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ASYMPTOTIC PROPERTIES IN FORWARD DIRECTIONS OF RESOLVENT KERNELS OF MAGNETIC SCHRÖDINGER OPERATORS IN TWO DIMENSIONS

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Abstract. We study the asymptotic properties in forward directions of resolvent kernels with spectral parameters in the lower half plane (unphysical sheet) of the complex plane for magnetic Schrödinger operators in two dimensions. The asymptotic formula obtained has an application to the problem of quantum resonances in magnetic scattering, and it is especially helpful in studying how the Aharonov–Bohm effect influences the location of resonances. Here we mention only the results without proofs.

1. Introduction

In this paper we derive the asymptotic formulas in forward directions of resolvent kernels (the Green functions) with spectral parameters in the lower half plane of the complex plane (unphysical sheet or 2nd sheet) for magnetic Schrödinger operators in two dimensions. The obtained results play an important role in studying quantum resonances in magnetic scattering, which are created near the positive real axis by trajectories trapped between scatters placed at large separation. We discuss this subject in some details in section 5 together with the motivation to analyze the asymptotic behavior in forward directions of resolvent kernels. There we explain how the Aharonov–Bohm effect influences the location of the resonances. The present paper is the preliminary step toward the study on the Aharonov–Bohm effect in resonances for magnetic scattering in two dimensions.

We always work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$ and write

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2} (-i\partial_j - a_j)^2, \quad \partial_j = \partial/\partial x_j,$$

for the magnetic Schrödinger operator with $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ as a vector potential. The magnetic field $b : \boldsymbol{R}^2 \to \boldsymbol{R}$ associated with A is defined by

$$b(x) = \nabla \times A(x) = \partial_1 a_2 - \partial_2 a_1$$

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and the quantity defined as the integral $\alpha = (2\pi)^{-1} \int b(x)\,dx$ is called the magnetic flux of b, where the integration with no domain attached is taken over the whole space. We often use this abbreviation.

We set up our problem. Let $b \in C_0^\infty(\boldsymbol{R}^2)$ be a given magnetic field with its flux α. We assume that b has support in some simply connected bounded domain $\mathcal{O}$ with the smooth boundary. For brevity, we further assume that

$$\operatorname{supp} b \subset \mathcal{O} \subset B = \{|x| < 1\} \tag{1.1}$$

with the origin in $\mathcal{O}$ as an interior point. We denote the exterior domain as $\Omega = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}$. Then we can take

$$A(x) = \alpha\,\Phi(x), \quad x \in \Omega, \tag{1.2}$$

as the vector potential corresponding to b, where $\Phi(x)$ is defined by

$$\Phi = \left(-x_2/|x|^2, x_1/|x|^2\right) = \left(-\partial_2 \log|x|, \partial_1 \log|x|\right). \tag{1.3}$$

In fact, Φ defines the δ–like magnetic field (solenoidal field)

$$\nabla \times \Phi = \Delta \log|x| = 2\pi\delta(x)$$

with center at the origin. This vector potential is often called the Aharonov–Bohm potential in physics literatures. Assumption (1.1) means that the field b is entirely shielded by the obstacle $\mathcal{O}$, although the corresponding vector potential A does not necessarily vanish over Ω.

We denote by $R(\zeta; T) = (T - \zeta)^{-1}$ the resolvent of an operator T acting on $L^2(\boldsymbol{R}^2)$ or $L^2(\Omega)$. On $L^2(\Omega)$, we consider the self–adjoint operator

$$H = H(A), \qquad \mathcal{D}(H) = H^2(\Omega) \cap H_0^1(\Omega), \tag{1.4}$$

under the zero Dirichlet boundary conditions. It is well known that H has no positive eigenvalues and the continuous spectrum occupied by $[0, \infty)$ is absolutely continuous. We further know that the resolvent

$$R(\zeta; H) = (H - \zeta)^{-1} : L^2(\Omega) \to L^2(\Omega), \quad \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta > 0,$$

is analytically continued from the upper half plane of the complex plane to a region in the lower half plane across the positive real axis where the continuous spectrum of H is located. Then $R(\zeta; H)$ with $\operatorname{Im}\zeta \le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$ in the sense that $\chi R(\zeta; H)\chi : L^2(\Omega) \to L^2(\Omega)$ is bounded for every $\chi \in C_0^\infty(\overline{\Omega})$, where $L^2_{\mathrm{comp}}(W)$ denotes the space of square integrable functions with compact support in the closure $\overline{W}$ of a region $W \subset \boldsymbol{R}^2$ and $L^2_{\mathrm{loc}}(W)$ denotes the space of locally square integrable functions over $\overline{W}$. This can be shown by an application of the complex scaling method ([5, 7, 10]). We use the same notation $R(\zeta; H)$ to denote this analytic function with values in operators from

$L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. In fact, we can show that $R(\zeta; H)$ admits the meromorphic continuation to the lower half plane $\{\zeta \in \boldsymbol{C} : \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta < 0\}$, but the argument here is restricted only to a neighborhood of the positive real axis.

We shall state the obtained results. Let $\lambda \gg 1$ be a large parameter. We set

$$D_0 = \left\{\zeta \in \boldsymbol{C} : |\operatorname{Re}\zeta - E_0| < E_0/2,\ |\operatorname{Im}\zeta| < e_0 (\log\lambda)/\lambda\right\} \tag{1.5}$$

for $E_0 > 0$ and $e_0 > 0$ fixed, and we write $k = \zeta^{1/2}$ for $\zeta \in D_0$, where the branch k is taken in such a way that $\operatorname{Re} k > 0$ for $\operatorname{Re}\zeta > 0$. We further define

$$\sigma(x; y) = \gamma(\hat{x}; \hat{y}) - \pi = \gamma(x; y) - \pi, \quad \hat{x} = x/|x|, \tag{1.6}$$

where $\gamma(\theta; \omega)$ denotes the azimuth angle from $\omega \in S^1$ to θ ($0 \le \gamma(\theta; \omega) < 2\pi$). We take μ to be

$$2/5 < \mu < 1/2 \left(< 1 - \mu\right) \tag{1.7}$$

close enough to 1/2. We use the notation $\lambda,\ k$ and μ with the meanings ascribed above throughout the entire discussion. Our aim is to study the asymptotic properties of the resolvent kernel $R(\zeta; H)(x, y)$ with $\zeta \in D_0$ when $|x - y| \gg 1$ with $|\sigma(x, y)| \ll 1$.

Theorem 1.1. *Assume that $\zeta \in D_0$ and that x and y fulfill*

$$\lambda/c \le |x|,\ |y| \le c\lambda, \qquad |\sigma(x, y)| \le c\lambda^{-(1-\mu)}$$

for some $c > 1$. Then the resolvent kernel $R(\zeta; H)(x, y)$ takes the asymptotic form

$$\begin{aligned} R(\zeta; H)(x, y) &= (i/4)\cos(\alpha\pi) e^{i\alpha(\gamma(\hat{x};\hat{y}) - \pi)} H_0(k|x - y|) \\ &+ e^{ik(|x|+|y|)} \left(|x| + |y|\right)^{-1/2} r_1(x, y; \zeta, \lambda), \end{aligned}$$

where $H_0(z) = H_0^{(1)}(z)$ denotes the Hankel function of the first kind and of order zero, and the remainder term r_1 is analytic in $\zeta \in D_0$ and obeys

$$\left|\partial_x^n \partial_y^m r_1\right| = O\left(\lambda^{\mu - 1/2 - (|n| + |m|)/2}\right) \tag{1.8}$$

uniformly in x, y and ζ.

Theorem 1.2. *Assume that $\zeta \in D_0$ and that x and y fulfill*

$$\lambda/c \le |x|,\ |y| \le c\lambda, \qquad \lambda^{-(1-\mu)}/c \le |\sigma(x, y)| \le c\lambda^{-\mu}$$

for some $c > 1$. *Define* $z_0 = z_0(x,y;\zeta)$ *and* $c_0(\zeta)$ *by*

$$z_0 = \Big(|x||y|/(|x|+|y|)\Big)^{1/2}|\sigma(x,y)|\zeta^{1/4} \tag{1.9}$$

$$c_0(\zeta) = (8\pi)^{-1/2}e^{i\pi/4}\zeta^{-1/4}. \tag{1.10}$$

Then $R(\zeta;H)(x,y)$ *behaves like*

$$\begin{aligned} R(\zeta;H)(x,y) &= (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) \\ &\pm\ c_0(\zeta)\frac{i\sin(\alpha\pi)}{\pi}\left(\frac{e^{ik|x-y|}}{|x-y|^{1/2}}\right)\left(\pi-(2\pi)^{1/2}e^{-i\pi/4}\int_0^{z_0} e^{is^2/2}\,ds\right) \\ &+\ e^{ik|x-y|}|x-y|^{-1/2}r_{\pm 2}(x,y;\zeta,\lambda) \end{aligned}$$

according as $\pm\sigma(x,y) > 0$, *where* $r_{\pm 2}$ *is analytic in* $\zeta \in D_0$ *and obeys*

$$\left|\partial_x^n\partial_y^m r_{\pm 2}\right| = O\left(\lambda^{-\mu-(|n|+|m|)\mu}\right) \tag{1.11}$$

uniformly in x, y *and* ζ.

Theorem 1.3. *Assume that* $\zeta \in D_0$ *and that* x *and* y *fulfill*

$$\lambda/c \le |x|,\ |y| \le c\lambda, \qquad |\sigma(x,y)| > \lambda^{-\mu}/c$$

for some $c > 1$. *Then*

$$\begin{aligned} R(\zeta;H)(x,y) &= (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) \\ &+\ e^{ik(|x|+|y|)}\,(|x|+|y|)^{-1/2}\,r_3(x,y;\zeta,\lambda), \end{aligned}$$

where r_3 *is analytic in* $\zeta \in D_0$ *and obeys*

$$\left|\partial_x^n\partial_y^m r_3\right| = |\sigma(x,y)|^{-1-(|n|+|m|)}\,O\left(\lambda^{-1/2-(|n|+|m|)}\right) \tag{1.12}$$

uniformly in x, y *and* ζ.

If we take account of the asymptotic formula

$$H_0(z) = (2/\pi)^{1/2}e^{-i\pi/4}\left(e^{iz}/z^{1/2}\right)\left(1+O(|z|^{-1})\right) \tag{1.13}$$

as $|z| \to \infty$, then the next corollary is obtained as a consequence of Theorem 1.2.

Corollary 1.1. *Under the same assumption and notation as in Theorem 1.2,* $R(\zeta;H)(x,y)$ *behaves like*

$$\begin{aligned} &R(\zeta;H)(x,y) = \\ &\quad c_0(\zeta)\left(\frac{e^{ik|x-y|}}{|x-y|^{1/2}}\right)\left(\cos(\alpha\pi)\mp(2/\pi)^{1/2}\,e^{i\pi/4}\sin(\alpha\pi)\int_0^{z_0} e^{is^2/2}\,ds\right) \end{aligned}$$

$$+ \; e^{ik|x-y|}|x-y|^{-1/2}\tilde{r}_{\pm 2}(x,y;\zeta,\lambda)$$

according as $\pm\sigma(x,y) > 0$, *where* $\tilde{r}_{\pm 2}$ *is analytic in* $\zeta \in D_0$ *and obeys the same bound as in (1.11).*

All the theorems, including Corollary 1.1, are proved in section 3 after summarizing the asymptotic properties of the resolvent kernel of the Aharonov–Bohm Hamiltonian with the δ–like (solenoidal) field with center at the origin in section 2. The resolvent kernel of the Aharonov–Bohm Hamiltonian is represented in terms of the contour integral in the complex plane, and the asymptotic properties are studied by use of the method of steepest descent in section 4. In section 5, we discuss the Aharonov–Bohm effect in resonances of magnetic scattering and state the two results without proofs. These results are based on the composition of two resolvent kernels. In the last section (section 6), we establish the asymptotic formula of the integral defined by the composition as an application of the theorems.

2. Aharonov–Bohm Hamiltonian

In this section, we mention the asymptotic properties of the resolvent kernel of the magnetic Schrödinger operator with one solenoid field as a series of propositions. The theorems in the previous section are proved on the basis of these propositions. The scattering system by one solenoidal field is known as one of the exactly solvable models in quantum mechanics. We refer to [1, 2, 3, 6, 9] for more detailed expositions.

We recall that $A(x) : \Omega \to \boldsymbol{R}^2$ is defined by (1.2). We write it as

$$A_\alpha(x) = \alpha\left(-x_2/|x|^2, x_1/|x|^2\right) = \alpha\Big(-\partial_2 \log|x|, \partial_1 \log|x|\Big), \tag{2.1}$$

when considered as a vector potential over $\boldsymbol{R}^2$. As stated in the previous section, A_α generates the δ–like field with center at the origin. We consider the energy operator

$$P = H(A_\alpha) = (-i\nabla - A_\alpha)^2 \tag{2.2}$$

on the space $L^2(\boldsymbol{R}^2)$. This operator governs the quantum particle moving in the solenoidal field $2\pi\alpha\delta(x)$ and is often called the Aharonov–Bohm Hamiltonian in physics literatures. The operator P is symmetric over the space $C_0^\infty\left(\boldsymbol{R}^2 \setminus \{0\}\right)$, but it is not necessarily essentially self–adjoint in $L^2(\boldsymbol{R}^2)$ because of the strong singularity at the origin of A_α. We know ([1, 6]) that it is a symmetric operator with type $(2,2)$ of deficiency indices, provided that α is not an integer. The self–adjoint extension is realized by imposing a boundary condition at the origin. Its Friedrichs extension denoted by the same notation P is obtained by imposing the boundary condition $\lim_{|x|\to 0} |u(x)| < \infty$ at the origin.

We calculate the generalized eigenfunction of the eigenvalue problem

$$P\,\varphi = E\,\varphi, \qquad \lim_{|x|\to 0} |\varphi(x)| < \infty,$$

with energy $E > 0$ as an eigenvalue. Since P is rotationally invariant, we work in the polar coordinate system (r, θ). Let U be the unitary mapping defined by

$$(Uu)(r,\theta) = r^{1/2}u(r\theta) : L^2(\boldsymbol{R}^2) \to L^2((0,\infty); dr) \otimes L^2(S^1).$$

We write $\sum_l$ for the summation ranging over all integers $l \in \boldsymbol{Z}$. Then U allows us to decompose P into the partial wave expansion

$$P \simeq UPU^* = \sum_l \oplus\, (P_l \otimes Id)\,, \tag{2.3}$$

where $P_l = -\partial_r^2 + (\nu^2 - 1/4)r^{-2}$ with $\nu = |l - \alpha|$ is self–adjoint in the space $L^2((0,\infty); dr)$ under the boundary condition $\lim_{r\to 0} r^{-1/2}|u(r)| < \infty$ at $r = 0$. Let $\varphi_0(x;\omega,E)$ be the plane wave defined by

$$\varphi_0(x;\omega,E) = \exp\left(iE^{1/2}x\cdot\omega\right), \quad E > 0, \tag{2.4}$$

with the incident direction $\omega \in S^1$ at energy E, where the notation $\cdot$ denotes the scalar product in $\boldsymbol{R}^2$. We recall that $\gamma(x;\omega)$ stands for the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$. Then the outgoing eigenfunction $\varphi_+(x;\omega,E)$ with ω as an incident direction is calculated as

$$\varphi_+(x;\omega,E) = \sum_l \exp(-i\nu\pi/2)\exp(il\gamma(x;-\omega))J_\nu(E^{1/2}|x|) \tag{2.5}$$

with $\nu = |l - \alpha|$, where $J_\nu(z)$ denotes the Bessel function of order ν. The eigenfunction φ_+ behaves like $\varphi_+(x;\omega,E) \sim \varphi_0(x;\omega,E)$ as $|x| \to \infty$ in the direction $-\omega$ $(x = -|x|\omega)$, and the difference $\varphi_+ - \varphi_0$ satisfies the outgoing radiation condition at infinity. On the other hand, the incoming eigenfunction $\varphi_-(x;\omega,E)$ is given by

$$\varphi_-(x;\omega,E) = \sum_l \exp(i\nu\pi/2)\exp(il\gamma(x;\omega))J_\nu(E^{1/2}|x|), \tag{2.6}$$

which behaves like $\varphi_- \sim \varphi_0(x;\omega,E)$ as $|x| \to \infty$ in the direction ω. The eigenfunctions $\varphi_\pm(x;\omega,E)$ admit the analytic extension

$$\varphi_\pm(x;\omega,\zeta) = \sum_l \exp(\mp i\nu\pi/2)\exp(il\gamma(x;\mp\omega))J_\nu(k|x|) \tag{2.7}$$

with $k = \zeta^{1/2}$ for $\zeta \in D_0$.

We calculate the resolvent kernel $R(\zeta;P)(x,y)$ with $\zeta \in D_0$ for the self–adjoint operator P, where D_0 is defined by (1.5). Let P_l be as in (2.3). Then the equation $(P_l-\zeta)\,u=0$ has $\left\{r^{1/2}J_\nu(kr), r^{1/2}H_\nu(kr)\right\}$ with Wronskian $2i/\pi$ as a pair of linearly independent solutions, where $H_\nu(z)=H_\nu^{(1)}(z)$ denotes the Hankel function of the first kind. Thus $(P_l-\zeta)^{-1}$ has the integral kernel

$$R(\zeta;P_l)(r,\rho)=(i\pi/2)\,r^{1/2}\rho^{1/2}J_\nu\left(k(r\wedge\rho)\right)H_\nu\left(k(r\vee\rho)\right)$$

with $\nu=|l-\alpha|$ again, where $r\wedge\rho=\min(r,\rho)$ and $r\vee\rho=\max(r,\rho)$. Hence the kernel $R(\zeta;P)(x,y)$ in question is given by

$$R(\zeta;P)(x,y)=(i/4)\sum_l e^{il(\theta-\omega)}J_\nu\left(k(|x|\wedge|y|)\right)H_\nu\left(k(|x|\vee|y|)\right), \tag{2.8}$$

where $x=(|x|\cos\theta,|x|\sin\theta)$ and $y=(|y|\cos\omega,|y|\sin\omega)$ in the polar coordinates. The resolvent $R(\zeta;P)$ with $\operatorname{Im}\zeta\le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\boldsymbol{R}^2)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$. Thus $R(\zeta;P)$ does not have any poles as a function with values in operators from $L^2_{\mathrm{comp}}(\boldsymbol{R}^2)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$. We can say that P with one solenoid $2\pi\alpha\delta(x)$ has no resonances. Here we do not discuss the possibility of resonances at zero energy.

We now state the asymptotic properties of $R(\zeta;P)(x,y)$ as a series of the propositions below. These proposition are verified in sections 4 after completing the proofs of Theorems 1.1, 1.2, 1.3 and Corollary 1.1 in section 3. Here we recall that μ is fixed as in (1.7) and $\sigma(x;y)$ is defined by (1.6).

Proposition 2.1. *Suppose that the same assumptions as in Theorem 1.1 are fulfilled. Then*

$$\begin{aligned}R(\zeta;P)(x,y) &= (i/4)\cos(\alpha\pi)e^{i\alpha(\gamma(\hat{x};\hat{y})-\pi)}H_0(k|x-y|)\\ &+ e^{ik(|x|+|y|)}\left(|x|+|y|\right)^{-1/2}e_1(x,y;\zeta,\lambda),\end{aligned}$$

where e_1 is analytic in $\zeta\in D_0$ obeys the same bound as in (1.8) uniformly in x, y and ζ.

Proposition 2.2. *Suppose that the same assumptions as in Theorem 1.2 are fulfilled. Define $z_0=z_0(x,y;\zeta)$ by (1.9) and $c_0(\zeta)$ by (1.10). Then $R(\zeta;P)(x,y)$ behaves like*

$$\begin{aligned}&R(\zeta;P)(x,y)=(i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|)\\ &\quad\pm\ c_0(\zeta)\frac{i\sin(\alpha\pi)}{\pi}\left(\frac{e^{ik|x-y|}}{|x-y|^{1/2}}\right)\left(\pi-(2\pi)^{1/2}e^{-i\pi/4}\int_0^{z_0}e^{is^2/2}\,ds\right)\\ &\quad+\ e^{ik|x-y|}|x-y|^{-1/2}e_{\pm2}(x,y;\zeta,\lambda)\end{aligned}$$

according as $\pm\sigma(x,y) > 0$*, where* $e_{\pm 2}$ *is analytic in* $\zeta \in D_0$ *and obeys the same bound as in (1.11) uniformly in* x*,* y *and* ζ*.*

Proposition 2.3. *Suppose that the same assumptions as in Theorem 1.3 are fulfilled. Then*

$$\begin{aligned} R(\zeta;P)(x,y) &= (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) \\ &+ e^{ik(|x|+|y|)}\left(|x|+|y|\right)^{-1/2} e_3(x,y;\zeta,\lambda), \end{aligned}$$

where e_3 *is analytic in* $\zeta \in D_0$ *and obeys the same bound as in (1.12) uniformly in* x*,* y *and* ζ*.*

Proposition 2.4. *Assume that* $\zeta = E + i\eta \in D_0$*. Denote by* $\varphi_+(x;\omega,E)$ *and* $\varphi_-(x;\omega,E)$ *the outgoing and incoming eigenfunctions of* P*, respectively. Then we have the following statements:*

(1) *If* x *and* y *fulfill* $\lambda/c \le |x| \le c\lambda$ *and* $1/c \le |y| \le c$ *for some* $c > 1$*, then*

$$R(\zeta;P)(x,y) = c_0(\zeta)e^{ik|x|}|x|^{-1/2}\left(\overline{\varphi}_-(y;\hat{x},\overline{\zeta}) + e_{-4}(x,y;\zeta,\lambda)\right),$$

where the complex conjugate $\overline{\varphi}_-(y;\omega,\overline{\zeta})$ *of* $\varphi_-(y;\omega,\overline{\zeta})$ *is analytic in* $\zeta \in D_0$*, and the remainder term* e_{-4} *is also analytic there and obeys*

$$\left|\partial_x^n \partial_y^m e_{-4}\right| = O\left(\lambda^{-1-|n|}\right)$$

uniformly in x*,* y *and* ζ*.*

(2) *If* x *and* y *fulfill* $1/c \le |x| \le c$ *and* $\lambda/c \le |y| \le c\lambda$ *for some* $c > 1$*, then*

$$R(\zeta;P)(x,y) = c_0(\zeta)e^{ik|y|}|y|^{-1/2}\left(\varphi_+(x;-\hat{y},\zeta) + e_{+4}(x,y;\zeta,\lambda)\right),$$

where e_{+4} *is analytic in* $\zeta \in D_0$ *and obeys* $\left|\partial_x^n \partial_y^m e_{+4}\right| = O\left(\lambda^{-1-|m|}\right)$ *uniformly in* x*,* y *and* ζ*.*

3. Proofs of Theorems 1.1, 1.2 and 1.3

In this section we prove Theorems 1.1, 1.2, 1.3 and Corollary 1.1, accepting the propositions mentioned in the previous section. We recall that $H = H(A)$ is the self–adjoint operator defined by (1.4) with $\mathcal{D}(H) = H^2(\Omega) \cap H_0^1(\Omega)$ and that $R(\zeta;H)(x,y)$ denotes the resolvent kernel of $R(\zeta;H)$ with ζ in the complex neighborhood D_0 defined by (1.5). Here we introduce a smooth non–negative cut–off function $\chi \in C_0^\infty[0,\infty)$ with the properties

$$(3.1) \qquad 0 \le \chi \le 1, \qquad \operatorname{supp}\chi \subset [0,2], \qquad \chi = 1 \ \text{ on } \ [0,1].$$

This function is often used in the future discussion without further references.

Lemma 3.1. *Let* $E_0 > 0$ *and* $e_0 > 0$ *be fixed in* D_0. *Then* $R(\zeta; H)$ *is analytic over* D_0 *as a function with values in operators from* $L^2_{\mathrm{comp}}(\Omega)$ *to* $L^2_{\mathrm{loc}}(\Omega)$.

Proof. The proof is based on the complex scaling method developed by [5, 7, 10], and the lemma follows as a particular case of such a general theory. The operator H is a long–range perturbation to the free Hamiltonian $-\Delta$, but the coefficients of H are analytic in Ω. The operator P defined by (2.2) is transformed into $e^{-2\theta}P$ under the group of dilations $x \to e^{\theta}x$. By assumption (1.1), $\mathcal{O} \subset \{|x| < 1\}$. Let $\Sigma = \{|x| < 8\}$. Since H has no positive eigenvalues, we can show by making use of the analytic dilation which leaves Σ invariant that there exists a complex neighborhood of E_0 in which the operator

$$j_0 R(\zeta; H) : L^2_{\mathrm{comp}}(\Sigma_0) \to L^2_{\mathrm{comp}}(\Sigma_0), \quad \Sigma_0 = \Sigma \setminus \mathcal{O},$$

is analytic as a function with values in bounded operators, where the multiplication by the characteristic function j_0 of Σ_0 is considered to be the restriction to $L^2_{\mathrm{comp}}(\Sigma_0)$ from $L^2_{\mathrm{loc}}(\Omega)$. The above neighborhood is independent of $\lambda \gg 1$, so that it contains D_0 for $\lambda \gg 1$. We assert that $R(\zeta; H)$ is analytic over D_0 as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. To see this, we set

$$u_0(x) = \chi(|x|/2), \quad u_1(x) = \chi(|x|/4), \quad v_0 = 1 - u_0, \quad v_1 = 1 - u_1$$

for the cut–off function $\chi \in C_0^\infty[0, \infty)$ with properties in (3.1). Recall that

$$R(\zeta; P) : L^2_{\mathrm{comp}}(\boldsymbol{R}^2) \to L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$$

depends analytically on ζ. If we regard the multiplication operator $f \mapsto v_1 f$ as the extension from $L^2(\Omega)$ to $L^2(\boldsymbol{R}^2)$, then $R(\zeta; P)v_1$ makes sense as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$, and similarly for $R(\zeta)v_0$. Since $v_0 v_1 = v_1$ and since $H = P$ over Ω, $R(\zeta; H) = R(\zeta; H)(u_1 + v_1)$ is decomposed into the sum of three terms

$$R(\zeta; H) = R(\zeta; H)u_1 + v_0 R(\zeta; P)v_1 - R(\zeta; H)[P, v_0]R(\zeta; P)v_1$$

at least for ζ with $\operatorname{Im}\zeta > 0$, where $[X, Y] = XY - YX$ denotes the commutator between two operators X and Y. The coefficients of $[P, v_0]$ have supports in Σ_0. Hence we see that

$$j_0 R(\zeta; H) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{comp}}(\Sigma_0)$$

depends analytically on $\zeta \in D_0$. Similarly we obtain the relation

$$R(\zeta; H) = u_1 R(\zeta; H) + v_1 R(\zeta; P) v_0 + v_1 R(\zeta; P)[P, v_0] R(\zeta; H)$$

on $L^2_{\mathrm{comp}}(\Omega)$. This yields the analytic dependence on ζ of the operator $R(\zeta; H) : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$ and the proof is complete. $\square$

We shall prove Theorems 1.1, 1.2, 1.3 and Corollary 1.1.

Proof of Theorem 1.1. We again set

$$u_0(x) = \chi(|x|/2), \quad u_1(x) = \chi(|x|/4), \quad v_0 = 1 - u_0, \quad v_1 = 1 - u_1$$

and fix $p,\ q \in \boldsymbol{R}^2$ ($|p|,\ |q| \gg 1$) as points having the properties in the theorem. If we further set $w_p(x) = \chi(|x - p|)$, then $w_p v_0 = w_p$ and $w_p v_1 = w_p$, and similarly for $w_q = \chi(|x - q|)$. The operator H coincides with P on the support of v_1. We compute

$$\begin{aligned} w_p R(\zeta; H) w_q &= w_p R(\zeta; P) w_q + w_p R(\zeta; P) \left(P v_1 - v_1 H \right) R(\zeta; H) w_q \\ &= w_p R(\zeta; P) w_q + w_p R(\zeta; P) [u_1, P] R(\zeta; H) w_q. \end{aligned}$$

Since $v_0 = 1$ on the support of ∇u_1 and since $H = P$ on the support of v_0, we repeat the above argument to get

$$\begin{aligned} w_p R(\zeta; H) w_q &= w_p R(\zeta; P) w_q \\ &\quad + w_p R(\zeta; P) [u_1, P] \Big(R(\zeta; P) + R(\zeta; H) [P, u_0] R(\zeta; P) \Big) w_q. \end{aligned}$$

Note that

$$w_p R(\zeta; P) [u_1, P] R(\zeta; P) w_q = w_p R(\zeta) u_1 w_q - w_p u_1 R(\zeta) w_q = 0$$

and hence we have

$$\begin{aligned} w_p R(\zeta; H) w_q &= w_p R(\zeta; P) w_q \\ &\quad + w_p R(\zeta; P) [u_1, P] R(\zeta; H) [P, u_0] R(\zeta; P) w_q. \end{aligned} \tag{3.2}$$

We apply Proposition 2.4 to the second operator on the right side. Since

$$e^{ik(|p|+|q|)} (|p||q|)^{-1/2} = e^{ik(|p|+|q|)} (|p| + |q|)^{-1/2} O(\lambda^{-1/2}),$$

Proposition 2.4 enables us to deal with the kernel of the second operator as a remainder term. Thus the theorem follows from Proposition 2.1. $\square$

Proof of Theorem 1.2. The proof of the theorem is almost the same as that of Theorem 1.1. If we make use of relation (3.2), then the theorem follows from Propositions 2.2 and 2.4. $\square$

Proof of Theorem 1.3. The proof of this theorem is also almost the same as that of Theorem 1.1. If we again make use of relation (3.2), then the theorem follows from Propositions 2.3 and 2.4. $\square$

Proof of Corollary 1.1. Assume that $\sigma(x,y)>0$. Then

$$\gamma(\hat{x};-\hat{y})=\gamma(\hat{x};\hat{y})-\pi=\sigma(x,y)=O(\lambda^{-\mu})$$

and hence it follows that

$$e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}=e^{-i\alpha\pi}\left(1+O(\lambda^{-\mu})\right).$$

By assumption, we also have $\lambda/c<|x-y|<c\lambda$ for some $c>1$. If we take into account the asymptotic form (1.13) of $H_0(z)$, then we have

$$\frac{i}{4}\,e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|)=\frac{e^{ik|x-y|}}{|x-y|^{1/2}}\left(c_0(\zeta)e^{-i\alpha\pi}+O(\lambda^{-\mu})\right).$$

This, together with Theorem 1.2, yields the desired asymptotic form for $\sigma(x,y)>0$. If $\sigma(x,y)<0$, then $\gamma(\hat{x};-\hat{y})=2\pi+\sigma(x,y)$, and the desired asymptotic form is obtained in the same way as above. Thus the proof is complete. □

4. Proofs of Propositions 2.1, 2.2, 2.3 and 2.4

This section is devoted to proving the propositions which have remained unproved in section 2. Most of the section is occupied by the proof of Propositions 2.2. Proposition 2.1 has been already established as [4, Proposition 3.1]. We skip the proof of this proposition. Propositions 2.3 and 2.4 are proved in a way similar to that in the proof of Proposition 2.2.

Before going into the proof, we begin by deriving the integral representation for the resolvent kernel $R(\zeta;P)(x,y)$ with $\zeta\in D_0$. The representation is based on the following formula

$$H_\nu(Z)J_\nu(z)=\frac{1}{i\pi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2}-\frac{Z^2+z^2}{2t}\right)I_\nu\left(\frac{Zz}{t}\right)\frac{dt}{t},\quad |z|\le|Z|,$$

for the product of Bessel functions ([14, p.439]), where $I_\nu(w)$ is defined by

$$I_\nu=\frac{1}{\pi}\left(\int_0^\pi e^{w\cos\rho}\cos(\nu\rho)\,d\rho-\sin(\nu\pi)\int_0^\infty e^{-w\cosh p-\nu p}\,dp\right)\tag{4.1}$$

with $\operatorname{Re}w\ge0$ ([14, p.181]), and the contour is taken to be rectilinear with corner at $\kappa+i0$, $\kappa>0$ being fixed arbitrarily. We apply to (2.8) this formula with $Z=k\,(|x|\vee|y|)$ and $z=k\,(|x|\wedge|y|)$, where $k=\zeta^{1/2}$. If we write $x=(|x|\cos\theta,|x|\sin\theta)$ and $y=(|y|\cos\omega,|y|\sin\omega)$ in the polar coordinates, then $R(\zeta;P)(x,y)$ is represented as

(4.2) $R(\zeta;P)(x,y)=$

$$\frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2}-\frac{\zeta\left(|x|^2+|y|^2\right)}{2t}\right)I_\nu\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t}$$

with $\nu = |l-\alpha|$, where $\psi = \theta - \omega$. If, in particular, $\alpha = 0$, then the resolvent $R(\zeta; H_0)$ of the free Hamiltonian H_0 has the kernel $(i/4)H_0(k|x-y|)$ represented as

$$\frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2}-\frac{\zeta\left(|x|^2+|y|^2\right)}{2t}\right) I_l\left(\frac{\zeta|x||y|}{t}\right)\frac{dt}{t},$$

where

$$I_{|l|}(w) = I_l(w) = (1/\pi)\int_0^\pi e^{w\cos\rho}\cos(l\rho)\,d\rho$$

by (4.1). The series $\sum_l e^{il\psi} I_l(w) = e^{w\cos\psi}$ is convergent by the Fourier expansion. Since

$$|x-y|^2 = |x|^2 + |y|^2 - 2|x||y|\cos\psi,$$

the kernel $(i/4)H_0(k|x-y|)$ has the representation

$$\frac{i}{4}H_0(k|x-y|) = \frac{1}{4\pi}\int_0^{\kappa+i\infty}\exp\left(\frac{t}{2}-\frac{\zeta|x-y|^2}{2t}\right)\frac{dt}{t}. \tag{4.3}$$

Proof of Proposition 2.2. We deal with only the case $\sigma(x;y) > 0$ in some details. A similar argument applies to the case $\sigma(x;y) < 0$ also, and we make only a brief comment on this case at the end of the proof.

We again write $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates and set $\psi = \theta - \omega$. We fix $M \gg 1$ large enough and take

$$\kappa = M^2 \log\lambda$$

in the contour of integral (4.2). Then we divide (4.2) into the sum of integrals over the following four intervals by a smooth partition of unity :

$$0 < t < \kappa, \quad 0 < s < 2\lambda/M, \quad \lambda/M < s < 2M\lambda, \quad s > M\lambda$$

for $t = \kappa + is$, and we evaluate the integral over each interval. The intervals are cut off by the smooth functions $\chi_0(s)$, $\chi_\infty(s)$ and $\chi_M(s)$ defined by

$$\chi_0 = \chi(Ms), \quad \chi_\infty = 1 - \chi(s/M), \quad \chi_M = \chi(s/M)(1-\chi(Ms)), \tag{4.4}$$

where $\chi \in C_0^\infty[0,\infty)$ is the cut–off function with properties in (3.1). The proof is long and is divided into several steps.

(1) We begin by evaluating the integral over the interval $s > M\lambda$ and show that it obeys the bound $O(\lambda^{-N})$ together with all the derivatives in x and y for any $N \gg 1$. To see this, we employ the formula

$$I_\nu(w) = \frac{e^{-i\nu\pi/2}}{\pi}\left\{\int_0^\pi \cos\left(\nu\rho - iw\sin\rho\right)\,d\rho - \sin(\nu\pi)\int_0^\infty e^{-iw\sinh p - \nu p}\,dp\right\}$$

for $\operatorname{Im} w \le 0$, which is obtained as an immediate consequence of the formula $I_\nu(w) = e^{-i\nu\pi/2} J_\nu(iw)$ ([14, p.176]). Since $\zeta = E + i\eta$ satisfies $|\eta| \le e_0 (\log \lambda)/\lambda$ for $\zeta \in D_0$, we note that

$$\operatorname{Im}(\zeta/t) = \left(\kappa^2 + s^2\right)^{-1} (\eta\kappa - Es) < 0$$

for $t = \kappa + is$ with $s > M\lambda$, provided that $\lambda \gg 1$. We insert $I_\nu(\zeta|x||y|/t)$ into the integral

$$\int_0^{\kappa+i\infty} \chi_\infty \left(\frac{\operatorname{Im} t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta\left(|x|^2 + |y|^2\right)}{2t}\right) I_\nu\left(\frac{\zeta|x||y|}{t}\right) \frac{dt}{t}$$

with $\nu = |l - \alpha|$, and we evaluate the resulting integral by use of partial integration for each l with $|l| < \lambda$. If $M \gg 1$, then

$$\left|\partial_t \left(t/2 - \zeta(|x|^2 + |y|^2)/2t \pm (\zeta|x||y|/t)\sin\rho\right)\right| > c > 0$$
$$\left|\partial_t \left(t/2 - \zeta(|x|^2 + |y|^2)/2t - (i\zeta|x||y|/t)\sinh p\right)\right| > c > 0$$

for $t = \kappa + is$ with $s > M\lambda$ uniformly in ρ, $0 < \rho < \pi$, and in p, $0 < p < 2$. If $p > 1$, then we use the relations $\left|\partial_t(t/2 - \zeta(|x|^2 + |y|^2)/2t)\right| > c > 0$ and

$$e^{-\nu p} \left(\partial_t e^{-i(\zeta|x||y|/t)\sinh p}\right) = \\ -\frac{1}{t} \frac{i(\zeta|x||y|/t)\sinh p}{i(\zeta|x||y|/t)\cosh p + \nu} \left(\partial_p e^{-i(\zeta|x||y|/t)\sinh p - \nu p}\right).$$

We take these relations into account to repeat the integration by parts. Then the sum of the integrals over l with $|l| < \lambda$ obeys $O(\lambda^{-N})$. To see that the sum over l with $|l| > \lambda$ is also negligible, we make use of the other representation formula

$$I_\nu(w) = \frac{(w/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^{1} e^{-w\rho} \left(1 - \rho^2\right)^{\nu - 1/2} d\rho \tag{4.5}$$

for $I_\nu(w)$ with $\nu \ge 0$ ([14, p.172]). Since $|x| + |y| = O(\lambda)$, we have $|w| = \left|\zeta|x||y|/t\right| = O(\lambda)/M$ for $s = \operatorname{Im} t > M\lambda$ and

$$\left|e^{-w\rho}\right| = O\left(e^{|\operatorname{Re}(\zeta|x||y|/t)|}\right) = O\left(e^{\lambda}\right), \quad |\rho| < 1.$$

By the Stirling formula, $\Gamma(\nu) \sim (2\pi)^{1/2} e^{-\nu} \nu^{\nu - (1/2)}$ for $\nu \gg 1$. Thus we can take $M \gg 1$ so large that

$$|w^\nu/\Gamma(\nu)| \le (1/2)^{|l|}, \quad \nu = |l - \alpha|,$$

for $|l| > \lambda$. Hence the sum of integrals over l with $|l| > \lambda$ also obeys $O(\lambda^{-N})$, and it follows that the integral over the interval $s > M\lambda$ is negligible.

(2) We evaluate the integral (4.2) over the other intervals $0 < t < \kappa$, $0 < s < 2\lambda/M$ and $\lambda/M < s < 2M\lambda$. We first note that $M \gg 1$ can be taken so large that

$$\operatorname{Re}(\zeta/t) = \left(\kappa^2 + s^2\right)^{-1}(E\kappa + \eta s) > 0, \quad \zeta = E + i\eta \in D_0, \tag{4.6}$$

over these intervals. We use formula (4.1) for $I_\nu(w)$ to calculate the series

$$I(w,\psi) = \sum_l e^{il\psi} I_\nu(w), \quad \nu = |l - \alpha|,$$

in the integrand of (4.2), where $w = \zeta|x||y|/t$. Then $I(w,\psi)$ is decomposed into the sum

$$I(w,\psi) = I_{\mathrm{fr}}(w,\psi) + e^{-w} I_{\mathrm{sc}}(w,\psi),$$

where

$$I_{\mathrm{fr}}(w,\psi) = (1/\pi)\sum_l e^{il\psi}\int_0^\pi e^{w\cos\rho}\cos(\nu\rho)\,d\rho,$$

$$I_{\mathrm{sc}}(w,\psi) = -\,(1/\pi)\sum_l e^{il\psi}\sin(\nu\pi)\int_0^\infty e^{-w(\cosh p - 1) - \nu p}\,dp.$$

The sum of the first series equals

$$I_{\mathrm{fr}}(w,\psi) = e^{i\alpha\psi} e^{w\cos\psi}, \quad |\psi| < \pi,$$

by the Fourier expansion. We compute the series

$$\begin{aligned}\sum_l e^{il\psi} e^{-\nu p}\sin(\nu\pi) &= \left\{\sum_{l\le[\alpha]} + \sum_{l\ge[\alpha]+1}\right\} e^{il\psi} e^{-\nu p}\sin(\nu\pi) \\ &= \sin(\alpha\pi)\,(-1)^{[\alpha]}\left\{\frac{e^{-\alpha p}\left(e^{i\psi}e^{p}\right)^{[\alpha]}}{1 + e^{-i\psi}e^{-p}} + \frac{e^{\alpha p}\left(e^{i\psi}e^{-p}\right)^{[\alpha]}}{1 + e^{-i\psi}e^{p}}\right\}\end{aligned}$$

for $|\psi| < \pi$, where the Gauss notation $[\alpha]$ denotes the greatest integer not exceeding α. Thus the second series takes the form

$$I_{\mathrm{sc}}(w,\psi) = -\frac{\sin(\alpha\pi)}{\pi}\,(-1)^{[\alpha]}\,e^{i[\alpha]\psi}\int_{-\infty}^{\infty} e^{-w(\cosh p - 1)}\frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\,dp \tag{4.7}$$

with $0 < \beta = \alpha - [\alpha] < 1$. It should be noted that the two relations hold true only for $|\psi| < \pi$. If $\psi = \pm\pi$, then the function $e^{i\alpha\psi}$ is not necessarily continuous and the denominator $e^p + e^{-i\psi}$ vanishes at $p = 0$. Since

$$\zeta(|x|^2 + |y|^2)/(2t) + \zeta|x||y|/t = \zeta(|x| + |y|)^2/(2t),$$

the resolvent kernel $R(\zeta;P)(x,y)$ under consideration admits the decomposition

$$R(\zeta;P)(x,y) = R_{\mathrm{fr}}(x,y;\zeta) + R_{\mathrm{sc}}(x,y;\zeta) + O\left(\lambda^{-N}\right) \tag{4.8}$$

for any $N \gg 1$, where

$$R_{\mathrm{fr}} = \frac{1}{4\pi} e^{i\alpha\psi} \int_0^{\kappa+i\infty} \chi\left(\frac{\operatorname{Im} t}{M\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right) \frac{dt}{t},$$

$$R_{\mathrm{sc}} = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \chi\left(\frac{\operatorname{Im} t}{M\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta(|x|+|y|)^2}{2t}\right) I_{\mathrm{sc}}\left(\frac{\zeta|x||y|}{t}, \psi\right) \frac{dt}{t}.$$

If α is an integer, then the integral $I_{\mathrm{sc}}(\zeta|x||y|/t, \psi)$ vanishes, and hence so does $R_{\mathrm{sc}}(x, y; \zeta)$. Let χ_∞ be defined in (4.4). Since $|x - y| < c\lambda$ by assumption, we have by partial integration that

$$\int_0^{\kappa+i\infty} \chi_\infty\left(\frac{\operatorname{Im} t}{M\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right) \frac{dt}{t} = O\left(\lambda^{-N}\right)$$

together with all the derivatives in x and y for any $N \gg 1$. This, together with (4.3), yields

$$R_{\mathrm{fr}}(x, y; \zeta) = (i/4)e^{i\alpha\psi} H_0(k|x-y|) + O\left(\lambda^{-N}\right).$$

We are dealing with the case $\sigma(x, y) > 0$. Hence

$$\psi = \gamma(\hat{x}; -\hat{y}) - \pi. \tag{4.9}$$

Thus the first term of the asymptotic formula is obtained from the relation

$$R_{\mathrm{fr}}(x, y; \zeta) = (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)} H_0(k|x-y|) + O\left(\lambda^{-N}\right). \tag{4.10}$$

(3) The second term is obtained from $R_{\mathrm{sc}}(x, y; \zeta)$. Let χ_0 be as in (4.4). We assert that the integral

$$\int_0^{\kappa+i\infty} \chi_0\left(\frac{\operatorname{Im} t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta(|x|+|y|)^2}{2t}\right) I_{\mathrm{sc}}\left(\frac{\zeta|x||y|}{t}, \psi\right) \frac{dt}{t} \tag{4.11}$$

obeys $O(\lambda^{-N})$ together with all the derivatives in x and y for any $N \gg 1$. By (4.6), we note that $\operatorname{Re}(\zeta|x||y|/t) > 0$ for $0 < t < \kappa$ and for $t = \kappa + is$ with $0 < s < 2M\lambda$. Since

$$\lambda^{-(1-\mu)}/c < \sigma(x, y) < c\lambda^{-\mu}, \quad c > 1,$$

by assumption, we have

$$\left|e^p + e^{-i\psi}\right| \ge \left|\sin\, \sigma(x, y)\right| \ge \lambda^{-(1-\mu)}/c \tag{4.12}$$

for another $c > 1$, and hence it follows from (4.7) that

$$\left|I_{\mathrm{sc}}(\zeta|x||y|/t, \psi)\right| = O(\lambda^{1-\mu})$$

uniformly in x, y, t and ζ. If $0 < t < \kappa$, then we have

$$\left|\exp\left(-\zeta(|x|+|y|)^2/(2t)\right)\right| = |t| O\left(\lambda^{-N}\right)$$

for any $N \gg 1$. This remains true for $t = \kappa + is$ with $0 < s < 2\lambda^{1-\delta}$, where δ, $0 < \delta \ll 1$, is taken small enough. If $t = \kappa + is$ fulfills $\lambda^{1-\delta} < s < 2\lambda/M$, then $|\zeta|x||y|/t| > \lambda/c$ and

$$\mathrm{Re}(\zeta/t)|x||y|(\cosh p - 1) > 0, \quad p \neq 0,$$

by (4.6). Hence we have

$$\int_{-\infty}^{\infty} e^{-w(\cosh p-1)} \left(1 - \chi(|p|)\right) \frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\, dp = O\left(\lambda^{-N}\right)$$

for $w = \zeta|x||y|/t$, because the stationary point $p = 0$ of the phase function $\cosh p - 1$ is outside the support of $1 - \chi(|p|)$ and the denominator $e^p + e^{-i\psi}$ is uniformly away from 0 there. If p is in the support of $\chi(|p|)$, then we have

$$\left|\partial_t \Big(t/2 - \zeta(|x| + |y|)^2/(2t) - (\zeta|x||y|/t)(\cosh p - 1)\Big)\right| > c > 0,$$

which allows us to make repeated use of partial integration in t. Thus we combine all the observations above to obtain that the integral defined by (4.11) obeys $O(\lambda^{-N})$, so that we have

$$R_{\mathrm{sc}}(x, y; \zeta) = \tilde{R}_{\mathrm{sc}}(x, y; \zeta) + O(\lambda^{-N}), \tag{4.13}$$

where

$$\tilde{R}_{\mathrm{sc}} = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \chi_M\left(\frac{\mathrm{Im}\, t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta(|x| + |y|)^2}{2t}\right) I_{\mathrm{sc}}\left(\frac{\zeta|x||y|}{t}, \psi\right) \frac{dt}{t}$$

and χ_M is defined in (4.4). We note that the cut–off function $\chi_M(s/\lambda)$ has support in $(\lambda/M, 2M\lambda)$ and $\chi_M(s/\lambda) = 1$ on $[2\lambda/M, M\lambda]$.

(4) We analyze the behavior as $\lambda \to \infty$ of $\tilde{R}_{\mathrm{sc}}(x, y; \zeta)$ defined above. Let $R_0(x, y; \zeta)$ be defined by

$$R_0 = \frac{1}{4\pi} \int_0^{\kappa+i\infty} \chi_M\left(\frac{\mathrm{Im}\, t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta\left(|x| + |y|\right)^2}{2t}\right) I_0\left(\frac{\zeta|x||y|}{t}, \psi\right) \frac{dt}{t},$$

where

$$I_0(w, \psi) = -\frac{\sin(\alpha\pi)}{\pi}(-1)^{[\alpha]} e^{i[\alpha]\psi} \int \chi(\lambda^\mu|p|) e^{-w(\cosh p-1)} \left(\frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\right) dp.$$

Then we assert that

$$\tilde{R}_{\mathrm{sc}}(x, y; \zeta) = R_0(x, y; \zeta) + O\left(\lambda^{-N}\right) \tag{4.14}$$

for any $N \gg 1$, where the remainder estimate $O(\lambda^{-N})$ holds true for all the derivatives in x and y. To prove this, we consider the integral

$$\int e^{-w(\cosh p-1)} \Big(1 - \chi(\lambda^\mu|p|)\Big) \frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\, dp$$

with $w = \zeta|x||y|/t$, where $t = \kappa + is$ with $\lambda/M < s < 2M\lambda$. Since $|w| = |\zeta|x||y|/t| \sim \lambda$ and $0 < \mu < 1/2$, we see by repeated use of partial integration that this integral obeys $O(\lambda^{-N})$ for any $N \gg 1$. We note that

$$\exp\left(\frac{t}{2} - \frac{\zeta(|x|+|y|)^2}{2t}\right) = e^{ik|x|} \exp\left(\frac{t}{2}\left(1 - \frac{ik(|x|+|y|)}{t}\right)^2\right) e^{ik|y|}$$

is of polynomial growth in λ. Hence (4.14) follows immediately. This, together with (4.13), yields

$$R_{\rm sc}(x, y; \zeta) = R_0(x, y; \zeta) + O\left(\lambda^{-N}\right). \tag{4.15}$$

(5) We apply the method of steepest descent to the integral $R_0(x, y; \zeta)$ defined in step (4). The phase function is transformed

$$(t/2)\Big(1 - ik(|x| + |y|)/t\Big)^2 \longrightarrow ik(|x| + |y|)\tau^2/(2(1-\tau))$$

under the change of the variable $\tau = 1 - ik(|x| + |y|)/t$. The line $t = \kappa + is$ with $\lambda/M < s < 2M\lambda$ is transformed into a certain curve in the complex plane. The stationary point $\tau = 0$ does not necessarily lie on this contour. We deform the contour of the integral suitably in the complex domain by analyticity and use the method of steepest decent in a neighborhood of $\tau = 0$. If we write $k = \zeta^{1/2} = |k|e^{i\theta}$ for $\zeta \in D_0$, then $|\theta| \le c(\log\lambda)/\lambda$. We further deform a small real interval around $\tau = 0$ by analyticity into the smooth contour defined by $z = \tau e^{iL(\log\lambda)/\lambda}$, $|\tau| \ll 1$, in a complex neighborhood of $\tau = 0$. We can take $L \gg 1$ so large that $\mathrm{Im}\Big(kz^2\Big) > 0$ for $z \neq 0$. This enables us to apply the stationary phase method ([8, Theorem 7.7.5]) to the resulting integral. If we take account of (4.12) and of the relation

$$\left(\frac{\partial}{\partial\tau}\right)^m I_0(w, \psi) = O\left(\lambda^{(m+1)(1-2\mu)}\right), \quad w = \frac{\zeta|x||y|}{t} = \left(\frac{ik|x||y|}{|x|+|y|}\right)(\tau - 1),$$

around $\tau = 0$ and if we recall that $\mu > 2/5$ in (1.7) (and hence $2-6\mu < -\mu$), then we get the asymptotic formula

$$R_0(x, y; \zeta) = c_0(\zeta)\left(\frac{e^{ik(|x|+|y|)}}{(|x|+|y|)^{1/2}}\right)\Big(I_0(w_0, \psi) + O(\lambda^{-\mu})\Big), \tag{4.16}$$

where $w_0 = -ik|x||y|/(|x| + |y|)$ and $c_0(\zeta)$ is defined by (1.10).

(6) We require the lemma below to analyze the behavior as $\lambda \to \infty$ of the integral $I_0(w_0, \psi)$. The lemma is verified after the proof of the proposition is complete (in step (8)).

Lemma 4.1. *Let $J(u,r)$ be the integral defined by*

$$J(u,r) = \int_{-\infty}^{\infty} e^{iup^2/2}(p+ir)^{-1}\,dp$$

for $u>0$ and $r \ne 0$. Set $v = u\,r^2 > 0$. If $r>0$, then

$$J(u,r) = e^{-iv/2}\left(-\pi\, i + (2\pi)^{1/2} e^{i\pi/4}\int_0^{v^{1/2}} e^{is^2/2}\,ds\right),$$

and if $r<0$, then $J(u,r) = -J(u,-r)$.

We recall that $I_0(w_0,\psi)$ is defined by

$$I_0(w_0,\psi) = -\frac{\sin(\alpha\pi)}{\pi}(-1)^{[\alpha]}e^{i[\alpha]\psi}\int \chi(\lambda^\mu|p|)e^{-w(\cosh p-1)}\left(\frac{e^{(1-\beta)p}}{e^p+e^{-i\psi}}\right)dp$$

with $w = w_0 = -ik|x||y|/(|x|+|y|)$. Since $\psi = \theta-\omega = \sigma(x,y)-\pi$ by (4.9) and $\sigma(x,y) = O(\lambda^{-\mu})$, we have

$$e^{i[\alpha]\psi} = e^{i[\alpha](\sigma-\pi)} = (-1)^{[\alpha]}\left(1+O(\lambda^{-\mu})\right)$$

and $e^{-i\psi} = e^{-i(\sigma-\pi)} = -e^{-i\sigma}$. Thus we have

$$\frac{e^{(1-\beta)p}}{e^p+e^{-i\psi}} = \frac{e^{(1-\beta)p}}{e^p-e^{-i\sigma}} = \frac{1+O(p)}{p+i\sigma+O(p^2)+O(\sigma^2)} = \frac{1}{p+i\sigma}+O(1).$$

Since

$$-w_0(\cosh p-1) = \Big(ik|x||y|/(2(|x|+|y|))\Big)p^2\left(1+O(p^2)\right)$$

for $|p| \ll 1$ and since

$$k = \zeta^{1/2} = (E+i\eta)^{1/2} = E^{1/2}+O(|\eta|) = E^{1/2}+O\left((\log\lambda)/\lambda\right) \tag{4.17}$$

for $\zeta \in D_0$, it follows that

$$\exp(-w_0(\cosh p-1)) = \exp\left(\frac{iE^{1/2}|x||y|}{2(|x|+|y|)}p^2\right)\left(1+p^2O(\log\lambda)+p^4O(\lambda)\right)$$

for $|p| < 2\lambda^{-\mu}$. By (1.7), $\mu > 2/5 > 1/3$, and hence we have

$$e^{-w_0(\cosh p-1)}\left(\frac{e^{(1-\beta)p}}{e^p+e^{-i\psi}}\right) = \exp\left(\frac{iE^{1/2}|x||y|}{2(|x|+|y|)}p^2\right)\left(\frac{1}{p+i\sigma}+O(1)\right).$$

As shown in step (3),

$$\int \exp\left(i\frac{E^{1/2}|x||y|}{2(|x|+|y|)}p^2\right)\Big(1-\chi(\lambda^\mu|p|)\Big)\Big(p+i\sigma\Big)^{-1}dp = O\left(\lambda^{-N}\right).$$

Thus Lemma 4.1 with $r = \sigma = \sigma(x, y) > 0$ yields

$$\int \chi(\lambda^\mu|p|)e^{-w_0(\cosh p-1)}\left(\frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\right) dp = J(u, \sigma) + O(\lambda^{-\mu}),$$

where $u = u(x, y; \zeta) = E^{1/2}|x||y|/(|x| + |y|)$ with $E = \mathrm{Re}\,\zeta$. By (4.17), we have

$$v(x, y; \zeta) = u(x, y; \zeta)|\sigma|^2 = z_0(x, y; \zeta)^2 + O\left((\log \lambda)/\lambda^{2\mu}\right),$$

where $z_0 = z_0(x, y; \zeta)$ is defined by (1.9). Hence Lemma 4.1, together with (4.15) and (4.16), implies that $R_{\mathrm{sc}}(x, y; \zeta)$ takes the asymptotic form

$$\begin{aligned} R_{\mathrm{sc}} \;&=\; c_0(\zeta)\frac{i\sin(\alpha\pi)}{\pi}\left(\frac{e^{-iz_0^2/2}e^{ik(|x|+|y|)}}{(|x| + |y|)^{1/2}}\right) \times \\ &\qquad \left(\pi - (2\pi)^{1/2}e^{-i\pi/4}\int_0^{z_0} e^{is^2/2}\, ds + O\left(\lambda^{-\mu}\right)\right). \end{aligned}$$

(7) We look at the numerator $e^{-iz_0^2/2}e^{ik(|x|+|y|)}$. We recall from (1.9) that

$$z_0 = z_0(x, y; \zeta) = \Big(|x||y|/(|x| + |y|)\Big)^{1/2}|\sigma(x, y)|^2k^{1/2}.$$

Since

$$\begin{aligned} |x - y|^2 \;&=\; |x|^2 + |y|^2 - 2|x||y|\cos(\pi - \sigma) \\ &=\; (|x| + |y|)^2 - |x||y|\sigma^2 + |x||y|O(\sigma^4), \end{aligned}$$

we have

$$ik|x - y| = ik(|x| + |y|) - iz_0^2/2 + O(\lambda^{1-4\mu}),$$

and hence it follows that

$$e^{-iz_0^2/2}e^{ik(|x|+|y|)} = e^{ik|x-y|}\left(1 + O(\lambda^{1-4\mu})\right) = e^{ik|x-y|}\left(1 + O(\lambda^{-\mu})\right).$$

We also have

$$(|x| + |y|)^{-1/2} = |x - y|^{-1/2}\left(1 + O(\sigma^2)\right) = |x - y|^{-1/2}\left(1 + O(\lambda^{-2\mu})\right).$$

Thus $R_{\mathrm{sc}}(x, y; \zeta)$ behaves like

$$\begin{aligned} R_{\mathrm{sc}} \;&=\; c_0(\zeta)\frac{i\sin(\alpha\pi)}{\pi}\left(\frac{e^{ik|x-y|}}{|x - y|^{1/2}}\right) \times \\ &\qquad \left(\pi - (2\pi)^{1/2}e^{-i\pi/4}\int_0^{z_0} e^{is^2/2}\, ds + O\left(\lambda^{-\mu}\right)\right). \end{aligned}$$

This yields the second term of the asymptotic formula. Since $\sigma = \sigma(x, y) > \lambda^{1-\mu}/c$ by assumption, we have the relation

$$\partial_x^n\partial_y^m\Big(\left(e^p - e^{-i\sigma}\right)^{-1} - (p + i\sigma)^{-1}\Big) = O(\lambda^{-(|n|+|m|)\mu})$$

for $|p| < 2\lambda^\mu$. If we take this relation into account, then we see that the remainder term $e_{+2}(x, y; \zeta)$ fulfills (1.11) uniformly in x, y and ζ. Thus the desired asymptotic form is obtained. If $\sigma(x, y) < 0$, then we make use of Lemma 4.1 with $r < 0$ and we obtain that $R_{\rm sc}(x, y; \zeta)$ behaves like

$$\begin{aligned} R_{\rm sc} &= -c_0(\zeta)\frac{i\sin(\alpha\pi)}{\pi}\left(\frac{e^{ik|x-y|}}{|x-y|^{1/2}}\right) \times \\ &\qquad \left(\pi - (2\pi)^{1/2}e^{-i\pi/4}\int_0^{z_0} e^{is^2/2}\,ds + O\left(\lambda^{-\mu}\right)\right). \end{aligned}$$

(8) It remains to prove Lemma 4.1. We define

$$J(v) = \int_{-\infty}^{\infty} e^{ivp^2/2}(p^2+1)^{-1}\,dp, \quad v = ur^2.$$

Assume that $r > 0$. Then the integral $J(u, r)$ is calculated as

$$\begin{aligned} J(u, r) &= (1/2)\int_{-\infty}^{\infty} e^{iup^2/2}\left((p+ir)^{-1} - (p-ir)^{-1}\right)\,dp \\ &= -i\int e^{iup^2/2} r(p^2+r^2)^{-1}\,dp = -i\,J(v) \end{aligned}$$

by making a change of variable $p \mapsto r\,p$. We use the formula

$$\int_{-\infty}^{\infty} e^{ivp^2/2}\,dp = (2\pi/v)^{1/2}e^{i\pi/4}$$

and compute

$$J'(v) = (i/2)\int e^{ivp^2/2}\,dp - (i/2)J(v) = (i/2)(2\pi/v)^{1/2}e^{i\pi/4} - (i/2)J(v).$$

Since $J(0) = \pi$, the solution to the equation above is given by

$$\begin{aligned} J(v) &= e^{-iv/2}\left(J(0) + (\pi/2)^{1/2}e^{i3\pi/4}\int_0^v t^{-1/2}e^{it/2}\,dt\right) \\ &= e^{-iv/2}\left(\pi + (2\pi)^{1/2}e^{i3\pi/4}\int_0^{v^{1/2}} e^{is^2/2}\,ds\right). \end{aligned}$$

If $r < 0$, then $J(u, r) = i\,J(v)$. This proves the lemma and the proof of the proposition is complete. $\square$

As already stated, Proposition 2.1 has been proved as Proposition 3.1 in [4]. We should note that decomposition (4.8) holds true only for $|\psi| < \pi$. If $\psi = \pm\pi$, then $e^{i\alpha\psi}$ is not necessarily continuous and the denominator in (4.7) vanishes at $p = 0$. The leading term of the asymptotic form in forward directions in Proposition 2.1 comes from the cancellation of these two singularities. Propositions 2.3 and 2.4 are proved in a way similar to that in the proof of Proposition 2.2. In fact, the proof is much easier, so we give only a sketch for it. We use the notation with the same meaning ascribed in the proof of Proposition 2.2.

Proof of Proposition 2.3. We start with the integral representation (4.2) for $R(\zeta, P)(x, y)$. This admits the decomposition

$$R(\zeta; P)(x, y) = R_{\mathrm{fr}}(x, y; \zeta) + R_{\mathrm{sc}}(x, y; \zeta) + O\left(\lambda^{-N}\right),$$

and $R_{\mathrm{fr}}(x, y; \zeta)$ and $R_{\mathrm{sc}}(x, y; \zeta)$ behave like

$$R_{\mathrm{fr}} = (i/4)e^{i\alpha\psi} H_0(k|x-y|) + O\left(\lambda^{-N}\right), \quad R_{\mathrm{sc}} = R_0(x, y; \zeta) + O\left(\lambda^{-N}\right),$$

where

$$R_0 = \frac{1}{4\pi}\int_0^{\kappa+i\infty} \chi_M\left(\frac{\operatorname{Im} t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta\left(|x|+|y|\right)^2}{2t}\right) I_0\left(\frac{\zeta|x||y|}{t}, \psi\right)\frac{dt}{t},$$

$$I_0(w, \psi) = -\frac{\sin(\alpha\pi)}{\pi}(-1)^{[\alpha]}e^{i[\alpha]\psi}\int \chi(\lambda^{\mu}|p|)e^{-w(\cosh p-1)}\frac{e^{(1-\beta)p}}{e^p + e^{-i\psi}}\,dp.$$

We further have

$$R_0(x, y; \zeta) = c_0(\zeta)\left(\frac{e^{ik(|x|+|y|)}}{(|x|+|y|)^{1/2}}\right)\Big(I_0(w_0, \psi) + O(\lambda^{-\mu})\Big),$$

where $w_0 = -ik|x||y|/(|x|+|y|)$. We note that

$$\psi = \theta - \omega = \gamma(\hat{x}; -\hat{y}) - \pi, \quad |\psi| < \pi,$$

for $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates. Thus we get the leading term

$$R_{\mathrm{fr}}(x, y; \zeta) = (i/4)e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)}H_0(k|x-y|) + O(\lambda^{-N}).$$

The behavior of $I_0(w_0, \psi)$ is analyzed by the method of steepest descent without using Lemma 4.1. By assumption, $|\sigma(x, y)| > \lambda^{-\mu}/c$, $c > 1$, for $0 < \mu < 1/2$, and

$$\left|\partial_p^n\left(e^p + e^{-i\psi}\right)^{-1}\right| = \left|\partial_p^n\left(e^p - e^{-i\sigma}\right)^{-1}\right| = O\left(|\sigma(x, y)|^{-(n+1)}\right).$$

This allows us to apply the method of steepest descent to the integral $I_0(w_0, \psi)$ with $|w_0| \sim \lambda$, and we get the bound

$$I_0(w_0, \psi) = |\sigma(x, y)|^{-1}\, O\left(\lambda^{-1/2}\right).$$

Hence

$$R_{\mathrm{sc}}(x, y; \zeta) = \left(\frac{e^{ik(|x|+|y|)}}{(|x|+|y|)^{1/2}}\right)|\sigma(x, y)|^{-1}\, O\left(\lambda^{-1/2}\right).$$

If we note that

$$\left|\partial_x^n \partial_y^m \left(e^p - e^{-i\sigma}\right)^{-1}\right| = |\sigma(x,y)|^{-1-(|n|+|m|)}\, O\left(\lambda^{-(|n|+|m|)}\right)$$

for $|p| < 2$, then we see that the remainder term $e_3(x, y; \zeta)$ satisfies (1.12) uniformly in x, y and ζ. This completes the proof. □

Proof of Proposition 2.4. We give only a sketch for the proof of statement (1). By assumption, $\lambda/c < |x| < c\lambda$ and $1/c < |y| < c$ for some $c > 1$. Then we can show that $R(\zeta; P)(x, y)$ behaves like

$$R(\zeta; P)(x,y) = \tilde{R}(x,y;\zeta) + O\left(\lambda^{-N}\right)$$

for any $N \gg 1$, where $\tilde{R}(x, y; \zeta)$ is defined by the integral

$$\frac{1}{4\pi}\int_0^{\kappa+i\infty} \chi_M\left(\frac{\operatorname{Im} t}{\lambda}\right) \exp\left(\frac{t}{2} - \frac{\zeta|x|^2}{2t}\right) \exp\left(-\frac{\zeta|y|^2}{2t}\right) I\left(\frac{\zeta|x||y|}{t}, \psi\right) \frac{dt}{t}$$

and $I(w, \psi)$ is defined by $I(w, \psi) = \sum_l e^{il\psi} I_\nu(w)$ with $w = \zeta|x||y|/t$. We write

$$\exp(t/2 - \zeta|x|^2/2t) = e^{ik|x|} \exp\left((t/2)(1 - ik|x|/t)^2\right)$$

and make a change of variable $t \mapsto \tau = 1 - ik|x|/t$ as in the proof of Proposition 2.2. Then we see by the method of steepest descent that $\tilde{R}(x, y; \zeta)$ takes the asymptotic form

$$\tilde{R} = c_0(\zeta) e^{ik|x|} |x|^{-1/2} \exp(ik|y|^2/2|x|)\left(I(k|y|/i, \psi) + O(\lambda^{-1})\right).$$

We note that $\exp(ik|y|^2/2|x|) = 1 + O(\lambda^{-1})$. Since $I_\nu(z/i) = e^{-i\nu\pi/2} J_\nu(z)$ by formula and since

$$e^{il\psi} = e^{il(\theta-\omega)} = e^{il\gamma(\hat{x};\hat{y})} = e^{-il\gamma(\hat{y};\hat{x})},$$

we have

$$\begin{aligned} I\left(k|y|/i, \psi\right) &= \sum_l e^{il\psi} I_\nu(k|y|/i) \\ &= \sum_l e^{-il\gamma(\hat{y};\hat{x})} e^{-i\nu\pi/2} J_\nu(k|y|) = \overline{\varphi}_-(y; \hat{x}, \overline{\zeta}) \end{aligned}$$

by (2.7). Thus we get the desired asymptotic form. □

5. AB effect in resonances for magnetic scattering

In quantum mechanics, a vector potential is said to have a direct significance to particles moving in a magnetic field. This quantum effect is called the Aharonov–Bohm effect (AB effect) and is known as one of the most remarkable quantum phenomena ([3]). We have studied the AB effect through resonances of a simple scattering system in two dimensions in the previous work [11], where the system consists of three scatterers, one bounded obstacle and two scalar potentials with compact supports at large separation, and the magnetic field is completely shielded by the obstacle placed between the two supports of the scalar potentials. The magnetic field does not influence particles from a classical mechanical point of view, but quantum particles are influenced by the corresponding vector potential which does not necessarily vanish outside the obstacle. We have shown that the resonances are generated near the positive axis by the trajectories oscillating between the two supports of the scalar potentials and that their locations heavily depend on the magnetic flux of the field. Our motivation of the present work lies in studying what happens in the case of several obstacles. In particular, the system of two obstacles yields a two dimensional model of scattering by toroidal solenoids in three dimensions. The result depends on the location of the obstacles as well as on the fluxes.

We begin by making a review on the results obtained in [11]. We write

$$H(A,V) = (-i\nabla - A)^2 + V = \sum_{j=1}^{2} (-i\partial_j - a_j)^2 + V, \quad \partial_j = \partial/\partial x_j,$$

for the Schrödinger operator with the scalar potential $V : \boldsymbol{R}^2 \to \boldsymbol{R}$ and the vector potential $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. Let $b \in C_0^\infty(\boldsymbol{R}^2)$ be as in section 1 and let A be defined by (1.2). We recall that the field $b = \nabla \times A$ has α as its magnetic flux and the support $\operatorname{supp} b$ is contained in some simply connected bounded obstacle $\mathcal{O}$. For $d \in \boldsymbol{R}^2$, we set

$$d_- = -\kappa_- d, \quad d_+ = \kappa_+ d, \quad \kappa_\pm > 0, \quad \kappa_- + \kappa_+ = 1,$$

so that $d_+ - d_- = d$. The distance $|d| \gg 1$ is regarded as a large parameter with the direction $\hat{d} = d/|d|$ fixed. Let $V_\pm \in C_0^\infty(\boldsymbol{R}^2)$. Then we define

$$V_d(x) = V_{-d}(x) + V_{+d}(x) = V_-(x - d_-) + V_+(x - d_+) \tag{5.1}$$

and consider the operator

$$H_d = H(A, V_d) = (-i\nabla - A)^2 + V_d$$

over the exterior domain $\Omega = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}$. We denote by the same notation H_d the self–adjoint realization obtained by imposing the zero Dirichlet boundary conditions $\mathcal{D}(H_d) = H^2(\Omega) \cap H_0^1(\Omega)$.

We know that the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2(\Omega) \to L^2(\Omega), \quad \zeta \in \boldsymbol{C}, \quad \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta > 0,$$

is meromorphically continued from the upper half plane of the complex plane to the lower half plane across the positive real axis where the continuous spectrum of H_d is located. Then $R(\zeta; H_d)$ with $\operatorname{Im}\zeta \le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$. The resonances of H_d are defined as the poles of $R(\zeta; H_d)$ in the lower half plane (unphysical sheet). In [11], we have studied how the resonances are generated near the real axis by the trajectories oscillating between the two centers d_- and d_+ as $|d| \to \infty$ and how the AB effect influences the location of the resonances. The obtained result is formulated in terms of the backward amplitudes by the potentials $V_\pm$. Let $H_0 = -\Delta$ be the free Hamiltonian and let $H_\pm$ be the Schrödinger operator defined by

$$H_\pm = H_0 + V_\pm = -\Delta + V_\pm, \quad \mathcal{D}(H_0) = \mathcal{D}(H_\pm) = H^2(\boldsymbol{R}^2).$$

We denote by $f_\pm(\omega \to \theta; E)$ the amplitude for scattering from the incident direction $\omega \in S^1$ to the final one θ at energy $E > 0$ for the pair $(H_0, H_\pm)$. These amplitudes admit the analytic extensions $f_\pm(\omega \to \theta; \zeta)$ in a complex neighborhood of the positive real axis as a function of E.

We now fix $E_0 > 0$ and take a complex neighborhood

$$D_d = \left\{ \zeta : |\mathrm{Re}\,\zeta - E_0| < \delta_0 E_0,\ |\mathrm{Im}\,\zeta| < (1 + 2\delta_0)\, E_0^{1/2} \left(\frac{\log |d|}{|d|} \right) \right\} \tag{5.2}$$

for δ_0, $0 < \delta_0 \ll 1$, small enough. We define

$$h(\zeta; d) = \left(\frac{e^{2ik|d|}}{|d|} \right) f_-(-\hat{d} \to \hat{d}; \zeta) f_+(\hat{d} \to -\hat{d}; \zeta) \cos^2(\alpha\pi) \tag{5.3}$$

over D_d, where the branch $k = \zeta^{1/2}$ is again taken in such a way that $\mathrm{Re}\, k > 0$ for $\mathrm{Re}\,\zeta > 0$. Assume that $f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) \neq 0$ and that α is not a half integer. Then the equation

$$h(\zeta; d) = 1 \tag{5.4}$$

has the solutions

$$\left\{ \zeta_j(d) \right\}, \quad \zeta_j(d) \in D_d, \quad \mathrm{Re}\,\zeta_1 < \mathrm{Re}\,\zeta_2 < \cdots < \mathrm{Re}\,\zeta_{N_d},$$

such that $\zeta_j(d)$ behaves like

$$\mathrm{Im}\,\zeta_j(d) \sim -E_0^{1/2} (\log |d|)/|d|, \quad \mathrm{Re}\,(\zeta_{j+1}(d) - \zeta_j(d)) \sim 2\pi E_0^{1/2}/|d|,$$

for $|d| \gg 1$. The following theorem has been established as [11, Theorem 1.1].

Theorem 5.1. *Let the notation be as above. Assume that α is not a half integer and $f_\pm(\pm\hat{d} \to \mp\hat{d}; E_0) \neq 0$ at energy $E_0 > 0$. Then we can take $\delta_0 > 0$ so small that the neighborhood D_d defined by (5.2) has the following* *property: For any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that for $|d| > d_\varepsilon$, H_d has the resonances*

$$\left\{\zeta_{\mathrm{res},j}(d)\right\}, \quad \zeta_{\mathrm{res},j}(d) \in D_d, \quad \operatorname{Re}\zeta_{\mathrm{res},1}(d) < \cdots < \operatorname{Re}\zeta_{\mathrm{res},N_d}(d)$$

in the neighborhood

$$\left\{\zeta \in \boldsymbol{C} : |\zeta - \zeta_j(d)| < \varepsilon/|d|\right\}$$

and the resolvent $R(\zeta; H_d)$ is analytic over $D_d \setminus \left\{\zeta_{\mathrm{res},1}(d), \cdots, \zeta_{\mathrm{res},N_d}(d)\right\}$ as a function with values in operators from $L^2_{\mathrm{comp}}(\Omega)$ to $L^2_{\mathrm{loc}}(\Omega)$.

We can explain intuitively how reasonable (5.4) is as an approximate relation to determine the location of the resonances. Recall that $\varphi_0(x;\omega,E)$ defined by (2.4) denotes the plane wave with ω as an incident direction at energy $E > 0$. We write $x_\pm$ for $x_\pm = x - d_\pm$. The incident plane wave $\varphi_0(x_-; -\hat{d}, E)$ takes the form

$$f_-(-\hat{d} \to \hat{d}; E)\left(e^{iE^{1/2}|x_-|}/|x_-|^{1/2}\right)$$

after scattered into the direction $\hat{d} = d/|d|$ by the potential V_{-d}, and the scattered wave hits the other potential V_{+d}. Since $|x_-|$ behaves like

$$|x_-| = |x - d_-| = |d + x_+| = |d| + \hat{d}\cdot x_+ + O\left(|d|^{-1}\right)$$

around d_+, the scattered wave behaves like the plane wave

$$\left(e^{iE^{1/2}|d|}/|d|^{1/2}\right) f_-(-\hat{d} \to \hat{d}; E)\varphi_0(x_+; \hat{d}, E)$$

when it arrives at the support of V_{+d}, provided that the vector potential vanishes identically. If, however, it does not necessarily vanish, then the wave function undergoes a change of the phase factor by the AB quantum effect. We consider the particle moving from d_- to d_+ under the assumption that the center $d_\pm$ of $\operatorname{supp} V_{\pm d}$ is located on the x_1 axis. We distinguish between the trajectories passing over $x_2 > 0$ and $x_2 < 0$ to denote the former and latter trajectories by l_+ and l_-, respectively. The vector potential $A(x)$ defined by (1.2) satisfies the relation $A(x) = \alpha\,\nabla\,\gamma(x)$ for the azimuth angle $\gamma(x)$ from the positive x_1 axis. Then the AB effect causes the change in the phase factor of the wave function, which is given by the line integral

$$\int_{l_\pm} A(x)\cdot dx = \alpha \int_{l_\pm} \Phi(x)\cdot dx = \mp\alpha\pi$$

along $l_\pm$. The factor $\cos(\alpha\pi)$ is generated from the sum of $e^{i\alpha\pi}$ and $e^{-i\alpha\pi}$. Thus the scattered wave takes

$$\left(e^{iE^{1/2}|d|}/|d|^{1/2}\right) f_-(-\hat{d} \to \hat{d}; E)\cos(\alpha\pi)\varphi_0(x_+; \hat{d}, E)$$

as an approximate form, when it hits the support of V_{+d}. A similar argument applies to the plane wave $\varphi_0(x_+;\hat{d},E)$ after scattered into the direction $-\hat{d}$ by the potential V_{+d}, so that it again returns to the support of V_{-d}, taking the approximate form $h(E;d)\varphi_0(x_-;-\hat{d},E)$. Hence the contribution from the trapping effect between d_- and d_+ is described by the series

$$\left(\sum_{n=1}^{\infty} h(E;d)^n\right)\varphi_0(x_-;-\hat{d},E).$$

For example, the term with $h(E;d)^n$ describes the contribution from the trajectory oscillating n times. This is the reason why the location of the resonances is approximately determined by the relation $h(\zeta;d)=1$. If α is a half integer, then $\cos(\alpha\pi)$ vanishes by cancellation, and the resonances can be shown to be generated by the second longest trajectory oscillating between $\mathcal{O}$ and $\operatorname{supp} V_{\pm d}$. Thus the location of the resonances heavily depends on the magnetic flux α, even if the magnetic field vanishes in Ω.

As stated above, our motivation comes from a generalization of the above result to the case of several obstacles. We consider the case of two obstacles. The results depend on the location of the obstacles. We discuss the two typical cases. One is the case when the obstacles are placed horizontally along the segment joining the centers d_- and d_+ of the scalar potentials $V_{\pm d}$, and the other is the case when the obstacles are placed vertically to the segment. We are going to discuss the two results above in details in the separate works [12, 13].

We set up our problem more precisely. Let $b_\pm \in C_0^\infty(\boldsymbol{R}^2)$ be a given magnetic field with the flux $\alpha_\pm = (2\pi)^{-1}\int b_\pm(x)\,dx$. We assume that the support of $b_\pm$ satisfies

$$\operatorname{supp} b_\pm \subset \mathcal{O}_\pm \subset B = \{|x|<1\} \tag{5.5}$$

for some simply connected bounded obstacle $\mathcal{O}_\pm$, where $\mathcal{O}_\pm$ is assumed to have the origin as an interior point and the smooth boundary $\partial\mathcal{O}_\pm$. We can take the vector potential $A_\pm(x)$ associated with $b_\pm$ to fulfill

$$A_\pm(x) = \alpha_\pm \Phi(x)$$

over $\Omega_\pm = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_\pm$, where $\Phi(x)$ is defined by (1.3). For $\rho_\pm \in \boldsymbol{R}^2$ with $|\rho_\pm| \gg 1$, we set

$$\Omega_\rho = \boldsymbol{R}^2 \setminus \left(\overline{\mathcal{O}}_{-\rho} \cup \overline{\mathcal{O}}_{+\rho}\right), \qquad \mathcal{O}_{\pm\rho} = \{x : x-\rho_\pm \in \mathcal{O}_\pm\}.$$

and define

$$A_\rho(x) = A_{-\rho}(x) + A_{+\rho}(x) = A_-(x-\rho_-) + A_+(x-\rho_+).$$

Let $V_d(x)$ be defined by (5.1). Then we consider the self–adjoint operator

$$H_{\rho,d} = H(A_\rho, V_d), \qquad \mathcal{D}\left(H_{\rho,d}\right) = H^2(\Omega_\rho) \cap H_0^1(\Omega_\rho), \tag{5.6}$$

in $L^2(\Omega_\rho)$ under the zero Dirichlet boundary conditions. We again denote by $f_\pm(\omega \to \theta; E)$ the scattering amplitude for the pair $(H_0, H_\pm)$.

We first deal with the horizontal case when the four centers $d_\pm$ and $\rho_\pm$ are located as follows :

$$d_- = (-\kappa_- d, 0), \quad \rho_\pm = (\pm\kappa_0 d, 0), \quad d_+ = (\kappa_+ d, 0), \tag{5.7}$$

for $d \gg 1$, where

$$\kappa_\pm > 0, \quad \kappa_- + \kappa_+ = 1, \quad -\kappa_- < -\kappa_0 < 0 < \kappa_0 < \kappa_+.$$

Here the distance $|d| = |d_+ - d_-|$ is identified d and the direction of $d_+ - d_-$ is fixed as $\omega_1 = (1, 0)$. We write $H_{d,\mathrm{hor}}$ for the self–adjoint operator $H_{\rho,d}$ defined by (5.6) with $d_\pm$ and $\rho_\pm$ as above. We define the angle ψ_0 by

$$\cos\psi_0 = \left(\frac{\kappa_- - \kappa_0}{\kappa_- + \kappa_0}\right)^{1/2} \left(\frac{\kappa_+ - \kappa_0}{\kappa_+ + \kappa_0}\right)^{1/2} < 1, \quad 0 < \psi_0 < \pi/2, \tag{5.8}$$

and set

$$\pi_0 = \left(1 - \frac{\psi_0}{\pi}\right)\cos\Big((\alpha_+ + \alpha_-)\pi\Big) + \frac{\psi_0}{\pi}\cos\Big((\alpha_+ - \alpha_-)\pi\Big). \tag{5.9}$$

We further define

$$h_{\mathrm{hor}}(\zeta; d) = \left(\frac{e^{2ikd}}{d}\right) f_-(-\omega_1 \to \omega_1; \zeta) f_+(\omega_1 \to -\omega_1; \zeta)\pi_0^2$$

over D_d, where D_d is the complex neighborhood defined by (5.2). If $\pi_0 \neq 0$ and $f_\pm(\pm\omega_1 \to \mp\omega_1; E_0) \neq 0$, then the resonances in D_d of $H_{d,\mathrm{hor}}$ is approximately determined by the solutions to the equation $h_{\mathrm{hor}}(\zeta; d) = 1$ in the same sense as in Theorem 5.1.

We move to the vertical case when the four centers $d_\pm$ and $\rho_\pm$ are located as follows :

$$d_- = (-\kappa_- d, 0), \quad \rho_\pm = (0, \pm\kappa d^{1/2}), \quad d_+ = (\kappa_+ d, 0), \quad \kappa > 0. \tag{5.10}$$

We write $H_{d,\mathrm{vert}}$ for the self–adjoint operator $H_{\rho,d}$ defined by (5.6) with $d_\pm$ and $\rho_\pm$ as above. We define

$$\pi_\pm(\zeta) = (1 - I_0(\zeta))\cos\Big((\alpha_+ + \alpha_-)\pi\Big) + I_0(\zeta)\exp\Big(\pm i(\alpha_+ - \alpha_-)\pi\Big)$$

over D_d, where

$$I_0(\zeta) = (2/\pi)^{1/2} e^{-i\pi/4} \int_0^\tau e^{it^2/2}\, dt, \quad \tau = \tau(\zeta) = \kappa\left(\frac{1}{\kappa_-} + \frac{1}{\kappa_+}\right)^{1/2} \zeta^{1/4},$$

and the contour is taken to be the segment from 0 to τ. We further define

$$h_{\mathrm{vert}}(\zeta; d) = \left(\frac{e^{2ikd}}{d}\right) f_-(-\omega_1 \to \omega_1; \zeta) f_+(\omega_1 \to -\omega_1; \zeta)\pi_+(\zeta)\pi_-(\zeta).$$

If $f_\pm(\pm\omega_1 \to \mp\omega_1; E_0) \neq 0$ and $\pi_\pm(E_0) \neq 0$, then the resonances in D_d of $H_{d,\mathrm{vert}}$ is approximately determined by the solutions to the equation $h_{\mathrm{vert}}(\zeta; d) = 1$ in the same sense as in Theorem 5.1. The critical case is that the width $\rho = |\rho_+ - \rho_-|$ is comparable to $d^{1/2}$. If, for example, we consider a system with the total flux vanishing ($\alpha_+ + \alpha_- = E0$), then the AB effect is not observed when $\rho \ll d^{1/2}$ or $d^{1/2} \ll \rho \ll d$.

6. An application to compositions

In this section we consider the composition of two resolvent kernels as an application of Theorems 1.1 and 1.2. The obtained result is used in analyzing the behaviors along forward directions of the resolvent kernel of the magnetic Schrödinger operator $H_{d,\mathrm{hor}}$ in the horizontal case above.

We set up the problem precisely. Let the four centers $d_\pm$ and $\rho_\pm = (\pm\kappa_0 d, 0)$ be as in (5.7), and let

$$\Omega_{\pm\rho} = \boldsymbol{R}^2 \setminus \overline{\mathcal{O}}_{\pm\rho}, \quad \mathcal{O}_{\pm\rho} = \{x : x - \rho_\pm \in \mathcal{O}_\pm\},$$

for $\mathcal{O}_\pm$ as in (5.5). We consider the magnetic Schrödinger operator

$$H_{\pm\rho} = H(A_{\pm\rho}), \quad \mathcal{D}(H_{\pm\rho}) = H^2(\Omega_{\pm\rho}) \cap H_0^1(\Omega_{\pm\rho}),$$

over $\Omega_{\pm\rho}$ under the zero Dirichlet boundary conditions and denote by

$$R(\zeta; H_{\pm\rho})(x, y), \quad (x, y) \in \Omega_{\pm\rho} \times \Omega_{\pm\rho},$$

the resolvent kernel of $H_{\pm\rho}$ for $\zeta \in D_0$, where $A_{\pm\rho}(x) = \alpha_\pm \Phi(x - \rho_\pm)$ and D_0 is defined by (1.5) with

$$\lambda = |\rho_+ - \rho_-| = 2\kappa_0 d \sim d.$$

We take a nonnegative function $\varphi_1 \in C^\infty(\boldsymbol{R})$ such that

$$\varphi_1 = 1 \ \text{on} \ (-\infty, -\kappa_0 d/2], \quad \varphi_1 = 0 \ \text{on} \ [\kappa_0 d/2, \infty), \quad \varphi_1^{(n)} = O(d^{-n}),$$

and we set

$$\varphi_2(z_2) = \chi\left(|z_2|/d^{1-\mu}\right)$$

for the cut–off function $\chi \in C_0^\infty[0, \infty)$ with properties in (3.1). Then we define the integral $G_d(x, y; \zeta)$ by

$$G_d = 2\int R(\zeta; H_{+\rho})(x, z)(\partial_1\varphi_1)(z_1)\varphi_2(z_2)\partial_1 R(\zeta; H_{-\rho})(z, y)\, dz, \tag{6.1}$$

where $z = (z_1, z_2)$ and $\partial_1 R(\zeta; H_{-\rho})(z, y) = \partial R(\zeta; H_{-\rho})(z, y)/\partial z_1$. Our aim here is to analyze the behavior as $d \to \infty$ of the integral $G_d(x, y; \zeta)$. The obtained result is formulated as the following proposition.

Proposition 6.1. *Let the notation be as above and let*

$$B_{\pm d} = \left\{ |x - d_\pm| < 1 \right\}$$

for $d_\pm$ as in (5.7). Define the angle ψ_0, $0 < \psi_0 < \pi/2$, through (5.8) and π_0 by (5.9). If $(x, y) \in B_{+d} \times B_{-d}$, then the integral $G_d = G_d(x, y; \zeta)$ defined by (6.1) behaves like

$$G_d(x, y; \zeta) = \left(\frac{e^{ik|x_1 - y_1|}}{|x_1 - y_1|^{1/2}} \right) \left(c_0(\zeta)\pi_0 + O\left(d^{-(1/2-\mu)} \right) \right)$$

uniformly in x, y and $\zeta \in D_0$, where $c_0(\zeta)$ is defined by (1.10).

Remark 6.1. If we define $G_d(x, y; \zeta)$ by

$$G_d = -2 \int R(\zeta; H_{-\rho})(x, z)(\partial_1 \varphi_1)(z_1)\varphi_2(z_2)\partial_1 R(\zeta; H_{+\rho})(z, y)\, dz$$

for $(x, y) \in B_{-d} \times B_{+d}$, then the same asymptotic formula as above is obtained.

The proposition above shows how the constant π_0 appears in the asymptotic formula obtained from the composition of the two resolvent kernels. Before going to the proof, we first discuss the two particular cases of Corollary 1.1 in the way adapted to the proof of the proposition. In what follows $R(\zeta; H)(x, y)$ denotes the resolvent kernel of H in Corollary 1.1.

Case 1. Assume that $x = (x_1, x_2)$ and $y = (y_1, y_2)$ fulfill

$$\lambda/c < -y_1, x_1 < c\lambda, \quad |y_2| = O(1), \quad \lambda^\mu/c < |x_2| < c\lambda^{1-\mu} \tag{6.2}$$

for some $c > 1$. Then $\pm\sigma(x, y) > 0$ according as $\pm x_2 > 0$, and we have

$$|x - y| = |x_1 - y_1| + x_2^2/(2|x_1 - y_1|) + O(\lambda^{-\mu}),$$
$$|y| = |y_1| \left(1 + O(\lambda^{-2})\right), \quad |x| = |x_1| \left(1 + O(\lambda^{-2\mu})\right).$$

Hence it follows that $|x| + |y| = |x_1 - y_1| \left(1 + O(\lambda^{-2\mu})\right)$. We further have

$$\sigma(x, y) = \left(x_2/|x_1| \right) \left(1 + O(\lambda^{-2\mu})\right).$$

Let $z_0(x, y; \zeta)$ be defined by (1.9). If we define

$$z_-(x, y; \zeta) = \left(|y_1|/(|x_1||x_1 - y_1|) \right)^{1/2} |x_2| \zeta^{1/4}, \tag{6.3}$$

then it follows that $z_0 = z_- \left(1 + O(\lambda^{-2\mu})\right)$. Under (6.2), $R(\zeta; H)(x, y)$ behaves like

$$R(\zeta; H)(x, y) = c_0(\zeta) \frac{e^{ik|x_1 - y_1|}}{|x_1 - y_1|^{1/2}} \exp\left(\frac{ikx_2^2}{2|x_1 - y_1|}\right) \times \left(\cos(\alpha\pi) \mp (2/\pi)^{1/2} e^{i\pi/4} \sin(\alpha\pi) \int_0^{z_-} e^{is^2/2}\, ds\right) + e^{ik|x_1 - y_1|} |x_1 - y_1|^{-1/2} O\left(\lambda^{-\mu}\right) \tag{6.4}$$

according as $\pm x_2 > 0$.

Case 2. Assume that $x = (x_1, x_2)$ and $y = (y_1, y_2)$ fulfill

$$\lambda/c < -y_1, x_1 < c\lambda, \quad |x_2| = O(1), \quad \lambda^\mu/c < |y_2| < c\lambda^{1-\mu} \tag{6.5}$$

for some $c > 1$, and hence $\pm\sigma(x, y) > 0$ according as $\pm y_2 > 0$. If we define

$$z_+(x, y; \zeta) = \Big(|x_1|/(|y_1||x_1 - y_1|)\Big)^{1/2} |y_2| \zeta^{1/4}, \tag{6.6}$$

then it follows that $z_0 = z_+ \left(1 + O(\lambda^{-2\mu})\right)$. Under (6.5), $R(\zeta; H)(x, y)$ behaves like

$$R(\zeta; H)(x, y) = c_0(\zeta) \frac{e^{ik|x_1 - y_1|}}{|x_1 - y_1|^{1/2}} \exp\left(\frac{iky_2^2}{2|x_1 - y_1|}\right) \times \left(\cos(\alpha\pi) \mp (2/\pi)^{1/2} e^{i\pi/4} \sin(\alpha\pi) \int_0^{z_+} e^{is^2/2}\, ds\right) + e^{ik|x_1 - y_1|} |x_1 - y_1|^{-1/2} O\left(\lambda^{-\mu}\right) \tag{6.7}$$

according as $\pm y_2 > 0$.

Proof of Proposition 6.1. To prove the proposition, we use Theorem 1.1 and Corollary 1.1 with $\lambda = |\rho_+ - \rho_-| = 2\kappa_0 d$. We deal with the case $(x, y) \in B_{+d} \times B_{-d}$ only. A similar argument applies to the other case $(x, y) \in B_{-d} \times B_{+d}$ also. Note that $|z_1| < \lambda/4 = \kappa_0 d/2$ for $z_1 \in \operatorname{supp} \partial_1 \varphi_1$, and hence $y_1 < z_1 < x_1$. The proof is long and is divided into several steps.

(1) We first summarize the asymptotic properties of $\partial_1 R(\zeta; H_{-\rho})(z, y)$ and $R(\zeta; H_{+\rho})(x, z)$. Among the four properties mentioned below, the first and second ones are immediate consequences of Theorem 1.1, and the third and fourth ones follow from Cases 1 and 2 discussed above.

- If $|y_2| = O(1)$ and $|z_2| = O(d^\mu)$, then

$$\partial_1 R(\zeta; H_{-\rho})(z, y) = \frac{e^{ik|z_1 - y_1|}}{|z_1 - y_1|^{1/2}} \left(ikc_0(\zeta) \cos(\alpha_-\pi) + O(d^{-(1/2-\mu)})\right).$$

- If $|x_2| = O(1)$ and $|z_2| = O(d^\mu)$, then

$$R(\zeta; H_{+\rho})(x,z) = \frac{e^{ik|x_1-z_1|}}{|x_1-z_1|^{1/2}}\left(c_0(\zeta)\cos(\alpha_+\pi) + O(d^{-(1/2-\mu)})\right).$$

- If $y_2 = O(1)$ and $d^\mu/c \le |z_2| \le cd^{1-\mu}$, then

$$\begin{aligned}\partial_1 R(\zeta; H_{-\rho})(z,y) &= ikc_0(\zeta)\frac{e^{ik|z_1-y_1|}}{|z_1-y_1|^{1/2}}\exp\left(\frac{ikz_2^2}{2|z_1-y_1|}\right)\times \\ &\quad \left(\cos(\alpha_-\pi) \mp (2/\pi)^{1/2}\, e^{i\pi/4}\sin(\alpha_-\pi)\int_0^{u_-} e^{is^2/2}\,ds\right) \\ &\quad +\ e^{ik|z_1-y_1|}|z_1-y_1|^{-1/2}O\left(d^{-\mu}\right)\end{aligned}$$

according as $\pm z_2 > 0$, where $u_- = u_-(z,y;\zeta)$ is defined by

$$u_- = z_-(z-\rho_-, y-\rho_-;\zeta) = \left(\frac{|y_1+\kappa_0 d|}{|z_1-y_1||z_1+\kappa_0 d|}\right)^{1/2}|z_2|\zeta^{1/4}. \tag{6.8}$$

- If $x_2 = O(1)$ and $d^\mu/c \le |z_2| \le cd^{1-\mu}$, then

$$\begin{aligned}R(\zeta; H_{+\rho})(x,z) &= c_0(\zeta)\frac{e^{ik|x_1-z_1|}}{|x_1-z_1|^{1/2}}\exp\left(\frac{ikz_2^2}{2|x_1-z_1|}\right)\times \\ &\quad \left(\cos(\alpha_+\pi) \mp (2/\pi)^{1/2}e^{i\pi/4}\sin(\alpha_+\pi)\int_0^{u_+} e^{it^2/2}\,dt\right) \\ &\quad +\ e^{ik|x_1-z_1|}|x_1-z_1|^{-1/2}O\left(d^{-\mu}\right)\end{aligned}$$

according as $\pm z_2 > 0$, where $u_+ = u_+(x,z;\zeta)$ is defined by

$$u_+ = z_+(x-\rho_+, z-\rho_+;\zeta) = \left(\frac{|x_1-\kappa_0 d|}{|x_1-z_1||z_1-\kappa_0 d|}\right)^{1/2}|z_2|\zeta^{1/4}. \tag{6.9}$$

The asymptotic formulas above allow us to decompose $\partial_1 R(\zeta; H_{-\rho})(z,y)$ into the sum of the three terms

$$\partial_1 R(\zeta; H_{-\rho})(z,y) = G^-_{\cos}(z,y;\zeta) \mp G^-_{\sin}(z,y;\zeta) + G^-_{\pm\mathrm{rem}}(z,y;\zeta)$$

according as $d^\mu/c \le \pm z_2 \le cd^{1-\mu}$, where

$$\begin{aligned}G^-_{\cos} &= ikc_0(\zeta)\cos(\alpha_-\pi)\frac{e^{ik|z_1-y_1|}}{|z_1-y_1|^{1/2}}\exp\left(\frac{ikz_2^2}{2|z_1-y_1|}\right) \\ G^-_{\sin} &= ikc_0(\zeta)(2/\pi)^{1/2}e^{i\pi/4}\sin(\alpha_-\pi)\times \\ &\qquad \frac{e^{ik|z_1-y_1|}}{|z_1-y_1|^{1/2}}\times\exp\left(\frac{ikz_2^2}{2|z_1-y_1|}\right)\int_0^{u_-} e^{is^2/2}\,ds,\end{aligned}$$

and the remainder term $G^-_{\pm\mathrm{rem}}(z,y;\zeta)$ obeys

$$G^-_{\pm\mathrm{rem}} = e^{ik|x_1-z_1|}|z_1-y_1|^{-1/2}O\left(d^{-\mu}\right).$$

The kernel $R(\zeta; H_{+\rho})(x,z)$ also admits a similar decomposition

$$R(\zeta; H_{+\rho})(x,z) = G^+_{\cos}(x,z;\zeta) \mp G^+_{\sin}(x,z;\zeta) + G^+_{\pm\mathrm{rem}}(x,z;\zeta)$$

with natural modifications. By definition, we have the following relations

$$\begin{aligned} G^-_{\cos}(z_1,-z_2,y;\zeta) &= G^-_{\cos}(z_1,z_2,y;\zeta) \\ G^-_{\sin}(z_1,-z_2,y;\zeta) &= G^-_{\sin}(z_1,z_2,y;\zeta) \end{aligned} \tag{6.10}$$

and similarly for $G^+_{\cos}(x,z;\zeta)$ and $G^+_{\sin}(x,z;\zeta)$.

(2) We set $\tilde{\varphi}_2(z_2) = \chi\left(|z_2|/d^\mu\right)$. Since $\mu < 1-\mu$, we have $\tilde{\varphi}_2\varphi_2 = \tilde{\varphi}_2$ for $d \gg 1$, so that $\varphi_2(z_2)$ admits the decomposition

$$\varphi_2(z_2) = \tilde{\varphi}_2(z_2) + \left(1-\tilde{\varphi}_2(z_2)\right)\varphi_2(z_2) = \tilde{\varphi}_2(z_2) + \varphi_3(z_2).$$

If we note that $e^{ik|x_1-z_1|}e^{ik|z_1-y_1|} = e^{ik|x_1-y_1|}$, then it is easy to see that

$$\begin{aligned} &\int\!\!\int R(\zeta; H_{+\rho})(x,z)(\partial_1\varphi_1)(z_1)\tilde{\varphi}_2(z_2)\partial_1 R(\zeta; H_{-\rho})(z,y)\,dz \\ &\quad = e^{ik|x_1-y_1|}O(d^{-1+\mu}) = \left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right) O\left(d^{-(1/2-\mu)}\right). \end{aligned}$$

Since $1/2 - 2\mu \le -1/2+\mu$, we can also easily see that

$$\begin{aligned} &\int\!\!\int G^+_{\pm\mathrm{rem}}(x,z;\zeta)(\partial_1\varphi_1)(z_1)\varphi_3(z_2)\partial_1 R(\zeta; H_{-\rho})(z,y)\,dz \\ &\quad = e^{ik|x_1-y_1|}O\left(d^{-2\mu}\right) = \left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right) O\left(d^{-(1/2-\mu)}\right) \end{aligned}$$

and similarly for the integral

$$\int\!\!\int R(\zeta; H_{+\rho})(x,z)(\partial_1\varphi_1)(z_1)\varphi_3(z_2)G^-_{\pm\mathrm{rem}}(z,y;\zeta)\,dz.$$

If we take (6.10) into account, then it follows that

$$\begin{aligned} &\int\!\!\int G^+_{\cos}(x,z;\zeta)(\partial_1\varphi_1)(z_1)\varphi_3(z_2)\big(\operatorname{sgn} z_2\big)G^-_{\sin}(z,y;\zeta)\,dz = 0, \\ &\int\!\!\int \big(\operatorname{sgn} z_2\big)G^+_{\sin}(x,z;\zeta)(\partial_1\varphi_1)(z_1)\varphi_3(z_2)G^-_{\cos}(z,y;\zeta)\,dz = 0. \end{aligned}$$

Thus the leading term comes from the two integrals $I_{\mathrm{cc}}(x,y;\zeta)$ and $I_{\mathrm{ss}}(x,y;\zeta)$ defined by

$$\begin{aligned} I_{\mathrm{cc}} &= 2\int\!\!\int G^+_{\cos}(x,z;\zeta)(\partial_1\varphi_1)(z_1)\varphi_3(z_2)G^-_{\cos}(z,y;\zeta)\,dz, \\ I_{\mathrm{ss}} &= 2\int\!\!\int G^+_{\sin}(x,z;\zeta)(\partial_1\varphi_1)(z_1)\varphi_3(z_2)G^-_{\sin}(z,y;\zeta)\,dz. \end{aligned}$$

(3) We analyze the behavior as $d \to \infty$ of the integral $I_{\mathrm{cc}}(x, y; \zeta)$ defined above. We compute

$$\int \varphi_3(z_2) \exp\left(\frac{ikz_2^2}{2|z_1 - y_1|}\right) \exp\left(\frac{ikz_2^2}{2|x_1 - z_1|}\right) dz_2$$
$$= \int (\varphi_2(z_2) - \tilde{\varphi}_2(z_2)) \exp\left(\frac{ikz_2^2|x_1 - y_1|}{2|x_1 - z_1||z_1 - y_1|}\right) dz_2.$$

We make a change of variable $z_2 \mapsto d^{1-\mu}t$ and the method of steepest descent to the first integral with φ_2 on the right side. If we note that

$$\left(\frac{|x_1 - y_1|}{|x_1 - z_1||z_1 - y_1|}\right) z_2^2 \sim d^{1-2\mu}t^2,$$

then we see that the integral behaves like

$$(2\pi/k)^{1/2} e^{i\pi/4} \Big(|x_1 - z_1||z_1 - y_1|/|x_1 - y_1|\Big)^{1/2} \Big(1 + O(d^{-(1-2\mu)})\Big).$$

On the other hand, the second integral with $\tilde{\varphi}_2$ is easily seen to obey the bound $O(d^\mu)$. Since $\int (\partial_1\varphi_1)(z_1)\, dz_1 = -1$ and since

$$-2ik(2\pi)^{1/2}k^{-1/2}e^{i\pi/4} = (8\pi)^{1/2}e^{-i\pi/4}k^{1/2} = 1/c_0(\zeta),$$

we obtain the asymptotic form

$$I_{\mathrm{cc}} = c_0(\zeta)\cos(\alpha_+\pi)\cos(\alpha_-\pi)\left(\frac{e^{ik|x_1-y_1|}}{|x_1 - y_1|^{1/2}}\right)\Big(1 + O(d^{-(1/2-\mu)})\Big) \tag{6.11}$$

for the integral $I_{\mathrm{cc}}(x, y; \zeta)$.

(4) We analyze the behavior of $I_{\mathrm{ss}}(x, y; \zeta)$. As in the previous step, we first consider the integral in the variable z_2. We again decompose $\varphi_3 = \varphi_2 - \tilde{\varphi}_2$ and define the integral $I = I(x, z_1, y; \zeta)$ as

$$I = \int \varphi_2(z_2) \exp\left(\frac{ikz_2^2|x_1 - y_1|}{2|x_1 - z_1||z_1 - y_1|}\right) \times \left[\int_0^{u_-} e^{is^2/2}\, ds \int_0^{u_+} e^{it^2/2}\, dt\right] dz_2, \tag{6.12}$$

where $u_- = u_-(z, y; \zeta)$ and $u_+ = u_+(x, z; \zeta)$ are defined by (6.8) and (6.9), respectively. The integral associated with $\tilde{\varphi}_2$ is easily seen to obey $O(d^\mu)$. We consider the integral in the brackets. We set

$$v_-(z, y; \zeta) = \operatorname{Re} u_-(z, y; \zeta) > 0, \quad v_+(x, z; \zeta) = \operatorname{Re} u_+(x, z; \zeta) > 0.$$

Then $v_\pm = O\left(d^{1/2-\mu}\right)$ and

$$|\operatorname{Im} u_-(z, y; \zeta)| + |\operatorname{Im} u_+(x, z; \zeta)| = O\left(d^{-1/2-\mu}\log d\right)$$

for $z_2 \in \operatorname{supp} \varphi_2$ (and hence $|z_2| \le 2d^{1-\mu}$). Hence

$$\int_0^{u_-} e^{is^2/2}\, ds \int_0^{u_+} e^{it^2/2}\, dt = \int_0^{v_-} e^{is^2/2}\, ds \int_0^{v_+} e^{it^2/2}\, dt + O\left(d^{-1/2-\mu}\log d\right).$$

Thus we have

$$I = \int \varphi_2(z_2) \exp\left(\frac{ikz_2^2|x_1-y_1|}{2|x_1-z_1||z_1-y_1|}\right) \times \left[\int_0^{v_-} e^{is^2/2}\, ds \int_0^{v_+} e^{it^2/2}\, dt\right] dz_2 + O\left(d^{\mu}\right).$$

We consider the integral in the brackets in the polar coordinate system. We define the angles $\theta_\pm(x, z_1, y)$ through the relations

$$\text{(6.13)} \qquad \cos\theta_\pm = v_\pm \left(v_+^2 + v_-^2\right)^{-1/2}, \quad 0 < \theta_\pm < \pi/2, \quad \theta_+ + \theta_- = \pi/2,$$

and the functions $p_-(\theta; z_1, y)$ and $p_+(\theta; x, z_1)$ by

$$p_\pm(\theta) = v_\pm / \cos\theta, \quad 0 < \theta < \pi/2.$$

Then the integral in question is calculated as follows :

$$\begin{aligned} &\int_0^{v_-} e^{is^2/2}\, ds \times \int_0^{v_+} e^{it^2/2}\, dt \\ &= \int_0^{\theta_-} \left(\int_0^{p_-(\theta)} e^{ir^2/2} r\, dr\right) d\theta + \int_0^{\theta_+} \left(\int_0^{p_+(\theta)} e^{ir^2/2} r\, dr\right) d\theta \\ &= -i \int_0^{\theta_-} \left(e^{ip_-(\theta)^2/2} - 1\right) d\theta - i \int_0^{\theta_+} \left(e^{ip_+(\theta)^2/2} - 1\right) d\theta \\ &= -i \int_0^{\theta_-} e^{ip_-(\theta)^2/2}\, d\theta - i \int_0^{\theta_+} e^{ip_+(\theta)^2/2}\, d\theta + i\pi/2. \end{aligned}$$

(5) We now set

$$\begin{aligned} w_-(z_1, y) &= \left(\frac{|y_1 + \kappa_0 d|}{|z_1 - y_1||z_1 + \kappa_0 d|}\right)^{1/2}, \\ w_+(x, z_1) &= \left(\frac{|x_1 - \kappa_0 d|}{|x_1 - z_1||z_1 - \kappa_0 d|}\right)^{1/2}, \\ w(x, z_1, y) &= \left(\frac{|x_1 - y_1|}{|x_1 - z_1||z_1 - y_1|}\right)^{1/2}. \end{aligned}$$

We further define

$$I_0(x, z_1, y; \zeta) = (i\pi/2) \int \varphi_2(z_2) \exp\left(ikz_2^2 w^2/2\right) dz_2,$$

$$I_\pm(x, z_1, y; \zeta) \quad = \quad \int_0^{\theta_\pm} J_\pm(x, z_1, y, \theta; \zeta)\, d\theta,$$

where

$$J_\pm = -i \int \varphi_2(z_2) \exp\left((ikz_2^2/2)\left(w^2 + w_\pm^2/\cos^2\theta\right)\right) dz_2.$$

By (6.8) and (6.9), $w_\mp$ are related to $v_\mp = \operatorname{Re} u_\mp$ through the relation

$$v_\mp = w_\mp |z_2| \left(\operatorname{Re} k^{1/2}\right) \tag{6.14}$$

and the integral I admits the decomposition

$$I(x, z_1, y; \zeta) = I_0(x, z_1, y; \zeta) + I_+(x, z_1, y; \zeta) + I_-(x, z_1, y; \zeta) + O\left(d^\mu\right).$$

After making a change of variable $z_2 \mapsto d^{1-\mu} t$, we apply the method of steepest descent to the integrals I_0 and $J_\pm$. Then we have

$$I_0 = (2\pi)^{1/2} k^{-1/2} e^{-i\pi/4} (-\pi/2) w^{-1} \left(1 + O(d^{-(1-2\mu)})\right)$$

$$J_\pm = (2\pi)^{1/2} k^{-1/2} e^{-i\pi/4} \left(w^2 + w_\pm^2/\cos^2\theta\right)^{-1/2} \left(1 + O(d^{-(1-2\mu)})\right)$$

uniformly in θ, $0 < \theta < \theta_\pm$. We calculate the integral

$$J_{\pm 0}(x, z_1, y) = \int_0^{\theta_\pm} \left(w^2 + w_\pm^2/\cos^2\theta\right)^{-1/2} d\theta.$$

The integrand equals

$$\left(w^2 + w_\pm^2/\cos^2\theta\right)^{-1/2} = w^{-1} \left((w_\pm^2/w^2 + 1) - \sin^2\theta\right)^{-1/2} \cos\theta$$

and hence

$$\begin{aligned} J_{\pm 0} &= w^{-1} \int_0^{\theta_\pm} \left((w_\pm^2/w^2 + 1) - \sin^2\theta\right)^{-1/2} \cos\theta\, d\theta \\ &= w^{-1} \int_0^{s_\pm} \left((w_\pm^2/w^2 + 1) - s^2\right)^{-1/2} ds \\ &= w^{-1} \arcsin\left(s_\pm (w_\pm^2/w^2 + 1)^{-1/2}\right) \end{aligned}$$

with $s_\pm = \sin\theta_\pm$. We combine these results to obtain that the integral $I(x, z_1, y; \zeta)$ under consideration behaves like

$$I = (2\pi)^{1/2} k^{-1/2} e^{-i\pi/4} w^{-1} \left((-\pi/2 + \psi_- + \psi_+) + O(d^{-(1-2\mu)})\right) + O\left(d^\mu\right),$$

where

$$\psi_\pm = \arcsin\left(s_\pm (w_\pm^2/w^2 + 1)^{-1/2}\right). \tag{6.15}$$

We assert that

$$\psi_+ + \psi_- = \psi_0 + O\left(d^{-1}\right) \tag{6.16}$$

for ψ_0 in the proposition. We proceed with the arguments, accepting this relation as proved. Then we have

$$I = (2\pi)^{1/2}k^{-1/2}e^{-i\pi/4}\left(\frac{|x_1-z_1||z_1-y_1|}{|x_1-y_1|}\right)^{1/2}\times \\ \left((-\pi/2+\psi_0)+O(d^{-(1/2-\mu)})\right).$$

(6) We are now in a position to see the asymptotic behavior of the integral $I_{\mathrm{ss}} = I_{\mathrm{ss}}(x,y;\zeta)$ in question. The integral takes the form

$$\begin{aligned} I_{\mathrm{ss}} &= 2ik\,\sin(\alpha_+\pi)\sin(\alpha_-\pi)c_0^2(\zeta)e^{i\pi/2}(2/\pi)e^{ik|x_1-y_1|}\times \\ &\quad \int(|x_1-z_1||z_1-y_1|)^{-1/2}(\partial_1\varphi_1)(z_1)\left(I(x,z_1,y;\zeta)+O(d^{\mu})\right)\,dz_1 \end{aligned}$$

and behaves like

$$I_{\mathrm{ss}} = C_{ss}\sin(\alpha_+\pi)\sin(\alpha_-\pi)c_0^2(\zeta)\left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right)\left(1+O(d^{-(1/2-\mu)})\right)$$

where

$$C_{ss} = -2ik(2\pi)^{1/2}k^{-1/2}e^{i\pi/4}\left(-1+2\psi_0/\pi\right) = \left(1/c_0(\zeta)\right)\left(-1+2\psi_0/\pi\right).$$

Hence we have

$$I_{\mathrm{ss}} = c_0(\zeta)\left(-1+2\psi_0/\pi\right)\sin(\alpha_+\pi)\sin(\alpha_-\pi)\times \\ \left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right)\left(1+O(d^{-(1/2-\mu)})\right).$$

We combine this relation with (6.11). A simple computation yields

$$\cos(\alpha_+\pi)\cos(\alpha_-\pi)+(-1+2\psi_0/\pi)\sin(\alpha_+\pi)\sin(\alpha_-\pi)=\pi_0$$

for π_0 in the proposition. Thus we get the desired asymptotic form of $G_d(x,y;\zeta)$.

(7) It remains to prove (6.16). From (6.13) and (6.14), we recall that $v_\pm = w_\pm|z_2|\left(\mathrm{Re}\,k^{1/2}\right)$ and

$$\sin\theta_\mp = \cos\theta_\pm = w_\pm\left(w_+^2+w_-^2\right)^{-1/2}.$$

We also recall the definition of $\psi_\pm$ from (6.15). If we make use of the relation $s_\pm = \sin\theta_\pm$, then

$$\sin\psi_\pm = \frac{s_\pm w}{\left(w^2+w_\pm^2\right)^{1/2}} = \frac{w_\mp w}{\left(w_+^2+w_-^2\right)^{1/2}\left(w^2+w_\pm^2\right)^{1/2}}$$

and hence

$$\cos\psi_\pm = \left(\frac{w_\pm^2\left(w^2+w_+^2+w_-^2\right)}{\left(w_+^2+w_-^2\right)\left(w^2+w_\pm^2\right)} \right)^{1/2}.$$

A direct computation shows

$$\cos(\psi_+ + \psi_-) = \cos\psi_+\cos\psi_- - \sin\psi_+\sin\psi_- = \frac{w_+w_-}{\left(w^2+w_+^2\right)^{1/2}\left(w^2+w_-^2\right)^{1/2}}.$$

By definition,

$$w_+w_- = \left(\frac{|x_1-\kappa_0 d||y_1+\kappa_0 d|}{|x_1-z_1||z_1-\kappa_0 d||z_1-y_1||z_1+\kappa_0 d|} \right)^{1/2}$$

and

$$w^2+w_+^2 = \frac{|y_1-\kappa_0 d|}{|z_1-y_1||z_1-\kappa_0 d|}, \quad w^2+w_-^2 = \frac{|x_1+\kappa_0 d|}{|x_1-z_1||z_1+\kappa_0 d|}.$$

Hence it follows from (5.8) that

$$\cos\left(\psi_+ + \psi_-\right) = \left(\frac{|y_1+\kappa_0 d||x_1-\kappa_0 d|}{|y_1-\kappa_0 d||x_1+\kappa_0 d|} \right)^{1/2} = \cos\left(\psi_0 + O(d^{-1})\right),$$

because $(x,y) \in B_{+d} \times B_{-d}$. Thus (6.16) is established and the proof of the proposition is now complete. □

The next proposition is more easily verified than Proposition 6.1. We skip a proof.

Proposition 6.2. *Let the notation be as in Proposition 6.1. Assume that* $\tilde\varphi_1 \in C_0^\infty(\boldsymbol{R}^1)$ *has support in* $|z_1| < \kappa_0 d/2$ *and obeys* $|\tilde\varphi_1| = O(d^{-1-\nu})$ *for some* $\nu > 0$ *and that* $\varphi_2(z_2)$ *preserves the same properties. Define the integral*

$$\tilde G_d(x,y;\zeta) = \int R(\zeta; H_{+\rho})(x,z)\tilde\varphi_1(z_1)\varphi_2(z_2)R(\zeta;H_{-\rho})(z,y)\,dz$$

for $(x,y) \in B_{\pm d} \times B_{\mp d}$. *Then one has*

$$\tilde G_d(x,y;\zeta) = e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}O\left(d^{-\nu}\right)$$

uniformly in $x,\ y$ *and* $\zeta \in D_0$.

We end the paper by discussing the other special cases of Corollary 1.1. These results are used in the resonance problem for $H_{d,\mathrm{vert}}$ in the vertical case. Let $R(\zeta; H_{\pm\rho})(x,y)$ be the resolvent kernel of $H_{\pm\rho}$ defined with $\rho_\pm = (0, \pm\kappa d^{1/2})$ as in (5.10). Define D_0 by (1.5) with $\lambda = d$, and set

$$\sigma_\pm(x,y) = \sigma(x-\rho_\pm, y-\rho_\pm)$$

for $\sigma(x,y)$ defined by (1.6). If $(x,y) \in B_{+d} \times B_{-d}$, then $\sigma_-(x,y) > 0$ and

$$\sigma_-(x,y) = \kappa\,(1/\kappa_- + 1/\kappa_+)\,d^{-1/2} + O(d^{-1}).$$

On the other hand, if $(x,y) \in B_{-d} \times B_{+d}$, then $\sigma_-(x,y) < 0$ and

$$\sigma_-(x,y) = -\kappa\,(1/\kappa_- + 1/\kappa_+)\,d^{-1/2} + O(d^{-1}).$$

By (1.9), we see that $z_0(x-\rho_-, y-\rho_-;\zeta)$ behaves like

$$z_0(x-\rho_-, y-\rho_-;\zeta) = \tau(\zeta) + O(d^{-1/2}),$$

for $(x,y) \in B_{\pm d} \times B_{\mp d}$, where

$$\tau(\zeta) = \kappa\Big(1/\kappa_- + 1/\kappa_+\Big)^{1/2}\zeta^{1/4}. \tag{6.17}$$

Similarly we have

$$z_0(x-\rho_+, y-\rho_+;\zeta) = \tau(\zeta) + O(d^{-1/2}).$$

As a consequence of Corollary 1.1, we can obtain the following proposition.

Proposition 6.3. *Let the notation be as above and let $\rho_\pm = (0, \pm\kappa d^{1/2})$. Assume that $\zeta \in D_0$. Then $R(\zeta; H_{\pm\rho})(x,y)$ takes the following asymptotic form:*

$$R(\zeta; H_{-\rho})(x,y) =$$
$$c_0(\zeta)\left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right)\left(\cos(\alpha_-\pi) \mp (2/\pi)^{1/2}\,e^{i\pi/4}\sin(\alpha_-\pi)\int_0^\tau e^{is^2/2}\,ds\right)$$
$$+\ e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}O\left(d^{-\mu}\right)$$

for $(x,y) \in B_{\pm d} \times B_{\mp d}$, and

$$R(\zeta; H_{+\rho})(x,y) =$$
$$c_0(\zeta)\left(\frac{e^{ik|x_1-y_1|}}{|x_1-y_1|^{1/2}}\right)\left(\cos(\alpha_+\pi) \pm (2/\pi)^{1/2}\,e^{i\pi/4}\sin(\alpha_+\pi)\int_0^\tau e^{is^2/2}\,ds\right)$$
$$+\ e^{ik|x_1-y_1|}|x_1-y_1|^{-1/2}O\left(d^{-\mu}\right)$$

for $(x,y) \in B_{\pm d} \times B_{\mp d}$, where $\tau = \tau(\zeta)$ is defined by (6.17).

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