Math. J. Okayama Univ. 57 (2015), 79–84

# ON MODEL STRUCTURE FOR COREFLECTIVE SUBCATEGORIES OF A MODEL CATEGORY

TADAYUKI HARAGUCHI

### 1. INTRODUCTION

Let  $\mathbf{C}$  be a coreflective subcategory of a cofibrantly generated model category  $\mathbf{D}$ . In this paper we show that under suitable conditions  $\mathbf{C}$  admits a cofibrantly generated model structure which is left Quillen adjunct to the model structure on  $\mathbf{D}$ . As an application, we prove that well-known convenient categories of topological spaces, such as k-spaces, compactly generated spaces, and  $\Delta$ -generated spaces [3] (called numerically generated in [12]) admit a finitely generated model structure which is Quillen equivalent to the standard model structure on the category **Top** of topological spaces.

2. Coreflective subcategories of a model category

Let **D** be a cofibrantly generated model category [7, 2.1.17] with generating cofibrations I, generating trivial cofibrations J and the class of weak equivalences  $W_{\mathbf{D}}$ . If the domains and codomains of I and J are finite relative to I-cell [7, 2.1.4], then **D** is said to be finitely generated.

Recall that a subcategory  $\mathbf{C}$  of  $\mathbf{D}$  is said to be coreflective if the inclusion functor  $i: \mathbf{C} \to \mathbf{D}$  has a right adjoint  $G: \mathbf{D} \to \mathbf{C}$ , so that there is a natural isomorphism  $\varphi: \operatorname{Hom}_{\mathbf{D}}(X, Y) \to \operatorname{Hom}_{\mathbf{C}}(X, GY)$ . The counit of this adjunction  $\epsilon: GY \to Y \ (Y \in \mathbf{D})$  is called the coreflection arrow.

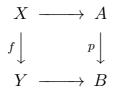
**Theorem 2.1.** Let  $\mathbf{C}$  be a coreflective subcategory of a cofibrantly generated model category  $\mathbf{D}$  which is complete and cocomplete. Suppose that the unit of the adjunction  $\eta: X \to GX$  is a natural isomorphism, and that the classes Iand J of cofibrations and trivial cofibrations in  $\mathbf{D}$  are contained in  $\mathbf{C}$ . Then  $\mathbf{C}$  has a cofibrantly generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and  $W_{\mathbf{C}}$  as the class of weak equivalences, where  $W_{\mathbf{C}}$  is the class of all weak equivalences contained in  $\mathbf{C}$ . If  $\mathbf{D}$  is finitely generated, then so is  $\mathbf{C}$ . Moreover, the adjunction  $(i, G, \varphi): \mathbf{C} \to \mathbf{D}$  is a Quillen adjunction in the sense of [7, 1.3.1].

*Proof.* It suffices to show that **C** satisfies the six conditions of [7, 2.1.19] with respect to I, J and  $W_{\mathbf{C}}$ . Clearly, the first condition holds because

Mathematics Subject Classification. Primary 55U40; Secondary 55U35.

Key words and phrases. model category, Quillen equivalence, numerically generated space.

 $W_{\mathbf{C}}$  satisfies the two out of three property and is closed under retracts. To see that the second and the third conditions hold, let  $I_{\mathbf{C}}$ -cell and  $J_{\mathbf{C}}$ -cell be the collections of relative *I*-cell and *J*-cell complexes contained in  $\mathbf{C}$ , respectively. Since  $I_{\mathbf{C}}$ -cell and  $J_{\mathbf{C}}$ -cell are subcollections of the collections of relative *I*-cell and *J*-cell complexes in  $\mathbf{D}$ , respectively, the domains of *I* and *J* are small relative to  $I_{\mathbf{C}}$ -cell and  $J_{\mathbf{C}}$ -cell, respectively. The rest of the conditions are verified as follows. Let  $f: X \to Y$  be a map in  $\mathbf{C}$ . Since  $\eta: X \to GX$  is isomorphic for  $X \in \mathbf{D}$ , f is *I*-injective in  $\mathbf{C}$  if and only if it is *I*-injective in  $\mathbf{D}$ . Similarly, f is *J*-injective in  $\mathbf{C}$  if and only if it is *J*-injective in  $\mathbf{D}$ . Let f be an *I*-cofibration in  $\mathbf{D}$ . Then it has the left lifting property with respect to all *I*-injective maps in  $\mathbf{C}$ . Suppose we are given a commutative diagram



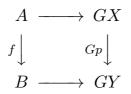
where p is I-injective in **D**. Then there is a relative I-cell complex  $g: X \to Z$ [7, 2.1.9] such that f is a retract of g by [7, 2.1.15]. Since g is an I-cofibration in **D**, there is a lift  $Z \to A$  of g with respect to p. Then the composite  $Y \to Z \to A$  is a lift of f with respect to p. Therefore f is an I-cofibration in **D**. Similarly, f is a J-cofibration in **C** if and only if it is a J-cofibration in **D**. Thus we have the desired inclusions

- $J_{\mathbf{C}}$ -cell  $\subseteq W_{\mathbf{C}} \cap I_{\mathbf{C}}$ -cof,
- $I_{\mathbf{C}}$ -inj  $\subseteq W_{\mathbf{C}} \cap J_{\mathbf{C}}$ -inj, and
- either  $W_{\mathbf{C}} \cap I_{\mathbf{C}}$ -cof  $\subseteq J_{\mathbf{C}}$ -cof or  $W_{\mathbf{C}} \cap J_{\mathbf{C}}$ -inj  $\subseteq I_{\mathbf{C}}$ -inj.

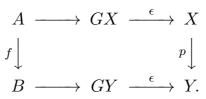
Here  $I_{\mathbf{C}}$ -inj and  $I_{\mathbf{C}}$ -cof denote, respectively, the classes of *I*-injective maps and *I*-cofibrations in  $\mathbf{C}$ , and similarly for  $J_{\mathbf{C}}$ -inj and  $J_{\mathbf{C}}$ -cof. Therefore  $\mathbf{C}$ is a cofibrantly generated model category by [7, 2.1.19].

It is clear, by the definition, that  $\mathbf{C}$  is finitely generated if so is  $\mathbf{C}$ .

Finally, to prove that  $(i, G, \varphi)$  is a Quillen adjunction, it suffices to show that  $G: \mathbf{D} \to \mathbf{C}$  is a right Quillen functor, or equivalently, G preserves J-injective maps in  $\mathbf{D}$  by [7, 1.3.4] and [7, 2.1.17]. Let  $p: X \to Y$  be a J-injective map in  $\mathbf{D}$ . Suppose there is a commutative diagram



where  $f \in J$ . Then we have a commutative diagram



Since p is J-injective in  $\mathbf{D}$ , there is a lift  $h: B \to X$  of f. Thus we have a lift  $Gh \circ \eta: B \cong GB \to GX$  of f with respect to Gp. Therefore  $Gp: GX \to GY$  is J-injective in  $\mathbf{C}$ . Similarly, we can show that G preserves I-injective maps in  $\mathbf{C}$ , and so G preserves trivial fibrations in  $\mathbf{C}$ . Hence  $(i, G, \varphi)$  is a Quillen adjunction.

We turn to the case of pointed categories [7, p.4]. Let  $\mathbf{D}_*$  be the pointed category associated with  $\mathbf{D}$ , and let  $U: \mathbf{D}_* \to \mathbf{D}$  be the forgetful functor. We denote by  $I_+$  and  $J_+$  the classes of those maps  $f: X \to Y$  in  $\mathbf{D}_*$  such that  $Uf: UX \to UY$  belongs to I and J, respectively. Then we have the following. (Compare [7, 1.1.8], [7, 1.3.5], and [7, 2.1.21].)

**Theorem 2.2.** Let **D** be a cofibrantly (resp. finitely) generated model category, and let **C** be a coreflective subcategory satisfying the conditions of Theorem 2.1. Then the pointed category  $\mathbf{C}_*$  has a cofibrantly (resp. finitely) generated model structure, with generating cofibrations  $I_+$  and generating trivial cofibrations  $J_+$ , such that the induced adjunction  $(i_*, G_*, \varphi_*): \mathbf{C}_* \to \mathbf{D}_*$  is a Quillen adjunction.

We also have the following Proposition.

**Proposition 2.3.** Suppose  $\mathbf{C}$  and  $\mathbf{D}$  satisfy the conditions of Theorem 2.1. Suppose, further, that the coreflection arrow  $\epsilon \colon GY \to Y$  is a weak equivalence for any fibrant object Y in  $\mathbf{D}$ . Then the adjunctions  $(i, G, \varphi) \colon \mathbf{C} \to \mathbf{D}$ and  $(i_*, G_*, \varphi_*) \colon \mathbf{C}_* \to \mathbf{D}_*$  are Quillen equivalences.

Proof. Let X be a cofibrant object in **C** and Y a fibrant object in **D**. Let  $f: X \to Y$  be a map in **D**. Then we have  $\varphi f = Gf \circ \eta: X \cong GX \to GY$ . Since f coincides with the composite  $X \xrightarrow{\varphi f} GY \xrightarrow{\epsilon} Y$  and  $\epsilon$  is a weak equivalence in **D**,  $\varphi f$  is a weak equivalence in **C** if and only if f is a weak equivalence in **D**. It follows by [7, 1.3.17] that that the induced adjunction  $(i_*, G_*, \varphi_*)$  is a Quillen equivalence.

## 3. On a model structure of the category $\mathbf{NG}$

In [12] we introduced the notion of numerically generated spaces which turns out to be the same notion as  $\Delta$ -generated spaces introduced by Jeff Smith (cf. [3]). Let X be a topological space. A subset U of X is numerically open if for every continuous map  $P: V \to X$ , where V is an open subset of Euclidean space,  $P^{-1}(U)$  is open in V. Similarly, U is numerically closed if for every such map P,  $P^{-1}(U)$  is closed in V. A space X is called a numerically generated space if every numerically open subset is open in X.

Let **NG** denote the full subcategory of **Top** consisting of numerically generated spaces. Then the category **NG** is cartesian closed [12, 4.6]. To any X we can associate the numerically generated space topology, denoted  $\nu X$ , by letting U open in  $\nu X$  if and only if U is numerically open in X. Therefore we have a functor  $\nu: \mathbf{Top} \to \mathbf{NG}$  which takes X to  $\nu X$ . Clearly, the identity map  $\nu X \to X$  is continuous. By the results of [7, §3] the following holds.

**Proposition 3.1.** The functor  $\nu$ : Top  $\rightarrow$  NG is a right adjoint to the inclusion functor i: NG  $\rightarrow$  Top, so that NG is a coreflective subcategory of Top.

A continuous map  $f: X \to Y$  between topological spaces is called a weak homotopy equivalence in **Top** if it induces an isomorphism of homotopy groups

$$f_* \colon \pi_n(X, x) \to \pi_n(Y, f(x))$$

for all n > 0 and  $x \in X$ . Let I be the set of boundary inclusions  $S^{n-1} \to D^n$ ,  $n \ge 0$ , J the set of inclusions  $D^n \times \{0\} \to D^n \times I$ , and  $W_{\text{Top}}$  the class of weak homotopy equivalences. The standard model structure on **Top** can be described as follows.

**Theorem 3.2** ([7, 2.4.19]). There is a finitely generated model structure on **Top** with I as the set of generating cofibraitons, J as the set of generating trivial cofibrations, and  $W_{\text{Top}}$  as the class of weak equivalences.

The category **NG** is complete and cocomplete by [12, 3.4]. A space X is numerically generated if and only if  $\nu X = X$  holds. Thus the unit of the adjunction  $\eta: X \to \nu X$  is a natural homeomorphism. Moreover, since CW-complexes are numerically generated spaces by [12, 4.4], the classes I and J are contained in **NG**. Let  $W_{\mathbf{NG}}$  be the class of maps  $f: X \to Y$ in **NG** which is a weak equivalence in **Top**. Since the coreflection arrow  $\nu Y \to Y$ , given by the identity of  $Y \in \mathbf{Top}$ , is a weak equivalence (cf. [12, 5.4]), we have the following by Theorem 2.1 and Proposition 2.3.

**Theorem 3.3.** The category NG has a finitely generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and  $W_{NG}$  as the class of weak equivalences. Moreover the adjunction  $(i, \nu, \varphi): NG \to Top$  is a Quillen equivalence.

We turn to the case of pointed spaces. Let  $\mathbf{Top}_*$  be the category of pointed topological spaces. By [7, 2.4.20], there is a finitely generated model structure on the category  $\mathbf{Top}_*$ , with generating cofibrations  $I_+$  and generating

trivial cofibrations  $J_+$ . Then we have the following by Theorem 2.2 and Proposition 2.3.

**Corollary 3.4.** There is a finitely generated model structure on the category  $\mathbf{NG}_*$  of pointed numerically generated spaces, with generating cofibrations  $I_+$  and generating trivial cofibrations  $J_+$ . Moreover, the inclusion functor  $i_*: \mathbf{NG}_* \to \mathbf{Top}_*$  is a Quilen equivalence.

*Remark.* (1) The argument of Theorem 3.3 can be applied to the subcategories **K** of k-spaces and **T** of compactly generated spaces. Similarly, the argument of Corollary 3.4 can be applied to the pointed categories  $\mathbf{K}_*$  and  $\mathbf{T}_*$ . Compare [2.4.28], [2.4.25], [2.4.26] of [7].

(2) Let **Diff** be the category of diffeological spaces (cf. [8]). In [12] we introduced a pair of functors  $T : \mathbf{Diff} \to \mathbf{Top}$  and  $D : \mathbf{Diff} \to \mathbf{Top}$ , where T is a left adjoint to D, and showed that the composite TD coincides with  $\nu : \mathbf{Top} \to \mathbf{NG}$ . Thus  $\mathbf{NG}$  can be embedded as a full subcategory into  $\mathbf{Diff}$ . It is natural to ask whether  $\mathbf{Diff}$  has a model category structure with respect to which the pair (T, D) gives a Quillen adjuntion between  $\mathbf{Top}$  and  $\mathbf{Diff}$ .

Let I be the unit interval, and let  $\lambda \colon \mathbf{R} \to I$  be the smashing function, that is, a smooth function such that  $\lambda(t) = 0$  for  $t \leq 0$  while  $\lambda(t) = 1$  for  $t \geq 1$ . Let  $\tilde{I}$  denote the unit interval equipped with the quotient diffeology  $\lambda_*(D_{\mathbf{R}})$ , where  $D_{\mathbf{R}}$  is the standard diffeology of  $\mathbf{R}$ . In [5] we introduce a finitely generated model category structure on **Diff** with the boundary inclusions  $\partial \tilde{I}^{n-1} \to \tilde{I}^n$  as generating cofibrations, and with the inclusions  $\partial \tilde{I}^{n-1} \times \tilde{I} \cup \tilde{I}^n \times \{0\} \to \tilde{I}^n \times \tilde{I}$  as generating trivial cofibrations. Its class of weak equivalences consists of those smooth maps  $f \colon X \to Y$  inducing an isomorphism  $f_* \colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$  for every  $n \geq 0$  and  $x_0 \in$ X. Here, the homotopy set  $\pi_n(X, x_0)$  is defined to be the set of smooth homotopy classes of smooth maps  $(\tilde{I}^n, \partial \tilde{I}^n) \to (X, x_0)$ .

It is expected that with respect to the model structure on **Diff** described above, the pair (T, D) induces a Quillen adjunction between **Top** and **Diff**.

Acknowledgements. I would like to express my sincere gratitude to my supervisor Kazuhisa Shimakawa. He introduced me to the project of building a homotopy theory on the category **NG**. This project was not completed without him. He carefully read this paper, helped me with the English and corrected many errors. In order that I might acquire a doctor's degree, he had supported me for a long time.

Next I would like to thank Referee. He proposed that there exists a general framework from proofs of Theorem 3.3 and Corollary 3.4. As a result, we have Theorem 2.1, Theorem 2.2, and Proposition 2.3.

#### TADAYUKI HARAGUCHI

### References

- C. Berger and I. Moerdijk. Axiomatic homotopy theory for operads. Comment. Math. Helv., 78(4):805-831, 2003.
- [2] S. E. Crans. Quillen closed model structures for sheaves. J. Pure Appl. Algebra , 101(1):35-57, 1995.
- [3] D. Dugger, Notes on Delta-generated spaces, available at http://www.uoregon.edu/ddugger/delta.html
- [4] W. G. Dwyer and J. Spalinski. Homotopy theories and model categories, Handbook of Algebraic Topology, Elsevier, 1995, 73-126.
- [5] T. Haraguchi and K. Shimakawa, A model structure on the category of diffeological spaces, arXiv:1311.5668v2
- [6] P. S. Hirschhorn, Model categories and their localizations, Mathematical Surveys and Monographs, 99, American Mathematical Society, 2003.
- [7] M. Hovey, Model categories, Mathematical Surveys and Monographs, 63, American Mathematical Society, Providence, RI, 1999.
- [8] P. Iglesias-Zemmour, Diffeology, CNRS, Marseille, France, and The Hebrew University of Jerusalem, Israel.
- [9] J. W. Milnor. Topology from the differentiable viewpoint. Princeton Landmarks.
- [10] D. G. Quillen, Homotopical Algebra, SLNM 43, Springer, Berlin (1967).
- [11] D. G. Quillen, Rational homotopy theory, Ann. of Math. 90 (1969), 205-295.
- [12] K. Shimakawa, K. Yoshida, and T. Haraguchi, Homology and cohomology via enriched bifunctors, arXiv:1010.3336v1.
- [13] E. H. Spanier, Algebraic Topology, McGraw-Hill, New York (1966).

TADAYUKI HARAGUCHI DEPARTMENT OF GENERAL EDUCATION OITA NATIONAL COLLEGE OF TECHNOLOGY 1666, OAZAMAKI, OITA-SHI, OITA, 870-0152, JAPAN *e-mail address*: t-haraguchi@oita-ct.ac.jp