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Acoustic radiation force on a rigid cylinder in a focused Gaussian beam

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ABSTRACT- The acoustic radiation force resulting from a 2D focused Gaussian beam incident on cylindrical objects in an inviscid fluid is investigated analytically. The incident and the reflected sound fields are expressed in terms of cylindrical wave functions and a weighting parameter, describing the beam shape and its location relative to the particle. Our main interest here is to study the possibility of using Gaussian beams for axial and lateral handling of rigid cylindrical particles by exerting attractive forces, towards the beam source and axis, respectively. Results have been presented for Gaussian beams with different waist sizes and wavelengths and it has been shown that the interaction of a focused Gaussian beam with a rigid cylinder can result in attractive axial and lateral forces under specific operational conditions. Results have also revealed that attractive axial forces generally occur when the backscattering amplitude is suppressed. The results provided here may provide a theoretical basis for development of single-beam acoustic handling devices.

KEYWORDS: Acoustic radiation force, Gaussian beam, acoustic manipulation

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1. INTRODUCTION

In his pioneering work in the early 1990s, Wu showed the feasibility of acoustic entrapment of very fine Latex particles (in order of 270 μm) using two collimated focused 3.5 MHz Gaussian ultrasonic beams, propagating along opposite directions [1]. Wu had also shown that the same configuration could be used for transportation of Latex particles by moving one of the broadband focusing transducers. This innovative acoustic trapping tool was considered as a suitable alternative when other customary trapping devices, such as optical or electromagnetic tweezers are inapplicable. In the years since then, many experimental and theoretical studies have been conducted for the calculation of the exerted radiation force and investigating the feasibility of acoustic entrapment. While the experimental activities have been mainly concerned with the application of standing waves and Gaussian beams [1-7], a better understanding of the nature of the problem has been gained through the development of a number of mathematical models using King's derivation of the exerted radiation force and the standard wave decomposition method [8-19]. The problem of acoustic radiation force for Gaussian beams, on the other hand, has often been dealt with using ray acoustics method; see for example [20]. In the next paragraphs, we shall provide a review of these two methods and the application of Gaussian beams for acoustic manipulation of fine particles.

The problem of acoustic radiation force on rigid and elastic spherical/cylindrical particles in an unfocused sound field has been the subject of many studies [15-31]. The very first studies on acoustic radiation force are those by King [8], Embleton [21], Gor'kov [22] and Nyborg [23], in which, several theoretical models for the calculation of the radiation force on spherical particles in different acoustical fields had been developed [24]. The problem of acoustic radiation force on a rigid cylinder was first investigated by Awatani [25]. This study was then more elaborated by Hasegawa et al [26] to take into account the elasticity of the target cylinder. In a later study, Wu et al [27] presented an analytical model for the acoustic radiation load on a rigid cylinder suspended in a plane progressive or standing-wave. The acoustic radiation

pressure resulting from a plane wave incident upon spherical and cylindrical shells was also studied by Hasegawa et al [28] and it was shown that the differences between the frequency dependence of acoustic radiation pressure on spherical and cylindrical shells are more pronounced than those between solid spheres and cylinders. Wei et al [29] calculated the acoustic radiation force on a compressible cylinder in a standing wave. More recently, the time-averaged acoustic pressure acting on cylinders suspended in an inviscid/viscous fluid, due to a standing sound field has been investigated by Wang and Dual [30] using finite element and lattice Boltzman methods. The same authors have recently presented a theoretical model for the calculation of the mean force and the torque on a rigid cylinder of arbitrary size in a low-viscosity fluid, and compared their results with numerical simulations carried out using a finite volume method [31].

Focused acoustic beams have been used in many applications, such as acoustic levitation [32, 33], acoustic microscopy and imaging [34, 35], medical diagnosis [36], non-destructive inspection of materials [37], etc. Despite the wide practical applications of Gaussian beams, only a few mathematical models have been developed for acoustic scattering of such beams by spherical and cylindrical particles. The same also applies to the mathematical modelling of the acoustic radiation force exerted on different types of particles. Some of the available studies are reviewed here. Acoustic radiation force on a small compressible sphere in different types of focused beams, such as focused piston field, has been studied by Wu and Du [38]. Wu, also, developed a simple model using ray acoustics approach for the calculation of the acoustic radiation force on an absorbing disk in a focused beam [39]. Acoustic radiation force on a rigid sphere in the close vicinity of a circular vibrating piston has also been studied by Hasegawa et al [40]. In all the above studies, the particles were positioned on the axis of the beam and the effects of off-axial/centre scattering were not considered. In some recent experimental and theoretical investigations [20, 41-43], it has been shown that Gaussian beams can effectively be used for axial and transverse entrapment of miro- and nano-scale particles. Preliminary analytical studies of Lee et al [20] and Lee and Shung [41] using ray acoustics approach also

showed the feasibility of acoustic tweezing and trapping of arbitrarily located spherical objects using a highly focused high frequency Gaussian beam. These studies have shown that Gaussian beams can be used for trapping particles in both axial and transverse directions, although the latter is expectedly much easier. These theoretical findings were also experimentally confirmed using a single focused Gaussian ultrasound beam, for entrapment of very fine lipid droplets (in order of 125 μm) [41-43]. Besides the transverse entrapment, it was also shown that a high frequency focused beam could be used to laterally move micro-droplets towards the focus point. As a promising result, Lee et al have shown that the proposed acoustic technique offers entrapment over a much wider spatial range than the equivalent optical devices [42, 43].

Gaussian beams have received considerable attention in electromagnetics, optics, as well as acoustics, because of their interesting features. From a practical point of view, the technique for manufacturing Gaussian transducers are relatively easy and available, as opposed to other types of beams, such as Bessel beam, Bessel- Laguerre beam, etc. Additionally, from a mathematical point of view, Gaussian beams possess some interesting characteristics which make them an appropriate choice for many applications [44]: Firstly, the wave-front of Gaussian beams behaves like a plane wave in the vicinity of the beam waist, but gradually converts into a spherical wave beyond the Rayleigh zone; secondly, it has no maxima and minima in the near-field, which is characteristics of the Fresnel field of a piston transducer; thirdly, the energy of the beam is chiefly confined within a finite divergence angle, and also the beam energy in the far-field is localized in a single beam free of diffraction lobes (characteristics of Fraunhofer field of piston transducers [45]). Also, with the application for acoustic handling in mind, one of the most important advantageous of using focused-beams, such as Gaussian beams, for manipulation of particles (as a single-beam device), instead of standing wave, is that the operational restrictions associated with the short spacing between the transducers in standing-wave acoustic manipulation techniques can be alleviated, as explained in Refs. [42, 43].

Motivated by the recent experimental studies on the viability of the development of single-beam acoustic tweezers using Gaussian beams [42, 43], we provide a theoretical model for the

calculation of the radiation force resulting from a 2D focused-Gaussian beam on rigid cylindrical objects. A better understanding of the interaction of acoustic Gaussian beams by different particles may help us to improve the performance of the existing acoustic trapping techniques and develop more advanced particle-handling devices. The remainder of the paper is structured as follows: In the next section, we shall present the mathematical modelling of the problem. The incident Gaussian field is represented in the cylindrical coordinate system in terms of Bessel functions of integer orders and some weighting coefficients. The scattered field is modelled and the unknown scattering modal coefficients are determined for a rigid cylindrical object. A rigorous mathematical modelling is provided for the calculation of the radiation force experienced by cylindrical objects in complex sound fields. Section 3 is devoted to numerical results and discussions for acoustic radiation force and far-field acoustic pressure. Finally, Section 4 concludes the paper and provides some ideas for future work.

2. MATHEMATICAL ANALYSIS

Consider the scattering of a Gaussian beam by an infinitely long, immovable, acoustically rigid cylindrical particle of radius a. The surrounding fluid medium is assumed infinite, lossless, with density ρ and speed of sound c. A schematic of the problem is shown in Fig. 1. Two Cartesian coordinate systems are used to describe the problem: (X, Y) corresponds to the coordinate system of the beam, centred at the origin of the beam, and (x, y) is used for the cylindrical particle, see Fig. 1. A cylindrical coordinate system (r, θ) is also introduced for describing the reflected sound field. The axial and transverse distances between the beam centre and the particle are denoted by (x_0, y_0), respectively. In Section 2.1, we shall provide the mathematical modelling of the incident Gaussian beam in cylindrical coordinate system. The mathematical modelling and calculation of the exerted radiation force will be dealt with in Section 2.2.

2.1. Incident Gaussian field

The problem of scattering of laser and electromagnetic Gaussian beams by cylindrical objects has been the subject of various studies [46, 47]. Here, we shall investigate the same problem in an acoustic context. Let us first focus on the derivation of the incident Gaussian beam in its own coordinate system (X, Y), see Fig. 1. The incident Gaussian beam, ϕ_i, with complex coordinate system, i.e. having amplitude and phase distribution, satisfies the following two-dimensional Helmholtz equation [46-49],

$$\frac{\partial^2 \phi_i}{\partial X^2} + \frac{\partial^2 \phi_i}{\partial Y^2} + k^2 \phi_i = 0, \tag{1}$$

where $k = \omega/c$ is the wavenumber, ω is the angular frequency. An incident Gaussian beam with spatial distribution of $\phi_i(X = 0, Y) = \phi_0 \mathrm{e}^{-\beta^2 Y^2}$ and beam waist of W_0 (in the focal plane), can be expressed in a Cartesian coordinate system as [50],

$$\phi_i(X, Y) = \frac{\phi_0}{2\beta\sqrt{\pi}} e^{-j\omega t} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{k_Y^2}{4\beta^2} + j\mathbf{k}\cdot\mathbf{X}} dk_Y, \tag{2}$$

where ϕ_0 is the beam amplitude, $j = \sqrt{-1}$, the wavenumber is defined as $\mathbf{k} = (k_X, k_Y)$, with k_X and k_Y being the longitudinal and transverse wavenumbers ($k^2 = k_X^2 + k_Y^2$), and $\mathbf{X} = (X, Y)$. The parameter $1/|\beta|$ corresponds to the incident wave beam width. The complex beam value β^2 can also be expressed in a more standard way, as $\beta^2 = \frac{k}{2q(X)}$, where the complex beam parameter $q(X) = X + q_0 - jX_R$, is a generalized radius curvature and controls both the phase and amplitude distribution in the transversal plane. The imaginary part of q is known as the beam focal point (confocal parameter), defined as $X_R = \frac{1}{2} k W_0^2$, while q_0 can shift the waist position in the axial direction. The focal point range is also known as the Rayleigh range (X_R) of the Gaussian beam, characterizing the Lorentzian profile of intensity along the axis. The near-

and far-field of the beam can, subsequently, be defined by $X < X_R$ and $X > X_R$, respectively. Furthermore, the phase of a Gaussian beam propagating in free space is given by [50]

$$kX + \frac{1}{2X}k\left[1 + \left(\frac{X_R}{X}\right)^2\right]^{-1}\rho^2 - \tan^{-1}\left(\frac{X}{X_R}\right), \tag{3}$$

where $\rho = (X^2 + Y^2)^{1/2}$, the first term is the phase term of a plane wave, the second term a phase shift in the transverse position, and the last term is the Gouy phase shift, which controls the phase shift within and in the vicinity of the Rayleigh range (X_R), and is unique to Gaussian beams [50].

Equation (2) gives the potential field of the incident Gaussian beam in a Cartesian coordinate system. However, in order to satisfy the appropriate boundary conditions on a cylindrical surface, it is more suitable to rewrite the above potential field in the cylindrical coordinate system (r, θ). Using a simple Cartesian coordinate system translation and change of variables $x = r\cos\theta$, $y = r\sin\theta$, Eq. (2) can be rewritten as:

$$\phi_i(r,\theta) = \frac{\phi_0}{2\beta\sqrt{\pi}}e^{-j\omega t}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{k_Y^2}{4\beta^2}}\mathrm{e}^{j(k_X x_0 - k_Y y_0)}\mathrm{e}^{jkr\cos(\theta-\alpha)}dk_Y, \tag{4}$$

where $\alpha = \sin^{-1}(k_Y/k)$. The above equation can be further simplified by expressing the last exponential term (plane wave term) in terms of Bessel functions of integer order, using Jacobi–Anger expansion, as

$$\mathrm{e}^{jkr\cos(\theta-\alpha)} = \sum_{m=-\infty}^{\infty} j^m \mathrm{e}^{jm(\theta-\alpha)}J_m(kr), \tag{5}$$

where J_m is the Bessel function of the first kind. Finally, substituting Eq. (5) into (4), the incident Gaussian sound field can be represented in the following from,

$$\phi_i(r,\theta) = \phi_0 e^{-j\omega t} \sum_{m=-\infty}^{\infty} j^m g_m(k,q,x_0,y_0)\, J_m(kr)\, \mathrm{e}^{jm\theta},$$

(6)

where the complex weighting functions, g_m , are obtained from the integral equation,

$$g_m(k,q,x_0,y_0) = \frac{1}{2\beta\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{k_Y^2}{4\beta^2} + j(k_X x_0 - k_Y y_0) - jm\alpha} dk_y.$$

(7)

In the limiting case, when the beam waist is very large, the weighting coefficient, g_m, tends to unity, and Eq. (6) will reduce to a simple plane progressive wave. Also, one must note that the above integral representation of the incident beam includes both the radiated field ($|k_Y| \le k$) and the evanescent field ($|k_Y| > k$). Restricting ourselves to only the radiated field, the above integral equation becomes,

$$g_m(k,q,x_0,y_0) = \left(\frac{2\pi}{qk}\right)^{-1/2} \int_{-\pi/2}^{\pi/2} \cos\alpha\, \mathrm{e}^{-\frac{1}{2}kq\sin^2\alpha + jk[x_0\cos\alpha - y_0\sin\alpha] - jm\alpha} d\alpha,$$

(8)

where q is used as a shorthand for $q(X=0)$.

To proceed with the derivation of the radiation force, one first needs to describe the scattered field around the object. The sound field reflected from the surface of the cylindrical object can be expressed as,

$$\phi_s(r,\theta) = \phi_0 e^{-j\omega t} \sum_{m=-\infty}^{\infty} j^m g_m(k,q,x_0,y_0)\, x_m\, H_m^{(1)}(kr)\, e^{jm\theta},$$

(9)

where $H_m^{(1)} = J_m(\cdot) + jY_m(\cdot)$ is the Hankel function of the first kind and x_m are the unknown scattering coefficients, to be determined from the appropriate boundary conditions at the surface of the object. In this paper, the cylindrical particle is assumed to be immovable and

acoustically rigid, which means the normal velocity on the particle is zero (Neumann condition), that is $\partial(\phi_i + \phi_s)/\partial r\big|_{r=a} = 0$) [51]. Upon substitution of Eqs. (6) and (9) into this boundary condition, we obtain $x_m = -J_m'(ka)/H_m^{(1)'}(ka)$, where the prime denotes derivative with respect to the argument. The case of an acoustically soft cylinder will also be discussed later in Sec. 3.

Finally, the far-field sound scattered by a cylindrical object can be expressed in terms of the form function $f_\infty(ka, \theta)$, defined as [52],

$$f_\infty(ka, \theta) = \left(\frac{2r_\infty}{a}\right)^{\frac{1}{2}} \mathrm{e}^{-jkr_\infty} \frac{p_s}{p_0}, \tag{10}$$

where r_∞ denotes a large radial distance from the particle, $p_s = -j\omega\rho\phi_s$ is the scattered pressure field, and p_0 is the pressure amplitude of the incident wave.

2.2. Acoustic radiation force

The concept of time-averaged acoustic force acting on particles, due to the interaction with a sound field has been used in many applications such as ultrasonic imaging, elastography, and contactless particle handling. The time-averaged force acting on a particle immersed in an infinite and ideal fluid is given by [8, 26, 28]:

$$\mathbf{F} = -\langle \iint_{S_0} \rho(\mathrm{v}_r \hat{\mathbf{r}} + \mathrm{v}_\theta \widehat{\boldsymbol{\theta}}) \mathrm{v}_r dS \rangle + \langle \iint_{S_0} \frac{\rho}{2} |\mathbf{v}^2| \hat{\mathbf{r}} dS \rangle - \langle \iint_{S_0} \frac{\rho}{2c^2} \dot{\psi}^2 \hat{\mathbf{r}} dS \rangle, \tag{11}$$

where S_0 is the boundary at its equilibrium position, $\mathbf{v} = -\nabla\psi$ is the first-order velocity of the fluid, $\psi = Re(\phi_i + \phi_s)$ is the first-order velocity of the fluid at the boundary, v_r and v_θ are, respectively, the radial and tangential components of the velocity at the surface, the radial and tangential unit vectors are denoted by $\hat{\mathbf{r}}$ and $\widehat{\boldsymbol{\theta}}$, respectively, and $\langle \quad \rangle$ represents time average. Since the particle is illuminated by a 2D Gaussian beam, i.e. no z-direction force components, the

exerted force can be written as, $\mathbf{F} = \mathrm{F}_x\hat{\mathbf{x}} + \mathrm{F}_y\hat{\mathbf{y}}$. An inspection of the above equation suggests that the radiation force exerted on a unit length of the cylinder in the x-and y- directions can be broken down into four components, as

$$F_i = F_{r,i} + F_{\theta,i} + F_{r\theta,i} + F_{t,i}, \qquad i = \{x, y\} \tag{12}$$

Noting that $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$ and $\widehat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}$, the axial component of the exerted radiation force in Eq. (11) can be obtained from

$$F_{r,x} = -\frac{1}{2}a\rho\,\langle\int_0^{2\pi}\left(\frac{\partial\psi}{\partial r}\right)^2_{r=a}\cos\theta\,d\theta\rangle\,, \tag{13-1}$$

$$F_{\theta,x} = \frac{\rho}{2a}\langle\int_0^{2\pi}\left(\frac{\partial\psi}{\partial\theta}\right)^2_{r=a}\cos\theta\,d\theta\rangle\,, \tag{13-2}$$

$$F_{r\theta,x} = \rho\,\langle\int_0^{2\pi}\left(\frac{\partial\psi}{\partial r}\right)_{r=a}\left(\frac{\partial\psi}{\partial\theta}\right)_{r=a}\sin\theta\,d\theta\rangle\,, \tag{13-3}$$

$$F_{t,x} = -\frac{a\rho}{2c^2}\langle\int_0^{2\pi}\left(\frac{\partial\psi}{\partial t}\right)^2_{r=a}\cos\theta\,d\theta\rangle\,. \tag{13-4}$$

In a similar way, the force component of the transverse radiation force, are given by

$$F_{r,y} = -\frac{1}{2}a\rho\,\langle\int_0^{2\pi}\left(\frac{\partial\psi}{\partial r}\right)^2_{r=a}\sin\theta\,d\theta\rangle\,, \tag{14-1}$$

$$F_{\theta,y} = \frac{\rho}{2a}\langle\int_0^{2\pi}\left(\frac{\partial\psi}{\partial\theta}\right)^2_{r=a}\sin\theta\,d\theta\rangle\,, \tag{14-2}$$

$$F_{r\theta,y} = -\rho\,\langle\int_0^{2\pi}\left(\frac{\partial\psi}{\partial r}\right)_{r=a}\left(\frac{\partial\psi}{\partial\theta}\right)_{r=a}\cos\theta\,d\theta\rangle\,, \tag{14-3}$$

$$F_{t,y} = -\frac{a\rho}{2c^2}\left\langle \int_0^{2\pi} \left(\frac{\partial \psi}{\partial t}\right)^2_{r=a} \sin\theta \, d\theta \right\rangle .$$

(14-4)

To further simplify the derivation of the radiation force, we may express the complex quantities as the summation of the real and imaginary parts; specifically, $x_m = \alpha_m + j\beta_m$, and $g_m = g_m^r + jg_m^j$. Substituting Eqs. (6) and (9) into the total potential field, we obtain,

$$\psi(r,\theta) = \phi_0 \sum_{m=-\infty}^{\infty} (R_m \cos m\theta + Q_m \sin m\theta),$$

(15)

in which

$$R_m = \begin{cases} [\mathfrak{M}_m \cos\omega t + \mathfrak{N}_m \sin\omega t] & (m = 4k) \\ [\mathfrak{M}_m \sin\omega t - \mathfrak{N}_m \cos\omega t] & (m = 4k+1) \\ -[\mathfrak{M}_m \cos\omega t + \mathfrak{N}_m \sin\omega t] & (m = 4k+2)' \\ -[\mathfrak{M}_m \sin\omega t - \mathfrak{N}_m \cos\omega t] & (m = 4k+3) \end{cases}$$

(16-1)

$$Q_m = \begin{cases} [\mathfrak{M}_m \sin\omega t - \mathfrak{N}_m \cos\omega t] & (m = 4k) \\ -[\mathfrak{M}_m \cos\omega t + \mathfrak{N}_m \sin\omega t] & (m = 4k+1) \\ -[\mathfrak{M}_m \sin\omega t - \mathfrak{N}_m \cos\omega t] & (m = 4k+2)' \\ [\mathfrak{M}_m \cos\omega t + \mathfrak{N}_m \sin\omega t] & (m = 4k+3) \end{cases}$$

(16-2)

where $k = 0, \pm1, \pm2, \cdots,$ and $\mathfrak{M}_m$ and $\mathfrak{N}_m$ are defined as

$$\mathfrak{M}_m = g_m^r U_m - g_m^j V_m, \quad (17\text{-}1)$$

$$\mathfrak{N}_m = g_m^j U_m + g_m^r V_m, \quad (17\text{-}2)$$

and

$$U_m = (1+\alpha_m)J_m(kr) - \beta_m Y_m(kr), \quad (18\text{-}1)$$

$$V_m = \beta_m J_m(kr) + \alpha_m Y_m(kr). \quad (18\text{-}2)$$

Using time averaging, *i.e.* $\frac{1}{T}\int_t^{t+T} \begin{Bmatrix} \cos^2(\omega t) \\ \sin^2(\omega t) \end{Bmatrix} dt = \frac{1}{2}$, one can readily obtain the following time average relations,

$$\langle R_m R_{m+1}\rangle = -\tfrac{1}{2}(\mathfrak{M}_m\mathfrak{N}_{m+1} - \mathfrak{M}_{m+1}\mathfrak{N}_m),$$

$$\langle R_m Q_{m+1}\rangle = -\tfrac{1}{2}(\mathfrak{M}_m\mathfrak{M}_{m+1} + \mathfrak{N}_{m+1}\mathfrak{N}_m),$$

$$\langle Q_m R_{m+1}\rangle = \tfrac{1}{2}(\mathfrak{M}_m\mathfrak{M}_{m+1} + \mathfrak{N}_{m+1}\mathfrak{N}_m),$$

$$\langle Q_m Q_{m+1}\rangle = -\tfrac{1}{2}(\mathfrak{M}_m\mathfrak{N}_{m+1} - \mathfrak{M}_{m+1}\mathfrak{N}_m). \tag{19}$$

To find an analytical expression for the radiation force, the following orthogonality relations for trigonometric functions must be used in Eqs. (13) and (14):

$$\tfrac{1}{\pi}\int_0^{2\pi} \cos n\theta \cos m\theta \cos\theta \, d\theta = \begin{cases} 1; & n+m=1 \\ 1/2; & |n-m|=1, \\ 0; & otherwise \end{cases} \tag{20-1}$$

$$\tfrac{1}{\pi}\int_0^{2\pi} \sin n\theta \sin m\theta \cos\theta \, d\theta = \begin{cases} 1/2; & |n-m|=1 \\ 0; & otherwise \end{cases}, \tag{20-2}$$

$$\tfrac{1}{\pi}\int_0^{2\pi} \cos n\theta \sin m\theta \sin\theta \, d\theta = \begin{cases} 1; & n=1, m=0 \\ 1/2; & n-m=1 \\ -1/2; & n-m=-1 \\ 0; & otherwise \end{cases}. \tag{20-3}$$

Finally, substituting Eqs. (15) through (18) into (13) and (14), and making use of the above orthogonality and time averaging relations, we obtain:

$$F_i = ES_c Y_{p,i}, \quad i = \{x, y\} \tag{21}$$

where $E = \frac{1}{2}\rho k^2 |\phi_0|^2$ is the energy density, S_c is the cross-sectional area, and $Y_{p,x}$ and $Y_{p,y}$ are dimensionless radiation force parameters, given by:

$$\begin{aligned} Y_{p,x} = -\frac{1}{2ka}\sum_{m=-\infty}^{\infty} \{ & \left(g_m^r g_{m+1}^r - g_m^i g_{m+1}^i\right)[\alpha_m + \alpha_{m+1} + 2(\alpha_m\alpha_{m+1} + \beta_m\beta_{m+1})] \\ & - \left(g_m^i g_{m+1}^r - g_{m+1}^i g_m^r\right)[\beta_m - \beta_{m+1} + 2(\beta_m\alpha_{m+1} + \beta_{m+1}\alpha_m)] \\ & + \left(g_m^r g_{m-1}^r - g_m^i g_{m-1}^i\right)[\alpha_m + \alpha_{m-1} + 2(\alpha_m\alpha_{m-1} + \beta_m\beta_{m-1})] \\ & - \left(g_m^i g_{m-1}^r - g_{m-1}^i g_m^r\right)[\beta_m - \beta_{m-1} + 2(\beta_m\alpha_{m-1} + \beta_{m-1}\alpha_m)]\}, \end{aligned}$$

(23)

$$Y_{p,y} = -\frac{1}{2ka}\sum_{m=-\infty}^{\infty}\{(g_m^i g_{m+1}^r - g_{m+1}^i g_m^r)[\alpha_m + \alpha_{m+1} + 2(\alpha_m\alpha_{m+1} + \beta_m\beta_{m+1})]$$
$$+ (g_m^r g_{m+1}^r - g_m^i g_{m+1}^i)[\beta_m - \beta_{m+1} + 2(\beta_m\alpha_{m+1} + \beta_{m+1}\alpha_m)]$$
$$- (g_m^i g_{m-1}^r - g_{m-1}^i g_m^r)[\alpha_m + \alpha_{m-1} + 2(\alpha_m\alpha_{m-1} + \beta_m\beta_{m-1})]$$
$$- (g_m^r g_{m-1}^r - g_m^i g_{m-1}^i)[\beta_m - \beta_{m-1} + 2(\beta_m\alpha_{m-1} + \beta_{m-1}\alpha_m)]\}.$$

(24)

The above radiation force relations can also be used for other types of incident beams, provided that they can be expressed as series of weighted plane wave components, similar to Eq. (6). Finally, in a limiting case, for a planar progressive wave (*i.e.* very large waist), one can show that g_m tend to unity and for the unknown scattering coefficient we will have $x_m = x_{-m}$. In such case, the wave equation, and thus the radiation force equation, is reduced to an axisymmetric one, i.e. $m = 0, 1, 2, \ldots$. It can now be readily shown that $Y_{p,y} = 0$ and the axial radiation force is given by the following well-known relation [26, 28]:

$$Y_{p,x} = -\frac{2}{ka}\sum_{m=0}^{\infty}[\alpha_m + \alpha_{m+1} + 2(\alpha_m\alpha_{m+1} + \beta_m\beta_{m+1})].$$

(25)

3. NUMERICAL RESULTS AND DISCUSSIONS

The axial and transverse acoustic radiation forces on a rigid cylinder in a Gaussian beam will be numerically studied in this section. The main objective here is to examine the feasibility of using Gaussian beams for generating an acoustic pulling force on cylindrical objects. Results will be presented for a rigid cylinder submerged in water at atmospheric condition ($c = 1490$ m/s, $\rho = 1000$ kg/m^3). To calculate the exerted axial/radial radiation forces using Eqs. (23) and (24), one first needs to evaluate the weighting coefficients, g_m, from Eq. (8). It has been shown before [53] that an asymptotical solution of Eq. (8) can be obtained for highly focused Gaussian beams (*i.e.* when most of the energy carried by the beam is confined within a finite divergence angle). In this paper, however, we need to calculate the radiation force caused by different Gaussian beams. Therefore, we shall instead evaluate g_m by performing a numerical integration using the trapezoidal method with a very small integration step length ($d\alpha = 0.001$) to ensure the convergence of the integration at different frequencies and beam widths. The numerical error due to the g_m integration is therefore quite negligible. As mentioned earlier, the integration is limited only to the radiated field ($|k_Y| \leq k$). Calculations are performed with a maximum truncation number of $N_{\max} = 70$ to ensure the proper convergence of the solution at high frequencies.

To fully describe the problem, the following six parameters are required: wavelength ($\lambda = 2\pi/k$), beam's basic parameters ($q_0,\ W_0$), particle size (a) and beam's position relative to the particle (x_0, y_0). Because of the large number of parameters involved in this modelling, we shall initially try to vary as many parameters as possible, see Figs. 2 and 3. The results presented here will be used in the rest of the paper for a more in depth study of the radiation force. Figure 2 shows the effects of the axial (x_0) and transverse (y_0) position of the particle on the exerted axial radiation force due to a Gaussian beam with $q = 0.7 - j0.2$, operating at $\lambda = 0.5a$ (Fig. 2-a; $W_0 = 0.178\ a$), and $\lambda = 1.0a$ (Fig. 2-b; $W_0 = 0.252\ a$). Results are provided for four axial locations $x_0/a = 0, 2, 5, 10$. For a Gaussian beam with small wavelength ($\lambda = 0.5a$), Fig.

2-a, results have shown that the axial radiation force peaks when the particle is placed at the beam centre ($x_0 = 0, y_0 = 0$), and gradually decreases with x_0 and y_0. More importantly, it has been observed that when the particle is located at $x_0/a = 0$ within the lateral range of $1.51 < |y_0/a| < 2.4$, the axial radiation force becomes negative, i.e. the particle can be pulled towards the source (shown by thick red line). The exerted axial radiation force tends to zero as the particle entirely leaves the beam domain (large $|y_0/a|$). The axial radiation force results at a lower frequency ($\lambda = 1.0a$) is presented in Fig. 2-b. Results show that increasing the beam wavelength (decreasing frequency) leads to the emergence of attractive axial forces over a number of y_0-ranges. Numerical evaluations have also shown that the effective y_0-ranges, over which the axial radiation force becomes negative, moves further away from the beam axis as the wavelength is increased (with constant Rayleigh range).

Figure 3 presents results for the transverse radiation force (y-direction) acting on an off-axially located rigid cylinder within a Gaussian beam with $q = 0.7 - j0.2$ at different axial locations $x_0/a = 0, 2, 5, 10$ and two wavelengths at $\lambda = 0.5a$ (Fig. 3-a; $W_0 = 0.178\,a$), and $\lambda = 1.0a$ (Fig. 3-b; $W_0 = 0.252\,a$). Results show that the exerted transverse force generally repels the object away from the beam axis, where the sound intensity is maximum, except for particles located on the waist plane and laterally within a certain distance from the beam centre, shown by thick red lines. The above observations suggest that negative axial and lateral radiation forces can be produced generally when a focused Gaussian beam is concentrated on the very top (or bottom) of the particle. To complete the discussion, it is worth mentioning that the problem of lateral movement of spherical particles using Gaussian beams, to pull them towards the beam axis has been previously reported by other authors [41-43]. The magnitude of such attractive transverse forces is understandably greater than the negative axial forces, as seen in Figs. 2 and 3. Using such a lateral force, combined with a negative axial force, one can move an object towards the beam origin.

As shown in Fig. 2, attractive axial forces can be achieved using focused Gaussian beams under special conditions and over some certain y_0-ranges. In Figs. 4 through 6, we shall further study the effects of beam basic parameters, q_0 and W_0, on the emergence of negative axial forces and the extent of the effective y_0-ranges. Results are presented over $0.25 < \lambda/a < 4$ and for beams with $W_0/a = 0.15, 0.5, 1$ and $q_0/a = 0.7, 1, 2$. Figure 4 shows the regions of negative axial forces when the object is positioned on the beam waist plane, $x_0 = 0$. The following observations have been made: Firstly, results show that the effective y_0-ranges moves laterally away almost linearly with λ, particularly for beams with small waist. However, it can also be seen that at some frequencies the beam is unable to produce any negative axial force. Secondly, results have shown that the magnitude of the exerted attractive axial force increases with q_0. Also, for beams with large waists, attractive axial forces can be achieved at higher wavelengths (lower frequencies). Thirdly, the effective y_0-ranges broaden as W_0 increases and negative axial forces can occur over multiple y_0- ranges. Additionally, our numerical studies have shown that attractive axial forces can still be achieved if the particle is located downstream of the beam focal plane ($x_0 > 0$). Figure 5 and 6 provide results for a rigid cylinder located axially away from the beam waist, at $x_0 = 0.5a$, and $x_0 = 1a$, respectively. Results have shown that producing negative radiation forces using a Gaussian beam with small waist, $W_0/a = 0.15$ (Figs. 5-a 5-d and 5-g, Figs. 6-a 6-d and 6-g) becomes almost impossible if the object is located off the beam waist plane. For Gaussian beams with large W_0, however, attractive axial forces can still be achieved, but they now generally occur at much lower wavelengths. Finally, results have shown that the magnitude of the exerted negative axial forces reduce with x_0 and will eventually disappear at large x_0.

It has been shown before, for the case of Bessel beams [54, 55], that the emergence of the negative radiation forces can be related to the far-field scattered field. Marston, in the case of elastic spheres illuminated by Bessel beams, has shown that the regions where $Y_{p,x}$ is negative with a significant magnitude tend to occur where the back-scattered amplitude is suppressed [54, 55], which was explained to be related to the resonance frequencies of the particle. We shall

also briefly study this matter here for Gaussian beams. Figure 7 presents the back-scattered form-function results ($\theta = \pi$), using Eq. (10) versus q_0, for a cylinder located on the focal plane ($x_0 = 0$), for three beam waist sizes, $W_0/a = 0.15, 0.5$ and 1.0, at three wavelengths, $\lambda/a = 1, 2, 3.5$, over some specific y_0-ranges (chosen from Fig. 4). The thick red lines show where the axial radiation force reverses in direction and becomes attractive. Comparisons of the near-field radiation force results (Fig. 4) and the far-field form-function results (Fig. 7) have shown that the back-scattered amplitude is very small over the effective q_0 region, and reaches its minimum where the maximum pulling force is obtained. However, unlike the reasoning in Marston [54] and Azarpeyvand [55], this cannot be related to the resonances of the particle since the cylinder is assumed to be acoustically rigid. Also, from the results of the ($W_0/a = 0.15$, $\lambda/a = 2.0, y_0/a = 2.7$) or ($W_0/a = 1.0$, $\lambda/a = 1.0, y_0/a = 2.1$) cases, one can conclude that this condition is necessary, but not sufficient.

To better understand the performance of the proposed single-beam acoustic tweezer for more realistic cases, it is important to consider particles with different boundary conditions. Acoustic manipulation of soft cylinders using a single Gaussian beam has also been investigated as part of this work. In this case, the appropriate boundary condition is $p = \rho\dot{\psi} = 0$, which leads to $x_m = -J_m(ka)/H_m^{(1)}(ka)$. It has, however, been shown by performing several numerical simulations that the exerted axial acoustic radiation force for a soft cylinder is always repulsive and no acoustic attraction force could be obtained. The results in this paper were limited only to rigid/soft immoveable particles. The effects of particle's compressibility, dissipative losses (for liquid, elastic, viscoelastic, and porous particles), when illuminated by a focused Gaussian beam will be investigated in a separate study. Finally, in order to check the overall validity of our model, Eq. (23) is evaluated for a Gaussian beam with a very wide waist, *i.e.* $g_m = 1$, and results are compared with those obtained using the standard radiation force formulation, Eq. (25). Results presented in Fig. 8 show an excellent agreement between the results obtained using Eqs. (23) and (25).

4. CONCLUSIONS

The problem of acoustic manipulation of cylindrical objects using a single focused Gaussian beam has been investigated. To help better understand the underlying mechanism of the emergence of acoustic pulling forces we have performed several simulations and have developed a number of criteria. It has been shown that a highly focused Gaussian beam can be used to produce negative axial radiation forces on rigid cylinders, when the beam is focused on the very top or bottom of the particle. It has also been found that Gaussian beams cannot produce pulling forces on acoustically soft cylinders. Furthermore, the comparison of the far-field and near-field results has shown that the negative axial forces with significant magnitude tend to take place when the backscattering amplitude is considerably suppressed. The promising observations made in this paper open several new avenues for research in this area, such as, acoustic manipulation of spherical or irregular-shaped particles of different mechanical properties using Gaussian beams, and the application of other Gaussian-type beams for acoustic handling purposes, such as Hermite-Gaussian and Laguerre-Gaussian beams.

REFERENCES

1) J. Wu, Acoustical tweezers, Journal of the Acoustical Society of America, 89 (1991) 2140-2143.

2) T. Lilliehorn, U. Simu, M. Nilsson, M. Almqvist, T. Stepinski, T. Laurell, J. Nilsson, S. Johansson, Trapping of microparticles in the near field of an ultrasonic transducer, Ultrasonics, 43 (2005) 293-303.

3) D. Bazou, L.A. Kuznetsova, W. Terence Coakley, Physical environment of 2-D animal cell aggregates formed in a short pathlength ultrasound standing wave trap, Ultrasound in Medicine & Biology, 31 (2005) 423-430.

4) A. Haakea, J. Dual, Contactless micromanipulation of small particles by an ultrasound field excited by a vibrating body, Journal of the Acoustical Society of America, 117 (2005) 2752-2760.

5) M. Saito, T. Daian, K. Hayashi, S. Izumida, Fabrication of a polymer composite with periodic structure by the use of ultrasonic waves, Journal of Applied Physics, 83 (1998) 3490-3494.

6) L. Gherardini, C.M. Cousins, J.J. Hawkes, J. Spengler, S. Radel, H. Lawler, B. Devcic-Kuhar, M. Groschl, A new immobilisation method to arrange particles in a gel matrix by ultrasound standing waves, Ultrasound in Medicine & Biology, 31 (2005) 261-272.

7) F.G. Mitri, F.H. Garzon, D.N. Sinha, Characterization of acoustically engineered polymer nanocomposite metamaterials using x-ray microcomputed tomography, Review of Scientific Instruments, 82 (2011) 034903-034908.

8) L.V. King, On the acoustic radiation pressure on spheres, Proceedings of the Royal Society A, 147 (1935) 212-240.

9) T. Hasegawa, T. Kido, T. Iizuka, C. Matsuoka, A general theory of Rayleigh and Langevin radiation pressures, Acoustical Science and Technology, 21 (2000) 145-152.

10) A.P. Zhuk, Radiation force acting on a cylindrical particle in a sound field, International Applied Mechanics, 22 (1986) 689-693.

11) F.G. Mitri, Acoustic radiation force on cylindrical shells in a plane standing wave, Journal of Physics A: Mathematical and General, 38 (2005) 9395-9404.

12) F.G. Mitri, Theoretical calculation of the acoustic radiation force acting on elastic and viscoelastic cylinders placed in a plane standing or quasi-standing wave field, European Physical Journal B, 44 (2005) 71-78.

13) F.G. Mitri, Radiation force acting on an absorbing cylinder placed in an incident plane progressive acoustic field, Journal of Sound and Vibration, 284 (2005) 494-502.

14) F.G. Mitri, Theoretical and experimental determination of the acoustic radiation force acting on an elastic cylinder in a plane progressive wave - far-field derivation approach, New Journal of Physics, 8 (2006) art. No. 138.

15) F.G. Mitri, Z.E.A. Fellah, Theoretical calculation of the acoustic radiation force on layered cylinders in a plane standing wave-comparison of near- and far-field solutions, Journal of Physics A: Mathematical and General, 39 (2006) 6085-6096.

16) F.G. Mitri, Z.E.A. Fellah, Acoustic radiation force on coated cylinders in plane progressive waves, Journal of Sound and Vibration, 308 (2007) 190-200.

17) F.G. Mitri, Axial time-averaged acoustic radiation force on a cylinder in a nonviscous fluid revisited, Ultrasonics, 50 (2010) 620-627.

18) F.G. Mitri, Z.E.A. Fellah, Axial acoustic radiation force of progressive cylindrical diverging waves on a rigid and a soft cylinder immersed in an ideal compressible fluid, Ultrasonics, 51 (2011) 523-526.

19) J. Wang, J. Dual, Theoretical and numerical calculation of the acoustic radiation force acting on a circular rigid cylinder near a flat wall in a standing wave excitation in an ideal fluid, Ultrasonics, 52 (2012) 325-332.

20) J. Lee, K. Ha, K.K. Shung, A theoretical study of the feasibility of acoustical tweezers: Ray acoustics approach, Journal of the Acoustical Society of America, 117 (2005) 3273-3280.

21) T.F.W. Embleton, Mean force on a sphere in a spherical sound field. I. (Theoretical), Journal of the Acoustical Society of America, 26 (1938) 40-45.

22) L.P. Gor'kov, On the forces acting on a small particles in an acoustical field in an ideal fluid, Soviet physics-Doklady, 6 (1962) 773-775.

23) W.L. Nyborg, Radiation pressure on a small rigid sphere, Journal of the Acoustical Society of America, 42 (1967) 947-952.

24) G.C. Gaunaurd, H. Uberall, Acoustics of finite beams, The Journal of the Acoustical Society of America, 63 (1978) 5-16.

25) J. Awatani, Study on acoustic radiation pressure (VI), radiation pressure on a cylinder, Memoirs of the Institute of Scientific and Industrial Research, Osaka University, 12 (1955) 95-102.

26) T. Hasegawa, K. Saka, N. Inoue, K. Matsuzawa, Acoustic radiation force experienced by a solid cylinder in a plane progressive sound field, Journal of the Acoustical Society of America, 83 (1988) 1770-1775.

27) J. Wu, G. Du, S.S. Work, D.M. Warshaw, Acoustic radiation pressure on a rigid cylinder: An analytical theory and experiments, Journal of the Acoustical Society of America, 87 (1990) 581-586.

28) T. Hasegawa, Y. Hino, A. Annou, H. Noda, M. Kato, Acoustic radiation pressure acting on spherical and cylindrical shells, Journal of the Acoustical Society of America, 93 (1993) 154-161.

29) W. Wei, D.B. Thiessen, P.L. Marston, Acoustic radiation force on a compressible cylinder in a standing wave, Journal of the Acoustical Society of America, 116 (2004) 201-208.

30) J. Wang, J. Dual, Numerical simulations for the time-averaged acoustic forces acting on rigid cylinders in ideal and viscous fluids, Journal of Physics A: Mathematical and General, 42 (2009) 285502.

31) J. Wang, J. Dual, Theoretical and numerical calculations for the time-averaged acoustic force and torque acting on a rigid cylinder of arbitrary size in a low viscosity fluid, Journal of the Acoustical Society of America, 129 (2011) 3490-3501.

32) E.G. Lierke, Acoustic levitation-A comprehensive survey of principles and applications, Acustica, 82 (1996) 220–237.

33) K. Yasuda, T. Kamakura, Acoustic radiation force on micrometer-size particles, Applied Physics Letters, 71 (1997) 1771–1773.

34) R. Lemons, C.F. Quate, Acoustic microscopy: biomedical applications, Science, 188 (1975) 905-911.

35) K. Nightingale, M.S. Soo, R. Nightingale, G. Trahey, Acoustic radiation force impulse imaging: in vivo Demonstration of clinical feasibility, Ultrasound in Medicine & Biology, 28 (2002) 227–235.

36) A.P. Sarvazyan, O.V. Rudenko, S.D. Swanson, J.B. Fowlkes, S.Y. Emelianov, Shear wave elasticity imaging: a new ultrasonic technology of medical diagnostics, Ultrasound in Medicine & Biology, 24 (1998) 1419-1435.

37) L. Schmerr, J.-S. Song, Ultrasonic Nondestructive Evaluation Systems: Models and Measurements, Springer, 2007.

38) J. Wu, G. Du, Acoustic radiation pressure on a small compressible sphere in a focused beam, Journal of the Acoustical Society of America, 87 (1990) 997-1003.

39) J. Wu, Calculation of acoustic radiation force generated by focused beams using the ray acoustics approach, Journal of the Acoustical Society of America, 97 (1995) 2747-2750.

40) T. Hasegawa, T. Kido, S. Takeda, N. Inoue, K. Matsuzawa, Acoustic radiation force on a rigid sphere in the near field of a circular piston vibrator, Journal of the Acoustical Society of America, 88 (1990) 1578-1583.

41) J. Lee, K.K. Shung, Radiation forces exerted on arbitrarily located sphere by acoustic tweezer, Journal of the Acoustical Society of America, 120 (2006) 1084-1094.

42) J. Lee, S.-Y. Teh, A. Lee, H.H. Kim, C. Lee, K.K. Shung, Single beam acoustic trapping, Applied Physics Letters, 95 (2009) 073701.

43) J. Lee, S.-Y. Teh, A. Lee, H.H. Kim, C. Lee, K.K. Shung, Transvers acoustic trapping using a Gaussian focused, Ultrasound in Medicine & Biology, 36 (2010) 350-355.

44) J. Wu, G. Du, Acoustic streaming generated by a focused Gaussian beam and finite amplitude tonebursts, Ultrasound in Medicine & Biology, 19 (1993) 167-176.

45) G. Du, M.A. Breazeale, Theoretical description of a focused Gaussian ultrasonic beam in a nonlinear medium, Journal of the Acoustical Society of America, 81 (1987) 51-57.

46) S. Kozaki, A new Expression for the scattering of a Gaussian beam by a conducting cylinder, IEEE Transaction on Ultrasonics, Ferroelectrics and Frequency Control, vol. AP-30 (1982) 881-887.

47) A.Z. Elsherbeni, M. Hamid, G. Tian, Iterative scattering of a Gaussian beam by an array of circular conducting and dielectric cylinders, Journal of Electromagnetic Waves and Applications, 7 (1993) 1323–1342.

48) G.A. Deschamps, Gaussian beam as a bundle of complex rays, Electronics Letters, 7 (1971) 684-685.

49) O.A. Sapozhnikov, An exact solution to the Helmholtz equation for a quasi-Gaussian beam in the form of a superposition of two sources and sinks with complex coordinates, Acoustical Physics, 58 (2012) 41-47.

50) K. IIzuka, Elements of photonics, vol. I, Wiley-Interscience, New York, 2002.

51) D. Pierce, Acoustics: An introduction to its physical principals and applications, American Inst. Physics, New York, 1991.

52) F. Honarvar and A.N. Sinclair, Acoustic wave scattering from transversely isotropic cylinders, Journal of the Acoustical Society of America, 100 (1996) 57-63.

53) Z.-S. Wu, H.-Y. Li, Debye series of scattering by a multi-layered cylinder in an off-axis 2D Gaussian beam, Chinese Physics Letters, 25 (2008) 1672-1675.

54) P.L. Marston, Axial radiation force of a Bessel beam on a sphere and direction reversal of the force, Journal of the Acoustical Society of America, 120 (2006) 3518-3524.

55) M. Azarpeyvand, Acoustic radiation force of a Bessel beam on a porous sphere, Journal of the Acoustical Society of America, 131 (2012) 4337-4348.

CAPTIONS

FIG. 1. Problem geometry and the schematic of the coordinate systems used in the problem.

FIG. 2. Axial radiation forces on a rigid cylinder in a Gaussian beam with $q = 0.7 - j0.2$ (a) $\lambda = 0.5a, W_0 = 0.178\ a$ (b) $\lambda = 1.0a, W_0 = 0.252\ a$.

FIG. 3. Transverse radiation forces on a rigid cylinder in a Gaussian beam with $q = 0.7 - j0.2$ (a) $\lambda = 0.5a, W_0 = 0.178\ a$ (b) $\lambda = 1.0a, W_0 = 0.252\ a$.

FIG. 4. Contour plots of negative radiation forces on a rigid cylinder located at $x_0/a = 0$, in terms of λ and y_0, at different $W_0/a = 0.15, 0.5, 1$ and $q_0/a = 0.5, 1, 2$.

FIG. 5. Contour plots of negative radiation forces on a rigid cylinder located at $x_0/a = 0.5$, in terms of λ and y_0, at different $W_0/a = 0.15, 0.5, 1$ and $q_0/a = 0.5, 1, 2$.

FIG. 6. Contour plots of negative radiation forces on a rigid cylinder located at $x_0/a = 1.0$, in terms of λ and y_0, at different $W_0/a = 0.15, 0.5, 1$ and $q_0/a = 0.5, 1, 2$.

FIG. 7. Back-scattered form-functions, f_∞ versus q_0/a, for three waist sizes $W_0/a = 0.15, 0.5$ and 1.0, at different wavelengths and lateral positions chosen from Fig. 4 ($x_0/a = 0$).

FIG. 8. Comparison of results obtained using Eq. (23) for a Gaussian beam with a large waist and Eq. (25).