidimensional region Ω. Taking the curl in both sides of the equation $(1a)$ and taking into account that

$$\left\{ u_1 = \frac{\partial \psi}{\partial y}, \quad u_2 = -\frac{\partial \psi}{\partial x}, \right. \tag{2}$$

which follows from $(1b)$, with ψ the stream function and u_1, u_2, the two components of the velocity, we get:

$$\left\{ \begin{array}{ll} \nabla^2 \psi \qquad\qquad\qquad = -\omega & (a) \\ \frac{\partial \omega}{\partial t} - \frac{1}{Re}\nabla^2 \omega + \mathbf{u} \cdot \nabla \omega = f_\omega & (b) \end{array} \right. \tag{3}$$

where ω is the vorticity ($\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$). These are the Navier-Stokes equations in the Stream function-vorticity formulation. The incompressibility condition $(1b)$, by (2) is automatically satisfied, and the pressure does not appear any more, which is a great advantage with respect to the primitive variables formulation.

2.2 Velocity-Vorticity Formulation

Taking the curl in

$$\omega = -\nabla \times \boldsymbol{u} \tag{4}$$

and using the identity $\nabla \times \nabla \times \mathbf{a} = -\nabla^2 \mathbf{a} + \nabla(\nabla \cdot \mathbf{a})$ and $(1b)$, a velocity Poisson equation results:

$$\nabla^2 \mathbf{u} = -\nabla \times \boldsymbol{\omega}. \tag{5}$$

Two Poisson equations for the velocity components are obtained, which together with the equation for the vorticity gives us

$$\left\{ \begin{array}{ll} \frac{\partial u_1}{\partial t} + \nabla^2 u_1 = -\frac{\partial \omega}{\partial y} & (a) \\ \frac{\partial u_2}{\partial t} + \nabla^2 u_2 = \frac{\partial \omega}{\partial x} \,. & (b) \\ \frac{\partial \omega}{\partial t} - \frac{1}{Re}\nabla^2 \omega + \mathbf{u} \cdot \nabla \omega = f_\omega & (c) \end{array} \right. \tag{6}$$

These are de Navier-Stokes equations in the Velocity-vorticity Formulation.

3 The Numerical Scheme.

Next, we are going to describe the numerical method used for solving the Navier-Stokes equations in both formulations. For the time derivative appearing in the vorticity equation in both schemes, the following second order approximation is used:

$$\frac{\partial f}{\partial t}(\mathbf{x}, (n+1)\Delta t) = \frac{3f^{n+1} - 4f^n + f^{n-1}}{2\Delta t} \tag{7}$$

where $\mathbf{x} \,\epsilon\, \Omega$, $n \geq 1$, Δt denotes the time step, and $f^r \approx f(\mathbf{x}, r\Delta t)$, assuming f is smooth enough.

3.1 Stream function-Vorticity formulation

Speaking about the Stream function-vorticity formulation, it can be observed that the following nonlinear elliptic system has to be solved at each time level:

$$\left\{ \begin{array}{ll} \nabla^2 \psi = -\omega, \quad \psi|_\Gamma = \psi_{bc}; & (a) \\ \alpha\omega - \nu\nabla^2 \omega + \mathbf{u} \cdot \nabla \omega = f_\omega, \quad \omega|_\Gamma = \omega_{bc}\,, & (b) \end{array} \right. \tag{8}$$

where $\alpha = \frac{3}{2\Delta t}$, $\nu = \frac{1}{Re}$ and $f_\omega = \frac{4\omega^n - \omega^{n-1}}{2\Delta t}$. To obtain (ψ^1, ω^1), the first subinterval is divided into M subintervals, and a first order scheme, such as Euler, is applied to each of the M subintervals.

Next, we define R_ω by:

$$R_\omega(\omega, \psi) \equiv \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega - f_\omega \; . \tag{9}$$

System (8) results equivalent to:

$$\left\{ \begin{array}{lcl} \nabla^2\psi & = -\omega & in\ \Omega\,, \quad \psi = \psi_{bc} \quad on\ \ \Gamma \\ R_\omega(\omega,\psi) & = \;\; 0 & in\ \Omega \quad\ \omega|_\Gamma = \omega_{bc} \end{array} \right. \tag{10}$$

Now, for solving this system at time level (n+1) the following fixed point iterative process [1] is used:

Given $\omega^{n,0} = \omega^n$, $\psi^{n,0} = \psi^n$ *solve until convergence in* ω *and* ψ

$$\left\{ \begin{array}{l} \nabla^2\psi^{n,m+1} = -\omega^{n,m} \quad in\ \Omega, \\ \psi^{n,m+1} = \psi_{bc}^{n,m+1}\ on\ \Gamma \\ (\alpha I - \nu\nabla^2)\omega^{n,m+1} = (\alpha I - \nu\nabla^2)\omega^{n,m} - \rho_\omega R_\omega(\omega^m, \psi^{n,m+1})\ in\ \Omega, \\ \omega^{n,m+1} = \omega_{bc}^{n,m+1}\ on\ \Gamma,\ \rho_\omega > 0. \end{array} \right. \tag{11}$$

and then, take $(\omega^{n+1}, \psi^{n+1}) = (\omega^{n,m+1}, \psi^{n,m+1})$.

3.2 The Velocity-Vorticity Formulation

In the case of the Velocity-Vorticity formulation, what we do is the following:

For the time derivatives appearing in the vorticity equation (7) is used, and the following semidiscretized system is obtained, in Ω,

$$\left\{ \begin{array}{ll} \frac{\partial u_1}{\partial t} + \nabla^2 u_1 & = -\frac{\partial\omega}{\partial y} \\ \frac{\partial u_2}{\partial t} + \nabla^2 u_2 & = \frac{\partial\omega}{\partial x}, \quad \mathbf{u}^{n+1}|_\Gamma = \mathbf{u}_{bc} \\ R_\omega(\omega, \mathbf{u}) = \mathbf{0}, & \omega|_{\mathbf{\Gamma}} = \omega_{\mathbf{bc}} \end{array} \right. \tag{12}$$

where

$$R_\omega(\omega, \mathbf{u}) \equiv \alpha\omega - \nu\nabla^{\mathbf{2}}\omega + \mathbf{u}\cdot\nabla\omega - \mathbf{f}_\omega \; , \tag{13}$$

Using again the fixed point iterative method described above , we get:

Given $\omega^{n,0} = \omega^n$, $u_1^{n,0} = u_1^n$, $u_2^{n,0} = u_2^n$ *solve until convergence on* ω, u_1 and u_2

$$\begin{cases} \frac{\partial u_1^{n,m+1}}{\partial t} + \nabla^2 u_1^{n,m+1} = -\frac{\partial \omega^{n,m}}{\partial y} \\ \frac{\partial u_2^{n,m+1}}{\partial t} + \nabla^2 u_2^{n,m+1} = \frac{\partial \omega^{n,m}}{\partial x}, \qquad \mathbf{u}^{n,m+1}|_\Gamma = \mathbf{u}_{bc}^{n,m+1} \\ (\alpha I - \nu \nabla^2)\omega^{n,m+1} = (\alpha I - \Delta)\omega^m - \rho_\omega R_\omega(\omega^{n,m}, \mathbf{u}^{n,m+1}), \\ \quad \rho_\omega > 0, \ \omega^{n,m+1}|_\Gamma = \omega_{bc}^{n,m}. \end{cases} \tag{14}$$

and then, take $(\omega^{n+1}, u_1^{n+1}, u_2^{n+1}) = (\omega^{n,m+1}, u_1^{n,m+1}, u_2^{n,m+1})$.

4 Numerical experiments

The numerical experiments take place in rectangular domains $\Omega = (a,b) \times (c,d)$, in connection with the lid-driven cavity problem. The boundary condition of $\mathbf{u}$ is given by $\mathbf{u} = (1,0)$ at the moving boundary $y = d$ and $\mathbf{u} = (0,0)$ elsewhere.

A translation of the boundary condition in terms of the velocity primitive variable $\mathbf{u}$ to the $\psi - \omega$ variables has to be performed when using the Stream function-vorticity formulation. Following [8], $\psi = 0$ is chosen on Γ, and by Taylor expansion of $(8a)$ on the boundary, with h_x and h_y the space steps, one obtains:

$$\begin{cases} \omega(0,y,t) = -\frac{1}{2h_x^2}[8\psi(h_x,y,t) - \psi(2h_x,y,t)] + O(h_x^2) \\ \omega(a,y,t) = -\frac{1}{2h_x^2}[8\psi(a-h_x,y,t) - \psi(a-2h_x,y,t)] + O(h_x^2) \\ \omega(x,0,t) = -\frac{1}{2h_y^2}[8\psi(x,h_y,t) - \psi(x,2h_y,t)] + O(h_y^2) \\ \omega(x,b,t) = -\frac{1}{2h_y^2}[8\psi(x,b-h_y,t) - \psi(x,b-2h_y,t)] - \frac{3}{h_y} + O(h_y^2). \end{cases} \tag{15}$$

Now, for the Velocity-vorticity formulation, the boundary conditions are given by:

$$\begin{cases} \mathrm{u}_1 = 0, \ u_2 = 0, \ \omega = \frac{\partial u_2}{\partial x} \text{ on } \Gamma_x = a \\ \mathrm{u}_1 = 0, \ u_2 = 0, \ \omega = \frac{\partial u_2}{\partial x} \text{ on } \Gamma_x = b \\ \mathrm{u}_1 = 0, \ u_2 = 0, \ \omega = -\frac{\partial u_1}{\partial y} \text{ on } \Gamma_y = c \\ \mathrm{u}_1 = 1, \ u_2 = 0, \ \omega = \frac{\partial u_1}{\partial x} \text{ on } \Gamma_y = d \end{cases} \tag{16}$$

In Figure 1 we show results obtained for $Re = 1000$, $A = 1/2$, $Hx = Hy = 1/128$, $dt = .01$ arriving to $T = 100$ (reaching the steady state), and using the Stream Function-vorticity formulation. Left, we show the graphics of the Stream Function, and right the graphics of the isovorticity contours. We did a mesh independence study and results agree well.

In Figure 2, results are reported for $Re = 1000$ and the same parameters described above, but with aspect ratio $A = 2$.

Now, in the Figure 3, we report results again for $Re = 1000$ and with the same value of the parameters, but now the aspect ratio is $A = 3$.

Next, in Figure 4, results for $Re = 5000$, $A = 2$, $Hx = Hy = 1/512$, $dt = .001$ arriving to $T = 100$.

In Figures 5 and 6, we report results for $Re = 10000$, $Hx = Hy = 1/512$, $dt = .001$ arriving again to $T = 100$. In Figure 5 $A = 2$ and in Figure 6 $A = 3$.

All the above results were obtained using the Stream function-vorticity formulation. And as one can notice, the mesh is not very fine, even for $Re = 10000$ and dt is not so small.

In Figures 7 and 8 we sow results using the Velocity-vorticity formulation. In Figure 7 for $Re = 1000$, $A = 1/2$, $Hx = Hy = 1/256$, $dt = .0001$, $T = 100$. Comparing the results obtained with the Stream Function-vorticity formulation, shown in Figure 1, it can be observed that in Figure 1 more contours appear, and the values of Hx and Hy are the half of the latter and dt ten times higher.

In Figure 8 for $Re = 5000$, $A = 2$, $Hx = Hy = 1/512$ $dt = .0001$, and $T = 100$ also. Figure 4, obtained with the Stream Function-vorticity formulation results much more better, and dt is again is again ten times higher.

Results obtained with the Stream Function-vorticity formulation were obtained in much more less time, using a coarser mesh and more contours appear in the graphics. We could not arrive with the Velocity-vorticity formulation to results for $Re = 10000$ because of the finer mesh needed and also because the dt has to be much more smaller and too much computer time and memory is needed in this last case.

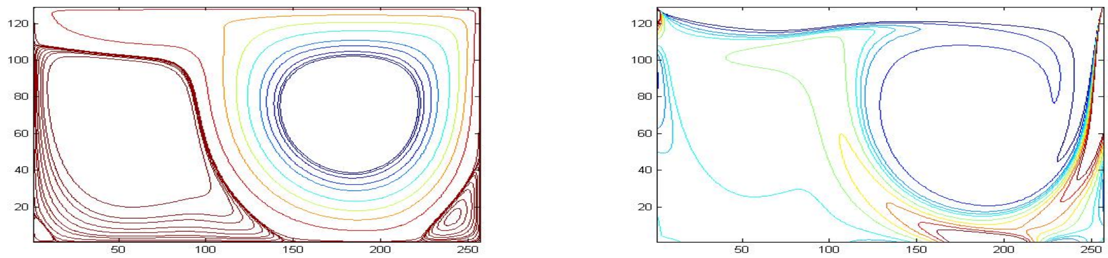

Figure 1: Stream function, left, and isovorticity contours, right, for $Re = 1000$ $A = 1/2$, $dt = .01$, $Hx = Hy = 1/128$, using the Stream function-vorticity formulation.

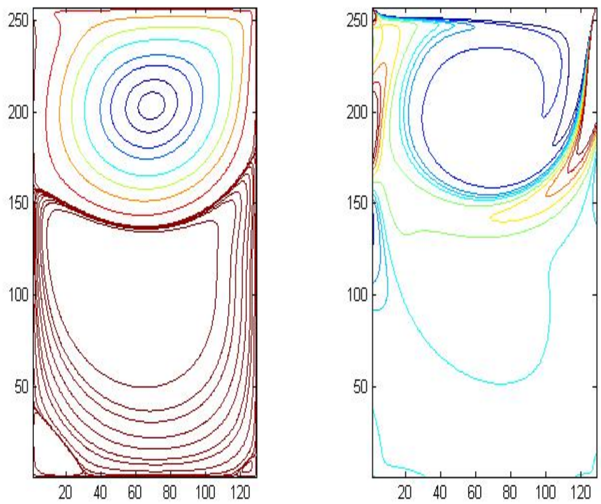

Figure 2: Stream function, left, and isovorticity contours, right, for $Re = 1000$ $A = 2$, $dt = .01$, $Hx = Hy = 1/128$, using the Stream-function-vorticity formulation.

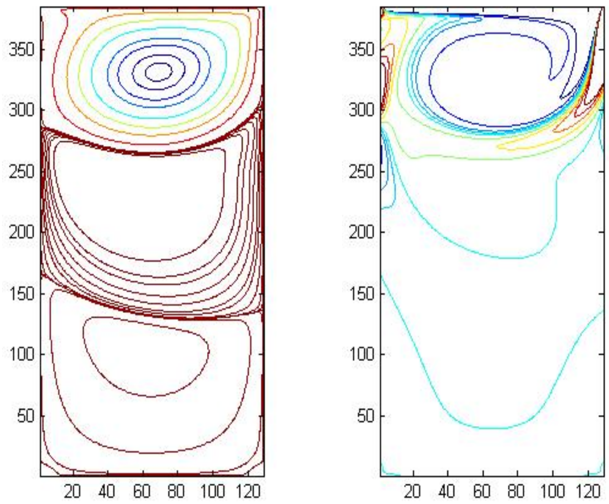

Figure 3: Stream function, left, and isovorticity contours, right, for $Re = 1000$ $A = 3$, $dt = .01$, $Hx = Hy = 1/128$, using the Stream function-vorticity formulation.

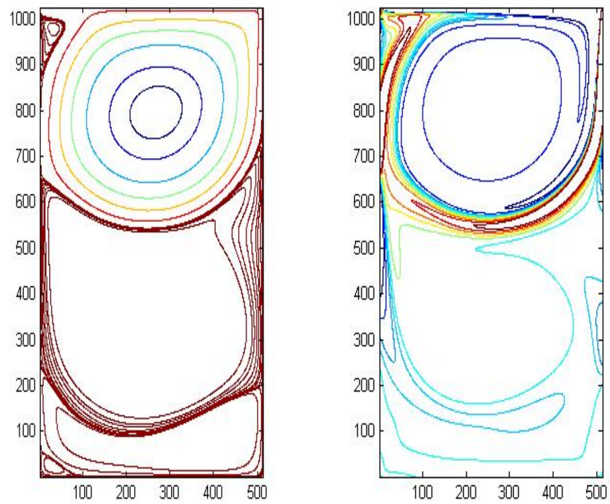

Figure 4: Stream function, left, and isovorticity contours, right, for $Re = 5000$ $A = 2$, $dt = .001$, $Hx = Hy = 1/512$, using the Stream function-vorticity formulation.

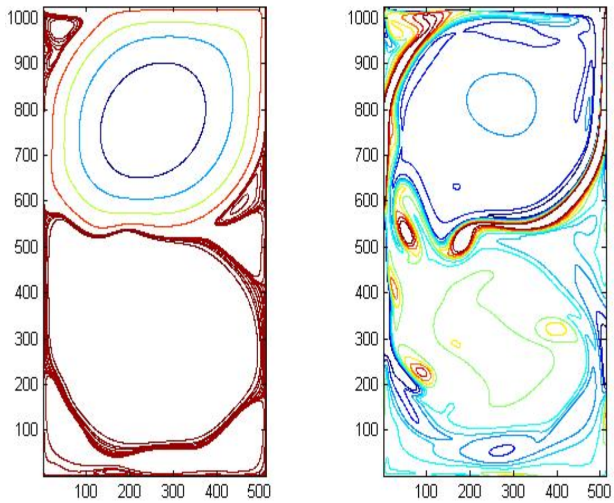

Figure 5: Stream function, left, and isovorticity contours, right, for $Re = 10000$ $A = 2$, $dt = .001$, $Hx = Hy = 1/512$, using the Stream-function-vorticity formulation.

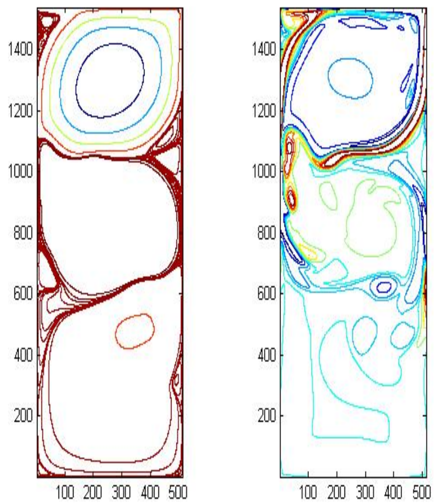

Figure 6: Stream function, left, and isovorticity contours, right, for $Re = 10000$ $A = 3$, $dt = .001$, $Hx = Hy = 1/512$, using the Stream function-vorticity formulation.

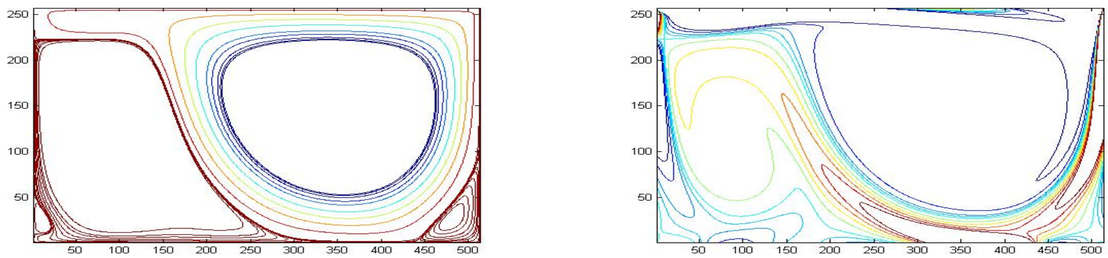

Figure 7: Stream function, left, and isovorticity contours, right, for $Re = 1000$ $A = 1/2$, $dt = .001$, $Hx = Hy = 1/256$, using the Velocity-vorticity formulation.

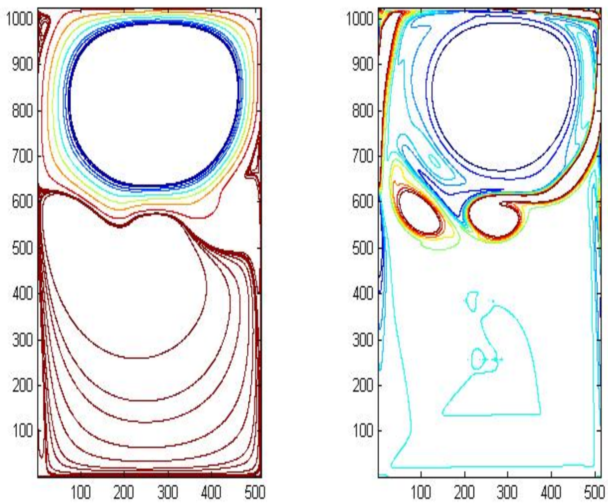

Figure 8: Stream function, left, and isovorticity contours, right, for $Re = 5000$ $A = 2$, $dt = .0001$, $Hx = Hy = 1/512$, using the Velocity-vorticity formulation.

5 Conclusions

We are presenting efficient numerical methods for solving the Navier-Stokes equations in the Stream funtion-vorticity and the Velocity-vorticity formulations. Both formulations provide turbulent flow patterns of the cavity for a wide range of Reynolds numbers. New results of flow profiles in conditions not included in the literature are presented ([7]).

We note that you for this range of numbers, stream flow patterns agree with those reported in the literature ([9]). However, the instrumentation for the Velocity-vorticity formulation is more complicated for effects of maintaining numerical stability. The time step, and the mesh size must be drastically reduced compared with those used in the Stream function-vorticity formulation. This results in a big cost of computing time and memory. In the Velocity-vorticity formulation the cavity flow lines are reproduced (not as well as with the Stream function-vorticity formulation using a coarser mesh size).

The numerical procedure applied to the Stream Function-vorticity formulation is not as good for the Velocity-vorticity formulation, however, the way it behaves, through the discretization parameters, and the order of discretization, gives us another point of view of the behavior of flows under different numerical methods and different formulations. The difficulty of the Velocity-vorticity formulation is reinforced through some works such as [9] who with a very different more sophisticated method reported driven cavity flows for moderate Reynolds numbers, lower than ours.

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