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Vertex-transitive digraphs with extra automorphisms that preserve the natural arc-colouring

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Dedicated to the memory of Dan Archdeacon

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Abstract

In a Cayley digraph on a group G, if a distinct colour is assigned to each arc-orbit under the left-regular action of G, it is not hard to show that the elements of the left-regular action of G are the only digraph automorphisms that preserve this colouring. In this paper, we show that the equivalent statement is not true in the most straightforward generalisation to G-vertex-transitive digraphs, even if we restrict the situation to avoid some obvious potential problems. Specifically, we display an infinite family of 2-closed groups G, and a G-arc-transitive digraph on each (without any digons) for which there exists an automorphism of the digraph that is not an element of G (it is an automorphism of G). Since the digraph is G-arc-transitive, the arcs would all be assigned the same colour under the colouring by arc-orbits, so this digraph automorphism is colour-preserving.

1 Introduction

A *Cayley digraph* $\mathrm{Cay}(G;S)$ on a group G with connection set $S \subset G$, is the digraph whose vertices are elements of G, with an arc from g to gs if and only if $s \in S$. If we want to consider Cayley graphs rather than digraphs, we insist that $S = S^{-1}$, so that whenever $s \in S$ we have the arc from gs to $(gs)s^{-1} = g$ as well as the arc from g to gs. By convention, we assume $1 \notin S$; this avoids the situation where our digraphs have loops at every vertex, although this assumption is in no way material to the results we present here.

Observe that for any Cayley digraph $\mathrm{Cay}(G;S)$, there are certain natural automorphisms that come from the group action. The most obvious is the *left-regular representation of* G, denoted by G_L. It was first observed by Sabidussi in [13, Lemma 4] that for every element $g \in G$, left-multiplication of all vertices by g is an automorphism of $\mathrm{Cay}(G;S)$, so that there is a regular copy of G sitting inside the automorphism group of the Cayley digraph. (In fact, as Sabidussi observed, Cayley digraphs on G are characterised by the property of having a regular subgroup isomorphic to G in their automorphism groups.)

The other natural Cayley digraph automorphism that can arise from the group action, arises when there is an automorphism α of G such that $\alpha(S) = S$. The set of all such automorphisms forms a subgroup of the automorphism group of G, usually denoted by $\mathrm{Aut}(G,S)$.

A *normal Cayley digraph* (defined by Xu in [16]) is a Cayley digraph for which these group automorphisms are the only digraph automorphisms. Thus, in a normal Cayley digraph $\Gamma = \mathrm{Cay}(G;S)$, we have $\mathrm{Aut}(\Gamma) = G_L \rtimes \mathrm{Aut}(G;S)$. These are very special digraphs. With some very small exceptions, it is never the case that all Cayley digraphs (or even all connected Cayley digraphs) on a group will be normal, since if $|G| = n$, K_n is a (connected) Cayley digraph on G with automorphism group S_n, and for $n \geq 5$, it is not possible to write S_n as $G_L \rtimes \mathrm{Aut}(G;S)$.

It therefore makes sense to look for natural ways to restrict the full automorphism group of a given Cayley digraph, so that all of the automorphisms subject to this restriction, do come from the group actions. One approach that has been proposed is the *CCA problem*, [6]. In this approach, the authors begin by defining a natural colouring on the arcs of any Cayley digraph, using the elements of S as the colours. The arc from g to gs is assigned the colour s.

Definition 1.1 *An automorphism α of an arc-colored digraph Γ is said to be* colour-preserving, *if the colour of every arc e is the same as the colour of $\alpha(e)$. A Cayley digraph* $\mathrm{Cay}(G;S)$ *is called* CCA *(for Cayley Colour Automorphism) if the colour-preserving automorphisms of the naturally coloured* $\mathrm{Cay}(G;S)$ *all arise from the group actions; that is, they all lie in $G_L \rtimes \mathrm{Aut}(G;S)$.*

A very straightforward argument was used in [15, Theorem 4-8] to show that all Cayley digraphs have an even stronger property than being CCA: specifically, all of their colour-preserving automorphisms come from G_L. This makes it easy to show that Cayley digraphs are CCA.

Proposition 1.2 *[15, Theorem 4-8] Every Cayley digraph is CCA.*

Notice that if $\alpha \in \mathrm{Aut}(G;S)$, even though S is fixed setwise by α, the only way that α will be colour-preserving is if for each $s \in S$, $\alpha(s) = s$. Thus, the restriction to colour-preserving automorphisms is stronger than we would really like, since it may eliminate many automorphisms that do come from the group action, as well as (we hope) those that do not. The authors therefore propose a related problem.

Definition 1.3 *An automorphism α of an arc-coloured digraph Γ is* colour-permuting *if whenever arcs e_1, e_2 of Γ have the same colour, so do $\alpha(e_1)$, $\alpha(e_2)$ (but the colour of $\alpha(e_1)$ might not be the same as the colour of e_1).*

Clearly every colour-preserving automorphism is also colour-permuting. Thus, in a Cayley digraph, the elements of G_L are colour-permuting automorphisms (since they are colour-preserving). Additionally, the elements of $\mathrm{Aut}(G;S)$ are all colour-permuting, even if they are not colour-preserving.

Definition 1.4 *A Cayley digraph* $\mathrm{Cay}(G;S)$ *is called* strongly CCA *if every colour-permuting automorphism of naturally coloured* $\mathrm{Cay}(G;S)$ *lies in $G_L \rtimes \mathrm{Aut}(G;S)$.*

Since it is easy to see that the group of colour-preserving automorphisms is normal in the group of colour-permuting automorphisms, the following result is also straightforward.

Proposition 1.5 *[4, Lemma 2.1] Every Cayley digraph is strongly CCA.*

Our aim in this paper is to generalise the CCA problem to vertex-transitive digraphs that are not Cayley digraphs. We will show that the status of vertex-transitive digraphs that are not Cayley digraphs is not so straightforward. To this end, we will present an infinite family of vertex-transitive digraphs that are not Cayley digraphs that have automorphisms that are not elements of a minimal transitive subgroup M of their automorphism group, amongst other properties. These automorphisms are still related to M, since they lie in its normaliser. We think that this family of digraphs may also have other interesting properties deserving of further study.

2 How should the CCA property be generalised to vertex-transitive digraphs?

Observe that in the CCA (or strongly CCA) problem, in the "natural arc-colouring" being studied, each colour class is precisely the orbit of some arc of a Cayley digraph on a group G, under the regular action of G.

There is an obvious generalisation of this colouring to vertex-transitive digraphs.

Definition 2.1 *Given a digraph* Γ *and a vertex-transitive subgroup* $G \leq \mathrm{Aut}(\Gamma)$*, the natural arc-colouring assigns a different colour to each arc-orbit of* G*: two arcs get the same color if and only if they lie in the same orbit of* G*.*

With this colouring, it is natural to study the closely-related problem on vertex-transitive digraphs: when do all colour-preserving (or more strongly, all colour-permuting) automorphisms of the digraph, come from the group action? However, even with the new colouring, we need to define what it means in this context for an automorphism to come from the group action.

Recall that we defined a Cayley digraph to have the CCA (respectively, strongly CCA) property if all colour-preserving (respectively, colour-permuting) digraph automorphisms lie in $G_L \rtimes \mathrm{Aut}(G;S)$. Observe that the elements of $\mathrm{Aut}(G) \setminus \mathrm{Aut}(G;S)$ are not automorphisms of the Cayley digraph $\mathrm{Cay}(G;S)$ at all. Therefore, it is equivalent to define a Cayley digraph to have the CCA (respectively, strongly CCA) property if all colour-preserving (respectively, colour-permuting) digraph automorphisms lie in $G_L \rtimes \mathrm{Aut}(G)$. This version of the definition proves easier to generalise to the vertex-transitive situation.

Clearly, the natural arc-colouring is preserved by every element of G, so G will take the role of G_L from the problem for Cayley digraphs. Let H be the stabiliser in G of some vertex of the digraph Γ. As before, $\mathrm{Aut}(G;H)$ will denote all automorphisms of G that fix H setwise. The action of G on Γ is isomorphic to the action of G on the left cosets of H; we identify these two actions. Automorphisms of Γ that come from the group action are those that lie in $G \rtimes \mathrm{Aut}(G;H)$. In fact, to be an automorphism of Γ, an automorphism of G must fix not only H, but the collection of left cosets of H that are neighbours of H. Similar to the Cayley digraph situation, every automorphism of G that is an automorphism of Γ, will permute the orbits of the edges (or arcs) under the action of G, so will be colour-permuting (if not colour-preserving).

It is tempting to immediately define the G-VTCA property analogously to the CCA property, but in some sense this could be considered "cheating." Let Γ be a G-vertex-transitive graph. We don't want to have a situation where there is some $G' \geq G$ that gives rise to precisely the same arc-colouring as G, and all of the digraph automorphisms of Γ lie in $G' \rtimes \mathrm{Aut}(G'; H')$ even though they do not lie in $G \rtimes \mathrm{Aut}(G; H)$. (Here H' is the stabiliser of a vertex of Γ under the action of G'.) In essence, this would mean that we erred in choosing G rather than G' as the group that gave rise to this colouring.

Definition 2.2 *The permutation group G acting on the set Ω is* 2-closed*, if there is no permutation group $G' > G$ such that the orbits of G' on $\Omega \times \Omega$ are the same as the orbits of G on $\Omega \times \Omega$. (The "2" comes from the fact that we are looking at the orbits of G on ordered pairs of elements of Ω.) The* 2-closure *of G is the largest group $G' \geq G$ that has the same orbits as G on $\Omega \times \Omega$. It is denoted by $G^{(2)}$.*

Equivalently, it can be shown that a permutation group is 2-closed if and only if it is the automorphism group of some colour digraph.

Observe that the regular action of any group G is 2-closed, since it is the automorphism group of every connected Cayley colour digraph on G (with the natural edge-colouring), as was shown in [15, Theorem 4-8] and mentioned previously. However, it is sometimes possible to find a group G that is not 2-closed, acting vertex-transitively on a digraph Γ. In such a case, as noted above, the 2-closure of G has the same orbits on arcs as G, and hence induces the same colouring on the arcs. Every element of $G^{(2)}$ is therefore a colour-preserving automorphism, but there is no reason to believe in general that the elements of $G^{(2)} \setminus G$ will lie in $\mathrm{Aut}(G; H)$. Before turning to an example of this, we will need some terminology.

Definition 2.3 *Let G be a transitive permutation group with invariant partition $\mathcal{B}$. By $G/\mathcal{B}$, we mean the subgroup of $S_{\mathcal{B}}$ induced by the action of G on $\mathcal{B}$, and by $\mathrm{fix}_G(\mathcal{B})$ the kernel of this action. Thus $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$ where $g/\mathcal{B}(B_1) = B_2$ if and only if $g(B_1) = B_2$, $B_1, B_2 \in \mathcal{B}$, and $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$.*

Example 2.4 *Let $p \geq 11$ be an odd prime, $\alpha = p + 1 \in \mathbb{Z}_{p^3}^*$ of order p^2, and $G = \mathbb{Z}_{p^3} \rtimes_\alpha \mathbb{Z}_{p^2}$. Then G is a minimal transitive subgroup of the automorphism group of a vertex-transitive digraph but is not 2-closed. Additionally, G is not normal in $G^{(2)}$.*

Proof: By [8, Proposition 3.5] or [11, Theorem 6] there exists a non-Cayley vertex-transitive digraph Γ of order p^4 whose automorphism group contains G as a transitive subgroup. (That the automorphism groups of these digraphs contains G as a transitive subgroup is not in the statements of either [8, Proposition 3.5] or [11, Theorem 6], but is given in the proofs of these results. Additionally, these results are only proven for graphs, but can be easily modified to give examples of such digraphs.)

More specifically, there is a G-invariant partition consisting of p blocks of size p^3, which are the orbits of the normal cyclic subgroup $\langle\rho\rangle$. We can identify the vertices with the elements of $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ so that for every $(i,j) \in \mathbb{Z}_p \times \mathbb{Z}_{p^3}$, we have $\rho((i,j)) = (i, j+1)$, and $G = \langle\rho,\tau\rangle$, where $\tau((i,j)) = (i+1,(p+1)j)$.

Observe that $|G| = p^5$, so if there were a smaller transitive subgroup it would have to be regular, contradicting the fact that this graph is non-Cayley. Hence G is a minimal transitive subgroup of $\mathrm{Aut}(\Gamma)$. It only remains to show that $G^{(2)} \neq G$ and that G is not normal in $G^{(2)}$. Now, G admits an invariant partition $\mathcal{B}$ consisting of p^3 blocks of size p formed by the orbits of $\langle\rho^{p^2}\rangle$; that is, each $B \in \mathcal{B}$ has the form $\{(i, j+kp^2) : 0 \le k \le p-1\}$, for some $(i,j) \in \mathbb{Z}_p \times \mathbb{Z}_{p^3}$. Observe that $\tau^p((i,j)) = (i,(p+1)^p j) = (i,(p^2+1)j)$, so $\tau^p \in \mathrm{fix}_G(\mathcal{B})$. Thus, $G/\mathcal{B} \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ is regular, and so $\mathrm{fix}_G(\mathcal{B})$ has order p^2. It then follows by [1, Lemma 2] that there is a nontrivial G-invariant partition $\mathcal{C} \succeq \mathcal{B}$ with $\rho^{p^2}|_C \le G^{(2)}$ for every $C \in \mathcal{C}$. Since $\mathcal{C}$ is nontrivial, it consists of at least p blocks. Hence $\mathrm{fix}_{G^{(2)}}(\mathcal{B})$ has order at least p^p, $|G^{(2)}| \ge p^{p+3} > p^5$ since p is odd, so $G^{(2)} \neq G$. Finally, it cannot be the case that $G^{(2)} \le G \rtimes \mathrm{Aut}(G)$ as $|\mathrm{Aut}(G)| = (p-1)p^5$ by [12], and so $|G \rtimes \mathrm{Aut}(G)| = (p-1)p^{10} < p^{p+3}$ as $p \ge 11$. □

The above result also holds for odd primes $p < 11$, but the proof is more complicated. We chose the simpler proof as our intention is to only show the existence of a G as in the result.

Definition 2.5 *We will say that a G-vertex-transitive digraph Γ has the* G-VTCA property *if G is a 2-closed group, and every automorphism of Γ that preserves the natural arc-colouring lies in $G \rtimes \mathrm{Aut}(G;H)$. Similarly, it has the* strong G-VTCA property*, if every automorphism of Γ that permutes the natural arc-colouring lies in $G \rtimes \mathrm{Aut}(G;H)$.*

As observed in Proposition 1.5, all Cayley digraphs are strongly CCA, and the only colour-preserving automorphisms are elements of G_L. The goal of this paper is to show that the corresponding situation for vertex-transitive digraphs is much more interesting and complex. In particular, we aim to construct an infinite family of G-vertex-transitive digraphs that may or may not have the G-VTCA property, but for which there are some colour-preserving digraph automorphisms that come from $\mathrm{Aut}(G;H)$ rather than from G.

Since it was already shown in [6, Theorem 1.6] that there exist Cayley graphs that do not even have the (weaker) CCA property, one way to construct such digraphs would be to find a group G that acts vertex-transitively on a digraph Γ, and also contains elements that reverse each arc of Γ, so that every arc is part of a digon, and both arcs in the digon have the same colour. This would be a very unsatisfying construction. Morally, such a structure is really a graph, not a digraph. We will insist in our construction that there exists some arc (u,v) in each of our digraphs, such that the reverse arc (v,u) is either not in the digraph, or at least has a different colour from (u,v).

There is one additional property that we would like our example to have. Namely, it is often the case that in the automorphism group of a digraph there is a chain of subgroups $G_1 < G_2 < \cdots < G_r$ and each of the G_i are transitive and 2-closed. Sabidussi [14, Theorem 2] has shown that every vertex-transitive graph (and the same construction works equally well for digraphs) can be constructed as a coset digraph of a transitive subgroup of its automorphism group. If the transitive subgroup is chosen to be regular, then the coset digraph construction is the same as the Cayley digraph construction. So when constructing vertex-transitive digraphs there are often many choices of which transitive subgroup to use in the coset digraph construction, but it is most natural to use a minimal transitive subgroup.

3 Adapted Marušič-Scappellato graphs

Our construction will involve orbital digraphs of $\mathrm{SL}(2,q)$, where $q = p^m$, p is any odd prime and $m \geq 2$. The first related construction of this type was introduced by Marušič and Scappellato [9, 10], with a related construction involving $\mathrm{SL}(2,p)$ being introduced in [2]. These constructions produced graphs with various interesting properties, but neither existing construction has three significant properties that we seek. First, we will construct digraphs rather than graphs; second, we will ensure that the resulting digraphs are not Cayley digraphs (since we know that Cayley digraphs will be CCA); and third, we will ensure that there will be an automorphism of the digraph that does not come directly from the appropriate vertex-transitive group actions (although it will normalise these).

In our construction, we require the concept of an *orbital digraph*.

Definition 3.1 *Let G be any permutation group acting on a set Ω. Let $O_0, O_1, \ldots, O_k$ be the orbits of G on $\Omega \times \Omega$, and assume O_0 is the diagonal orbit $\{(\omega,\omega) : \omega \in \Omega\}$. Then there are k* orbital digraphs *of G: $\Gamma_1, \ldots, \Gamma_k$. Each of these digraphs has for its vertices the elements of Ω. For every $1 \leq i \leq k$, there is an arc from u to v in Γ_i if and only if $(u,v) \in O_i$. In other words, each of the orbital digraphs of G has for its arcs the orbit of some particular arc under the action of G.*

Construction 3.2 *Let $G_q = \mathrm{SL}(2,q)$ be the special linear group of dimension two over the field $\mathrm{GF}(q)$ where $q = p^m$, p is any odd prime, and $m \geq 2$. The vertices of our digraph will be the non-zero two-dimensional vectors over $\mathrm{GF}(q)$. Observe that the one-dimensional subspaces with the zero vector removed, form a G_q-invariant partition $\mathcal{B}$ of these vertices. We call the elements of this partition the* blocks *of $\mathcal{B}$, and remark that they are also called projective points. We choose the arc $((1,0),(0,1))$ (note that $(1,0)$ and $(0,1)$ do not lie in the same block of $\mathcal{B}$), and let Γ_q be the orbital digraph of G_q that contains this arc.*

It is clear that the group G_q has a vertex-transitive action on Γ_q. In fact, the action of G_q on the blocks of $\mathcal{B}$ is doubly-transitive. Note that $\mathrm{fix}_{G_q}(\mathcal{B})$ consists of the identity matrix I and the matrix

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(no other scalar matrix lies in $G_q = \mathrm{SL}(2, q)$).

Observe that every arc of Γ_q goes from one element of $\mathcal{B}$ to another; since $\mathcal{B}$ forms an invariant partition under G_q, it is not possible for an element of G_q to take $((1, 0), (0, 1))$ to an arc both of whose endpoints lie in a single block of $\mathcal{B}$.

Also note that since Γ_q is a single orbital digraph of G_q, every arc of Γ_q has a single colour in our colouring, so any automorphism of the digraph will be colour-preserving.

We now show that Γ_q is indeed a digraph rather than a graph; in fact, that for every arc (u, v) of Γ_q, the arc (v, u) is not in Γ_q.

Lemma 3.3 *[5, Theorem 8.3] A Sylow* 2*-subgroup of* $\mathrm{SL}(2, q)$ *is generalised quaternion when* q *is odd.*

Corollary 3.4 *There is a unique subgroup of order* 2 *in* $\mathrm{SL}(2, q)$ *when* q *is odd. Furthermore, the unique element of order* 2 *is* $-I$*.*

Lemma 3.5 *For every arc* (u, v) *in* Γ_q*, the arc* (v, u) *is not in* Γ_q*.*

PROOF: Since Γ_q is an orbital digraph of G_q, the group G_q acts transitively on the arcs of Γ_q. Thus, if for some arc (u, v) of Γ_q the arc (v, u) were also in Γ_q, there must be some element g of G_q that maps (u, v) to (v, u). Clearly, the element g must have even order.

By Corollary 3.4, some power of g is $-I$. Since $-I$ fixes every block of $\mathcal{B}$, it cannot map (u, v) to (v, u). However, $-I$ also does not fix any vertex of Γ_q. Since every power of g maps (u, v) to either (u, v), or (v, u), this is a contradiction. □

Next we show that G_q is 2-closed.

Proposition 3.6 *The group* $G_q = \mathrm{SL}(2, q)$ *in its action on the non-zero two-dimensional vectors over* $\mathrm{GF}(q)$*, is the automorphism group of a colour digraph, so is* 2*-closed.*

PROOF: It is well-known that the nonzero elements of any finite field form a cyclic group under multiplication. Let α be a generator for this group. Define Γ'_q to be the digraph obtained by adding to Γ_q (all of whose arcs have been assigned a single colour, say black) all arcs of the form $(u, \alpha u)$ where u is a nonzero 2-dimensional vector, and assigning a new colour (red, say) to all of these arcs. We claim that G_q is the automorphism group of this digraph.

It should be clear that $G_q \leq \mathrm{Aut}(\Gamma'_q)$. Since Γ_q is an orbital digraph of G_q, its arcs are clearly preserved under the action of G_q, while the red arcs are preserved because multiplying a vector u by the scalar α commutes with the action of any element of G_q (scalar matrices lie in the centre of $\mathrm{SL}(2,q)$). We must show that Γ'_q has no other automorphisms. We will do this by showing that $|\operatorname{Aut}(\Gamma'_q)| = q(q^2-1) = |G_q|$.

Observe that if we ignore the black arcs, the digraph of red arcs consists of $q+1$ disjoint directed cycles, each of length $q-1$. Furthermore, these connected components are precisely the blocks of $\mathcal{B}$. Thus, the automorphism group of Γ'_q must be imprimitive, with these blocks forming an invariant partition. Since $G_q \leq \operatorname{Aut}(\Gamma'_q)$, we see that $\operatorname{Aut}(\Gamma'_q)$ is vertex-transitive, so by the orbit-stabiliser theorem, its order will be q^2-1 times the order of the subgroup that fixes some vertex. We choose to consider the subgroup that fixes the vertex $(1,0)$.

Note that since G_q acts arc-transitively on Γ_q, the neighbours in Γ_q of $(1,0)$ are the images of $(0,1)$ under the subgroup of G_q that fixes $(1,0)$. The matrices of $\mathrm{SL}(2,q)$ that fix $(1,0)$ have the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where b is any element of $\mathrm{GF}(q)$. Thus, the outneighbours in Γ_q of $(1,0)$ are precisely the elements $(b,1)$. Since $G_q \leq \operatorname{Aut}(\Gamma'_q)$ has this same action on the black arcs of Γ'_q, we see that the orbit of $(0,1)$ under the subgroup of $\operatorname{Aut}(\Gamma'_q)$ that fixes $(1,0)$, has length q. To complete the proof, it will suffice (using the orbit-stabiliser theorem) to show that the subgroup of $\operatorname{Aut}(\Gamma'_q)$ that fixes both $(1,0)$ and $(0,1)$ is trivial.

Consider an arbitrary automorphism β of Γ'_q that fixes both $(1,0)$ and $(0,1)$. Since there is a unique directed cycle of red arcs containing any vertex, we see that every vertex of the form $(x,0)$ and every vertex of the form $(0,y)$ must also be fixed by β (where $x,y \in \mathrm{GF}(q)$). We have seen above that $(1,0)$ has precisely one black outneighbour in each of the blocks $B \in \mathcal{B}$. Consider an arbitrary such black outneighbour of $(1,0)$, say $(b,1)$. Using the fact that Γ_q is an orbital digraph of G_q, it is not hard to see that $(b,1)$ must have a unique black inneighbour in the block $B \in \mathcal{B}$ that contains $(0,1)$, and that this inneighbour is $(0,b^{-1})$. In particular, this shows that each of the $q-1$ black outneighbours of $(1,0)$ (not counting $(0,1)$) has a distinct black inneighbour from B. This shows that every black outneighbour of $(1,0)$ is fixed by β, which implies that the directed red cycles containing each of these vertices are all fixed pointwise by β. There are no other vertices in Γ'_q, so β is trivial, completing the proof. □

We still have to show that Γ_q admits some automorphism that is not an element of G_q.

Proposition 3.7 *The map α defined by $\alpha((x,y)) = (x^p, y^p)$ is an automorphism of Γ_q, and does not lie in G_q.*

Proof: We see that α fixes $(1,0)$ and fixes the outneighbours of $(1,0)$ setwise, since any element of the form $(y,1)$ maps to $(y^p,1)$. Thus, these arcs are preserved. Since

Γ_q is an orbital digraph of G_q, any other arc of Γ_q has the form $(\beta(1,0), \beta(0,1))$, where $\beta \in G_q$. In particular, if

$$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the arc is $((a,c),(b,d))$. Under α, this arc maps to $((a^p, c^p), (b^p, d^p))$. Since

$$\begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

is also in $G_q = \mathrm{SL}(2,q)$, we see that $\alpha((a,c),(b,d))$ is also an arc of Γ_q. Thus α is indeed an automorphism of Γ_q.

Since $q = p^r$ with $r > 1$, α is not the identity, but α fixes both $(0,1)$ and $(1,0)$. Therefore α does not lie in G_q. $\square$

Finally, in order to show that $\mathrm{SL}(2,q)$ contains no proper transitive subgroups in its action on the 2-dimensional nonzero vectors, we will need Dickson's Classification of the subgroups of $\mathrm{PSL}(2,q)$ [7, Hauptsatz 8.27].

Theorem 3.8 *A subgroup of* $\mathrm{PSL}(2,q)$ *with* $q = p^r$ *and* p *a prime is one of the following groups:*

1. *An elementary abelian p-group of order p^m with $m \le r$.*
2. *A cyclic group of order z where z is a divisor of $q-1$ or $q+1$ if $p = 2$, and a divisor of $(q-1)/2$ or $(q+1)/2$ if $p > 2$.*
3. *A dihedral group of order $2z$ where z is as in (2).*
4. *A semi-direct product of an elementary abelian p-group of order p^m and a cyclic group of order t where t is a divisor of $p^{\gcd(m,r)} - 1$.*
5. *A group isomorphic to A_4 where if $p = 2$ then r is even.*
6. *A group isomorphic to S_4 if $p^{2^r} - 1 \equiv 0 \pmod{16}$.*
7. *A group isomorphic to A_5 if $p^r(p^{2^r} - 1) \equiv 0 \pmod 5$.*
8. *A group isomorphic to* $\mathrm{PSL}(2,p^m)$ *where m divides r.*
9. *A group isomorphic to* $\mathrm{PGL}(2,p^m)$ *where $2m$ divides r.*

Proposition 3.9 *Let p be an odd prime, $r \ge 2$, and $q = p^r$. The action of* $\mathrm{SL}(2,q)$ *on the 2-dimensional nonzero vectors over* $\mathrm{GF}(q)$ *contains no proper transitive subgroups.*

PROOF: Let G_q denote the action of $\mathrm{SL}(2,q)$ on the 2-dimensional nonzero vectors over $\mathrm{GF}(q)$. Clearly there are q^2-1 nonzero 2-dimensional vectors over $\mathrm{GF}(q)$. Also, as $\mathrm{fix}_{G_q}(\mathcal{B}) = \langle -I \rangle$, we have $G_q/\mathcal{B} = \mathrm{SL}(2,q)/\mathcal{B} \cong \mathrm{PSL}(2,q)$ and is transitive of degree $(q^2-1)/2$. Hence $\mathrm{PSL}(2,q)$ contains a subgroup of order a multiple of $(q^2-1)/2$. Inspecting Dickson's Classification of subgroups of $\mathrm{PSL}(2,q)$, we see that the only subgroup of $\mathrm{PSL}(2,q)$ whose order is a multiple of $(q^2-1)/2$ is $\mathrm{PSL}(2,q)$ itself. Thus the only subgroups of G_q of order a multiple of q^2-1 are either $\mathrm{PSL}(2,q)$ or G_q. As $\mathrm{PSL}(2,q)$ has even order and so has elements of order 2 which are not $-I$, we see by Corollary 3.4 that $\mathrm{PSL}(2,q)$ is not a subgroup of G_q, and so the only transitive subgroup of G_q is G_q itself. □

These results combine to prove our main theorem.

Theorem 3.10 *The digraphs Γ_q of Construction 3.2 are $\mathrm{SL}(2,q)$-vertex-transitive, with the following additional properties:*

1. *for every arc in Γ_q, the reverse arc is not in Γ_q; and*
2. *Γ_q has a colour-preserving automorphism that does not lie in $\mathrm{SL}(2,q)_L$.*

Furthermore, this action of $\mathrm{SL}(2,q)$ is 2-closed, and $\mathrm{SL}(2,q)$ contains no proper transitive subgroups.

PROOF: The digraphs are $\mathrm{SL}(2,q)$-vertex-transitive by construction. Lemma 3.5 implies (1). Proposition 3.7 implies (2), and Proposition 3.6 implies that $\mathrm{SL}(2,q)$ in its actions on the non-zero two-dimensional vectors over $\mathrm{GF}(q)$ is a 2-closed group. Finally, $\mathrm{SL}(2,q)$ contains no proper transitive subgroups by Proposition 3.9. □

We close this section with two obvious problems that the work in this paper demonstrates are worth considering.

Problem 3.11 *Are there digraphs Γ which are not graphs and which are not G-VTCA digraphs for some transitive 2-closed group $G \leq \mathrm{Aut}(\Gamma)$ that contains no proper 2-closed minimal transitive subgroups?*

Problem 3.12 *Are there digraphs Γ which are not graphs and which are not strongly G-VTCA digraphs for some transitive 2-closed group $G \leq \mathrm{Aut}(\Gamma)$ that contains no proper 2-closed minimal transitive subgroups?*

4 Natural edge-colouring of vertex-transitive graphs

In this section we explain how the concepts that we study for digraphs can be naturally extended to graphs. We start by defining CCA and strongly CCA property for Cayley graphs, first introduced in [6].

Definition 4.1 *In the* natural edge-colouring *of the edges of a Cayley graph* $\mathrm{Cay}(G, S)$*, we color the edges* $x - xs_1$ *and* $x - xs_2$ $(s_1, s_2 \in S)$ *with the same colour if and only if* $s_2 = s_1^{-1}$*. Similarly to the definition for coloured digraphs, we say that an automorphism* α *of an edge-coloured graph* Γ *is colour-preserving if the colour of every edge* e *is the same as the colour of* $\alpha(e)$*. Analogously, we say that an automorphism* α *of an edge-coloured graph* Γ *is colour-permuting if whenever edges* e_1*,* e_2 *of* Γ *have the same colour, so do* $\alpha(e_1)$*,* $\alpha(e_2)$ *(but the colour of* $\alpha(e_1)$ *might not be the same as the colour of* e_1*). We say that a Cayley graph* $\mathrm{Cay}(G; S)$ *is CCA (respectively strongly CCA) if every colour-preserving (respectively colour-permuting) automorphism of a naturally coloured graph* $\mathrm{Cay}(G; S)$ *lies in* $G_L \rtimes \mathrm{Aut}(G; S)$*.*

As observed in Proposition 1.5, every Cayley digraph is strongly CCA. The situation with graphs is much more interesting and much more complicated. It is an interesting and open problem to classify Cayley graphs that have the CCA property. Most of the progress to date has lain in the direction of determining the groups on which all connected Cayley graphs have the CCA property, and the groups for which this is not true. For more details regarding these topics, see [3, 6].

We extend the definition of G-VTCA digraph to graphs as follows.

Definition 4.2 *Given a graph* Γ *and a vertex-transitive subgroup* $G \leq \mathrm{Aut}(\Gamma)$*, the natural edge-colouring assigns a different colour to each edge-orbit of* G*: two edges get the same color if and only if they lie in the same orbit of* G*.*

Definition 4.3 *We will say that a* G*-vertex-transitive graph* Γ *has the* G-VTCA property*, if* G *is a* 2*-closed group, and every automorphism of* Γ *that preserves the natural edge-colouring lies in* $G \rtimes \mathrm{Aut}(G; H)$*. Similarly, it has the* strong G-VTCA property*, if every automorphism of* Γ *that permutes the natural edge-colouring lies in* $G \rtimes \mathrm{Aut}(G; H)$*.*

We end with the natural graph analogues of our previous problems for digraphs.

Problem 4.4 *Are there graphs* Γ *which are not* G*-VTCA graphs for some transitive* 2*-closed group* $G \leq \mathrm{Aut}(\Gamma)$ *that contains no proper* 2*-closed minimal transitive subgroups?*

Problem 4.5 *Are there graphs* Γ *which are not strongly* G*-VTCA graphs for some transitive* 2*-closed group* $G \leq \mathrm{Aut}(\Gamma)$ *that contains no proper* 2*-closed minimal transitive subgroups?*

References

[1] E. Dobson, Isomorphism problem for Cayley graphs of $\mathbb{Z}_p^3$, *Discrete Math.* **147** (1995), no. 1-3, 87–94.

[2] E. Dobson, Some non-normal Cayley digraphs of the generalized quaternion group of certain orders, *Electron. J. Combin.* **10** (2003), #R31, 7 pp.

[3] E. Dobson, A. Hujdurović, K. Kutnar and J. Morris, On color-preserving automorphisms of Cayley graphs of odd square-free order, *J. Algebraic Combin.* (2016), 1–16.

[4] M. L. Fiol, M. A. Fiol and J. L. A. Yebra, When the arc-colored line digraph of a Cayley colored digraph is again a Cayley colored digraph, *Ars Combin.* **34** (1992), 65–73.

[5] D. Gorenstein, *Finite groups*, Harper & Row Publishers, New York, 1968.

[6] A. Hujdurović, K. Kutnar, D. W. Morris and J. Morris, On colour-preserving automorphisms of Cayley graphs, *Ars Math. Contemp.* **11** (2016), no. 1, 189–213.

[7] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[8] D. Marušič, *Vertex transitive graphs and digraphs of order p^k*, Cycles in graphs (Burnaby, B.C., 1982), North-Holland Math. Stud., vol. 115, North-Holland, Amsterdam, 1985, pp. 115–128.

[9] D. Marušič and R. Scapellato, Characterizing vertex-transitive pq-graphs with an imprimitive automorphism subgroup, *J. Graph Theory* **16** (1992), no. 4, 375–387.

[10] D. Marušič and R. Scapellato, Imprimitive representations of $\mathrm{SL}(2, 2^k)$, *J. Combin. Theory Ser. B* **58** (1993), no. 1, 46–57.

[11] B. D. McKay and C. E. Praeger, Vertex-transitive graphs which are not Cayley graphs I, *J. Austral. Math. Soc. Ser. A* **56** (1994), no. 1, 53–63.

[12] F. Menegazzo, Automorphisms of p-groups with cyclic commutator subgroup, *Rend. Sem. Mat. Univ. Padova* **90** (1993), 81–101.

[13] G. Sabidussi, On a class of fixed-point-free graphs, *Proc. Amer. Math. Soc.* **9** (1958), 800–804.

[14] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426–438.

[15] A. T. White, *Graphs, groups and surfaces*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, North-Holland Mathematics Studies, No. 8.

[16] M.-Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998), no. 1-3, 309–319, Graph theory (Lake Bled, 1995).

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