数理解析研究所講究録 第2114巻 2019年 92-95

RECENT RESULTS ON SEQUENTIAL OPTIMALITY THEOREMS FOR CONVEX OPTIMIZATION PROBLEMS

JAE HYOUNG LEE AND GUE MYUNG LEE

ABSTRACT. In this brief note, we review sequential optimality theorems in [5]. We give two kinds of sequential optimality theorems for a convex optimization problem, which are expressed in terms of sequences of ϵ-subgradients and subgradients of involved functions.

1. INTRODUCTION

Consider the following convex programming problem

$$\begin{aligned} \text{(CP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leqq 0, \ i = 1, \ldots, m, \end{aligned}$$

where $\overline{\mathbb{R}} = [-\infty, +\infty]$ and f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, are proper lower semi-continuous convex functions.

New sequential Lagrange multiplier conditions characterizing optimality without any constraint qualification for convex programs have been presented in terms of the subgradients and the ϵ-subgradients ([2, 3, 4]).

In this paper, we review sequential optimality results in [5]. We give two kinds of sequential optimality theorems for a convex optimization problem, which are expressed in terms of sequences of ϵ-subgradients and subgradients of involved functions. The involved functions of the problem are proper, lower semi-continuous and convex functions.

2. PRELIMINARIES

Let us give some notations and preliminary results which will be used throughout this thesis.

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y \rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set A in $\mathbb{R}^n$ is convex whenever $\mu a_1 + (1 - \mu) a_2 \in A$ for all $\mu \in [0, 1]$, $a_1, a_2 \in A$.

Date: January 4, 2018.

1991 *Mathematics Subject Classification.* 90C22, 90C25, 90C46.

Key words and phrases. sequential optimality theorems, ϵ-subgradient, convex optimization problems.

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$, and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by dom$f$, that is, $\mathrm{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The epigraph of f, epi$f$, is defined as $\mathrm{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leqq r\}$, and f is said to be convex if epi$f$ is convex.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leqq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \mathrm{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

More generally, for any $\epsilon \geqq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n : \langle x^*, y - x\rangle \leqq f(y) - f(x) + \epsilon,\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \mathrm{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We say that f is a lower semicontinuous function if $\liminf_{y \to x} f(y) \geqq f(x)$ for all $x \in \mathbb{R}^n$.

As usual, for any proper convex function g on $\mathbb{R}^n$, its conjugate function $g^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) : x \in \mathbb{R}^n\}$ for any $x^* \in \mathbb{R}^n$.

We recall a version of the Brondsted-Rockafellar theorem which was established in [6].

Proposition 2.1. [1, 6] **(Brondsted-Rockafellar Theorem)** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then for any real number $\epsilon > 0$ and any $x^* \in \partial_\epsilon f(\bar{x})$ there exist $x_\epsilon \in \mathbb{R}^n$, $x_\epsilon^* \in \partial f(x_\epsilon)$ such that*

$$\|x_\epsilon - \bar{x}\| \leqq \sqrt{\epsilon}, \quad \|x_\epsilon^* - x^*\| \leqq \sqrt{\epsilon} \quad \text{and} \quad |f(x_\epsilon) - \langle x_\epsilon^*, x_\epsilon - \bar{x}\rangle - f(\bar{x})| \leqq 2\epsilon.$$

3. Sequential Optimality Theorems

The following theorem is a sequential optimality result for (CP), which is expressed sequences of ϵ-subgradients of involved functions. The involved functions of the problem are proper, lower semi-continuous and convex functions.

Theorem 3.1. [5] *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \ldots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \ldots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of (CP);*

(ii) *there exist $\delta_k \geqq 0$, $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $\xi_k \in \partial_{\delta_k} f(\bar{x})$ and $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that*

$$\lim_{k \to \infty} (\xi_k + \zeta_k) = 0, \quad \lim_{k \to \infty} (\delta_k + \gamma_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} (\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) = 0.$$

Theorem 3.2. [5] *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \dots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \dots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$. Assume that* $\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i)^*$ *is closed. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP);

(ii) *there exist $\gamma_k \geqq 0$, $\lambda_i^k \geqq 0$, $i = 1, \dots, m$, $\xi \in \partial f(\bar{x})$, $\zeta_k \in \partial_{\gamma_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$ such that*

$$\xi + \lim_{k \to \infty} \zeta_k = 0, \quad \lim_{k \to \infty} \gamma_k = 0 \text{ and } \lim_{k \to \infty} (\sum_{i=1}^m \lambda_i^k g_i)(\bar{x}) = 0.$$

The following theorem is a sequential optimality result for (CP), which involve only the subgradients at nearby points to a minimizer of (CP). It is established by Proposition 2.1(a version of Brondsted-Rockafellar Theorem) and Theorem 3.1

Theorem 3.3. [5] *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \dots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \dots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP);

(ii) *there exist $x_k \in \mathbb{R}^n$, $\lambda_i^k \geqq 0$, $i = 1, \dots, m$, $\bar{\xi}_k \in \partial f(x_k)$, $\bar{\zeta}_k \in \partial(\sum_{i=1}^m \lambda_i^k g_i)(x_k)$ such that*

$$\lim_{k \to \infty} x_k = \bar{x}, \quad \lim_{k \to \infty} (\bar{\xi}_k + \bar{\zeta}_k) = 0,$$

$$\textit{and} \quad \lim_{k \to \infty} \left[f(x_k) + (\sum_{i=1}^m \lambda_i^k g_i)(x_k) - f(\bar{x}) \right] = 0.$$

The following theorem is a sequential optimality result for (CP), which involve only the subgradients at nearby points to a minimizer of (CP). It is established by Proposition 2.1(a version of Brondsted-Rockafellar Theorem) and Theorem 3.2

Theorem 3.4. [5] *Let f, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, \dots, m$, be proper lower semi-continuous convex functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, \dots, m\} \neq \emptyset$ and let $\bar{x} \in A$. Assume that $A \cap \mathrm{dom} f \neq \emptyset$ and* $\mathrm{epi} f^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^n \lambda_j g_i)^*$ *is closed. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an optimal solution of* (CP);

(ii) *there exist $x_k \in \mathbb{R}^n$, $\lambda_i^k \geqq 0$, $i = 1, \dots, m$, $\bar{\xi} \in \partial f(\bar{x})$, $\bar{\zeta}_k \in \partial(\sum_{i=1}^m \lambda_i^k g_i)(x_k)$ such that*

$$\lim_{k \to \infty} x_k = \bar{x}, \ \bar{\xi} + \lim_{k \to \infty} \bar{\zeta}_k = 0 \textit{ and } \lim_{k \to \infty} (\sum_{i=1}^m \lambda_i^k g_i)(x_k) = 0.$$

References

[1] A. Brondsted and R. T. Rockafellar, On the subdifferential of convex functions, *Proc. Amer. Math. Soc.*, **16** (1965), 605-611.

[2] V. Jeyakumar, Z. Y. Wu, G. M. Lee and N. Dinh, Liberating the subgradient optimality conditions from constraint qualifications, *J. Global Optim.*, **36** (2006), 127-137.

[3] V. Jeyakumar, G. M. Lee and N. Dinh, New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs, *SIAM J. Optim.*, **14** (2003), 534-547.

[4] J. H. Lee and G. M. Lee, On sequential optimality conditions for robust multiobjective convex optimizatiojn problems, *Linear and Nonlinear Analysis*, **2** (2015), 221-246.

[5] J. H. Lee and G. M. Lee, On sequential optimality theorems for convex optimization problems, *Linear and Nonlinear Analysis*, **3** (2017), 155-170.

[6] L. Thibault, Sequential convex subdifferential calculus and sequential Lagrange multipliers, *SIAM. J. Control Optim.*, **35** (1997), 1434-1444.

Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

E-mail address: mc7558@naver.com

Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

E-mail address: gmlee@pknu.ac.kr