VIII International Conference on Computational Methods for Coupled Problems in Science and Engineering
COUPLED PROBLEMS 2019
E. Oñate, M. Papadrakakis and B. Schrefler (Eds)

KINETIC CONSISTENT ALGORITHM FOR INCOMPRESSIBLE CONDUCTIVE FLUID

B. CHETVERUSHKIN*, A. SAVELIEV† AND V. SAVELIEV†

*Keldysh Institute of Applied Mathematics
Russian Academy of Science
4 Miusskaya sq., 125047 Moscow, Russia
e-mail: office@keldysh.ru, web page: http://www.keldysh.ru/

†Immanuil Kant Baltic Federal University
14 Al. Nevskogo, 236041 Kaliningrad, Russia
e-mail: Valeri.Saveliev@desy.de, Andrey.Saveliev@desy.de - Web page: http://www.kantiana.ru

Key words: Kinetic Methods, Kinetic Consistent Methods, High Performance Computing

Abstract. A kinetic model based on the single-particle Boltzmann-like distribution function is used to describe magneto gas dynamic phenomena. Along with the original kinetic consistent gas dynamic model, a simplified version that is more convenient for numerical implementation is considered and justified. Numerical results for a number of problems are presented, especially for the incompressible conductive fluid.

1 INTRODUCTION

The magneto gas dynamics (MGD) and magneto hydro dynamic (MHD) equations with allowance for dissipative effects (molecular and magnetic viscosity, heat conduction and others) are a mathematical models for describing technological processes and processes in fundamental science [1, 2, 3]. The modelling of many important problems requires computing technologies making use increasing capabilities of modern parallel high performance computers. Unfortunately, the use of high and ultrahigh performance of computing systems is hampered by difficulties associated with the adaptation of algorithms and software to the architecture of computers with extra-massive parallelism [4].

The novel kinetic consistent model of magneto gas and magneto hydro dynamic processes is proposed. The mathematical models are based on single-particle Maxwellian distribution function [5]:

$$f_{0\mathrm{M}} = \frac{\rho(t, \mathbf{x})}{(2\pi RT)^{\frac{3}{2}}} e^{-\frac{(\boldsymbol{\xi} - \mathbf{u} - i\mathbf{v}_{\alpha}^{\mathbf{2}})}{2RT}}, \tag{1}$$

here t is time, $\mathbf{x}$ is the spatial coordinate, $\boldsymbol{\xi}$ is the molecular velocity, ρ is the density, $\mathbf{u}$ is the macroscopic velocity, T is the temperature, R is the gas constant, and $\mathbf{v}_\alpha$ is the Alfven velocity given by

$$\mathbf{v}_\alpha = \frac{\mathbf{B}}{\sqrt{4\pi\rho}}\,, \tag{2}$$

where $\mathbf{B}$ is the magnetic intensity and i is the imaginary unit.

In contrast to the classical locally Maxwellian statistical distribution function, the exponent in function (1) has the term $i\mathbf{v}_\alpha$, which describe the relation between the charged particles in the electromagnetic field on the level of kinetic theory [6, 7].

The momentum of f_{0M}^j gives the macroscopic variables including the magnetic field:

$$\rho(t,\mathbf{x}) = \int m f_{0\mathrm{M}} \mathrm{d}\boldsymbol{\xi}\,, \tag{3}$$

$$\mathbf{u}(t,\mathbf{x}) = \frac{1}{\rho}\int m\boldsymbol{\xi} f_{0\mathrm{M}} \mathrm{d}\boldsymbol{\xi}\,, \tag{4}$$

$$E = \int \frac{1}{2} m\boldsymbol{\xi}^2 f_{0\mathrm{M}} \mathrm{d}\boldsymbol{\xi}\,, \tag{5}$$

$$\mathbf{B} = \frac{1}{\sqrt{\mu\rho}}\int m\boldsymbol{\xi}^* f_{0\mathrm{M}} \mathrm{d}\boldsymbol{\xi}\,, \tag{6}$$

here $E = \rho\frac{u^2}{2} + \epsilon + \frac{B^2}{8\pi}$ is the total energy, ϵ is internal energy, $\boldsymbol{\xi}^*$ is the complex conjugate of the molecular velocity $\boldsymbol{\xi}$ and the magnetic intensity $\mathbf{B}$ is defined as integral with conjugated velocity.

By using function (1) and a procedure similar for deriving the Euler equations from the Boltzmann one, the ideal magneto gas dynamic equations can be obtained from the kinetic equation. In view of these possibilities of describing magneto gas dynamic phenomena, we construct a model of dissipative magneto gas dynamic processes by using approaches developed for deriving the kinetic consistent gas dynamic system in [9].

2 MODEL OF MAGNETO GAS DYNAMIC PROCESSES

We can deriving the kinetic consistent magneto gas dynamic system of equations like kinetic consistent gas dynamic equations, consider the balance equation [6]

$$\frac{\tilde{f}^{j+1} - f_{0\mathrm{M}}^j}{\tau^*} + \xi_k \frac{\partial f_{0\mathrm{M}}^j}{\partial x_k} = \frac{\partial}{\partial x_k}\frac{\tau^*}{2}\xi_k\xi_p\frac{\partial f_{0\mathrm{M}}^j}{\partial x_p}\,, \tag{7}$$

where $\tilde{f}^{j+1}$ is the distribution function describing the behavior of an ensemble of charged particles and the magnetic intensity at the time $t = t^{j+1}$.

In a similar manner, kinetic consistent magneto gas dynamic system of equations are obtained by multiplying the balance equation (8) by the summation invariants and integration the results over all molecular velocities. However, there is a difference from the procedure for deriving the quasi gas dynamic system. In multiplication by m, $m\boldsymbol{\xi}, m\xi^2/2$, we specify τ as the time between molecular collisions as in the case of the quasi gas dynamic system. When (7) is multiplied by $m\boldsymbol{\xi}^*$, τ_{M} is defined from magneto dynamic processes τ_M, the value of τ^* will be discussed later.

As result of multiplication, whole technique is described in [6, 9], is derived the system of equations:

$$\frac{\partial \rho}{\partial t} + \frac{\tau}{2}\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x_i}\rho u_i = \frac{\partial}{\partial x_i}\left(\frac{\tau}{2}\frac{\partial}{\partial x_k}\Pi_{ik}\right), \tag{8}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\tau}{2}\frac{\partial^2}{\partial t^2}\rho u_i + \frac{\partial}{\partial x_k}\Pi_{ik} = \frac{\partial}{\partial x_k}\Pi^{\mathrm{D}}_{ik} + \frac{\partial}{\partial x_k}\left[\left(\frac{\tau}{2}\frac{\partial}{\partial x_k}\Pi_{ik}\right)u_k\right], \tag{9}$$

$$\frac{\partial E}{\partial t} + \frac{\tau}{2}\frac{\partial^2 E}{\partial t^2} + \frac{\partial F_i}{\partial x_i} = \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}\Pi^{\mathrm{D}}_{ik}u_k + \frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\left(E + p + \frac{B^2}{8\pi}\right)\left(\frac{\tau}{2}\frac{\partial}{\partial x_k}\Pi_{ik}\right)\right], \tag{10}$$

$$\frac{\partial B_i}{\partial t} + \frac{\tau_{\mathrm{M}}}{2}\frac{\partial^2 B_i}{\partial t^2} + \frac{\partial}{\partial x_k}\Pi^{B}_{ik} = \frac{\partial}{\partial x_k}\Pi^{\mathrm{DB}}_{ik}, \tag{11}$$

$$\mathrm{div}\mathbf{B} = 0\,. \tag{12}$$

The left hand side of this system, including (8)-(12), like the ideal magneto gas dynamic equations. The dissipative terms on the right hand side of (8)-(12) are given by

$$\Pi_{ik} = \left(p + \frac{B^2}{8\pi}\right)\delta_{ik} + \rho u_i u_k - \frac{B_i B_k}{4\pi}, \tag{13}$$

$$\begin{aligned}
\Pi^{\mathrm{D}}_{ik} = &\frac{\tau}{2}\left[p\frac{\partial u_i}{\partial x_k} + p\frac{\partial u_k}{\partial x_i} - \frac{2}{3}p\frac{\partial u_m}{\partial x_m}\delta_{ik}\right] \\
&+ \frac{\tau}{2}\left[\left(\frac{B^2}{8\pi}\delta_{mk} - \frac{B_m B_k}{4\pi}\right)\frac{\partial u_i}{\partial x_m} + \left(\frac{B^2}{8\pi}\delta_{im} - \frac{B_i B_m}{4\pi}\right)\frac{\partial u_k}{\partial x_m} - \left(\frac{B^2}{8\pi}\delta_{ik} - \frac{B_i B_k}{4\pi}\right)\frac{\partial u_m}{\partial x_m}\right] \\
&+ \frac{\tau}{2}\left[\frac{B_m}{4\pi}\left(-B_k\frac{\partial u_i}{\partial x_m} - B_i\frac{\partial u_k}{\partial x_m} + B_n\frac{\partial u_n}{\partial x_m}\delta_{ik}\right)\right] \\
&+ \frac{\tau}{2}\left[\rho u_i u_m\frac{\partial u_k}{\partial x_m} + u_i\frac{\partial p}{\partial x_k} + u_i\frac{\partial}{\partial x_k}\frac{B^2}{8\pi} - u_i\frac{\partial}{\partial x_m}\frac{B_m B_k}{4\pi}\right] + \frac{\tau}{2}\left[u_m\frac{\partial p}{\partial x_m} + \gamma p\frac{\partial u_m}{\partial x_m}\right]\delta_{ik} \\
&+ \frac{\tau}{2}\left[\frac{1}{4\pi}\left(B_n^2\frac{\partial u_m}{\partial x_m} - B_n B_m\frac{\partial u_n}{\partial x_m} + B_n u_m\frac{\partial B_n}{\partial x_m}\right)\right]\delta_{ik} \\
&+ \frac{\tau}{2}\left[\frac{1}{4\pi}\left(-B_i B_k\frac{\partial u_m}{\partial x_m} + B_i B_m\frac{\partial u_k}{\partial x_m} - B_i u_m\frac{\partial B_k}{\partial x_m}\right)\right] \\
&+ \frac{\tau}{2}\left[\frac{1}{4\pi}\left(-B_k B_i\frac{\partial u_m}{\partial x_m} + B_k B_m\frac{\partial u_i}{\partial x_m} - B_k u_m\frac{\partial B_i}{\partial x_m}\right)\right],
\end{aligned} \tag{14}$$

$$
\begin{aligned}
Q_i^{\mathrm{D}} &= \frac{\tau}{2}\left[\frac{5}{2}p\frac{\partial}{\partial x_i}\frac{p}{\rho}\right] + \frac{\tau}{2}\left[\frac{5}{2}\left(\frac{B^2}{8\pi}\delta_{ik} - \frac{B_iB_k}{4\pi}\right)\frac{\partial}{\partial x_k}\frac{p}{\rho}\right] \\
&+ \frac{\tau}{2}\left[\frac{3}{2}\left(p\delta_{ik} + \frac{B^2}{8\pi}\delta_{ik} - \frac{B_iB_k}{4\pi}\right)\frac{\partial}{\partial x_k}\frac{B^2}{8\pi\rho} - \left(p + \frac{B^2}{8\pi}\right)\frac{\partial}{\partial x_k}\frac{B_iB_k}{4\pi\rho} - \frac{B_iB_k}{4\pi\rho}\frac{\partial}{\partial x_k}\frac{B^2}{8\pi}\right] \\
&+ \frac{\tau}{2}\left[\rho u_iu_k\frac{\partial}{\partial x_k}\frac{3}{2}\frac{p}{\rho}\right] + \frac{\tau}{2}\left[\rho u_iu_k\left(p + \frac{B^2}{8\pi}\right)\frac{\partial}{\partial x_k}\frac{1}{\rho} - u_iB^2\frac{\partial u_k}{\partial x_k}\right] \\
&+ \frac{\tau}{2}\left[u_i\frac{B_m}{4\pi}\left(B_m\frac{\partial u_k}{\partial x_k} - B_k\frac{\partial u_m}{\partial x_k} + u_k\frac{\partial B_m}{\partial x_k}\right)\right] + \frac{\tau}{2}\left[\frac{1}{2}\rho u_iu_k\left(\frac{B^2}{8\pi}\frac{\partial}{\partial x_k}\frac{1}{\rho} - \frac{1}{\rho}\frac{\partial}{\partial x_k}\frac{B^2}{8\pi}\right)\right] \\
&+ \frac{\tau}{2}\left[\frac{B_iB_m}{4\pi}\left(-u_k\frac{\partial u_m}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_m} - \frac{1}{\rho}\frac{\partial}{\partial x_m}\frac{B^2}{8\pi} + \frac{1}{\rho}\frac{\partial}{\partial x_k}\frac{B_mB_k}{4\pi}\right)\right], \qquad (15)
\end{aligned}
$$

$$
\Pi_{ik}^{\mathrm{DB}} = \frac{\tau_M}{2}\left[\frac{1}{\rho}\left(p + \frac{B^2}{8\pi}\right)\left(\frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i}\right)\right] + \Pi_{ik}^{\mathrm{DB}^*}, \qquad (16)
$$

$$
\begin{aligned}
\Pi_{ik}^{\mathrm{DB}^*} &= \frac{\tau_{\mathrm{M}}}{2}\left[\left(p + \frac{B^2}{8\pi}\right)\left(B_i\frac{\partial}{\partial x_k}\frac{1}{\rho} - B_k\frac{\partial}{\partial x_i}\frac{1}{\rho}\right)\right] \\
&+ \frac{\tau_{\mathrm{M}}}{2}\left[u_kB_i\frac{\partial u_m}{\partial x_m} - u_kB_m\frac{\partial u_i}{\partial x_m} + u_ku_m\frac{\partial B_i}{\partial x_m}\right] \\
&+ \frac{\tau_{\mathrm{M}}}{2}\left[B_iu_m\frac{\partial u_k}{\partial x_m} + \frac{B_i}{\rho}\frac{\partial p}{\partial x_k} + \frac{B_i}{\rho}\frac{\partial}{\partial x_k}\frac{B^2}{8\pi} - \frac{B_i}{\rho}\frac{\partial}{\partial x_m}\frac{B_kB_m}{4\pi}\right] \\
&+ \frac{\tau_{\mathrm{M}}}{2}\left[-u_iB_k\frac{\partial u_m}{\partial x_m} + u_iB_m\frac{\partial u_k}{\partial x_m} - u_iu_m\frac{\partial B_k}{\partial x_m}\right] \\
&+ \frac{\tau_{\mathrm{M}}}{2}\left[-B_ku_m\frac{\partial u_i}{\partial x_m} - \frac{B_k}{\rho}\frac{\partial p}{\partial x_i} - \frac{B_k}{\rho}\frac{\partial}{\partial x_i}\frac{B^2}{8\pi} + \frac{B_k}{\rho}\frac{\partial}{\partial x_m}\frac{B_iB_m}{4\pi}\right]. \qquad (17)
\end{aligned}
$$

The full kinetic consistent gas dynamic system of equations and magneto gas dynamic system of equations are represent more physical model, based on the relations to the kinetic Boltzmann equation and have the representation of many real physical processes which now under investigations. We can inphasise two substantial differences from their classical counterparts:

- First, the kinetic consistent magneto gas dynamic system of equations are hyperbolic type, what ensured by presence of second order derivatives in time [7], despite the presence of dissipation terms. By applying this approach, we can implement a three level explicit scheme with a better stability conditions than that in explicit schemes fore parabolic equations. This scheme has been successfully used to modeling magneto gas dynamic problem on high performance computing system [10, 11].

- Second, the continuity equation involves a dissipative term absent from the Euler and Navier–Stokes equations. Recall that, in combination with the second order derivative in time of density, the difference from the classical continuity equation consists in terms of the second order of smallness in the Knudsen number [6, 13]

However, for the practical purposes, the cumbersomeness of the kinetic consistent gas dynamic system and even more of its magneto gas dynamic version (8)-(12) make them complicated for the numerical analysis and computing. Let us describe how, relying on the kinetic consistent gas dynamic system and (8)-(17), a more compact models of gas dynamic and magneto gas dynamic processes can be obtained with the preservation of the potentials of the kinetic consistent gas dynamic description for efficient calculations on high performance computing systems.

3 COMPACT KINETIC CONSISTENT MAGNETO GAS DYNAMIC SYSTEM OF EQUATIONS

First, we note the relation between the kinetic consistent magneto gas dynamic system (9)–(13) and the classical equations of dissipative magneto gas dynamic system. Due to the cumbersomeness of system (9)–(17), this relation has not yet been analysed completely and continued. For this reason, we use an analogy with the kinetic consistent gas dynamic system, which differs from the Navier–Stokes equations in terms of the second order of smallness in the Kn number. Schematically, this relation can be expressed as

$$\mathrm{QGS} = \mathrm{NS} + \mathcal{O}(Kn^2)\,. \tag{18}$$

The closeness of the solutions to two systems has been confirmed by numerous numerical experiments and a theoretical analysis [6, 11] This analysis is based on the fact that the solution of the balance equation

$$\frac{f^{j+1} - f_0^j}{\tau} + \xi_k \frac{\partial f_0}{x_k} = \frac{\partial}{\partial x_k} \frac{\tau}{2} \xi_k \xi_p \frac{\partial f_0^j}{\partial x_p}\,, \tag{19}$$

where f_0 is given by (1), differs by terms of order Kn^2 from the solution of the Bhatnagar–Gross–Krook equation [6]

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} = \frac{1}{\tau}\left(f_0 - f\right)\,. \tag{20}$$

First, we consider a compact version of the kinetic consistent gas dynamic system with dissipative terms, physically this terms appears due to smoothing the solution over the mean free path, which follows from the method of deriving the balance equation [6, 13]. It is appear as the contribution made by the additional velocity $\mathbf{w}$, which appears on the right-hand sides of the continuity, momentum, and energy equations

$$w_k = \frac{\tau}{\rho} \frac{\partial}{\partial x_i}\left(\rho u_k u_i + p\delta_{ik}\right)\,. \tag{21}$$

For this purpose, we follow a method similar to that used in continuum mechanics to derive similar equations, taking into account the gas dynamic velocity and molecular transport:

$$\frac{\partial\rho}{\partial t}+\frac{\tau}{2}\frac{\partial^2\rho}{\partial t^2}+\mathrm{div}\rho\left(\mathbf{u}-\mathbf{w}\right)=0\,, \tag{22}$$
$$\frac{\partial\rho\mathbf{u}}{\partial t}+\frac{\tau}{2}\frac{\partial^2\rho\mathbf{u}}{\partial t^2}+\mathrm{div}\left[\left(\mathbf{u}-\mathbf{w}\right)\times\mathbf{u}\rho\right]+\mathrm{div}p=\mathrm{div}P_{\mathrm{NS}}\,, \tag{23}$$
$$\frac{\partial E}{\partial t}+\frac{\tau}{2}\frac{\partial^2 E}{\partial t^2}+\mathrm{div}\left[\left(E+p\right)\left(\mathbf{u}-\mathbf{w}\right)\right]=\mathrm{div}\mathbf{q}+\mathrm{div}\left(P_{\mathrm{NS}}\mathbf{u}\right)\,, \tag{24}$$

where P_{NS} is the Navier-Stokes viscous stress tensor and $\mathbf{q}$ is the heat flux with components $q_i=\kappa\frac{\partial T}{\partial x_i}$.

System (22)-(24) differs from the Navier–Stokes equations in that its left-hand side involves the generalized velocity $\mathbf{u}-\mathbf{w}$ rather than $\mathbf{u}$. Recall that this system of equations is derived by applying the phenomenological approach used in continuum mechanics for obtaining corresponding equations. Specifically, the original kinetic consistent gas dynamic system of equations (obtained from the balance equation (19) can be represented in the form

$$\frac{\partial\rho}{\partial t}+\frac{\tau}{2}\frac{\partial^2\rho}{\partial t^2}+\mathrm{div}\rho\left(\mathbf{u}-\mathbf{w}\right)=0\,, \tag{25}$$
$$\frac{\partial\rho\mathbf{u}}{\partial t}+\frac{\tau}{2}\frac{\partial^2\rho\mathbf{u}}{\partial t^2}+\mathrm{div}\left[\rho\left(\mathbf{u}-\mathbf{w}\right)\times\mathbf{u}\right]+\mathrm{div}p=\mathrm{div}P^*\,, \tag{26}$$
$$\frac{\partial E}{\partial t}+\frac{\tau}{2}\frac{\partial^2 E}{\partial t^2}+\mathrm{div}\left[\left(E+p\right)\left(\mathbf{u}-\mathbf{w}\right)\right]=\mathrm{div}\mathbf{q}^*+\mathrm{div}\left(P^*\mathbf{u}\right)\,, \tag{27}$$

where the tensor P is given by the expression

$$P^*=P_{\mathrm{NS}}+\rho\mathbf{u}\times\left[\mathbf{w}-\frac{\tau}{\rho}\mathbf{u}\mathrm{div}\left(\rho\mathbf{u}\right)\right]+\tau\left[\mathbf{u}\mathrm{div}p+\gamma p\mathrm{div}\mathbf{u}\right]\,, \tag{28}$$
$$\mathbf{q}^*=\mathbf{q}+\tau\left[\rho\left(\mathbf{u}\mathrm{div}\epsilon-\frac{\mathrm{p}}{\rho^2}\mathbf{u}\mathrm{div}\rho\right)\right]\mathbf{u}\,, \tag{29}$$

here γ is the ratio of specific heat.

Let us estimate the values of the additional terms involved in P^* and $\mathbf{q}^*$ in the two-dimensional case. The components of the tensor P^* (for the equation describing the momentum along the OX axis) are

$$\begin{aligned}P_{xx}=&\left[\left(\frac{4}{3}u+\zeta\right)+\tau\gamma p+\tau\rho u_x^2\right]\frac{\partial u_x}{\partial x}+2\tau u_x\frac{\partial p}{\partial x}\\&+\left(-\frac{2}{3}u+\zeta+\tau\gamma p\right)\frac{\partial u_y}{\partial y}+\tau u_y\frac{\partial p}{\partial y}+\tau p u_x u_y\frac{\partial u_x}{\partial y}\,,\end{aligned} \tag{30}$$
$$P_{yx}=\left(\mu+\tau p u_y^2\right)\frac{\partial u_x}{\partial y}+\mu\frac{\partial u_x}{\partial x}+\tau\rho u_x u_y\frac{\partial u_x}{\partial x}\,, \tag{31}$$

where

$$P_{\mathrm{NS}xx} = \left(\frac{4}{3}u + \zeta\right)\frac{\partial u_x}{\partial x} + \left(-\frac{2}{3}u + \zeta\right)\frac{\partial u_y}{\partial y}, P_{\mathrm{NS}yx} = \mu\frac{\partial u_x}{\partial y} + \mu\frac{\partial u_x}{\partial x}\,. \tag{32}$$

and ζ is the bulk viscosity.

Consider two typical cases where the dissipative terms of order $\mathcal{O}(Kn)$ involved in components P (28)-(29) of the tensor P^*, which can determine the flow structure. First, we consider the case of a boundary layer on a surface perpendicular to the OY direction. The viscous terms become comparable with the convective ones only if they contain the derivatives $\frac{\partial u_x}{\partial y}$ [14]. Since u_y in the boundary layer is much smaller than u_x in order of magnitude the convective terms can be comparable only with the viscous ones involved in P^*_{yx} , i.e. with the components P_{NS} (28). Similarly, in the boundary layer, the work of the dissipative terms div $(P\mathbf{u})$ in the energy equation is determined by the work of the viscous forces. In turn, the additional terms involved in $\mathbf{q}^*$ (29) are small as compared with $\mathbf{q}$. This is associated with the fact that the gradients of ϵ and ρ in the OY direction are multiplied by small values of u_y.

Consider the case of flow with small Re number, when the viscous terms have the same order of magnitude as the convective terms. As a rule, such flows correspond to low Mach numbers $\mathcal{M}$, so the gas, let alone the fluid, is nearly incompressible.

In this case, the value of $div\mathbf{u}$ involved in (28) is close to zero. Let us estimate the other terms in (28) in order of magnitude:

$$\left[-\rho\mathbf{u} \times \frac{\tau}{\rho}\mathbf{u}\mathrm{div}\,(\rho\mathbf{u}) + \tau u \mathrm{div} p\right] = \frac{\tau\rho V^3}{L}\,, \tag{33}$$

where V is the characteristic velocity and L is the characteristic length of the problem.

The term $\rho\,\mathbf{u} \times \mathbf{w}$ has the same order of magnitude. Taking into account the relation between μ, τ and the speed of sound, a similar estimate for the components of P_{NS} is given by

$$[P_{\mathrm{NS}}] = \frac{\tau\rho c^2 V}{L}\,. \tag{34}$$

Thus, in the case under consideration, the value of P^* is determined nearly completely by the components of P_{NS}.

Now, we will analyse the relation between the energy equation (24), which was obtained using phenomenological definition, and the complete kinetic consistent gas dynamic energy equation (27), which was derived from the balance equation. From (24), the work of P^* is determined by P_{NS}. Let us estimate the order of magnitude of the P_{NS} terms involved in $\mathbf{q}^*$ (27), $\mathbf{q} = \kappa\,\mathrm{grad}T$, and $\tau\mathbf{u}\left[\rho\left(\mathbf{u}\mathrm{div}\epsilon - \frac{p}{\rho^2}\mathbf{u}\mathrm{div}\rho\right)\right]\mathbf{u}$ between the thermal conductivity κ and viscosity ν , and between T and the speed of sound c:
the first term has the order of magnitude

$$[q] = \frac{\tau\rho c^4}{L}\,, \tag{35}$$

the second term is given by

$$\left[\tau\rho\left(\mathbf{u}\nabla\epsilon - \frac{p}{\rho^2}\mathbf{u}\nabla\rho\right)\right] = \frac{\tau\rho c^2 V^2}{L}\,. \tag{36}$$

Taking into account $[c] \gg [u]$, we conclude that the value of $\mathbf{q}^*$ is then determined by the heat flux $\mathbf{q}$.

In contrast to the analysis of the relation between the kinetic consistent gas dynamic system (25)-(27) and the Navier–Stokes equations, which differ by terms of the second order of smallness in the Kn number, the difference between the kinetic consistent gas dynamic system and its simplified version (22)-(24) is analysed by studying the situation when the viscosity and thermal conductivity have a large effect on the gas dynamic flows.

This conclusion is confirmed by computations, some of which are presented in the next section.

Next let's consider the induction equation in term of compact version of the kinetic consistent magneto gas dynamic system of equations. The first dissipative term in the right hand side of (16) has a main contribution

$$\left[\frac{\tau_{\mathrm{m}}}{2}\frac{1}{\rho}\left(p + \frac{B^2}{8\pi}\right)\left(\frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i}\right)\right] \gg \left[\Pi_{ik}^{\mathrm{DB}^*}\right]\,. \tag{37}$$

Since

$$\mathrm{rot}\mathbf{B} = \frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i},\; \nu_{\mathrm{m}} = \frac{\tau_{\mathrm{m}}}{2\rho}\left(p + \frac{B^2}{8\pi}\right)\,. \tag{38}$$

where ν_m has the meaning of magnetic viscosity.

Remark about definition of τ_{m}. By using (38) we can determine τ_{m} in terms of the magnetic viscosity ν_{m}:

$$\tau_{\mathrm{m}} = \frac{2\rho\nu_{\mathrm{m}}}{p + \frac{B^2}{8\pi}}\,, \tag{39}$$

taking to account

$$\nu_{\mathrm{m}} = \frac{c^2}{4\pi\sigma(T,\rho)}\,, \tag{40}$$

where c is the speed of light and σ is the conductivity.

Finally the induction equation can be represented in the form

$$\frac{\partial\mathbf{B}}{\partial t} - \mathrm{rot}\left(\mathbf{u}\times\mathbf{B}\right) = \nu_{\mathrm{m}}\mathrm{rot}\mathbf{B} + \frac{\partial}{\partial x_i}\Pi_{ik}^{\mathrm{DB}^*}\,. \tag{41}$$

In view of (41) we can say that the induction equation of kinetic consistent magneto gas dynamic equations differs from the corresponding classical magneto gas dynamic equations by a small quantity. It has been shown previously that the numerical results based on system (8)-(17) nearly do not differ from those obtained with classical models [12].

The construction of a system for describing magneto gas dynamic processes is based on physical factors that, along with molecular transport and the magnetic field have the influence of the additional velocity $\mathbf{w}$ associated with smoothing the solution over the mean free path. According to (32), $\mathbf{w}$ is given by

$$w_k = \frac{1}{\rho}\frac{\partial}{\partial x_i}\left[\left(p + \frac{B^2}{8\pi}\right)\delta_{ik} + \rho u_i u_k - B_i B_k\right]. \tag{42}$$

This velocity, together with $\mathbf{u}$, affects the all magneto gas dynamic parameters and finally the compact system can be represented:

$$\frac{\partial \rho}{\partial t} + \frac{\tau}{2}\frac{\partial^2 \rho}{\partial t^2} + \mathrm{div}\rho\left(\mathbf{u} - \mathbf{w}\right) = 0\,, \tag{43}$$

$$\frac{\partial \rho\mathbf{u}}{\partial t} + \frac{\tau}{2}\frac{\partial^2 \rho\mathbf{u}}{\partial t^2} + \mathrm{div}\left[\rho\left(\mathbf{u} - \mathbf{w}\right)\times\mathbf{u} + B_k B_p\right] + \mathrm{div}\left(p + \frac{B^2}{8\pi}\right) = \mathrm{div}P_{\mathrm{NS}}\,, \tag{44}$$

$$\frac{\partial E}{\partial t} + \frac{\tau}{2}\frac{\partial^2 E}{\partial t^2} + \mathrm{div}\left[\left(E + p + \frac{B^2}{8\pi}\right)\left(\mathbf{u} - \mathbf{w}\right)\right] = \mathrm{div}\mathbf{q} + \mathrm{div}\left(P_{\mathrm{NS}}\mathbf{u}\right)\,, \tag{45}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \frac{\tau_{\mathrm{m}}}{2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = \mathrm{rot}\left(\mathbf{u} - \mathbf{w}\right)\times\mathbf{B} + \mathrm{rot}\nu_{\mathrm{m}}\mathrm{rot}\mathbf{B} \tag{46}$$

$$\mathrm{div}\mathbf{B} = 0\,. \tag{47}$$

where P_{NS} is the viscous stress tensor, $\mathbf{q}$ is the flux vector, ν_{m} is the magnetic viscosity and τ_{m} is determined in terms of ν_{m} given by (40)

4 MODELING RESULTS WITH COMPACT MAGNETO GAS DYNAMIC SYSTEM

Below, we present results obtained using compact systems (43)-(47). All computations relied on three-level explicit schemes with new values of the gas dynamic variables and the magnetic field at the time $t = t^{j+1}$ determined from known values at $t = t^j$ and $t = t^{j-1}$. All spatial derivatives were approximated at the central time level $t = t^j$ up to $\mathcal{O}(n^2)$.

Figure 1 presents the results obtained for the Orszag–Tang test, i.e., for the evolution of a vortex in a magnetic field [17].The initial data and the grid for the two-dimensional problem were the same as in [18]. The figure displays the density, pressure, magnetic pressure, and kinetic at the time t = 0.5. The numerical results agree with those obtained earlier for this problem.

Second we consider solutions of hydro dynamic problems based on system (43)-(47) without magnetic field. The results of the incompressible flow in a 3D cavity with a moving upper lid [16]. To simulate an isotropic incompressible (liquid), we used the equation of state

$$p = p_0 + \beta\left(\rho - \rho_0\right)\,, \tag{48}$$

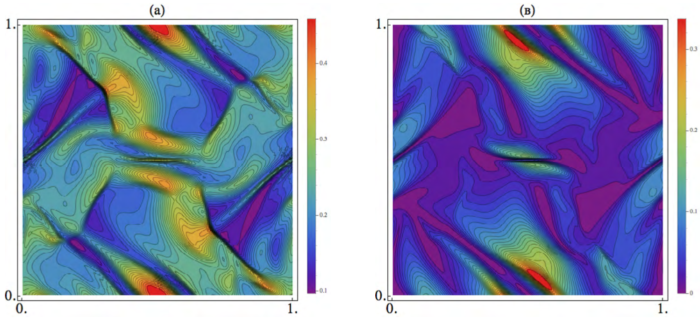

Figure 1: Orszang-Tang magnetic vortex: (a) dencity, (b) magnetic pressure at $t = 0.5$.

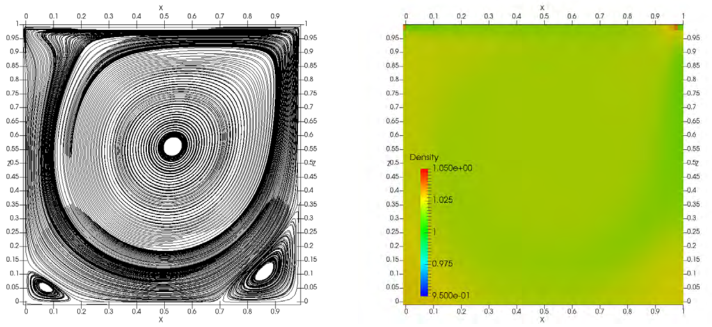

Figure 2: Result of modeling of the incompressible flow with Re =1000, (a) stream, (b) density

where p_0 and ρ_0 are the reference values of the pressure and density and β is a sufficiently large coefficient ensuring a large variation in pressure under minor variations in density. Figure 2 shows streamlines at $Re = 1000$, including detailed representations in the cavity corners, where vortex regions are formed. The plots do not exhibit any differences between the reference results of [16] and the results obtained on the basis of system (43)-(47).

The results of magneto hydro dynamic drive simulation is shown on Fig. 3. The representative results of magneto hydro dynamic flow in magneto hydro dynamic drive with caverna for the liquid Sodium flow. Liquid Sodium at temperature 600 K is placed in a vessel with dimensions 0.3×0.3 m^2. A stream of liquid Sodium is initiated by the magneto hydro dynamic drive on top of the vessel with a tube with dimension 0.02×0.02 m^2. The velocity provided by the magneto hydro dynamics drive is 36 m/s. The analysis of the results shows the possibility of modeling of liquid flow with the kinetic consistent magneto hydro dynamic models.

5 CONCLUSIONS

The compact models (25)-(27) and ((43)-(47) for describing gas dynamic and magneto gas dynamic flows, respectively, are based on kinetic models, which lead to the kinetic consistent gas dynamic system, and on phenomenological ideas. The use of phenomeno-

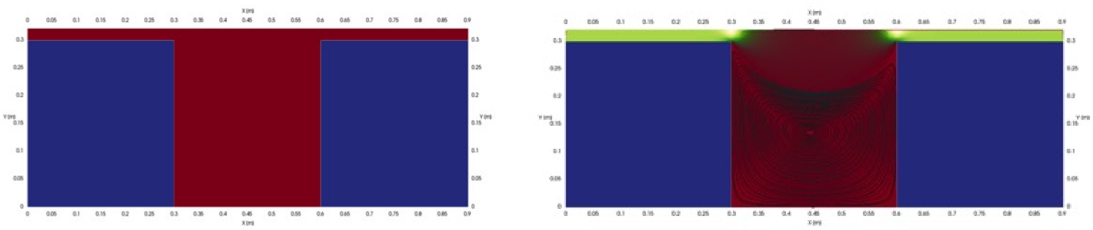

Figure 3: The flow of Sodium Liquid in the magneto hydro dynamic drive with caverna: (a) initial conditions of stream of liquid, (b) stream of liquid at steady state

logical approaches, which are visual from the point of view of physical concepts, makes the systems not only more compact in comparison with their pure kinetic consistent gas dynamic versions, but also better justified in terms of macroscopic views. Their validity was confirmed not only by theoretical estimates, but also by test computations.

The compact models preserve all positive properties intrinsic to previously proposed pure kinetic consistent gas dynamic models. Additionally, they provide a visual and simple description, which is an important factor in numerical simulation.

ACKNOWLEDGEMENTS

This work was supported by the Russian Science Foundation, grant no. 19-11-00032.

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