A composite material with inextensible–membrane–type interface[1]

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Abstract: We consider a model of a composite material with "inextensible membrane type" interface conditions. An analytic solution of a stationary heat conduction problem in an unbounded doubly periodic 2D composite whose matrix and inclusions consist of isotropic temperature dependent materials is given. In the case when the conductive properties of the inclusions are proportional to that of the matrix, the problem is transformed into a fully linear boundary value problem for doubly periodic analytic functions. The solution makes it possible to calculate the average properties over the unit cell and discuss the effective conductivity of the composite. We present numerical examples to indicate some peculiarities of the solution.

Keywords: Nonlinear 2D doubly periodic composite material; effective properties of the composite; non-ideal contact condition; numerical analysis

1 Introduction

The importance and applications of composite materials are increasing very fast in the last decades. In part, this is due to their very flexible potentialities to distinguish and use a great set of different physical properties which may be described through different meanings (e.g. within mechanical, thermal and electrical senses).

Thus, it is important to propose, understand and analyse different models of composite materials in view of their potential use in different applications. This can be achieved having in mind the unique character of the

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detailed microstructure of the composites, but their different properties are also significatively dependent on the boundary conditions which describe their interfaces. Namely, in a composite material we typically have several reinforcements such as particles, flakes and fibers which are embedded in a matrix of metals, polymers or ceramics. Although the shape is determined by the matrix when holding all that items, it is the combination and interaction of all them that determine the overall physical properties of the matrix. There, the boundary conditions play a crucial role. Moreover, it is also natural to allow different conditions on different parts of the boundary. Sometimes, it is also relevant to consider the boundary conditions as transmission conditions in some bounded manifold in the interior of the domain. This allows the proposal materials to have thermal and electrical conduction of very different strengths.

From the mathematical point of view, all that need to be formulated in appropriate function spaces within which the boundary conditions must agree (e.g. by using trace theorems, etc.). So, when proposing a model, it should exist a compromise and interaction between the needs from the applications and the convenient theoretical setting; see, e.g., [1, 2, 6, 7, 10, 11, 12, 13, 16].

In the case of randomly distributed components, effective properties of such composites were successfully studied, for example, in [3, 14, 15, 17, 21, 23, 26, 28], while analytical and numerical results for composites with a periodic structure can be found in [4, 6, 9, 18, 24, 25]. An extensive and complete overview of the employed methods can be found in the fundamental work [22].

Bearing all this in mind, in the present paper we are proposing a model of a composite material which is also determined by some general boundary conditions (and so incorporating different scenarios). We construct an exact solution for the unbounded doubly periodic nonlinear composite under specific assumptions on material properties of the components. Namely, we consider the static thermal conductivity problem of unbounded 2D anisotropic composite materials with circular non-overlapping inclusions in the square unit periodicity cell geometrically forming a doubly periodic structure. We suppose that each component of the composite is imperfectly embedded in the matrix. Namely, the boundary interface conditions are the so-called "inextensible membrane type". It allows very flexible properties on the interfaces upon the choice of different sequences of fixed parameters on the boundary. The conductivities of the matrix and the inclusions depend on the temperature. The key assumption is that ratios of the component conductivities are independent of the temperature. The external flux is assumed to be arbitrarily oriented with respect to the composite symmetry. We determine the temperature and flux distributions and derive the effective conductivity of such composites.

The spaces where we will consider the problem are convenient to prove the existence and uniqueness of a corresponding solution (upon some conditions). This will allow us to understand also the general properties of such solution and to interpret their most useful properties as it concerns the mechanical and physical behaviour. In particular, this will be also done with the help of some software. In our case, we will be particularly concerned with the description of the effective conductivity tensor of a steady-state heat conduction problem in 2D unbounded doubly periodic composite materials with temperature dependent conductivities.

2 Formulation of the problem

We consider a lattice in the complex plane (identified with $\mathbb{C} \cong \mathbb{R}^2$), with complex variables being denoted by $z := x + \imath y$ (for real numbers x and y). As the representative cell, we take the unit square

$$Q_{(0,0)} := \left\{ z = t_1 + \imath t_2 \in \mathbb{C} : -\frac{1}{2} < t_p < \frac{1}{2},\, p = 1, 2 \right\}.$$

Let $\mathcal{E} := \bigcup\limits_{m_1,m_2} \{m_1 + \imath m_2\}$ be the set of the lattice points, where $m_1, m_2 \in \mathbb{Z}$. The cells corresponding to the points of the lattice $\mathcal{E}$ will be denoted by

$$Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + \imath m_2 := \left\{ z \in \mathbb{C} : z - m_1 - \imath m_2 \in Q_{(0,0)} \right\}.$$

It is considered the situation when mutually disjoint disks (i.e., inclusions) of (possible) different radii $D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$ with the boundaries $\partial D_k := \{z \in \mathbb{C} : |z - a_k| = r_k\}$ (for $k = 1, 2, \ldots, N$) are located inside the cell $Q_{(0,0)}$ and periodically repeated in all the cells $Q_{(m_1,m_2)}$. Let us denote by

$$D_0 := Q_{(0,0)} \setminus \left(\bigcup_{k=1}^{N} D_k \cup \partial D_k \right)$$

the connected domain obtained by removing of the inclusions from the cell $Q_{(0,0)}$.

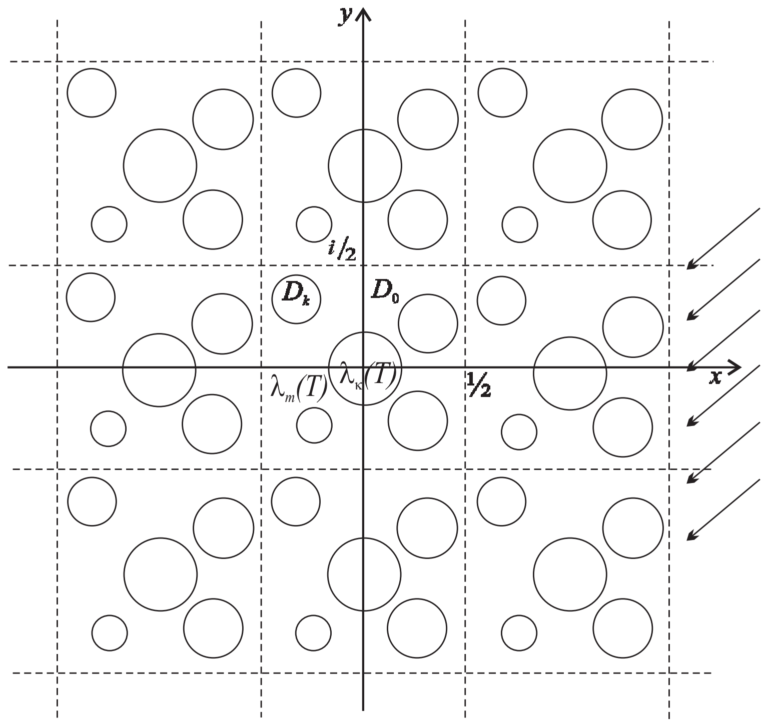

Figure 1: 2D double periodic composite with inclusions.

We investigate the entire infinite composite where the matrix and inclusions occupy domains

$$D_{matrix} = \bigcup_{m_1,m_2} ((D_0 \cup \partial Q_{(0,0)}) + m_1 + \imath m_2)$$

and

$$D_{inc} = \bigcup_{m_1,m_2} \bigcup_{k=1}^{N} (D_k + m_1 + \imath m_2)$$

with thermal conductivities $\lambda_m = \lambda_m(T)$ and $\lambda_k = \lambda_k(T)$, respectively. Here, the temperature T is defined in the whole $\mathbb{R}^2$. We assume that the conductivities λ_m, λ_k $(k = 1, \ldots, N)$ are continuous bounded positive functions on $\mathbb{R}$.

We search for the steady-state distribution of the temperature and heat flux within such a composite. The problem is equivalent to determining the function $T = T(x, y)$ satisfying the nonlinear differential equations

$$\nabla(\lambda_m(T)\nabla T) = 0, \ (x, y) \in D_{matrix}, \tag{2.1}$$

$$\nabla(\lambda_k(T)\nabla T) = 0, \ (x, y) \in D_{inc}. \tag{2.2}$$

We assume that the non-ideal contact conditions on the boundaries between the matrix and inclusions are satisfied:

$$T_m(t) = T_k(t), \tag{2.3}$$

$$\lambda_m(T_m(t))\frac{\partial T_m(t)}{\partial n} - \lambda_k(T_k(t))\frac{\partial T_k(t)}{\partial n} = \gamma_k(T_k(t))\frac{\partial T_k(t)}{\partial s}, \qquad t \in \bigcup_{m_1,m_2} \partial D_k, \tag{2.4}$$

where γ_k is a given function (the so-called resistance coefficient), the vector $n = (n_1, n_2)$ is the outward unit normal vector to ∂D_k, the vector s is the outward unit tangent vector to ∂D_k, and

$$T_m(t) := \lim_{z \to t, z \in D_0} T(z), \qquad T_k(t) := \lim_{z \to t, z \in D_k} T(z).$$

Let us mention that the usually adopted ideal contact conditions consist in demanding the continuity of the temperature and the thermal flux. Here, we use a relaxation of one of these conditions and allow certain discontinuities. Namely, in (2.3), according to the Fourier's law, we assume that the thermal flux jump across the boundary is proportional to the thermal flux of an inclusion within the tangent direction, caused by heat flow around the inclusions.

It is worth mentioning that our boundary value problem can be used for the characterization of other physical or mechanical processes. We meet analogous boundary contact conditions in classical problems of solid mechanics for elastic media known as "inextensible membrane type" conditions. For more details, we refer to [5] where different types of boundary contact conditions are described.

We assume that the average flux vector of intensity A is directed at an angle θ to axis Ox (see Fig. 1) which does not coincide, in general, with the orientation of the periodic cell. This gives the following conditions

$$\int_{\partial Q^{(top)}_{(m_1,m_2)}} \lambda_m(T) T_y dt = -A \sin\theta, \tag{2.5}$$

$$\int_{\partial Q^{(right)}_{(m_1,m_2)}} \lambda_m(T) T_x dt = -A\cos\theta. \tag{2.6}$$

Note that, in general, the flux is not periodic. However, since there are no sources and sinks in the composite, the energy conservation law dictates

$$\int\limits_{\partial\, Q_{(m_1,m_2)}} \lambda_m(T) \frac{\partial T}{\partial n} dt = 0. \tag{2.7}$$

This, in turns, allows us to replace conditions (2.5) and (2.6) with those defined on the opposite sides of the cell.

3 Reformulation of the problem

To solve the problem, we use the Kirchhoff transformation (cf. [20]) and introduce new continuous functions f_m and f_k ($k = 1, \ldots, N$)

$$f_m(T) = \int\limits_0^T \lambda_m(\xi)\, d\xi, \quad f_k(T) = \int\limits_0^T \lambda_k(\xi)\, d\xi. \tag{3.1}$$

Then, using the representations (3.1) and changing the dependent variables as

$$u_m(x,y) = f_m(T_m(x,y)), \quad u_k(x,y) = f_k(T_k(x,y)), \tag{3.2}$$

the original equations (2.1) and (2.2) can be transformed into the Laplace equations

$$\Delta u_m = 0, \quad (x,y) \in D_{matrix}, \tag{3.3}$$

$$\Delta u_k = 0, \quad (x,y) \in D_{inc}. \tag{3.4}$$

The functions f_m and f_k are monotonic increasing functions of temperature and, therefore, there exist their inverses f_m^{-1} and f_k^{-1}. The contact conditions (2.3) and (2.4) can be rewritten now as follows:

$$u_m = F_k(u_k), \quad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{3.5}$$

$$\frac{\partial u_m}{\partial n} - \frac{\partial u_k}{\partial n} = \frac{\gamma_k(T_k)}{\lambda_k(T_k)} \frac{\partial u_k}{\partial s}, \quad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{3.6}$$

where the functions

$$F_k(\xi) := f_m(f_k^{-1}(\xi)) \tag{3.7}$$

are defined for all $\xi \in \mathbb{R}$. Note that, in general, the functions u_m and u_k may have different values on the interface ∂D_k. The derivative of F_k can be computed as follows

$$F_k'(\xi) = \frac{f_m'(f_k^{-1}(\xi))}{f_k'(f_k^{-1}(\xi))} = \frac{\lambda_m(T_k)}{\lambda_k(T_k)}, \tag{3.8}$$

where $\xi = f_k(T_k)$. If we suppose that $\frac{\lambda_m(T_k)}{\lambda_k(T_k)}$ and $\frac{\gamma_k(T_k)}{\lambda_k(T_k)}$ are proportional nonlinear coefficients, namely,

$$\lambda_m(T_k) = c_k \lambda_k(T_k), \quad \gamma_k(T_k) = d_k \lambda_k(T_k), \tag{3.9}$$

then all functions F_k are linear:

$$F_k(\xi) = h_k + c_k \xi. \tag{3.10}$$

This property is satisfied for any $T \in \mathbb{R}$ by some positive real constants c_k. From (3.1) we have $f_m(0) = 0$ and $f_k(0) = 0$, and, therefore, $h_k = 0$. Thus, conditions (3.5)-(3.6) become

$$u_m = c_k u_k, \quad (x, y) \in \bigcup_{m_1, m_2} (\partial D_k + m_1 + \imath m_2), \tag{3.11}$$

$$\frac{\partial u_m}{\partial n} - \frac{\partial u_k}{\partial n} = d_k \frac{\partial u_k}{\partial s}, \quad (x, y) \in \bigcup_{m_1, m_2} (\partial D_k + m_1 + \imath m_2). \tag{3.12}$$

Note that

$$\int_{\Gamma} \frac{\partial u_m}{\partial n} dt = 0, \quad \Gamma \subset D_{matrix}, \tag{3.13}$$

for any closed curve Γ in the matrix. Moreover, since there is no source (sink) inside the composite (neither in the matrix nor in any inclusion), the same condition is satisfied for any closed simply connected curve within the inclusion

$$\int_{\Gamma_k} \frac{\partial u_k}{\partial n} dt = 0, \quad \Gamma_k \subset D_k. \tag{3.14}$$

Finally, the conditions (2.5) and (2.6) are transformed into the following:

$$\int_{\partial Q^{(top)}_{(m_1,m_2)}} u_{my} dt = -A\sin\theta, \tag{3.15}$$

$$\int_{\partial Q^{(right)}_{(m_1,m_2)}} u_{mx} dt = -A\cos\theta. \tag{3.16}$$

Let us introduce new harmonic functions

$$\tilde{u}_k(x,y) = c_k u_k(x,y) \tag{3.17}$$

inside the inclusions. Then, the transmission conditions (3.5) and (3.6) become

$$u_m = \tilde{u}_k, \quad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{3.18}$$

$$\frac{\partial u_m}{\partial n} - \frac{1}{c_k}\frac{\partial \tilde{u}_k}{\partial n} = \frac{d_k}{c_k}\frac{\partial \tilde{u}_k}{\partial s}, \quad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2). \tag{3.19}$$

4 Solution of the problem

We will solve the problem (3.3), (3.4), (3.15), (3.16), (3.18), (3.19) using the same approach as in [9]. We shall now outline some basic facts which we apply.

Let us introduce complex potentials $\varphi(z)$ and $\varphi_k(z)$ which are analytic in D_0 and D_k, and continuously differentiable in the closures of D_0 and D_k, respectively, by using the following relations

$$u(z) = \begin{cases} \operatorname{Re}(\varphi(z) + Bz), \ z \in D_{matrix}, \\ \frac{2c_k}{c_k+1}\operatorname{Re}\varphi_k(z), \ z \in D_{inc}, \end{cases} \tag{4.1}$$

where B is an unknown constant belonging to $\mathbb{C}$. Besides, we assume that the real part of φ is doubly periodic in D_0, i.e.,

$$\operatorname{Re}\varphi(z+1) - \operatorname{Re}\varphi(z) = 0, \qquad \operatorname{Re}\varphi(z+\imath) - \operatorname{Re}\varphi(z) = 0.$$

It is shown in [18] that φ is a single-valued function in D_{matrix}. The harmonic conjugate to u is a function v which has the following form:

$$v(z) = \begin{cases} \operatorname{Im}\left(\varphi(z) + Bz\right), \ z \in D_{matrix}, \\ \\ \frac{2c_k}{c_k+1} \operatorname{Im} \varphi_k(z), \ z \in D_{inc}, \end{cases} \tag{4.2}$$

with the same unknown constant B.

For the determination of the flux $\nabla u(x, y)$, we introduce the derivatives of the complex potentials:

$$\begin{aligned} \psi(z) &:= \frac{\partial \varphi}{\partial z} = \frac{\partial u_m}{\partial x} - \imath \frac{\partial u_m}{\partial y} - B, \qquad z \in D_0, \\ \psi_k(z) &:= \frac{\partial \varphi_k}{\partial z} = \frac{c_k+1}{2c_k}\left(\frac{\partial \tilde{u}_k}{\partial x} - \imath \frac{\partial \tilde{u}_k}{\partial y}\right), \quad z \in D_k. \end{aligned} \tag{4.3}$$

Applying the Cauchy-Riemann equations $\frac{\partial u_m}{\partial n} = \frac{\partial v_m}{\partial s}$, $\frac{\partial \tilde{u}_k}{\partial n} = \frac{\partial \tilde{v}_k}{\partial s}$ the equality (2.4) can be written as

$$\frac{\partial v_m}{\partial s}(t) - \frac{1}{c_k}\frac{\partial \tilde{v}_k}{\partial s}(t) = \frac{d_k}{c_k}\frac{\partial \tilde{u}_k}{\partial s}(t), \qquad |t - a_k| = r_k. \tag{4.4}$$

Integrating the last equality on s, we arrive at the relation

$$v_m(t) - \frac{1}{c_k}\tilde{v}_k(t) = \frac{d_k}{c_k}\tilde{u}_k(t) + C, \tag{4.5}$$

where C is an arbitrary constant. We put $C = 0$ since the imaginary part of the function φ is determined up to an additive constant which does not impact on the form of u. Using (4.2), we have

$$\operatorname{Im}\varphi(t) = -\operatorname{Im} Bt + \frac{2}{c_k+1}\operatorname{Im}\varphi_k(t) + \frac{2d_k}{c_k+1}\operatorname{Re}\varphi_k(t). \tag{4.6}$$

Using now (4.1), we are able to write the equality (2.3) in the following form:

$$\operatorname{Re}\varphi(t) = -\operatorname{Re} Bt + \frac{2c_k}{c_k+1}\operatorname{Re}\varphi_k(t). \tag{4.7}$$

Adding the relation (4.7) and (4.6) multiplied by $\imath$, and using $\operatorname{Re}\varphi_k = \frac{\varphi_k + \overline{\varphi_k}}{2}$, $\operatorname{Im}\varphi_k = \frac{\varphi_k - \overline{\varphi_k}}{2\imath}$, $t - a_k = \frac{r_k^2}{\overline{t - a_k}}$, we rewrite the conditions (2.3) and (2.4) in terms of the complex potentials $\varphi(z)$ and $\varphi_k(z)$:

$$\varphi(t) = (1 + \imath\mu_k)\varphi_k(t) - (\rho_k - \imath\mu_k)\overline{\varphi_k(t)} - Bt, \tag{4.8}$$

where

$$\rho_k = \frac{1-c_k}{c_k+1},\ \mu_k = \frac{d_k}{c_k+1}. \tag{4.9}$$

Representing the function φ_k in the form $\varphi_k(z) = \sum\limits_{l=0}^{\infty} \alpha_k(z-a_k)^l$, $|z-a_k| \le r_k$, and by using the relation $t = \frac{r_k^2}{t-a_k} + a_k$ on the boundary $|t-a_k| = r_k$, one get

$$[\overline{\varphi(t)}]' = -\left(\frac{r_k}{t-a_k}\right)^2 \overline{\varphi'(t)}. \tag{4.10}$$

Thus, after differentiating (4.8), we arrive at the following *R-linear boundary value problem* on each contour $|t-a_k| = r_k$:

$$\psi(t) \quad = \quad (1+\imath\mu_k)\psi_k(t) + (\rho_k - \imath\mu_k)\left(\frac{r_k}{t-a_k}\right)^2 \overline{\psi_k(t)} - B \tag{4.11}$$

with the unknown constant $B = B_1 + \imath B_2$. An algorithm for solving such R-linear boundary value problem is developed and described in detail in [18]. We use this approach in our computations.

5 Effective conductivity

We assume that the effective conductivity tensor Λ_e depends on the average temperature $\langle T\rangle$ and is defined in the following way:

$$\langle\lambda(T)\nabla T\rangle = \Lambda_e(\langle T\rangle)\langle\nabla T\rangle, \quad \text{or} \quad R_e(\langle T\rangle)\langle\lambda(T)\nabla T\rangle = \langle\nabla T\rangle, \tag{5.1}$$

where $R_e = \Lambda_e^{-1}$ is the effective resistance tensor. A similar definition to (5.1) has been used in [27]. Here, the operator $\langle\cdot\rangle$ is defined as

$$\langle f\rangle = \iint\limits_{Q_{(m_1,m_2)}} f(x,y)\,dxdy.$$

Note that definition (5.1) needs further justification as the question arises whether the approach is invariant with respect to the averaging cell. We will discuss this issue later during the computations.

We represent all elements involved in (5.1) in terms of solutions u_m and u_k of the problem (3.3)-(3.6). For the total flux in the x-direction, we have

$$\begin{aligned}\iint\limits_{Q_{(m_1,m_2)}} \lambda(T)\frac{\partial T}{\partial x}\,dxdy &= \iint\limits_{D_0+m_1+\imath m_2} \lambda_m(T_m)\frac{\partial T_m}{\partial x}\,dxdy \\ &\quad + \sum_{k=1}^{N}\iint\limits_{D_k+m_1+\imath m_2} \lambda_k(T_k)\frac{\partial T_k}{\partial x}\,dxdy \\ &= \iint\limits_{Q_{(m_1,m_2)}} \frac{\partial u_m}{\partial x}\,dxdy + \sum_{k=1}^{N}\iint\limits_{D_k+m_1+\imath m_2} \frac{\partial u_k}{\partial x}\,dxdy.\end{aligned}$$

Using the first Green's formula

$$\iiint\limits_{U} (\psi\Delta\varphi + \nabla\varphi\cdot\nabla\psi)\,dV = \oint\limits_{\partial U} \psi\,(\nabla\varphi\cdot\mathbf{n})\,dS \tag{5.2}$$

with $\psi = x$ or $\psi = y$ and $\varphi(x,y) = u_m$ in D_0 (or $\varphi(x,y) = u_k$ in the respective domain D_k) and formulas (3.3), (3.4), (3.6), (3.9) and (3.17), we obtain

$$q_x := \iint\limits_{Q_{(m_1,m_2)}} \lambda(T)\frac{\partial T}{\partial x}\,dxdy = -A\cos\theta - \sum_{k=1}^{N}\frac{d_k}{c_k}\oint\limits_{\partial(D_k+m_1+\imath m_2)} x\frac{\partial\tilde{u}_k}{\partial s}\,ds,$$

and similarly

$$q_y := \iint\limits_{Q_{(m_1,m_2)}} \lambda(T)\frac{\partial T}{\partial y}\,dxdy = -A\sin\theta - \sum_{k=1}^{N}\frac{d_k}{c_k}\oint\limits_{\partial(D_k+m_1+\imath m_2)} y\frac{\partial\tilde{u}_k}{\partial s}\,ds.$$

According to the Cauchy–Riemann condition $\frac{\partial\tilde{u}_k}{\partial s} = -\frac{\partial\tilde{v}_k}{\partial n}$, we get

$$\begin{aligned} q_x &= -A\cos\theta + \sum_{k=1}^{N}\frac{d_k}{c_k}\oint\limits_{\partial(D_k+m_1+\imath m_2)} x\frac{\partial\tilde{v}_k}{\partial n}\,ds \\ &= -A\cos\theta + \sum_{k=1}^{N}\frac{d_k}{c_k}\iint\limits_{D_k+m_1+\imath m_2}\frac{\partial\tilde{v}_k}{\partial x}\,dxdy\end{aligned}$$

$$
\begin{aligned}
q_y &= -A\sin\theta + \sum_{k=1}^{N} \frac{d_k}{c_k} \oint\limits_{\partial(D_k+m_1+\imath m_2)} y\frac{\partial \tilde{v}_k}{\partial n}\, ds \\
&= -A\sin\theta + \sum_{k=1}^{N} \frac{d_k}{c_k} \iint\limits_{D_k+m_1+\imath m_2} \frac{\partial \tilde{v}_k}{\partial y}\, dxdy.
\end{aligned}
$$

Thus,

$$
\langle \lambda(T)\,\nabla T\rangle = [q_x, q_y]^\top, \tag{5.3}
$$

where the components q_x and q_y can be found via a solution ψ_k (cf. also the notations (4.3)) of the R-linear conjugation problem (4.11) as

$$
\imath q_x + q_y = -\imath A e^{-\imath\theta} + 2\sum_{k=1}^{N} \mu_k \iint\limits_{D_k+m_1+\imath m_2} \psi_k(z) dxdy.
$$

Or, using the mean value theorem for harmonic functions, we get

$$
\imath q_x + q_y = -\imath A e^{-\imath\theta} + 2\pi \sum_{k=1}^{N} \mu_k r_k^2 \psi_k(a_k).
$$

Due to Gauss-Ostrogradsky formula and the boundary condition (2.3), the components of the term $\langle \nabla T\rangle$ in (5.1) are defined as

$$
\begin{aligned}
\iint\limits_{Q_{(m_1,m_2)}} \frac{\partial T}{\partial x}\, dxdy &= \iint\limits_{D_0+m_1+\imath m_2} \frac{\partial T_m}{\partial x}\, dxdy + \sum_{k=1}^{N} \iint\limits_{D_k+m_1+\imath m_2} \frac{\partial T_k}{\partial x}\, dxdy \\
&= \oint\limits_{\partial D_0+m_1+\imath m_2} T_m(s)\cos(n_s, e_i)\, ds \\
&\quad + \sum_{k=1}^{N} \oint\limits_{\partial D_k+m_1+\imath m_2} [T_k(s) - T_m(s)]\cos(n_s^k, e_i)\, ds \\
&= \oint\limits_{\partial D_0+m_1+\imath m_2} T_m(s)\cos(n_s, e_i)\, ds \\
&= \oint\limits_{\partial D_0+m_1+\imath m_2} f_m^{-1}(u_m(x,y))\cos(n_s, e_i)\, ds,
\end{aligned}
$$

where n_s and n_s^k are the outward unit normal vectors to $\partial D_0 + m_1 + \imath m_2$ and $\partial D_k + m_1 + \imath m_2$, respectively, and e_i is the basis vector. Analogously,

$$\iint\limits_{Q_{(m_1,m_2)}} \frac{\partial T}{\partial y}\, dxdy = \oint\limits_{\partial D_0 + m_1 + \imath m_2} f_m^{-1}(u_m(x,y)) \cos(n_s, e_j)\, ds.$$

Finally, the average temperature is given by

$$\langle T \rangle = \iint\limits_{Q_{(m_1,m_2)}} T(x,y)\, dxdy$$

$$= \iint\limits_{D_0 + m_1 + \imath m_2} f_m^{-1}(u_m(x,y))\, dxdy + \sum_{k=1}^{N} \iint\limits_{D_k + m_1 + \imath m_2} f_k^{-1}(u_k(x,y))\, dxdy. \quad (5.4)$$

Note that it is more convenient to first compute the components of the effective resistance tensor R_e from the second formula in (5.1) and then find the effective conductivity tensor $\Lambda_e = R_e^{-1}$.

6 Numerical analysis

The algorithm mentioned above is realized in the Maple 14 software.

In our computations we consider a composite where four inclusions are situated inside the cell $Q_{(0,0)}$ with the centers (defined in the notations of complex variables): $a_1 = -0.18 + 0.2\imath$, $a_2 = 0.33 - 0.34\imath$, $a_3 = 0.33 + 0.35\imath$, $a_4 = -0.18 - 0.2\imath$. The radii of the inclusions are the same $r_k = R = 0.145$ (cf. Fig. 2). Thus, the volume fraction of the inclusions for such composite is $\nu = 4\pi R^2 = 0.2642$. For this choice, the inclusion boundaries are situated very close to each other (the minimal distance is 0.02).

Further, we suppose to have the heat flows in the x-direction ($\theta = 0$) with intensity $A = -1$. We choose the conductivities functions

$$\lambda_m(T) = \sin T + 2, \quad \lambda_k(T) = 10 \cdot (\sin T + 2)$$

which are positive periodic proportional with the constant $c_k = 0.1$ (cf. Fig. 3), and $\gamma_k(T)$ with the constant $d_k = 0; 2; 10$ (cf. (3.9)).

Note that the following statement is true.

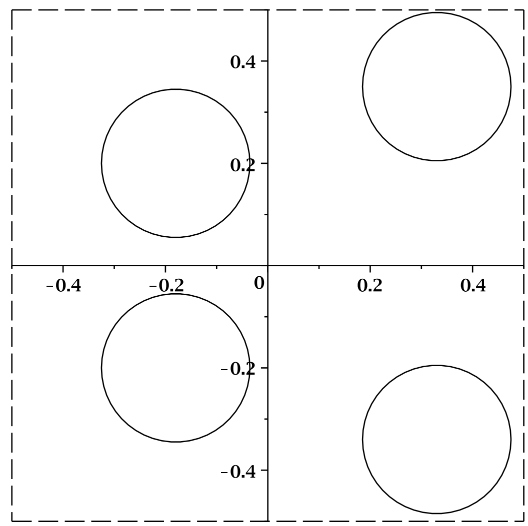

Figure 2: Configuration of the unit cell with four inclusions considered in computation

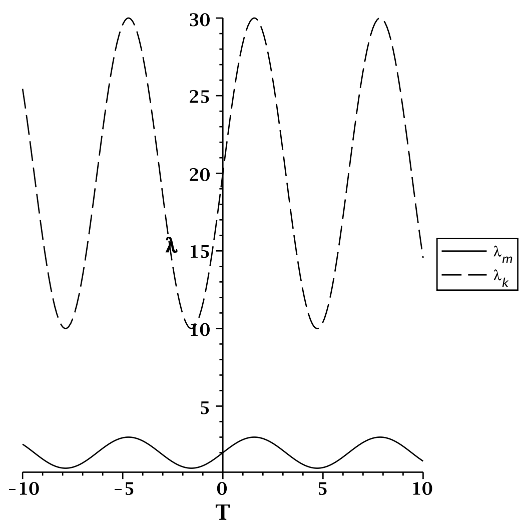

Figure 3: The functions λ_m and λ_k.

Theorem 6.1 *Let λ_m, λ_k and γ_k be periodic functions with the same period*

τ satisfying (3.9)*. Then the effective conductivity tensor Λ_n defined in* (5.1) *is also a periodic matrix function with the same period τ.*

Proof. First note that if T_n is a solution of the boundary value problem (2.1)-(2.6) then $T_n + \tau$ is also a solution of this problem and, for any unit cell $Q_{(m_1,m_2)}$, we have

$$\langle T_n+\tau\rangle = \langle T_n\rangle+\tau, \quad \nabla(T_n+\tau) = \nabla T_n, \quad \text{and} \quad \lambda_n(T_n+\tau) = \lambda_n(T_n). \tag{6.1}$$

Now recall that by definition (5.1) we have

$$\langle\lambda_n(T_n+\tau)\nabla(T_n+\tau)\rangle = \Lambda_n(\langle T_n+\tau\rangle)\langle\nabla(T_n+\tau)\rangle. \tag{6.2}$$

Applying (6.1) in (6.2), we obtain

$$\langle\lambda_n(T_n)\nabla T_n\rangle = \Lambda_n(\langle T_n\rangle+\tau)\langle\nabla T_n\rangle.$$

Since the tensor Λ_n is uniquely defined in (5.1), we get

$$\Lambda_n(\langle T_n\rangle+\tau) = \Lambda_n(\langle T_n\rangle) \tag{6.3}$$

for any $\langle T_n\rangle$. □

Note that in the linear case the temperature is defined up to an arbitrary additive constant, and this constant is not involved in the determination of the effective conductivity of a composite material. Generally speaking, this is not the case for nonlinear problems, and the additive constant, appearing during the stage of solving the linear problem (3.3)-(3.16), influences on the computation of the effective conductivity tensor of the equivalent nonlinear composite.

Here, we evaluate the effective resistance tensor R_e by the interpolation method and then we get $\Lambda_e = R_e^{-1}$. Two equivalent procedures are used for obtaining a discrete data set of the effective resistance tensor components $R_e^x, R_e^y, R_e^{xy}, R_e^{yx}$ suggested in [19].

(i) First, one can solve the corresponding linear boundary value problem in a doubly periodic domain preserving its uniqueness by any appropriately chosen condition (for example, here we impose that the function $u = u_*$ satisfies the condition $u_*(0) = 0$). Then, to evaluate the properties of the composite material, one can compute the average temperature and the effective resistivity for each particular unit cell presenting the data as the functional relationship $R_e = R_e(\langle T\rangle)$.

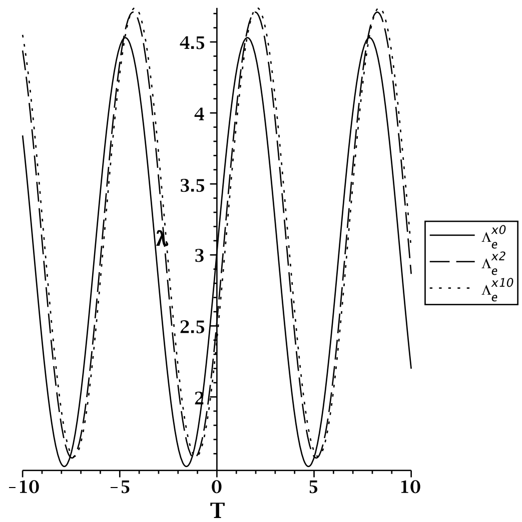

Figure 4: Component Λ_e^x of the effective conductivity tensor Λ_e computed by combination of two methods for $d_k = 0; 2; 10$.

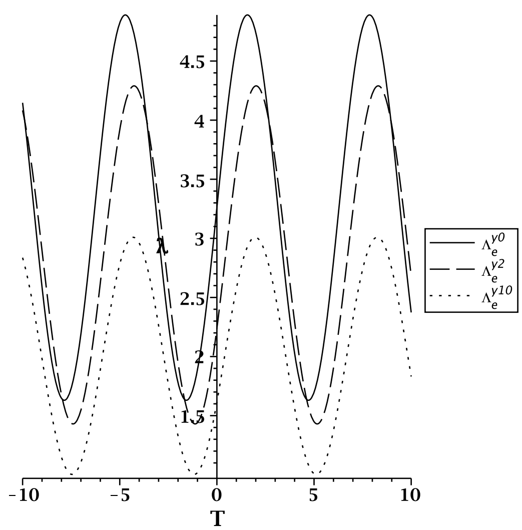

Figure 5: Component Λ_e^y of the effective conductivity tensor Λ_e computed by combination of two methods for $d_k = 0; 2; 10$.

Tracing cells belonging to the strip, we get a set of points which are not dense enough at the top of the sinusoid considered in Theorem 6.1. Therefore, we used a second method.

(ii) Namely, we consider an arbitrary cell in the original domain and build a set of solutions to the linear boundary value problem in the form $u = u_* + C$, where C is an arbitrary constant. Then, for every constant C, the components of the effective resistance tensor, R_e, and the average temperature, $\langle T \rangle$, are functions of the parameter C. Changing the value of C continuously from $-\infty$ to ∞, one receives the sought for the effective conductivity tensor of the composite as a continuous function of the average temperature. It is clear that this procedure does not depend on the chosen cell.

Naturally, for the conductivities of the composite components analyzed in this example and for the fact proved in Theorem 6.1, the nonlinear character of the relationship is observed only within the finite interval of the parameter C.

Note that both methods allow one to determine two components of the effective resistance tensor R_e for each one of the two orthogonal flux directions. Thus considering $\theta = 0$, we define $R_e^x = R_e^x(\langle T \rangle)$ and $R_e^{yx} = R_e^{yx}(\langle T \rangle)$, and choosing $\theta = \pi/2$ we find $R_e^y = R_e^y(\langle T \rangle)$ and $R_e^{xy} = R_e^{xy}(\langle T \rangle)$. As a result, the entire tensor $R_e(\langle T \rangle)$ is defined.

In view of having a more accurate interpolation procedure, we use both suggested methods and calculate 141 data $(R_e^x, R_e^{yx}, R_e^{xy}, R_e^y)$ with respect to the average temperature $\langle T \rangle \in [-10, 10]$. For the chosen configuration it guarantees a computational error less than 10^{-6}. The components Λ_e^x and Λ_e^y of the effective conductivity tensor $\Lambda_e = R_e^{-1}$ are presented in Fig. 4 and Fig. 5, respectively, for different $d_k = 0; 2; 10$.

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