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New Properties for Certain Generalized Ces´aro Integral Operator

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***Abstract*— In this work, we obtain the order of convexity of the integral operator which is a generalization to Ces´aro integral operator. Furthermore, some other properties of the integral operator by using the concept of the norm and pre-Schwarzian derivatives are obtained.**

***Keywords*— Analytic function; pre-Shwarzian derivatives; Ces´aro integral operator; starlike function; convex function.**

I. INTRODUCTION

The Ces´aro operator C acts formally on the power series $f(z)=\sum_{k=0}^{\infty} a_k(f)z^k$ as

$$C[f](z)=\frac{1}{z}\int_0^z \frac{f(t)}{1-t}\,dt \qquad (1.1)$$

the classical Ces´aro means play an important role in geometric function theory (see [2], [3],[4],[5]).

Let H denote the class of all analytic functions in the open unit disk $U=\{z\in\mathbb{C}: |z|<1\}$ of complex plane.

Let A denote the class of functions $f\in H$ normalized by $f(0)=0,\ f'(0)=1$.

Also, let S denote the class of all univalent functions in A.

A function f belonging to A is said to be starlike of order α in U if it satisfies

$$f\in S^*(\alpha) \Leftrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, \qquad (z\in U),$$

for some $\alpha(0\le\alpha<1)$

Further, a function f belonging to A is said to be convex in U if it satisfies

$$f\in K(\alpha) \Leftrightarrow \Re\left\{\frac{zf''(z)}{f'(z)}+1\right\}>\alpha, \qquad (z\in U),$$

for some $\alpha(0\le\alpha<1)$.

A function f belonging to A is said to be the class $R(\alpha)$ iff

$$\Re\{f'(z)\}>\alpha, \qquad (z\in U),$$

for some $\alpha(0\le\alpha<1)$.

Very recently, Frasin and Jahangiri [6] defined the family $B(\mu,\alpha)$, for some $(\mu\ge 0, 0\le\alpha<1)$, so that it consists of functions $f\in A$ satisfying the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|<1-\alpha\ , \qquad (z\in U) \qquad (1.2)$$

The family $B(\mu,\alpha)$ is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well known ones. For example,

$B(1,\alpha)\equiv S^*(\alpha)$, and $B(0,\alpha)\equiv R(\alpha)$.

Another interesting subclass is the special case $B(2,\alpha)\equiv B(\alpha)$, which has been introduced by Frasin and Darus [7] .

Let $f:U\to C$ be analytic and locally univalent. The pre-Schwarzian derivative (or nonlinearity) T_f to f is defined by

$$T_f=\frac{f''}{f'}\ .$$

Also, with respect to the Hornich operation, the quantity

$$\|T_f\|=\sup_{z\in U}(1-|z|^2)|T_f|,$$

can be regarded as a norm on the space of uniformly locally univalent analytic functions $f \in U$.

It is known that $T_f < \infty$ if and only if f is uniformly locally univalent.

It is well-known that from Becker's univalence criterion [8]: every analytic function f in U with $\|T_f\| \leq 1$ is in fact univalent in U. Conversely, $\|T_f\| \leq 6$ holds if f univalent.

Consider the general integral operator defined by the formula:

$$C[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z) = \frac{1}{z}\int_0^z \left(\frac{f_1(t)}{1-t}\right)^{\frac{1}{\beta_1}} ... \left(\frac{f_m(t)}{1-t}\right)^{\frac{1}{\beta_m}} dt, \quad (z \in U\text{-}\{0\}) \quad (1.3),$$

where $\beta_i \in \mathbb{C} - \{0\}, \forall i = 1, ..., m$, and the functions $f_i(z)$ are in $B(\mu, \alpha)$. It is clear that when $\beta_1 = 1$ and $\beta_j = 0$, $j = 2, ..., m$ the integral operator (1.3) reduces to Ces´aro integral operator (1.1).

In this paper we will study some general properties for function

$$zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z) = \int_0^z \left(\frac{f_1(t)}{1-t}\right)^{\frac{1}{\beta_1}} ... \left(\frac{f_m(t)}{1-t}\right)^{\frac{1}{\beta_m}} dt, \quad (z \in U\text{-}\{0\}).$$

For the purpose this work, we shall make use of the following lemmas.

Lemma 1.1 [1]

Let the analytic function f be regular in the disk with $f(0) = 0$. If $|f(z)| \leq 1$, for all $(z \in U)$ then

$|f(z)| \leq |z|, \quad (z \in U).$

The equality can hold only if $f(z) = \varepsilon z$, where $|\varepsilon| = 1$.

Lemma 1.2 Let the analytic and locally univalent f in U. Then

(*i*) If $\|T_f\| \leq 1$, then f is univalent, and

(*ii*) If $\|T_f\| \leq 2$, then f is bounded.

The part (i) is due to Becker [8] and sharpness of the constant 1 is due to Becker and Pommerenke [9]. The part (ii) is obvious (see [10], Corollary 2.4). Note also that, recently, Kari and Per Hag [12] gave a necessary and sufficient condition for f ∈ S to have a John disk as the image in terms of the preSchwarzian derivative of f.

Also, the norm estimates for typical subclasses of univalent functions are investigated by many authors. See for example ([10], and so on).

Lemma 1.3 [11]

Let $0 \leq \alpha < 1$ and $f \in S$.

(i) If f is starlike of order α, then $\|T_f\| \leq 6 - 4\alpha$, and

(*ii*) If f is convex of order α, *then* $\|T_f\| \leq 4(1-\alpha)$.

The constants are sharp.

II. Main Results

Theorem 2.1

Let $f_i \in A$, be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$, for all $i = 1, 2, ..., m$. If $|f_i(z)| \leq M$, $0 \leq |z| < \frac{1}{2}$, $(M \geq 1, z \in U)$, then

$$zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z) = \int_0^z \left(\frac{f_1(t)}{1-t}\right)^{\frac{1}{\beta_1}} ... \left(\frac{f_m(t)}{1-t}\right)^{\frac{1}{\beta_m}} dt,$$

is convex of order δ,

where

$$\delta = 1 - \sum_{i=1}^{m} \frac{1}{|\beta i|}((2-\alpha)M^{\mu-1} + 1),$$

and

$$\sum_{i=1}^{m} \frac{1}{|\beta i|}((2-\alpha)M^{\mu-1} + 1) < 1, \quad \beta_i \in C - \{0\},$$

For all i=1,2,…,m.

Proof:

From the definition of the operator (1.3), we have

$$zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z) = \int_0^z \prod_{i=1}^{m} \left(\frac{f_i(t)}{1-t}\right)^{\frac{1}{\beta_i}} dt,$$

For $f_i \in B(\mu, \alpha)$. It is easy to see that

$$(zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z))' = \prod_{i=1}^{m} \left(\frac{f_i(t)}{1-t}\right)^{\frac{1}{\beta_i}}. \quad (2.1)$$

Differentiating both sides of (2.1) logarithmically, we obtain

$$\frac{(zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z))''}{(zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z))'} = \sum_{i=1}^{m} \frac{1}{\beta_i}\left(\frac{f_i'(z)}{f_i(z)} + \frac{1}{1-z}\right),$$

which readily shows that

$$\left|\frac{z(zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z))''}{(zC[f_1, f_2, ..., f_m]_{\beta_1, \beta_2, ..., \beta_m}(z))'}\right| \leq \sum_{i=1}^{m} \frac{1}{|\beta_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + \left|\frac{z}{1-z}\right|\right),$$

$$= \sum_{i=1}^{m} \frac{1}{|\beta_i|}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right| \left|\left(\frac{f_i(z)}{z}\right)^{\mu-1}\right| + \left|\frac{z}{1-z}\right|\right). (2.2)$$

Since $|f_i(z)| \le M, (z \in U,\ i \in \{1,2,\ ...,\ m\})$, applying the Schwarz lemma, we obtain

$$\left|\frac{f_i(z)}{z}\right| \le M,\ (z \in U,\ i \in \{1,2,\ ...,\ m\}).$$

Therefore, from (2.2), we obtain

$$\left|\frac{z(zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'}\right| \le$$

$$\sum_{i=1}^{m}\frac{1}{|\beta_i|}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right|M^{\mu-1}+1\right). \qquad (2.3)$$

From (2.3) and (1.2), we see that

$$\left|\frac{z(zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'}\right|$$

$$\le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left(\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}-1\right|+1\right)M^{\mu-1}+1\right)$$

$$\le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left((2-\alpha)M^{\mu-1}+1\right) \le 1-\delta .$$

This completes the proof.

Theorem 2.2

Let $f_i \in A$, for all $i = 1, 2, ..., m$. Suppose that $zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)$ is locally univalent in U.

(1) If $\left[\|T_{f_i}\| + 2\right] \le |\beta_i|$. $\qquad$ (2.4)

Then $zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)$, is univalent in U.

(2) If $\left[\|T_{f_i}\| + 2\right] \le 2\,|\beta_i|$. $\qquad$ (2.5)

Then $zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)$, is univalent in U.

Proof:

Since $\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| = \sup_{z\in U}(1-|z|^2)\left|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right|$. We obtain

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\|$$

$$= \sup_{z\in U}(1-|z|^2)\left|\frac{z(zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'}\right|.$$

$$\le \sup_{z\in U}(1-|z|^2)\sum_{i=1}^{m}\frac{1}{|\beta_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+\left|\frac{z}{1-z}\right|\right)$$

$$\le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left[\|T_{f_i}\| + 2\right].$$

From (2.4),and applying Lemma 1.2 we get

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left[\|T_{f_i}\| + 2\right] \le 1 .$$

Then $T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}$ is univalent in U.

Also, from (2.5), and applying Lemma 1.2, we get

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left[\|T_{f_i}\| + 2\right] \le 2 .$$

Then $T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}$ is bounded in U.

Theorem 2.3

Let $f_i \in S$, for all $i = 1, 2,..., m$.

(1) If f_i are starlike of order α_i, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 4\sum_{i=1}^{m}\frac{1}{|\beta_i|}(1-\alpha_i) .$$

(2) If f_i are convex of order α_i, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 2\sum_{i=1}^{m}\frac{1}{|\beta_i|}(3-2\alpha_i) .$$

Proof: The results follow from (2.6) and by using Lemma 1.3.

Corollary 2.1

Let $f_i \in S$, for all $i = 1, 2,\dots,m$.

(1) If f_i are starlike of order α, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 4(1-\alpha)\sum_{i=1}^{m}\frac{1}{|\beta_i|} .$$

(2) If f_i are convex of order α, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 2(3-2\alpha)\sum_{i=1}^{m}\frac{1}{|\beta_i|} .$$

Corollary 2. Let $f_1 \in S$.

(1) If f_1 are starlike of order α, then

$$\left\|T_{zC[f_1]_{\beta_1}(z)}\right\| \le \frac{4(1-\alpha)}{|\beta_1|} .$$

(2) If f_1 are convex of order α, then

$$\left\|T_{zC[f_1]_{\beta_1}(z)}\right\| \le \frac{2(3-2\alpha)}{|\beta_1|} .$$

III. Conclusions

We conclude this study with some suggestions for future research; one direction is to obtain the order of convexity of the integral operator. Another direction would be studying other properties of the integral operator.

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