Title	Hyperbolic Eisenstein Series on n-dimensional Hyperbolic Spaces (Analytic and Arithmetic Theory of Automorphic Forms)
Author(s)	Irie, Yosuke
Citation	数理解析研究所講究録 = RIMS Kokyuroku (2019), 2100: 151-164
Issue Date	2019-01
URL	http://hdl.handle.net/2433/251801
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録 第2100巻 2019年 151-164

Hyperbolic Eisenstein Series on n-dimensional Hyperbolic Spaces

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1 Introduction

The hyperbolic Eisenstein series is the Eisenstein series associated to hyperbolic fixed points, or equivalently a primitive hyperbolic element of Fuchsian groups of the first kind. It was first introduced by S. S. Kudla and J. J. Millson [7] in 1979 as an analogue of the ordinary Eisenstein series associated to a parabolic fixed point. They established an explicit construction of the harmonic 1-form dual to an oriented closed geodesic on an oriented Riemann surface M of genus greater than 1. Furthermore, they proved the meromorphic continuation of the hyperbolic Eisenstein series to all of $\mathbb{C}$ and gave the location of the possible poles when M is compact. After that, they generalized the results of [7] to compact n-dimensional hyperbolic manifold and its totally geodesic hyperbolic $(n-k)$-manifolds. In [7, 8], they constructed the hyperbolic Eisenstein series by averaging certain smooth closed k-form.

Following Kudla and Millson's point of view, the scalar-valued analogue of the hyperbolic Eisenstein series is defined in [1, 2], and [6]. It is defined as follows. Let $\mathbb{H}^2 := \{z = x+iy \in \mathbb{C} \mid x, y \in \mathbb{R}, y > 0\}$ be the upper-half plane and $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ a Fuchsian group of the first kind acting on $\mathbb{H}^2$ by the fractional linear transformations. Then the quotient $\Gamma\backslash\mathbb{H}^2$ is a hyperbolic Riemann surface of finite volume. Let $\gamma \in \Gamma$ be a primitive hyperbolic element and $\Gamma_\gamma = \langle\gamma\rangle$ be its centralizer group in Γ. Consider the coordinates

$x = e^{\rho}\cos\theta$ and $y = e^{\rho}\sin\theta$. In this setting, the hyperbolic Eisenstein series associated to γ is defined by the series

$$E_{\mathrm{hyp},\gamma}(z,s) := \sum_{\eta\in\Gamma_\gamma\backslash\Gamma} (\sin\theta(A\eta z))^s, \tag{1}$$

where $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$ and A is an element in $\mathrm{PSL}(2,\mathbb{R})$ such that $A\gamma A^{-1} = \left(\begin{smallmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1}\end{smallmatrix}\right)$ for some $a(\gamma) \in \mathbb{R}$ with $|a(\gamma)| > 1$. The hyperbolic Eisenstein series (1) converges for any $z \in \mathbb{H}^2$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and defines a Γ-invariant function where it converges. Furthermore, it is known that the hyperbolic Eisenstein series $E_{\mathrm{hyp},\gamma}(z,s)$ satisfies the following differential equation

$$(-\Delta + s(s-1))E_{\mathrm{hyp},\gamma}(z,s) = s^2 E_{\mathrm{hyp},\gamma}(z,s+2),$$

where Δ is the hyperbolic Laplace-Beltrami operator.

J. Jorgenson, J. Kramer and A.–M. v. Pippich [6], in 2010, proved that the hyperbolic Eisenstein series is a square integrable function on $\Gamma\backslash\mathbb{H}^2$ and obtained the spectral expansion associated to the hyperbolic Laplace-Beltrami operator $-\Delta$ precisely. It is given as follows. Let

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

be the eigenvalues of $-\Delta$ and e_m the eigenfunction corresponding to λ_m. Let $\mathfrak{D} \subset \mathbb{N}$ be an index set for a complete orthogonal system of eigenfunctions $\{e_m\}_{m\in\mathfrak{D}}$. We denote a cusp of $\Gamma\backslash\mathbb{H}^2$ by ν and the ordinary Eisenstein series associated to the cusp ν by $E_\nu(z,s)$. Then the spectral expansion of the hyperbolic Eisenstein series $E_{\mathrm{hyp},\gamma}(z,s)$ is given by

$$\begin{aligned} E_{\mathrm{hyp},\gamma}(z,s) = &\sum_{m\in\mathfrak{D}} a_{m,\gamma}(s)e_m(z) \\ &+ \frac{1}{4\pi}\sum_{\nu:\text{cusps}}\int_{-\infty}^{\infty} a_{1/2+i\mu,\gamma}(s)E_\nu(z,1/2+i\mu)d\mu. \end{aligned} \tag{2}$$

Then this series converges absolutely and locally uniformly. The coefficients $a_{m,\gamma}(s)$ and $a_{1/2+i\mu,\gamma}(s)$ are given by

$$\begin{aligned} a_{m,\gamma}(s) = \sqrt{\pi}\cdot \frac{\Gamma((s-1/2+\mu_m)/2)\Gamma((s-1/2-\mu_m)/2)}{\Gamma(s/2)^2} \\ \times \int_{\widetilde{L}_\gamma} e_m(z)d\sigma \end{aligned} \tag{3}$$

and

$$a_{1/2+i\mu,\gamma}(s) = \sqrt{\pi} \cdot \frac{\Gamma((s-1/2+i\mu)/2)\Gamma((s-1/2-i\mu)/2)}{\Gamma(s/2)^2} \times \int_{\widetilde{L}_\gamma} E_\nu(z, 1/2+i\mu) d\sigma, \quad (4)$$

where $\mu_m^2 = \frac{1}{4} - \lambda_m$ and $\widetilde{L}_\gamma$ is the closed geodesic corresponding to γ. Furtheremore, they proved the meromorphic continuation of $E_{\text{hyp},\gamma}(z,s)$ to the whole complex plane $\mathbb{C}$. They also derived the location of the possible poles and residues from the spectral expansion (2) and the meromorphic continuation.

In our previous paper [3], we defined the hyperbolic Eisenstein series for a loxodromic element of the cofinite Kleinian groups acting on 3-dimensional hyperbolic space and proved the results analogous to [6]. We also in [4] consider the asymptotic behavior of the hyperbolic Eisenstein series for the degeneration of 3-dimensional hyperbolic manifolds and obtain the results corresponding to [1].

Our purpose in this article is to define a generalization of the hyperbolic Eisenstein series (1) for the n-dimensional hyperbolic spaces and prove the spectral expansion of it.

2 Preliminaries

2.1 The hyperboloid model of the hyperbolic n-space

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional real vector space and $\mathbf{e}_i$ $(1 \leq i \leq n+1)$ be the standard basis of $\mathbb{R}^{n+1}$. For any vector $\mathbf{x} \in \mathbb{R}^{n+1}$, we write the coordinate representation in standard basis of $\mathbb{R}^{n+1}$ as

$$\mathbf{x} = (x_1, x_2, ..., x_{n+1}).$$

We consider the Lorentzian inner product $(\ , \ )$ on $\mathbb{R}^{n+1}$. It is defined for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+1}$ as follows.

$$(\mathbf{x}, \mathbf{y}) := x_1 y_1 + x_2 y_2 + ... + x_n y_n - x_{n+1} y_{n+1}.$$

The inner product space $\mathbb{R}^{n+1}$ together with the Lorentzian inner product $(\ , \ )$ is called Lorentzian $(n+1)$-space and is also denoted by $\mathbb{R}^{n,1}$. The norm in $\mathbb{R}^{n+1}$ associated with (,) is defined to be the complex number

$$||\mathbf{x}|| = (\mathbf{x}, \mathbf{x})^{\frac{1}{2}},$$

where $||\mathbf{x}||$ is either positive real number, zero, or positive imaginary. This norm is also called the Lorentzian norm. A function $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lorentz transformation if and only if

$$(\phi(\mathbf{x}), \phi(\mathbf{y})) = (\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^{n+1}$.

We define the hyperbolic n-space as the hyperboloid model. Let $\mathcal{F}^n \subset \mathbb{R}^{n+1}$ be the sphere of unit imaginary radius, i.e.

$$\mathcal{F}^n := \{\ \mathbf{x} \in \mathbb{R}^{n+1} \mid ||\mathbf{x}|| = -1\ \}.$$

Then $\mathcal{F}^n$ is disconnected. The subset of all $\mathbf{x} \in \mathcal{F}^n$ such that $x_{n+1} > 0$ (resp. $x_{n+1} < 0$) is called the *positive* (resp. *negative*) sheet of $\mathcal{F}^n$. The *hyperboloid model* of hyperbolic n-space is defined as the positive sheet of $\mathcal{F}^n$. We denote it by $\mathcal{F}^n_+$. Then, for two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{F}^n_+$, the hyperbolic distance between $\mathbf{x}$ and $\mathbf{y}$ is written as follows.

$$\cosh d_{\mathcal{F}^n_+}(\mathbf{x}, \mathbf{y}) = -(\mathbf{x}, \mathbf{y}),$$

where $(,)$ is the Lorentzian inner product. This hyperbolic distance function defines a hyperbolic metric on $\mathcal{F}^n_+$.

2.2 The upper-half space model

We introduce another model of hyperbolic n-space, namely upper-half space model. Let U^n be the upper-half space of $\mathbb{R}^n$ i.e.

$$U^n = \{\ \mathbf{x} \in \mathbb{R}^n \mid x_n > 0\ \}.$$

The hyperbolic line element and the hyperbolic volume element of U^n associated to d_{U^n} are given as

$$\frac{|d\mathbf{x}|}{x_n} \quad \text{and} \quad \frac{dx_1 \cdots dx_n}{{x_n}^n}.$$

Then the hyperbolic Laplace-Beltrami operator associated with the hyperbolic line element is given by

$$\Delta = x_n^2 \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) - (n-2) x_n \frac{\partial}{\partial x_n}.$$

2.3 Orthogonal group $O(n, 1)$

A real $(n+1) \times (n+1)$ matrix A is said to be Lorentzian if and only if the corresponding linear transformation $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is Lorentzian. The set of all Lorentzian matrices forms a group with the ordinary matrix multiplication. We let

$$G = O(n,1) := \left\{ g \in GL(n+1, \mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} 1_n & \\ & -1 \end{pmatrix} g = \begin{pmatrix} 1_n & \\ & -1 \end{pmatrix} \right\}$$

be the orthogonal group of signature $(n, 1)$. Here 1_n denotes the $n \times n$ unit matrix. Then any element of G is a Lorentzian matrix and G is naturally isomorphic to the group of all Lorentz transformations of $\mathbb{R}^{n+1}$. Immediately, G acts $\mathcal{F}^n$ transitively and preserves the Lorentz inner product so that we can naturally identify G with the isometry group of $\mathcal{F}^n$.

Let K be the stabilizer of $\mathbf{e}_{n+1}$ in G. Then K is a maximal compact subgroup of G. By definition, the determinant of $g \in G$ is equal to $+1$ or -1. We denote the connected component of G (resp. K) containing the unit element by G_0 (resp. K_0). Then G_0 acts on $\mathcal{F}^n_+$ transitively and naturally identifies the orientation preserving isometries on $\mathcal{F}^n_+$. K_0 is the stabilizer of $\mathbf{e}_{n+1}$ in G_0 and a maximal compact subgroup of G_0. Then the quotient space G_0/K_0 is naturally identified with $\mathcal{F}^n_+$.

2.4 Eisenstein series associated to cusps

Let $\Gamma \subset G_0$ be a cofinite discrete subgroup of G_0 and $\zeta \in \mathbb{R}^{n-1} \cup \{\infty\}$ be a cusp. We define the stabilizer-group of ζ by

$$\Gamma_\zeta := \{M \in \Gamma \mid M\zeta = \zeta\}.$$

Choose $A \in G_0$ such that $A\zeta = \infty$. For any $\mathbf{x} \in U^n$, we write its coordinates

$$\mathbf{x} = (x_1, ..., x_n).$$

Then, for any $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$, the Eisenstein series associated to ζ is defined as

$$E_\zeta(\mathbf{x}, s) := \sum_{M \in \Gamma_\zeta \backslash \Gamma} x_n(AM\mathbf{x})^s.$$

The Eisenstein series $E_\zeta(\mathbf{x}, s)$ converges absolutely and locally uniformly for any $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$ and it defines a Γ-invariant function where it converges. Furthermore, It satisfies the following differential equation

$$(-\Delta - s(n-1-s))E_\nu(\mathbf{x}, s) = 0,$$

if s is not a pole of $E_\zeta(\mathbf{x}, s)$.

The Eisenstein series $E_\zeta(\mathbf{x}, s)$ has no poles in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \frac{n-1}{2}\}$ except possibly finitely many points in the semi-open interval $(\frac{n-1}{2}, n-1]$ on the real line.

2.5 Domain of Laplace-Beltrami operator

Let $\Gamma \subset G_0$ be a cofinite subgroup of G_0. We denote by $L^2(\Gamma \backslash U^n)$ the set of all Γ-invariant measurable functions $f : U^n \to \mathbb{C}$ which satisfy

$$\int_{\mathcal{F}_\Gamma} |f|^2 \, dv < \infty,$$

where $\mathcal{F}_\Gamma$ denotes a fundamental domain of Γ. For $f, g \in L^2(\Gamma \backslash U^n)$, the function $f\bar{g}$ is Γ-invariant. Hence the definition

$$\langle f, g \rangle := \int_{\mathcal{F}_\Gamma} f\bar{g} \, dv \tag{5}$$

makes sense and $\langle \cdot, \cdot \rangle$ is an inner product on $L^2(\Gamma \backslash U^n)$. The space $L^2(\Gamma \backslash U^n)$ is a Hilbert space through the inner product $\langle \cdot, \cdot \rangle$. For any $f \in L^2(\Gamma \backslash U^n)$, we have the following proposition.

Proposition 2.1. Every $f \in L^2(\Gamma \backslash U^n)$ has the following spectral expansion associated to $-\Delta$

$$\begin{aligned} f(\mathbf{x}) &= \sum_{m \in \mathfrak{D}} \langle f, e_m \rangle e_m(\mathbf{x}) \\ &+ \frac{1}{4\pi} \sum_{\nu : \text{cusps}} \int_{-\infty}^{\infty} \left\langle f, E_\nu \left(\cdot, \frac{n-1}{2} + it \right) \right\rangle \cdot E_\nu \left(\mathbf{x}, \frac{n-1}{2} + it \right) dt, \end{aligned} \tag{6}$$

where $\mathfrak{D} \subset \mathbb{N}$ is an index set for a complete orthonormal set of eigenfunctions $(e_n)_{n \in \mathfrak{D}}$ for $-\Delta$ in $L^2(\Gamma \backslash U^n)$ and $\langle f, E_\nu(\cdot, \frac{n-1}{2} + it) \rangle$ is defined by $\int_{\mathcal{F}_\Gamma} f(\mathbf{y})\overline{E_\nu(\mathbf{y}, \frac{n-1}{2} + it)}dv(\mathbf{y})$. The series of the right hand side of (6) converges in the norm of the $L^2(\Gamma\backslash U^n)$.

Besides, if $f \in C^{l_0}(\Gamma\backslash U^n) \cap L^2(\Gamma\backslash U^n)$ for a positive integer $l_0 > 0$ such that $l_0 > \frac{n}{2}$ and $-\Delta^l f \in L^2(\Gamma\backslash U^n)$ for any $0 \leq l \leq \lfloor \frac{n+1}{4} \rfloor + 1$, the spectral expansion (6) of f converges uniformly and locally uniformly on $\Gamma\backslash U^n$. Especially, if f and $-\Delta^l f$ are smooth and bounded on $\Gamma\backslash U^n$ for any $0 \leq l \leq \lfloor \frac{n+1}{4} \rfloor + 1$, the spectral expansion (6) of f converges uniformly and locally uniformly on $\Gamma\backslash U^n$.

Proof. See [10]. □

3 Hyperbolic Eisenstein series

Let $V \subset \mathbb{R}^{n+1}$ be a vector subspace of $\dim V = k$ such that $(\mathbf{x}, \mathbf{x}) > 0$ for any $\mathbf{x} \in V$. We denote by $V^\perp$ the orthogonal complement space of V. The dimension of $V^\perp$ is $n - k + 1$. Then $\mathcal{F}_+^n \cap V^\perp$ is a hyperbolic $(n - k)$-plane. Let $\sigma = \sigma_V \in O(n + 1)$ be the involution such that

$$\sigma = \begin{cases} -1 & \text{on} \quad V \\ 1 & \text{on} \quad V^\perp. \end{cases}$$

Then $\mathcal{F}_+^n \cap V^\perp$ is the fixed point set of σ in $\mathcal{F}_+^n$. Let G_σ be the centralizer of σ in G_0 i.e.

$$G_\sigma = \{ \ g \in G_0 \mid \sigma g \sigma = g \ \}.$$

Let $\Gamma \subset G_0$ be a cofinite discrete subgroup i.e. the quotient $\Gamma\backslash\mathcal{F}_+^n$ has finite volume and Γ_σ be the intersection of Γ with G_σ. We assume that $\sigma\Gamma\sigma = \Gamma$ and $\Gamma\backslash(\mathcal{F}_+^n \cap V^\perp)$ is compact.

Without loss of generality, we may assume the vector subspace V and $V^\perp$ in $\mathbb{R}^{n+1}$ as follows.

$$\begin{aligned} V &= \{ \ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_i = 0, \quad k+1 \leq i \leq n+1 \ \} \\ V^\perp &= \{ \ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_i = 0, \quad 1 \leq i \leq k \ \}. \end{aligned}$$

Then the intersection $\mathcal{F}_+^n \cap V^\perp$ is identified with

$$D_\sigma = \{ \ \mathbf{x} \in U^n \mid \mathbf{x} = (0, \cdots, 0, x_{k+1}, \ldots, x_n), \ x_n > 0 \ \}.$$

We introduce the partial polar coordinate on U^n. It is defined as follows. If $2 \leq k \leq n-1$,

$$\begin{cases} x_1 = e^{\rho} \cos\varphi_0 \sin\varphi_1, \\ \vdots \\ x_i = e^{\rho} \cos\varphi_0 \cdots \cos\varphi_{i-1} \sin\varphi_i, \ \ 2 \leq i \leq k-1, \\ \vdots \\ x_k = e^{\rho} \cos\varphi_0 \cdots \cos\varphi_{k-2} \cos\varphi_{k-1}, \\ x_{k+1} = x_{k+1}, \\ \vdots \\ x_{n-1} = x_{n-1}, \ \text{and} \\ x_n = e^{\rho} \sin\varphi_0, \end{cases} \tag{7}$$

where

$$\left(\begin{array}{l} \rho = \log\sqrt{x_1^2 + \cdots + x_k^2 + x_n^2}, \\ 0 < \varphi_0 < \dfrac{\pi}{2}, \\ -\dfrac{\pi}{2} < \varphi_i < \dfrac{\pi}{2}, \ \ 1 \leq i \leq k-2, \ \text{and} \\ 0 \leq \varphi_{k-1} < 2\pi. \end{array} \right.$$

If $k = 1$,

$$\begin{cases} x_1 = e^{\rho} \cos\varphi_0, \\ x_2 = x_2, \\ \vdots \\ x_{n-1} = x_{n-1}, \ \text{and} \\ x_n = e^{\rho} \sin\varphi_0, \end{cases} \tag{8}$$

where

$$\left(\begin{array}{l} \rho = \log\sqrt{x_1^2 + x_n^2}, \\ 0 < \varphi_0 < \pi. \end{array} \right.$$

Under above coordinates, we define the generalized hyperbolic Eisenstein series associated to σ as follows.

Definition 3.1. Let $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$. Then the hyperbolic Eisenstein series associated to the involution σ is defined as follows.

$$E_\sigma(\mathbf{x}, s) := \sum_{\eta \in \Gamma_\sigma \backslash \Gamma} (\sin \varphi_0(\eta \mathbf{x}))^s. \tag{9}$$

Let $d_{\mathrm{hyp}}(\mathbf{x}, D_\sigma)$ be the hyperbolic distance from $\mathbf{x}$ to D_σ. Then we have

$$\sin \varphi_0(\mathbf{x}) \cdot \cosh(d_{\mathrm{hyp}}(\mathbf{x}, D_\sigma)) = 1$$

for any $\mathbf{x} \in U^n$. Using this formula, we can write the Eisenstein series associated to σ as

$$E_\sigma(\mathbf{x}, s) = \sum_{\eta \in \Gamma_\sigma \backslash \Gamma} \cosh(d_{\mathrm{hyp}}(\eta \mathbf{x}, D_\sigma))^{-s}. \tag{10}$$

Definition 3.2. Let $T > 0$ be a positive real number. Then we define the counting function associated to σ as follows.

$$N_\sigma(T; \mathbf{x}, D_\sigma) := \sharp\{\eta \in \Gamma_\sigma \backslash \Gamma \mid d_{\mathrm{hyp}}(\eta \mathbf{x}, D_\sigma) < T\}, \tag{11}$$

where $\sharp$ is the cardinality of the set.

By using the counting function defined above, we can write the hyperbolic Eisenstein series associated to σ as the Stieltjes integrals, namely

$$E_\sigma(\mathbf{x}, s) = \int_0^\infty \cosh(u)^{-s} dN_\sigma(u; \mathbf{x}, D_\sigma). \tag{12}$$

Proposition 3.3. The hyperbolic Eisenstein series associated to σ converges absolutely and locally uniformly for any $\mathbf{x} \in U^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$. It satisfies the following differential shift equation

$$(-\Delta + s(s-n+1))E_\sigma(\mathbf{x}, s) = s(s-n+k+1)E_\sigma(\mathbf{x}, s+2). \tag{13}$$

4 Spectral expansion

Lemma 4.1. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$, the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ is bounded as a function of $\mathbf{x} \in \Gamma \backslash U^n$. If Γ is not cocompact and ν is a cusp such that $\nu = A(x_n\infty)$ for some $A \in G$, then we have the estimate

$$|E_\sigma(\mathbf{x}, s)| = O(x_n(A^{-1}\mathbf{x})^{-\mathrm{Re}(s)}) \tag{14}$$

as $P \to \nu$.

Lemma 4.2. Let $\langle \cdot, \cdot \rangle$ be the inner product in $L^2(\Gamma\backslash U^n)$ and ψ be the real-valued, smooth, bounded function on $\mathcal{F}_\Gamma = \Gamma\backslash U^n$. Assume $\varepsilon > 0$ to be the sufficiently small. Then we have the following estimate

$$\begin{aligned}&\langle E_\sigma(\mathbf{x}, s), \psi\rangle \\ &\quad = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \left(\int_{\Gamma_\sigma\backslash D_\sigma} \psi(\mathbf{x}) dv + O(\varepsilon)\right) \cdot \frac{\Gamma\left((s-n+1)/2\right)\Gamma\left(k/2\right)}{\Gamma\left((s-n+k+1)/2\right)}\end{aligned}$$

as $s \to \infty$.

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots$ be the eigenvalues of $-\Delta$ and e_m the eigenfunction corresponding to λ_m. Let $\mathfrak{D} \subset \mathbb{N}$ be an index set for a complete orthogonal system of eigenfunctions $\{e_m\}_{m\in\mathfrak{D}}$. Then the following theorem holds.

Theorem 4.3. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n-1$, the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ admits the following spectral expansion.

$$\begin{aligned}E_\sigma(\mathbf{x}, s) &= \sum_{m\in\mathfrak{D}} a_{m,\sigma}(s) e_m(\mathbf{x}) \\ &\quad + \frac{1}{4\pi} \sum_{\nu:\text{cusps}} \int_{-\infty}^{\infty} a_{\frac{n-1}{2}+i\mu,\sigma}(s) E_\nu\left(\mathbf{x}, \frac{n-1}{2} + i\mu\right) d\mu, \end{aligned}\tag{15}$$

where E_ν is the ordinary Eisenstein series associated to the cusp ν. Then this series converges absolutely and locally uniformly. The coefficients $a_{m,\sigma}(s)$ and $a_{\frac{n-1}{2}+i\mu,\sigma}(s)$ are given by

$$\begin{aligned}a_{m,\sigma} &= \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \Gamma\left(\frac{k}{2}\right) \\ &\quad \times \frac{\Gamma\left(\left(s - \frac{n-1}{2} + \mu_m\right)/2\right)\Gamma\left(\left(s - \frac{n-1}{2} - \mu_m\right)/2\right)}{\Gamma\left(s/2\right)\Gamma\left(\left(s-n+k+1\right)/2\right)} \\ &\quad \times \int_{\Gamma_\sigma\backslash D_\sigma} e_m dv_2\end{aligned}\tag{16}$$

and

$$a_{\frac{n-1}{2}+i\mu,\sigma} = \frac{1}{2}\mathrm{vol}(S^{k-1}) \cdot \Gamma\left(\frac{k}{2}\right)$$
$$\times \frac{\Gamma\left(\left(s - \frac{n-1}{2} + i\mu\right)/2\right)\Gamma\left(\left(s - \frac{n-1}{2} - i\mu\right)/2\right)}{\Gamma\left(s/2\right)\Gamma\left(\left(s-n+k+1\right)/2\right)}$$
$$\times \int_{\Gamma_\sigma\backslash D_\sigma} E_\nu\left(\mathbf{x}, \frac{n-1}{2} + i\mu\right) dv_2, \quad (17)$$

where $\mu_m^2 = (\frac{n-1}{2})^2 - \lambda_m$ and dv_2 is the hyperbolic volume element restricted on $\Gamma_\sigma\backslash D_\sigma$. In addition, $\mathrm{vol}(S^{k-1})$ denotes the Euclidean volume of the unit $(k-1)$-dimensional sphere

$$S^{k-1} := \{\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}^k \mid |x|^2 = x_1^2 + \cdots + x_k^2 = 1\}.$$

Proof. The hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ is a bounded and smooth function on $\Gamma\backslash U^n$ by Definition 3.1 and Lemma 4.1. Since the hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ satisfies the differential sift equation (13), $-\Delta^l E_\sigma(\mathbf{x}, s)$ is bounded and smooth on $\Gamma\backslash U^n$ for any $0 \le l \le \lfloor\frac{n+1}{4}\rfloor + 1$. Hence, from Proposition 2.1, the hyperbolic Eisenstein series has the spectral expansion (15) and it converges absolutely and locally uniformly.

In order to give the coefficients $a_{m,\gamma}(s)$ and $a_{\frac{n-1}{2}+i\mu,\gamma}(s)$, we calculate the inner product $\langle E_\sigma, e_m\rangle$, which converges by asymptotic bound proved in Lemma 4.2. From the differential equation (13), we have

$$\lambda_m a_{m,\sigma}(s) = \lambda_m\langle E_{\mathrm{hyp},\sigma}, e_m\rangle = \langle E_{\mathrm{hyp},\sigma}, \lambda_m e_m\rangle = \langle -\Delta E_{\mathrm{hyp}}, \sigma, e_m\rangle$$
$$= -s(s-n+1)a_{m,\sigma}(s) + s(s-n+k+1)a_{m,\sigma}(s+2).$$

It implies the relation

$$a_{m,\sigma}(s+2) = \frac{s(s-n+1)+\lambda_m}{s(s-n+k+1)}a_{m,\sigma}(s).$$

For μ_m with $\mu_m^2 = (\frac{n-1}{2})^2 - \lambda_m$, we set the function $g(s)$ by

$$g(s) = \frac{\Gamma((s-\frac{n-1}{2}+\mu_m)/2)\Gamma((s-\frac{n-1}{2}-\mu_m)/2)}{\Gamma(s/2)\Gamma((s-n+k+1)/2)}.$$

Then $g(s)$ satisfies the relation

$$g(s+2) = \frac{s(s-n+1)+\lambda_m}{s(s-n+k+1)} g(s).$$

The quotient $a_{m,\sigma}(s)/g(s)$ is invariant under $s \mapsto s+2$. It is bounded in a vertical strip. Therefore the quotient $a_{m,\sigma}(s)/g(s)$ is constant. We obtain this constant by comparing the order of $a_{m,\sigma}(s)$ as $s \to \infty$ with that of $g(s)$ using Lemma 4.2 and Stirling's asymptotic formula for the gamma function. □

We derive the meromorphic continuation of $E_\sigma(\mathbf{x}, s)$ to all complex plane $\mathbb{C}$ and the possible poles and residues from the spectral expansion.

Theorem 4.4. The hyperbolic Eisenstein series $E_\sigma(\mathbf{x}, s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. The possible poles of the continued function are located at the following points.

(a) $s = \frac{n-1}{2} \pm \mu_m - 2n'$, where $n' \in \mathbb{N}$ and $\mu_m^2 = \left(\frac{n-1}{2}\right)^2 - \lambda_m$ for the eigenvalue λ_m, with residues

$$\begin{aligned}\operatorname{Res}_{s=\frac{n-1}{2}\pm\mu_m-2n'}&\Big[E_\sigma(\mathbf{x}, s)\Big] \\ &= \frac{1}{2}\operatorname{vol}(S^{k-1}) \cdot \frac{(-1)^{n'}\Gamma(k/2)\Gamma(\pm\mu_m - n')}{n'! \cdot \Gamma((\frac{n-1}{2} \pm \mu_m - 2n')/2)^2} \\ &\qquad \times \int_{\Gamma_\sigma \backslash D_\sigma} e_m(\mathbf{x}) dv_2 \cdot e_m(\mathbf{x}).\end{aligned}$$

(b) $s = \rho_\nu - 2n'$, where $n' \in \mathbb{N}$ and $\omega = \rho_\nu$ is a pole of the Eisenstein series $E_\nu(\mathbf{x}, \omega)$ with $\mathrm{Re}(\rho_\nu) < \frac{n-1}{2}$, with residues

$$\begin{aligned}&\operatorname{Res}_{s=\rho_\nu-2n'}\Big[E_\sigma(\mathbf{x}, s)\Big] \\ &= \frac{1}{2}\operatorname{vol}(S^{k-1}) \cdot \sum_{j=0}^{m} \frac{(-1)^j\Gamma(k/2)\Gamma(\rho_\nu - 2n' + j - (n-1)/2)}{j! \cdot \Gamma((\rho_\nu - 2n')/2)\Gamma((\rho_\nu - 2n' + k + 1)/2)} \\ &\times \sum_{\nu:\text{cusps}} \Big[\mathrm{CT}_{\omega=\rho_\nu-2n'+2j} E_\nu(\mathbf{x}, \omega) \cdot \int_{\Gamma_\sigma\backslash D_\sigma} \operatorname{Res}_{\omega=\rho_\nu-2n'+2j} E_\nu(\mathbf{x}, \omega) dv_2 \\ &\qquad + \operatorname{Res}_{\omega=\rho_\nu-2n'+2j} E_\nu(\mathbf{x}, \omega) \cdot \int_{\Gamma_\sigma\backslash D_\sigma} \mathrm{CT}_{\omega=\rho_\nu-2n'+2j} E_\nu(\mathbf{x}, \omega) dv_2\Big],\end{aligned}$$

where $\mathrm{CT}_\omega E_\nu(\mathbf{x},\omega)$ denotes the constant term of the Laurent expansion of the Eisenstein series E_ν at ω and $m \in \mathbb{N}$ is the real number such that $\frac{n-1}{2} - 2 - 2m + 2n' < \mathrm{Re}(\rho_\nu) \leq \frac{n-1}{2} - 2m + 2n'$.

(c) $s = n-1-\rho_\nu - 2n'$, where $n' \in \mathbb{N}$ and $\omega = \rho_\nu$ is a pole of the Eisenstein series $E_\nu(\mathbf{x},\omega)$ with $\mathrm{Re}(\rho_\nu) \in (\frac{n-1}{2}, n-1]$, with residues

$$\begin{aligned}
\mathrm{Res}_{s=n-1-\rho_\nu-2n'}\Big[E_\sigma(\mathbf{x},s)\Big] &= \frac{1}{2}\mathrm{vol}(S^{k-1}) \\
\times \sum_{j=m-\lfloor\frac{n-1}{4}\rfloor}^{m} & \frac{(-1)^j\Gamma(k/2)\Gamma((n-1)/2-\rho_\nu-2n'+j)}{j!\cdot\Gamma((n-1-\rho_\nu-2n')/2)\Gamma((-\rho_\nu-2n'+k)/2)} \\
\times \sum_{\nu=1}^{h}\Big[&\mathrm{CT}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)\cdot\int_{\Gamma_\sigma\backslash D_\sigma}\mathrm{Res}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)dv_2 \\
&+\mathrm{Res}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)\cdot\int_{\Gamma_\sigma\backslash D_\sigma}\mathrm{CT}_{\omega=\rho_\nu+2n'-2j}E_\nu(\mathbf{x},\omega)dv_2\Big],
\end{aligned}$$

where $\mathrm{CT}_\omega E_\nu(\mathbf{x},\omega)$ denotes the constant term of the Laurent expansion of the Eisenstein series E_ν at ω and $m \in \mathbb{N}$ is the real number such that $\frac{n-1}{2} + 2m - 2n' < \mathrm{Re}(\rho_\nu) \leq \frac{n-1}{2} + 2m - 2n' + 2$.

Remark 4.5. The poles given in (a), (b), and (c) might coincide in parts. If it is in the case, the corresponding residues have to be the sum added the each residue.

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