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ENERGY-MOMENTUM TIME INTEGRATIONS OF A NON-ISOTHERMAL TWO-PHASE DISSIPATION MODEL FOR FIBER-REINFORCED MATERIALS BASED ON A VIRTUAL POWER PRINCIPLE

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Abstract. Fiber-reinforced plastics (FRP) are composite materials made of an isotropic polymer matrix reinforced with organic or inorganic fibers. The major contribution to the internal dissipation in FRP is due to the isotropic matrix material. However, internal dissipation in the fibers has to be taken into account if Carbon or Kevlar fibers, respectively, are applied [1], or by using organic fibers, applied in the automotive industry [2]. Therefore, we introduce in this presentation a new two-phase dissipation model at finite strains with independent viscoelastic behaviour in the matrix and the fibers.

This model is based on the multiplicative split of the deformation gradient $\boldsymbol{F} := \boldsymbol{F}_e \boldsymbol{F}_v$ of the composite in an elastic and viscous deformation gradient $\boldsymbol{F}_\mathrm{e}$ and $\boldsymbol{F}_v$, respectively, as well as on the multiplicative split of the fiber deformation gradient $\boldsymbol{F}_F := \boldsymbol{F}_F^e \boldsymbol{F}_F^v$ in an elastic and viscous fiber deformation gradient $\boldsymbol{F}_F^e$ and $\boldsymbol{F}_F^v$, respectively. However, the time-dependent matrix behaviour depends directly on the principal invariants of the symmetric elastic right Cauchy-Green tensor $\boldsymbol{C}_\mathrm{e} := [\boldsymbol{F}_\mathrm{e}]^T \boldsymbol{F}_\mathrm{e}$. Analogously, the trace of the elastic fiber right Cauchy-Green tensor $\boldsymbol{C}_F^e := [\boldsymbol{F}_F^e]^T \boldsymbol{F}_F^e$ determines the time evolution of the viscous fiber deformation.

In the internal power of a mixed principle of virtual power, we consider free energy functions $\varPsi_M^{\mathrm{vis}}$ and $\varPsi_F^{\mathrm{vis}}$, depending on matrix and fiber invariants. In the external power, the non-negative internal dissipation with respect to a positive-definite viscosity tensor or a positive fiber viscosity parameter, respectively, is introduced. In this way, we derive the viscous evolution equations by variation. The internal power also includes mixed fields for the thermo-elastic matrix and fiber behaviour. Algorithmic terms in the virtual external power provide energy-momentum time integrations of this two-phase model.

1 INTRODUCTION

In Reference [3], a mixed finite element formulation for fiber-reinforced continua is presented, which is derived from a principle of virtual power. The reason for choosing this principle is the comfortable derivation of energy-momentum schemes of higher-order for coupled problems and mixed finite elements. This can be explained by the Simo-Taylor-Pister functional

$$\varPi_{\mathrm{STP}}(\boldsymbol{\varphi}, \tilde{J}, p) := \int_{\mathscr{B}_0} \varPsi^{\mathrm{iso}}(\boldsymbol{\varphi})\,\mathrm{d}V + \int_{\mathscr{B}_0} \varPsi^{\mathrm{vol}}(\tilde{J})\,\mathrm{d}V - \int_{\mathscr{B}_0} p\, G(\boldsymbol{\varphi}, \tilde{J})\mathrm{d}V \tag{1}$$

with $G(\boldsymbol{\varphi}, \tilde{J}) := \tilde{J} - \det(\nabla\boldsymbol{\varphi})$ in Reference [4]. This Hu-Washizu functional introduces an independent volume dilatation field $\tilde{J}$ and volume pressure field p, during a deformation $\boldsymbol{\varphi}$ of a reference configuration $\mathscr{B}_0$ with the isochoric strain energy function $\varPsi^{\mathrm{iso}}$ and the volumetric strain energy function $\varPsi^{\mathrm{vol}}$. The symbol ∇ denotes the partial derivative with respect to any material point $\boldsymbol{X} \in \mathscr{B}_0$. The energy-momentum time integration of this functional requires the preservation of the balance law corresponding to the strain energy density $\varPsi(\boldsymbol{\varphi}, \tilde{J}) := \varPsi^{\mathrm{iso}}(\boldsymbol{\varphi}) + \varPsi^{\mathrm{vol}}(\tilde{J})$ on a time step $[t_n, t_{n+1}]$, given by

$$\int_{\mathscr{B}_0} \varPsi(\boldsymbol{\varphi}_{t_{n+1}}, \tilde{J}_{t_{n+1}})\,\mathrm{d}V - \int_{\mathscr{B}_0} \varPsi(\boldsymbol{\varphi}_{t_n}, \tilde{J}_{t_n})\,\mathrm{d}V = \int_{t_n}^{t_{n+1}}\int_{\mathscr{B}_0} \left[\frac{\partial \varPsi}{\partial \boldsymbol{\varphi}} \cdot \dot{\boldsymbol{\varphi}} + \frac{\partial \varPsi}{\partial \tilde{J}}\,\dot{\tilde{J}}\right] \mathrm{d}V\mathrm{d}t \tag{2}$$

We denote by the symbol '·' a single tensor contraction. This balance law can be easily satisfied with the weak form

$$\int_{t_n}^{t_{n+1}}\int_{\mathscr{B}_0} \delta p\, \dot{G}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{J}})\,\mathrm{d}t = 0 \quad \text{with} \quad \dot{G}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{J}}) := \dot{\tilde{J}} - \overline{\det(\nabla\boldsymbol{\varphi})}^{\cdot} \tag{3}$$

by substituting the pressure p for the test function δp. Hence, it is more appropriate to enforce the constraint G on the velocity level as on the variable level, if an energy-momentum time integration of higher-order is the goal of the simulation. The weak form in Eq. (3) can be directly derived by the principle of virtual power in Reference [3].

2 THE PRINCIPLE OF VIRTUAL POWER

In Reference [3], damping in the fibers is neglected. Therefore, in this paper, we take into account a two-face model of damping, which assume an independent viscoelastic behaviour of the fibers and the matrix. According to Reference [3], we start with the kinetic power functional

$$\dot{\mathcal{T}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) := \int_{\mathscr{B}_0} [\rho_0\, \boldsymbol{v} - \boldsymbol{p}] \cdot \dot{\boldsymbol{v}}\,\mathrm{d}V - \int_{\mathscr{B}_0} \dot{\boldsymbol{p}} \cdot [\boldsymbol{v} - \dot{\boldsymbol{\varphi}}]\mathrm{d}V + \int_{\mathscr{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{\varphi}}\,\mathrm{d}V \tag{4}$$

where $\boldsymbol{v}$ and $\boldsymbol{p}$ denotes the Lagrangian velocity and momentum field, respectively, and ρ_0 the mass density in the initial configuration $\mathscr{B}_0$. A superimposed dot indicates a time derivative. Recall that according to Reference [3], the variation has to be performed with respect to the independent fields in the argument of the power functionals. This means, the variation is performed with respect to temporally continuous time rate fields and temporally discontinuous Lagrange multiplier fields. This variation is indicated by the symbol 'δ_*' [3]. The exception is the temporally discontinuous field $\tilde{\Theta}$ in the internal power functional

$$\begin{aligned}
\dot{\Pi}^{\text{int}} := {} & \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{\boldsymbol{F}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\boldsymbol{C}}_v, \dot{\tilde{C}}_V, \dot{\tilde{C}}_F, \dot{C}_F^v, \dot{\Theta}, \dot{\eta}, \tilde{\Theta}, \boldsymbol{P}, \boldsymbol{S}, S_V, S_F) \\
= {} & \frac{1}{2}\int_{\mathscr{B}_0}\Big[2\,\frac{\partial \Psi_M}{\partial \tilde{\boldsymbol{C}}} + S_V\,\boldsymbol{A}^{\text{vol}} + S_F\,\boldsymbol{A}_0 - \boldsymbol{S}\Big] : \dot{\tilde{\boldsymbol{C}}}\,\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0}\Big[2\,\frac{\partial \Psi_M}{\partial \tilde{C}_V} - S_V\Big]\dot{\tilde{C}}_V\,\mathrm{d}V \\
& + \frac{1}{2}\int_{\mathscr{B}_0}\Big[2\,\frac{\partial \Psi_F}{\partial \tilde{C}_F} - S_F\Big]\dot{\tilde{C}}_F\,\mathrm{d}V + \int_{\mathscr{B}_0}\Big[\tilde{\boldsymbol{F}}\boldsymbol{S} - \boldsymbol{P}\Big] : \dot{\tilde{\boldsymbol{F}}}\,\mathrm{d}V + \int_{\mathscr{B}_0}\frac{\partial \Psi_M}{\partial \boldsymbol{C}_v} : \dot{\boldsymbol{C}}_v\,\mathrm{d}V \\
& + \int_{\mathscr{B}_0}\boldsymbol{P} : \nabla\dot{\boldsymbol{\varphi}}\,\mathrm{d}V - \int_{\mathscr{B}_0}\dot{\eta}\Big[\tilde{\Theta} - \Theta\Big]\mathrm{d}V + \int_{\mathscr{B}_0}\Big[\eta + \frac{\partial \Psi}{\partial \Theta}\Big]\dot{\Theta}\,\mathrm{d}V + \int_{\mathscr{B}_0}\frac{\partial \Psi_F}{\partial C_F^v}\,\dot{C}_F^v\,\mathrm{d}V
\end{aligned} \tag{5}$$

which is no Lagrange multiplier, but replaces the time rate of a thermal displacement field. This is possible due to the fact, that the free energy does not depend on a thermal displacement field, but on the temperature field Θ. We denote the independent entropy density field by η, which is associated with the free energy $\Psi := \Psi_M + \Psi_F$ of the n_{dim}-dimensional body $\mathscr{B}_0$, additively split in the matrix function Ψ_M and the fiber function Ψ_F. The prescribed structural tensor of the fibers is denoted by $\boldsymbol{A}_0 = \hat{\boldsymbol{A}}_0(\boldsymbol{X})$, and

$$\boldsymbol{A}^{\text{vol}} := \frac{1}{n_{\text{dim}}}\,\det(\tilde{\boldsymbol{C}})^{\frac{1}{n_{\text{dim}}}}\,\tilde{\boldsymbol{C}}^{-1} \tag{6}$$

denotes the volumetric structural tensor with respect to the independent right Cauchy-Green tensor $\tilde{\boldsymbol{C}}$. We indicate by the superscript '-1' the inverse of a tensor, and by the symbol ':' a double tensor contraction. The independent scalar-valued fields S_F and S_V designate the volumetric and fiber stress, which are energy-conjugated to the fiber strain $\tilde{C}_F$ and the volumetric strain $\tilde{C}_V$. The total second Piola-Kirchhoff stress tensor is denoted by the independent second-order tensor field $\boldsymbol{S}$. In contrast to Reference [3], we introduce the independent deformation gradient field $\tilde{\boldsymbol{F}}$ and the independent first Piola-Kirchhoff stress field $\boldsymbol{P}$. In the last term of Eq. (5), we introduce an independent viscous internal variable C_F^v of the fibers, which is scalar-valued, in contrast to the symmetric internal variable tensor field $\boldsymbol{C}_v$ of the matrix material.

Conservative and non-conservative external force terms and Dirichlet as well as Neumann boundary terms are included in the external power functional. We take into account the conservative, mass-specific gravitational force $\boldsymbol{B}$ and the traction load $\bar{\boldsymbol{T}}$ on the Neumann boundary $\partial_T\mathscr{B}_0$. A transient inward boundary heat flux is introduced as thermal Neumann load $\bar{Q}$ on the boundary $\partial_Q\mathscr{B}_0$. The heat conduction law in the considered fiber-reinforced body $\mathscr{B}_0$ is given by

$$\boldsymbol{Q} := -\left[\frac{k_{F_0} - k_0}{\tilde{C}_F}\,\boldsymbol{A}_0 + k_0\,\tilde{\boldsymbol{C}}^{-1}\right]\nabla\Theta \tag{7}$$

where $k_{F_0} := \det(\tilde{\boldsymbol{F}})\,k_F$ and $k_0 := \det(\tilde{\boldsymbol{F}})\,k$ denote the material conductivity coefficients of the fibers and the matrix, respectively (cp. Reference [5]). The corresponding dissipation by conduction of heat is given by

$$D^{\mathrm{cdu}} := -\frac{1}{\Theta}\,\nabla\Theta\cdot\boldsymbol{Q} \tag{8}$$

The viscous driving force in the matrix material, defined by the tensor

$$\boldsymbol{\Sigma}_v := \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v \tag{9}$$

with respect to the positive-definite viscosity tensor

$$\mathbb{V}(\boldsymbol{C}_v) := \frac{1}{4}\left(V_{\mathrm{vol}} - \frac{V_{\mathrm{dev}}}{n_{\mathrm{dim}}}\right)\boldsymbol{C}_v^{-1}\otimes\boldsymbol{C}_v^{-1} + \frac{V_{\mathrm{dev}}}{4}\,\mathbb{I}^{\mathrm{sym}} : \boldsymbol{C}_v^{-1}\otimes\boldsymbol{C}_v^{-1} \tag{10}$$

is also introduced in the external power functional. The associated deviatoric and volumetric viscosity parameters are designated by $V_{\mathrm{dev}} > 0$ and $V_{\mathrm{vol}} > 0$, respectively. As Dirichlet boundaries, we take into account a heat sink of the body by means of a constant ambient temperature Θ_∞ and the corresponding boundary heat flux λ on the boundary $\partial_\Theta\mathscr{B}_0$. A temperature open-loop control on the boundary $\partial_{\dot{\Theta}}\mathscr{B}_0$ is realized by a prescribed temperature time evolution $\bar{\Theta}(t)$ with the corresponding boundary entropy h. And finally, a mechanical Dirichlet boundary $\partial_\varphi\mathscr{B}_0$ for introducing a bearing with the reaction force $\boldsymbol{R}$ is also taken into account. These external forces and boundaries leads to the external power functional

$$\begin{aligned}
\dot{\Pi}^{\mathrm{ext}} :=\; & \dot{\Pi}^{\mathrm{ext}}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\boldsymbol{C}}_v, \dot{\tilde{C}}_V, \dot{\tilde{C}}_F, \dot{C}_F^v, \dot{\Theta}, \tilde{\Theta}) \\
=\; & \int_{\mathscr{B}_0}\frac{1}{2}\,\bar{\boldsymbol{S}} : \dot{\tilde{\boldsymbol{C}}}\,\mathrm{d}V + \int_{\mathscr{B}_0}\frac{1}{2}\,\bar{S}_V\,\dot{\tilde{C}}_V\,\mathrm{d}V + \int_{\mathscr{B}_0}\frac{1}{2}\,\bar{S}_F\,\dot{\tilde{C}}_F\,\mathrm{d}V \\
& - \int_{\mathscr{B}_0}\rho_0\boldsymbol{B}\cdot\dot{\boldsymbol{\varphi}}\,\mathrm{d}V - \int_{\partial_T\mathscr{B}_0}\bar{\boldsymbol{T}}\cdot\dot{\boldsymbol{\varphi}}\,\mathrm{d}A + \int_{\partial_Q\mathscr{B}_0}\frac{\tilde{\Theta}}{\Theta}\,\bar{Q}\,\mathrm{d}A \\
& + \int_{\mathscr{B}_0}\frac{1}{\Theta}\,\nabla\tilde{\Theta}\cdot\boldsymbol{Q}\,\mathrm{d}V + \int_{\mathscr{B}_0}\frac{\tilde{\Theta}}{\Theta}\left(D^{\mathrm{cdu}} + D^{\mathrm{int}}\right)\mathrm{d}V + \int_{\mathscr{B}_0}\dot{\boldsymbol{C}}_v : \boldsymbol{\Sigma}_v\,\mathrm{d}V \\
& + \int_{\partial_\Theta\mathscr{B}_0}\lambda\left[\tilde{\Theta} - \Theta_\infty\right]\mathrm{d}A - \int_{\partial_{\dot{\Theta}}\mathscr{B}_0}h\left[\dot{\Theta} - \dot{\bar{\Theta}}\right]\mathrm{d}A - \int_{\partial_\varphi\mathscr{B}_0}\boldsymbol{R}\cdot\left[\dot{\boldsymbol{\varphi}} - \dot{\bar{\boldsymbol{\varphi}}}\right]\mathrm{d}A \\
& + \int_{\mathscr{B}_0}\dot{C}_F^v\,\Sigma_F^v\,\mathrm{d}V + \int_{\mathscr{B}_0}\bar{M}_F^v\left[L_F(\dot{\tilde{C}}_F) - L_F^v(\dot{C}_F^v)\right]\mathrm{d}V
\end{aligned} \tag{11}$$

where the first four rows with integral termes in Eq. (11) are also applied in Reference [3]. The first row with space integrals introduces algorithmic stresses for achieving an energy-momentum scheme. In this paper, however,

$$D^{\text{int}} := D_M^{\text{int}} + D_F^{\text{int}} \tag{12}$$

includes beside the internal dissipation

$$D_M^{\text{int}} := \dot{\boldsymbol{C}}_v : \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v \geq 0 \tag{13}$$

of the matrix material also the new internal dissipation D_F^{int} of the fibers. The last row of Eq. (11) is new and includes the viscous time evolution of the fiber material in the first term, and in the last term, an algorithmic stress in order to achieve the energy-momentum time integration of the viscoelastic fiber material. According to Reference [3], the space-time weak formulation follows from the incremental principle of virtual power

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \Big[\delta_* \dot{\mathcal{T}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) + \quad & \delta_* \dot{\Pi}^{\text{int}}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{\boldsymbol{F}}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\boldsymbol{C}}_v, \dot{\tilde{C}}_V, \dot{\tilde{C}}_F, \dot{C}_F^v, \dot{\Theta}, \dot{\eta}, \tilde{\Theta}, \boldsymbol{P}, \boldsymbol{S}, S_V, S_F) \\ + \quad & \delta_* \dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{\boldsymbol{C}}}, \dot{\boldsymbol{C}}_v, \dot{\tilde{C}}_V, \dot{\tilde{C}}_F, \dot{C}_F^v, \dot{\Theta}, \tilde{\Theta}) \Big] \, \mathrm{d}t = 0 \end{aligned} \tag{14}$$

As mentioned above, the viscoelastic constitutive formulation of the fibers is new, and therefore not described in Reference [3]. Therefore, we summarize this model in Section 3.

3 THE VISCOELASTIC CONSTITUTIVE MODEL OF THE FIBERS

In Reference [6] is already shown, that the derivation of a viscoelastic material model for fibers from a three-dimensional constitutive model for isotropic materials is successful. Therefore, we derive a fiber viscoelasticity formulation from the constitutive model of the matrix material in Reference [3].

3.1 The free energy

The idea is based on the well-known multiplicative split of the deformation gradient in a viscous and an elastic product tensor. In the case of the fiber deformation, we assume

$$\boldsymbol{F}_F = \boldsymbol{F}_F^e \boldsymbol{F}_F^v \tag{15}$$

with the fiber deformation gradients

$$\boldsymbol{F}_F := \boldsymbol{a}_t \otimes \boldsymbol{a}_0 \qquad \boldsymbol{F}_F^e := \boldsymbol{a}_t \otimes \bar{\boldsymbol{a}} \qquad \boldsymbol{F}_F^v := \bar{\boldsymbol{a}} \otimes \boldsymbol{a}_0 \tag{16}$$

defined by the stretched fiber direction vector $\boldsymbol{a}_t$ in the tangent space $T_x\mathscr{B}_t$ at the current configuration $\mathscr{B}_t$, the fiber direction vector $\bar{\boldsymbol{a}}$ in the linear space $\mathscr{V}_F$ ('intermediate fiber configuration') and the unit direction vector $\boldsymbol{a}_0$ with $\|\boldsymbol{a}_0\| = 1$ in the tangent space $T_X\mathscr{B}_0$ at the initial configuration $\mathscr{B}_0$ (see Figure 1). The symbol $\otimes$ denotes the standard dyadic

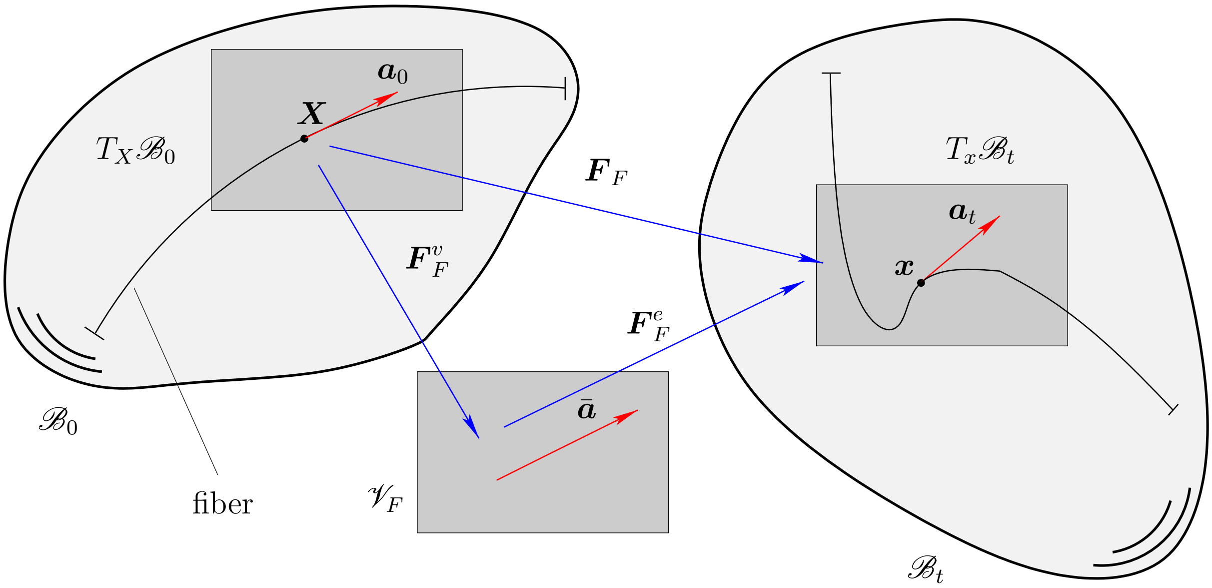

Figure 1: Fiber configurations with tangent spaces.

tensor product. In principle, we assume a generalized Maxwell element with an elastic branch and a viscoelastic branch. In the case of large deformations, we therefore define the fiber free energy

$$\boxed{\Psi_F = \Psi_F^{\text{ela}} + \Psi_F^{\text{vis}}} \tag{17}$$

The elastic free energy depends on the squared fiber stretch $\tilde{C}_F$, given by

$$\tilde{C}_F := \boldsymbol{a}_t \cdot \boldsymbol{a}_t = \boldsymbol{a}_0\, \boldsymbol{F}^T \cdot \boldsymbol{F} \boldsymbol{a}_0 = \boldsymbol{C} : \boldsymbol{a}_0 \otimes \boldsymbol{a}_0 = \boldsymbol{C} : \boldsymbol{A}_0 = \boldsymbol{C}_F : \boldsymbol{I} \tag{18}$$

with the structural tensor $\boldsymbol{A}_0 := \boldsymbol{a}_0 \otimes \boldsymbol{a}_0$ of the fiber family, the right Cauchy-Green tensor $\boldsymbol{C} := \boldsymbol{F}^T \boldsymbol{F}$ of the body $\mathscr{B}_0$, the unity metric tensor $\boldsymbol{I}$ with respect to $\mathscr{B}_0$, and the right Cauchy-Green tensor

$$\boldsymbol{C}_F := \boldsymbol{F}_F^T \boldsymbol{F}_F = [\boldsymbol{C} : \boldsymbol{A}_0]\, \boldsymbol{A}_0 \tag{19}$$

of the fiber family. The superscript 'T' denotes the transposition of a second-order tensor. Hence, we define a free energy function $\Psi_F^{\text{ela}} := \hat{\Psi}_F^{\text{ela}}(\tilde{C}_F)$. The viscoelastic free energy depends on the elastic fiber stretch

$$C_F^e := \boldsymbol{C}_F^e : \bar{\boldsymbol{I}} = [\boldsymbol{F}_F^e]^T\, \boldsymbol{F}_F^e : \bar{\boldsymbol{I}} = [\boldsymbol{F}_F^v]^{-T}\, \boldsymbol{F}^T \boldsymbol{F}\, [\boldsymbol{F}_F^v]^{-1} : \bar{\boldsymbol{I}} = [\boldsymbol{F}_F^v]^{-T}\, \boldsymbol{C}\, [\boldsymbol{F}_F^v]^{-1} : \bar{\boldsymbol{I}} \tag{20}$$

where $\bar{\boldsymbol{I}}$ designates the unity metric tensor of the linear space $\mathscr{V}_F$. By using the index notation of tensor products, we arrive at

$$C_F^e = \boldsymbol{C}^T : [\boldsymbol{F}_F^v]^{-1}\, [\boldsymbol{F}_F^v]^{-T} = \boldsymbol{C}^T : [\boldsymbol{C}_F^v]^{-1} = \boldsymbol{C}\, [\boldsymbol{C}_F^v]^{-1} : \boldsymbol{I} \tag{21}$$

where the second-order tensor $\boldsymbol{C}_F^v$ denotes the viscous right Cauchy-Green tensor

$$\boldsymbol{C}_F^v = [\boldsymbol{F}_F^v]^T \, \boldsymbol{F}_F^v = [\boldsymbol{a}_0 \otimes \bar{\boldsymbol{a}}] \cdot [\bar{\boldsymbol{a}} \otimes \boldsymbol{a}_0] = [\bar{\boldsymbol{a}} \cdot \bar{\boldsymbol{a}}] \, \boldsymbol{A}_0 =: C_F^v \, \boldsymbol{A}_0 \tag{22}$$

The scalar-valued field C_F^v denotes the new viscous internal variable of the fiber family. The inverse of the viscous right Cauchy-Green tensor of the fiber family is defined by

$$[\boldsymbol{C}_F^v]^{-1} := [\boldsymbol{F}_F^v]^{-1} \, [\boldsymbol{F}_F^v]^{-T} \tag{23}$$

where the inverse of the deformation gradient $\boldsymbol{F}_F^v$ satisfies the relation

$$\boldsymbol{a}_0 = [\boldsymbol{F}_F^v]^{-1} \, \bar{\boldsymbol{a}} =: \bar{\lambda}_F \, [\boldsymbol{a}_0 \otimes \bar{\boldsymbol{a}}] \cdot \bar{\boldsymbol{a}} = \bar{\lambda}_F \, [\bar{\boldsymbol{a}} \cdot \bar{\boldsymbol{a}}] \, \boldsymbol{a}_0 = \bar{\lambda}_F \, C_F^v \, \boldsymbol{a}_0 \tag{24}$$

Thus, the multiplier $\bar{\lambda}_F$ denotes the inverse of the viscous internal variable, and Eq. (23) reads

$$[\boldsymbol{C}_F^v]^{-1} = \frac{1}{(C_F^v)^2} \, [\boldsymbol{a}_0 \otimes \bar{\boldsymbol{a}}] \cdot [\bar{\boldsymbol{a}} \otimes \boldsymbol{a}_0] = \frac{1}{(C_F^v)^2} \, [\boldsymbol{a}_0 \otimes \boldsymbol{a}_0] \, C_F^v = \frac{1}{C_F^v} \, \boldsymbol{A}_0 \tag{25}$$

Employing Eq. (25) in Eq. (21), we obtain the elastic fiber stretch

$$C_F^e = \boldsymbol{C} \, \frac{1}{C_F^v} \, \boldsymbol{A}_0 : \boldsymbol{I} = [\boldsymbol{C} : \boldsymbol{A}_0] \, \frac{1}{C_F^v} = \frac{\tilde{C}_F}{C_F^v} \tag{26}$$

Accordingly, we define the viscous free energy function $\Psi_F^{\mathrm{vis}} := \hat{\Psi}_F^{\mathrm{vis}}(C_F^e) = \hat{\Psi}_F^{\mathrm{ela}}(\tilde{C}_F[C_F^v]^{-1})$, which means we apply the same free energy function as in the elastic branch, but with a different argument and generally different material constants (cf. Reference [7]).

3.2 The viscous dissipation and evolution equation

According to the Clausius-Plank inequality and the definition of the entropy, the viscous internal dissipation in the fibers is given by

$$D_F^{\mathrm{int}} := \frac{1}{2} \, S_F \, \dot{\tilde{C}}_F - \dot{\Psi}_F^{\mathrm{ela}} - \dot{\Psi}_F^{\mathrm{vis}} = \tilde{C}_F \, S_F \, \frac{\dot{\tilde{C}}_F}{2 \, \tilde{C}_F} - \dot{\Psi}_F^{\mathrm{ela}} - \dot{\Psi}_F^{\mathrm{vis}} = M_F \, L_F - \dot{\Psi}_F^{\mathrm{ela}} - \dot{\Psi}_F^{\mathrm{vis}} \geqslant 0 \tag{27}$$

Here, we introduced the Mandel stress $M_F := \tilde{C}_F \, S_F$, energy-conjugated to the fiber strain rate L_F. The time rate of the free energy pertaining to the elastic branch takes the form

$$\dot{\Psi}_F^{\mathrm{ela}} = 2 \, \frac{\partial \Psi_F^{\mathrm{ela}}}{\partial \tilde{C}_F} \, \frac{\dot{\tilde{C}}_F}{2} = 2 \, \tilde{C}_F \, \frac{\partial \Psi_F^{\mathrm{ela}}}{\partial \tilde{C}_F} \, L_F =: M_F^{\mathrm{ela}} \, L_F \tag{28}$$

and the time rate of the free energy associated with the viscoelastic branch reads

$$\dot{\Psi}_F^{\mathrm{vis}} = \frac{\partial \Psi_F^{\mathrm{vis}}}{\partial C_F^e} \, \dot{\overline{C_F^e}} = 2 \, C_F^e \, \frac{\partial \Psi_F^{\mathrm{vis}}}{\partial C_F^e} \left[\frac{\dot{\tilde{C}}_F}{2 \, C_F} - \frac{\dot{C}_F^v}{2 \, C_F^v} \right] =: M_F^{\mathrm{vis}} \, [L_F - L_F^v] \tag{29}$$

Consequently, the viscous dissipation in the fibers are given by

$$D_F^{\text{int}} := \left[M_F - M_F^{\text{ela}} - M_F^{\text{vis}}\right] L_F + M_F^{\text{vis}}\, L_F^v \geqslant 0 \tag{30}$$

The Clausius-Plank inequality can be fulfilled by setting

$$M_F := M_F^{\text{ela}} + M_F^{\text{vis}} \tag{31}$$

and by defining the viscous evolution equation

$$\boxed{M_F^{\text{vis}} \doteq \Sigma_F^v} \qquad \text{with} \quad \Sigma_F^v := V_F\, L_F^v \tag{32}$$

as viscous driving force associated with the viscosity constant $V_F > 0$. The viscous fiber dissipation then takes the form $D_F^{\text{vis}} = V_F (L_F^v)^2 \geqslant 0$. Eq. (32) represents an ordinary differential equation for the viscous internal variable C_F^v, which reveals the equivalent form of the viscous fiber dissipation

$$M_F^{\text{vis}}\, L_F^v = \frac{C_F^e}{C_F^v} \frac{\partial \Psi_F^{\text{vis}}}{\partial C_F^e} \dot{C}_F^v = -\frac{\partial \Psi_F^{\text{vis}}}{\partial C_F^v} \dot{C}_F^v \quad \text{and} \quad V_F (L_F^v)^2 = \frac{V_F}{(2\, C_F^v)^2} \dot{C}_F^v\, \dot{C}_F^v \tag{33}$$

By taking into account Eq. (33), the viscous evolution equation can be written as

$$Y_F \doteq \frac{V_F}{(2\, C_F^v)^2} \dot{C}_F^v \qquad \text{with} \quad Y_F := -\frac{\partial \Psi_F^{\text{vis}}}{\partial C_F^v} \tag{34}$$

as non-equilibrium stress of the fiber family.

4 THE VISCOUS ALGORITHMIC FIBER STRESS

The energy-momentum time integration of the viscoelastic fiber formulation requires the exact fulfillment of the gradient theorem

$$\Psi^{\text{vis}}_{F_{t_{n+1}}} - \Psi^{\text{vis}}_{F_{t_n}} = \int_{t_n}^{t_{n+1}} \dot{\Psi}_F^{\text{vis}}\, \mathrm{d}t \equiv \int_0^1 \overset{\circ}{\overline{\Psi_F^{\text{vis}}}}\, \mathrm{d}\alpha \tag{35}$$

on each time step $[t_n, t_{n+1}]$, or on the normalized time interval $[0, 1]$, respectively (compare Reference [3]). We indicate by a superimposed $\circ$ the derivative with respect to $\alpha \in [0, 1]$. We satisfy this constraint by means of the algorithmic Mandel stress $\bar{M}_F^v$ in the external power of the variational formulation.

Remark 4.1 *In Reference [3], the energy-momentum time integration of the viscous free energy of the matrix material is only based on the algorithmic elastic Mandel stress, which modifies the equations of motion only. This leads in the isothermal case to an unmodified viscous evolution equation of the matrix material. Motivated by References [8],in which the viscous evolution equation is modified by a discrete derivative, we here modify also the viscous Mandel stress of the fibers. In this way, the equations of motion as well as the viscous evolution equation are modified. Note that, in contrast to References [8],the modification in this paper is higher-order accurate, and not restricted to second-order accurate energy-momentum time integrations.*

As described in detail in Reference [9], we determine the variational parameter field $\bar{M}_F^v$ by using a separate constrained variational problem. Here, we define the Lagrange functional

$$\mathcal{F}_F^v(\bar{M}_F^v, \mu_F^v) := \mu_F^v \, \mathcal{G}_F^v(\bar{M}_F^v) + \int_0^1 F_F^v(\bar{M}_F^v) \, \mathrm{d}\alpha \tag{36}$$

with the Lagrange multiplier μ_F^v and the definitions

$$F_F^v(\bar{M}_F^v) := \frac{(\bar{M}_F^v)^2}{2} \qquad \text{and} \qquad \mathcal{G}_F^v(\bar{M}_F^v) := \Psi^{\text{vis}}_{F_{t_{n+1}}} - \Psi^{\text{vis}}_{F_{t_n}} - \int_0^1 G_F^v(\bar{M}_F^v) \, \mathrm{d}\alpha \tag{37}$$

with the constraint function

$$G_F^v(\bar{M}_F^v) := \left[M_F^{\text{vis}} + \bar{M}_F^v\right] [L_F - L_F^v] \tag{38}$$

A functional minimization subject to the constraint $\mathcal{G}_F^v = 0$ leads to the Euler-Lagrange equations

$$\frac{\delta_* \mathcal{F}_F^v}{\delta_* \bar{M}_F^v} \equiv \int_0^1 \frac{\partial F_F^v}{\partial \bar{M}_F^v} \, \delta_* \bar{M}_F^v \, \mathrm{d}\alpha - \mu_F^v \int_0^1 \frac{\partial G_F^v}{\partial \bar{M}_F^v} \, \delta_* \bar{M}_F^v \, \mathrm{d}\alpha = 0 \qquad \frac{\delta_* \mathcal{F}_F^v}{\delta_* \mu_F^v} \equiv \mathcal{G}_F^v = 0 \tag{39}$$

where

$$\frac{\partial F_F^v}{\partial \bar{M}_F^v} = \bar{M}_F^v \qquad \frac{\partial G_F^v}{\partial \bar{M}_F^v} = L_F - L_F^v \tag{40}$$

These Euler-Lagrange equations lead to the algorithmic Mandel stress function

$$\boxed{\bar{M}_F^v := \mu_F^v \, [L_F - L_F^v]} \tag{41}$$

with the Lagrange multiplier

$$\mu_F^v := \frac{\Psi^{\text{vis}}_{F_{t_{n+1}}} - \Psi^{\text{vis}}_{F_{t_n}} - \displaystyle\int_0^1 M_F^{\text{vis}} \, [L_F - L_F^v] \, \mathrm{d}\alpha}{\displaystyle\int_0^1 [L_F - L_F^v]^2 \, \mathrm{d}\alpha} \tag{42}$$

Note that the Lagrange multiplier μ_F^v depends on the material point $\boldsymbol{X} \in \mathscr{B}_0$, but not on the normalized time $\alpha \in [0, 1]$.

5 THE WEAK FORMULATION

With the following exceptions, the weak formulation in Reference [3] applies also in this paper by taking into account the internal dissipation in Eq. (12) and the free energy in Eq. (17). First, in this paper, we obtain the weak equation of motion

$$\begin{aligned} &\int_{t_{n+1}}^{t_n} \int_{\mathscr{B}_0} \delta_* \dot{\boldsymbol{\varphi}} \cdot [\dot{\boldsymbol{p}} - \rho_0 \boldsymbol{B}] \, \mathrm{d}V \, \mathrm{d}\alpha \; + \int_{t_{n+1}}^{t_n} \int_{\mathscr{B}_0} \boldsymbol{P} : \nabla (\delta_* \dot{\boldsymbol{\varphi}}) \, \mathrm{d}V \, \mathrm{d}t \\ = \; &\int_{t_{n+1}}^{t_n} \int_{\partial_T \mathscr{B}_0} \delta_* \dot{\boldsymbol{\varphi}} \cdot \bar{\boldsymbol{T}} \, \mathrm{d}V \, \mathrm{d}\alpha + \int_{t_{n+1}}^{t_n} \int_{\partial_\varphi \mathscr{B}_0} \delta_* \dot{\boldsymbol{\varphi}} \cdot \boldsymbol{R} \, \mathrm{d}V \, \mathrm{d}t \end{aligned} \tag{43}$$

where the weak stress power term is formulated with the independent first Piola-Kirchhoff stress tensor. Second, the fiber stress equation includes the algorithmic viscous Mandel stress, so that we arrive at the weak form

$$\int_{t_{n+1}}^{t_n} \int_{\mathscr{B}_0} \delta_* \dot{\tilde{C}}_F \left[2\, \frac{\partial \Psi_F}{\partial \tilde{C}_F} + \bar{S}_F + \frac{\bar{M}_F^v}{\tilde{C}_F} - S_F \right] \mathrm{d}V \, \mathrm{d}t = 0 \tag{44}$$

where the algorithmic fiber stress $\bar{S}_F$ fulfills the gradient theorem with respect to Ψ_F^{ela} (see Reference [3]). As we introduce the deformation gradient and the first Piola-Kirchhoff stress tensor as independent fields, we obtain the additional weak forms

$$\int_{t_{n+1}}^{t_n} \int_{\mathscr{B}_0} \delta_* \boldsymbol{P} : \left[\dot{\tilde{\boldsymbol{F}}} - \nabla \dot{\boldsymbol{\varphi}} \right] \mathrm{d}V \, \mathrm{d}t = 0 \qquad 0 = \int_{t_{n+1}}^{t_n} \int_{\mathscr{B}_0} \delta_* \dot{\tilde{\boldsymbol{F}}} : \left[\tilde{\boldsymbol{F}} \boldsymbol{S} - \boldsymbol{P} \right] \mathrm{d}V \, \mathrm{d}t \tag{45}$$

Finally, we obtain the weak form of the new viscous evolution equation of the fiber family, given by

$$\int_{t_{n+1}}^{t_n} \int_{\mathscr{B}_0} \delta_* \dot{C}_F^v \left[-Y_F - \frac{\bar{M}_F^v}{2\, C_F^v} + \frac{V_F}{2\, C_F^v}\, L_F^v \right] \mathrm{d}V \, \mathrm{d}t = 0 \tag{46}$$

Note that a further advantage of the existence of the additional algorithmic stress $\bar{M}_F^v$ besides the algorithmic stress $\bar{S}_F$ is, that each free energy part of the fiber family possesses its own algorithmic modification. Hence, neglecting one part of the fiber free energy allows for the neglection of the corresponding algorithmic stress term.

6 THE DISCRETE VISCOUS EVOLUTION EQATION OF THE FIBERS

Usually, viscous evolution equations are solved at each Gaussian quadrature point in space, because the missing boundary conditions suggests to solve a viscous evolution equation pointwise as initial boundary value problem. But, due to the derivation of the viscous evolution equation as additional weak form in Eq. (46), it is also possible to solve the viscous evolution equation of the fibers as elementwise field equation, analogous to the fiber stress equation in Eq. (44). In this way, we obtain one matrix differential equation in time for each finite element, instead one scalar differential equation for each Gaussian quadrature point of a finite element. For higher order finite elements with many Gaussian quadrature points, this can be more efficient. Hence, the discretization of the viscous fiber evolution equation, in this paper, is distinct from the matrix evolution equation in Reference [3] in two ways:

1. the existence of an algorithmic stress in the viscous evolution equation,
2. the solution procedure by using spatially discretized discrete equations.

We discretize the viscous evolution equation in Eq. (46) on the parent domain $\mathscr{B}_\square$ associated with each finite element in space as well as with respect to the normalized time

$\alpha \in [0,1]$ on each time step $[t_n, t_{n+1}]$ with the time step size $h_n = t_{n+1} - t_n$ in time. Hence, we discretize the weak form

$$\int_0^1 \int_{\mathscr{B}_\square} \delta_* \overline{\overset{\circ}{C_F^v}} \left[-Y_F - \frac{\bar{M}_F^v}{2\, C_F^v} + \frac{V_F}{2\, C_F^v}\, L_F^v \right] \mathrm{d}V_\square \, \mathrm{d}\alpha = 0 \tag{47}$$

with the viscous strain rate

$$L_F^v = \frac{\overline{\overset{\circ}{C_F^v}}}{2\, h_n\, C_F^v} \tag{48}$$

As we aim at a higher-order accurate space-time approximation, we employ for the test function the Galerkin approximation

$$\delta_* \overline{\overset{\circ}{C_F^v}}\, (\boldsymbol{\chi}, \alpha) = \sum_{J=1}^{k} \sum_{B=1}^{\tilde{n}_{\text{node}}} \tilde{M}^J(\alpha) \tilde{N}_B(\boldsymbol{\chi})\, [\tilde{C}_F^v]_J^B = \left[\tilde{\mathbf{c}}(\alpha) \boxtimes \tilde{\mathbf{N}}(\boldsymbol{\chi}) \right]^{\mathsf{T}} \tilde{\mathbf{C}}_F^v \tag{49}$$

where $\tilde{M}^J(\alpha)$ denotes a one-dimensional Lagrange polynomial of degree $k-1$ with respect to α (see Reference [3]), and $\tilde{N}_B(\boldsymbol{\chi})$ the standard spatial shape functions with respect to the considered spatial parent domain (tetrahedral or hexahedral finite elements, for instance). In a matrix notation, we combine these temporal shape functions in the column vector $\tilde{\mathbf{c}}(\alpha)$, and the spatial shape functions in the column vector $\tilde{\mathbf{N}}(\boldsymbol{\chi})$. The nodal values of the variations of the internal variable field are combined in the column vector $\tilde{\mathbf{C}}_F^v$. Thereby, the symbol $\boxtimes$ denotes the standard Kronecker matrix product, and the superscript 'T' the transposition of a matrix. The temporally continuous viscous internal variable field is then approximated by

$$C_F^v(\boldsymbol{\chi}, \alpha) = \sum_{I=1}^{k+1} \sum_{A=1}^{\tilde{n}_{\text{node}}} M^I(\alpha) \tilde{N}_A(\boldsymbol{\chi})\, [C_F^v]_I^A = \left[\mathbf{c}(\alpha) \boxtimes \tilde{\mathbf{N}}(\boldsymbol{\chi}) \right]^{\mathsf{T}} \mathbf{C}_F^v + \left[M^1(\alpha) \boxtimes \tilde{\mathbf{N}}(\boldsymbol{\chi}) \mathbf{c}_F^v \right] \tag{50}$$

The polynomials $M^I(\alpha)$ denote the one-dimensional Lagrange polynomials of degree k with respect to α (see also Reference [3]). In the matrix notation, the unknown nodal values of the internal variable field are then included in the column vector $\mathbf{C}_F^v$, and the prescribed initial values in the column vector $\mathbf{c}_F^v$. The k polynomials $M^I(\alpha)$, $I = 2, \ldots, k+1$, are combined in the column vector $\mathbf{c}(\alpha)$. Employing Eqs. (49) and (50) in the weak form in Eq. (47), we arrive at the matrix equation

$$\boxed{\mathbf{Y}^{\text{mat}} + \mathbf{Y}^{\text{alg}} = \boldsymbol{\Sigma}} \tag{51}$$

The material stress vector $\mathbf{Y}^{\text{mat}}$ includes the viscous non-equilibrium stress, derived from the viscous free energy. The algorithmic stress vector $\mathbf{Y}^{\text{alg}}$ includes the algorithmic Mandel stress $\bar{M}_F^v$, and the viscous driving force $\boldsymbol{\Sigma}$ the viscosity constant V_F and the viscous strain rate L_F^v. For variational consistency, we apply exactly k Gauss points in time (see [3, 9]).

7 THE NUMERICAL EXAMPLE

As numerical examples, we consider examples from References [3, 9] for a direct comparison. These include a fiber-reinforced turbine rotor, whose mesh is derived from Reference [10]. In the numerical examples, we demonstrate the above introduced Dirichlet and Neumann boundary conditions as well as the obtained energy-momentum consistency.

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