数理解析研究所講究録 第2129巻 2019年 34-40

Positive flow-spines and contact structures — a short summary

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1 Introduction

In this report, we discuss a relationship between positive flow-spines and contact structures of 3-manifolds.

Let M be a closed oriented 3-manifold. A *contact structure* on M is a totally non-integrable plane field in TM. There is a well-known relationship between open book decompositions of M and contact structures on M, called the *Giroux correspondence* [3]. On the other hand, a *flow-spine*, defined by the first author [6], of M is a special kind of spine, which defines a non-singular flow on M in such a way that the flow is transverse to the spine, and the flow in the complement of the spine is diffeomorphic to a constant flow in an open ball. We say that a contact structure is *supported* by a flow-spine if the flow of a Reeb vector field for the contact structure is defined by the flow-spine. The following is the main theorem of this report.

Theorem. (1) *Every positive flow-spine of a* 3*-manifold supports a unique contact structure up to isotopy; and*

(2) *Every contact structure on a* 3*-manifold is supported by a positive flow-spine.*

This report is adapted from the talk at 2019 Intelligence of Low-dimensional Topology held in Research Institute for Mathematical Sciences, Kyoto University. The details will be given in the forthcoming paper [7].

2 Preliminaries

Let M be an oriented, smooth 3-manifold. A *positive contact structure* on M is a transversely orientable 2-plane field ξ on M, given as the kernel of a 1-form (called a *contact form*) α on M, where α satisfies $\alpha \wedge d\alpha > 0$. In this paper we will omit the word "positive" for simplicity. The pair (M, ξ) is called a *contact* 3-*manifold*. We denote by $\mathrm{Cont}(M)$ the set of contact structures on M. Two contact structures $\xi_0, \xi_1 \in \mathrm{Cont}(M)$ are said to be *isotopic* if there exists a 1-parameter family of contact structures connecting them. Two contact 3-manifolds $(M; \xi)$ and $(M'; \xi')$ are said to be *contactomorphic* if there exists a diffeomorphism $f : M \to M'$ such that $f_*(\xi) = \xi'$. Contact geometry has no local invariants due to the following theorem.

Theorem 2.1 (Darboux's theorem)**.** *Let α be a contact form on an oriented* 3*-manifold M, and let p be a point in M. Then there exists a chart $(U; x, y, z)$ (called a Darboux chart) around p such that $p = (0, 0, 0)$ and $\alpha|_U = dz + xdy$.*

The next theorem claims that there are no non-trivial deformation of contact structures on M: it is especially useful when we prove that two contact structures are isotopic.

Theorem 2.2 (Gray's stability [4])**.** *Let $\{\xi_t\}_{t \in [0,1]}$ be a smooth family of contact structures on a closed oriented* 3*-manifold M. Then there exists an isotopy $\{\psi_t\}_{t \in [0,1]}$ of M such that $(\psi_t)_*(\xi_0) = \xi_t$ for each $t \in [0, 1]$.*

For a contact form α on an oriented 3-manifold M, the *Reeb vector field* R_α on M is defined by $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. We also call R_α a Reeb vector field of the contact structure $\xi = \ker \alpha$. The flow generated by R_α is called the *Reeb flow* of α (or a Reeb flow of ξ). A contact structure ξ on M is said to be *overtwisted* if there exists a disk D embedded in M such that ∂D is everywhere tangent to ξ and the framing of D along ∂D coincides with that of ξ. Otherwise ξ is said to be *tight*. For a discussion of the basic theory of contact 3-manifolds, we refer the reader to [9] and [2].

A 2-dimensional polyhedron P in M is called a *flow-spine* if there exists a non-singular flow $\Phi = \{\varphi_t\}_{t \in \mathbb{R}}$ on M such that

1. for each point of P, there exists a positive chart $(U; x, y, z)$ of M around the point such that $(U, U \cap P)$ is diffeomorphic (by an orientation-preserving diffeomorphism) to one of the four models shown in Figure 1, where the flow Φ on U is generated by the vector field $\partial/\partial z$; and

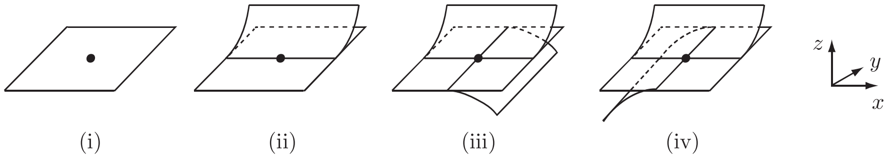

Figure 1: The local models of a flow-spine.

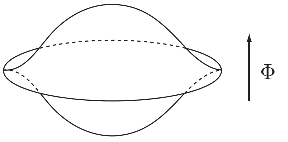

Figure 2: The complement of a flow-spine.

2. P is a *spine*, that is, $M \setminus P$ is an open 3-ball, and the flow Φ in $M \setminus P$ is diffeomorphic to a constant flow in an open ball, see Figure 2.

A point of P whose neighborhood is shaped on the model (iii) (resp. (iv)) in Figure 1 is called a *vertex of ℓ-type* (resp. *r-type*), and we denote the set of vertices of P by $V(P)$. The set of points whose neighborhoods are shaped on the models (ii), (iii) or (v) in Figure 1 is called the *singular set* of P, and we denote it by $S(P)$. Each component of $S(P) \setminus V(P)$ is called an *edge*. Each component of $P \setminus S(P)$ is called a *region*. We remark that if a flow-spine P contains an edge e diffeomorphic to a circle, then P has no vertices. Moreover, it is known that in that case, the ambient 3-manifold M is diffeomorphic to $S^2 \times S^1$, and the set of regions of P consists of a single disk and a single annulus. We also remark that if P contains at least one vertex, then every edge of P is diffeomorphic to an open interval. See [6] and [1] for the details.

A flow-spine P is said to be *positive* if $V(P)$ is non-empty and P has no point of the model (iv) in Figure 1. In the above setting, we say that the flow Φ is *carried* by P. A contact structure ξ on M is said to be *supported* by a flow-spine P if a Reeb flow of ξ is carried by P. We note that when a contact structure ξ on M is supported by a flow-spine P, $M \setminus P$ is an ultimately large Darboux chart.

The following is our main theorem.

Theorem 2.3. *Let M be a closed oriented 3-manifold. Then the following holds:*

(1) *For any positive flow-spine $P \subset M$, there exists a unique contact structure on M supported by P up to isotopy.*

(2) *For any contact structure ξ on M, there exists a flow-spine of M that supports ξ.*

The above theorem implies that the map

$$\{\text{positive flow-spines of } M\}/\text{isotopy} \ \to \ \text{Cont}(M)/\text{isotopy}$$

that takes a positive flow-spine P (up to isotopy) to a contact structure ξ (up to isotopy) whose Reeb flow is carried by P is a well-defined surjective map. We remark that we cannot remove the positivity condition for flow-spines from Theorem 2.3. In fact, the following holds.

Theorem 2.4. *Suppose that M admits a tight contact structure. Then at least one of the following holds:*

(1) *There exists a flow-spine of M that does not support any contact structure; or*

(2) *There exists a flow-spine of M supporting two contact structures that are not contactomorphic.*

3 Summary of the proof of Theorem 2.3

First we briefly explain the following two notions, which play key roles in the proof of Theorem 2.3 (1).

- the *admissibility* condition for flow-spines, and
- a *reference* 1-*form* associated with a flow-spine.

Let P be a flow-spine of a closed oriented 3-manifold M. Let Φ be a non-singular flow on M carried by P. Let $R_1, \dots, R_n$ be the regions of P. Equip each region R_i of P with the orientation compatible with the orientation of M and the direction of Φ. Let $\bar{R}_i$ be the metric completion of R_i with the path metric inherited from a Riemannian metric on R_i. Let $\kappa_i : \bar{R}_i \to M$ be the natural extension of the inclusion $R_i \hookrightarrow M$. Assign an orientation to each edge of P in an arbitrary way.

Definition. P is said to be *admissible* if there exists an assignment of real numbers $x_1, \dots, x_m$ to the edges $e_1, \dots e_m$, respectively, of P such that for any $i \in \{1, \dots, n\}$

$$\sum_{\tilde{e}_j \subset \partial \bar{R}_i} \varepsilon_{ij} x_j > 0,$$

where $\tilde{e}_j$ is an open interval or a circle on $\partial \bar{R}_i$ such that $\kappa_i|_{\tilde{e}_j} : \tilde{e}_j \to e_j$ is a homeomorphism, and $\varepsilon_{ij} = 1$ if the orientation of e_j coincides with that of $\kappa_i(\tilde{e}_j)$ induced from the orientation of R_i and is $\varepsilon_{ij} = -1$ otherwise.

The proof of the following proposition is given by the combinatorics of the *DS-diagram* (see for instance [5], [6] and [8]) corresponding to a positive flow-spine.

Proposition 3.1. *Every positive flow-spine satisfies the admissibility condition.*

Let P be a positive flow-spine of a closed oriented 3-manifold M. A *reference* 1-*form* η on M associated with P is roughly defined as follows. We consider a compact neighborhood of each of vertices, edges and regions of P. On a compact neighborhood $R_i \times [0,1]$ of each region R_i of P, the 1-form is defined as $\eta \fallingdotseq dt_i$, where t_i is the parameter on $[0,1]$, see Figure 3. The 1-form η is exactly dt_i outside of a neighborhood of $\partial R_i \times [0,1]$. We define η on the neighborhoods of vertices and edges in a similar way. Figure 3 shows the 1-form η on the neighborhood $\mathrm{Nbd}(v_j)$ of a vertex v_j, and the gluing map from $R_i \times [0,1]$ to $\mathrm{Nbd}(v_j)$. Finally, we extend η from $\mathrm{Nbd}(P)$ to the whole M using the product structure $D^2 \times [0,1]$ defined by a non-singular flow carried by P. Since each vertex of P is of ℓ-type, η turns out to be a *confoliation*, i.e. $\eta \wedge d\eta \geq 0$.

Proof sketch of Theorem 2.3 (1). Let M be a closed oriented 3-manifold and $P \subset M$ be a positive flow-spine. Since P is admissible by Proposition 3.1, we can assign real numbers to the edges of P satisfying the condition in the definition of the admissibility. Then we can find a 1-form β on a neighborhood of $S(P)$ in P such that $d\beta > 0$ and $\int_{\partial R_i} \beta$ is

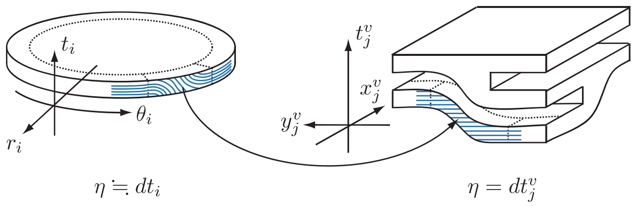

Figure 3: The 1-form η.

the real number given by the assignment. We extend β to the whole P by using Stokes' Theorem in a similar way as in Thurston-Winkelnkemper's construction [10], and then extend that to a neighborhood $\mathrm{Nbd}(P)$ of P in a natural way. Set $\alpha := \alpha_R = \beta + R\eta$, where $R > 0$. Then we have

$$\alpha \wedge d\alpha = \beta \wedge d\beta + R(\eta \wedge d\beta + \beta \wedge d\eta) + R^2 \eta \wedge d\eta.$$

We can show that $\beta \wedge d\beta = 0$, $\beta \wedge d\eta = 0$ and $\eta \wedge d\beta > 0$, thus α_R is a contact form on $\mathrm{Nbd}(P)$. The Reeb vector field R_α is positively transverse to P provided that R is sufficiently large. We extend such a contact form α to the whole M in a natural way. The contact structure $\xi = \ker \alpha$ is then supported by P. The uniqueness of the contact structures supported by P up to isotopy is due to Gray's stability. □

Proof sketch of Theorem 2.3 (2). Let M be a closed oriented 3-manifold and ξ be a contact structure on M. By Giroux [3], there exists an open book decomposition of M whose pages are transverse to a Reeb flow of ξ. We then construct a positive flow-spine P from a finite number of pages and adding more regions according to its monodromy vector field. By Theorem 2.3 (1), we know that there exists a contact form whose Reeb vector field is carried by P. We may choose such a contact form so that it is supported by the open book decomposition. The contact structure thus obtained is isotopic to ξ by the Giroux correspondence. Consequently, ξ is carried by P. □

4 Complexity of contact 3-manifolds

For a contact 3-manifold (M, ξ), we define the *complexity* $c(M, \xi)$ of (M, ξ) to be the minimum number of vertices of any positive flow-spine that supports (M, ξ). By Theorem 2.3 (2), $c(M, \xi)$ is well-defined for any contact 3-manifold (M, ξ). Since there exist only finitely may flow-spines, which are actually simple polyhedra equipped with some additional structures, of a given number of vertices, the complexity c is a finite-to-one invariant.

There exists exactly one positive flow-spine P, called a *positive abalone*, with a single vertex. The left-hand side in Figure 4 shows a neighborhood of $S(P)$ in P. This is a spine of S^3. The right-hand side in Figure 4 depicts the metric completion of $M \setminus P$ with the path metric inherited from a Riemannian metric on M. The pattern, which is a 3-regular graph, on the bounadry of the 3-ball comes from the singular set $S(P)$ of P. We can show that after moving the boundary-pattern of the 3-ball by an isotopy, the constant

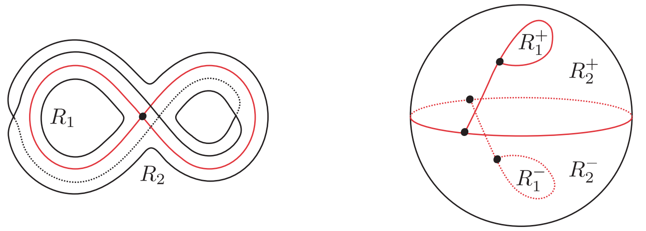

Figure 4: The positive abalone P and the metric completion of $M \setminus P$.

vertical flow on the 3-ball defines a non-singular flow on S^3 whose orbits form the Seifert fibration of S^3 with a regular fiber a trefoil. This is the Reeb flow of the contact form $(2(x_1dy_1 - y_1dx_1) + 3(x_2dy_2 - y_2dx_2))|_{S^3}$, where x_1, y_1, x_2, y_2 are the standard coordinates of $\mathbb{R}^4$ and S^3 is the unit sphere in $\mathbb{R}^4$. The kernel of this form is contactomorphic to the standard contact structure $\xi_{\rm std}$ on S^3. Consequently, P supports the standard contact structure on S^3. In other words, $c(M, \xi) = 1$ if and only if (M, ξ) is contactomorphic to $(S^3, \xi_{\rm std})$.

There exists exactly three positive flow-spines with two vertices, and we can check that they respectively support $(S^3, \xi_{\rm std})$, $(\mathbb{RP}^3, \xi_{\rm tight})$ and $(L(3,2), \xi_{\rm tight})$, where $\xi_{\rm tight}$ is the unique tight contact structure on $\mathbb{RP}^3$ or $L(3,2)$. Thus $c(M, \xi) = 2$ if and only if (M, ξ) is contactomorphic to $(\mathbb{RP}^3, \xi_{\rm tight})$ or $(L(3,2), \xi_{\rm tight})$.

It seems that any positive flow-spine with at most 3 vertices supports a tight contact structure. On the other hand, there exists a positive flow-spine of S^3 with 5 vertices supporting an overtwisted contact structure. It is interesting to determine whether there exists a positive flow-spine with 4 vertices supporting an overtwisted contact structure. It is also an interesting problem to give a criterion for the tightness of contact structures in terms of supporting positive flow-spines.

Acknowledgments

The second author is supported by JSPS KAKENHI Grant Number 19K03499, 17H06128 and Keio University Academic Development Funds for Individual Research. The third author is supported by JSPS KAKENHI Grant Numbers 15H03620, 17K05254, 17H06463, and JST CREST Grant Number JPMJCR17J4. The fourth author is supported by JSPS KAKENHI Grant Number 18H05827. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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