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Existence and non-existence of harmonic functions under integrable conditions

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1. INTRODUCTION

The purpose of this notes is to introduce a recent development of existence and non-existence of harmonic functions u under the integrability conditions $u \in L^p(M)$ for $p = 1, 2$ on a connected smooth Riemannian manifold M without boundary. [15, 13, 7]. We say that M enjoys \mathcal{F} -Liouville property if

$$\Delta u = 0, u \in \mathcal{F} \implies u \equiv \text{constant}$$

here Δ is the distributional Laplacian. Among various extensions, the most robust Liouville property is the L^2 -Liouville property; namely,

Theorem 1 ([18, 16]). *Any complete Riemannian manifolds enjoys the L^2 -Liouville property.*

This extends easily to $p \in (1, \infty)$, even for certain Dirichlet forms provided proper distance functions [17, 12, 9].

In contrast, there are counter examples of complete Riemannian manifolds for the L^1 -Liouville property [2, 11, 10]. In this notes, we study first the L^2 -Liouville property of incomplete manifolds and then next the L^1 -Liouville property of manifolds with ends. More precisely, in Section 1 we will learn the L^2 -Liouville property via it's relationship with the essential self-adjointness of the Laplacian, which plays an important role in the theory of quantum mechanics; and next, in Section 2 we introduce new classes of manifolds which guarantee the existence and non-existence of non-trivial L^1 harmonic functions, which is related to the mean exit time of Brownian motion of M to infinity.

2. L^2 -LIOUVILLE PROPERTY AND THE ESSENTIAL SELFADJOINTNESS OF THE LAPLACIAN

The Laplacian is called essentially selfadjoint if it's restriction to the set $C_0^\infty(M)$ of smooth functions with compact support has the unique selfadjoint extension in L^2 . This is equivalent to:

$$(1) \quad (\Delta + \lambda)u = 0, u \in L^2, \lambda < 0 \implies u \equiv 0,$$

here $\int u \Delta u \leq 0$ for any $u \in C_0^\infty(M)$.

The Laplacian of any complete manifold is essentially selfadjoint [1, 16]). Let us point out that Gaffney [3] proved the essential selfadjointness of the Laplacian Δ starting from a larger domain than $C_0^\infty(M)$. Both the L^2 -Liouville property and the essential selfadjointness of complete manifolds is a direct consequence of the Caccioppoli type inequality (associated to the L^2 -Liouville property): For any $0 < r_1 < r_2$

$$\int_{B_{r_1}} |du|^2 \leq \frac{C}{(r_2 - r_1)^2} \int_{B(r_2) \setminus B(r_1)} |u - \lambda|^2, \quad \forall \lambda \in \mathbb{R},$$

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where $B_r = \{x \mid \rho(x) < r\}$ and $\rho(x)$ is the distance from any fixed point $x_0 \in M$. The Caccioppoli inequality is a consequence of the existence of the sequence of cut-off functions:

$$\chi_{r_2, r_1}(x) = \left(\frac{\rho(x) - r_1}{r_2 - r_1} \wedge 1 \right)_+$$

Note that χ_{r_2, r_1} solves

$$(2) \quad \begin{cases} |\nabla u(x)| = \frac{1}{r_2 - r_1}, & x \in B_{r_2} \setminus \overline{B_{r_1}} \\ u(x) = 1, & x \in B_{r_1} \\ u(x) = 0, & x \in M \setminus B_{r_2}. \end{cases}$$

This robust approach has been used to prove the same conclusion for certain Dirichlet forms [17, 12, 9].

The L^2 -Liouville property and the essential selfadjointness of Δ are related as in

Theorem 2 ([13]). *For a general Riemannian manifold, the essential selfadjointness of Δ yields the L^2 -Liouville property, and these two properties are equivalent if M has infinite volume and if M enjoys Poincaré’s inequality: there exists $\lambda > 0$ such that*

$$(3) \quad \int u^2 \leq \lambda \int |du|^2, \quad \forall u \in W_0^1(M).$$

Let us take a closer look at this relationship in the case of model manifolds [5]:

Definition 1. We call $M_\sigma = (0, \infty) \times \mathbb{S}^{n-1}$ a *model manifold* if it’s Riemannian metric has the form:

$$dr^2 + \sigma(r)^2 d\theta^2$$

where $\sigma(r) \in C^\infty([0, \infty))$ such that $\sigma(r) > 0$ for $r > 0$, $\sigma(0) = 0$, and $\sigma'(0) = 0$.

Note that M is incomplete¹. By Weyl’s criteria, Δ of a model manifold M_σ is essentially selfadjoint if and only if $n \geq 4$. We say (a general Riemannian manifold) M is stochastically complete if the heat kernel (minimal positive fundamental solution of the heat equation) k satisfies

$$\int k(t, x, y) \mu(dx) = 1, \quad \forall t > 0, \forall y \in M.$$

If a model manifold M_σ is stochastically incomplete, then Friedrich’s extension of Δ has discrete spectrum; hence, M_σ enjoys Poincaré’s inequality (3). It is known that M_σ is stochastically complete if and only if

$$\int_0^\infty \frac{V(r)}{S(r)} dr = \infty,$$

where $S(r) = C\sigma^{n-1}$ and $V(r) = \int_0^r S(t)dt$. As a stochastically incomplete manifold needs to have infinite volume, we conclude that the L^2 -Liouville property of a stochastically incomplete model manifold M_σ fails if and only if $n = 2, 3$.

Recall that the condition $n = 2, 3$ corresponds to the non-polarity of the Cauchy boundary $\partial_C M = \overline{M} \setminus M$, where \overline{M} is the completion of M with respect to the Riemannian distance, associated with the $\text{Cap}_{2,2}$ defined as

$$\text{Cap}_{2,2}(\partial_C M) = \begin{cases} \inf_{u \in \mathcal{F}} \|u\|_{W^{2,2}}^2, & \mathcal{F} \neq \emptyset \\ \infty, & \mathcal{F} = \emptyset, \end{cases}$$

where $\mathcal{F} = \{u \in C^\infty(M) \mid u \geq 1 \text{ on a neighborhood of } \partial_C M\}$, $\|u\|_{W^{2,2}}^2 = \|u\|^2 + \|du\|^2 + \|\Delta u\|^2$ and $\|\cdot\|$ is the L^2 -norm. In contrast, by Bergman’s result, M has the L^2 -Liouville property if $\sigma(r) = r$ (the Euclidean case) for any $n \geq 2$. In summary, those observations, made in [13], suggest that in order to break the L^2 -Liouville property, M needs to have both a not too small singularity (in the sense that $\partial_C M$ is not polar) and an *ample end* which we will define explicitly next.

The relationship between the $\text{Cap}_{2,2}(\partial_C M)$ and the essential self-adjointness of the Laplacian should be compared with the following weaker but more complete relationship between $\text{Cap}_{1,2}(\partial_C M)$ and the

¹Usually, M_σ includes the pole, namely, $M_\sigma = [0, \infty) \times \mathbb{S}^{n-1}$.

Markov uniqueness of the Laplacian ². The capacity $\text{Cap}_{1,2}$ of the Cauchy boundary $\partial_C M = \overline{M} \setminus M$ is defined as

$$\text{Cap}_{1,2}(\partial_C M) = \begin{cases} \inf_{u \in \mathcal{F}} \|u\|_{W^{1,2}}^2, & \mathcal{F} \neq \emptyset \\ \infty, & \mathcal{F} = \emptyset. \end{cases}$$

Then

Theorem 3 ([6]). *For a general weighted Riemannian manifold M ,*

$$\text{Cap}_{1,2}(\partial_C M) = 0 \implies \Delta \text{ is Markov unique} \implies M \text{ is stochastically complete.}$$

If $\text{Cap}_{1,2}(\partial_C M)$ is finite, then

$$\text{Cap}_{1,2}(\partial_C M) = 0 \iff \Delta \text{ is Markov unique.}$$

3. EXISTENCE AND NON-EXISTENCE OF NON-TRIVIAL INTEGRABLE HARMONIC FUNCTIONS

3.1. Positive and negative results for the L^1 -Liouville property. Let us collect criteria which implies the L^1 -Liouville property.

Theorem 4 ([10]). *Let M be complete and $x_0 \in M$. Let r denote the distance from x_0 .*

$$(4) \quad \text{Ric}(x) \geq -C(1 + r^2(x)) \implies L^1\text{-Liouville property}$$

Note that the curvature condition (4) yields the stochastic completeness of M .

Theorem 5 ([14]). *Any model manifold has the L^1 -Liouville property.*

The manifold in this theorem is allowed to be stochastically incomplete. Next example by Chung shows that the stochastic completeness does not yield the L^1 -Liouville property:

Example 1 ([2]). Let $M = \mathbb{R} \times \mathbb{S}^1$ with parametrization (r, θ) , $-\infty < r < \infty$ and $0 \leq \theta \leq 2\pi$ with the Riemannian metric $ds^2 = \sigma(r)^2(dr^2 + d\theta^2)$, where

$$\sigma(r) = \frac{1}{(r \log r)^2}, \quad |r| > 2$$

Then, $m(M) < \infty$ and M is complete since $\int_2^\infty \sqrt{\sigma} = \infty$. The function $H(r, \theta) = r$ is harmonic ($\Delta r = \sigma^{-2}(r)(\frac{\partial}{\partial r})^2 r = 0$) and is integrable since $\int_2^\infty \sigma(r)r \, dr < \infty$.

However, Grigoryan showed:

Theorem 6 ([4]). *If M is stochastically complete, then every positive super-harmonic function $u \in L^1$ is constant.*

3.2. New results. Inspired by the observation in the previous section, we study the existence and the non-existence of non-trivial integrable harmonic functions for manifolds with ends:

Definition 2 (Ends and Manifold with ends). An open set $E \subset M$ is called an end if it is connected, unbounded, and ∂E is compact. We assume ∂E is smooth. We call $E_\sigma = \{x \in M_\sigma \mid r(x) < 1\}$ a model end. A manifold with ends is a smooth connected manifold which is a disjoint union of finite number of end and a compact set K . If all ends of a manifold with ends M are model end then we call M a manifold with model ends.

Definition 3. Let $K \subset M$ be a compact non-polar set. A function h on M is called an Evans potential of K if

$$\begin{cases} \Delta h(x) = 0, & x \in M \setminus K \\ h(x) = 0, & x \in K \\ h(x) \rightarrow \infty, & x \rightarrow \infty. \end{cases}$$

The minimal and positive solution e to the following boundary value problem is called the equilibrium potential of K

$$\begin{cases} \Delta e(x) = 0, & x \in M \setminus K \\ e(x) = 1, & x \in K. \end{cases}$$

² Recall that a selfadjoint operator in L^2 is called Markovian if the associated L^2 -semigroup satisfies the Markov property:

$$0 \leq u \leq 1, u \in L^2 \implies 0 \leq T_t u \leq 1, \quad \forall t > 0,$$

A symmetric operator is called Markov unique if it has a unique Markovian extension.

Definition 4 ([7]). We say that M is narrow or ample, respectively, if there is a compact non-polar set $K \subset M$ such that its Evans potential h is in $L^1(M)$ or its equilibrium potential e is in $L^1(M)$, respectively. For an end E , we take h or e , respectively, to be the Evans potential or equilibrium potential on \overline{E} with $K = \partial E$. We say that M is moderate if it is not ample nor narrow.

The (minimal and positive) Green function G of M is defined as

$$G(x, y) = \int_0^\infty k(t, x, y) dt, \quad x, y \in M.$$

Note that it is allowed that $G \equiv \infty$ (for instance, $M = \mathbb{R}^n$ with $n = 1, 2$), and if not, then

$$\Delta G(\cdot, x) = -\delta_x.$$

We also note

- the integrability of $e = e_K$ and $G = G(x, \cdot)$ are independent of the choice of $K \subset M$ and $x \in M$;
- e is integrable if and only if so is G .

The former is a consequence of the maximum principle and local Harnack inequality, and the latter follows from the fact that e and G are obtained as the limit of the equilibrium potentials e_n and the Green functions G_n of an exhaustion $\{\Omega_n\}$ of M with the Dirichlet boundary condition.

Proposition 1 ([7, 4]). *The following assertions are equivalent.*

- (1) M is ample.
- (2) $G(x, \cdot) \in L^1(M)$, $\exists/\forall x \in M$.
- (3) $\tau_M(x) < \infty$, $\exists/\forall x \in M$.
- (4) *There exists an integrable non-trivial super-harmonic function on M .*

A manifold M is called parabolic if $G \equiv \infty$. Hansen and Netuka [8] showed that M has an Evans potential if and only if it is parabolic. By Fubini’s lemma, M is ample if and only if the mean exit time τ_M of Brownian motion on M starting from $x \in M$ to escape to ∞ is finite, that is,

$$\tau_M(x) = \int_M \int_0^\infty k(t, x, y) m(dy) < \infty.$$

Recall that the stochastic completeness means that the life time of Brownian motion on M is finite almost surely. Combining those facts together, we have the following implications:

- (5) narrow \implies parabolic \implies stochastically complete \implies not ample

Hereafter, let M be a manifold with at least two ends otherwise stated explicitly. Note that such M can be decomposed into a disjoint union of two ends as $M = E_1 \cup \overline{E}_2$.

Proposition 2 ([7]). *Let $M = E_1 \cup \overline{E}_2$.*

- (1) E_1 and E_2 are ample $\implies M$ is ample.
- (2) $m(E_1) < \infty$ and E_2 is ample $\implies M$ is ample.
- (3) E_1 is not ample and $m(E_1) = \infty \implies M$ is not ample.

A model manifold is stochastically complete if and only if it is not ample [5]; however, it is not true in general if M is not a model manifold. Indeed, by Proposition 2,

Example 2 ([7]). Let $M = E_1 \cup \overline{E}_2$, where E_2 is not stochastically complete. Then

$$E_1 \text{ is not ample and } m(E_1) = \infty \implies M \text{ is not ample and not stochastically complete.}$$

Recently, Pessoa, Pigola, and Setti [15] obtained the same conclusion under a different assumption:

Example 3 (Example 35 [15]). Let $M = E_1 \cup \overline{E}_2$, where E_2 is complete and not stochastically complete. Then

$$\begin{aligned} E_1 \text{ is complete and non-parabolic and enjoys a parabolic Harnack inequality.} \\ \implies M \text{ is not ample and not stochastically complete.} \end{aligned}$$

The idea of Example 3 is to get a lower bound of the Green function G via the parabolic Harnack inequality so that $G(x, \cdot)$ is not integrable for $x \in M$.

We state the main results in [7]:

Theorem 7. *Let $M = E_1 \cup \overline{E}_2$.*

- (1) If E_1 is narrow and E_2 is ample, then M admits a positive integrable harmonic function H such that $\sup H = \infty$.
- (2) If E_1 and E_2 are both narrow, and if M enjoys Poincaré's inequality for functions with 0-mean, then M admits an integrable harmonic function H such that $\inf H = -\infty$ and $\sup H = \infty$.

Theorem 8. Let M be a manifold with model end(s), and let N be the number of the end(s). Then, M enjoys the L^1 -Liouville property if one of the following conditions is satisfied.

- (1) $N = 1$.
- (2) $N \geq 2$, and each end is ample or moderate.
- (3) $N \geq 2$, only one end is narrow, and the other ends are moderate.

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