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On n-normal operators

by

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Abstract

Let T be a bounded linear operator on a complex Hilbert space. T is said to be n-normal if $T^*T^n = T^nT^*$, where T^* is the dual operator of T. First we explain the study of n-normal of operators given by S.A. Alzuraiqi and A.B. Patel. Next we show our results of n-normal operators.

1 n-normal operator (Alzuraiqi and Patel's results)

First we explain Alzuraiqi and Patel's results of n-normal operators.

Definition 1.1 *A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be n-normal if $T^*T^n = T^nT^*$.*

This definition, may be, first appeared in "S. A. Alzuraiqi and A. B. Patel, *On n-normal operators*, General Math. Notes, **1**(2010), 61-73".

Theorem 1.1 (Characterization) *T is n-normal if and only if T^n is normal.*

Proof. Proof is clear from $T^{*n}T^n = T^{*n-1}T^nT^* = \cdots = T^nT^{*n}$ and converse is from Fuglede-Putnam's Theorem, i.e., since $T \cdot T^n = T^n \cdot T$. $\quad \therefore \quad T^* \cdot T^n = T^n \cdot T^*$.

It's clear that if T satisfies $T^n = 0$, then T is n-normal, that is, n-nilpotent operator is n-normal.

Theorem 1.2 (Fundamental results) *Let T be n-normal. Then*

(1) *T^* is n-normal;*
(2) *T^m is n-normal for all $m \in \mathbb{N}$;*
(3) $^\exists T^{-1} \implies T^{-1}$ *is n-normal;*
(4) *$\mathcal{M}$ is reducing subspace of $T \implies T_{|\mathcal{M}}$ is n-normal.*

Proof. Proof is clear.

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Theorem 1.3 *$T - z$ is n-normal for every $z \in \mathbb{C}$ $\rightleftharpoons$ T is normal.*

Proof. For $z \neq 0$, since $(T-z)^*(T-z)^n - (T-z)^n(T-z)^* = 0$,

$$\therefore \sum_{k=1}^{n-1} (-1)^k \binom{m}{k} z^k (T^*T^{n-k} - T^{n-k}T^*) = 0.$$

$$(-1)^{n-1}(T^*T - TT^*) = -\sum_{k=1}^{n-2} (-1)^k \frac{\binom{m}{k} z^k}{z^{n-1}} (T^*T^{n-k} - T^{n-k}T^*).$$

Theorem 1.4 *T is n-normal and $(n+1)$-normal $\Longrightarrow$ T is $(n+2)$-normal.*

Proof. Since $T^*T^n = T^nT^*$ and $T^*T^{n+1} = T^{n+1}T^*$, hence we have
$T \cdot T^*T^n \cdot T = T \cdot T^nT^* \cdot T$, $T \cdot T^*T^{n+1} = T^{n+1}T^* \cdot T$.
$\therefore$ $T^{n+2}T^* = T^*T^{n+2}$.

Theorem 1.5 *Let T be n-normal and $(n+1)$-normal.*
Either T or T^* is injective $\Longrightarrow$ T is normal.

Proof. Let T be injective. Since

$$T^n \cdot TT^* = T^{n+1}T^* = T^*T^{n+1} = T^*T^n \cdot T = T^n \cdot T^*T$$

and hence $T^n(TT^* - T^*T) = 0$. Therefore, T is normal.

It is similar in the case that T^* is injective.

Theorem 1.6 *T is n-normal and quasinilpotent $\Longrightarrow$ $T^n = 0$.*

Proof. Since T^n is normal and $\sigma(T) = \{0\}$, $\|T^n\| = r(T^n)$. Since $\sigma(T^n) = \{0\}$, $T^n = 0$.

Theorem 1.7 *T is n-normal and T is partial isometry, $\Longrightarrow$ T is $(n+1)$-normal.*

Proof. Since $TT^*T = T$, i.e., $TT^*T^n = T^n = T^nT^*T$ $\therefore$ $T^{n+1}T^* = T^*T^{n+1}$.

Theorem 1.8 *For T, let $F = T^n + T^*$ and $G = T^n - T^*$. Then*
T : n-normal $\Longleftrightarrow$ $FG = GF$.

Proof. Proof is clear from $FG = T^{2n} + T^*T^n - T^nT^* - T^{*2}$, $GF = T^{2n} - T^*T^n + T^nT^* - T^{*2}$.

- Next let D be bounded open disk of $\mathbb{C}$, $L^2(D, \mathcal{H})$ be Hilbert space, and $W^2(D,\mathcal{H})$ be Sobolev space. Then, it holds $W^2(D,\mathcal{H}) \subset L^2(D,\mathcal{H})$.

They assumed the following property:

(1) $$\sigma(T) \bigcap (-\sigma(T)) = \emptyset.$$

Theorem 1.9 *Let T be 2-normal and satisfy* **(1)**. *Then*

$z - T \;:\; W^2(D,\mathcal{H}) \longrightarrow L^2(D,\mathcal{H})$ *is one-to-one for every* $z \in \mathbb{C}$.

- In this case, T is automatically **invertible**.

2 Our results (2-normal operator)

Single-valued extension property

• We say that an operator T has the *single-valued extension property at* λ (SVEP at λ) if for every open set U containing λ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U.

- T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.
- A normal operator S has SVEP and
- S has SVEP $\Longrightarrow$ $p(S)$ has SVEP for any polynomial $p(\cdot)$
- $p(S)$ has SVEP for some polynomial $p(\cdot)$ $\Longrightarrow$ S has SVEP

Therefore we have following result.

Theorem 2.1 *T is n-normal* $\Longrightarrow$ *T has* SVEP.

For T, we set the following property:

(**2**) $$\sigma(T) \bigcap (-\sigma(T)) \subset \{0\}.$$

- (**2**) is OK for **NOT** invertible operators.

Lemma 2.1 *Let T satisfy* (**2**)*. z is an isolated point of $\sigma(T)$ $\Longrightarrow$ z^2 is an isolated point of $\sigma(T^2)$.*

Proof. If $z = 0$, then the proof is easy. If $z \neq 0$, then it follows from $T^2 - z^2 = (T+z)(T-z)$ and **(2)**, because $-z \notin \sigma(T)$.

Theorem 2.2 *Let T be 2-normal and satisfy* (**2**)*. Then $\sigma(T) = \sigma_a(T)$.*

Proof. Let $z \in \sigma_r(T)$. Then $^{\exists}x \neq 0$; $T^* x = \overline{z}x$. Since $T^{*2}x = \overline{z}^2 x$ and T^{*2} : normal. $\therefore$ $T^2 x = z^2 x$ From this, it is clear that $z \in \sigma_p(T)$ by (**2**).

Theorem 2.3 *Let T be 2-normal and satisfy* (**2**)*.*
(1) $Tx = z \cdot x$, $Ty = w \cdot y$. $z \neq w$ $\Longrightarrow$ $\langle x, y \rangle = 0$.
(2) *Let $\{x_n\}$, $\{y_n\}$ be the sequences of unit vectors in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - w)y_n \to 0$ $(n \to \infty)$. $z \neq w \Longrightarrow \lim_{n \to \infty} \langle x_n, y_n \rangle = 0$.*

Proof. If $z^2 = w^2$, then $z = w$ or $z = -w$. Hence, it is easy.

Since T^2 is normal, it's easy. So, we have the following corollary.

Corollary 2.4 *Let T be 2-normal and satisfy* (**2**)*.*
$z, w \in \sigma_p(T)$ is $z \neq w$ $\Longrightarrow$ $\ker(T - z) \perp \ker(T - w)$.

Theorem 2.5 *Let T be 2-normal and satisfy* (**2**)*.*
$0 \neq z \in \sigma_p(T)$ $\Longrightarrow$ $\ker(T - z) = \ker(T^2 - z^2) = \ker(T^{*2} - \overline{z}^2) = \ker(T^* - \overline{z})$ *and hence* $\ker(T - z)$ *is a reducing subspace for T.*

Next we study Weyl's theorem. For T, the *Weyl spectrum* $\omega(T)$ is defined by

$$\omega(T) = \bigcap_{K \in \mathcal{C}(\mathcal{H})} \sigma(T + K),$$

where $\mathcal{C}(\mathcal{H})$ is the set of all compact operators on $\mathcal{H}$. Let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of finite multiplicity of T. We say that *Weyl's theorem holds for T* if $\omega(T) = \sigma(T) - \pi_{00}(T)$. J.V. Baxley showed the following result.

Theorem 2.6 **(Baxley)** *Let T satisfy the following condition*

C-1: *"If $\{z_n\}$ is an infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of T and $\{x_n\}$ is any sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge." Then*

$$\sigma(T) - \pi_{00}(T) \subset \omega(T).$$

If T is a 2-normal operator satisfying (**2**), then T satisfies the condition C-1 by Corollary 2.4. Hence we have the following result by Theorem 2.6.

Theorem 2.7 *T is a 2-normal operator satisfying* (**2**) $\Longrightarrow \sigma(T) - \pi_{00}(T) \ \subset\ \omega(T)$.

For the converse inclusion, we show the following result.

Theorem 2.8 *T is a 2-normal operator satisfying* (**2**) $\Longrightarrow \omega(T) \ \subset\ \sigma(T) - \big(\pi_{00}(T) - \{0\}\big)$.
If T satisfies (**1**), then T is invertible and $0 \notin \sigma(T)$. Hence we have the following result by Theorems 2.7 and 2.8.

Theorem 2.9 *T is 2-normal operator satisfying* (**1**) $\Longrightarrow\ \omega(T) = \sigma(T) - \pi_{00}(T)$. *That is, Weyl's theorem holds for T.*

- Let z be an isolated point of $\sigma(T)$. Then let

$$E_T(\{z\}) := \frac{1}{2\pi i}\int_{\partial D} (\lambda - T)^{-1}\, d\lambda,$$

where D is a nice closed disk centered at z and

$$H_0(T - z) := \{x \in \mathcal{H} \mid \lim_{n\to\infty} \|(T - z)^n x\|^{\frac{1}{n}} = 0\,\}.$$

Theorem 2.10 *Let T be a 2-normal operator satisfying* (**2**) *and z be an isolated point of $\sigma(T)$.*

(1) *If $z = 0$, then $H_0(T) = \ker(T^2)$.*

(2) *If $z \neq 0$, then $H_0(T - z) = \ker(T - z)$ and $E_T(\{z\})$ is self-adjoint.*

Theorem 2.11 *Let T be 2-normal and satisfy*

(3) $$m\big(\sigma(T) \bigcap (-\sigma(T))\big) = 0,$$

where m is the planer Lebesgue measure. Then

$z - T \,:\, W^2(D, \mathcal{H}) \longrightarrow L^2(D, \mathcal{H})$ *is one-to-one for* $\forall\ z \in \mathbb{C}$.

- **(3)** is weaker than **(2)**; $\sigma(T) \cap (-\sigma(T))) \subset \{0\}$.

3 n-normal operator

In this section, we show spectral properties of n-normal operators. Recall that, for $n \in \mathbb{N}$, T is said to be n-normal if $T^*T^n = T^nT^*$. First we extend Proposition 2.19 of [1] as follows:

Theorem 3.1 *The following statements are equivalent*:
(1) $T - t$: *n-normal for all $t \geq 0$.*

(2) T *is normal.*
(3) $T - z$ *is n-normal for all* $z \in \mathbb{C}$.

Proof. Proof follows from equation of Theorem 1.3.

$$(-1)^{n-1}(T^*T - TT^*) = -\sum_{k=1}^{n-2}(-1)^k \frac{\binom{m}{k} t^k}{t^{n-1}}(T^*T^{n-k} - T^{n-k}T^*).$$

For an n-normal operator $T \in B(\mathcal{H})$, we set the following property:

$$\text{(4)} \qquad \sigma(T) \bigcap \left(\bigcup_{j=1}^{n-1} e^{\frac{2j\pi}{n} i} \sigma(T) \right) \subset \{0\}.$$

Then we continue with the following lemma.

Lemma 3.2 *Let* T *satisfy* **(4)**. z *is an isolated point of* $\sigma(T)$

$\Longrightarrow$ z^n *is isolated point of* $\sigma(T^n)$.

If T is n-normal, then T^n is normal by Theorem 1.2. Hence, by Lemma 3.2, we have the following results. The proofs are similar to the proofs of Theorem 2.2,Theorem 2.3,Theorem 2.4, Theorem 2.6 and Corollary 2.5. So the proofs are omitted.

Theorem 3.3 *Let* T *be n-normal and satisfy* (**3**). *Then* T *is isoloid.*

Theorem 3.4 *Let* T *be n-normal and satisfy* (**4**). *Then* $\sigma(T) = \sigma_a(T)$.

Theorem 3.5 *Let* T *be n-normal and satisfy* (**3**).
(1) $Tx = z \cdot x$ and $Ty = w \cdot y, \quad z \neq w \implies \langle x, y \rangle = 0$.
(2) $\{x_n\}, \{y_n\}$ *is the sequences of unit vectors in* $\mathcal{H}$ *such that* $(T - z)x_n \to 0$ *and* $(T - w)y_n \to 0$ $(n \to \infty)$ $z \neq w \implies \lim_{n\to\infty} \langle x_n, y_n \rangle = 0$.

Corollary 3.6 *Let* T *be n-normal and satisfy* (**3**). $z \neq w \implies \ker(T - z) \perp \ker(T - w)$.

Theorem 3.7 *Let* T *be n-normal and satisfy* (**3**)
z is non-zero eigenvalue of $T \implies$

$$\ker(T - z) = \ker(T^n - z^n) = \ker(T^{*n} - \overline{z}^n) = \ker(T^* - \overline{z}).$$

$\therefore$ $\ker(T - z)$ is reducing subspace for T.

Theorem 3.8 T *is an n-normal operator satisfying* (**3**) $\implies$

$$\sigma(T) - \pi_{00}(T) \subset \omega(T) \subset \sigma(T) - \big(\pi_{00}(T) - \{0\}\big)$$

Moreover, T *invertible* $\implies \sigma(T) - \pi_{00}(T) = \omega(T)$, *that is, Weyl's theorem holds for* T.

T is scalar order $m \iff {}^{\exists}\Phi : C_0^m(\mathbb{C}) \to B(\mathcal{H})\,;\ \Phi(z) = T$.

T is subscalar order $m \iff T_{\mathcal{M}} \sim S$: scalar order m on $\mathcal{M}$.

Theorem 3.9 *Let T be n-normal. $\sigma(T)$ is contained in an angle $< \dfrac{2\pi}{n}$ with vertex in the origin, i.e., $\exists\theta_1 \in [0, 2\pi)$;*

$$\sigma(T) \subset W = \left\{re^{i\theta} : 0 < r, \theta_1 < \theta < \theta_1 + \frac{2\pi}{n}\right\} \implies T \text{ is subscalar of order } 2.$$

Proof is too long ! Please see [3].

Corollary 3.10 *Under same hypothesis of Theorem 3.9,*
$\sigma(T)^\circ \neq \emptyset \implies T$ *has non-trivial invariant subspace.*

$z \in \sigma(T)$, $n \in \mathbb{N}$ and $\zeta := \exp(\frac{2\pi i}{n})$. We say that T has property (n) at z if

$$\zeta^k \cdot z \notin \sigma(T) \quad \text{for } k = 1, \cdots, n-1.$$

Theorem 3.11 *T is n-normal. Then* (i) $H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$.
(ii-1) $z \neq 0 \implies H_0(T - z) = \ker(T - z)$.
(ii-2) $z \neq 0$ *and T has property (n) at z* $\implies H_0(T - z) = \ker(T - z) = \ker((T - z)^*)$.

Proof is too long ! Please see [3].

4 (n, m)-normal operator

We begin with the definition of (n, m)-normal operators.

Definition 4.1 *For $n, m \in \mathbb{N}$, T is said to be (n, m)-normal if*

$$T^{*m}T^n = T^nT^{*m}.$$

From the definition, it is clear that T is (n, m)-normal if and only if T is (m, n)-normal. Let $T \in B(\mathcal{H})$ be (n, m)-normal. Then the followings hold clearly:

Theorem 4.1 *Let T be (n, m)-normal. Then*
(1) T^* is (m, n)-*normal.*
(2) ${}^{\exists}T^{-1} \implies T^{-1}$ is (n, m)-*normal.*
(3) *$S \in B(\mathcal{H})$ is $S \sim T \implies S$ is (n, m)-normal.*
(4) *$\mathcal{M}$ is closed subspace of $\mathcal{H}$ which reduces T.*
$\implies T_{|\mathcal{M}}$ *is (n, m)-normal on $\mathcal{M}$.*

Lemma 4.2 (1) *T is (n,m)-normal $\Longrightarrow$ T^k is normal, where k is the least common multiple of n and m.*
(2) *T^n is normal $\Longrightarrow$ T is (n,m)-normal for every m.*

Proof. (1) Let $k = n \cdot j$ and $k = m \cdot \ell$. If T is (n,m)-normal, then

$$T^{*k}T^k = \overbrace{T^{*m} \cdots T^{*m}}^{\ell} \cdot \overbrace{T^n \cdots T^n}^{j} = T^n \cdots T^n \cdot T^{*m} \cdots T^{*m} = T^k T^{*k}.$$

Hence T^k is normal.
(2) Since T^n is normal and $T^m \cdot T^n = T^n \cdot T^m$, it follows from Fuglede-Putnam's theorem that $T^{*m} \cdot T^n = T^n \cdot T^{*m}$. Hence, T is (n,m)-normal.

Theorem 4.3 *T is quasi-nilpotent and (n,m)-normal $\Longrightarrow$ T* is nilpotent.

Proof. Since $\sigma(T) = \{0\}$, we have $\sigma(T^k) = \{0\}$ for every $k \in \mathbb{N}$.

Let k be the least common multiple of n and m.

Then, by Lemma 4.2, T^k is normal. Hence $T^k = 0$.

Theorem 4.4 *Let T and S be commuting (n,m)-normal operators $\Longrightarrow$ TS is (k,j)-normal for every $j \in \mathbb{N}$ and the least common multiple k of n and m.*

Proof. Since k is the least common multiple of n and m, by Lemma 4.2, $(TS)^k$ is normal. Since $(TS)^k$ commutes with $(TS)^j$ for every $j \in \mathbb{N}$. By Fuglede-Putnam's theorem, it holds

$$(TS)^{*j}(TS)^k = (TS)^k(TS)^{*j}.$$

Hence TS is (k,j)-normal for every $j \in \mathbb{N}$.

Theorem 4.5 *Let T be (n,m)-normal and $(n+1,m)$-normal.*

Either T or T^* is injective $\Longrightarrow$ T is m-normal.

Proof. Let T be injective. Since T is (n,m)-normal and $(n+1,m)$-normal, it holds

$$T^{n+1}T^{*m} = T^{*m}T^{n+1} = (T^{*m}T^n)T = T^nT^{*m}T.$$

$\therefore$ $T^n(TT^{*m} - T^{*m}T) = 0$. Since T is injective,

$\therefore$ $TT^{*m} = T^{*m}T$ and $T^*T^m = T^mT^*$ $\therefore$ T : m-normal.

Theorem 4.6 *For T, let $F = T^n + T^{*m}$ and $G = T^n - T^{*m}$. Then*
T is (n,m)-normal $\Longleftrightarrow$ F commutes with G.

Proof. By $FG = T^{2n} - T^nT^{*m} + T^{*m}T^n - T^{*2m}$ and $GF = T^{2n} + T^nT^{*m} - T^{*m}T^n - T^{*2m}$, hence $FG = GF$ if and only if $T^{*m}T^n = T^nT^{*m}$. It completes the proof.

Theorem 4.7 *For T, let $A = T^nT^{*m}$, $F = T^n + T^{*m}$ and $G = T^n - T^{*m}$. Then T (n, m)-normal $\Longrightarrow A$ commutes with F and G.*

Proof. Since T is (n, m)-normal, we have

$$AF = T^nT^{*m}(T^n + T^{*m}) = T^nT^nT^{*m} + T^{*m}T^nT^{*m} = FA.$$

Similarly we have $AG = GA$.

Theorem 4.8 *Let T be an invertible (n, m)-normal operator. Then T and T^{-1} have a common nontrivial closed invariant subspace.*

Proof. Let k be the least common multiple of n and m. Then by Lemma 4.2, $\therefore$ T^k : normal. Hence T^{-k} : also normal.
$\therefore$ T^k and T^{-k} have NO hypercyclic vector. $\therefore$ T and T^{-1} have NO hypercyclic vector.
$\therefore$ T and T^{-1} have a common nontrivial closed invariant subspace.

- T is polaroid $\Longleftrightarrow$ If z is an isolated point of $\sigma(T)$, then z is a simple pole.

Theorem 4.9 *Let T be (n, m)-normal. Then*
(1) *T is isoloid and polaroid.*
Let z be an isolated point of $\sigma(T)$. Then
(2-1) *if $z = 0$, then $H_0(T) = E_T(\{0\}) = \ker(T^{nm}) = \ker(T^{*nm})$.*
(2-2) *if $z \neq 0$, then $H_0(T - z) = E_T(\{z\}) = \ker(T - z)$.*

Since normal operator is decomposable and has SVEP, we have following results.

Theorem 4.10 *Let T be (n, m)-normal. Then*
(1) *T is decomposable.*
(2) *f is analytic on $\sigma(T)$ and not constant on each domain*
$\Longrightarrow$ *Weyl's theorem holds for $f(T)$.*

Finally, we show results of the direct sum and the tensor product. The proof is easy.

Theorem 4.11 *Let T, S be (n, m)-normal. Then $T \oplus S$ and $T \otimes S$ is (n, m)-normal on $\mathcal{H} \oplus \mathcal{H}$ and $\mathcal{H} \overline{\otimes} \mathcal{H}$, respectively.*

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