Complexity and Conformal Field Theory

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We study circuit and state complexity in the universal setting of (1+1)-dimensional conformal field theory and unitary transformations generated by the stress-energy tensor. We provide a unified view of assigning a cost to circuits based on the Fubini-Study metric and via direct counting of the stress-energy tensor insertions. In the former case, we iteratively solve the emerging integro-differential equation for sample optimal circuits and discuss the sectional curvature of the underlying geometry. In the latter case, we recognize that optimal circuits are governed by Euler-Arnold type equations and discuss relevant results for three well-known equations of this type in the context of complexity.

Introduction– One of the most interesting recent results in black hole physics are arguably holographic complexity proposals. They are conjectured relations between volumes [1–3] or actions [4, 5] in anti-de Sitter gravity [6–8] and quantum information notions of complexity of states and circuits in dual quantum field theories (QFTs).

Complexity in its native quantum computing setting concerns hardness of approximating a given unitary transformation using circuits composed from gates acting only on a limited number of qubits (circuit complexity) or approximating a desired quantum state using such circuits acting on a simple state (state complexity). Holographic complexity proposals have motivated references [9, 10] to properly embed circuit and state complexity in the QFT setting, which had not been done before, and led to many recent developments and motivates the present letter.

These works view the preparation of a unitary operator U or, upon acting on a reference state $|R\rangle$, also (target) state preparation $|T\rangle = U|R\rangle$ in a continuous way as a path-ordered exponential

$$U(\tau) = \overleftarrow{\mathcal{P}} e^{-i \int_0^\tau Q(\gamma)\mathrm{d}\gamma}. \quad (1)$$

with $U(\tau = 1)$ being equal to some desired and reachable unitary U. In the equation above, the Hermitian operator $Q(\gamma)\mathrm{d}\gamma$ is a single layer of the circuit parametrized by the parameter γ that constructs the operator U. The key idea used in [9, 10] to *define* complexity in a QFT appeared earlier in [11–13] as a way of bounding complexity of discrete circuits acting on qubits. The relevant definition assigns a cost to $Q(\tau)$ which reflects the decomposition of $Q(\tau)$ into more elementary building blocks or *gates*, each with a specified cost of use, and minimizes the sum of the contributions from all layers of the circuit subject to appropriate initial and final conditions.

Most of the studies in the literature to date were concerned with free QFTs and claimed optimality of circuits with $Q(\tau)$ being at most quadratic in underlying bosonic or fermionic operators. Such studies, despite their intrinsic simplicity, could be fine-tuned to nevertheless reproduce several predictions of holographic complexity proposals, see [9, 10, 14–18].

In the eyes of the authors, studies of complexity in QFT are, in fact, quite similar to the historic development of entanglement entropy of quantum fields. The latter quantity also arose in connection with black hole physics [19–21] and later became an independent research subject with a strong quantum gravity component. One of the most fruitful seeds of progress in this much more established discipline originated from the studies of entanglement entropy in the setting of (1+1)-dimensional conformal field theories (CFTs_{1+1}). This includes in particular the universal result for single interval entanglement entropy in the vacuum state [21, 22]. Drawing a parallel from how the field of entanglement entropy in QFT has developed, which includes also the matching of the results of [21, 22] by the holographic entanglement entropy [23], CFTs_{1+1} should provide an ideal setting for accelerating our understanding of the notion of complexity in QFTs and the holographic complexity proposals.

This vision is largely shared by [24–29], which considered this problem in various ways and directly motivate our approach. The universality of CFTs_{1+1} stems from their stress-energy tensor generating the Virasoro algebra and in the present letter we will be concerned with unitary circuits obtained from the exponentiation of the stress-energy tensor operator. To this end, we will adopt the setting of [25, 29], but focus on a different way of assigning a cost to the involved operations. We focus on a CFT_{1+1} defined on a Lorentzian cylinder, whose circle has a unit radius and is parametrized by the coordinate σ. We will also restrict ourselves to one copy of the Virasoro algebra. This means we will assign complexities to unitary circuits of the form (1) on a representation of the group of diffeomorphisms on the circle, where an operator $U(\tau)$ corresponds to a group element $f(\tau, \sigma)$ that maps the circle to itself. One should hence think of $f(\tau, \sigma)$ as representing a sequence of diffeomorphisms of σ interpolating between the identity $f(0, \sigma) = \sigma$ and a desired one $f(1, \sigma) \equiv f(\sigma)$. Note that we will generally ignore terms stemming from the central extension of the group, as these would lead to additional complex phase-factors in (1), and complex phases are generally considered not to be relevant for physical notions of complexity (a problem faced in [25, 29]). An infinitesimal layer of the circuit is generated by

$$Q(\tau) = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, T(\sigma)\epsilon(\tau,\sigma) \tag{2}$$

where $T(\sigma)$ is the right-moving component of the stress-energy tensor operator and $\epsilon(\tau,\sigma)$ is an element of the Lie-algebra defined via

$$\epsilon(\tau, f(\tau,\sigma)) = \dot{f}(\tau,\sigma). \tag{3}$$

In the present letter we discuss two viable instances of cost functions. The first emerges by taking an energy eigenstate $|h\rangle$ and evaluating the Hilbert space distance traversed by the circuit defined by (1) and (2). This is the Fubini-Study complexity defined in [9]. The second instance realizes the approach of [10] and arises from treating $T(\sigma)$ as a one-parameter set of elementary contributions to each circuit layer and minimizing the L_2-norms of $\epsilon(\tau,\sigma)$ or $\epsilon'(\tau,\sigma)$ or a combination of both, averaged over all circuit layers.

Our approach is the first study of complexity in a generic (including large central charge c) CFT_{1+1} that 1) does not assign cost to trivial phase factors (see the discussion in [29]), 2) ends up with a well-posed variational problem for determining a transformation between two arbitrary unitaries generated by the insertions of $T(\sigma)$, and 3) sheds light on the underlying geometry of circuits by probing its sectional curvatures. While we intend to present our main results in a concise and self-contained manner in this letter, some further discussions will be postponed to our upcoming paper [30].

Cost functions and complexity– The Fubini-Study metric a.k.a. fidelity susceptibility arises from considering an overlap between two nearby states in the Hilbert space, see for example [31] for a review. It is attractive from the point of view of holography and the largely open problem of physical interpretation of holographic complexity proposals, since it is known how the overlap between at least certain states in holographic QFTs manifest itself on the gravity side [26]. For a family of states $|\psi(\tau)\rangle$ parametrized by τ, we can define

$$|\langle\psi(\tau)|\psi(\tau+d\tau)\rangle| \approx 1 - G_{\tau\tau}(\tau)d\tau^2 + \mathcal{O}(d\tau^3) \tag{4}$$

where $G_{\tau\tau} \geq 0$ is the Fubini-Study-metric. Assume $|\psi(\tau)\rangle$ is a path on the space of states parametrized by unitary operators acting on an initial state $|h\rangle$,

$$|\psi(\tau)\rangle \equiv U(\tau)\,|h\rangle\,. \tag{5}$$

The Fubini-Study metric $G_{\tau\tau}$ becomes then the variance of $Q(\tau)$ evaluated in the state $|\psi(\tau)\rangle$ [25] or, upon introducing

$$\tilde{Q}(\tau) = U(\tau)^\dagger Q(\tau) U(\tau) \tag{6}$$

equivalently the variance of $\tilde{Q}(\tau)$ evaluated in the state $|h\rangle$: $\langle h|\,\tilde{Q}^2(\tau)\,|h\rangle = \langle\psi(\tau)|Q^2(\tau)|\psi(\tau)\rangle$ by definition, and similarly for the one-point function. Note that the applications of the operators $U(\tau)$ in (6), using (2), essentially causes a conformal transformation of the stress-energy tensor. Using the well-known transformation law and ignoring the Schwarzian term leading to an irrelevant for us phase factor, we can write [25]

$$\tilde{Q}(\tau) = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, T(\sigma)\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}. \tag{7}$$

Each trajectory through state space $|\psi(\tau)\rangle$ parametrized by $\tau \in [0,1]$ can now be assigned the total cost $\mathrm{L_{FS}}$

$$\mathrm{L_{FS}} = \int_0^1 d\tau \sqrt{G_{\tau\tau}(\tau)} \tag{8}$$

and complexity arises as its minimum subject to the appropriate initial and final condition [9]. We should note here that the present discussion is completely general and concerns complexity of state $|T\rangle = U\,|h\rangle$ given a reference state $|R\rangle = |h\rangle$. Alternatively, one can view it as a definition of circuit complexity associated with a circuit representation of a unitary U in which one decomposes $\tilde{Q}(\tau)$ into elementary transformations. This is somewhat similar to the notion of circuit complexity explored in [28].

One can alternatively define an a priori inequivalent notion of complexity based on a variance of $Q(\tau)$ in the state $|h\rangle$ (instead of $|\psi(\tau)\rangle$ as so far), which would be a more faithful realization of the approach [10]. What we mean by that is that the cost of one layer in the Fubini-Study metric depends not only on $\epsilon(\tau,\sigma)$ from a given layer, but also on what all previous layers do through the two-point function of T in the *evolved* state $|\psi(\tau)\rangle$. On the other hand, in the approach of [10] the cost of each layer depends only on $\epsilon(\tau,\sigma)$ as the insertions of T would be assigned the same weight at each layer.

The discussion so far was completely general and now it is time to specialize to the case of interest, i.e. circuits defined by (7). Our choice for $|h\rangle$, as in [25, 29], is that of an energy eigenstate in the CFT_{1+1} corresponding, via the operator-state correspondence, to a (quasi-)primary of the chiral dimension h. To evaluate (8) explicitly, we follow [25, 29] and write the variance of $\tilde{Q}$ as a bi-local integral over circle

$$\mathrm{L} = \int_0^1 \frac{d\tau}{2\pi}\sqrt{\iint_0^{2\pi} d\sigma d\kappa \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}\frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)}\Pi(\sigma-\kappa)}, \tag{9}$$

where Π corresponds to a connected correlator of the stress-energy tensor in the state $|h\rangle$ [32]

$$\begin{aligned}\Pi(\sigma-\kappa) &= \langle h|T(\sigma)T(\kappa)|h\rangle - \langle h|T(\sigma)|h\rangle\,\langle h|T(\kappa)|h\rangle \\ &= \frac{c}{32\sin^4[(\sigma-\kappa)/2]} - \frac{h}{2\sin^2[(\sigma-\kappa)/2]}.\end{aligned} \tag{10}$$

As usual when studying geodesic motion, by requiring affine parametriztion we can move from a *length-functional* (8) to an *energy-functional* where, essentially, the square-root in (8) and (9) is removed. To do so, we note that (9) clearly corresponds to a geodesic problem in infinite dimensions, where summation over indices has been replaced by integration over variables $\sigma, \kappa,\ f(\tau,\sigma)$ has taken over the role of the coordinate $X^\sigma(\tau)$, and the expression $\frac{\Pi(\sigma-\kappa)}{f'(\tau,\sigma)f'(\tau,\kappa)}$ takes on the role of the metric $g_{\sigma\kappa}(X(\tau))$. The partial integro-differential equation (IDE) of motion for the circuit $f(\tau,\sigma)$ extremizing (9) then reads

$$\int_0^{2\pi} d\sigma\left(-\Pi(\sigma-\kappa)\frac{d}{d\tau}\left(\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)f'(\tau,\kappa)}\right)+\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}\partial_\kappa\left(\Pi(\sigma-\kappa)\frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)^2}\right)\right)\equiv 0. \tag{11}$$

As expected, this equation is of second order in derivatives with respect to the parameter τ. This is adequate for a boundary value problem in which we envision being given an initial and final condition, $f(0,\sigma)$ and $f(1,\sigma)$, and finding the shortest circuit $f(\tau,\sigma)$ connecting these two maps. This is a notable contrast to the geometric actions studied in [25, 29], which lead to equations first order in τ, in which generally only one initial value $f(0,\sigma)$ needs to be provided to fix a solution.

Given our motivation and derivation, it is the most natural for us to equate the integration kernel Π with the connected stress-energy two-point function in the state $|h\rangle$ (10), however we have kept our discussion more generic for a reason. Broadly speaking, our goal in this paper is to define geodesic motion on the Virasoro-group, and this has of course already been accomplished in the framework of Euler-Arnold-type partial differential equations (PDEs) such as the Korteweg-de Vries (KdV), Camassa-Holm (CH), or Hunter-Saxton (HS) equations [33]. They were already discussed in the context of QFT complexity in [25, 29], see also [34]. In order to show how our functional (9) and equation (11) fit into this more general framework, we note that according to our complexity definition (9), the distance between the identity map $f=\sigma$ and a map $f=f_1(\sigma)$ is identical to the distance between $f=\sigma$ and the inverse map $F_1(\sigma)=f_1^{-1}(\sigma)$. This is easy to show by using invariance under a change of affine parameter $\tau\to s=1-\tau$ and invariance under conformal transformations $f(\tau,\sigma)\to G(f(\tau,\sigma))$. Hence, replacing the circuit $f(\tau,\sigma)$ by the inverse circuit $F(\tau,\sigma)$ in (9) and using the identity [25]

$$\epsilon(\tau,\sigma)=-\frac{\dot{F}(\tau,\sigma)}{F'(\tau,\sigma)}, \tag{12}$$

we can write the inner product in (9) entirely in terms of the kernel Π and the Lie-algebra-element ϵ. Note that if $A=B\cdot C$, then $A^{-1}=C^{-1}\cdot B^{-1}$, hence replacing $f(\tau,\sigma)$ by $F(\tau,\sigma)$ in (9) corresponds to switching from a left- to a right-invariant metric on the Lie-group. This would yield exactly the alternative complexity definition based on the variance of Q (not $\tilde{Q}$) in the state $|h\rangle$ discussed below equation (8).

Now, while our derivation above would suggest Π to be the stress-energy two-point function in the state $|h\rangle$ [32], another choice of

$$\Pi(\sigma-\kappa)=a\,\delta(\sigma-\kappa)+b\,\delta''(\sigma-\kappa) \tag{13}$$

with $\delta(\sigma-\kappa)$ being Dirac's delta function would allow us to obtain the CH equation ($a=b=1$), the HS equation ($a=0, b=1$), and the KdV equation ($a=1, b=0$), ignoring as before the term coming from the central extension. The choice (13) can be seen through the lenses of [10] as assigning directly a spatially uniform cost to individual insertions of $T(\sigma)$ via $Q(\tau)$ defined in (2). In a sense, our complexity functional (9) corresponds to a generalization of the inner products that led to these well studied integrable PDEs. Likewise, while these PDEs provide valid choices for a definition of complexity associated with the Virasoro group, they may be also regarded as simpler models for the physics encoded in the optimization problem behind the Fubini-Study complexity defined by IDE (11) supplemented with (10).

In the Fubini-Study case, in order to assign a well-defined finite value to (9) despite the pole of the two-point function, we can make use of the method of *differential regularisation* [35, 36]. This means we will (implicitly) phrase the divergent terms of the two-point-function in (9) as derivatives of more mildly-divergent terms, $\Pi\equiv\partial^2\hat{\Pi}$, and then carry the derivatives over onto the test-function $\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}\frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)}$ via integration by parts. The immediate physical consequence of this is that the metric will be degenerate. For example, if $\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}=$ const, the integrals in (9) will vanish when applying derivatives to this term in the implementation of differential regularisation. Although the cause of some technical problems, in our eyes this degeneracy of the metric is a desirable feature as it makes sure that transformations which only change the reference state by a complex phase will be assigned zero cost in terms of complexity. For this reason, we believe that the HS equation ($a = 0$ in (13)) will be a more realistic model of CFT-complexity than the KdV equation studied in [25, 29]. We will discuss these issues in more detail in [30].

Optimal circuits for Fubini-Study– The exact solutions and integrability properties of the KdV, CH and HS equations are well studied [33], and hence in this section we will focus again on the concrete integro-differential equations of motion (11) wherein we choose Π given by (10). For this, apart from trivial circuits such as $f(\tau,\sigma) = f(\tau+\sigma)$ leading to a vanishing cost, we do not know any exact solutions. However, for boundary conditions of the form $f(0,\sigma) = \sigma$, $f(1,\sigma) = \sigma + \frac{\epsilon}{m}\sin(m\sigma)$ with $m = 1, 2, 3, \ldots$ and $\epsilon \ll 1$, it is possible to iteratively construct a circuit $f(\tau,\sigma)$ satisfying (11) order by order in ϵ. As an example, for $m = 1$ this yields

$$\begin{aligned} f(\tau,\sigma) =& \sigma + \epsilon\tau\sin(\sigma) \\ &+ \epsilon^2 \frac{c\tau^2 - c\tau + 20h\tau^2 - 20h\tau}{4(c+8h)}\sin(2\sigma) + ... \end{aligned} \tag{14}$$

and the complexity can be evaluated by plugging this into the functional (9).

In order to gain a qualitative understanding of the geometry of our system without having to tediously calculate individual (approximate) circuits, we will now proceed to calculate the sectional curvatures $K(u,v)$ at the identity map $f(\sigma) = \sigma$ for tangent-vectors of the form $u = sin(m\sigma)$, $v = \sin(n\sigma)$, $m, n = 1, 2, 3, \ldots$ and $m \neq n$. Following the analogy with finite dimensional geodesic motion discussed above, first and second derivatives of the metric can then be defined as functional derivatives, e.g. $\frac{\partial g_{\sigma\kappa}}{\partial x^\eta} \rightarrow \frac{\delta}{\delta f(\eta)} \frac{\Pi(\sigma-\kappa)}{f'(\sigma)f'(\kappa)}$ $= -\Pi(\sigma-\kappa)\left(\frac{\delta'(\sigma-\eta)}{f'(\sigma)^2 f'(\kappa)} + \frac{\delta'(\kappa-\eta)}{f'(\sigma)f'(\kappa)^2}\right)$. In order to calculate the sectional curvatures, commonly defined as

$$K(u,v) = \frac{R_{\sigma\kappa\eta\omega}u^\sigma u^\eta v^\kappa v^\omega}{(u\cdot u)(v\cdot v) - (u\cdot v)^2}, \tag{15}$$

we would still need an analogue of the inverse metric which is needed in the definition of the Riemann-tensor $R_{\sigma\kappa\eta\omega}$. Note that in (15) the scalar product is taken using the metric $g_{\sigma\kappa}$ and we assume Einstein's summation formula involving integration. The inverse metric in question strictly speaking does not exist because our metric is degenerate. However, for circuits of the type (14) which satisfy the condition $\int d\sigma \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} = 0$ for all τ, it is possible to show that the equation of motion (11) is left invariant adding a non-zero constant to the two-point function Π. This creates a metric which is invertible and yet has identical (i.e. independent of the added constant) sectional curvatures to the original metric in the tangent-planes spanned by $u = sin(m\sigma), v = \sin(n\sigma)$. We find

$$K(u,v) = \frac{3}{\pi^2(m+n)}\left(\frac{(2m+n)(m+2n)}{24h + c(m+n-1)(m+n+1)} - \frac{(2m-n)(m+n)^2}{m\left(24h + c\left(m^2-1\right)\right)}\right) \text{ for } m > n. \tag{16}$$

The most important qualitative features of this result can be summarized briefly: For $0 \leq h/c < \frac{4}{13}$, all $K(u,v)$ are negative, for $h/c \rightarrow +\infty$, all $K(u,v)$ are positive, and for generic $h > 0, c > 0$, only a finite number of $K(u,v)$ for small m, n will be positive, while the countably infinitely many remaining ones will be negative. This means that in a sense, the *generic* curvature felt by the geodesic curves we are investigating will be negative unless $h/c \rightarrow \infty$. This is important, as in the study of Euler-Arnold-type geodesic equations negative sectional curvatures (in most directions) are related to a strong dependence of geodesics on initial conditions, and hence a certain instability of the geodesic problem [37]. For models of complexity, the necessity of negative sectional curvatures has been discussed in detail in [38]. Curiously, the metrics leading to the KdV and CH equations are known to lead to sectional curvatures of non-definite sign [39, 40], while the geometry underlying the HS equation is positively curved [41]. Furthermore, it was shown in [41] that the geometry underlying the HS equation maps the group-manifold to an open subset of an L^2-sphere. This however implies geodesic incompleteness: Geodesics can leave the space of invertible maps on the circle in finite affine parameter τ. From the point of view of the HS equation as a wave equation, this is related to the well known phenomenon of wave breaking, however, from a complexity point of view this phenomenon is harder to interpret. It would be fascinating to develop a better understanding of the conditions on a generic Π under which equations of the form (11) do or do not allow for such wave-breaking in finite time τ.

Summary and outlook– We addressed the problem of complexity of unitary operators resulting from exponentiation of the right-moving (or equivalently left-moving) component of the stress-energy tensor operator in CFT_{1+1}, see (1) and (2). There are two important features of the complexity notions we consider that we want to highlight. Firstly, they lead to equations of motion second order in derivatives of the circuit parameter, which allows to search for optimal circuits connecting two elements of the Virasoro group. Secondly, they are based on counting nontrivial (i.e. neglecting phase factors) elementary operations with non-negative weights. It would be interesting, see [42] for a related discussion, to see if there are circumstances under which the neglected phase factor can be interpreted as a geometric phase [43].

We primarily focused on the definition of complexity based on the Fubini-Study distance (4), as it seems to be potentially easier to embed in holography and has other attractive properties. Furthermore, it naturally leads to the novel Euler-Arnold type nonlinear integro-differential equation (11). Despite the rather intricate form of this equation capturing geodesic motion in an infinitely-dimensional space, we were able to find approximate solutions interpolating between the identity and its perturbation containing a single Fourier mode, see (14) for an explicit example. The leading term there reproduces the result of [24, 26] for complexity change under infinitesimal conformal transformations, which is a double integral of two test functions (related to $\dot{f}$) integrated against the stress-energy two-point function as integration kernel. Higher order terms in (14) are new predictions originating from the nonlinear nature of the equations of motion. We also probed the manifold of circuits via evaluating its sectional curvatures at the identity and found that for the Fubini-Study metric in physically relevant cases it is negative in most directions, see (16).

Subsequently, we looked at another possibility of defining circuit complexity that is based on explicit counting of appearances of the stress-energy tensor operator. This approach can be thought of as originating from a putative state in which the correlation function of the stress-energy tensor is ultra-local in the sense of (13) and it would certainly be exciting to pursue this analogy further, perhaps along the lines of [44]. Optimal circuits in this case would be exactly described by the KdV, CH or HS equations, which were suggested as models for complexity already in [25, 29]. However, as we discussed above, these equations and their underlying inner products induce a geometry on the group manifold that may have undesirable properties from a complexity point of view, such as positive sectional curvatures or the possibility of geodesic incompleteness [41]. We will comment in more details on these issues in our upcoming paper [30].

Regarding open problems, the one that we find particularly intriguing is realizing circuits given by (1) and (2) in holography, and their relation to holographic complexity proposals [1–5]. An encouraging hint in this direction is the agreement noted in [26] between the Fubini-Study complexity in CFT_{1+1} between the vacuum and an infinitesimal conformal transformation of the vacuum and the result of the complexity = volume proposal reported in [24].

One possible way forward is to understand the circuit (1) with $Q(\tau)$ given by (2) as being realized by placing a CFT_{1+1} in a curved geometry in which τ plays the role of the physical time. This brings a very close parallel with the path-integral optimization program [28, 45, 46], which was, however, predominantly presented in the context of non-unitary circuits originating from the Euclidean time evolution. Another interesting issue to return to in the future is the question raised in [28] of permissible cost functions being covariant functionals of the underlying metric. One can view [25] as a step towards obtaining a geometric notion of complexity in the sense outlined above, however, as recently pointed out in [29], there are problems with this approach.

Finally, it is clearly important to see if inclusion of primary operators and their descendants in circuits containing the stress-energy tensor can lead to short-cuts, see [34] for the role of short-cuts in the complexity of time evolution. This is especially interesting in the context of understanding holographic complexity proposals.

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