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SUPPLEMENTED MORPHISMS

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ABSTRACT. In the present paper, left R-modules M and N are studied under the assumptions that $\mathrm{Hom}_R(M,N)$ is supplemented. It is shown that $\mathrm{Hom}(M,N)$ is ($\oplus$, $\mathcal{G}^*$, amply)-supplemented if and only if N is ($\oplus$, $\mathcal{G}^*$, amply)-supplemented. Some applications to cosemisimple modules, refinable modules and UCC-modules are presented. Finally, the relationship between the Jacobson radical $J[M,N]$ of $\mathrm{Hom}_R(M,N)$ and $\mathrm{Hom}_R(M,N)$ is supplemented are investigated. Let M be a finitely generated, self-projective left R-module and $N \in Gen(M)$. We show that if $\mathrm{Hom}(M,N)$ is supplemented and N has GD2 then $\mathrm{Hom}(M,N)/J(M,N)$ is semisimple as a left E_M-module.

1. Introduction

Throughout this article, all rings are associative with unity, and all modules are unital left modules. Let R be a ring. If $_RM$ and $_RN$ are modules, we use the following notations: $E_M = \mathrm{End}(M_R)$. If $N \subseteq M$, then $N \leq M$, $N \ll M$, $N \leq_d M$ and $Rad(M)$ denote N is a submodule of M, N is a small submodule of M, N is a direct summand of M and the Jacobson radical of M, respectively.

We recall the fundamental terminology for our paper. Let U be a submodule of an R-module M. A submodule V of M is called *supplement* of U in M if V is a minimal element in the set of submodules L of M with $U + L = M$. V is a supplement of U if and only if $U + V = M$ and $U \cap V$ is small in V. An R-module M is *supplemented* if every submodule of M has a supplement in M. The module M is *amply supplemented* if, for any submodules A and B of M with $M = A + B$, there exists a supplement P of A such that $P \leq B$.

For the other definitions in this note, we refer to [1], [14] and [17].

In the present paper, we establish an order-preserving bijective correspondence between the sets of coclosed left R-submodules of N and coclosed left E_M-submodules of $\mathrm{Hom}_R(M,N)$. This concept is extremely useful in analyzing the structure of the endomorphism ring of a supplemented module. For instance, by definitions of supplemented modules, one easily checks that there is no any direct implication between the notions supplemented modules and when $\mathrm{Hom}_R(M,N)$ is supplemented. But we prove that if

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M is a finitely generated, self-projective left R-module and $N \in Gen(M)$, then $\mathrm{Hom}(M,N)$ is ($\oplus$, amply,$\mathcal{G}^*$)-supplemented if and only if N is ($\oplus$, amply,$\mathcal{G}^*$)-supplemented.

Beidar and Kasch [2] defined and studied substructures, the singular ideal $\Delta(M,N)$ and the co-singular ideal $\nabla(M,N)$, of $\mathrm{Hom}_R(M,N)$ such as:

$$\begin{aligned} \Delta(M,N) &= \{f \in \mathrm{Hom}_R(M,N) : Ker(f) \leq^e M\} \\ \nabla(M,N) &= \{f \in \mathrm{Hom}_R(M,N) : Im(f) \ll N\}. \end{aligned}$$

The other substructure, radical, of $Hom(M,N)$ was introduced and studied by Kasch-Mader [11] and Nicholson-Zhou [15]. They have shown that:

$$\begin{aligned} J(M,N) &= \{\alpha \in \mathrm{Hom}_R(M,N) : 1_M - \alpha\beta \in Aut(M), \forall \beta \in \mathrm{Hom}_R(N,M)\} \\ &= \{\alpha \in \mathrm{Hom}_R(M,N) : 1_N - \beta\alpha \in Aut(N), \forall \beta \in \mathrm{Hom}_R(N,M)\} \\ &= \{\alpha \in \mathrm{Hom}_R(M,N) : \alpha\beta \in J(E_M), \forall \beta \in \mathrm{Hom}_R(N,M)\} \\ &= \{\alpha \in \mathrm{Hom}_R(M,N) : \beta\alpha \in J(E_N), \forall \beta \in \mathrm{Hom}_R(N,M)\}. \end{aligned}$$

Thus, we have $J[M,M] = J(E_M)$, which is similar to well known notion $J[R,R] = J(R)$. For the other new properties of these substructures, we refer to [12], [13] and [18]. Let M be a finitely generated, self-projective left R-module and $N \in Gen(M)$. We show that if $\mathrm{Hom}_R(M,N)$ is supplemented, then $\mathrm{Hom}_R(M,N)/\nabla(M,N)$ is semisimple as a left E_M-module.

2. Results.

Let M and N be R-modules. If there is an epimorphism $f : M^{(\Lambda)} \longrightarrow N$ for some set Λ, then N is said to be *an M-generated* module, denoted by $N \in Gen(M)$, (see [17]). We denote

$$N_M = M\mathrm{Hom}(M,N) = \{\Sigma_{i=1}^k m_i f_i : m_i \in M,\ f_i \in \mathrm{Hom}(M,N)\}.$$

Clearly, if $N_M = N$ then N is M-generated.

Lemma 2.1. (1) *Let N and M be two left R-modules. Then N is an M-generated R-module if and only if, for all non-zero R-homomorphism $f : N \to K$, there exists $h : M \to N$ such that $hf \neq 0$.*

(2) *If N_1 and N_2 are M-generated modules with $N = N_1 + N_2$, then N is also M-generated.*

Proof. Clear. □

The class of supplemented modules under Hom need not closed under taking factor modules, in general.

Proposition 2.2. *Let M be a P-projective module and $P \in Gen(M)$. If* $\mathrm{Hom}(M,P)$ *is supplemented then every homomorphic image of P is again supplemented under* Hom.

Proof. Let X be a submodule of P. We will prove that $\mathrm{Hom}(M,P/X)$ is a supplemented E_M-module. Let A be a submodule of $\mathrm{Hom}(M,P/X)$. For every element $f \in A$, there exists $g \in \mathrm{Hom}(M,P)$ such that $gs = f$, where $s : P \to P/X$ is the canonical projection. Let B be the set of all $h \in \mathrm{Hom}(M,P)$ such that h extends an elements in A. It is a simple matter to prove that B is a submodule of $\mathrm{Hom}(M,P)$. Since $\mathrm{Hom}(M,P)$ is supplemented, there exists a submodule C of $\mathrm{Hom}(M,P)$ such that C is minimal for the property $\mathrm{Hom}(M,P) = B + C$. Let $D = \{fs \mid f \in C\}$. It is clear that D is a submodule of $\mathrm{Hom}(M,P/X)$ and $\mathrm{Hom}(M,P/X) = A + D$. Let E be a submodule of $\mathrm{Hom}(M,P/X)$ contained in D such that $\mathrm{Hom}(M,P/X) = A + E$. Therefore

$$\mathrm{Hom}(M,P) = \mathrm{Hom}(M,X) + B + F,$$

where $F = \{f \in C \mid fs \in E\}$ and it is a submodule of C. But $\mathrm{Hom}(M,X) \leq B$. Then $\mathrm{Hom}(M,P) = B+F$. Since $F \leq C$, we have $F = C$. Consequently, D is a supplement of A in $\mathrm{Hom}(M,P/X)$. Hence $\mathrm{Hom}(M,P/X)$ is a supplemented E_M-module. □

Let $K \subset L \subset M$. Recall that K is said to be *cosmall* of L in M if $L/K \ll M/K$ and we denote it by $K \overset{cs}{\hookrightarrow} L$. A submodule L of the module M is called *co-closed* in M if $K \overset{cs}{\hookrightarrow} L$ implies $K = L$.

Lemma 2.3. *Let $K \subset L \subset M$. Then $K \overset{cs}{\hookrightarrow} L$ if and only if, for any submodule X of M, $M = L + X$ implies $M = K + X$.*

Proof. It is well known. □

Let M be an R-module and $X, Y \leq M$. In [3], the notion of β^* relation on submodules X, Y of M, denoted by $X\beta^*Y$, is defined such as $X\beta^*Y$ if and only if $(X+Y)/Y \ll M/Y$ and $(X+Y)/X \ll M/X$. We notice that β^* is an equivalence relation by [3, Lemma 2.2].

Lemma 2.4. *Let M be an R-module and $X, Y \leq M$. Then $X\beta^*Y$ if and only if for each $A \leq M$ such that $M = X + Y + A$ then $M = X + A$ and $M = Y + A$*

Proof. See [3, Theorem 2.3]. □

Proposition 2.5. *Let M be a finitely generated self-projective R-module and $N \in Gen(M)$. Then the following conditions hold.*
(1) *For every $K, L \leq N$,* $\mathrm{Hom}(M,K+L) = \mathrm{Hom}(M,K) + \mathrm{Hom}(M,L)$.
(2) *For every $I \leq \mathrm{Hom}(M,N)$, $I = \mathrm{Hom}(M,MI)$.*
(3) *If $K \leq N$, then $K_M\beta^*K$ and $K_M \overset{cs}{\hookrightarrow} K$ in N.*
(4) *Let $K \leq L \leq N$.*
  a) If $K \overset{cs}{\hookrightarrow} L$ in N, then $\mathrm{Hom}(M,K) \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,L)$ *in* $\mathrm{Hom}(M,N)$.

b) If $K\beta^*L$*, then* $\mathrm{Hom}(M,K)\beta^*\mathrm{Hom}(M,L)$.

(5) *Let* $A, B \leq \mathrm{Hom}(M,N)$.
a) If $A \overset{cs}{\hookrightarrow} B$ *in* $\mathrm{Hom}(M,N)$*, then* $MA \overset{cs}{\hookrightarrow} MB$ *in* N.
b) If $A\beta^*B$*, then* $(MA)\beta^*(MB)$.
(6) *If* $K \leq_{cc} N$*, then* $K \in Gen(M)$.

Proof. (1) and (2) have been shown in [17, 18.4].
(3) For two submodules X and Y of M with $X \leq Y$, it is clear that $X\beta^*Y$ and $X \overset{cs}{\hookrightarrow} Y$ in M are equivalent. Let $K \leq N$ and $N = K + L$ for some $L \leq N$. Since $N \in Gen(M)$, we can obtain that

$$\begin{aligned} N &= M\mathrm{Hom}(M,N) \\ &= M\mathrm{Hom}(M,K) + M\mathrm{Hom}(M,L) \subseteq K_M + L \end{aligned}$$

by (1). It follows that $N = K_M + L$. By Lemma 2.3, we can obtain that $K_M \overset{cs}{\hookrightarrow} K$ in N.
(4) Let $K \overset{cs}{\hookrightarrow} L$ in N (or $K\beta^*L$, respectively) and $\mathrm{Hom}(M,N) = \mathrm{Hom}(M,L)+A$ for some $A \leq \mathrm{Hom}(M,N)$. Since $N \in Gen(M)$, we can obtain that

$$\begin{aligned} N &= M\mathrm{Hom}(M,N) \\ &= M\mathrm{Hom}(M,L) + MA \subseteq L + MA \end{aligned}$$

by (1). We note that $MA \leq N$. Then $N = L + MA$.
(a) Since $K \overset{cs}{\hookrightarrow} L$ in N, by Lemma 2.3, we have $N = K + MA$. By (1) and (2),

$$\begin{aligned} \mathrm{Hom}(M,N) &= \mathrm{Hom}(M,K) + \mathrm{Hom}(M,MA) \text{ ..... } (*) \\ &= \mathrm{Hom}(M,K) + A. \end{aligned}$$

(b)By Lemma 2.4, $N = L+MA$ implies that $N = K + MA$. By (1),(2) and the equation (*), we can obtain that

$$\begin{aligned} \mathrm{Hom}(M,N) &= \mathrm{Hom}(M,K) + \mathrm{Hom}(M,MA) \\ &= \mathrm{Hom}(M,K) + A. \end{aligned}$$

Next, we assume that $\mathrm{Hom}(M,N) = \mathrm{Hom}(M,K) + H$ for some $H \leq \mathrm{Hom}(M,N)$. Since $N \in Gen(M)$, we can obtain that

$$N = M\mathrm{Hom}(M,N) = M\mathrm{Hom}(M,K) + MH.$$

By Lemma 2.4, we have $N = L + MH$. Then, by (1) and (2),

$$\begin{aligned} \mathrm{Hom}(M,N) &= \mathrm{Hom}(M,L) + \mathrm{Hom}(M,MH) \\ &= \mathrm{Hom}(M,L) + H. \end{aligned}$$

They imply that $\mathrm{Hom}(M,K)\beta^*\mathrm{Hom}(M,L)$ by [3, Theorem 2.3].
(5) We only give a proof of (a). Let $A \overset{cs}{\hookrightarrow} B$ in $\mathrm{Hom}(M,N)$ and let $N =$

$MB + L$ for some $L \leq N$. By (1) and (2),

$$\begin{aligned}\mathrm{Hom}(M, N) &= \mathrm{Hom}(M, MB) + \mathrm{Hom}(M, L)\\ &= B + \mathrm{Hom}(M, L).\end{aligned}$$

Since $A \overset{cs}{\hookrightarrow} B$ in $\mathrm{Hom}(M, N)$ and $N \in Gen(M)$, we can obtain that $\mathrm{Hom}(M, N) = A + \mathrm{Hom}(M, L)$ and so

$$\begin{aligned}N &= M\mathrm{Hom}(M, N)\\ &= M(A + \mathrm{Hom}(M, L)) \subseteq MA + L_M \subseteq MA + L.\end{aligned}$$

Thus $N = MA + L$. By Lemma 2.3, $MA \overset{cs}{\hookrightarrow} MB$ in N.
(6) Assume that $K \leq_{cc} N$. Since $K_M \overset{cs}{\hookrightarrow} K$ in N by (3), we obtain $K = K_M$, that is $K \in Gen(M)$ □

The proof of the following theorem can be seen also from [5, Corollary 4.2], and we give the proof for the sake of completeness.

Theorem 2.6. *Let M be a finitely generated self-projective R-module and $N \in Gen(M)$. If K is a co-closed submodule of N, then* $\mathrm{Hom}(M, K)$ *is coclosed in* $\mathrm{Hom}(M, N)$*, and, conversely, if A is a co-closed submodule of* $\mathrm{Hom}(M, K)$ *in* $\mathrm{Hom}(M, N)$ *then MA is coclosed in N. Furthermore, there exists a bijection between the direct summands of N and the direct summands of* $\mathrm{Hom}(M, N)$.

Proof. Let K be a coclosed submodule of N and $A \overset{cs}{\hookrightarrow} \mathrm{Hom}(M, K)$ in $\mathrm{Hom}(M, N)$. By Proposition 2.5 (5), $MA \overset{cs}{\hookrightarrow} M\mathrm{Hom}(M, K) = K_M$ in $N = M\mathrm{Hom}(M, N)$. By Proposition 2.5 (3) and [4, 3.2], we can obtain that $MA \overset{cs}{\hookrightarrow} K$ in N. Since K is a coclosed submodule of N, $MA = K_M = K$ and so $\mathrm{Hom}(M, K) = \mathrm{Hom}(M, MA) = A$ by Proposition 2.5 (3). This implies that $\mathrm{Hom}(M, K)$ is a coclosed submodule of $\mathrm{Hom}(M, N)$.

For converse, let A be a coclosed submodule of $\mathrm{Hom}(M, N)$ and $L \overset{cs}{\hookrightarrow} MA$ in N. By Proposition 2.5 (3) and (4), $\mathrm{Hom}(M, L) \overset{cs}{\hookrightarrow} A = \mathrm{Hom}(M, MA)$ in $\mathrm{Hom}(M, N)$. Since A is a coclosed submodule of $\mathrm{Hom}(M, N)$, we can obtain that $A = \mathrm{Hom}(M, L)$ and so $MA = M\mathrm{Hom}(M, L) = L_M \subseteq L$. It follows that $L = MA$. Hence MA is a coclosed submodule of N. □

Corollary 2.7. *Let M be a finitely generated, self-projective left R-module and $N \in Gen(M)$.*
(1) $\mathrm{Hom}(M, N)$ *is supplemented if and only if N is supplemented.*
(2) $\mathrm{Hom}(M, N)$ *is $\oplus$-supplemented if and only if N is $\oplus$-supplemented.*
(3) $\mathrm{Hom}(M, N)$ *is amply supplemented if and only if N is amply supplemented.*

Proof. (1) It follows from [5, Corollary 4.1(ii)]. (2) This is similar to (3).
(3) Assume that N is an amply supplemented module and let $I \leq \mathrm{Hom}(M,N)$. By [10, Proposition 1.5], let K be a coclosure of MI in N, i.e., $K \overset{cs}{\hookrightarrow} MI$ in N and K is coclosed in N. By hierarchy, there exists a supplemented submodule X of K such that $X \overset{cs}{\hookrightarrow} K$ in N. It follows that $X = K$, and so K is supplemented and $K \in Gen(M)$ by Proposition 2.5(6). By (1), $\mathrm{Hom}(M,K)$ is supplemented. Now, by Proposition 2.5(2) and (4), we can obtain that $\mathrm{Hom}(M,K) \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,MI) = I$ in $\mathrm{Hom}(M,N)$. This implies that $\mathrm{Hom}(M,N)$ is amply supplemented.

For converse, assume that $\mathrm{Hom}(M,N)$ is amply supplemented. Let $L \leq N$. Then $\mathrm{Hom}(M,L) \leq \mathrm{Hom}(M,N)$ and we assume that I is a coclosure of $\mathrm{Hom}(M,L)$ in $\mathrm{Hom}(M,N)$ by [10, Proposition 1.5]. Hence $I \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,L)$ in $\mathrm{Hom}(M,N)$ and I is coclosed in $\mathrm{Hom}(M,N)$. By Theorem 2.6, NI is coclosed in N. Since $MI \in Gen(M)$ by Proposition 2.5(6), we can obtain that there is a supplemented submodule I' of I such that $I' \overset{cs}{\hookrightarrow} I$ in $\mathrm{Hom}(M,N)$. Then $I' = I$ and $I = \mathrm{Hom}(M,MI)$ is supplemented. By Theorem 2.6, MI is supplemented. On the other hand, we can obtain that $MI \overset{cs}{\hookrightarrow} M\mathrm{Hom}(M,L) = L_M$ in N by Proposition 2.5(5). But we know that $L_M \overset{cs}{\hookrightarrow} L$ in N by Proposition 2.5(3), whence $MI \overset{cs}{\hookrightarrow} L$ in N by [4, 3.2]. This implies that N is amply supplemented. □

We have the following corollary.

Corollary 2.8. *Let H be a hollow projective module and K be a finitely H-generated module. Then K is a supplemented module and* $\mathrm{Hom}(H,K)$ *is supplemented.*

In [3], the authors used the β^* equivalence relation to define the class of $\mathcal{G}^*$-lifting modules and the class of $\mathcal{G}^*$-supplemented modules. M is called *$\mathcal{G}^*$-lifting* if, for each X of M, there exists a direct summand D of M such that $X\beta^*D$, and M is *$\mathcal{G}^*$-supplemented* if, for each X submodule of M, there exists a supplement S of M such that $X\beta^*S$.

By [3, Theorem 3.6], we have the following hierarchy:
lifting $\Rightarrow$ $\mathcal{G}^* -$ lifting $\Leftrightarrow$ $H -$ supplemented $\Rightarrow$ $\mathcal{G}^* -$ supplemented $\Rightarrow$ supplemented .

Theorem 2.9. *Let M be a finitely generated self-projective R-module and $N \in Gen(M)$. Then;*
(a) N is $\mathcal{G}^*$-lifting (H-supplemented) if and only if $\mathrm{Hom}_R(M,N)$ *is $\mathcal{G}^*$-lifting (H-supplemented) as an E_M-module.*
(b) N is $\mathcal{G}^*$-supplemented if and only if $\mathrm{Hom}_R(M,N)$ *is $\mathcal{G}^*$-supplemented as an E_M-module.*

Proof. (*a*) This is similar to (b).
(b) Assume that N is a $\mathcal{G}^*$-supplemented module. Let $I \subset \mathrm{Hom}_R(M,N)$ be an E_M-submodule. Then MI is a submodule of N. Since N is $\mathcal{G}^*$-supplemented, there exists a supplement submodule, say A, in N such that $A\beta^*(MI)$. Hence there exists $W \leq N$ such that $N = A + W$ and A is minimal with respect to this property. We show that $\mathrm{Hom}(M,A)$ is a supplement of $\mathrm{Hom}(M,W)$ in $\mathrm{Hom}(M,N)$ and $\mathrm{Hom}(M,A)\beta^* I$. By Proposition 2.5,

$$\begin{aligned}\mathrm{Hom}(M,N) &= \mathrm{Hom}(M,A+W)\\ &= \mathrm{Hom}(M,A) + \mathrm{Hom}(M,W).\end{aligned}$$

Let $\mathrm{Hom}(M,N) = S + \mathrm{Hom}(M,W)$ for $S \subseteq \mathrm{Hom}(M,A)$. Then

$$\begin{aligned}N = M\mathrm{Hom}(M,N) &= M(S + \mathrm{Hom}(M,W))\\ &= MS + M\mathrm{Hom}(M,W) \subseteq MS + W_M \subseteq N\end{aligned}$$

by Proposition 2.5. Minimality of A implies that $A = MS$. By Proposition 2.5, we obtain that $\mathrm{Hom}(M,A) = S$. Hence $\mathrm{Hom}(M,A)$ is a supplement of $\mathrm{Hom}(M,W)$ in $\mathrm{Hom}(M,N)$. On the other hand, $A\beta^*(MI)$ implies that $\mathrm{Hom}(M,A)\beta^*\mathrm{Hom}(M,MI) = I$ by Proposition 2.5. Hence $\mathrm{Hom}_R(M,N)$ is $\mathcal{G}^*$-supplemented as an E_M-module.

Conversely, assume that $\mathrm{Hom}_R(M,N)$ is $\mathcal{G}^*$-supplemented as an E_M-module. Let $X \leq N$. Then $\mathrm{Hom}(M,X) \subset \mathrm{Hom}(M,N)$. Since $\mathrm{Hom}_R(M,N)$ is $\mathcal{G}^*$-supplemented as an E_M-module, there exists a supplement submodule, say I, in $\mathrm{Hom}(M,N)$ such that $I\beta^*\mathrm{Hom}(M,X)$. Hence there exists $Y \leq \mathrm{Hom}(M,N)$ such that $\mathrm{Hom}(M,N) = I + Y$ and I is minimal with respect to this property. We show that MI is a supplement of MY and $(MI)\beta^* X$. By Proposition 2.5;

$$\begin{aligned}N = M\mathrm{Hom}(M,N) &= M(I+Y)\\ &= MI + MY.\end{aligned}$$

Let $N = K + MY$ for $K \subseteq MI$. Then

$$\begin{aligned}N = M\mathrm{Hom}(M,N) &= M\mathrm{Hom}(M,K+MY)\\ &= M\mathrm{Hom}(M,K) + M\mathrm{Hom}(M,MY)\\ &\subseteq K_M + MY \subseteq K + MY \subseteq N\end{aligned}$$

by Proposition 2.5. Minimality of MI implies that $K = MI$. By Proposition 2.5, we can also obtain that $(MI)\beta^* X$. Hence N is $\mathcal{G}^*$-supplemented. □

Recall that an R-module M is said to be *cosemisimple* if all simple modules are M-injective. By [4, 3.8], M is a cosemisimple module iff every submodule of M is coclosed in M.

Theorem 2.10. *Let M be a a finitely generated, self-projective R-module. Then the following cases are equivalent for the module $N \in Gen(M)$.*

(1) $\mathrm{Hom}(M,N)$ *is cosemisimple.*
(2) N *is cosemisimple.*

An R-module M is said to be *refinable* if, for any submodules U, V of M with $M = U + V$, there exits a direct summand D of M with $D \subset U$ and $M = D + V$ ([4]). A ring R is called *left refinable* if ${}_RR$ is a refinable module.

Theorem 2.11. *Let M be a finitely generated, self-projective left R-module and $N \in Gen(M)$. Then:*
(1) $\mathrm{Hom}(M,N)$ *is refinable if and only if N is refinable.*
(2) *If N is a refinable module, then the following are equivalent.*
  (*i*) *N is $\oplus$-supplemented*
  (*ii*) $\mathrm{Hom}(M,N)$ *is $\oplus$-supplemented*
  (*iii*) *N is supplemented.*
  (*iv*) $\mathrm{Hom}(M,N)$ *is supplemented.*

Proof. We only prove (1). The rest is clear.
(1)(:⇒) Let $U, V \leq N$ with $N = U + V$. Then

$$\begin{aligned}\mathrm{Hom}(M,N) &= \mathrm{Hom}(M,U+V)\\ &= \mathrm{Hom}(M,U) + \mathrm{Hom}(M,V)\end{aligned}$$

by Proposition 2.5. Since $\mathrm{Hom}(M,N)$ is refinable as a left E_M-module, there exists a direct summand D of $\mathrm{Hom}(M,N)$ such that $\mathrm{Hom}(M,N) = D + \mathrm{Hom}(M,V)$. By Theorem 2.6, we can obtain that MD is a direct summand of N. Now, it is easy to see that $N = MD + V$.
(:⇐)Assume that N is a refinable module. Let ${}_SI, {}_SJ \subset \mathrm{Hom}(M,N)$ with $\mathrm{Hom}(M,N) = I + J$. By Proposition 2.5, we have $I = \mathrm{Hom}(M,MI)$ and $J = \mathrm{Hom}(M,MJ)$. We also note that MI and MJ are submodule of N and $N = MI + MJ$. Since N is a refinable module, there exists a direct summand D of N such that $N = D + MJ$. By Theorem 2.6, we can obtain that $\mathrm{Hom}(M,D)$ is a direct summand of $\mathrm{Hom}(M,N)$. Now, it is easy to see that $\mathrm{Hom}(M,N) = \mathrm{Hom}(M,D) + J$. □

As a consequence, we have the following result (see [4, 11.28]):

Corollary 2.12. *Let M be a finitely generated, self-projective left R-module. Then the following cases are equivalent.*
(1) *E_M is left refinable.*
(2) *M is refinable.*

Recall that an R-module M is said to be *a unique coclosure module*, denoted by UCC, if every submodule of M has a unique coclosure in M (see [7]). By [4, 21.3], M is a UCC module if and only if, given $N \subseteq M$, there exists a coclosure N' of N such that $N' \subseteq L$ whenever $L \overset{cs}{\hookrightarrow} N$ in M.

Theorem 2.13. *Let M be a finitely generated, self-projective left R-module and $N \in Gen(M)$. Then the following cases are equivalent.*
(1) $\mathrm{Hom}(M,N)$ *is a UCC module.*
(2) N *is a UCC module.*

Proof. (1) $\Rightarrow$ (2) Let $A \leq N$. Then $\mathrm{Hom}(M,A) \leq \mathrm{Hom}(M,N)$. Since $\mathrm{Hom}(M,N)$ is a UCC module, there exists a coclosure, say K, of $\mathrm{Hom}(M,A)$ such that $K \subseteq L$ whenever $L \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,A)$ in $\mathrm{Hom}(M,N)$, i.e. $K \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,A)$ in $\mathrm{Hom}(M,N)$ and K is coclosed in $\mathrm{Hom}(M,N)$. By Proposition 2.5 (5) and Theorem 2.6, we can obtain that $MK \overset{cs}{\hookrightarrow} A$ in N and MK is coclosed in N. It implies that MK is a coclosure of A in N. On the other hand, $K \subseteq L$ implies $MK \subseteq ML$ and $L \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,A)$ in $\mathrm{Hom}(M,N)$ implies that $ML \overset{cs}{\hookrightarrow} A$ in N. Hence N is a UCC-module.
(2) $\Rightarrow$ (1) Let ${}_SI \subset \mathrm{Hom}(M,N)$. Then, by Proposition 2.5 (2), $I = \mathrm{Hom}(M,MI)$ and MI is a submodule of N. Since N is a UCC-module, there exists a coclosure K of MI in N such that $K \subset L$ whenever $L \overset{cs}{\hookrightarrow} MI$ in N. Since K is a coclosure of MI in N, we have $K \overset{cs}{\hookrightarrow} MI$ in N and K is coclosed in N. By Proposition 2.5 (4) and Theorem 2.6, we can obtain that $MK \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,MI) = I$ in $\mathrm{Hom}(M,N)$ and $\mathrm{Hom}(M,K)$ is coclosed in $\mathrm{Hom}(M,N)$. They imply that $\mathrm{Hom}(M,K)$ is a coclosure of I in $\mathrm{Hom}(M,N)$. On the other hand, $K \subset L$ implies that $\mathrm{Hom}(M,K) \subset \mathrm{Hom}(M,L)$ and $L \overset{cs}{\hookrightarrow} MI$ in N implies that $ML \overset{cs}{\hookrightarrow} \mathrm{Hom}(M,MI) = I$ in $\mathrm{Hom}(M,N)$ by Proposition 2.5 (4). Hence $\mathrm{Hom}(M,N)$ is a UCC module. □

3. The substructure $\nabla(M,N)$

In this section, we study the concept of the substructure $\nabla(M,N)$.

Theorem 3.1. *Let M be a finitely generated, self-projective left R-module and $N \in Gen(M)$. If* $\mathrm{Hom}(M,N)$ *is supplemented as a left E_M-module, then* $\mathrm{Hom}(M,N)/\nabla(M,N)$ *is semisimple as a left E_M-module.*

Proof. Let $\overline{A} = A/\nabla(M,N) \leq \mathrm{Hom}(M,N)/\nabla(M,N)$. There exists $B \leq \mathrm{Hom}(M,N)$ such that $\mathrm{Hom}(M,N) = A + B$ and $A \cap B \ll B$. Then

$$\mathrm{Hom}(M,N)/\nabla(M,N) = A/\nabla(M,N) + (B + \nabla(M,N))/\nabla(M,N).$$

For any $f \in A \cap B$, we note that $E_M f \leq A \cap B$ and so $E_M f \ll \mathrm{Hom}(M,N)$. Now we show that $f \in \nabla(M,N)$. Let $H \leq N$ with $N = Imf + H$. By Proposition 2.5 (1), we can obtain that

$$\mathrm{Hom}(M,N) = \mathrm{Hom}(M,f(M)) + \mathrm{Hom}(M,H).$$

It follows that $E_M f + \mathrm{Hom}(M,H) = \mathrm{Hom}(M,N)$ and hence $\mathrm{Hom}(M,H) = \mathrm{Hom}(M,N)$. On the other hand, $N = M\mathrm{Hom}(M,N) = M\mathrm{Hom}(M,H) \leq H$ since $N \in Gen(M)$. Therefore $N = H$, i.e. $Imf \ll N$. Hence $f \in \nabla(M,N)$. Thus

$$\mathrm{Hom}(M,N)/\nabla(M,N) = A/\nabla(M,N) \oplus (B + \nabla(M,N))/\nabla(M,N),$$

as desired. □

Recall that;

(D2) For any submodule A of M for which M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

(GD2) For any submodule A of M for which M/A is isomorphic to M, then A is a direct summand of M.

A module M is called *discrete* (respectively, *generalized discrete*) if M satisfies (D1) and (D2) (respectively, (D1) and (GD2)). Let p be a prime number. Then $M_{\mathbb{Z}} = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ is generalized discrete but not discrete.

Lemma 3.2. ([9, Lemma 3.1]) *Let M and N be R-modules. If N satisfies (GD2), then $\nabla(M,N) \subset J[M,N]$.*

Proof. Let $\beta \in \nabla(M,N)$ and $f \in \mathrm{Hom}(N,M)$. Then

$$Im\beta + Im(1_N - \beta f) = N.$$

Let $\eta := 1_N - \beta f$. Since $Im\beta \ll N$, we have $Im(\eta) = N$. It follows that $N \cong N/Ker(\eta)$. By (GD2), we have $Ker(\eta)$ is a direct summand of N. Since $Ker(\eta) \leq Im\ (\beta)$, we can obtain that $Ker(\eta) \ll N$. Hence $Ker(\eta) = 0$. Now η is an isomorphism. Thus $\beta \in J[M,N]$. □

The next result extends Mohammed and Müller [14, Theorem 5.4].

Corollary 3.3. *Let M be a finitely generated, self-projective left R-module and $N \in Gen(M)$. If* $\mathrm{Hom}(M,N)$ *is supplemented and N satisfies GD2 then* $\mathrm{Hom}(M,N)/J[M,N]$ *is semisimple as a left E_M-module.*

Proof. It is clear from Theorem 3.1 and Lemma 3.2. □

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