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Constructive Analysis of Two Dimensional Fermi Systems at Finite Temperature

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Zusammenfassung

Wir betrachten ein 2-dimensionales verdünntes Fermigas Modell mit kurzreichweitigen Wechselwirkung. Wir zeigen nicht-störungstheoretisch, dass die renormierte Störungsentwicklung des Modells bei positiven Temperaturen endlichen Konvergenzradius hat. Der Konvergenzradius schrumpft logarithmisch gegen null wenn die Energieskala gegen die Infrarotgrenz geht. Durch eine detaillierte Analyse des Beitrags der Leitersummen stellen wir fest, dass der Konvergenzradius auch von dem Vorzeichen der Kopplungskonstanten g abhängt. Es wird außerdem gezeigt, dass die Selbstenergie eine einmal stetig differenzierbare Funktion $\mathbb{C}^1$ im analytischen Bereich des Modells ist. Zur Untersuchung dieses fermionischen Modells wird die mathematische Renormierunggruppe, die in der konstruktiven Quantenfeldtheorie entwickelt wurde, verwendet. Die Fermifläche wird durch Konterterme fixiert. Die Baumentwicklung wird eingeführt, durch denen Anwendung die Konvergenz der Störungsentwicklung nachgewiesen werden kann. Diese Beweismethode wird angewendet, um ein halbgefülltes Hubbard-Modell auf einer Doppelschicht Graphen mit lokalen Wechselwirkungen zu konstruktieren.

Abstract

We consider a dilute Fermion system in continuum two spatial dimensions with short-range interaction. We prove nonperturbatively that at low temperature the renormalized perturbation expansion has non-zero radius of convergence. The convergence radius shrinks when the energy scale goes to the infrared cutoff. The shrinking rate of the convergence radius is established to be dependent of the sign of the coupling constant g by a detailed analysis of the so-called ladder contributions. We prove further that the self-energy of the model is uniformly of $\mathbb{C}^1$, but not $\mathbb{C}^2$ in the analytic domain of the theory. The proofs are based on renormalization of the Fermi surface and multiscale analysis employing mathematical renormalization group technique. Tree expansion is introduced to reorganize perturbation expansion nicely. Finally we apply these techniques to construct a half-filled Hubbard model on honeycomb bilayer lattice with local interaction.

Contents

Chapter 1

Introduction

For many years, Landau Fermi liquid (FL) theory [Lan59] was appeared to be congenial to explain the interacting many fermions systems, at least for scenarios where temperatures are not too low or the systems are in their normal phases. At very low temperature, however, Landau's Fermi liquid theory breaks down, and a number of potential instabilities appear. For example, the BCS instability for the formation of Cooper pairs leads to superconductivity in two or three dimensions. Here we call dimension the dimension of space only. But in the past three decades, Landau's Fermi liquid theory has been engaged with a range of problems very different from those it began with, in particular, the discovery of high temperature superconductivity in cuprate compounds. Their normal phase shows apparent discrepancy between theoretical prediction and experimental observation. This discovery challenges the view of Fermi liquid theory. One began to doubt the correctness of Fermi liquid theory or the existence of Fermi liquid in two dimensions. Do interacting Fermi systems in two dimensions resemble more the Fermi liquid in three dimensions, or the Luttinger liquid in one dimension which has a linear dispersion relation and is exactly solvable by bosonization? Anderson [And90a, And90b, And95] suggested that a two dimensional Fermi gas behaviors similarly to a one dimensional Luttinger liquid. But his answer was not wholly right. Rigorously speaking, the answer should depend on the shape of the Fermi surface.

We present a brief review of rigorous mathematical studies of interacting Fermi systems accomplished until now. In dimension one there is no extended Fermi surface and Fermi systems have been proved to be Luttinger liquid by Benfatto and Gallaotti [BG95], which exhibits anomalous decay exponents. In dimension two, in a series of papers [FKT00, FKT04h, FKT04i, FKT04j, FKT04c, FKT04d, FKT04f, FKT04e, FKT04g, FKT04a, FKT04b] by

Feldman, Knörrer and Trubowitz the construction of two dimensional Fermi liquid for a class of non-parity invariant Fermi surface has been completed in great details. In these papers the convergence of perturbation expansion of the connected Green's functions in term of the coupling constant at zero temperature has been proved. The Fermi liquid behavior was established in a traditional sense, namely the existence of a jump in the particle number density at the Fermi surface. Systems with non-parity invariant Fermi surface can be realized by applying a sufficiently high magnetic field, which breaks the Cooper pairs and prevent superconductivity. The other approach was based on Salmhofer's criterion for Fermi liquid behavior at positive temperature [Sal98a, Sal99]. One has to study whether given models satisfy Salmhofer's criterion or not. The criterion are as follows

1. The renormalized perturbation expansion in the coupling constant g for the Green functions and the counterterm function converges in a region $\mathcal{R} = \{(g, \beta) : |g| \log \beta < \text{const}\}$.

2. The self-energy is $\mathbb{C}^2$ in k, and its second derivative is uniformly bounded on compact subsets of $\mathcal{R}$.

3. The counterterm function is $\mathbb{C}^{k_0}$ in $\mathbf{k}$, with $k_0 > d$, and there is a unique solution to the inverse equation (1.3).

We emphasize that these restrictions are sufficient but not necessary to ensure Fermi liquid behavior. Condition 1 defines a critical temperature $T_c \propto e^{-\frac{C}{|g|}}$, above which the perturbation expansion converges. In other words, temperature plays a role as a cutoff which prevents Cooper pair formation. Condition 2 requires the regularity of the self-energy. If it holds, we can do a first order Taylor expansion of the self-energy around the Fermi surface. We obtain a finite wave function renormalization

$$Z(k) = 1 + i\partial_{k_0}\Sigma(0, \mathbf{\Pi}(\mathbf{k})) \tag{1.1}$$

and a finite correction to the Fermi velocity. The Taylor reminder term vanishes quadratically in the distance between $(k_0, \mathbf{k})$ and $(0, \mathbf{\Pi}(\mathbf{k}))$, where $\mathbf{\Pi}$ projects the momentum onto the Fermi surface. In dimension two, above conditions have been verified for system with Jellium band relation and on-site interaction by Disertori and Rivasseau in [DR00a, DR00b]. An extension of this model into three spatial dimensions is completed recently by the same authors [DRJ01]. For Hubbard model at low filling, the Fermi surface becomes more and more circular as the filling factor goes away from the half filling, Fermi liquid was estimated there by Benfatto, Giuliani and

Mastropietro [BGM06], and independently by Pedra [Ped05], was proved the result for Hubbard models with parity invariant, more general convex but not necessarily rotation invariant Fermi surfaces. Models showing non Fermi liquid behavior, such as Hubbard model at half filling has been confirmed by Rivasseau, Afchain and Magnen in [Riv02, SA05b, SA05a]. Moreover, by combining the result of [BGM06] and [Riv02, SA05b, SA05a] we conjecture that there is a crossover between Fermi liquid and non Fermi liquid behavior while varying the chemical potential. Remeber that Anderson suggested a Luttinger liquid behavior, thus this might explain the controversy on the nature of Fermi systems in their normal phase.

In this dissertation we intend to construct two interacting Fermions systems. Our rigorous mathematical construction consists of estimating the regularity properties of the Euclidean Green's functions and of the self-energy.

Our first model describes many Fermions in two dimensional continuum space with short-range interaction under a dilutness assumption. We state the hypotheses on the free dispersion relation $E(\mathbf{k})$ (shifted by the chemical potential). We assume that $E(\mathbf{k})$ has the form

$$E(\mathbf{k}) = \mathbf{k}^2 - \frac{\nu}{\beta} \tag{1.2}$$

where ν is a constant of order one, and β is the inverse temperature. At low temperature, its Fermi surface is a circle with much smaller radius ($r \ll 1$) than the one of Jellium band structure, which is of order one. This distinction may have very important effects on the low temperature properties of the system.

Our second model is a lattice model based on bilayer graphene, which is a material of great interest at present. The experimental realization of ultra-thin graphitic devices including monolayer structure [NGM$^+$04], known as graphen, has attached many attentions. In particular, because of its dispersion relation, which resembles the one of massless Dirac fermions in two space dimensions, graphen shows many exciting phenomena, e.g. the anomalous integer quantum Hall effect. In [GM10] Giuliani and Mastropietro has constructed rigorously a low temperature two dimensional Hubbard model on the monolayer honeycomb lattice with on-site interaction. Bilayer graphene consists of two graphene monolayers, typically arranged in the Bernal stacking arrangement. Our proposal is to construct a Hubbard model on bilayer graphene at half filling with on-site interaction. In its simplest case with nearest neighbour hoping only, it gives rise to a band structure with two bands touching quadratically at two non-equivalent points in the first Brillouin zone. This band structure can be seen as a special case of our dilute

Fermi gas model with $\nu = 0$. On the other hand, inclusion of a next nearest neighbor hopping between two layers gives rise to a Liftshiftz transition.

In order to study fundamental issues, such as the existence of non-perturbative effects, we represent our models in field theory formalism, which is the best tool to do that. In this formalism the usual second quantized form of the Hamiltonian with creation and annihilation operators are replaced by the anti-commuting fermionic fields with spin indices and arguments in space-time. The strategy of our rigorous mathematical study of interacting Fermi systems bases on renormalization of the Fermi surface and on multi-scale analysis employing mathematical renormalization group technique.

It is well known that many coefficients in naive perturbation expansion of the unrenormalized Green's function in term of the coupling constant diverge. The reason for this divergence is the deformation of the Fermi surface while the interaction turns on. In other words, interacting and non-interacting Fermi surface do not, in general, agree. Renormalization of the Fermi surface becomes necessary. To do so, we use counterterm which is bilinear in fermionic fields, that is, it is of mass type. The counterterm is used to compensate the shift between the non-interacting and the interacting Fermi surface, and the perturbation expansion will be implemented on the fixed interacting Fermi surface. There still remains a question how to justify putting in the counterterm function. We write the dispersion relation $E(\mathbf{k})$ of the free model in the form $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k})$ and try to choose the counterterm $\delta e(\mathbf{k})$, which becomes parts of the interaction, as a suitable function of the original interaction, so that the Fermi surface of the interacting system is the set $S_g = \{\mathbf{k} \in \mathbb{R}^2 \mid e(\mathbf{k}) = 0\}$. We should prove that $\delta e(\mathbf{k}, g)$ and $e(\mathbf{k})$ can be chosen with same differentiability and the equation

$$E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k}, \mathbf{g}) \tag{1.3}$$

can be solved with respect to $e(\mathbf{k})$, given $E(\mathbf{k})$ and g. (1.3) is called the inverse equation. For rotation invariant systems, such as the Jellium model [DR00a, DR00b] and the models considered in this work, the counterterm is fixed to be a constant or vanishes. Thus (1.3) is easy to solve. But this is far from straightforward for a non-spherical Fermi surface, such as the Hubbard model, where the counterterms are functions. The inverse equation (1.3) becomes a highly nontrivial problem. In [FST96,FST98,FST99,FST00] by Feldman, Salmhofer and Trubowitz, where they discussed this inversion problem at the level of perturbation theory, an inversion theorem has been proved, which shown a unique solution to (1.3), i.e. a one-to-one map between the interacting and the non interacting Fermi surface. In reality, if the Fermi surface is fixed, the band structure changes, and vice versa. Hence,

instead of fixing the Fermi surface by using the counterterm we can alternatively adjust the Fermi surface during the renormalization group flow. This avoids counterterms and the associated inversion problem. Instead we have to check that the modified Fermi surface satisfies the same regularity properties of the free one. This is also not a easy task. A number of constructions has used this technique, e.g. [BGM03,Ped05,GM10]. A detailed explanation of Fermi surface flow can be found in [Sal07a].

In addition to the renormalization of the Fermi surface, multiscale analysis and discrete renormalization group flow technique will be heavily used in our construction. Multiscale rather than single scale constructive analysis is necessary because of the singularity of the propagator located at the Fermi surface, which gives rise to an infrared divergence problem. Hence the generating functional integral defining the Green's functions can not be treat in one piece. The basis idea of the multiscale analysis is to slice the propagator according to the size of its denominator. With these, we write the propagator as a sum of regular quantities. Integrating out a certain scale is the implementation of integrating out the fluctuation. The task becomes to control the limit of this sequence of integration. The discrete renormalization group technique is a tool to control that limit. It is an extention of the renormalization group of Wilson to long-range behavior governed by extended singularity. The perturbation expansion of the renormalization group map at each scale bases on some kind of tree expansion, which keeps a large fraction of the theory in unexpanded determinants. Tree expansion is much better than the usual Feynman expansion, which would be simply diverge at large order because there are two many Feynman graphs at large order. Tree expansion is the core of any constructive method.

The rest of this dissertation is divided into six parts. In chapter two we define our dilute Fermi gas and bilayer graphene models explicitly. Afterward our main results will be stated. Chapter three provides the necessary mathematical tools for our construction. We will represent the models in their functional integral forms. The generator of the connected, amputated Green's function, which we call effective action, will be defined. Its graphical representation will be discussed. We prove the convergence of the perturbation expansion of the effective action in the coupling constant in chapter four. In chapter five we study the regularity properties of the self-energy and its derivatives with respect to the frequency and momentum, which are crucial for establishing Fermi liquid behavior. In chapter six we extend our method to calculate the decay property of the response functions. Chapter seven contains the construction of Hubbard model for bilayer graphene.

Chapter 2

The Models and Results

In this chapter we define two models, a dilute Fermi gas model in continuum two dimensional space and a lattice model, which describe bilayer graphene. Our main results are presented subsequently.

2.1 Dilute Fermi gas

We consider an interacting fermions system in continuum two spatial dimensions without external potential. Given a side length $L \in \mathbb{R}$, an inverse temperature β and a periodic boundary condition, the configuration space is $\Lambda' = (\mathbb{R}/\beta\mathbb{R}) \times \Lambda$, $\Lambda = \left(\mathbb{R}^2/L\mathbb{R}^2\right)$, whose elements are given by $x = (\tau, \mathbf{x})$. $\tau = it$ is imaginary time, $\mathbf{x} = (x_1, x_2)$ is the two dimensional spatial component. We introduce a momentum space $\mathcal{B}' = \mathcal{M}_{k_0} \times \Lambda^*$, $\Lambda^* = \left(L\mathbb{R}^2/\mathbb{R}^2\right)$ is dual to Λ. The elements are $k = (k_0, \mathbf{k})$, with $k_0 = \frac{\pi}{\beta}(2n+1) \in \mathcal{M}_{k_0}$, $n \in \mathbb{Z}$, k_0s are called the Matsubara frequencies. Remark that only odd frequencies appear, because of anti-periodicity due to fermions. $k_i = 2\pi n/L$, $i = 1, 2$, $n \in \mathbb{Z}$.

A model for weakly interacting fermions at low temperature is characterized in term of a single free dispersion relation $E(\mathbf{k})$ (shifted by the chemical potential) on Λ^*, an ultraviolet cutoff smooth function $U(k)$ and an interaction V. In its second quantized form the Hamiltonian is defined as

$$H = -\sum_{\sigma} \int d^2\mathbf{k}\, E(\mathbf{k})\, c^\dagger_\sigma(\mathbf{k}) c_\sigma(\mathbf{k}) + V(c^\dagger, c), \tag{2.1}$$

where $c^\dagger$ and c are fermionic creation or annihilation operator, respectively. $\sigma = \pm$ denotes the spin of the fermions.

We state the hypotheses on the dispersion relation $E(\mathbf{k})$. Under a dilutness assumption the dispersion relation has the form

$$E(\mathbf{k}) = \mathbf{k}^2 - \frac{\nu}{\beta}, \tag{2.2}$$

where ν is a constant of order one. ν is chosen, such that at low temperature, $\nu/\beta \ll 1$ is a very small number. In (2.2) we put the mass $m = \frac{1}{2}$ to simplify the notation. Similar to the Jellium model constructed in [DR00a, DR00b] the non-interacting Fermi surface, which is by definition

$$S_0 = \{\mathbf{k} \in \Lambda^* \mid E(\mathbf{k}) = 0\}, \tag{2.3}$$

is rotational invariant. The main distinction between the Jellium model and the model discussed in this work situates at the extent of the Fermi surface. By dilute Fermi gas, we will show that the Fermi surface is so small, so that some significant properties of the Jellium model will lose, which may change the behavior of the system drastically at low temperature.

The ultraviolet cutoff function $U(k)$ is defined as a smooth function on $\mathcal{B}'$, which satisfies $0 \leq U(k) \leq 1$, for all $k \in \mathcal{B}'$. We assume that $U(k)$ vanishes for $k_0^2 + E^2(\mathbf{k}) > \epsilon_u^2$, where ϵ_u is the ultraviolet cutoff energy. The present of this ultraviolet cutoff function is necessary, it makes the Fourier transformed kernel of the propagator well defined. In fact, in order to preserve the physical positivity one needs this ultraviolet cutoff only on momentum. We restrict the cutoff both on momentum and on Matsubara frequency only for convenience.

The interaction V between the fermions is assumed to be a two-body interaction, which is short-range:

$$V(c^\dagger, c) = g \sum_{\sigma,\sigma'} \int d\mathbf{x} \int d\mathbf{y}\; v(\mathbf{x}-\mathbf{y})\; c_\sigma^\dagger(\mathbf{x}) c_{\sigma'}^\dagger(\mathbf{y}) c_\sigma(\mathbf{x}) c_{\sigma'}(\mathbf{y}) \tag{2.4}$$

where g is a small coupling constant. It can be either positive or negative. Due to short-range, the kernel of the interaction has to satisfy

$$\int d\mathbf{x}\; |\mathbf{x}|^2 |v(\mathbf{x})| < 1. \tag{2.5}$$

The model is studied in grand canonical ensemble. In a quantum field theory, the time evolved creation and annihilation operators are

$$\bar{\varphi}_\sigma(x) = e^{H\tau} c_\sigma^\dagger(\mathbf{x}) e^{-H\tau}, \qquad \varphi_\sigma(x) = e^{H\tau} c_\sigma(\mathbf{x}) e^{-H\tau}. \tag{2.6}$$

Note that $\bar{\varphi}_\sigma(x)$ is neither the complex conjugate, nor adjoint, of $\varphi_\sigma(x)$. The observables $\mathcal{O}$ are represented by polynomials of φ and $\bar{\varphi}$. The expected

value of an observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \operatorname{Tr}(e^{-\beta H} \mathcal{O}), \tag{2.7}$$

where $Z = \operatorname{Tr} e^{-\beta H}$. Note that we have shifted the chemical potential into $E(\mathbf{k})$, hence there is no $-\mu N$ in the exponential function.

The low temperature properties of the system are described by its correlation functions. The $2n$-point Green's functions are defined as the expectation values of

$$G_{2n,L}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \langle \mathbb{T} \varphi(x_1) \cdots \varphi(x_n) \bar{\varphi}(y_n) \cdots \bar{\varphi}(y_1) \rangle, \tag{2.8}$$

where $\mathbb{T}$ designates the time ordering operator, defined for fermion by

$$\mathbb{T} \varphi(x_1) \cdots \varphi(x_n) \bar{\varphi}(y_n) \cdots \bar{\varphi}(y_1) = \varphi(x_{\pi(1)}) \cdots \varphi(x_{\pi(n)}) \bar{\varphi}(y_{\pi(n+1)}) \cdots \bar{\varphi}(y_{\pi(2n)}),$$

where π is a permutation that obeys $\tau_{\pi(i)} \geq \tau_{\pi(i+1)}$ for all $1 \leq i < 2n$.

The question arises now whether these correlation functions (2.8) have a finite thermodynamic limit and whether a perturbation expansion in the coupling constant g, which can be used to get asymptotic behavior of the system, has non-zero radius of convergence. These are the main problems we want to treat in detail in this work.

Since the interaction modifies the Fermi surface, renormalization is necessary. We realize it by adding a counterterm $\delta e(\mathbf{k})$. Details will be represented in the following chapters. The main results on dilute Fermi gas model are

Theorem 1. *Let the spatial dimension $d = 2$. We assume that the dispersion relation is given by (2.2), V is a short-range interaction at a positive small temperature T. Then, there are positive constants C_1, C_2, C_3, and a suitable constant counterterm $|\delta e| < C_3|g|$, such that for all*

$$|g|^2 \log \beta < C_1, \quad \textit{if} \quad g > 0 \tag{2.9}$$

or

$$|g| \log \beta < C_2, \quad \textit{if} \quad g < 0, \tag{2.10}$$

the following hold:

1. *The renormalized $2n$-points Green's functions $G_{2n,L}$ converge uniformly in the thermodynamic limit to a translation invariant function G_{2n} that is analytic in the bare coupling constant g.*

2. *The self-energy $\Sigma(k)$ of the system is uniformly of $\mathbb{C}^1$. Moreover, the second derivative of the self-energy satisfies*

$$\left|\frac{\partial^2}{\partial k_i \partial k_j}\Sigma(k)\right| \leq C_4, \qquad C_5\beta \leq \left|\frac{\partial^2}{\partial k_0^2}\Sigma(k)\right| \leq C_6\beta, \tag{2.11}$$

where i and j take the values $1, 2$, and C_4, C_5, C_6 are some constants.

3. *Let $\Sigma(k) = \sum_{n=1}^{\infty}\Sigma_n(g^n, k)$, there are positive constants C_7, C_8, such that*

$$\left|\frac{\partial_{k_0}^2\Sigma_{n\geq 3}(k_0, \mathbf{k})}{\partial_{k_0}^2\Sigma_{n=2}(k_0, 0)}\right| \leq \begin{cases} C_7|g| < 1, & \text{if} \quad g > 0, \\ C_8|g|\log\beta < 1, & \text{if} \quad g < 0. \end{cases} \tag{2.12}$$

Remark. The Fermi surface of the interacting system is fixed to be a circle because of the spherical symmetry. The counterterm δe has to be a constant, and just a correction to the chemical potential. The shift of the Fermi surface is at most $O(|g|)$ to S_0. We don't have to treat the difficult inverse problem, which relates to study the differentiability properties of the counterterm.

The sign of the coupling constant g plays a crucial role in our renormalization analysis. The restrictions (2.9) and (2.10) on the coupling constant are obtained by a detailed analysis of the second order contributions. For an attractive interaction, the restriction $|g|\log\beta < C_2$ removes possible physical instabilities, for instance, the Cooper pairing, which leads to BCS instability. We show that the logarithm is due to the particle-particle ladder contribution. On the other side, the restriction is replaced by $g^2\log\beta < C_2$ for a repulsive interaction, the coupling constant remains bounded and decreases during the renormalization flow. We get an asymptotic free theory in the infrared limit. In fact for $g > 0$ a restriction that $g < C$ is actually sufficient to prove the analytic properties of the Green's function. (2.9) is required to guarantee the regularity of the self-energy.

In comparison with [DR00a, DR00b], in which a proof of Fermi liquid behavior was given for the Jellium model with on-site interaction, we can not entrench the Fermi liquid behavior for our model in sense of [Sal99]. The reason for this is the β like divergence of the second derivative of the self-energy with respect to the frequencies. The self-energy is actually not $\mathbb{C}^2$. The geometric reason for this is its small Fermi surface, which leads to the absent of a volume improvement coming from overlapping graphs.

The last part of the theorem shows that for a choice of the constants, the higher order contribution to the second derivative of the self-energy with respect to the frequencies is much smaller in comparison with its second

order contribution. In this sense, a second order approximation is essential for studying the properties of the self-energy.

As a corollary of our construction we can show further that in the analysis domain of g the correlation of the fermionic bilinears $\mathcal{B}(x)$ defined in Chapter 6 satisfies

$$\begin{aligned} & |\langle \mathcal{B}(x)\,;\mathcal{B}(y)\rangle| = |\langle \mathcal{B}(x)\mathcal{B}(y)\rangle - \langle \mathcal{B}(x)\rangle\langle \mathcal{B}(y)\rangle| \\ \leq \; & \frac{C}{|x-y|^2}. \end{aligned} \tag{2.13}$$

where C is a suitable constant.

To prove theorem 1 we will rewrite the model in its functional integral representation by using Grassmann variables. Effective action is introduced, which is the generator of connected amputated Green's functions. The theorem is proved by showing convergence of a suitable resummation of the weak coupling expansion for the effective action. Renormalization group and multiscale techniques are applied.

2.2 Bilayer graphene

In this section we introduce our second model. We consider a graphene lattice $\Lambda = \mathbb{G}/L\mathbb{G}$, where $L \in \mathbb{N}$ is the side length and $\mathbb{G}$ is the hexagonal lattice with basis $a_1 = \frac{1}{2}\left(3, \sqrt{3}\right)$ and $a_2 = \frac{1}{2}\left(3, -\sqrt{3}\right)$. The corresponding basis vectors in the reciprocal lattice Λ^* are $b_1 = \frac{2\pi}{3}\left(1, \sqrt{3}\right)$ and $b_2 = \frac{2\pi}{3}\left(1, -\sqrt{3}\right)$. Λ is a bipartite lattice including two sublattices A and B that are triangular Braviais lattices.

A graphite bilayer consists of two graphene monolayers including non-equivalent sites A_l, B_l and A_u, B_u in the bottom and top layers, respectively. They are typically arranged in the Bernel(AB)-stacking arrangement, those atom sites (top layer B_u sites and bottom layer A_l sites) do not have a neighbor in the opposite layer (Figure 2.1).

The tight-binding Hamiltonian for bilayer graphene at half filling with interaction in second quantized form can be written as

$$H = H_0 + V, \tag{2.14}$$

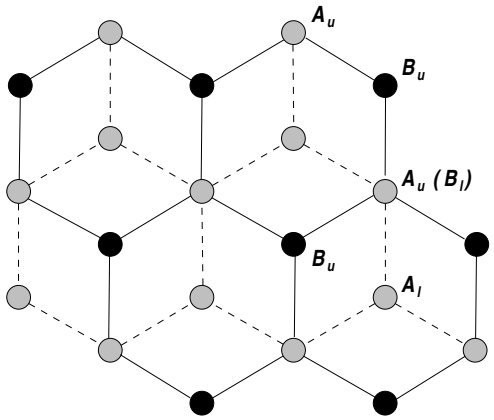

Figure 2.1: Schematic of a graphite bilayer arranged in the Bernel-stacking arrangement. Bonds in the top layer A_u, B_u are indicated by solid lines and in the bottom layer A_l, B_l by dashed lines. Top layer B_u and bottom layer A_l do not have their neighbors in the opposite layer.

where

$$\begin{aligned} H_0 \;=\; & -\gamma_0 \sum_{\sigma=\pm,\rho=l,u} \sum_{i=1}^{3} \sum_{\mathbf{x}\in\Lambda} \left(a^{\dagger}_{\rho,\sigma}(\mathbf{x}) b_{\rho,\sigma}(\mathbf{x}+\mathbf{u}_i) + b^{\dagger}_{\rho,\sigma}(\mathbf{x}+\mathbf{u}_i) a_{\rho,\sigma}(\mathbf{x}) \right) \\ & -\gamma_1 \sum_{\sigma=\pm} \sum_{\mathbf{x}\in\Lambda} \left(a^{\dagger}_{u,\sigma}(\mathbf{x}) b_{l,\sigma}(\mathbf{x}) + b^{\dagger}_{l,\sigma}(\mathbf{x}) a_{u,\sigma}(\mathbf{x}) \right) \\ & -\gamma_3 \sum_{\sigma=\pm} \sum_{i=1}^{3} \sum_{\mathbf{x}\in\Lambda} \left(a^{\dagger}_{l,\sigma}(\mathbf{x}) b_{u,\sigma}(\mathbf{x}+\mathbf{u}_i) + b^{\dagger}_{u,\sigma}(\mathbf{x}+\mathbf{u}_i) a_{l,\sigma}(\mathbf{x}) \right). \end{aligned} \tag{2.15}$$

represents hopping and

$$\begin{aligned} V = g \sum_{\rho=l,u} \sum_{\mathbf{x}\in\Lambda_\rho} \Bigg[& \left(a^{\dagger}_{\rho,\uparrow}(\mathbf{x}) a_{\rho,\uparrow}(\mathbf{x}) - \frac{1}{2} \right) \left(a^{\dagger}_{\rho,\downarrow}(\mathbf{x}) a_{\rho,\downarrow}(\mathbf{x}) - \frac{1}{2} \right) \\ & + \sum_{i=1}^{3} \left(b^{\dagger}_{\rho,\uparrow}(\mathbf{x}+\mathbf{u}_i) b_{\rho,\uparrow}(\mathbf{x}+\mathbf{u}_i) - \frac{1}{2} \right) \left(b^{\dagger}_{\rho,\downarrow}(\mathbf{x}+\mathbf{u}_i) b_{\rho,\downarrow}(\mathbf{x}+\mathbf{u}_i) - \frac{1}{2} \right) \Bigg] \end{aligned}$$

represents the on-site density density interaction. $a^{\dagger}_{\rho,\sigma}(\mathbf{x})$ and $a_{\rho,\sigma}(\mathbf{x})$ are creation or annihilation fermionic operators with spin index $\sigma = \pm$, layer index $\rho = u, l$ and site index $\mathbf{x} = (x_1, x_2) \in \Lambda$ on sublattice A, satisfying periodic boundary conditions in $\mathbf{x}$. An equivalent definition is used for $b^{\dagger}_{\rho,\sigma}(\mathbf{x})$ and $b_{\rho,\sigma}(\mathbf{x})$ on sublattice B. The vectors $\mathbf{u}_i$ are defined as

$$\mathbf{u}_1 = (1, 0)\,, \qquad \mathbf{u}_2 = \big(-\frac{1}{2}, \frac{\sqrt{3}}{2} \big), \qquad \mathbf{u}_3 = \big(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \big). \tag{2.16}$$

We denote γ_0 as the intralayer nearest neighbor hopping energy. The strength of interlayer coupling between A_u and B_l lattice sites is parametrized by γ_1. The next-nearest neighbor interlayer couplings are the $A_l \leftrightarrow B_u$ hops parametrized by γ_3, which is called trigonal warping. A number of experimental measurements give the values of these parameters. $\gamma_0 \approx 0.4$ eV, $\gamma_1 \approx 0.3$ eV and $\gamma_3 \approx 0.1 \sim 0.3$ eV.

g is the initial coupling constant of the on-site density density interaction. It can be either positive or negative. The role of $\frac{1}{2}$'s subtracted from the density $a^\dagger_{\rho,\sigma}(\mathbf{x})a_{\rho,\sigma}(\mathbf{x})$ is to keep the bilayer graphene at half filling. A significant property of the Hamiltonian (2.14) resides in its particle-hole symmetric. It is invariant under exchange

$$a^\dagger_{\rho,\sigma}(\mathbf{x}) \longleftrightarrow a_{\rho,\sigma}(\mathbf{x}), \qquad b^\dagger_{\rho,\sigma}(\mathbf{x}+\mathbf{u}_i) \longleftrightarrow -b_{\rho,\sigma}(\mathbf{x}+\mathbf{u}_i). \tag{2.17}$$

A direct consequence of this property is that the Fermi surface stays fixed as interaction turns on. As a result, renormalization of the Fermi surface becomes unnecessary.

We now restrict the vectors $\mathbf{k} \in \Lambda^*$ to the first Brillouin zone:

$$\mathcal{B}_L = \{\mathbf{k} = \frac{n_1}{L}b_1 + \frac{n_2}{L}b_2 :\ 0 \le n_1, n_2 \le L-1\}. \tag{2.18}$$

Given a periodic function $f : \Lambda \to \mathbb{R}$, its Fourier transform $\hat{f} : \mathcal{B}_L \to \mathbb{R}$ is defined as:

$$\hat{f}(\mathbf{k}) = \int_{\mathbf{x}\in\Lambda} d\mathbf{x}\ e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}), \tag{2.19}$$

the inverse relation is

$$f(\mathbf{x}) = \int_{\mathbf{k}\in\mathcal{B}_L} d\mathbf{k}\ e^{i\mathbf{k}\cdot\mathbf{x}} \hat{f}(\mathbf{k}), \tag{2.20}$$

where we abbreviate

$$\int_{\mathbf{k}\in\mathcal{B}_L} d\mathbf{k}\ \hat{f}(\mathbf{k}) = \frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_L} \hat{f}(\mathbf{k}), \qquad \int_{\mathbf{x}\in\Lambda} d\mathbf{x}\ f(\mathbf{x}) = \sum_{\mathbf{x}\in\Lambda} f(\mathbf{x}), \tag{2.21}$$

and $\mathbf{k}\cdot\mathbf{x} = k_1x_1 + k_2x_2$. The corresponding creation or annihilation operators in the momentum space are given by

$$a^{(\dagger)}_{\rho,\sigma}(\mathbf{k}) = \int_{\mathbf{x}\in\Lambda} d\mathbf{x}\ e^{\mp i\mathbf{k}\cdot\mathbf{x}}\ \hat{a}^{(\dagger)}_{\rho,\sigma}(\mathbf{x}), \quad b^{(\dagger)}_{\rho,\sigma}(\mathbf{k}) = \int_{\mathbf{x}\in\Lambda} d\mathbf{x}\ e^{\mp i\mathbf{k}\cdot\mathbf{x}}\ \hat{b}^{(\dagger)}_{\rho,\sigma}(\mathbf{x}), \tag{2.22}$$

they satisfy the anticommutate relations

$$\{a^{(\dagger)}_{\rho,\sigma}(\mathbf{k}), a^{(\dagger)}_{\rho',\sigma'}(\mathbf{k}')\} = |\Lambda|\,\delta_{\rho,\rho'}\delta_{\sigma,\sigma'}\delta_{\mathbf{k},\mathbf{k}'}, \quad \{b^{(\dagger)}_{\rho,\sigma}(\mathbf{k}), b^{(\dagger)}_{\rho',\sigma'}(\mathbf{k}')\} = |\Lambda|\,\delta_{\rho,\rho'}\delta_{\sigma,\sigma'}\delta_{\mathbf{k},\mathbf{k}'}$$

and $\{a^{\dagger}_{\rho,\sigma}(\mathbf{k}), a_{\rho',\sigma'}(\mathbf{k}')\} = 0$, $\{a^{(\dagger)}_{\rho,\sigma}(\mathbf{k}), b^{(\dagger)}_{\rho',\sigma'}(\mathbf{k}')\} = 0$, which are periodic over Λ^*. With these definitions, we introduce a vector state

$$\Xi^{\dagger}_{\sigma}(\mathbf{k}) = \left(a^{\dagger}_{l,\sigma}(\mathbf{k}), b^{\dagger}_{u,\sigma}(\mathbf{k}), a^{\dagger}_{u,\sigma}(\mathbf{k}), b^{\dagger}_{l,\sigma}(\mathbf{k})\right) \tag{2.23}$$

to write the non-interacting Hamiltonian (2.15) in momentum space as

$$H_0 = \sum_{\sigma} \int_{\mathbf{k}\in\mathcal{B}_L} d\mathbf{k}\, \Xi^{\dagger}_{\sigma}(\mathbf{k})\, H_0(\mathbf{k})\, \Xi_{\sigma}(\mathbf{k}), \tag{2.24}$$

with

$$H_0(\mathbf{k}) = -\begin{pmatrix} 0 & \gamma_3 A(\mathbf{k}) & 0 & \gamma_0 A^*(\mathbf{k}) \\ \gamma_3 A^*(\mathbf{k}) & 0 & \gamma_0 A(\mathbf{k}) & 0 \\ 0 & \gamma_0 A^*(\mathbf{k}) & 0 & \gamma_1 \\ \gamma_0 A(\mathbf{k}) & 0 & \gamma_1 & 0 \end{pmatrix} \tag{2.25}$$

and

$$A(\mathbf{k}) = 1 + 2e^{-i\frac{3}{2}k_1}\cos\frac{\sqrt{3}}{2}k_2 \tag{2.26}$$

is a complex number. The spectrum of this free Hamiltonian consists of four bands,

$$E^{\pm}_{\alpha} = \pm\sqrt{\frac{\gamma_1^2}{2} + \left(\gamma_0^2 + \frac{\gamma_3^2}{2}\right)|A(\mathbf{k})|^2 + (-1)^{\alpha}\sqrt{F}}, \tag{2.27}$$

with

$$F = \frac{1}{4}(\gamma_1^2 - \gamma_3^2|A(\mathbf{k})|^2)^2 + \gamma_0^2|A(\mathbf{k})|^2(\gamma_1^2 + \gamma_3^2|A(\mathbf{k})|^2) + \gamma_0^2\gamma_1\gamma_3(A(\mathbf{k})^3 + A^*(\mathbf{k})^3).$$

where $\pm$ refers to the conduction and valence band, and $\alpha = 1, 2$ to the lower energy associated with A sublattices at the bottom layer and B sublattices at the top layer and high energy band associated with A sublattices at the top layer and B sublattices at the bottom layer.

In the simplest case including only the nearest neighbor approximation $\gamma_3 = 0$, the low energy bands becomes

$$E^{\pm}_1(\mathbf{k}) = \pm\frac{\gamma_1}{2}\left(\sqrt{1 + \frac{4\gamma_0^2}{\gamma_1^2}|A(\mathbf{k})|^2} - 1\right) \tag{2.28}$$

which vanish at $\mathbf{k}_F = (\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})$. These two points are called Fermi points. Around them we observe that the spectrums is quadratic

$$E^{\pm}_1(\mathbf{k}' + \mathbf{k}_F) = \pm c_1|\mathbf{k}'|^2 + R(\mathbf{k}') \tag{2.29}$$

with c_1 as a constant, and $|R(\mathbf{k}')| < c_2|\mathbf{k}'|^3$. This makes the model resemble the dilute Fermi gas model, where the radius of the Fermi surface shrinks to zero.

If we take into the trigonal warping term ($\gamma_3 \neq 0$), for small energies, a Lifshitz transition occurs whereby the Fermi surface breaks into four points with linear spectrum around them. Details will be represented in the following chapters.

We summarize our results based on bilayer graphene in the following theorem:

Theorem 2. *We consider bilayer graphene at half filling with on-site interaction. Then there are positive constants C_1, C_2, C_3, such that for all*

$$\begin{array}{lll} |g| < C_1 \quad \textit{and temperature} \quad \beta^{-1} > e^{-\frac{C_2}{|g|}}, & \textit{if} \quad \gamma_3 = 0, \\ |g| < C_3, & \textit{if} \quad \gamma_3 \neq 0, \end{array} \tag{2.30}$$

the renormalized $2n$-points Green's functions $G_{2n,L}$ converge uniformly in the thermodynamic limit to a function G_{2n} that is analytic in the bare coupling constant g.

Remark. The theorem implies that without trigonal warping the system is stable above an exponentially small temperature. When trigonal warping turns on, the parabolic degeneracy of the Fermi point splits into Dirac points. This is very similar to the monolayer graphene model with Dirac point like Fermi surface. Thus the system is expected to be stable provided the coupling is small enough, depending on the size of the trigonal warping term. In other words, there are no true symmetry breaking states as the temperature goes down to zero.

Analogous to the proof of theorem 1 we prove this theorem by using renormalization group and multiscale techniques. Moreover, we make the following statement. The renormalization flow derived from the no trigonal warping case stops at a energy scale, where either the perturbation expansion diverges or the dispersion relation deviations from the quadratic form. In the former case the system enters a symmetry breaking phase. While in the later case if the coupling constant obtained at this scale is smaller than a critical value, it can be used as an initial coupling constant for the new flow associated with the Dirac points. Thus we conclude that for $|g| < C_3$, the perturbation expansion in g converges, even the temperature goes to zero, for $C_3 < |g| < C_1$, it converges only for temperature $\beta^{-1} > e^{-\frac{C_2}{|g|}}$.

Chapter 3

Techniques

The purpose of this chapter is to provide the necessary mathematical tools for the construction of the models. We introduce Grassmann algebra, which is an algebraically more convenient way to represent the system. The system will be reexpressed by its functional Grassmann integral obtained from the standard Lie product formula. Since the Fermi surface of non-interacting and of interacting system do not agree, renormalization becomes necessary. Therefore counterterm technique is introduced. We define the renormalized effective action, which is the generator of the connected, amputated Euclidean Green's functions. Its regularity shall be proven. The last part of this chapter is concerned with the graphical representation techniques. Feynman graph expansion and tree expansion for the effective action are discussed. This largely reproduces works done in [Sal99, SW00, FKT02].

3.1 The functional integral representation

In this section we introduce some basis definitions about Grassmann integration. We need them to represent our model as a functional integral.

Let us consider a finite dimensional Grassmann algebra, which is a set of anti-commuting Grassmann variables ψ_i, with i an index belonging to some finite set $\mathcal{N}$. They obey

$$\{\psi_i, \psi_j\} = \psi_i\psi_j + \psi_j\psi_i = 0, \tag{3.1}$$

for all $i, j \in \mathcal{N}$. In particular $\psi_i^2 = 0$ for all $i \in \mathcal{N}$.

The Grassmann integral is defined to be a unique linear map and

$$\int d\psi_i = 0, \qquad \int \psi_i d\psi_i = 1. \tag{3.2}$$

$d\psi$ anticommute also between themselves and with all ψ variables. More generally, we have

$$\int \psi_1 \cdots \psi_m \, d\psi_n \cdots d\psi_1 = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases} \tag{3.3}$$

A main advantage of the Grassmann representation is that a determinant of any n by n matrix can be expressed as a Grassmann Gaussian integral over $2n$ independent Grassmann variables, named as $\bar{\psi}_1, \ldots, \bar{\psi}_n, \psi_1, \ldots, \psi_n$. Note that $\bar{\psi}$ is not the complex conjugate of ψ, it is just another variable that is totally independent of ψ. The formula

$$\det M = \int \prod_{i=1}^{n} d\bar{\psi}_i d\psi_i \; e^{-\sum_{i,j} \bar{\psi}_i M_{ij} \psi_j} \tag{3.4}$$

holds for any matrix M. There is no positivity requirement for M like that for ordinary Gaussian measure with commuting variables. There are also normalized Grassmann Gaussian measure which may be expressed formally as

$$d\mu_C = \frac{\prod_{i=1}^{n} d\bar{\psi}_i d\psi_i \; e^{-\sum_{i,j} \bar{\psi}_i C_{ij}^{-1} \psi_j}}{\int \prod_{i=1}^{n} d\bar{\psi}_i d\psi_i \; e^{-\sum_{i,j} \bar{\psi}_i C_{ij}^{-1} \psi_j}}, \tag{3.5}$$

which are characterized by their covariance

$$\int d\mu_C(\psi) \; \bar{\psi}_i \psi_j = C_{ij}. \tag{3.6}$$

For the moment this is all we need for the Grassmann formulation of our model. More algebraic properties of the Grassmann algebra can be found in [Sal99].

Let us now introduce the functional integral as a Grassmann integral for dilute Fermi gas model. In order to get a finite dimensional Grassmann integral, we introduce a spacing in the imaginary time direction ϵ_τ, such that $n_\tau = \frac{\beta}{2\epsilon_\tau} \in \mathbb{N}$ is a large number. Let n_τ be even, and $\mathcal{T} = \{\tau = n\varepsilon_\tau : n \in \mathbb{Z}, -\frac{n_\tau}{2} \leq n \leq \frac{n_\tau}{2}\}$, we denote $\Lambda' = \mathcal{T} \times \left(\mathbb{R}^2/L\mathbb{R}^2\right)$, $X = (\tau, \mathbf{x}, \sigma) = (x, \sigma) \in \Gamma' = \Lambda' \times \{-1, 1\}$. Let $\mathcal{A}$ be the Grassmann algebra generated by $\bar{\psi}(X)$, $\psi(X)$, which are labeled by $X = (x, \sigma) \in \Gamma'$, we write also $\psi(X) = \psi_\sigma(\tau, \mathbf{x})$. We require that the fields ψ and $\bar{\psi}$ are anti-periodic with respect to translation of τ by β

$$\psi_\sigma(\tau, \mathbf{x}) = -\psi_\sigma(\tau + \beta, \mathbf{x}). \tag{3.7}$$

The elements of $\mathcal{A}$ are polynomials

$$\mathcal{V}(\psi) = \sum_{m,\bar{m}\geq 0} \int_{\Gamma'} d^{\bar{m}}\underline{X}' d^m \underline{X}\, v_{\bar{m},m}\left(\underline{X}', \underline{X}\right) \bar{\psi}^{\bar{m}}\left(\underline{X}'\right) \psi^m\left(\underline{X}\right), \tag{3.8}$$

where we denote

$$\int_{\Gamma'} dX F(X) = \sum_{\sigma} \epsilon_\tau \sum_{\tau\in\mathcal{T}} \int_{\Lambda} d^2\mathbf{x}\, F(x,\sigma), \tag{3.9}$$

$\underline{X} = (X_1, \ldots, X_m)$ and $\psi^m(\underline{X}) = \psi(X_1)\cdots\psi(X_m)$. The coefficient function $v_{\bar{m},m}(\underline{X}', \underline{X})$ is chosen to be antisymmetric under permutation of the X or X' variables separately, because any other part of it would cancel out in (3.8). The sums over m, $\bar{m}$ are finite sums because of the nilpotency of the Grassmann variables. We call $\mathcal{V}(\psi)$ even, if $m+\bar{m}$ is even. If $\mathcal{V}(\psi)$ is even, it commutes with all other elements of the Grassmann algebra.

We briefly discuss the norm imposed on $\mathcal{V}$, which is an element of the even subalgebra of $\mathcal{A}$. For $h > 0$, we define the seminorm $\|\mathcal{V}\|_h$ by

$$\|\mathcal{V}\|_h = \sum_{\bar{m},m\geq 1} |v_{\bar{m},m}|_{1,\infty}\, h^{\bar{m}+m}, \tag{3.10}$$

where $|v_{\bar{m},m}|_{1,\infty}$ is defined as

$$|v_{\bar{m},m}|_{1,\infty} = \max_i \sup_{X_i} \int \prod_{j\neq i}^{\bar{m}+m} dX_j\ \left|v_{\bar{m},m}(X_1, \ldots, X_{\bar{m}+m})\right|. \tag{3.11}$$

Note that the term with $\bar{m} = m = 0$ is left out from (3.10), then the constant polynomials have zero norm values.

We now turn to the partition function $Z_{\Lambda'}$. The standard Grassmann integral representation is obtained by applying the Lie product formula

$$e^{-\beta(H_\Lambda - \mu N_\Lambda)} = \lim_{n_\tau\to\infty} \left(e^{-\epsilon_\tau(H_0-\mu N_\Lambda)} e^{-\epsilon_\tau V}\right)^{n_\tau} \tag{3.12}$$

to the trace for $Z_{\Lambda'}$. For a finite space Λ', all operators are just finite-dimensional matrices, hence the right hand side of (3.12) converges. The derivation of the functional integral is discussed in [Sal99], we just state the result here. (3.12) implies

$$Z_{\Lambda'} = \lim_{n_\tau\to\infty} Z_{n_\tau,\Lambda'} \tag{3.13}$$

with

$$Z_{n_\tau,\Lambda'} = \mathcal{N}_{\Lambda'} \int \prod_{X\in\Gamma'} d\bar{\psi}(X) d\psi(X)\ e^{S_{\Lambda'}(\bar{\psi},\psi)}, \tag{3.14}$$

where $\mathcal{N}_{\Lambda'}$ is a normalization factor that depends on ϵ_τ, n_τ and L, and where

$$S_{\Lambda'}(\bar{\psi},\psi) = \sum_\sigma \int_{\mathcal{T}} d\tau \int_\Lambda d\mathbf{x} \left(\bar{\psi}_\sigma(\tau,\mathbf{x})\partial_\tau\psi_\sigma(\tau,\mathbf{x}) - H_\Lambda(\bar{\psi},\psi)\right). \quad (3.15)$$

Here we have abbreviated $\int_{\mathcal{T}} d\tau \;\; F(\tau) = \epsilon_\tau \sum_{\tau\in\mathcal{T}} F(\tau)$, and $\partial_\tau\psi_\sigma(\tau) = \epsilon_\tau^{-1}(\psi_\sigma(\tau+\epsilon_\tau) - \psi_\tau(\tau))$. The sum over τ runs over $\mathcal{T}$, with anti-periodic boundary condition [Sal99]. For $n_\tau < \infty$, $L < \infty$, (3.14) is a finite dimensional Grassmann integral. Moreover, the action $S_{\Lambda'}$ can be rewritten in the form

$$S_{\Lambda'}(\bar{\psi},\psi) = S_{2,\Lambda'}(\bar{\psi},\psi) - V(\psi), \quad (3.16)$$

with

$$S_{2,\Lambda'}(\bar{\psi},\psi) = -\int_{\Gamma'} dX \int_{\Gamma'} dX' \; \bar{\psi}_{\sigma'}(x') \; A(X,X') \; \psi_\sigma(x). \quad (3.17)$$

Assume that $\tilde{E}(\mathbf{x},\mathbf{x}')$ is the Fourier transform of the dispersion relation $E(\mathbf{k})$. The operator $A(X,X') = \delta_{\sigma\sigma'}(-\partial_\tau\delta_\Lambda(\mathbf{x}-\mathbf{x}') + \tilde{E}(\mathbf{x},\mathbf{x}')\delta(\tau-\tau'))$ is invertible because of the anti-periodicity condition imposed on the fields, which removes the zero mode of the discrete time derivative. $A(X,X')$ is estimated by the dispersion relation of the model in the momentum space. The Fourier transformation of ψ and $\bar{\psi}$ are

$$\psi_\sigma(k) = \int_{\Lambda'} dx \; e^{-ikx}\psi_\sigma(x), \qquad \bar{\psi}_\sigma(k) = \int_{\Lambda'} dx \; e^{-ikx}\bar{\psi}_\sigma(x) \quad (3.18)$$

where, for $k = (k_0,\mathbf{k})$, $kx = k_0\tau + \mathbf{k}\cdot\mathbf{x}$. The momentum k is in $\mathcal{B}' = \mathcal{M}_{k_0,n_\tau} \times \left(L\mathbb{R}^2/\mathbb{R}^2\right)$, with

$$\mathcal{M}_{k_0,n_\tau} = \{k_0 = \frac{\pi}{\beta}(2n+1) : n\in\mathbb{Z}, -\frac{n_\tau}{2} \le n \le \frac{n_\tau}{2}\}, \quad (3.19)$$

is a set of Matsubara frequencies. We denote

$$\int_{\mathcal{B}'} dk F(k) = \frac{1}{\beta} \sum_{k_0\in\mathcal{M}_{k_0,n_\tau}} \int d^2\mathbf{k} \; F(k_0,\mathbf{k}) \quad (3.20)$$

where $\int d^2\mathbf{k} = L^{-2}\int d^2\mathbf{k}$. Then the inverse Fourier transform is $\psi_\sigma(x) = \int_{\mathcal{B}'} dk \; e^{ikx} \; \psi_\sigma(k)$. In the dilute Fermi gas model we assume that the non-interacting dispersion relation including the chemical potential has the form

$$E(\mathbf{k}) = \mathbf{k}^2 - \frac{\nu}{\beta}, \quad (3.21)$$

where ν is a constant of order 1. Denoting $\delta_{k,-k'} = \delta_{\mathbf{k},-\mathbf{k}'}\epsilon_\tau^{-1}\delta_{k_0,-k_0'}$, and

$$\hat{k}_0 = \frac{1}{i\epsilon_\tau}(e^{i\epsilon_\tau k_0} - 1), \tag{3.22}$$

with this the Fourier transformation of the operator A in $S_{2,\Lambda'}$ is given by

$$A(k,\sigma,k',\sigma') = \delta_{k,-k'}\delta_{\sigma\sigma'}(i\hat{k}_0 - E(\mathbf{k})), \tag{3.23}$$

since A is invertible, the propagator $C(X,X') = A^{-1}(X,X')$ exists, it is a skew symmetric matrix, $C(X,X') = -C(X',X)$, its Fourier transform is

$$C(k) = \delta_{k,-k'}\delta_{\sigma\sigma'}\frac{U(k)}{i\hat{k}_0 - E(\mathbf{k})}, \tag{3.24}$$

where an ultraviolet cutoff function $U(k)$ has been inserted to preserve the physical positivity.

For a non-interacting system, where $V = 0$, the partition function is

$$\begin{aligned} Z_{n_\tau,\Lambda'} &= \mathcal{N}_{\Lambda'}\int\prod_{X\in\Gamma'} d\bar{\psi}(X)d\psi(X)\; e^{S_{2,\Lambda'}(\bar{\psi},\psi)} \\ &= \mathcal{N}_{\Lambda'}\det A \end{aligned} \tag{3.25}$$

which is non zero for positive temperature. For an interacting system, the interaction term V is an element of the even Grassmann subalgebra,

$$V(\psi) = \int dX_1dX_2dX_3dX_4\; v\,(X_1,X_2,X_3,X_4)\,\bar{\psi}(X_1)\psi(X_2)\bar{\psi}(X_3)\psi(X_4).$$

In the present work, we consider a two-body interaction, the interaction kernel is given by

$$v\,(X_1,X_2,X_3,X_4) = g\delta\,(X_1,X_2)\,\delta\,(X_3,X_4)\,\delta(\tau_1-\tau_3)v(x_1-x_3), \tag{3.26}$$

where $\delta\,(X,X') = \delta\,(\tau-\tau')\,\delta\,(\mathbf{x}-\mathbf{x}')\,\delta_{\sigma,\sigma'}$, $\delta(\tau-\tau') = \epsilon_\tau^{-1}\delta_{\tau,\tau'}$. g is the coupling constant. It can be either positive or negative. Hence

$$V(\psi) = g\sum_{\sigma,\sigma'}\int d\mathbf{x}d\mathbf{y}\; v(\mathbf{x}-\mathbf{y})\bar{\psi}_\sigma(\mathbf{x})\psi_\sigma(\mathbf{x})\bar{\psi}_{\sigma'}(\mathbf{y})\psi_{\sigma'}(\mathbf{y}). \tag{3.27}$$

Putting (3.16), (3.23) and (3.27) into (3.17), the interacting partition function becomes

$$Z_{n_\tau,\Lambda'} = \mathcal{N}_{\Lambda'}\det A\int d\mu_C(\psi)\; e^{-V(\psi)}, \tag{3.28}$$

where $d\mu_C$ is a linear functional (Grassmann Gaussian measure) defined by

$$d\mu_C(\psi) = (\det A)^{-1} \prod_{X\in\Gamma'} d\bar{\psi}(X) d\psi(X) e^{-\int_{\Gamma'} dX \int_{\Gamma'} dX' \; \bar{\psi}_{\sigma'}(x') \; C^{-1}(X,X') \; \psi_\sigma(x)},$$

the measure is normalized, $\int d\mu_C(\psi) = 1$, and it is characterized by

$$\int d\mu_C(\psi) \; \bar{\psi}(X')\psi(X) = C(X, X'). \tag{3.29}$$

We are interested in the $2n$-point Euclidean Green's functions or Schwinger function of the system determined by C and V. The constant $\mathcal{N}_{\Lambda'} \det A$ in $Z_{n_\tau,\Lambda'}$ drops out of all correlation functions and can therefore be omitted. We can define the Green's functions by the following Grassman Gaussian integral

$$\begin{aligned} & G_{2n,n_\tau,\Lambda'}(X'_1, \ldots, X'_n, X_1, \ldots, X_n) \\ = \; & \frac{1}{Z_{n_\tau,\Lambda'}} \int d\mu_C(\psi) \prod_{i=1}^{n} \bar{\psi}(X'_i) \prod_{i=1}^{n} \psi(X_i) \; e^{-V(\psi)}, \end{aligned} \tag{3.30}$$

where $Z_{n_\tau,\Lambda'} = \int d\mu_C(\psi) \; e^{-V(\psi)}$. Let $(\phi(X))_{X\in\Gamma'}$ be a family of Grassmann generators. We define the Grassmann bilinear form as

$$(\bar{\psi}, \phi) = \int_{\Gamma'} dX \; \bar{\psi}(X)\phi(X), \tag{3.31}$$

which is antisymmetric: $(\bar{\psi}, \phi) = -(\phi, \bar{\psi})$. By doing so, the partition function with source term is defined as

$$S_{n_\tau,\Lambda'}(\phi) = \frac{1}{Z_{n_\tau,\Lambda'}} \int d\mu_C(\psi) \; e^{-V(\psi)+(\bar{\phi},\psi)+(\bar{\psi},\phi)}. \tag{3.32}$$

Let $\frac{\delta}{\delta\phi(X)} = \epsilon_\tau^{-1} \frac{\partial}{\partial\phi(X)}$, then

$$\begin{aligned} & G_{2n,n_\tau,\Lambda'}(X'_1, \ldots, X'_n, X_1, \ldots, X_n) \\ = \; & \frac{1}{S_{n_\tau,\Lambda'}(V)(0)} \left[\prod_{k=1}^{n} \frac{\delta}{\delta\phi(X_k)} \prod_{k=1}^{n} \frac{\delta}{\delta\bar{\phi}(X'_k)} S_{n_\tau,\Lambda'}(V)(\phi) \right]_{\phi=0}. \end{aligned} \tag{3.33}$$

$S_{n_\tau,\Lambda'}(V)(\phi)$ is called the generator of the Euclidean Green's functions.

It is convenient to study the connected correlation functions, which are the derivatives

$$\begin{aligned} & G_{c,2n,n_\tau,\Lambda'}(X'_1, \ldots, X'_n, X_1, \ldots, X_n) \\ = \; & \left[\prod_{k=1}^{n} \frac{\delta}{\delta\phi(X_k)} \prod_{k=1}^{n} \frac{\delta}{\delta\bar{\phi}(X'_k)} \log S_{n_\tau,\Lambda'}(V)(\phi) \right]_{\phi=0}. \end{aligned} \tag{3.34}$$

We call $C_{n_\tau,\Lambda'}(V)(\phi) = \log S_{n_\tau,\Lambda'}(V)(\phi)$ the generator of the connected Green's functions. Since $S_{n_\tau,\Lambda'}(V)(\phi)$ is the exponential of $C_{n_\tau,\Lambda'}(V)(\phi)$, one can construct all correlation functions from the connected ones.

Instead of $C_{n_\tau,\Lambda'}(\phi)$ we will study

$$W_{n_\tau,\Lambda'}(V)(\phi) = \log \int d\mu_C(\psi)\; e^{-V(\psi+\phi)}. \tag{3.35}$$

A shift in the measure shows that

$$W_{n_\tau,\Lambda'}(V)(\phi) = (\bar{\phi}, C^{-1}\phi) + C_{n_\tau,\Lambda'}(V)(C^{-1}\phi). \tag{3.36}$$

Comparing this to $C_{n_\tau,\Lambda'}(\phi)$, we see that apart from the explicit prefactor, this is a usual formula for $C_{n_\tau,\Lambda'}(\phi)$, only that the source term is replaced by $C^{-1}\phi$, so that the study of $W_{n_\tau,\Lambda'}(V)(\phi)$ is equivalent to that of $S_{n_\tau,\Lambda'}(V)(\phi)$. $W_{n_\tau,\Lambda'}(V)(\phi)$ is called the generator of the connected, amputated Green's functions, or the effective action.

We briefly discuss the limit $n_\tau \to \infty$ and the thermodynamic limit $L \to \infty$. In the limit $n_\tau \to \infty$ the time variable becomes continuous, $\hat{k}_0 \to k_0$, and the set of Matsubara frequencies becomes $\mathcal{M}_{k_0} = \{\frac{\pi}{\beta}(2l+1) : l \in \mathbb{Z}\}$. The Grassmann integral becomes infinite dimensional, which is a well defined object. See Appendix A of [FKT02]. Then one proves that the term in the right side of (3.35) is well defined at finite n_τ and L, as elements of a finite Grassmann integral. One can also prove that it has a well defined limit as $n_\tau \to \infty$, and afterward $L \to \infty$

$$W(V)(\phi) = \lim_{L\to\infty} \lim_{n_\tau\to\infty} W_{n_\tau,\Lambda'}(V)(\phi). \tag{3.37}$$

This is achieved by studying the perturbation expansion of (3.35) and by showing that it is uniformly convergent in n_τ and L. The proof of the existence of these limits can be found in [Sal99]. In order to simplify the discussions, we study the model directly in these limits, since the perturbation expansion has nothing to do with the details related with the finite values of n_τ and L.

$W(V)(\phi)$ has an important property which will be wildly used in the following. Let us consider the case where $C = C_1 + C_2$, $\psi = \psi_1 + \psi_2$, then

$$\begin{aligned} e^{W(V)(\phi)} &= \int d\mu_{C_2}(\psi_2) \int d\mu_{C_1}(\psi_1)\; e^{-V(\psi_1+\psi_2+\phi)} \\ &= \int d\mu_{C_2}(\psi_2)\; e^{V_1(\psi_2+\phi)}, \end{aligned} \tag{3.38}$$

with $V_1(\psi_2+\phi) = \log \int d\mu_{C_1}(\psi)\; e^{-V(\psi+\phi)}$. Thus integrating out ψ_1 associated with the propagator C_1 generates an effective action $V_1(\psi_2+\phi)$

for the remaining ψ_2 field. Generally, if the propagator is a sum of n terms, $C = \sum_{i=1}^{n} C_i$, with $C_i \geq 0$ for all $i \in \{1, \ldots, n\}$, the field is a sum $\psi = \sum_{i=1}^{n} \psi_i$ of n independent fields. Successive integration over $\psi_1, \ldots, \psi_n$ generates a sequence of $n+1$ effective actions $V_0, \ldots, V_{n+1}$, where $V_0 = -V$ is the initial interaction and

$$V_i = \log \int d\mu_{C_i}(\psi_i)\; e^{V_{i-1}(\psi_i + \phi)} \tag{3.39}$$

is obtained from V_{i-1} by integrating over the field ψ_i. (3.39) defines the renormalization group map. The sequence $(V_0, V_1, \ldots, V_{n+1})$ builds the renormalization group flow.

If the effective action $W(V)$ is a well-defined object, it is an element of even Grassmann subalgebra. It has the expansion

$$W(V)(\psi) = \sum_{m,\bar{m} \geq 0} \int d^{\bar{m}} \underline{X}' d^m \underline{X}\; w_{\bar{m},m}(\underline{X}', \underline{X})\; \bar{\psi}^{\bar{m}}(\underline{X}') \psi^m(\underline{X}), \tag{3.40}$$

whose kernel function $w_{\bar{m},m}(\underline{X}', \underline{X})$ is the connected, amputated $m + \bar{m}$ points Green's function.

The main task of this work is to prove the analytical properties of the effective action $W(V)(\phi)$. If the propagator C were really nice, $W(V)$ would be very easy to define rigorously. However, life is not so easy, due to the singularity of the propagator on the Fermi surface, we have to write the propagator as a sum of really nice propagators, and we control the limits of a sequence of well defined quantities. This makes the proof more cumbersome and difficult.

3.2 Renormalization of the Fermi surface

We are interested in whether the thermodynamic limit $G_{c,2n}$ of the connected Green functions exists and whether in an infinite volume the weak coupling expansion

$$G_{c,2n} = \sum_{p=0}^{\infty} g^p G_{c,2n,p} \tag{3.41}$$

converges for small g. As is well known, the most singular case is that of zero temperature, $T = 0$. At infinite volume, already the coefficients $G_{c,2n,p}$ diverge for $p \geq 3$ [FT90, FT91]. To remove this divergence one has to introduce an infrared cutoff $\epsilon > 0$ to make the infinite volume model well defined. At positive temperature or at finite volume case the expansion

obtained by expanding e^{gV} in g is convergent, but its radius of convergence shrinks to zero in the thermodynamic and zero temperature limit.

This divergence can be connected to the singularity of the propagator C. More precisely, powers of C are in general not locally integrable

$$\int dk \frac{1}{|ik_0 - E(\mathbf{k})|^\alpha} = \infty, \tag{3.42}$$

for $\alpha \geq 2$. By momentum conservation and Feynman rules, (3.42) implies that graphs, which contain a string of two legged insertions produce arbitrarily high powers of C and thus diverges [FT90, FT91].

The physical reason for this divergence of the unrenormalized Green functions has been well known for a long time [FT90, FST96]. One expects that the interaction produces a self-energy $\Sigma(g, k)$ such that the propagator behaves essentially as

$$C(k) = \frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(g, k)}. \tag{3.43}$$

We expand

$$\frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(g, k)} = \sum_{n=0}^{\infty} \frac{1}{ik_0 - E(\mathbf{k})} \left(\frac{\Sigma(g, k)}{ik_0 - E(\mathbf{k})} \right)^n, \tag{3.44}$$

by (3.42), no term of the right hand side, except $n = 0$, is locally integrable. The problem arises because we are attempting to expand the interacting propagator, which has a singularity at $k_0 = 0$ on the interacting Fermi surface $S_g = \{\mathbf{k} \in \Lambda^* : E(\mathbf{k}) + \Sigma(g, 0, \mathbf{k}) = 0\}$ in powers of the free propagator which has a singularity when $k_0 = 0$ and $\mathbf{k}$ on the free Fermi surface $S_0 = \{\mathbf{k} \in \Lambda^* : E(\mathbf{k}) = 0\}$.

The infrared divergence makes renormalization necessary. One way is to replace the free propagator in terms of the exact interacting propagator, and two legged insertions disappear in the graphs. But in practice, $\Sigma(g, k)$ is not known in advance, and it is expected to have less regularities than the free dispersion relation.

We do renormalization by using counterterms [Sal98b]. We write $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k})$, where $e(\mathbf{k})$ has the property that its zero set $\{\mathbf{k} \in \Lambda^* : e(\mathbf{k}) = 0\}$ coincides with the interacting Fermi surface

$$S_g = \{\mathbf{k} \in \Lambda^* : E(\mathbf{k}) + \Sigma(g, 0, \mathbf{k}) = 0\}. \tag{3.45}$$

Note that the condition $\{\mathbf{k} \in \Lambda^* : e(\mathbf{k}) = 0\} = S_g$ does not uniquely determine the decomposition $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k})$. It only forces

$$\delta e(\mathbf{k}) + \Sigma(g, 0, \mathbf{k}) = 0, \tag{3.46}$$

for all $\mathbf{k} \in S_g$. In other words, δe is constructed from the self-energy by evaluating at $k_0 = 0$ and $\mathbf{k} \in S_g$. For a dilute Fermi gas model one can always find a decomposition $E(\mathbf{k}) = e(\mathbf{k}) + \delta e(\mathbf{k})$ satisfying (3.46), because of the rotational symmetry. So that $\delta e(\mathbf{k})$ does not depend on $\mathbf{k}$ and thus is just a correction to the chemical potential. It is expected to have more regularities than the self-energy. Moreover, one can show that δe remains small for a small enough coupling constant g.

Now denoting the propagator with E by $C(E)$ and the one with e by $C(e)$, and the partition function Z_E and Z_e, we have, by shift formulas for Gaussian measures, the identity

$$\begin{aligned}
& \log \frac{1}{Z_E} \int d\mu_{C(E)}(\psi)\ e^{-V(\psi+\phi)} \\
= \ & \log \left(\frac{Z_e}{Z_E}\right)^2 \frac{1}{Z_e} \int d\mu_{C(e)}(\psi)\ e^{-V(\psi+\phi)} e^{(\bar{\psi},\delta e\psi)} \\
= \ & \log \left(\frac{Z_e}{Z_E}\right)^2 \frac{1}{Z_e}\ e^{-(\bar{\phi},\delta e\phi)} \int d\mu_{C(e)}(\psi)\ e^{-V(\psi+\phi)+(\bar{\psi}+\bar{\phi},\delta e\psi+\phi)}\ e^{-(\bar{\phi},\delta e\psi)-(\bar{\psi},\delta e\phi)} \\
= \ & \log \left(\frac{Z_e}{Z_E} e^{-(\bar{\phi},\delta e\phi)}\right)^2 \frac{1}{Z_e} \int d\mu_{C(e)}(\psi)\ e^{-V(\psi+\phi)+(\bar{\psi}+\bar{\phi},\delta e\psi+\phi)} \\
= \ & \log \left(\frac{Z_e}{Z_E} e^{-(\bar{\phi},\delta e\phi)}\right)^2 + \log \frac{1}{Z_e} \int d\mu_{C(e)}(\psi)\ e^{-V(\psi+\phi)+(\bar{\psi}+\bar{\phi},\delta e\psi+\phi)}. \qquad (3.47)
\end{aligned}$$

This identity is obtained by moving the $\delta e(\mathbf{k})$ from the propagator to the interaction, and leaving e in the propagator. The change in the normalization factor is irrelevant for any correlation function.

Physically, this procedure means that we shift the Fermi surface from the free surface S_0 to the interacting Fermi surface S_g. The deformation of the surface caused by the interaction is taken into account. δe compensates for all self-energy corrections that would move the Fermi surface under the interaction. Then we do the renormalized expansion at the fixed interacting Fermi surface S_g with a new interaction by putting the counterterm into the initial interaction,

$$\mathcal{V}(\psi) = -V(\psi) + \mathcal{K}(\psi), \qquad (3.48)$$

where $\mathcal{K}(\psi)$ has a bilinear form

$$\mathcal{K}(\psi) = \delta e \int dX \bar{\psi}(X)\psi(X). \qquad (3.49)$$

Now let us define the renormalized generating functional for the connected amputated Green functions

$$W(\mathcal{V})(\phi) = \log \frac{1}{Z} \int d\mu_C(\psi)\ e^{\mathcal{V}(\psi+\phi)} \qquad (3.50)$$

with $C = C(e)$, and Z is a constant such that $W(\mathcal{V})(0) = 0$. In the following sections the regularity of $W(\mathcal{V})(\phi)$ will be shown by imposing some conditions on g.

Remark. We conclude from the discussion that the unrenormalized Green functions diverge because it is wrong to assume that both the band structure and the Fermi surface stay fixed when the interaction is turned on. In reality, if the surface being fixed, the band structure changes, and vice versa. Hence instead of posing counterterm one can also adjust the Fermi surface dynamically in the flow. This method was described in detail in [Sal07b, PS03] and has been used applied in applied studies [HSFR01] and in different constructions [Ped05, GM10].

3.3 Bounds on propagators

In order to study the perturbation expansion of the effective action we need to know the asymptotic behavior of the propagator C. In the following we establish some bounds on the propagator.

We observe that the Gaussian integral of a monomial vanishes unless there are as many $\bar{\psi}$ as ψ in it,

$$\int d\mu_C(\psi) \prod_{k=1}^{m} \bar{\psi}(X_k) \prod_{l=1}^{n} \psi(Y_l) = \begin{cases} 0 & m \neq n, \\ (-1)^{\frac{m(m+1)}{2}} \det(C(X_k, Y_l))_{1 \leq k,l \leq m} & m = n. \end{cases}$$

To bound this determinant whose entries are C we need

Definition 1 (Gram bound, see [SW00])**.** *We assume that the propagator can be written as an inner product on some Hilbert spaces $\mathcal{H}$, that is*

$$\begin{aligned} &\forall X \in \Lambda, \exists f_X, g_Y \in \mathcal{H} : C(X,Y) = \langle f_X, g_Y \rangle \\ &\textit{and} \quad \exists \gamma_C > 0, |f_X| < \gamma_C, |g_X| < \gamma_C \end{aligned} \tag{3.51}$$

then for a $n \times n$ matrix $C(X_k, Y_l)_{1 \leq k,l \leq n}$ holds

$$|\det C(X_k, Y_l)_{1 \leq k,l \leq n}| \leq \prod_{k=1}^{n} |f_{X_k}| \, |g_{X_K}| \leq \gamma_C^{2n}. \tag{3.52}$$

Definition 2 (Determinant bound, see [PS08])**.** *Let C be an $n \times n$ matrix with matrix elements $C(X_k, X_l)_{k,l}$. A finite constant δ_C is called the determinant bound of C if for all $1 \leq k, l \leq n$*

$$\sup_{p_1,\ldots,p_n,q_1,\ldots,q_n \in B^{(n)}} \left| \det \left(\langle p_k, q_l \rangle \, C(X_k, X_l') \right)_{k,l} \right| \leq \delta_C^{2n}. \tag{3.53}$$

Here $B^{(n)}$ denotes the closed n-dimensional unit ball.

Note that if C has a Gram representation with a Gram constant γ_C then C also has a determinant bound $\delta_C = \gamma_C$. The main use of Gram representations is to bound determinants arising in the expansion. The reason for introducing the determinant bound is that at large Matsubara frequency the propagator C does not admit a Gram representation with a finite Gram constant due to the slow decay of the propagator C.

To show the convergence of perturbation series we also need to know the decay properties of the propagator C. We put $C(\tau, \mathbf{x}) = C(X_1, X_2)$, where $\tau = \tau_1 - \tau_2$ and $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, by a translation invariance. The decay bound α_C^ν is by definition

Definition 3 (Decay bound, see [PS08])**.** *Let $n = (n_0, n_1, n_2)$ be a multi-index of non-negative integers, and we write $|n| = \sum_{i=0}^{2} |n_i|$, then*

$$\alpha_C^\nu = \max_{n:|n|=\nu} \int_{-\beta/2}^{\beta/2} d\tau \int_\Lambda d^2\boldsymbol{x}\, |C(\tau, \mathbf{x})|\, |\tau|^{n_0} \prod_{i=1}^{2} |x_i|^{n_i}\,. \tag{3.54}$$

3.4 Graphical representations

Since we want to implement renormalization group map in a constructive way, we need a graphical representation to calculate the effective action non-perturbatively, with good properties for the convergence radius of the expansion. By a formal expansion series in g, we write

$$W(\mathcal{V})(\phi) = \sum_{p \geq 1} \frac{g^p}{p!}\, \mathcal{E}_p(\mathcal{V})(\phi), \tag{3.55}$$

for elements $\mathcal{V}_1, \ldots, \mathcal{V}_p$ of the even subalgebra,

$$\begin{aligned} & \mathcal{E}_p(\mathcal{V}_1, \ldots, \mathcal{V}_p)(\phi) \\ = \; & \left[\frac{\partial^p}{\partial g_1 \cdots \partial g_p} \log \int d\mu_C(\psi)\; e^{g_1 \mathcal{V}_1(\psi+\phi) + \cdots g_p \mathcal{V}_p(\psi+\phi)} \right]_{g_i = 0, \forall i \in \{1,\ldots,p\}} \\ & - \left[\frac{\partial^p}{\partial g^p} \log Z \right]_{g=0}. \end{aligned} \tag{3.56}$$

We may regard $g_1, \ldots, g_p$ as formal variables since they are only used to do combinatorics. The subtraction of Z removes all ϕ independent terms from $\mathcal{E}_p(\phi)$. Note that $\mathcal{E}_p(\mathcal{V}_1, \ldots, \mathcal{V}_p)(\phi)$ is linear in every $\mathcal{V}_i$, $i = 1, \ldots, p$. Because the derivative is evaluated at $g = 0$, the ϕ dependent parts of $\mathcal{E}_p(\mathcal{V}_1, \ldots, \mathcal{V}_p)$

can be replaced by

$$\left[\frac{\partial^p}{\partial g_1 \cdots \partial g_p} \log \int d\mu_C(\psi) \prod_{q=1}^{p} (1 + g_q \mathcal{V}_q(\psi + \phi))\right]_{g_i=0, \forall i \in \{1,\ldots,p\}}$$

$$= \left[\frac{\partial^p}{\partial g_1 \cdots \partial g_p} \log(1 + \sum_{\substack{Q \subset \mathbb{N}_p \\ Q \neq 0}} \int d\mu_C(\psi) \prod_{q \in Q} \mathcal{V}_q(\psi + \phi) \prod_{q \in Q} g_q)\right]_{g_i=0, \forall i \in \{1,\ldots,p\}} \tag{3.57}$$

for any $Q \subset \mathbb{N}_p$. Similarly, we can replace Z by (3.57) evaluated at $\phi = 0$. The Gaussian convolutions contained in the right hand side of (3.57) can be rewritten in term of the action of a Laplacian acting on p independent copies of the field ψ. The Laplace operator is defined as Δ_C

$$\Delta_C = \sum_{q,q'=1}^{p} \Delta_{C,qq'}, \tag{3.58}$$

where

$$\Delta_{C,qq'} = -\int dX' \int dX \, \frac{\partial}{\partial \bar{\psi}_q(X')} \, C(X, X') \frac{\partial}{\partial \psi_{q'}(X)}, \tag{3.59}$$

We have the identity

Lemma 1. *For all $Q \subset \mathbb{N}_p$ and $Q \neq 0$,*

$$\int d\mu_C(\psi) \prod_{q \in Q} \mathcal{V}_q(\psi + \phi) = e^{\Delta_C} \prod_{q \in Q} \mathcal{V}(\psi + \phi) \tag{3.60}$$

holds.

Proof. Using source fields η and $\bar{\eta}$, we rewrite

$$\mathcal{V}_q(\bar{\psi} + \bar{\phi}, \psi + \phi) = \left[\mathcal{V}_q\left(-\frac{\partial}{\partial \eta_q}, \frac{\partial}{\partial \bar{\eta}_q}\right) e^{(\bar{\eta}_q, \psi + \phi) + (\bar{\psi} + \bar{\phi}, \eta_q)}\right]_{\eta_q = \bar{\eta}_q = 0}, \tag{3.61}$$

then

$$\begin{aligned}
&\int d\mu_C(\psi) \prod_{q\in Q} \mathcal{V}_q(\psi+\phi) \\
&= \left[\int d\mu_C(\psi) \prod_{q\in Q} \mathcal{V}_q\left(-\frac{\partial}{\partial\eta_q}, \frac{\partial}{\partial\bar{\eta}_q}\right) e^{(\bar{\eta}_q,\psi+\phi)+(\bar{\psi}+\bar{\phi},\eta_q)}\right]_{\eta_q=\bar{\eta}_q=0,\forall q\in Q} \\
&= \left[\prod_{q\in Q} \mathcal{V}_q\left(-\frac{\partial}{\partial\eta_q}, \frac{\partial}{\partial\bar{\eta}_q}\right) \prod_{q\in Q}\int d\mu_C(\psi) e^{(\bar{\eta}_q,\psi+\phi)+(\bar{\psi}+\bar{\phi},\eta_q)}\right]_{\eta_q=\bar{\eta}_q=0,\forall q\in Q} \\
&= \left[\prod_{q\in Q} \mathcal{V}_q\left(-\frac{\partial}{\partial\eta_q}, \frac{\partial}{\partial\bar{\eta}_q}\right) e^{\sum_{q,q'\in Q}(\bar{\eta}_q, C\eta_{q'})+\sum_{q\in Q}(\bar{\eta}_q,\phi)+(\bar{\phi},\eta_q)}\right]_{\eta_q=\bar{\eta}_q=0,\forall q\in Q} \\
&= e^{\Delta_C} \prod_{q\in Q} \mathcal{V}_q(\psi+\phi). \qquad (3.62)
\end{aligned}$$

□

By this identity, we denote $\alpha(Q) = e^{\Delta_C} \prod_{q\in Q} \mathcal{V}_q(\psi+\phi)$, then (3.57) becomes

$$\frac{\partial^p}{\partial g_1 \cdots \partial g_p} \log(1 + \sum_{\substack{Q\subset\mathbb{N}_p \\ Q\neq 0}} \alpha(Q) \prod_{q\in Q} g_q). \qquad (3.63)$$

Before evaluating (3.63) we define what is a graph. Let $Q \subset \mathbb{N}$ be a finite set and we define

$$\mathcal{G}(Q) = \mathbb{P}\left(\{(i,j) \in Q\times Q : i<j\}\right), \qquad (3.64)$$

where $\mathbb{P}(A)$ denotes the set of all subset of A. A graph G is an element of $\mathcal{G}(Q)$. More precisely, G is a set of ordered pairs of $(i,j) \in Q\times Q$ with $i<j$. Now we see that $1 + \sum_{\substack{Q\subset\mathbb{N}_p \\ Q\neq 0}} \alpha(Q)$ takes the form of a polymer partition function, with the nonempty subset $\mathbb{N}_p$ as polymers and disjointness as the compatibility relation. Using the standard polymer formula for the logarithm of the partition function in [Sal99]

$$\log(1 + \sum_{\substack{Q\subset\mathbb{N}_p \\ Q\neq 0}} \alpha(Q)) = \sum_{m\geq 1} \frac{1}{m!} \sum_{\substack{I_1,\ldots,I_m\subset\mathbb{N}_p \\ I_i\neq\emptyset}} \mathcal{U}_c^{(m)}(I_1,\ldots,I_m) \prod_{l=1}^{m} \alpha_c(I_l) \qquad (3.65)$$

where $\mathcal{U}_c^{(m)}$ is the Ursell function, $\mathcal{U}_c^{(1)} = 1$, for $m \geq 2$

$$\mathcal{U}_c^{(m)}(I_1,\ldots,I_m) = \sum_{G\in\mathcal{G}_c(\mathbb{N}_m)} \prod_{\{i,j\}\in G} \gamma(i,j) \qquad (3.66)$$

with $\gamma(i,j) = -1$, if $I_i \cap I_j \neq \emptyset$ and 0 otherwise. $\mathcal{G}_c(\mathbb{N}_p)$ denotes the set of connected graphs on $\mathbb{N}_p$. $\alpha_C(\emptyset) = 1$, $\alpha_c(Q)$ is the uniquely map which satisfies

$$\alpha(Q) = \sum_{Q_0 \subset Q, \min Q \in Q_0} \alpha_c(Q_0)\alpha(Q \backslash Q_0), \tag{3.67}$$

where $\min Q$ stands for the smallest number contained in Q (See Lemma 1 in [SW00]). To simplify (3.63) we observe that for all $m \geq 2$, some g_i remain after differentiation because of the connectedness condition in $\mathcal{U}_c^{(m)}$, so that setting $g_i = 0$ picks out the term $m = 1$ from the sum,

$$\left[\frac{\partial^p}{\partial g_1 \cdots \partial g_p} \log(1 + \sum_{\substack{Q \subset \mathbb{N}_p \\ Q \neq 0}} \alpha(Q) \prod_{q \in Q} g_q) \right]_{g_i = 0, \forall i \in \{1,\ldots,p\}} = \alpha_c(\mathbb{N}_p). \tag{3.68}$$

which implies the final expression for $\mathcal{E}_p(\mathcal{V})$, that is

$$\mathcal{E}_p(\mathcal{V})(\phi) = \left(e^{\Delta_C}\right)_c \left(\prod_{q=1}^{p} \mathcal{V}_q(\psi + \phi) - \prod_{q=1}^{p} \mathcal{V}_q(\psi) \right) \tag{3.69}$$

the subscript c means that only connected parts of e^{Δ_C} have contributions to $\mathcal{E}_p(\mathcal{V})$.

3.4.1 Feynman graphs expansion

The expansion of e^{Δ} can be written in term of a sum over Feynman graphs. Because all $\Delta_{qq'}$ commute with one another,

$$\begin{aligned} e^{\Delta} &= \prod_{q=1}^{p} e^{\Delta_{qq}} \prod_{q<q'} (1 + e^{\Delta_{qq'} + \Delta_{q'q}} - 1), \\ &= \prod_{q=1}^{p} e^{\Delta_{qq}} \sum_{G \in \mathcal{G}(\mathbb{N}_p)} \prod_{\{q,q'\} \in G} (e^{\Delta_{qq'} + \Delta_{q'q}} - 1) \end{aligned} \tag{3.70}$$

where from the first to the second line multinomial theorem

$$\prod_{r \in R} (1 + a_r) = \sum_{S \subset R} \prod_{r \in S} a_r \tag{3.71}$$

with $R = \{(i,j) : i < j\}$, has been used. G denotes the sum over graphs on $\mathbb{N}_p$. Selecting only connected graphs we get

$$(e^{\Delta})_c = \prod_{q=1}^{p} e^{\Delta_{qq}} \sum_{G \in \mathcal{G}_c(\mathbb{N}_p)} \prod_{\{q,q'\} \in G} (e^{\Delta_{qq'} + \Delta_{q'q}} - 1) \tag{3.72}$$

with G now summed over connected graphs on $\mathbb{N}_p$. Applying (3.72) to $\prod_{q=1}^{p} \mathcal{V}_q(\psi+\phi)$ generates a sum over all connected Feynman graphs

$$\mathcal{E}_p(\mathcal{V}_1,\ldots,\mathcal{V}_p)(\phi) = \sum_{\bar{m},m\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X}\; w_{\bar{m},m,p}(\underline{X}',\underline{X})\; \bar{\phi}^{\bar{m}}(\underline{X}')\phi^m(\underline{X})$$

where

$$w_{\bar{m},m,p}(\underline{X}',\underline{X}) = \sum_{G\in\mathcal{G}_c(\mathbb{N}_p)} \mathrm{Val}(G), \tag{3.73}$$

with

$$\mathrm{Val}\, G = (-1)^{a_p} \int \prod_{i\in I} dX_i \prod_{q=1}^{p} v_q(\underline{X}_q,\underline{X}'_q) \prod_{l=(i,i')\in L_I} C(X_i,X'_{i'}), \tag{3.74}$$

where $a_p = 0,1$, I is set of all internal fields, L_I denotes the lines connecting the internal fields. The expansions (3.74) allows to write the functional integral as a sum of Feynman graphs which can be explicitly computed. But several problems will be arose in the practical applications. One problem above mentioned is that some Feynman graphs diverge when the cut offs introduced to regularize the theory are removed. Second, even if the Feynman graphs are bounded, the number of Feynman graphs proliferates at large order as $(p!)^2$, which can not lead to a convergence of the series. Hence the Feynman graph expansion is not suitable to get non-perturbation results and other methods must be used.

3.4.2 Tree expansion

Here we describe methods, which are suited to organize expansions such that analytical statements can be proven. We name the methods as the tree expansion. The tree expansion ultimately may be considered just as repacking Feynman graphs in some clever ways according to the underlining tree. They could also be seen as a kind of polymer expansion. The presented tree expansion was introduced by Salmhofer and Wieczerkowski in [SW00], which is actually a variant of the well-known cluster expansion techniques for fermions. Cluster expansion was found by Brydges and Kennedy [BK87]. It was redesigned into a more explicit formalism in [AR95]. The tree expansion is not the only way to prove the convergence. A ring expansion invented by Feldman, Knörr and Trubowitz in [FKT02] is also very useful.

The motivation for a tree expansion is due to the observation that for fermionic theories, because of the Pauli principle there are important cancellations among the Feynman graphs which make the behavior of the series

expansion at large order much better than what seems like at first sight. Mathematically this is because these graphs form a determinant and the size of a determinant is much less than what its permutation expansion usually suggests.

The determinant can be bounded by Gram estimates introduced in the previous section if the Gram representation exists. But in this way we cannot use the decay properties of the propagators in order to perform the coordinate integrations appearing in the expansions. What we need is to extract the minimal number of propagators which are connected with each other and built a spanning tree, to perform the integrations, leaving all other fields packed in the determinant. This is the main idea behind the tree expansion. The advantage of the tree formula is that, the species of trees does not proliferate at large orders, in contrast with the species of Feynman graphs. Indeed the number of trees is estimated by Cayley's theorem:

Theorem 3. *The number of labeled unordered trees with p vertices is p^{p-2}. The number of such trees with fixed coordination numbers d_i is $\frac{(p-2)!}{\prod_i (d_i-1)!}$.*

The theorem implies that there are only $p^{p-2} \leq (p-1)! e^{p-1}$ labeled trees on p vertices. $(p-1)!$ gets canceled by the $p!$. Thus this resummation might lead to a convergence proof.

We begin to introduce our tree formula. The resumming of a graphical expansion in term of trees can be realized in different ways. It depends on the choice of the tree and the interpolation parameters. To discuss the dependence we introduce some notations. For a p by p matrix H, and $Q \subset \mathbb{N}_p$, let $\Delta_Q[H] = \sum_{q,q' \in Q} H_{qq'} \Delta_{qq'}$. We abbreviate $\Delta_{\mathbb{N}_p}[H] = \Delta[H]$. $H_{qq'}$ is a weight factor associated to the line $\{q, q'\}$. H is always real and symmetry. The tree expansion introduced in [SW00] has the form

Theorem 4 (Theorem **3** in [SW00])**.** *Let H be a real symmetric matrix, and $H \geq 0$. Then*

$$\begin{aligned} & (e^{\Delta[H]})_c (\mathbb{N}_p) \\ = & \sum_{T \in \mathcal{T}_p} \prod_{\{q,q'\} \in T} H_{qq'} \left(\Delta_{qq'} + \Delta_{q'q} \right) \int_{[0,1]^{p-1}} d\mathbf{s} \sum_{\pi \in \prod(T)} \varphi(T, \pi, \mathbf{s}) \, e^{\Delta[H(T,\pi,\mathbf{s})]} \end{aligned} \tag{3.75}$$

where $\mathbf{s} = (s_1, \ldots, s_{p-1})$, $d\mathbf{s} = ds_1, \ldots, ds_{p-1}$, $\varphi(\pi, \mathbf{s}) \geq 0$, and $H(T, \pi, \mathbf{s})$ is a non-negative symmetric matrix with diagonal entries $(H(T, \pi, \mathbf{s}))_{qq} = H_{qq}$. The sum over π runs over a T-dependent set $\prod(T)$ of permutations π of $\mathbb{N}_p$

and

$$\int d\mathbf{s} \sum_{\pi \in \prod(T)} \varphi\left(T, \pi, \mathbf{s}\right) = 1. \tag{3.76}$$

(3.75) is a variant of Brydges-Battle-Federbush formula [BF82, BF84, Bat84, BF78, BK87]. Other variants of the tree formula can be found in [AR95]. In our case H is a matrix with all entries equal to one. The positivity of the interpolation matrix $H(T, \pi, \mathbf{s})$ is crucial in the constructive theory, which ensures that there exists a uniform optimal Gram estimate which does not dependent on the trees for the remaining determinant. Note that not all tree expansions have a positive matrix $H(T, \pi, \mathbf{s})$, which ensures the existence of a Gram representation for the determinant. For example, a rooted tree expansion [AR95] has no positive $H(T, \pi, \mathbf{s})$ matrix.

The tree is not yet the usual Gallavotti-Nicolo tree [GN85]. The Gallavotti-Nicolo tree arises in multiscale analysis, since it shows inclusion relations of the vertices in different scales. This means that the vertex function on a scale j can be expressed as a function of the initial coupling function directly. The Gallavotti-Nicolo tree was used in several constructions [DR00a, DR00b, Riv02, SA05b, BGM06, GM10].

To get a deep understanding of the tree expansion, we try to motivate the proof of (3.75). The proof of the present tree expansion theorem can be found in [SW00]. The idea behind the expansion is to make a Taylor expansion with integral remainder to interpolate between a coupled one (with parameter 1) and an uncoupled one (with parameter 0). For $0 \neq A \subset \mathbb{N}_p$, let

$$\tilde{\Delta}_{A,q}[H] = \sum_{q' \in A} H_{q'q} \left(\Delta_{qq'} + \Delta_{q'q}\right). \tag{3.77}$$

Then, if $H = H^T$,

$$\begin{aligned}
\Delta_Q[H^{(A,1)}] &= \Delta_Q[H] \\
\Delta_Q[H^{(A,0)}] &= \Delta_A[H] + \Delta_{Q\setminus A}[H] \\
\Delta_Q[H^{(A,s)}] &= \Delta_A[H] + \Delta_{Q\setminus A}[H] + s \sum_{q \in Q\setminus A} \sum_{q' \in A} H_{q'q} \left(\Delta_{qq'} + \Delta_{q'q}\right),
\end{aligned} \tag{3.78}$$

$s \in [0, 1]$. A Taylor expansion gives

$$e^{\Delta_Q[H]} = e^{\Delta_A[H] + \Delta_{Q\setminus A}[H]} + \sum_{q \in Q\setminus A} \sum_{q' \in A} H_{q'q} \left(\Delta_{qq'} + \Delta_{q'q}\right) \int_0^1 ds\, e^{\Delta_Q[H^{(A,s)}]}.$$

We have expressed $e^{\Delta_Q[H]}$ as a sum of two terms, in the first term the elements of A is decoupled from the rest of Q, in the second one a bound between the elements of A and the elements in Q is exhibited. This bound is also the tree line connected the two subgraphs. We iterative this procedure to expand $e^{\Delta_Q[H^{(A,s)}]}$, until all the vertices are connected, as a result we get the following lemma:

Lemma 2 (Lemma 10 in [SW00]). *Let $Q \subset \mathbb{N}_p$, $H = H^T$. For $r \geq 1$ let*

$$\mathcal{S}_r = \{\mathbf{q} = (q_1, \dots, q_r) : q_1 = \min Q, \forall i : q_i \in Q, q_i \neq q_j, if\ i \neq j\}, \quad (3.79)$$

Then for all $R \geq 1$,

$$e^{\Delta_Q[H]} = \sum_{r=1}^{R} \sum_{\mathbf{q} \in \mathcal{S}_r(Q)} e^{\Delta_{Q \setminus A_r}[H]} \int \prod_{w=1}^{r-1} ds_w \tilde{\Delta}_{A_w, q_{w+1}}[H_w] e^{\Delta_{A_r}[H_r]} + \mathcal{R}_{R+1} \quad (3.80)$$

with $A_w = \{q_1, \dots, q_w\}$, the H_r being defined recursively as $H_1 = H$, $H_{w+1} = H_w^{(A_w, s_w)}$, and a remainder terms

$$\mathcal{R}_{R+1} = \sum_{\mathbf{q} \in \mathcal{S}_{R+1}(Q)} \int_{w=1}^{R} ds_w \tilde{\Delta}_{A_w, q_{w+1}}[H_w] e^{\Delta_Q[H_{R+1}]} \quad (3.81)$$

This lemma can be proven by an induction on R. The statement for $R = 1$ is (3.80), with $A = \{q_1\}$. In the remainder term, the sum over $\mathbf{q} \in \mathcal{S}_{R+1}(Q)$ includes a sum over $q_{R+1} \notin A_R$. Let $A_{R+1} = A_R \cup \{q_{R+1}\}$, and $H_{R+2} = H_{R+1}^{A_{R+1}, s_{R+1}}$, the lemma follows.

If $R = |Q|$, $\mathcal{S}_{R+1} = \emptyset$, so the remainder term vanishes, and we get

$$e^{\Delta_Q[H]} = \sum_{J \subset Q, \min Q \in J} e^{\Delta_{Q \setminus J}[H]} \mathcal{K}(J), \quad (3.82)$$

where for $|J| = j$,

$$\mathcal{K}(J) = \sum_{\mathbf{q} \in \mathcal{S}_j(J)} \int_{i=1}^{j-1} ds_i \tilde{\Delta}_{\{q_1, \dots, q_i\}, q_{i+1}}[H_i] e^{\Delta_J[H_j]}. \quad (3.83)$$

By Lemma 1 in [SW00], follows $(e^{\Delta_Q[H]})_c = \mathcal{K}(Q)$. After some reformations we get the final expression (3.75).

In the following we show that the tree expansion ensures the the positivity of the interpolation matrix $H(\pi, T, \mathbf{s})$.

Lemma 3 (Lemma 9 in [SW00]). *Let H be a positive symmetric p by p matrix, $A \subset \mathbb{N}_p$ and $s \in [0,1]$. Then the interpolated matrices $H^{(A,s)}$ with entries $(H^{(A,s)})_{qq'} = sH_{qq'}$, if $q \notin A$, and $q' \in A$, or $q \in A$, and $q' \notin A$, and $(H^{(A,s)})_{qq'} = H_{qq'}$ otherwise, are positive.*

Proof. The diagonal elements of $H^{(A,s)}$ remain unchanged $(H^{(A,s)})_{qq} = H_{qq}$, and the matrix remains symmetric. Since the permutations of the rows or columns do not change the positivity properties of $H^{(A,s)}$, we denote $A_c = \mathbb{N}_p \backslash A$, we can write

$$H^{(A,s)} = \begin{pmatrix} H_{AA} & H_{AA_c} \\ H_{AA_c} & H_{A_cA_c} \end{pmatrix} = sH + (1-s) \begin{pmatrix} H_{AA} & 0 \\ 0 & H_{A_cA_c} \end{pmatrix} \tag{3.84}$$

The block diagonal matrix inherits positivity from H. Thus $H^{(A,s)}$ is positive. □

The positivity property of $H^{(A,s)}$ is crucial, since it allows to apply Gram's estimates:

Lemma 4. *Let $H^{(A,s)}$ be a positive symmetric matrix, with diagonal elements $(H^{(A,s)})_{qq} = H_{qq} \leq 1$. Then for a matrix, whose entries are given by $H^{(A,s)}_{qq'} C(X, X')$ has a Gram constant γ_C.*

Proof. Since $H^{(A,s)}$ is a positive symmetric matrix, then it is a Gram matrix with

$$0 \leq \det H^{(A,s)} \leq \prod_{q=1}^{p} H^{(A,s)}_{qq}. \tag{3.85}$$

Its Gram constant is 1. Using the fact that if A and B are Gram matrices with Gram constant a and b, and $C_{qq'} = A_{qq'} B_{qq'}$, then C is a Gram matrix with Gram constant ab. The lemma follows directly from this fact. □

3.5 Wick ordering

For some purposes we shall expand a polynomial $\mathcal{V} \in \mathcal{A}$ in the basis for the Grassmann algebra given by the Wick ordered monomials $: \bar{\psi}^{\bar{m}}(\underline{X}')\psi^{m}(\underline{X}) :_C$

$$V(\psi) = \sum_{m,\bar{m} \geq 0} \int d^{\bar{m}}\underline{X}' d^{m}\underline{X}\; v_{\bar{m},m}(\underline{X}', \underline{X}) \; : \bar{\psi}^{\bar{m}}(\underline{X}')\psi^{m}(\underline{X}) :_C \tag{3.86}$$

The Wick ordering is defined as

Definition 4. *Let* $\mathcal{W}(\eta,\psi) = e^{(\bar\eta,\psi)+(\bar\psi,\eta)-\int dX' \int dX\ \bar\eta(X')C(X,X')\eta(X)}$. *Wick ordering is a map:* $\mathcal{A}\to\mathcal{A}$, *that takes the value,* $:1:_C = 1$, *and for* $m+\bar m \geq 2$,

$$:\bar\psi^{\bar m}(\underline{X}')\psi^m(\underline{X}):_C = \left[\prod_{k=1}^m \frac{\partial}{\partial\bar\eta(X'_k)}\prod_{k=1}^m \frac{\partial}{\partial\eta(X_l)}\mathcal{W}(\eta,\psi)\right]_{\bar\eta=\eta=0} \tag{3.87}$$

It is easy to see that the Wick ordered monomial is an element of the even Grassmann algebra, and antisymmetric under permutations of the X or X' variables separately. The Wick ordering map is bijective.

An alternative formula for the Wick ordered monomial is contained in the following identity [Sal98a]: Let Δ_C be defined by (3.59), then

$$:\bar\psi^{\bar m}(\underline{X}')\psi^m(\underline{X}):_C = e^{-\Delta_C}\ \bar\psi^{\bar m}(\underline{X}')\psi^m(\underline{X}). \tag{3.88}$$

This formula will be used in the renormalization group analysis. We remark that in our tree expansion the Wick ordering does not play an important role. It simply drops the diagonal terms from our Laplacians. As a consequence, the norm parameter would be shifted by a constant,

$$\|\mu_{-C} * \mathcal{V}\|_h \leq \|\mathcal{V}\|_{h+\gamma_C} \tag{3.89}$$

At this point we have introduced most mathematical backgrounds for the constructions. In next chapter we can begin our renormalization group analysis for the effective action $W(\mathcal{V})$.

Chapter 4

The Renormalization Group Analysis

In this chapter we consider the effective action of the dilute Fermi gas. The key task is to prove that in perturbation theory the generating functional has a finite convergence radius in the coupling constant. We show that the convergence radius depends on the sign of the initial coupling constant by a detailed analysis of the so-called ladder contributions. The proof is based on multiscale expansions and graphical representation for the generating functional. Instead of replacing the usual Feynman graphs representation we will use tree formula to evaluate the fermionic expectations, since the bound obtained by using Feynman graphs would diverge at large order, due to the large number of the graphs. On the other hand, multiscale rather than single scale analysis becomes necessary, since the convergence radius shrinks when the energy scale goes to the infrared cutoff. We establish further that the shrinking rate of the convergence radius is logarithmic. In order to complete our program, the existence of counterterm will be estimated by requiring zero renormalized mass. In dilute Fermi gas model, where one has rotational symmetry, $\delta e < C|g|$, does not depend on $\mathbf{k}$ and is a correction to the chemical potential. In order to simplify the discussion, we study the model directly in the thermodynamical limit $L \to \infty$. The relevant estimated properties of the model leave unchanged.

4.1 Multiscale decomposition

We start to evaluate the effective action of the dilute Fermi gas model

$$\mathcal{W}(\mathcal{V})(\phi) = \log \frac{1}{Z} \int d\mu_C(\psi) e^{\mathcal{V}(\psi+\phi)}, \tag{4.1}$$

where $\mathcal{V}$ is given by (3.48), which contains the counterterm, and $C = C(e)$, ϕ is another set of Grassmann variables. For a model with non empty Fermi surface a single scale analysis is not possible because there is an infrared problem due to the singularity of the propagator $\frac{U(k)}{ik_0 - e(\mathbf{k})}$ on the Fermi surface. This singularity causes the norm of the propagator to be infinite. Consequently, it would not be easy to define $\mathcal{W}(\mathcal{V})(\phi)$ rigorously.

To analyze the singularity we begin to illustrate the multiscale integration of the effective action. The basic idea of the multiscale analysis is to scale the propagator. One writes C as a limit of regular propagators, and controls the limit of a sequence of well defined quantities. Our model has an ultraviolet cutoff $U(k)$, hence it is convenient to decompose the propagator as

$$\begin{aligned} C(k) &= C_>(k) + C_<(k) \\ &= \sum_{j=-\infty}^{0} \frac{f_j(k)}{ik_0 - e(\mathbf{k})} + \frac{f_1(k)}{ik_0 - e(\mathbf{k})}, \end{aligned} \tag{4.2}$$

where $C_>(k)$ and $C_<(k)$ have supports on the ultraviolet region and the infrared region, respectively. Correspondingly we decompose the field as

$$\psi = \psi^{(1)} + \sum_{j=-\infty}^{0} \psi^{(j)}, \tag{4.3}$$

where $\psi^{(1)}$ and $\psi^{(j)}$ are independent fields with supports on the ultraviolet and infrared region, respectively.

Let us now define the partition of unity given by the f_j. We choose a scale parameter M which eventually has to be chosen sufficiently large and a smooth function $h \in \mathbb{C}^\infty$ that takes values in $[0, 1]$, is identically 1, if $x < 1$ and 0 if $x > 2$. The function f may be constructed by setting $f(x) = h(x) - h(Mx)$, and obeys

$$\sum_{j=-\infty}^{0} f(M^{-2j}x) = 1 \tag{4.4}$$

for all $0 < x < 1$. Then for $j \leq 0$ the j-th scale function is defined as

$$f_j(k) = f(M^{-2j}(k_0^2 + e(\mathbf{k})^2)). \tag{4.5}$$

By construction $f_j(k)$ is identically one on

$$\{k \in \mathcal{B}' \mid \sqrt{\frac{2}{M}}\, M^j \leq |ik_0 - e(\mathbf{k})| \leq M^j\}. \tag{4.6}$$

The support of $f_j(k)$ is called the jth shell. By construction, it is contained in

$$\{k \in \mathcal{B}' \mid \sqrt{\frac{1}{M}}\, M^j \leq |ik_0 - e(\mathbf{k})| \leq \sqrt{2}M^j\}. \tag{4.7}$$

The momentum k is said to be of scale j if k lies in the jth shell. With this definition of the partition function f_j, $f_1(k)$ has to be chosen such that it obeys $f_1(k) = U(k) - \sum_{j=-\infty}^{0} f_j(k)$.

At finite temperature $|k_0| \geq \pi/\beta$, $\forall k_0 \in \mathcal{M}_{k_0}$, we define the infrared cutoff scale:

$$J = \max\left\{j \leq 0, M^{j-1} < \pi/\beta\right\}, \tag{4.8}$$

then we have the identity

$$C(k) = \sum_{j=J}^{0} C_j(k) + C_1(k) \tag{4.9}$$

with $C_j(k) = \frac{f_j(k)}{ik_0 - e(\mathbf{k})}$, and the Fourier transform of $C_j\,(k)$ is

$$C_j\left(X, X'\right) = \delta_{\sigma,\sigma'} \int dk\, \frac{e^{ik(x-x')}}{ik_0 - e\,(\mathbf{k})}\, f_j\,(k)\,. \tag{4.10}$$

4.2 Multiscale integration

In this section we implement the multiscale integration of the effective action. The treatment of the first renormalization group step i.e. the ultraviolet regime, is a simple application of the tree expansion. The infrared integral will be done by using discrete renormalization group flow. The expansion is written in term of a set of running coupling functions.

4.2.1 The ultraviolet integration

The definition of the Grassmann integration implies the following identity (additional principle):

$$\begin{aligned} \mathcal{W}(\mathcal{V})(\phi) &= \log \frac{1}{Z} \int d\mu_{C_<}(\psi^{(\leq 0)}) \int d\mu_{C_1}(\psi^{(1)}) e^{\mathcal{V}(\psi+\phi)} \\ &= \log \frac{1}{Z_0} \int d\mu_{C_<}(\psi^{(\leq 0)}) e^{\mathcal{V}^{(0)}(\psi^{(\leq 0)})}, \end{aligned} \tag{4.11}$$

where we have abbreviated $\psi^{(\leq 0)} = \psi^{(\leq 0)} + \phi$ in $\mathcal{V}^{(0)}(\psi^{(\leq 0)})$, which is the effective potential at scale 0, given by

$$\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \log \int d\mu_{C_1}(\psi^{(1)})\; e^{\mathcal{V}(\psi+\phi)}. \tag{4.12}$$

It is called the ultraviolet part of the integration as well. Z_0 is defined by the condition $W(\mathcal{V})(0) = 0$.

The analysis of the ultraviolet integration (4.12) is far easier than the infrared one because the propagator C_1 is really nice. It is easy to see that there is no relevant or marginal term on scale > 0, except those which are contained by contracting two fields associated with the same space time point in a vertex, i.e. the tadpoles. The integration can be done by applying a tree expansion for the effective action. However due to the slow decay of the Matsubara frequency, it is not easy to find a good Gram representation for C_1 to bound the determinant for this ultraviolet integration. As a possible circumvent one has to do multiscale expansion. However, an approach via determinant bound introduced in [PS08] is much simpler to do this integral, it makes the multiscale analysis unnecessary. The regularity properties of $\mathcal{V}^{(0)}$ are summarized in the following lemma:

Lemma 5. *There is a positive constant $\tilde{g}$, such that for $|g| < \tilde{g}$, $\mathcal{V}^{(0)}(\psi^{(\leq 0)})$ is analytic in $\mathcal{V}$, and it satisfies the following bound*

$$\|\mathcal{V}^{(0)}\|_h \leq 2\|\mathcal{V}\|_{h'}, \tag{4.13}$$

where h is a positive constant. δ_{C_1} is the determinant bound of the propagator C_1 and $h' = h + 3\delta_{C_1}$.

Proof. According to Theorem 1.3 in [PS08], δ_{C_1} is given by

$$\delta_{C_1} = 2\left(\int_{\Lambda^*} d^2\mathbf{k}\, |f_1(k)|\right)^{\frac{1}{2}}, \tag{4.14}$$

by Corollary 4.2 in [PS08] δ_{C_1} can be bounded by

$$\frac{1}{\sqrt{2}}\|f_1(k)\|_1 \leq \delta_{C_1} \leq 2\|f_1(k)\|_1. \tag{4.15}$$

Thus we find that δ_{C_1} is proportional to $\epsilon_u^{1/2}$ and is uniform in β. For our ultraviolet propagator C_1, for which $e(\mathbf{k}) > e_0 > 0$, its decay bound α_{C_1} is also uniform in β. By Corollary 4.4 in [PS08] $\alpha_{C_1} \leq c_\alpha e_0^{-3}$, where c_α is a constant, which is proportional to the volume of the support $f_1(k)$.

Let us define $\omega_{C_1} = \alpha_{C_1}\delta_{C_1}^{-2}$. We implement the integral (4.12)

$$\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \mu_{C_1} * \mathcal{V} + \sum_{p\geq 2} \frac{1}{p!}\, \mathcal{E}_p(\mathcal{V}). \tag{4.16}$$

By the same procedure as applied below for the infrared problem, which we postpone doning this until next section, we yield

$$\|\mathcal{V}^{(0)}\|_h \leq \|\mathcal{V}\|_{h''} + \sum_{p\geq 2} \omega_{C_1}^{p-1}\|\mathcal{V}\|_{h'}^p. \tag{4.17}$$

with $h'' = h + \delta_{C_1}$ and $h' = h + 3\delta_{C_1}$. Then if $\omega_{C_1}\|\mathcal{V}\|_{h'} < \frac{1}{2}$, namely for $|g| < \tilde{g}$, $\tilde{g}$ is a suitable constant,

$$\begin{aligned} \|\mathcal{V}^{(0)}\|_h &\leq \|\mathcal{V}\|_{h'}\left(1 + \frac{\omega_{C_1}\|\mathcal{V}\|_{h'}}{1 - \omega_{C_1}\|\mathcal{V}\|_{h'}}\right) \\ &\leq 2\|\mathcal{V}\|_{h'}. \end{aligned} \tag{4.18}$$

We observe that the integrating over $\psi^{(1)}$ only increases the norm parameter by a constant, which indicates the convergence of the effective potential obtained from the integration over large momentum. The most important point here is that all constants are independent of β. □

Corollary 1. *There is a positive constant $\tilde{g}$, such that for $|g| < \tilde{g}$, $\mathcal{V}^{(0)}$ has an expression as*

$$\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \sum_{m,\bar{m}\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X}\; v^{(0)}_{\bar{m},m}(\underline{X}', \underline{X})\; \bar{\psi}^{(\leq 0)\bar{m}}(\underline{X}')\psi^{(\leq 0)m}(\underline{X}),$$

the kernels satisfy the following bounds

$$|v^{(0)}_{\bar{m},m}|_{1,\infty} \leq C|g|^{\max\{1,m-1\}}, \tag{4.19}$$

where C is a suitable constant.

4.2.2 The infrared integration

Let us now consider the infrared integration, which shall be done in multiscale steps. Using the addition principle for the Grassmann Gaussian integral, $j = 0, -1, -2, \ldots, J$, the successive integration over $\psi^{(j)}$ generates a sequence of effective potential $\mathcal{V}^{(0)}, \ldots, \mathcal{V}^{(J-1)}$, where $\mathcal{V}^{(0)}$ is the initial effective interaction for the flow and for all j,

$$\mathcal{W}(\mathcal{V})(\phi) = \log \frac{1}{Z_j} \int d\mu_{C_{\leq j}}(\psi^{(\leq j)}) e^{\mathcal{V}^{(j)}(\psi^{(\leq j)})}. \tag{4.20}$$

Here Z_j is a constant such that $\mathcal{W}(\mathcal{V})(0) = 0$, and

$$\mathcal{V}^{(j)}(\psi^{(\leq j)}) = \log \int d\mu_{C_{j+1}}(\psi^{(j+1)}) e^{\mathcal{V}^{(j+1)}(\psi^{(\leq j+1)})} \tag{4.21}$$

is obtained from $\mathcal{V}^{(j+1)}$, where $\mathcal{V}^{(j+1)}$ is called the effective potential by integrating the fields with scale larger than $j+1$. The expansion of effective potential in the fields is a power series, whose convergence needs to be proven,

$$\mathcal{V}^{(j)}(\psi^{(\leq j)}) = \sum_{m,\bar{m}\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X}\; v^{(j)}_{\bar{m},m}(\underline{X}', \underline{X})\; \bar{\psi}^{(\leq j)\bar{m}}(\underline{X}')\psi^{(\leq j)m}(\underline{X}),$$

the coefficient function $v^{(j)}_{m,m}(\underline{X}', \underline{X})$ can be interpreted as the connected amputated Green function for the interaction $\mathcal{V}^{(0)}$ and the propagator $C_{\geq j+1}$ obtained from integration over field $\psi^{(\geq j+1)}$, and they are also the vertex function for the integration of the field $\psi^{(j)}$. Our main task is to prove that the integral (4.21) is well defined for all $J \leq j \leq 0$. It will be performed in perturbation theory through an iterative procedure.

We first consider the asymptotic behaviors of the propagator C_j at each scale, which is very important in our analysis. We have the following lemma:

Lemma 6. *The fermionic propagator $C_j(X, X')$, $J \leq j \leq 0$ given in (4.10) has determinant bound*

$$c'_\delta M^{\frac{j}{2}} \leq \delta_{C_j} \leq c_\delta M^{\frac{j}{2}}, \tag{4.22}$$

where c_δ, $c'_\delta > 0$.

Proof. According to Theorem 1.3 in [PS08], δ_{C_j} is given by

$$\delta_{C_j} = 2\left(\int_{\Lambda^*} d^2\mathbf{k}\, |f_j(k)|\right)^{\frac{1}{2}}. \tag{4.23}$$

f_j is a smooth functions with support on $\{\sqrt{\frac{1}{M}}\, M^j \leq |ik_0 - e(\mathbf{k})| \leq \sqrt{2}M^j\}$. Hence, we have $M^{j/2}$ for each spatial dimension after the integration. The upper and lower bounds follow from Corollary 4.2 in [PS08]. □

Corollary 2. *The fermionic propagator $C_j(X, X')$, $J \leq j \leq 0$ given in (4.10) has Gram bound*

$$c'_\gamma M^{\frac{j}{2}} \leq \gamma_{C_j} \leq c_\gamma M^{\frac{j}{2}}, \tag{4.24}$$

with c_γ, c'_γ are suitable constants.

Lemma 7. *Let n be a multi-index. $n = (n_0, n_1, n_2)$ and $|n| = |n_0| + |n_1| + |n_2|$. The fermionic propagator $C_j(X, X')$, $J \leq j \leq 0$ given in (4.10) has a decay bound*

$$\alpha^\nu_{C_j} \leq c_\alpha M^{-j\left(n_0 + \frac{n_1+n_2}{2}+1\right)}, \tag{4.25}$$

where $c_\alpha > 0$.

Proof. By integration by parts $|n|$ times in (4.10), we have

$$\left|(\tau M^j)^{n_0}\prod_{i=1}^{2}(x_i M^{\frac{j}{2}})^{n_i} C_j(\tau, \mathbf{x})\right| \leq cM^{2j}\sup_{k_0,k}\left|(M^j\partial_{k_0})^{n_0}\prod_{i=1}^{2}(M^{\frac{j}{2}}\partial_{k_i})^{n_i} C_j(k_0, \mathbf{k})\right|,$$

where c is a constant. We have used the fact that the support of the integral has a volume of order M^{2j}, namely one M^j comes from the frequency, the other M^j comes from the spatial volume. To prove the desired bound it can be prove that

$$\sup_{k_0,k}\left|\left(M^j\partial_{k_0}\right)^{n_0}\prod_{i=1}^{2}\left(M^{\frac{j}{2}}\partial_{k_i}\right)^{n_i}C_j\left(k_0,\mathbf{k}\right)\right|\leq cM^{-j}. \tag{4.26}$$

Then we get

$$\left|\left(\tau M^j\right)^{n_0}\prod_{i=1}^{2}\left(x_iM^{j/2}\right)^{n_i}C_j\left(\tau,\mathbf{x}\right)\right|\leq cM^j. \tag{4.27}$$

We introduce a decay factor

$$\rho\left(x\right)=\left(1+\left(\tau M^j\right)^2\right)\prod_{i=1}^{2}\left(1+(x_iM^{j/2})^2\right). \tag{4.28}$$

Then, by (4.27)

$$\rho\left(x\right)\left|\tau^{n_0}\prod_{i=1}^{2}x_i^{n_i}C_j\left(\tau,\mathbf{x}\right)\right|\leq c\;M^{-j\left(n_0+\frac{1}{2}\sum_{i=1}^{2}n_i\right)+j}. \tag{4.29}$$

Dividing $\rho\left(x\right)$ and integrating both sides over Λ' give the decay bound

$$\begin{aligned}\alpha_{C_j}^{\nu} &\leq c\;M^{-j\left(n_0+\frac{1}{2}\sum_{i=1}^{2}n_i\right)+j}\int d\tau\int d^2\mathbf{x}\frac{1}{\rho\left(x\right)}\\ &\leq c_\alpha M^{-j\left(n_0+\frac{1}{2}\sum_{i=1}^{2}n_i+1\right)}.\end{aligned} \tag{4.30}$$

where c_α is a constant. □

Power counting for vertex functions

As mentioned in previous chapter, when representing the fermionic perturbation series in term of Feynman graphs, one has a combinatorial problem, associated to the $p!$, but due to the fermionic antisymmetry there are sign cancellations that lead to convergence. However, for the dilute Fermi gas we have another problem, called divergence problem. To see this explicitly let us first take a look at the power counting of the vertex functions. We write

$$\begin{aligned}&\mathcal{V}^{(j-1)}(\psi^{(\leq j-1)})\\ =\;&\sum_{p=1}\frac{1}{p!}\sum_{m,\bar{m}\geq 1}\int_\Gamma d^{\bar{m}}\underline{X}'d^m\underline{X}\;v_{\bar{m},m,p}^{(j-1)}(\underline{X}',\underline{X})\bar{\psi}^{(\leq j-1)\bar{m}}\left(\underline{X}'\right)\psi^{(\leq j-1)m}\left(\underline{X}\right),\end{aligned}$$

with

$$v^{(j-1)}_{\bar{m},m,p}(\underline{X}',\underline{X}) = \sum_{G\in\mathcal{G}_c} \mathrm{Val}(G), \tag{4.31}$$

where the sum is over all connected amputated Feynman graphs with $m+\bar{m}$ external lines and p $\mathcal{V}^{(j)}$ vertices. $\mathrm{Val}(G)$ is evaluated by using the usual Feynman rules for a connected graph, which allows us to derive a naive upper bound on $v^{(j-1)}_{\bar{m},m,p}(\underline{X}',\underline{X})$.

Lemma 8. *Let G be a connected amputated graph with $m+\bar{m}$ external legs built up from generalized $m_i+\bar{m}_i$ legged vertices $v^{(j)}_{\bar{m}_i,m_i}$, with $|v^{(j)}_{\bar{m}_i,m_i}|_{1,\infty} < \infty$, each line of the graph has a propagator C_j with*

$$\gamma_{C_j} \le c_\gamma M^{j/2}, \qquad\qquad \alpha_{C_j} \le c_\alpha M^{-j}, \tag{4.32}$$

then

$$|\,\mathrm{Val}(G)|_{1,\infty} \le C\left(\prod_{i=1}^{p} |v^{(j)}_{\bar{m}_i,m_i}|_{1,\infty} M^{\frac{1}{2}(m_i+\bar{m}_i-4)j}\right) M^{-\frac{1}{2}(m+\bar{m}-4)j}. \tag{4.33}$$

where $C>0$ is a suitable constant.

Proof. To obtain the bound (4.33), we select an arbitrary spanning tree $T\in G$, i.e. a loop less subset of G that connects all the p vertices. It contains $p-1$ lines. Now the integrals over the space time coordinate of products of the propagators on the spanning tree can be bounded by $\alpha^{p-1}_{C_j}$, the product of the remaining propagators can be bounded by $\gamma^{\sum_{i=1}^{p}(m_i+\bar{m}_i)-2(p-1)-(m+\bar{m})}_{C_j}$, using (4.32), we get the power counting bound (4.33). □

If we iterativly apply the lemma for all scale $j<j'\le 0$, we yield a similar bound with $M^{\frac{1}{2}(m_i+\bar{m}_i-4)j}$ replaced by $M^{\frac{1}{2}(m_i+\bar{m}_i-4)(j-j')}$. This scale must be summed over, one yields

$$\sum_{j'=j+1}^{0} M^{\frac{1}{2}(m_i+\bar{m}_i-4)(j-j')} \sim \begin{cases} \le 1, & \text{if} \quad m_i+\bar{m}_i \ge 6. \\ |j|, & \text{if} \quad m_i+\bar{m}_i = 4. \\ M^{|j|}, & \text{if} \quad m_i+\bar{m}_i = 2. \end{cases} \tag{4.34}$$

It is obvious that vertices with more than six external fields produce a small factor $M^{-\frac{1}{2}(m_i+\bar{m}_i-4)}$, which becomes irrelevant in the renormalization flow. However, vertices with four fields diverge as $|j|$. The two legged vertices gives the exponentially growing factor $M^{|j|}$. In the language of the renormalization group these behaviors are called respectively, marginal and relevant, they are the source of the infrared divergence. This divergence problem can be cured by renormalizing both the two- and four legged vertices.

The localization operator

One way to overcome the divergence problem consists in defining a localization operator $\mathcal{L}$ acting on $\mathcal{V}^{(j)}$, in terms of which we write

$$\mathcal{V}^{(j)} = \mathcal{L}\mathcal{V}^{(j)} + (1-\mathcal{L})\mathcal{V}^{(j)}. \tag{4.35}$$

We denote $\mathcal{P}_{\bar{m},m}$ as a projection operator, which projects on terms with $m+\bar{m}$ fields from $\mathcal{V}^{(j)}(\psi^{(\leq j)})$,

$$\mathcal{P}_{\bar{m},m}(\mathcal{V}^{(j)})(\psi^{(\leq j)}) = \int d^{\bar{m}}\underline{X}' d^m \underline{X}\; v^{(j)}_{\bar{m},m}(\underline{X}',\underline{X})\bar{\psi}^{(\leq j)\bar{m}}\left(\underline{X}'\right)\psi^{(\leq j)m}\left(\underline{X}\right).$$

The localization operator is defined in the following way:

1. $\mathcal{L}\mathcal{P}_{\bar{m},m}(\mathcal{V}^{(j)})(\psi^{(\leq j)}) = 0$, if $\bar{m} > 2, m > 2$.

2. If $\bar{m} = 1, m = 1$,

$$\mathcal{L}\mathcal{P}_{1,1}(\mathcal{V}^{(j)})(\psi^{(\leq j)}) = \int dX' dX\; v^{(j)}_{1,1}(X',X)\delta_{\sigma'\sigma}\bar{\psi}^{(\leq j)}_{\sigma'}\left(\tau,\mathbf{x}'\right)\psi^{(\leq j)}_{\sigma}\left(\tau,\mathbf{x}\right). \tag{4.36}$$

   By time-translation invariance, which can be proven under the hypotheses of the main theorem, it follows that this definition is independent of the choice of the localization point τ.

3. If $\bar{m} = 2, m = 2$,

$$\begin{aligned} & \mathcal{L}\mathcal{P}_{2,2}(\mathcal{V}^{(j)})(\psi^{(\leq j)}) \\ = & \int dX_1\; dX_2\; dX_3\; dX_4\; v^{(j)}_{2,2}\left(X_1,X_2,X_3,X_4\right)\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} \\ & \cdot \bar{\psi}^{(\leq j)}_{\sigma_1}\left(\tau_4,\mathbf{x}_1\right)\bar{\psi}^{(\leq j)}_{\sigma_2}\left(\tau_4,\mathbf{x}_2\right)\psi^{(\leq j)}_{\sigma_3}\left(\tau_4,\mathbf{x}_3\right)\psi^{(\leq j)}_{\sigma_4}\left(\tau_4,\mathbf{x}_4\right), \end{aligned} \tag{4.37}$$

   Here we use spin $SU(2)$ invariance to restrict to one coupling. We choose τ_4 as the localization point.

Instead of acting on the fields, $\mathcal{L}$ can equivalently be defined by its action on the kernel of the effective potential in the following way:

1. $\mathcal{L}v^{(j)}_{\bar{m},m}(\underline{X}',\underline{X}) = 0$, if $\bar{m} > 2, m > 2$.

2. If $\bar{m} = 1, m = 1$,

$$\mathcal{L}v^{(j)}_{1,1}(X',X) = \delta(\tau'-\tau)\int d\tilde{\tau}'\; v^{(j)}_{1,1}(\tilde{X}',X)\delta_{\sigma'\sigma}, \tag{4.38}$$

   where we put $\tilde{X}' = (\tilde{\tau}',\mathbf{x}')$.

3. If $\bar{m} = 2, m = 2$,

$$\begin{aligned} & \mathcal{L}v_{2,2}^{(j)}(X_1, X_2, X_3, X_4) \\ = \; & \tilde{\delta}(\tau_4) \int d\tilde{\tau}_1 d\tilde{\tau}_2 d\tilde{\tau}_3 \; v_{2,2}^{(j)}(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, X_4)\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4}, \end{aligned} \quad (4.39)$$

where we put $\tilde{X}_i = (\tilde{\tau}_i, \mathbf{x}_i)$, and $\tilde{\delta}(\tau_4) = \delta(\tau_1 - \tau_4)\delta(\tau_2 - \tau_4)\delta(\tau_3 - \tau_4)$. Under this definition we write $\mathcal{L}\mathcal{V}^{(j)}\left(\psi^{(\leq j)}\right)$

$$\mathcal{L}\mathcal{V}^{(j)}(\psi^{(\leq j)}) = K_j + G_j, \quad (4.40)$$

where

$$K_j = \int dX' dX \; \delta(\tau' - \tau) \; \kappa_j(X', X) \; \bar{\psi}^{(\leq j)}(X')\psi^{(\leq j)}(X) \quad (4.41)$$

and

$$G_j = \int d^2\underline{X}' d^2\underline{X} \; \tilde{\delta}(\tau_4) \; g_j(\underline{X}', \underline{X}) \; \bar{\psi}^{(\leq j)}(X_1)\bar{\psi}^{(\leq j)}(X_2)\psi^{(\leq j)}(X_3)\psi^{(\leq j)}(X_4), \quad (4.42)$$

with

$$\kappa_j(X', X) = \int d\tilde{\tau}' \; v_{1,1}^{(j)}(\tilde{X}', X)\delta_{\sigma'\sigma}, \quad (4.43)$$

and

$$g_j(\underline{X}', \underline{X}) = \int d\tilde{\tau}_1 d\tilde{\tau}_2 d\tilde{\tau}_3 \; v_{2,2}^{(j)}(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, X_4)\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4}. \quad (4.44)$$

These functions (4.43) and (4.44) are called the running coupling functions of scale j. The L_1-norms of the coupling functions are denoted as

$$|\kappa_j| = \sup_X \int dX' \; \delta(\tau' - \tau) \; |\kappa_j(X', X)|, \quad (4.45)$$

and

$$|g_j| = \sup_{X_4} \int dX_1 dX_2 dX_3 \; \tilde{\delta}(\tau_4) \; |g_j(\underline{X}', \underline{X})| \quad (4.46)$$

If the localization point is chosen, we define also the renormalization operator $\mathcal{R} = 1 - \mathcal{L}$, we have

$$\begin{aligned} & \mathcal{R}\mathcal{P}_{1,1}(\mathcal{V}^{(j)}) \\ = \; & \int dX' dX \; v_{1,1}^{(j)}\left(X', X\right) \left(\bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau', \mathbf{x}'\right) - \bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau, \mathbf{x}'\right)\right) \psi_{\sigma}^{(\leq j)}\left(\tau, \mathbf{x}\right). \end{aligned} \quad (4.47)$$

By using the identity

$$\int_0^1 dtF'(t) = F'(0) + \int_0^1 dt\,(1-t)\,F''(t)\,, \tag{4.48}$$

we expand $\bar{\psi}_{\sigma'}^{(\leq j)}(\tau', \mathbf{x}') - \bar{\psi}_{\sigma'}^{(\leq j)}(\tau, \mathbf{x}')$ to the second order in $\tau' - \tau$, to get

$$\begin{aligned} & \bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau', \mathbf{x}'\right) - \bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau, \mathbf{x}'\right) \\ = \; & \left(\tau' - \tau\right) \frac{\partial}{\partial \tau} \bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau, \mathbf{x}'\right) + \left(\tau' - \tau\right)^2 \int_0^1 dt\,(1-t)\,\frac{\partial^2}{\partial \tau'(t)^2} \bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau'(t), \mathbf{x}'\right) \end{aligned}$$

where $\tau'(t) = \tau + t\,(\tau' - \tau)$. Similarly, we have

$$\begin{aligned} & \mathcal{RP}_{2,2}(\mathcal{V}^{(j)}) \\ = \; & \int dX_1\, dX_2\, dX_3\, dX_4\, v_{2,2}^{(j)}(X_1, X_2, X_3, X_4) \\ \cdot \; & \Big[\bar{\psi}_{\sigma_1}^{(\leq j)}(\tau_1, \mathbf{x}_1)\, \bar{\psi}_{\sigma_2}^{(\leq j)}(\tau_2, \mathbf{x}_2)\, \psi_{\sigma_3}^{(\leq j)}(\tau_3, \mathbf{x}_3)\, \psi_{\sigma_4}^{(\leq j)}(\tau_4, \mathbf{x}_4) \\ & - \bar{\psi}_{\sigma_1}^{(\leq j)}(\tau_4, \mathbf{x}_1)\, \bar{\psi}_{\sigma_2}^{(\leq j)}(\tau_4, \mathbf{x}_2)\, \psi_{\sigma_3}(\tau_4, \mathbf{x}_3)\, \psi_{\sigma_4}^{(\leq j)}(\tau_4, \mathbf{x}_4)\Big]\,. \end{aligned} \tag{4.49}$$

The term in the square bracket can be written as:

$$\begin{aligned} & \left(\bar{\psi}_{\sigma_1}^{(\leq j)}(\tau_1, \mathbf{x}_1) - \bar{\psi}_{\sigma_1}^{(\leq j)}(\tau_4, \mathbf{x}_1)\right) \bar{\psi}_{\sigma_2}^{(\leq j)}(\tau_2, \mathbf{x}_2)\, \psi_{\sigma_3}^{(\leq j)}(\tau_3, \mathbf{x}_3)\, \psi_{\sigma_4}^{(\leq j)}(\tau_4, \mathbf{x}_4) \\ + \; & \bar{\psi}_{\sigma_1}^{(\leq j)}(\tau_4, \mathbf{x}_1) \left(\bar{\psi}_{\sigma_2}^{(\leq j)}(\tau_2, \mathbf{x}_2) - \bar{\psi}_{\sigma_2}^{(\leq j)}(\tau_4, \mathbf{x}_2)\right) \psi_{\sigma_3}^{(\leq j)}(\tau_3, \mathbf{x}_3)\, \psi_{\sigma_4}^{(\leq j)}(\tau_4, \mathbf{x}_4) \\ + \; & \bar{\psi}_{\sigma_1}^{(\leq j)}(\tau_4, \mathbf{x}_1)\, \bar{\psi}_{\sigma_2}^{(\leq j)}(\tau_4, \mathbf{x}_2) \left(\psi_{\sigma_3}^{(\leq j)}(\tau_3, \mathbf{x}_3) - \psi_{\sigma_3}^{(\leq j)}(\tau_4, \mathbf{x}_3)\right) \psi_{\sigma_4}^{(\leq j)}(\tau_4, \mathbf{x}_4)\,, \end{aligned}$$

each term contains exactly one difference which may be written as a Taylor remainder,

$$\psi_{\sigma_j}^{(\leq j)}(\tau_j, \mathbf{x}_j) - \psi_{\sigma_j}^{(\leq j)}(\tau_4, \mathbf{x}_j) = (\tau_j - \tau_4) \int_0^1 dt\, \frac{\partial}{\partial \tau_j(t)} \psi_{\sigma_j}^{(\leq j)}(\tau_j(t), \mathbf{x}_j)\,,$$

where $\tau_j(t) = \tau_4 + t\,(\tau_j - \tau_4)$. In order to simplify the notations, we denote the Taylor remainder terms as

$$\hat{\partial}_\tau^1 F(\tau) = \int_0^1 dt\; \partial_\tau F(\tau), \qquad \hat{\partial}_\tau^2 F(\tau) = \int_0^1 dt\; (1-t) \partial_\tau^2 F(\tau). \tag{4.50}$$

With these the renormalized part of $\mathcal{V}^{(j)}$ becomes

$$\begin{aligned}
& \mathcal{R}\mathcal{V}^{(j)}\left(\psi^{(\leq j)}\right) \\
= & \int dX'dX\ v_{1,1}^{(j)}\left(X',X\right) \\
\cdot & \left[\left(\tau'-\tau\right)\hat{\partial}_{\tau}^{1}\bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau',\mathbf{x}'\right)+\left(\tau'-\tau\right)^{2}\hat{\partial}_{\tau}^{2}\bar{\psi}_{\sigma'}^{(\leq j)}\left(\tau',\mathbf{x}'\right)\right]\psi_{\sigma}^{(\leq j)}\left(\tau,\mathbf{x}\right) \\
+ & \int dX_1\ dX_2\ dX_3\ dX_4\ v_{2,2}^{(j)}\left(X_1,X_2,X_3,X_4\right) \\
\cdot & \Big[(\tau_1-\tau_4)\hat{\partial}_{\tau_1}^{1}\bar{\psi}_{\sigma_1}^{(\leq j)}\left(\tau_1',\mathbf{x}_1\right)\bar{\psi}_{\sigma_2}^{(\leq j)}\left(\tau_2,\mathbf{x}_2\right)\psi_{\sigma_3}^{(\leq j)}\left(\tau_3,\mathbf{x}_3\right)\psi_{\sigma_4}^{(\leq j)}\left(\tau_4,\mathbf{x}_4\right) \\
& +\bar{\psi}_{\sigma_1}^{(\leq j)}\left(\tau_4,\mathbf{x}_1\right)(\tau_2-\tau_4)\hat{\partial}_{\tau_2}^{1}\bar{\psi}_{\sigma_2}^{(\leq j)}\left(\tau_2',\mathbf{x}_2\right)\psi_{\sigma_3}^{(\leq j)}\left(\tau_3,\mathbf{x}_3\right)\psi_{\sigma_4}^{(\leq j)}\left(\tau_4,\mathbf{x}_4\right) \\
& +\bar{\psi}_{\sigma_1}^{(\leq j)}\left(\tau_4,\mathbf{x}_1\right)\bar{\psi}_{\sigma_2}^{(\leq j)}\left(\tau_4,\mathbf{x}_2\right)(\tau_3-\tau_4)\hat{\partial}_{\tau_3}^{1}\psi_{\sigma_3}^{(\leq j)}\left(\tau_3',\mathbf{x}_3\right)\psi_{\sigma_4}^{(\leq j)}\left(\tau_4,\mathbf{x}_4\right)\Big] \\
+ & \mathcal{P}_{\bar{m}\geq 3,m\geq 3}(\mathcal{V}^{(j)})(\psi^{(\leq j)}),
\end{aligned} \tag{4.51}$$

where $\tau_i'=\tau_i+t(\tau_i-\tau_4)$, and $\mathcal{P}_{\bar{m}\geq 3,m\geq 3}=\sum_{m,\bar{m}\geq 3}\mathcal{P}_{m,\bar{m}}$.

Bound for the effective potential

After introducing the localization operator, we treat the perturbation expansion of $\mathcal{V}^{(j-1)}$ in $\mathcal{V}^{(j)}$,

$$\mathcal{V}^{(j-1)}(\psi^{\leq j-1})=\log\int d\mu_{C_j}(\psi^{\leq j})\ e^{\mathcal{V}^{(j)}(\psi^{\leq j})} \tag{4.52}$$

with

$$\mathcal{V}^{(j)}(\psi^{(\leq j)})=\sum_{m,\bar{m}\geq 1}\int d^{\bar{m}}\underline{X}'d^{m}\underline{X}\ v_{\bar{m},m}^{(j)}(\underline{X}',\underline{X})\ \bar{\psi}^{(\leq j)\bar{m}}(\underline{X}')\psi^{(\leq j)m}(\underline{X}).$$

In order to avoid the tadpole contribution, we will study instead of (4.52) an equivalent sequence of Wick ordered effective potentials $:\mathcal{V}^{(j)}:_{C_{\leq j}}$

$$\mathcal{V}^{(j-1)}(\psi^{\leq j-1})=:\log\int d\mu_{C_j}(\psi^{\leq j})e^{:\mathcal{V}^{(j)}(\psi^{\leq j}):_{C_{\leq j}}}:_{-C_{\leq j-1}}. \tag{4.53}$$

In perturbation theory, the first order term of (4.53) gives

$$:\int d\mu_{C_j}(\psi^{(j)})\ :\mathcal{V}^{(j)}(\psi^{(\leq j)}):_{C_{\leq j}}:_{-C_{\leq j-1}}=\mathcal{V}^{(j)}(\psi^{(\leq j-1)}), \tag{4.54}$$

thus we can write

$$\mathcal{V}^{(j-1)}(\psi^{\leq j-1})-\mathcal{V}^{(j)}(\psi^{\leq j-1})=\sum_{p\geq 2}W_{p}^{(j-1)}(\mathcal{V}^{(j)}), \tag{4.55}$$

with

$$W_p^{(j-1)}(\mathcal{V}^{(j)}) = \frac{1}{p!} : \mathcal{E}_p^{(j-1)} \left(: \mathcal{V}^{(j)} :_{C_{\le j}}, \dots, : \mathcal{V}^{(j)} :_{C_{\le j}} \right) (\psi^{(\le j-1)}) :_{-C_{\le j-1}} .$$

Since we expect that the convergence radius depends on the sign of the running coupling functions, we have to express $\mathcal{V}^{(j-1)}$ in terms of G_j. We will evaluate the second order in K_j and G_j as explicitly as possible. K_j and G_j are defined in (4.42) and (4.43). We decompose $\mathcal{V}^{(j-1)}$ as follows. Denoting

$$\delta\mathcal{V}^{(i)} = \mathcal{V}^{(i)} - \mathcal{V}^{(i+1)}, \tag{4.56}$$

for all $j-1 \le i \le 0$, and $\delta\mathcal{V}^{(0)} = \mathcal{V}^{(0)}$, we write $\mathcal{V}^{(j-1)}$ as a sum over $\delta\mathcal{V}^{(i)}$,

$$\mathcal{V}^{(j-1)} = \sum_{i=j-1}^{0} \delta\mathcal{V}^{(i)} = \sum_{i=j-1}^{0} (\mathcal{L}\delta\mathcal{V}^{(i)} + \mathcal{R}\delta\mathcal{V}^{(i)}). \tag{4.57}$$

By (4.55), (4.57) reduces to

$$\mathcal{V}^{(j-1)} \;=\; \mathcal{V}_2^{(j-1)} + \sum_{i=j-1}^{0} (G_i - G_{i+1}) + \sum_{i=j-1}^{0} \mathcal{P}_{\ge 2,\ge 2} \left(\mathcal{R} W_{\ge 2}^{(i)}(\mathcal{V}^{(i+1)}) \right). \tag{4.58}$$

where $\mathcal{V}_2^{(j-1)} = \mathcal{P}_{1,1}(\mathcal{V}^{(j-1)})$ is the part of the effective potential at scale $j-1$ that is quadratic in $(\bar\psi, \psi)$. Inserting $\mathcal{V}^{(i+1)} = \mathcal{L}\mathcal{V}^{(i+1)} + \mathcal{R}\mathcal{V}^{(i+1)}$ and $\mathcal{L}\mathcal{V}^{(i+1)} = K_{i+1} + G_{i+1}$, using the fact

$$\mathcal{P}_{\ge 2,\ge 2} \left(W_2^{(i)}(K_{i+1}, K_{i+1}) \right) = \mathcal{P}_{\ge 3,\ge 3} \left(W_2^{(i)}(K_{i+1}, G_{i+1}) \right) = 0, \tag{4.59}$$

the last term of (4.58) can be further decomposed into

$$\begin{aligned} & \mathcal{P}_{\ge 2,\ge 2} \left(\mathcal{R} W_{\ge 2}^{(i)}(\mathcal{V}^{(i+1)}) \right) \\ = \;& \mathcal{P}_{2,2} \left(\mathcal{R} W_{\ge 2}^{(i)}(\mathcal{V}^{(i+1)}) \right) + \mathcal{P}_{\ge 3,\ge 3} \left(W_{\ge 2}^{(i)}(\mathcal{V}^{(i+1)}) \right) \\ = \;& \mathcal{P}_{2,2} \left(\mathcal{R} W_{\ge 2}^{(i)}(\mathcal{V}^{(i+1)}) \right) + \mathcal{P}_{3,3} \left(W_2^{(i)}(G_{i+1}, G_{i+1}) \right) + \mathcal{P}_{\ge 3,\ge 3} \left(\tilde R(\mathcal{V}^{(i+1)}) \right), \end{aligned} \tag{4.60}$$

where

$$\tilde R(\mathcal{V}^{(i+1)}) = 2W_2^{(i)}(\mathcal{L}\mathcal{V}^{(i+1)}, \mathcal{R}\mathcal{V}^{(i+1)}) + W_2^{(i)}(\mathcal{R}\mathcal{V}^{(i+1)}, \mathcal{R}\mathcal{V}^{(i+1)}) + W_{\ge 3}^{(i)}(\mathcal{V}^{(i+1)}).$$

The bounds of progapators imply the convergence of the perturbation expansion, as well as analyticity in $\|\cdot\|_h$ provided that $\omega_C \|\mathcal{V}\|_{h+3\gamma_C} < 1/2$ is

satisfied. For this purpose, we define $\omega_{C_i} = \alpha_{C_i}\gamma_{C_i}^{-2}$. By Corollary 2 and Lemma 7, we have $\omega_{C_i} \leq c_\omega M^{-2i}$, where $c_\omega = c_\alpha (c'_\gamma)^{-2}$ is a constant. We denote $h_i = \alpha\gamma_{C_i}$, where $\alpha > 0$ is a constant, for all $J \leq i \leq 0$, and consider the quantity $\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}}$, bounded by (4.58) and (4.60),

$$\begin{aligned}
\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} &\leq \omega_{C_{j-1}}\|\mathcal{V}_2^{(j-1)}\|_{h_{j-1}} + C_4 \sum_{i=j-1}^{0} |g_i - g_{i+1}| \\
&+ \sum_{i=j-1}^{0} \omega_{C_{j-1}}\|\mathcal{P}_{2,2}\left(\mathcal{R}W_{\geq 2}^{(i)}(\mathcal{V}^{(i+1)})\right)\|_{h_{j-1}} \\
&+ \sum_{i=j-1}^{0} \omega_{C_{j-1}}\|\mathcal{P}_{3,3}\left(W_2^{(i)}(G_{i+1}, G_{i+1})\right)\|_{h_{j-1}} \\
&+ \sum_{i=j-1}^{0} \omega_{C_{j-1}}\|\mathcal{P}_{\geq 3,\geq 3}\left(\tilde{R}(\mathcal{V}^{(i+1)})\right)\|_{h_{j-1}}, \qquad (4.61)
\end{aligned}$$

where

$$|g_i - g_{i+1}| = \sup_{X_4} \int dX_1 dX_2 dX_3 \tilde{\delta}(\tau_4) |g_i(X_1, X_2, X_3, X_4) - g_{i+1}(X_1, X_2, X_3, X_4)|$$

and $C_4 = c_\omega c_\gamma^4 \alpha^4$ is a constant. The first term on the right side of (4.61) will be bounded below, which relates to the renormalization condition. The second term associates to the flow of running coupling functions, we postpone analyzing this until next section. In this section we bound the other three terms separately. Their contributions will turns out to be small, and do not play significant roles in the flow.

$\omega_{C_{j-1}}\|\mathcal{P}_{\geq 3,\geq 3}\left(\tilde{R}(\mathcal{V}^{(i+1)})\right)\|_{h_{j-1}}$ term:

We consider first the last term of (4.61). The projection onto effective potential with more than 6 external fields induces a gain factor M^{j-i-1}, which can be seen in the following way,

$$\begin{aligned}
& \omega_{C_{j-1}}\|\mathcal{P}_{\geq 3,\geq 3}\left(\tilde{R}(\mathcal{V}^{(i+1)})\right)\|_{h_{j-1}} \\
= \;& \omega_{C_i}\left(\frac{\omega_{C_{j-1}}}{\omega_{C_i}}\right) \sum_{\bar{m},m\geq 3} \left|\mathcal{P}_{\bar{m},m}\left(\tilde{R}(\mathcal{V}^{(i+1)})\right)\right|_{1,\infty} \left(\frac{h_{j-1}}{h_i}\right)^{\bar{m}+m} h_i^{\bar{m}+m} \\
\leq \;& M^{j-i-1}\omega_{C_i} \sum_{\bar{m},m\geq 3} \left|\mathcal{P}_{\bar{m},m}\left(\tilde{R}(\mathcal{V}^{(i+1)})\right)\right|_{1,\infty} h_i^{\bar{m}+m} \\
\leq \;& M^{j-i-1}\omega_{C_i}\|\tilde{R}(\mathcal{V}^{(i+1)})\|_{h_i}. \qquad (4.62)
\end{aligned}$$

However M^{j-i-1} alone is not enough to show convergence. Further improvement is expected by bounding $\omega_{C_i}\|\tilde{R}(\mathcal{V}^{(i+1)})\|_{h_i}$. We write

$$\omega_{C_i}\|\tilde{R}(\mathcal{V}^{(i+1)})\|_{h_i} \leq \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3, \tag{4.63}$$

with

$$\begin{aligned}
\tilde{R}_1 &= 2\omega_{C_i}\|W_2^{(i)}(\mathcal{L}\mathcal{V}^{(i+1)}, \mathcal{R}\mathcal{V}^{(i+1)})\|_{h_i}, && (4.64)\\
\tilde{R}_2 &= \omega_{C_i}\|W_2^{(i)}(\mathcal{R}\mathcal{V}^{(i+1)}, \mathcal{R}\mathcal{V}^{(i+1)})\|_{h_i}, && (4.65)\\
\tilde{R}_3 &= \omega_{C_i}\|W_{\geq 3}^{(i)}(\mathcal{V}^{(i+1)})\|_{h_i}. && (4.66)
\end{aligned}$$

To bound these remainder terms we consider first the bound on

$$: \mathcal{E}_p^{(i)} \left(: \mathcal{V}^{(i+1)} :_{C_{\leq i+1}}, \ldots, : \mathcal{V}^{(i+1)} :_{C_{\leq i+1}} \right) (\psi^{(\leq i)}) :_{-C_{\leq i}} . \tag{4.67}$$

By using the tree expansion formula (3.75), we have

$$\begin{aligned}
& : \mathcal{E}_p^{(i)} \left(: \mathcal{V}^{(i+1)} :_{C_{\leq i+1}}, \ldots, : \mathcal{V}^{(i+1)} :_{C_{\leq i+1}} \right) (\psi^{(\leq i)}) :_{-C_{\leq i}} \\
= & \sum_{T \in \mathcal{T}_p} \prod_{\{q,q'\} \in T} \left(\Delta_{C_{i+1},qq'} + \Delta_{C_{i+1},q'q} \right) \\
& \cdot \int_{[0,1]^{p-1}} d\mathbf{s} \sum_{\pi \in \prod(T)} \varphi\left(T, \pi, \mathbf{s}\right) e^{\Delta_{C_{\leq i}, C_{i+1}}[H'(T,\pi,\mathbf{s})]} \prod_{q=1}^{p} \mathcal{V}^{(i+1)}(\psi^{\leq i+1}),
\end{aligned} \tag{4.68}$$

with $\Delta_{C_{\leq i}, C_{i+1}}[H'(T, \pi, \mathbf{s})]$ is the Laplacian to the propagator

$$\begin{aligned}
& D_{C_{\leq i}, C_{i+1}}\left((q, X), (q', X')\right) \\
= & \begin{cases} 0 & \text{if} \quad q = q' \\ C_{\leq i}\left((q, X), (q', X')\right) + H_{qq'}(T, \pi, \mathbf{s}) C_{i+1}\left((q, X), (q', X')\right) & \text{if} \quad q \neq q' \end{cases}
\end{aligned} \tag{4.69}$$

In order to express the dependence of the line on the vertices explicitly, we write $C(X, X') = C((q, X), (q, X'))$. Due to the Wick ordering the interpolation matrix $H'(T, \pi, \mathbf{s})$ in $\Delta_{C_{\leq i}, C_{i+1}}[H'(T, \pi, \mathbf{s})]$ is not positively defined. The Wick ordering drops the diagonal terms $\sum_q \Delta_{qq}$ from the Laplacian. This destroys the positivity of the matrix $H'[T, \pi, \mathbf{s}]$. However we can restore the positivity by adding and subtracting the diagonal terms,

$$\Delta_{C_{\leq i}, C_{i+1}}[H'(T, \pi, \mathbf{s})] = \Delta_{H(T,\pi,\mathbf{s})C_{i+1}} + \Delta_{C_{\leq i}} - \Delta_{C_{\leq i+1},qq}, \tag{4.70}$$

where $H_{qq'}(T, \pi, \mathbf{s}) C_{i+1}\left((q, X), (q', X')\right)$, $1 \leq q, q' \leq p$ is the propagator to the Laplacian $\Delta_{H(T,\pi,\mathbf{s})C_{i+1}}$. Because the matrix $H(T, \pi, \mathbf{s})$ is positive

with diagonal elements bounded by 1, the corresponding Gram constant $\gamma_{C_{i+1}}$ is independent of $T, \mathbf{s}$, and π. $\Delta_{C_{\le i}}$ is the Laplacian to the propagator $C_{\le i}((q,X),(q',X'))$, with $1 \le q, q' \le p$, whose Gram constant is γ_{C_i}. $\Delta_{C_{\le i+1},qq}$ relates to the propagator $C_{\le i+1}((q,X),(q',X'))$, if $q = q'$, 0 otherwise, whose Gram constant is $\gamma_{C_{i+1}}$. After applying the three Laplacians one after the other, the three Gram constants add up, so the Wick ordering changes the Gram constant from $\gamma_{C_{i+1}}$ to $3\gamma_{C_{i+1}}$.

Now we can bound $\| : \mathcal{E}_p^{(i)}\left(: \mathcal{V}^{(i+1)} :_{C_{\le i+1}}, \ldots, : \mathcal{V}^{(i+1)} :_{C_{\le i+1}}\right) :_{-C_{\le i}} \|_{h_i}$. We use the representation of $\mathcal{L}$ and $\mathcal{R}$ based on (4.36), (4.37) and (4.51), where some field variables in $\mathcal{V}_q^{(i+1)}(\psi^{(\le i+1)})$ have to be substituted with new ones containing possibly some derivatives. As we shall see, these new variables allow to get the right bounds. Before evaculating, it is convenient to introduce a new label $\tilde{n}$ to keep track of the localized or renormalized terms in $\mathcal{V}^{(i+1)}(\psi^{(\le i+1)})$. We write

$$\mathcal{V}^{(i+1)}(\psi^{(\le i+1)}) = \sum_{\tilde{n}\ge 1} \mathcal{V}_{\tilde{n}}^{(i+1)}(\psi^{(\le i+1)}), \tag{4.71}$$

where

$$\mathcal{V}_{\tilde{n}=1}^{(i+1)}(\psi^{(\le i+1)}) = \int dX' dX \; v_{1,1,1}^{(i+1)}(X',X)\delta_{\sigma'\sigma}\bar{\psi}_{\sigma'}^{(\le i+1)}(\tau,\mathbf{x}')\psi_{\sigma}^{(\le i+1)}(\tau,\mathbf{x}),$$

is the localized part of $\mathcal{V}^{(i+1)}(\psi^{(\le i+1)})$, that is quadratic in $\bar{\psi}$ and ψ. The corresponding renormalized parts are denoted as

$$\mathcal{V}_{\tilde{n}=2}^{(i+1)}(\psi^{(\le i+1)}) = \int dX' dX \; v_{2,1,1}^{(i+1)}(X',X)\,(\tau'-\tau)\hat{\partial}_{\tau}^{1}\bar{\psi}_{\sigma'}^{(\le i+1)}\left(\tau',\mathbf{x}'\right)\psi_{\sigma}^{(\le i+1)}(\tau,\mathbf{x}),$$

and

$$\mathcal{V}_{\tilde{n}=3}^{(i+1)}(\psi^{(\le i+1)}) = \int dX' dX \; v_{3,1,1}^{(i+1)}(X',X)\,(\tau'-\tau)^2\hat{\partial}_{\tau}^{2}\bar{\psi}_{\sigma'}^{(\le i+1)}(\tau',\mathbf{x}')\psi_{\sigma}^{(\le i+1)}(\tau,\mathbf{x}),$$

with a field carrying one or two derivatives, respectively. Similarly, we denote

$$\begin{aligned}\mathcal{V}_{\tilde{n}=4}^{(i+1)}(\psi^{(\le i+1)}) \;=\; & \int dX_1\, dX_2\, dX_3\, dX_4\; v_{4,2,2}^{(i+1)}\left(X_1,X_2,X_3,X_4\right)\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} \\ & \cdot \bar{\psi}_{\sigma_1}^{(\le i+1)}\left(\tau_4,\mathbf{x}_1\right)\bar{\psi}_{\sigma_2}^{(\le i+1)}\left(\tau_4,\mathbf{x}_2\right)\psi_{\sigma_3}^{(\le i+1)}\left(\tau_4,\mathbf{x}_3\right)\psi_{\sigma_4}^{(\le i+1)}\left(\tau_4,\mathbf{x}_4\right),\end{aligned}$$

is the quartic part of $\mathcal{L}\mathcal{V}^{(i+1)}$, the three renormalized term are denoted as

$$\begin{aligned}\mathcal{V}_{\tilde{n}=5}^{(i+1)}(\psi^{(\le i+1)}) \;=\; & \int dX_1\, dX_2\, dX_3\, dX_4\; v_{5,2,2}^{(i+1)}(X_1,X_2,X_3,X_4)(\tau_1-\tau_4) \\ & \cdot \quad \hat{\partial}_{\tau_1}^{1}\bar{\psi}_{\sigma_1}^{(\le i+1)}\left(\tau_1',\mathbf{x}_1\right)\bar{\psi}_{\sigma_2}^{(\le i+1)}\left(\tau_2,\mathbf{x}_2\right)\psi_{\sigma_3}^{(\le i+1)}\left(\tau_3,\mathbf{x}_3\right)\psi_{\sigma_4}^{(\le i+1)}\left(\tau_4,\mathbf{x}_4\right)\end{aligned}$$

$$\begin{aligned}\mathcal{V}^{(i+1)}_{\tilde{n}=6}(\psi^{(\leq i+1)}) &= \int dX_1\, dX_2\, dX_3\, dX_4\, v^{(i+1)}_{6,2,2}(X_1,X_2,X_3,X_4)(\tau_2-\tau_4)\\ &\cdot\ \bar{\psi}^{(\leq i+1)}_{\sigma_1}(\tau_4,\mathbf{x}_1)\,\hat{\partial}^1_{\tau_2}\bar{\psi}^{(\leq i+1)}_{\sigma_2}\left(\tau_2',\mathbf{x}_2\right)\psi^{(\leq i+1)}_{\sigma_3}(\tau_3,\mathbf{x}_3)\,\psi^{(\leq i+1)}_{\sigma_4}(\tau_4,\mathbf{x}_4)\end{aligned}$$

$$\begin{aligned}\mathcal{V}^{(i+1)}_{\tilde{n}=7}(\psi^{(\leq i+1)}) &= \int dX_1\, dX_2\, dX_3\, dX_4\, v^{(i+1)}_{7,2,2}(X_1,X_2,X_3,X_4)(\tau_3-\tau_4)\\ &\cdot\ \bar{\psi}^{(\leq i+1)}_{\sigma_1}(\tau_4,\mathbf{x}_1)\,\bar{\psi}^{(\leq i+1)}_{\sigma_2}(\tau_4,\mathbf{x}_2)\,\hat{\partial}^1_{\tau_3}\psi^{(\leq i+1)}_{\sigma_3}\left(\tau_3',\mathbf{x}_3\right)\psi^{(\leq i+1)}_{\sigma_4}(\tau_4,\mathbf{x}_4)\,,\end{aligned}$$

and finally for $\tilde{n}\geq 8$, $\mathcal{V}^{(i+1)}_{\tilde{n}}(\psi^{(\leq i+1)})$ are the usual vertices with $\bar{m}=m=\tilde{n}-5$ external fields

$$\mathcal{V}^{(i+1)}_{\tilde{n}\geq 8}(\psi^{(\leq i+1)})=\int d^{\bar{m}}\underline{X}'d^m\underline{X}\; v^{(i+1)}_{\tilde{n},\bar{m},m}(\underline{X}',\underline{X})\bar{\psi}^{(\leq i+1)\bar{m}}\left(\underline{X}'\right)\psi^{(\leq i+1)m}\left(\underline{X}\right).$$

With these definitions, the $\|\cdot\|_{h_{i+1}}$ norm on $\mathcal{V}^{(i+1)}$ can be rewritten as

$$\|\mathcal{V}^{(i+1)}\|_{h_{i+1}}=\sum_{\tilde{n}\geq 1}|v^{(i+1)}_{\tilde{n},\bar{m},m}|_{1,\infty}\,h_{i+1}^{\bar{m}+m}. \tag{4.72}$$

Let us now consider the action of the Laplacian on the product of polynomial

$$\prod_{\{q,q'\}\in T}\left(\Delta_{C_{i+1},qq'}+\Delta_{C_{i+1},q'q}\right)\prod_{q=1}^{p}\mathcal{V}^{(i+1)}_q(\psi^{(\leq i+1)}). \tag{4.73}$$

where $\mathcal{V}^{(i+1)}_q(\psi^{(\leq i+1)})=\sum_{\tilde{n}\geq 1}\mathcal{V}^{(i+1)}_{\tilde{n}_q}(\psi^{(\leq i+1)})$. For a fixed tree T, we define incidence numbers $d_q=\bar{\theta}_q+\theta_q$, $q=1,\ldots,p$. This means that there are d_q derivatives act on the q-th vertex, namely θ_q derivatives with respect to $\psi^{(i+1)}_q$ and $\bar{\theta}_q$ derivatives with respect to $\bar{\psi}^{(i+1)}_q$. Then the Laplacian gives a sum of monomials of degree $m_q-\theta_q+\bar{m}_q-\bar{\theta}_q$ for every vertex $\mathcal{V}^{(i+1)}_{\tilde{n}_q}$. Because the coefficient function $v^{(i+1)}_{\tilde{n},\bar{m},m}(\underline{X},\underline{X}')$ is totally antisymmetric, we have a combinatorial factor

$$\binom{m_q}{\theta_q}\theta_q!\binom{\bar{m}_q}{\bar{\theta}_q}\bar{\theta}_q!. \tag{4.74}$$

In addition to these monomials we obtain further a product of propagators, which build the spanning tree connecting all vertices

$$\prod_{\{q,q'\}\in T}\left[(\tau-\tau_{q0})^{\tilde{q}(v^{(i+1)}_{q,\tilde{n}})}\hat{\partial}_\tau^{q(X'_{q'})}\hat{\partial}_\tau^{q(X_q)}C_{i+1}\left((q,X),(q',X')\right)\right], \tag{4.75}$$

where τ_{q0} denotes the localization point, note that we have the freedom to choose the localization point. $\tilde{q}(v^{(i+1)}_{q,\tilde{n}})$, $q(X'_{q'})$ and $q(X_q)$ are constants,

equal to $0, 1$ or 2. The prefactor $(\tau - \tau_{q0})^{\tilde{q}(v^{(i+1)}_{q,\tilde{n}})}$ comes from the vertices $\mathcal{V}^{(i+1)}_{\tilde{n}_q}$ with $\tilde{n}_q = 2, 3, 5, 6, 7$, which has a field carrying derivatives. If this field $\psi^{(\leq i+1)}$ is an internal fields of $\mathcal{V}^{(i)}$, carrying derivative, the derivative will act on the propagator C_{i+1} connecting the field with another field. This is the reason for appearing $\hat{\partial}_\tau^{q(X'_{q'})}\hat{\partial}_\tau^{q(X_q)}$ in (4.75). In principle we are free to fix the localization points, that is the field containing the derivatives. We make a choice in a way, which avoids that a field affected by the derivative remains after acting the Laplacian. In other words, we put all such fields with derivatives to connect the vertices. It follows from this that the product of the distance on a scale $i+2$ and the derivative of the propagators on a scale $i+1$ gives a gain factor M^{-1} with respect to the contraction of fields without derivative. This extracted factor is called the renormalization gain. To see this explicitly, we note that the constraints on $q(X'_{q',r'})$, $q(X_{q,r})$ and $\tilde{q}(v^{(i+1)}_{q,\tilde{n}})$ imply that

$$\sum_{\{q,q'\}\in T} \big(q(X_{q,r}) + q'(X'_{q',r'})\big) = \sum_{q=1}^{p} \tilde{q}(v^{(i+1)}_{q,\tilde{n}}),$$

such that we get a renormalization gain

$$\prod_{q=1}^{p} M^{-(i+2)q(v^{(i+1)}_{q,\tilde{n}})} \prod_{\{q,q'\}\in T} M^{(i+1)(q(X_{q,r})+q'(X'_{q',r'}))},$$

which can be bounded by $\prod_{q=1}^{p} M^{-\gamma_q(\tilde{n}_q)}$, with

$$\gamma_q(\tilde{n}_q) = \begin{cases} 1, & \text{if} \quad \tilde{n}_q = 2, \\ 2, & \text{if} \quad \tilde{n}_q = 3, \\ 1, & \text{if} \quad \tilde{n}_q = 5, 6, 7, \\ 0, & \text{otherwise.} \end{cases} \tag{4.76}$$

We move this gain factor to the front of the coefficient function $v^{(i+1)}_{\tilde{n},\bar{m},m}$, where it comes from. We get finally the expression up to the monomials and the combinatorial factors

$$\sum_{\tilde{n}_1,\ldots,\tilde{n}_p} \prod_{q=1}^{p} \left[M^{-\gamma_q(\tilde{n}_q)} \int d^{\bar{m}_q}\underline{X}'_q d^{m_q}\underline{X}_q v^{(i+1)}_{\tilde{n}_q,\bar{m}_q,m_q}(\underline{X}'_q, \underline{X}_q) \right]$$
$$\cdot \sum_{T\in\mathcal{T}_p} \left[\prod_{\{q,q'\}\in T} C_{i+1}\big((q,X),(q',X')\big) \right]. \tag{4.77}$$

Now we turn our attention to the action of $e^{\Delta_{C_{\leq i},C_{\leq i+1}}[H']}$ on the remaining monomial with degree of $\sum_{q=1}^{p}(m_q-\theta_q+\bar{m}_q-\bar{\theta}_q)$. We prove firstly the following lemma. We denote $\xi=(q,X)$ and $\Gamma((q,X),(q',X'))=H_{qq'}C(X,X')$. Using the notation $\int d\xi F(\xi)=\sum_{q=1}^{p}\int dXF(q,X)$, we denote

$$\Delta[H]=-\int d\xi'\int d\xi\ \frac{\delta}{\delta\bar{\psi}(\xi')}\ \Gamma(\xi,\xi')\ \frac{\delta}{\delta\psi(\xi')}=\Delta_{\Gamma}. \tag{4.78}$$

Then

$$e^{\Delta[H]}\prod_{q=1}^{p}\bar{\psi}^{\bar{m}_q}(\underline{X}'_q)\prod_{q=1}^{p}\psi^{m_q}(\underline{X}_q)=e^{\Delta_{\Gamma}}\prod_{\xi'\in\bar{D}}\bar{\psi}(\xi')\prod_{\xi\in D}\psi(\xi), \tag{4.79}$$

where $\bar{D},D\subset\mathbb{N}_p\times\Gamma$. The lemma is

Lemma 9. *We have*

$$e^{\Delta_{\Gamma}}\prod_{\xi'\in\bar{D}}\bar{\psi}(\xi')\prod_{\xi\in D}\psi(\xi)=\sum_{A\subset D,\bar{A}\subset\bar{D}}\epsilon_{A\bar{A}}^{D\bar{D}}\ \det\Gamma_{A,\bar{A}}\prod_{\xi'\in\bar{D}\setminus\bar{A}}\bar{\psi}(\xi')\prod_{\xi\in D\setminus A}\psi(\xi).$$

where $\epsilon_{A\bar{A}}^{D\bar{D}}=\pm1$.

Proof. We expand

$$e^{\Delta_{\Gamma}}\prod_{\xi'\in\bar{D}}\bar{\psi}(\xi')\prod_{\xi\in D}\psi(\xi)=\sum_{l=0}^{\infty}\frac{1}{l!}\Delta_{\Gamma}^{l}\prod_{\xi'\in\bar{D}}\bar{\psi}(\xi')\prod_{\xi\in D}\psi(\xi), \tag{4.80}$$

we observe that the terms with $l>|D|$ are zero, thus for subsets $A\subset D$, $\bar{A}\subset\bar{D}$ and $|A|=|\bar{A}|=l$, $A=\{a_1,\dots,a_l\}$ with $a_1<\cdots<a_l$ and $\bar{A}=\{\bar{a}_1,\dots,\bar{a}_l\}$ with $\bar{a}_1<\cdots<\bar{a}_l$, the right hand side of (4.80) is given by

$$=\sum_{l=0}^{|D|}\sum_{\substack{A\subset D,\bar{A}\subset\bar{D}\\|A|=|\bar{A}|=l}}\epsilon_{A\bar{A}}^{D\bar{D}}\left(\frac{1}{l!}\Delta_{\Gamma}^{l}\prod_{\xi'\in\bar{A}}\bar{\psi}(\xi')\prod_{\xi\in A}\psi(\xi)\right)\prod_{\xi'\in\bar{D}\setminus\bar{A}}\bar{\psi}(\xi')\prod_{\xi\in D\setminus A}\psi(\xi)$$

where

$$\begin{aligned}\frac{1}{l!}\Delta_{\Gamma}^{l}\prod_{\xi'\in\bar{A}}\bar{\psi}(\xi')\prod_{\xi\in A}\psi(\xi) &= \pm\frac{1}{l!}\int\prod_{i=1}^{l}d\xi'_i d\xi_i\prod_{i=1}^{l}\left[\Gamma(\xi_i,\xi'_i)\frac{\delta}{\delta\bar{\psi}(\xi'_i)}\frac{\delta}{\delta\psi(\xi'_i)}\right]\\ &\quad\cdot\prod_{i=1}^{l}\bar{\psi}(\bar{a}_i)\prod_{i=1}^{l}\psi(a_i).\end{aligned}$$

The derivatives give

$$\begin{aligned} &= \pm\frac{1}{l!}\prod_{i=1}^{l}\int d\xi_i' d\xi_i \Gamma(\xi_i,\xi_i')\sum_{\sigma'}\prod_{i=1}^{l}\mathrm{sgn}(\sigma')\delta(\xi_i',\bar{a}_{\sigma'(i)})\sum_{\sigma}\mathrm{sgn}(\sigma)\prod_{i=1}^{l}\delta(\xi_i,a_{\sigma(i)}) \\ &= \pm\sum_{\sigma'}\mathrm{sgn}(\sigma')\prod_{i=1}^{l}\Gamma(\bar{a}_i,a_{\sigma'(i)}) \\ &= \det\Gamma_{\bar{A},A} \end{aligned} \tag{4.81}$$

where $\Gamma_{\bar{A},A}$ is the l by l matrix with entries $(\Gamma_{A,\bar{A}})_{i,j}=\Gamma_{\bar{a}_i,a_j}$. This compelets the proof. □

By above Lemma 9, the action of $e^{\Delta_{C_{\le i},C_{i+1}}[H'(T,\pi,\mathbf{s})]}$ gives a sum over subset A and $\bar{A}$ of determinants determined by A and $\bar{A}$. We denote $a_q=|A_q|$, and $\bar{a}_q=|\bar{A}_q|$. We use (3.76) to do the $\mathbf{s}$-integral and the sum over π, we are left with

$$\begin{aligned} &\| : \mathcal{E}_p^{(i)}\left(: \mathcal{V}^{(i+1)} :_{C_{\le i+1}},\ldots,: \mathcal{V}^{(i+1)} :_{C_{\le i+1}}\right) :_{-C_{\le i}} \|_{h_i} \\ \le\ & \sum_{\tilde{n}_1,\ldots,\tilde{n}_p}\sum_{T\in\mathcal{T}_p}\mathcal{S}(\tilde{n}_{q\in\mathbb{N}_p}) \\ & \cdot\sum_{\theta_1,\ldots,\theta p}\sum_{\bar{\theta}_1,\ldots,\bar{\theta}_p}\prod_{q=1}^{p}\binom{m_q}{\theta_q}\theta_q!\binom{\bar{m}_q}{\bar{\theta}_q}\bar{\theta}_q!\sum_{\substack{a_1,\ldots,a_p>0\\ \bar{a}_1,\ldots,\bar{a}_p>0}}\prod_{q=1}^{p}\binom{m_q-\theta_q}{a_q}\binom{\bar{m}_q-\bar{\theta}_q}{\bar{a}_q} \\ & \cdot\left(\det G^{C_{i+1}}_{T,\theta,a_q}+\det G^{C_{\le i}}_{T,\bar{\theta},a_q}+\det G^{C_{\le i+1},qq}_{T,\theta,a_q}\right)h_i^{\bar{m}_q-\bar{\theta}_q-\bar{a}_q+m_q-\theta_q-a_q}, \end{aligned} \tag{4.82}$$

where $G^{C_{i+1}}_{T,\theta,a_q}$, $G^{C_{\le i}}_{T,\bar{\theta},a_q}$, $G^{C_{\le i+1},qq}_{T,\theta,a_q}$ are a_q by $\bar{a}_q$ matrices. The elements of $G^{C_{i+1}}_{T,\theta,a_q}$ are of the form $H_{qq'}C_{i+1}((q,X),(q',X'))$, similarly, the elements of $G^{C_{\le i}}_{T,\bar{\theta},a_q}$ are given by $C_{\le i}((q,X),(q',X'))$, and $C_{\le i+1}((q,X),(q,X'))$ for the matrix $G^{C_{\le i+1},qq}_{T,\theta,a_q}$, q and q' label the vertices. The binomials $\binom{m_q-\theta_q}{a_q}$ come from the number of subsets A_q with $|A_q|=a_q$, and

$$\begin{aligned} \mathcal{S}(\tilde{n}_{q\in\mathbb{N}_p}) &= \sup_{T\in\mathcal{T}_p}\sup_{\tilde{X}}\max_i\prod_{q=1}^{p}M^{-\gamma_q(\tilde{n}_q)}\int d^{\bar{m}_q}\underline{X}_q'\int d^{m_q}\underline{X}_q\,\delta(\tilde{X},Z_i) \\ &\quad\cdot\prod_{\{q,q'\}\in T}\mathbf{C}_{i+1}((q,X),(q',X'))\prod_{q=1}^{p}|v^{(i+1)}_{\tilde{n}_q,\bar{m}_q,m_q}(\underline{X}_q',\underline{X}_q)| \end{aligned} \tag{4.83}$$

with

$$\mathbf{C}_{i+1}(X,X')=\max\{|C_{i+1}(X,X')|,|C_{i+1}(X',X)|\}. \tag{4.84}$$

The supremum over $\tilde{X}$ is the supremum in the definition of $|\cdot|_{1,\infty}$. Z_i denotes one of the coordinates in $\prod_{q=1}^{p} \int d^{\bar{m}_q} \underline{X}'_q \int d^{m_q} \underline{X}_q$ that is fixed to $\tilde{X}$ by the delta function. To perform the integrals in (4.83) we can arrange for the vertex, which contains $\tilde{X}$ as an argument, to be the root of the tree. Its $|\cdot|_{1,\infty}$ norm is obtained by fixing $\tilde{X}$. For other vertices in T, its $|\cdot|_{1,\infty}$ norms are obtained by holding fixed a coordinate X_{qT} in $\underline{X}_q$ and $\underline{X}'_q$, where X_{qT} is to be the tree line connecting the vertex with its ancester vertex (see [Bry86]). Using the summability of the propagator

$$\alpha_{C_{i+1}} = \max\{\sup_X \int |C_{i+1}(X,Y)|dY, \sup_X \int |C_{i+1}(Y,X)|dY\} \tag{4.85}$$

and the Gram bounds for the determinant, denoting $\omega_{C_{i+1}} = \alpha_{C_{i+1}} \gamma_{C_{i+1}}^{-2}$, recall $h_i = \alpha \gamma_{C_i}$, we have

$$\begin{aligned}
& \| : \mathcal{E}_p^{(i)} \left(: \mathcal{V}^{(i+1)} :_{C_{\le i+1}}, \dots, : \mathcal{V}^{(i+1)} :_{C_{\le i+1}} \right) :_{-C_{\le i}} \|_{h_i} \\
\le\ & \omega_{C_{i+1}}^{p-1} \gamma_{C_{i+1}}^{2(p-1)} \sum_{\tilde{n}_1,\dots,\tilde{n}_p} \prod_{q=1}^{p} \left[M^{-\gamma_q(\tilde{n}_q)} |v^{(i+1)}_{\tilde{n}_q, \bar{m}_q, m_q}|_{1,\infty} \right] \\
& \cdot \sum_{T \in \mathcal{T}_p} \sum_{\theta_1,\dots,\theta p} \sum_{\bar{\theta}_1,\dots,\bar{\theta}_p} \prod_{q=1}^{p} \binom{m_q}{\theta_q} \theta_q! \binom{\bar{m}_q}{\bar{\theta}_q} \bar{\theta}_q! \sum_{\substack{a_1,\dots,a_p>0 \\ \bar{a}_1,\dots,\bar{a}_p>0}} \prod_{q=1}^{p} \binom{m_q - \theta_q}{a_q} \binom{\bar{m}_q - \bar{\theta}_q}{\bar{a}_q} \\
& \cdot (3\gamma_{C_{i+1}})^{\bar{a}_q + a_q} (\alpha \gamma_{C_i})^{\bar{m}_q - \bar{\theta}_q - \bar{a}_q + m_q - \theta_q - a_q}.
\end{aligned} \tag{4.86}$$

The sum over a_q and $\bar{a}_q$ gives $\prod_{q=1}^{p} \left(3\gamma_{C_{i+1}} + \alpha\gamma_{C_i}\right)^{\bar{m}_q - \bar{\theta}_q + m_q - \theta_q}$. According to Cayley's theorem on the number of trees with fixed incidence numbers $d_1, \dots, d_p$, we have the tree number

$$\sum_{d_1,\dots,d_p} = \frac{(p-2)!}{(d_1-1)! \cdots (d_p-1)!}, \tag{4.87}$$

and the constraint

$$\sum_{q=1}^{p} d_q = \sum_{q=1}^{p} \left(\theta_q + \bar{\theta}_q\right) = 2\,(p-1)\,. \tag{4.88}$$

Because $\frac{\theta_q! \bar{\theta}_q!}{(d_q-1)!} \le \max\left\{\bar{\theta}_q, \theta_q\right\} \le 2^{\max\left\{\bar{\theta}_q, \theta_q\right\}}$, by using this bound, we sum over the d and the θ sums without the constraints. This gives

$$\sum_{\theta_q} \binom{m_q}{\theta_q} \left(3\gamma_{C_{i+1}} + \alpha\gamma_{C_i}\right)^{m_q - \theta_q} 2^{\theta_q} \gamma_{C_{i+1}}^{\theta_q} = \left(5\gamma_{C_{i+1}} + \alpha\gamma_{C_i}\right)^{m_q}. \tag{4.89}$$

The sum over $\tilde{n}_q$ gives

$$\begin{aligned} & \| : \mathcal{E}_p^{(i)} \left(: \mathcal{V}^{(i+1)} :_{C_{\leq i+1}}, \ldots, : \mathcal{V}^{(i+1)} :_{C_{\leq i+1}} \right) :_{-C_{\leq i}} \|_{h_i} \\ \leq \; & (p-2)!\, \omega_{C_{i+1}}^{p-1} \| \mathcal{V}_R^{(i+1)} \|_{h'_{i+1}}^p, \end{aligned} \tag{4.90}$$

with $h'_{i+1} = \left(5 + \frac{\alpha}{\sqrt{M}}\right) \gamma_{C_{i+1}}$, and

$$\begin{aligned} \| \mathcal{V}_R^{(i+1)} \|_{h'_{i+1}} &= \left| v_{1,1}^{(i+1)} \right|_{1,\infty} h_{i+1}'^2 + \frac{1}{M} \left| v_{1,1}^{(i+1)} \right|_{1,\infty} h_{i+1}'^2 + \frac{1}{M^2} \left| v_{1,1}^{(i+1)} \right|_{1,\infty} h_{i+1}'^2 \\ &+ \left| v_{2,2}^{(i+1)} \right|_{1,\infty} h_{i+1}'^4 + \frac{3}{M} \left| v_{2,2}^{(i+1)} \right|_{1,\infty} h_{i+1}'^4 \\ &+ \sum_{\bar{m}+m \geq 6} \left| v_{\bar{m},m}^{(i+1)} \right|_{1,\infty} h_{i+1}'^{\bar{m}+m}. \end{aligned} \tag{4.91}$$

Now we assume that there exist suitable constants M and α such that

$$\frac{h'_{i+1}}{h_{i+1}} = \frac{5}{\alpha} + \frac{1}{\sqrt{M}} = \tilde{A} < 1 \tag{4.92}$$

holds. With this the following inequalities hold,

$$\begin{aligned} |v_{1,1}^{(i+1)}|_{1,\infty} h_{i+1}'^2 &\leq \tilde{A}^2 |v_{1,1}^{(i+1)}|_{1,\infty} h_{i+1}^2, \\ |v_{2,2}^{(i+1)}|_{1,\infty} h_{i+1}'^4 &\leq \tilde{A}^4 |v_{2,2}^{(i+1)}|_{1,\infty} h_{i+1}^4, \\ \sum_{m,\bar{m} \geq 3} |v_{\bar{m},m}^{(i+1)}|_{1,\infty} h_{i+1}'^{\bar{m}+m} &\leq \tilde{A}^6 \sum_{m,\bar{m} \geq 3} |v_{\bar{m},m}^{(i+1)}|_{1,\infty} h_{i+1}^{\bar{m}+m}. \end{aligned} \tag{4.93}$$

Putting (4.93) into (4.91) and noting that $1/M < \tilde{A}^2 < 1$, we have

$$\begin{aligned} \| \mathcal{V}_R^{(i+1)} \|_{h'_{i+1}} &\leq (\tilde{A}^2 + \tilde{A}^4 + \tilde{A}^6) |v_{1,1}^{(i+1)}|_{1,\infty} h_{i+1}^2 + (\tilde{A}^4 + 3\tilde{A}^6) |v_{2,2}^{(i+1)}|_{1,\infty} h_{i+1}^4 \\ &\quad + \tilde{A}^6 \sum_{m,\bar{m} \geq 3} |v_{\bar{m},m}^{(i+1)}|_{1,\infty} h_{i+1}^{\bar{m}+m} \\ &\leq (\tilde{A}^2 + \tilde{A}^4) |v_{1,1}^{(i+1)}|_{1,\infty} h_{i+1}^2 + (\tilde{A}^4 + 2\tilde{A}^6) |v_{2,2}^{(i+1)}|_{1,\infty} h_{i+1}^4 \\ &\quad + \tilde{A}^6 \| \mathcal{V}^{(i+1)} \|_{h_{i+1}} \\ &\leq (\tilde{A}^2 + 2\tilde{A}^4 + 3\tilde{A}^6) \| \mathcal{V}^{(i+1)} \|_{h_{i+1}} \\ &\leq 6\tilde{A}^2 \| \mathcal{V}^{(i+1)} \|_{h_{i+1}}, \end{aligned} \tag{4.94}$$

and consequently we have

$$\begin{aligned} & \| W_p^{(i)}(\mathcal{V}^{(i+1)}) \|_{h_i} = \frac{1}{p!} \| : \mathcal{E}_p^{(i)}(: \mathcal{V}^{(i+1)} :_{C_{\leq i+1}}) :_{-C_{\leq i}} \|_{h_i} \\ \leq \; & (6\tilde{A}^2)^p \omega_{C_{i+1}}^{p-1} \| \mathcal{V}^{(i+1)} \|_{h_{i+1}}^p. \end{aligned} \tag{4.95}$$

For our further analysis we have to require that $6\tilde{A}^2 < 1$ holds.

Now everything is prepared for bounding $\tilde{R}_i$, using (4.95), $\tilde{R}_3$ can be bounded by

$$\tilde{R}_3 = \omega_{C_i} \|W_{\geq 3}^{(i)}(\mathcal{V}^{(i+1)})\|_{h_i} \leq \frac{K_{c3}}{M} \sum_{p\geq 3} (\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}})^p, \tag{4.96}$$

where $K_{c3} = 6^3 M^3 \tilde{A}^6$. $\omega_{C_i}/\omega_{C_{i+1}} \leq M^2$ has been used.

We turn to the terms $\tilde{R}_1$, $\tilde{R}_2$, which are bounded in a similarly way. We observe that

$$\begin{aligned} \|\mathcal{L}\mathcal{V}^{(i+1)}\|_{h'_{i+1}} &\leq \tilde{A}^2 |v_{1,1}^{(i+1)}|_{1,\infty} h_{i+1}^2 + \tilde{A}^4 |v_{2,2}^{(i+1)}|_{1,\infty} h_{i+1}^4 \\ &\leq 2\tilde{A}^2 \|\mathcal{V}^{(i+1)}\|_{h_{i+1}} \end{aligned} \tag{4.97}$$

and

$$\begin{aligned} \|\mathcal{R}\mathcal{V}^{(i+1)}\|_{h'_{i+1}} &\leq (\tilde{A}^4 + \tilde{A}^6) |v_{1,1}^{(i+1)}|_{1,\infty} h_{i+1}^2 + 3\tilde{A}^6 |v_{2,2}^{(i+1)}|_{1,\infty} h_{i+1}^4 \\ &\quad + \tilde{A}^6 \sum_{m,\bar{m}\geq 3} |v_{\bar{m},m}^{(i+1)}|_{1,\infty} h_{i+1}^{\bar{m}+m} \\ &\leq (\tilde{A}^4 + 3\tilde{A}^6) \|\mathcal{V}^{(i+1)}\|_{h_{i+1}} \\ &\leq 4\tilde{A}^4 \|\mathcal{V}^{(i+1)}\|_{h_{i+1}}. \end{aligned} \tag{4.98}$$

Then using the tree expansion we get

$$\tilde{R}_1 = 2\omega_{C_i} \|W_2^{(i)}(\mathcal{L}\mathcal{V}^{(i+1)}, \mathcal{R}\mathcal{V}^{(i+1)})\|_{h_i} \leq \frac{K_{c1}}{M} (\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}})^2, \tag{4.99}$$

where $K_{c1} = 16 M^3 \tilde{A}^6$, and

$$\tilde{R}_2 = \omega_{C_i} \|W_2^{(i)}(\mathcal{R}\mathcal{V}^{(i+1)}, \mathcal{R}\mathcal{V}^{(i+1)})\|_{h_i} \leq \frac{K_{c2}}{M} (\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}})^2 \tag{4.100}$$

where $K_{c2} = 16 M^3 \tilde{A}^8$. Putting all together we obtain finally

$$\begin{aligned} &\omega_{C_{j-1}} \|\mathcal{P}_{\geq 3, \geq 3} \left(\tilde{R}(\mathcal{V}^{(i+1)}) \right) \|_{h_{j-1}} \\ \leq\ & M^{j-i-1} \left(\frac{K_{c1} + K_{c2}}{M} (\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}})^2 + \frac{K_{c3}}{M} \sum_{p\geq 3} (\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}})^p \right). \end{aligned} \tag{4.101}$$

$\omega_{C_{j-1}} \| \mathcal{P}_{3,3} \left(W_2^{(i)}(G_{i+1}, G_{i+1}) \right) \|_{h_{j-1}}$ **term:**

The tree expansion gives

$$\begin{aligned} & \omega_{C_{j-1}} \left| \mathcal{P}_{3,3} \left(W_2^{(i)}(G_{i+1}, G_{i+1}) \right) \right|_{1,\infty} h_{j-1}^6 \leq 8\, \omega_{C_{j-1}} \alpha_{C_{i+1}} |g_{i+1}|^2 h_{j-1}^6 \\ \leq & \frac{8}{\alpha^2} \left(\frac{\omega_{C_{j-1}}}{\omega_{C_{i+1}}} \right) \left(\frac{h_{j-1}}{h_{i+1}} \right)^6 \omega_{C_{i+1}}^2 |g_{i+1}|^2 h_{i+1}^8 \\ \leq & M^{j-i-1} \frac{K_{c4}}{M} \, (\omega_{C_{i+1}} \| \mathcal{V}^{(i+1)} \|_{h_{i+1}})^2, \end{aligned} \tag{4.102}$$

where $K_{c4} = \frac{8}{\alpha^2}$, 8 is the combinatorial factor counting the number of second order graphs.

$\omega_{C_{j-1}} \| \mathcal{P}_{2,2} \left(\mathcal{R} W_{\geq 2}^{(i)}(\mathcal{V}^{(i+1)}) \right) \|_{h_{j-1}}$ **term:**

The renormalization transfer improves the power counting. We consider

$$\mathcal{P}_{2,2} \left(\mathcal{R} W_{\geq 2}^{(i)}(\mathcal{V}^{(i+1)}) \right) = \mu_{-C_{j-1}} * \mu_{C_{j-1}} * \mathcal{P}_{2,2} \left(\mathcal{R} W_{\geq 2}^{(i)}(\mathcal{V}^{(i+1)}) \right).$$

(4.51) shows that gradients created by the renormalization subtraction apply to fields. When this object is evaluated by Gaussian integration, it leads to the same effect as a gradient on a propagator. Hence we can extract an improvement M^{j-i-2} by consecutively using Gaussian convolution with C_{j-1} and $-C_{j-1}$, together with (4.95) we get

$$\begin{aligned} & \omega_{C_{j-1}} \| \mathcal{P}_{2,2} \left(\mathcal{R} W_{\geq 2}^{(i)}(\mathcal{V}^{(i+1)}) \right) \|_{h_{j-1}} \leq M^{j-i-1} \frac{3}{M} \omega_{C_i} \| W_{\geq 2}^{(i)}(\mathcal{V}^{(i+1)}) \|_{h_i} \\ \leq & M^{j-i-1} \frac{K_{c5}}{M} \sum_{p \geq 2} (\omega_{C_{i+1}} \| \mathcal{V}^{(i+1)} \|_{h_{i+1}})^p \end{aligned} \tag{4.103}$$

with $K_{c5} = 3 \cdot 6^2 \tilde{A}^4$.

Putting (4.61), (4.101), (4.102) and (4.103) all together, we get an expression of $\mathcal{V}^{(j-1)}$ in term of the running coupling functions g_i and $\mathcal{V}^{(i)}$ with $j \leq i \leq 0$:

$$\begin{aligned} & \omega_{C_{j-1}} \| \mathcal{V}^{(j-1)} \|_{h_{j-1}} \\ \leq & \omega_{C_{j-1}} \| \mathcal{V}_2^{(j-1)} \|_{h_{j-1}} + C_4 \sum_{i=j-1}^{0} |g_i - g_{i+1}| \\ & + \frac{K_{c1} + K_{c2} + K_{c4} + K_{c5}}{M} \sum_{i=j-1}^{0} M^{j-i-1} (\omega_{C_{i+1}} \| \mathcal{V}^{(i+1)} \|_{h_{i+1}})^2 \\ & + \frac{K_{c3} + K_{c5}}{M} \sum_{i=j-1}^{0} M^{j-i-1} \sum_{p \geq 3} (\omega_{C_{i+1}} \| \mathcal{V}^{(i+1)} \|_{h_{i+1}})^p. \end{aligned} \tag{4.104}$$

4.3 The flow of the running coupling functions

In order to estimate the role of the sign of the initial coupling constant playing in this convergence problem we study the flow of the running coupling functions g_j in detail. Applying localization operator on both side of (4.55), and projecting out the four fields part, we have

$$G_{j-1} = G_j + \beta_j(\mathcal{V}^{(j)}), \tag{4.105}$$

where the Beta function is given by

$$\beta_j(\mathcal{V}^{(j)}) = \mathcal{P}_{2,2}\left(2\mathcal{L}W_2^{(j-1)}(G_j, K_j) + \mathcal{L}W_2^{(j-1)}(G_j, G_j)\right) + \mathcal{P}_{2,2}\left(\mathcal{L}\tilde{R}(\mathcal{V}^{(j)})\right),$$

whose convergence follows directly from $\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \leq 1/2$. In this section we want to find the right behaviors of G_j as $j \to J$. Now we pay our atten-

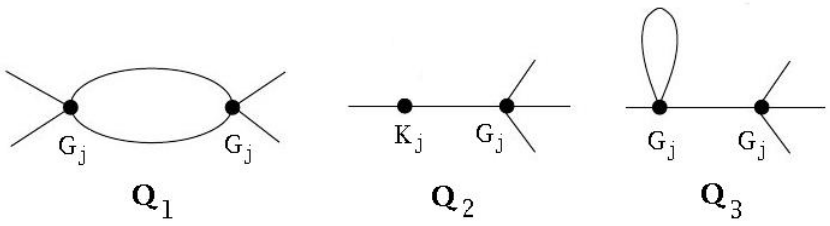

Figure 4.1: Connected four point graphs with two vertices.

tion to the second order contributions. The possible graph in the recursion relation (4.105) is the bubble Q_1 in Fig. (4.3); other possible connected four point graphs with two vertices are Q_2 and Q_3. The graph Q_2 contains a two point function, which according to Lemma 12 below produces a small factor M^j, thus the graph Q_2 becomes irrelevant in the flow. The graph Q_3 has an exponential small contribute to the running coupling flow, since it contains a tadpole, namely a local propagator $C_j(X, X)$, which is exponential small. To see this, we consider the Fourier transform of $C_j(X, X)$,

$$C_j(X, X) = \frac{1}{\beta}\sum_{k_0}\int d^2\mathbf{k}\,\frac{f_j(k)}{ik_0 - e(\mathbf{k})} = \frac{1}{\beta}\sum_{k_0}\int d^2\mathbf{k}\,\frac{f(M^{-2j}(k_0^2 + (\mathbf{k}^2 - \frac{\nu}{\beta})^2)}{ik_0 - \mathbf{k}^2 + \nu/\beta}.$$

Introducing polar coordinate in k, and substitute $t = -\mathbf{k}^2 + \frac{\nu}{\beta}$, we get

$$-\frac{1}{2\beta}\sum_{k_0}\int_{\nu/\beta}^{\infty} dt \int_0^{2\pi} d\theta\,\frac{f(M^{-2j}(k_0^2 + t^2))}{ik_0 + t}. \tag{4.106}$$

Substituting further $t = \rho\cos\phi$, $k_0 = \rho\sin\phi$,

$$C_j(X, X) = -\frac{1}{2\beta}\sum_{k_0}\int_0^{2\pi} d\theta \int d\rho\,\rho f(M^{-2j}\rho^2)\int d\phi\, e^{-i\phi}. \tag{4.107}$$

Then at zero temperature the integral vanishes. However for positive temperature, k_0, t are integrated over a finite region, above integral turns out to be exponential small. $\mathcal{P}_{2,2}\left(\mathcal{L}\tilde{R}(\mathcal{V}^{(j)})\right)$ contains a gain factor M^{-1} coming from the renormalized part of the effective potential. See (4.99), (4.100) and (4.96). By assuming $\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \leq C|g_j|$ (Lemma 11), this term is of order $M^{-1}|g_j|^2$.

Analysis above implies that the flow of g_j has the approximated form

$$g_{j-1} = g_j + \beta_j g_j^2 + O(g_j^2), \tag{4.108}$$

where $\beta_j = B_{pp}^{(j)} + B_{ph}^{(j)}$ denotes the bubble contributions, and power counting indicates that it is of order one. The remainder term is M^{-1} smaller than the bubble graph, thus the bubble terms establish the relevant properties of the running coupling functions.

The particle-particle bubble:

Now we begin to calculate the bubble terms explicitly. We calculate them by setting $q_0 = 0$ first, and let the spatial parts $\mathbf{q}$ tend to zero. For $q = 0$, by the symmetry $e(\mathbf{k}) = e(-\mathbf{k})$,

$$\begin{aligned} B_{pp}^{(j)}(0) &= -\frac{1}{\beta}\sum_{k_0}\int_{\mathbf{k}\in j-shell} d^2\mathbf{k}\, \frac{1}{ik_0 - e(\mathbf{k})}\frac{1}{-ik_0 - e(-\mathbf{k})} \\ &= \int_{\mathbf{k}\in j-shell} d^2\mathbf{k}\, \frac{1}{\beta}\sum_{k_0}\frac{1}{k_0^2 + e^2(\mathbf{k})} \\ &= -\int_{\mathbf{k}\in j-shell} d^2\mathbf{k}\, \frac{1}{2e(\mathbf{k})}\tanh\left(\frac{\beta e(\mathbf{k})}{2}\right). \end{aligned} \tag{4.109}$$

We substitute $u = \mathbf{k}^2 - \frac{\nu}{\beta}$, since $\tanh\left(\frac{\beta e(\mathbf{k})}{2}\right) \approx 1$, we have

$$\begin{aligned} B_{pp}^{(j)}(0) &\approx -\frac{\pi}{2}\int_{\sqrt{M^{-1}}M^j < u < \sqrt{2}M^j} du\frac{1}{u} \\ &\approx -\frac{\pi}{2}\log\sqrt{2M}. \end{aligned} \tag{4.110}$$

The particle-hole bubble:

We denote $K = \{\mathbf{k} : \mathbf{k} \in \text{j-shell} \cap \text{j-th shell shifted by } \mathbf{q}\}$, then

$$\begin{aligned} B_{ph}^{(j)}(q) &= -\frac{1}{\beta}\sum_{k_0}\int_{\mathbf{k}\in K} d^2\mathbf{k}\frac{1}{ik_0 - e(\mathbf{k})}\frac{1}{i(k_0+q_0) - e(\mathbf{k}+\mathbf{q})} \\ &= -\int_{\mathbf{k}\in K} d^2\mathbf{k}\ \frac{f_\beta(e(\mathbf{k})) - f_\beta(e(\mathbf{k}+\mathbf{q}))}{iq_0 + e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q})} \\ &= -\int_{\mathbf{k}\in K} d^2\mathbf{k}\frac{e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q})}{iq_0 + e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q})}\int_0^1 dt\frac{\partial f_\beta(e(\mathbf{k},\mathbf{q},t))}{\partial e} \\ &= \int_{\mathbf{k}\in K} d^2\mathbf{k}\frac{e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q})}{iq_0 + e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q})}\int_0^1 dt\delta_\beta(e(\mathbf{k},\mathbf{q},t))) \end{aligned}$$

where

$$e(\mathbf{k},\mathbf{q},t)) = te(\mathbf{k}) + (1-t)e(\mathbf{k}+\mathbf{q}), \tag{4.111}$$

and

$$\delta_\beta(e(\mathbf{k},\mathbf{q},t))) = -\frac{\partial f_\beta(e(\mathbf{k},\mathbf{q},t))}{\partial e}. \tag{4.112}$$

Note that

$$\delta_\beta(x) = \frac{\beta}{4\cosh^2\frac{\beta x}{2}}. \tag{4.113}$$

Then for $q_0 \neq 0$, $B_{ph}^{(j)}(q_0,\mathbf{q}) \to 0$ as $\mathbf{q} \to 0$. For $q_0 = 0$, consider

$$\begin{aligned} B_{ph}^{(j)}(0) &= \lim_{\mathbf{q}\to 0} B_{ph}(0.\mathbf{q}) \\ &= \int_{\sqrt{M^{-1}}M^j<|e(\mathbf{k})|<\sqrt{2}M^j} d^2\mathbf{k}\ \delta_\beta(e(\mathbf{k})) \\ &= \int_{\sqrt{M^{-1}}M^j<|e(\mathbf{k})|<\sqrt{2}M^j} d^2\mathbf{k}\ \frac{\beta}{4\cosh^2\frac{\beta e(\mathbf{k})}{2}} \\ &= \frac{\pi}{2}\left[\tanh\frac{\beta u}{2}\right]_{\sqrt{M^{-1}}M^j}^{\sqrt{2}M^j} = \frac{c\pi}{2} \end{aligned} \tag{4.114}$$

with $0 < c < 1$.

By (4.110) and (4.114), we conclude that by chosing M large enough, in other words, for a small enough energy shell, the explicit calculations of the particle-particle and particle-hole bubbles induce that $\beta_j < 0$. We denote the scale invariant part of β_j as $\tilde{\beta}$, thus the flow relation becomes approximately

$$g_{j-1} \approx g_j + \tilde{\beta} g_j g_{j-1}, \tag{4.115}$$

with $\tilde{\beta} < 0$, dividing by $g_j g_{j-1}$,

$$\frac{1}{g_j} = \frac{1}{g_{j-1}} + \tilde{\beta}, \tag{4.116}$$

with solution

$$g_j \approx \frac{g_0}{1 - \tilde{\beta}|j|g_0}, \tag{4.117}$$

the behavior of g_j can be summarized:

1. For a repulsive interaction $g_0 > 0$, g_j remains bounded and decreases to zero as $\beta \to \infty$, $g_j < g_0$ for all $J < j < 0$. Such a theory is called asymptotically free in the infrared limit. Physically, the interaction is screened at large distances.

2. For an attractive interaction $g_0 < 0$, $|g_j|$ grows as the energy scale is lowered, and becomes singular if β gets large. However, g_j remains finite and analytic in g_0 as long as $\tilde{\beta}|j|g_0 < 1$. We find an approximated behavior of g_j, i.e. there exists a constant C, such that $|g_j| \approx C|j||g_0|$ holds.

4.4 Analyticity of the effective action

In the last section, we have expressed the effective potential $\mathcal{V}^{(j-1)}$ in term of the running coupling functions g_i and $\mathcal{V}^{(i)}$. In this section we will prove the following result:

Lemma 10. *Suppose that there is a positive constant C_{2c}, such that for $J \leq j \leq 0$,*

$$|v_{1,1}^{(j)}|_{1,\infty} \leq \begin{cases} C_{2c} M^j |g_0|^2, & \text{if} \quad g_0 > 0, \\ C_{2c} M^j |g_0|, & \text{if} \quad g_0 < 0 \end{cases} \tag{4.118}$$

holds. Then there exist two positive constants $C_{0>}$ and $C_{0<}$ and suitable M and α, which satisfy inequality (4.121), such that, for all $g_0 > 0$, $g_0^2 \log \beta < C_{0>}$, or $g_0 < 0$, $|g_0| \log \beta < C_{0<}$, $h_j = \alpha \gamma_{C_j}$, the inequality

$$\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} < \frac{1}{2} \tag{4.119}$$

holds for all $J \leq j \leq 0$.

Proof. We prove the lemma by induction in j, with the statement of the lemma as the inductive hypothesis. The case with $j = 0$ is trivial. Let $j \leq -1$ and we assume that the statement holds, i.e. $\omega_{C_i}\|\mathcal{V}^{(i)}\|_{h_i} < \frac{1}{2}$, for all $j \leq i \leq 0$, (4.104) becomes

$$\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq \omega_{C_{j-1}}\|\mathcal{V}_2^{(j-1)}\|_{h_{j-1}} + C_4 \sum_{i=j-1}^{0} |g_i - g_{i+1}| + \frac{\tilde{K}}{M}\frac{1}{2}, \tag{4.120}$$

where $\tilde{K} = \frac{1}{2}(K_{c1} + K_{c2} + K_{c3} + K_{c4} + 2K_{c5}) \sum_{i=J-1}^{0} M^{J-i-1}$. It can be shown that there exist M and α, such that

$$\frac{\tilde{K}}{M} < 1 \tag{4.121}$$

holds. Moreover, (4.121) implies $6\tilde{A}^2 < 1$ holds.

1. For a repulsive interaction $g_0 > 0$: The first term of (4.120) can be easily bounded under the assumptions on (4.118) $v_{1,1}^{(j)}$,

$$\omega_{C_{j-1}}\|\mathcal{V}_2^{(j-1)}\|_{h_{j-1}} = \omega_{C_{j-1}}|v_{1,1}^{(j-1)}|_{1,\infty}h_{j-1}^2 \leq C_{2c}C_2|g_0|^2, \tag{4.122}$$

with $C_2 = c_\omega c_\gamma^2 \alpha^2$. Since g_j remains bounded and decreases as $j \to J$, the flow equation (4.105) implies that the second summand in (4.120) gives

$$\sum_{i=j-1}^{0} |g_i - g_{i+1}| \leq C_{4c}(|j| + 2)|g_0|^2, \tag{4.123}$$

with C_{4c} as a suitable constant. Thus there exists a constant $C_{>0}$, for $|g_0| < C_{>0}/(\log\beta)^{1/2}$, $\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq 1/2$ holds.

2. For an attractive interaction $g_0 < 0$, $|g_i - g_{i+1}| \leq C|g_0|$, then the second summand in (4.104) gives

$$\sum_{i=j-1}^{0} |g_i - g_{i+1}| \leq C_{4c}(|j| + 2)|g_0|, \tag{4.124}$$

(4.120) can be bounded by

$$\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq C_{2c}C_2|g_0| + C_4C_{4c}(|j| + 2)|g_0| + \frac{\tilde{K}}{M}\frac{1}{2}, \tag{4.125}$$

thus there exists M, α and $C_{<0}$, such that for $|g_0| \leq C_{<0}/\log\beta$, $\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq 1/2$ holds.

□

Remark. There is a weaker condition on the coupling constant for a repulsive interaction, which is temperature independent, namely $g_0 < C_{0>}$, $C_{0>}$ is a constant. This can be shown in the following way: we write

$$\mathcal{V}^{(j-1)} = \mathcal{V}_2^{(j-1)} + \mathcal{V}_a^{(j-1)} + \mathcal{V}_b^{(j-1)} + \mathcal{V}_c^{(j-1)}, \tag{4.126}$$

where

$$\mathcal{V}_a^{(j-1)} = G_j + \mathcal{P}_{2,2}\left(2W_2^{(j-1)}(G_j, K_j) + W_2^{(j-1)}(G_j, G_j)\right) + \mathcal{P}_{2,2}\left(\mathcal{L}\tilde{R}(\mathcal{V}^{(j)})\right),$$

$$\mathcal{V}_b^{(j-1)} = \mathcal{P}_{2,2}(\mathcal{R}\mathcal{V}^{(j)}) + \mathcal{P}_{\geq 3,\geq 3}(\mathcal{V}^{(j)}),$$

and

$$\mathcal{V}_c^{(j-1)} = \mathcal{P}_{3,3}\left(W_2^{(j-1)}(G_j, G_j)\right) + \mathcal{P}_{2,2}\left(\mathcal{R}\tilde{R}(\mathcal{V}^{(j)})\right) + \mathcal{P}_{\geq 3,\geq 3}\left(\tilde{R}(\mathcal{V}^{(j)})\right).$$

with

$$\tilde{R}(\mathcal{V}^{(j)}) = 2W_2^{(j-1)}(\mathcal{L}\mathcal{V}^{(j)}, \mathcal{R}\mathcal{V}^{(j)}) + W_2^{(j-1)}(\mathcal{R}\mathcal{V}^{(j)}, \mathcal{R}\mathcal{V}^{(j)}) + W_{\geq 3}^{(j-1)}(\mathcal{V}^{(j)}).$$

$\omega_{C_{j-1}}\|\mathcal{V}_a^{(j-1)}\|_{h_{j-1}}$ term:

Applying the localization operator on the both side of $\mathcal{V}_a^{(j-1)}$ we have

$$G_{j-1} = G_j + \mathcal{P}_{2,2}\left(2\mathcal{L}W_2^{(j-1)}(G_j, K_j) + \mathcal{L}W_2^{(j-1)}(G_j, G_j)\right) + \mathcal{P}_{2,2}\left(\mathcal{L}\tilde{R}(\mathcal{V}^{(j)})\right),$$

inserting this equation into $\mathcal{V}_a^{(j-1)}$, we get

$$\begin{aligned}
\mathcal{V}_a^{(j-1)} &= G_{j-1} - \mathcal{P}_{2,2}\mathcal{R}\left(2W_2^{(j-1)}(G_j, K_j) + W_2^{(j-1)}(G_j, G_j)\right), \\
&= G_{j-1} - \mu_{-C_{j-1}} * \mu_{C_{j-1}} * \mathcal{P}_{2,2}\mathcal{R}\left(2W_2^{(j-1)}(G_j, K_j) + W_2^{(j-1)}(G_j, G_j)\right) \\
&= G_{j-1} - \frac{3}{M}\,\mathcal{P}_{2,2}\left(2W_2^{(j-1)}(G_j, K_j) + W_2^{(j-1)}(G_j, G_j)\right).
\end{aligned} \tag{4.127}$$

the gain factor $\frac{3}{M}$ comes from Wick ordering. The last term has the bound

$$\begin{aligned}
& \omega_{C_{j-1}}\left|\mathcal{P}_{2,2}\left(2W_2^{(j-1)}(G_j, K_j) + W_2^{(j-1)}(G_j, G_j)\right)\right|_{1,\infty} h_{j-1}^4 \\
\leq\ & \omega_{C_{j-1}}\alpha_{C_j}\left(2|g_j||\kappa_j| + 18|g_j|^2 h_j^2\right) h_{j-1}^4 \\
\leq\ & \left(\frac{\omega_{C_{j-1}}}{\omega_{C_j}}\right)\left(\frac{h_{j-1}}{h_j}\right)^4 \omega_{C_j}^2\left(\frac{2}{\alpha^2}|g_j||\kappa_j|h_j^6 + \frac{18}{\alpha^4}|g_j|^2 h_j^8\right) \\
\leq\ & \left(\frac{2}{\alpha^2} + \frac{18}{\alpha^4}\right)(\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j})^2.
\end{aligned} \tag{4.128}$$

where the constants 2 and 18 are the combinatorial factors counting the number of graphs. Combining both contributions we obtain a final expression

$$\omega_{C_{j-1}}\|\mathcal{V}_a^{(j-1)}\|_{h_{j-1}} \leq C_4|g_{j-1}| + \frac{K_a}{M}(\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j})^2. \tag{4.129}$$

where $C_4 = c_\omega c_\gamma \alpha^4$ and $K_a = 3\left(\frac{2}{\alpha^2} + \frac{18}{\alpha^4}\right)$.

$\omega_{C_{j-1}}\|\mathcal{V}_b^{(j-1)}\|_{h_{j-1}}$ **term:**

Using the trick of Wick ordering it is easy to get the bound

$$\begin{aligned} & \omega_{C_{j-1}}\|\mathcal{V}_b^{(j-1)}\|_{h_{j-1}} \\ \leq\ & \frac{3}{M}\,\omega_{C_j}\|\mathcal{P}_{2,2}(\mathcal{V}^{(j)})\|_{h_j} + M^2\left(\frac{h_{j-1}}{h_j}\right)^6 \omega_{C_j}\|\mathcal{P}_{\geq 3,\geq 3}(\mathcal{V}^{(j)})\|_{h_j} \\ \leq\ & \frac{3}{M}\,\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j}. \end{aligned} \tag{4.130}$$

$\omega_{C_{j-1}}\|\mathcal{V}_c^{(j-1)}\|_{h_{j-1}}$ **term:**

This last term can be bounded by

$$\begin{aligned} & \omega_{C_{j-1}}\|\mathcal{V}_c^{(j-1)}\|_{h_{j-1}} \\ \leq\ & \omega_{C_{j-1}}\left|\mathcal{P}_{3,3}\left(W_2^{(j-1)}(G_j,G_j)\right)\right|_{1,\infty} h_{j-1}^6 + \omega_{C_{j-1}}\|\tilde{R}(\mathcal{V}^{(j)})\|_{h_{j-1}} \\ \leq\ & \frac{K_{1c}+K_{2c}+K_{3c}}{M}(\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j})^2 + \frac{K_{4c}}{M}\sum_{p\geq 3}(\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j})^p. \end{aligned} \tag{4.131}$$

where $K_{1c} = 8/\alpha^2$, $K_{2c} = 16M^3\tilde{A}^6$, $K_{3c} = 16M^3\tilde{A}^8$ and $K_{4c} = 216M^3\tilde{A}^6$. Putting all together we obtain

$$\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq \omega_{C_{j-1}}\|\mathcal{V}_2^{(j-1)}\|_{h_{j-1}} + C_4|g_{j-1}| + \frac{\tilde{K}'}{M}\frac{1}{2}, \tag{4.132}$$

where $\tilde{K}' = \frac{1}{2}(K_a + K_{1c} + K_{2c} + K_{3c} + K_{4c} + 6)$ is a constant. It is possible to choice M and α, such that

$$\frac{\tilde{K}'}{M} < 1 \tag{4.133}$$

holds. For $g_0 > 0$, g_j is finite and obeys $g_j < g_0$, for all $j < 0$. Then the condition $g_0 < C_{0>}$, with $C_{0>}$ as a constant, is sufficient for the convergence of the perturbation series. This result implies that for a sufficient small $g_0 > 0$, there might be no instability as the temperature goes to zero. However

the temperature independent condition on the initial coupling constant is not sufficient to guarantee the regularity properties of the self-energy. This will be shown in the next chapter. By (4.132) we have further

Lemma 11. *Suppose that there is a positive constant C_{2c}, such that for $J \leq j \leq 0$,*

$$|v^{(j)}_{1,1}|_{1,\infty} \leq \begin{cases} C_{2c}M^j|g_0|^2, & \text{if} \quad g_0 > 0, \\ C_{2c}M^j|g_0|, & \text{if} \quad g_0 < 0. \end{cases} \tag{4.134}$$

holds. There exist two positive constants $C_{g>}$ and $C_{g<}$ and suitable M and α, which satisfy inequality (4.133), such that for all $g_0 > 0$, $g_0^2 \log\beta < C_{0>}$, or $g_0 < 0$, $|g_0|\log\beta < C_{0<}$, the inequality

$$\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \leq \begin{cases} C_{g>}|g_0| & \text{if} \quad g_0 > 0, \\ C_{g<}|g_j| & \text{if} \quad g_0 < 0. \end{cases} \tag{4.135}$$

holds, for all $J \leq j \leq 0$.

Proof. The lemma is proved by induction in j, with the statement of the lemma as the inductive hypothesis. The case with $j = 0$ is trivial. Let $j \leq -1$ and we assume that the statement holds, i.e. $\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \leq C_{g>}|g_0|$, if $g_0 > 0$, or $\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \leq C_{g<}|g_j|$, if $g_0 < 0$ and $\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \leq \frac{1}{2}$, for all $j \leq i \leq 0$, we have

$$\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq \omega_{C_{j-1}}\|\mathcal{V}_2^{(j-1)}\|_{h_{j-1}} + C_4|g_{j-1}| + \frac{\tilde{K}'}{M}\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \tag{4.136}$$

1. For a repulsive interaction $g_0 > 0$: (4.136) reduced to

$$\begin{aligned} \omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} &\leq C_{2c}C_2|g_0|^2 + C_4|g_{j-1}| + \frac{\tilde{K}'}{M}C_{g>}|g_0| \\ &\leq \left(\frac{C_{2c}C_2 + C_4}{C_{g>}} + \frac{\tilde{K}'}{M}\right)C_{g>}|g_0|, \end{aligned} \tag{4.137}$$

where we have used the fact $|g_{j-1}|/g_0 < 1$. We find that there exists a constant $C_{g>}$, so that

$$C_{g>} > \frac{C_{2c}C_2 + C_4}{1 - \frac{\tilde{K}'}{M}} \tag{4.138}$$

holds, which implies the prefactor in (4.137) smaller than one. It follows that $\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq C_{g>}|g_0|$ holds.

2. For an attractive interaction $g_0 < 0$, (4.136) reduced to

$$\begin{aligned}\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} &\leq C_{2c}C_2|g_0| + C_4|g_{j-1}| + \frac{\tilde{K}'}{M}C_{g<}|g_j| \\ &\leq \left(\frac{C_{2c}C_2 + C_4}{C_{g<}} + \frac{\tilde{K}'}{M}\right) C_{g<}|g_{j-1}|, \quad (4.139)\end{aligned}$$

where we have used the fact $|g_0|/|g_{j-1}| < 1$ and $|g_j|/|g_{j-1}| < 1$. We find that there exists a constant $C_{g<}$, so that

$$C_{g<} > \frac{C_{2c}C_2 + C_4}{1 - \frac{\tilde{K}'}{M}} \quad (4.140)$$

holds. It follows that $\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq C_{g>}|g_{j-1}|$ holds.

□

The result about the analytical of the effective potential of scale j in the running coupling g_j shown above bases on the assumption on the two point vertex functions. We have to verify this. The assumption on $|v^{(j)}_{1,1}|_{1,\infty}$ corresponds to the problem of fixing the counterterm. We consider at first the flow of the two point functions. By using the tree formula we have for $v^{(j)}_{1,1}$,

$$\begin{aligned}&\left|v^{(j-1)}_{1,1} - v^{(j)}_{1,1}\right|_{1,\infty} \\ \leq\ & \sum_{p\geq 2}\frac{1}{p!}\,\omega^{p-1}_{C_j}\gamma^{2(p-1)}_{C_j}\sum_{m_1,\ldots,m_p}\sum_{\bar{m}_1,\ldots,\bar{m}_p}\prod_{q=1}^{p}|v^{(j)}_{\bar{m}_q,m_q}|_{1,\infty} \\ &\cdot\sum_{T\in\mathcal{T}_p}\sum_{\theta_1,\ldots,\theta p}\sum_{\bar{\theta}_1,\ldots,\bar{\theta}_p}\prod_{q=1}^{p}\binom{m_q}{\theta_q}\theta_q!\binom{\bar{m}_q}{\bar{\theta}_q}\bar{\theta}_q!\sum_{\substack{a_1,\ldots,a_p>0\\ \bar{a}_1,\ldots,\bar{a}_p>0}}\prod_{q=1}^{p}\binom{m_q-\theta_q}{a_q}\binom{\bar{m}_q-\bar{\theta}_q}{\bar{a}_q} \\ &\cdot\prod_{q=1}^{p}\left(3\gamma_{C_j}\right)^{\bar{a}_q+a_q}\delta_{\sum_{q=1}^{p}\bar{m}_q-\bar{\theta}_q-\bar{a}_q+m_q-\theta_q-a_q=2}\cdot \quad (4.141)\end{aligned}$$

The delta function at the last line ensures that only terms with two external fields are picked out. Taking into account the delta function and summing over a_q, $\bar{a}_q$,

$$\sum_{\substack{a_1,\ldots,a_p>0\\ \bar{a}_1,\ldots,\bar{a}_p>0}}\prod_{q=1}^{p}\left[\binom{m_q-\theta_q}{a_q}\binom{\bar{m}_q-\bar{\theta}_q}{\bar{a}_q}\left(3\gamma_{C_j}\right)^{\bar{a}_q+a_q}\right]\delta_{\sum_{q=1}^{p}\bar{m}_q-\bar{\theta}_q-\bar{a}_q+m_q-\theta_q-a_q=2}$$

gives $(3\gamma_{C_j})^{\sum_{q=1}^{p} \bar{m}_q - \bar{\theta}_q + m_q - \theta_q - 2}$ times a combinatorial factor

$$\binom{\sum_{q=1}^{p}(m_q-\theta_q)}{1}\binom{\sum_{q=1}^{p}(\bar{m}_q-\bar{\theta}_q)}{1} = \prod_{q=1}^{p}\left[(m_q-\theta_q)(\bar{m}_q-\bar{\theta}_q)\right], \quad (4.142)$$

using the fact $m_q - \theta_q \leq 2^{m_q-\theta_q}$, the combinatorial factor can be bounded by $2^{\sum_{q=1}^{p} \bar{m}_q - \bar{\theta}_q + m_q - \theta_q}$. It turns out that

$$\left| v_{1,1}^{(j-1)} - v_{1,1}^{(j)} \right|_{1,\infty} \leq \omega_{C_j}^{-1} (3\gamma_{C_j})^{-2} \sum_{p\geq 2} (\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j''})^p \quad (4.143)$$

with $h_j'' = 6\gamma_{C_j} + 2\gamma_{C_j} = 8\gamma_{C_j}$, Choosing $\alpha > 8$, so that $\|\mathcal{V}^{(j)}\|_{h_j''} \leq \|\mathcal{V}^{(j)}\|_{h_j}$, we arrive at

$$\left| v_{1,1}^{(j-1)} - v_{1,1}^{(j)} \right|_{1,\infty} \leq \omega_{C_j}^{-1} (3\gamma_{C_j})^{-2} \sum_{p\geq 2} (\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j})^p. \quad (4.144)$$

Lemma 12. *It is possible to choose the counterterm, such that*

$$|v_{1,1}^{(j)}|_{1,\infty} \leq \begin{cases} C_{2c} M^j |g_0|^2 & \text{if } g_0 > 0, g_0 < \frac{C_{0>}}{\sqrt{\log \beta}}. \\ C_{2c} M^j |g_0| & \text{if } g_0 < 0, |g_0| < \frac{C_{0<}}{\log \beta}. \end{cases} \quad (4.145)$$

for all $J - 1 \leq j \leq 0$, where C_{2c} is a suitable constant.

Proof. We have to show that there is a suitable counterterm κ, which should be a constant, so that the lemma holds. We consider the counterterm flow of $\kappa_j(X', X)$. See (4.43). Since the effective actions are constructed by modifying the interaction such that the Fermi surface of the interaction system stays fixed, this requires a special choice of the counterterm κ, so that we require, as a condition on κ that

$$\kappa_{J-1}(X', X) = \kappa + \sum_{i=J-1}^{0} (\kappa_i(X', X) - \kappa_{i+1}(X', X)) = 0, \quad (4.146)$$

where we denote $\kappa_1 = \kappa$. (4.146) means that the "renormalized mass has to be zero ". This is not a definition, but only an equation to be solved. From (4.146) we have further

$$\begin{aligned} \kappa_j(X', X) &= \kappa + \sum_{i=j}^{0} (\kappa_i(X', X) - \kappa_{i+1}(X', X)) \\ &= -\sum_{i=J-1}^{j-1} (\kappa_i(X', X) - \kappa_{i+1}(X', X)), \end{aligned} \quad (4.147)$$

which can be rewritten as

$$\int d\tilde{\tau}' v_{1,1}^{(j)}(\tilde{X}', X) = -\sum_{i=J-1}^{j-1} \int d\tilde{\tau}' \left(v_{1,1}^{(i)}(\tilde{X}', X) - v_{1,1}^{(i+1)}(\tilde{X}', X) \right). \quad (4.148)$$

By using (4.144), (4.148) induces that

$$\begin{aligned} |v_{1,1}^{(j)}|_{1,\infty} &\leq \sum_{i=J-1}^{j-1} |v_{1,1}^{(i)} - v_{1,1}^{(i+1)}|_{1,\infty} \\ &\leq \sum_{i=J-1}^{j-1} \omega_{C_{i+1}}^{-1} (3\gamma_{C_{i+1}})^{-2} \sum_{p\geq 2} \left(\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}} \right)^p. \end{aligned} \quad (4.149)$$

1. For $g_0 > 0$: we have $\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} \leq C_{g>} g_0$ and $\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} < \frac{1}{2}$, for all $J \leq j \leq 0$. Then (4.149) gives

$$|v_{1,1}^{(j)}|_{1,\infty} \leq 2C_{g>}^2 |g_0|^2 \sum_{i=J-1}^{j-1} \omega_{C_{i+1}}^{-1} (3\gamma_{C_{i+1}})^{-2} \leq C_{2c} M^j |g_0|^2. \quad (4.150)$$

   with a suitable constant C_{2c}.

2. For $g_0 < 0$: we have $\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} \leq C_{g<} |g_j|$ and $\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} < \frac{1}{2}$, $|g_j| \approx C|j|g_0$ and $|g_{j-1} - g_j| \leq C'_g |g_0|$, for all $J \leq j \leq 0$. Then (4.149) gives

$$\begin{aligned} |v_{1,1}^{(j)}|_{1,\infty} &\leq 2 \sum_{i=J-1}^{j-1} \omega_{C_{i+1}}^{-1} (3\gamma_{C_{i+1}})^{-2} (C_{g<} |g_j|)^2 \\ &\leq C_{2c} M^j |g_0|. \end{aligned} \quad (4.151)$$

This completes the proof. □

According to Lemma 12 one can choose a constant counterterm $\kappa = \delta e$, which obeys $\delta e < C|g|$. The first part of the Theorem 1, namely the analysity of the renormalized $2n$-points Green's function is a direct consequence of the Lemma 5 and the Lemma 10.

Chapter 5

The Regularity Analysis of the Self-energy

In this chapter we turn to the regularity properties of the self-energy of the dilute Fermi gas model. One expects that the interaction produces a self-energy $\Sigma(g,k)$ of the fermions such that the propagator becomes

$$C(k) = \frac{1}{ik_0 - E(\mathbf{k}) - \Sigma(g,k)}. \tag{5.1}$$

The problem is now to verify that $\Sigma(g,k)$ is a regular function, such that the interacting propagator indeed has the same integrability properties as that of the free one, but the singularity is at a different place, namely $E(\mathbf{k}) + \Sigma(g,k) = 0$.

This regularity problem turns out to be nontrivial. We will prove the regularity properties of the self-energy by using the renormalization group techniques. The results are summarized in the following theorem:

Theorem 5. *With the constraints on g, $|g|^2 \log\beta < C_{0>}$, for $g > 0$, or $|g| \log\beta < C_{0<}$, for $g < 0$, there exists a constant $c > 0$, such that*

1. $|\Sigma(k)| \leq c|g|$,
2. $|\partial_{k_\alpha}\Sigma(k)| \leq c|g|$, $\alpha = 1, 2$.
3. $|\partial_{k_\alpha}\Sigma(k)| \leq c$, $\alpha = 0$.
4. $|\partial_{k_\alpha}\partial_{k_\beta}\Sigma(k)| \leq c$, $\alpha, \beta = 1, 2$.
5. $|\partial_{k_0}^2\Sigma(k)| \leq c\beta$.

We make some remarks. For a repulsive interaction $g > 0$, the $|g|$ dependence of the self-energy and its first derivative with respect to the spatial momentum can be replaced by $|g|^2$.

The theorem proves that the self-energy is uniform of $\mathbb{C}^1$ in the domain analytic of the theory. However, the bound for the second derivative with respect to the frequency grows with β, which strongly suggests that the self energy is not uniform of $\mathbb{C}^2$ in the analytic domain. More precisely we will prove a lower bound for the amplitude of the sunset graph (see Figure 5)

$$|\frac{\partial^2}{\partial k_0^2}\Sigma_{p=2}(k_0, 0)| \geq C\beta, \tag{5.2}$$

in a special case of incoming momentum ($\mathbf{k} = 0$). This has an important consequence. In [Sal99] sufficient conditions for Fermi liquid behavior were given, the self-energy should be $\mathbb{C}^2$, but this is not the situation in the dilute Fermi gas model, so that a hypotheses that the dilute Fermi gas with short-range interaction at low temperature is Fermi liquid can not be verified. But the Fermi liquid behavior has not been definitively ruled out. Since in fact only divergence in the first derivative will change the asymptotic behavior of the propagator at small frequency or momentum, and rules out the Fermi liquid behavior. The second spatial derivative enters the curvature of the Fermi surface, and plays no significant role here, because the shift of the Fermi surface is small. More precisely, we do a first order Taylor expansion in $k - \Pi(k)$, to get

$$\Sigma(k) = k_0\partial_{k_0}\Sigma(0, \Pi(\mathbf{k})) + (\mathbf{k} - \Pi(\mathbf{k}))\nabla\Sigma(0, \Pi(\mathbf{k})) + \tilde{\Sigma}(k) \tag{5.3}$$

$\tilde{\Sigma}(k)$ is the Taylor remainder term. Since the first derivative still remains small in the analytic domain, so that we obtain a finite wave function renormalization

$$Z(k) = 1 + i\partial_{k_0}\Sigma(0, \Pi(\mathbf{k})), \tag{5.4}$$

which stays close to 1, and a finite correction to the Fermi velocity.

We observe that from the claim of Theorem 1 the counterterm which essentially restricts to the self-energy to the Fermi surface has more regularities than the self-energy itself. Therefore, it is more convenient to do renormalization using counterterms instead of putting the self-energy into the denominator of the propagator. Indeed we found that for any $\epsilon > 0$, the $1 + \epsilon$ derivative of the self-energy with respect to the frequency grows with β^ϵ, even after the renormalization.

Let us now discuss the divergence of the second derivative with respect to the frequency. The lack of a bound for the second frequency derivative

which diverges when the temperature goes to zero, is due to the geometric form of the Fermi surface. As proven in [FST96, FST99], if a graph G is two-legged, 1-PI, and has two external vertices, then G is a overlapping graph. A definition of overlapping graph can be found in [FST96]. Let T be a spanning tree for G, and l be a line not in T. Denotes the vertices at the ends of l by v and w. If $v = w$, the loop generated by l contains only the line l. If $v \neq w$, there is a unique path P_l in T from v to w. A graph is overlapping if for some choice of the spanning tree T, there are lines $l_T \in T$ and $l_1 \neq l_2 \notin T$ such that the loops generated by l_i both contain l_T. This property of G is independent of the choice of the spanning tree. It is straightforward to verify that the sunset graph shown in Figure 5 is overlapping. The significance of the notion of overlapping graphs is that the

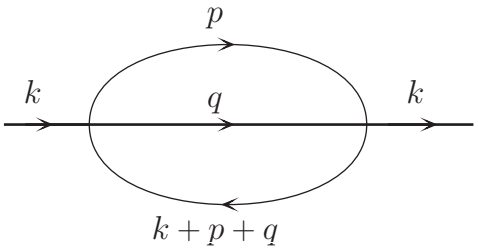

Figure 5.1: Example for overlapping graph: the sunset graph.

value of any overlapping graph, for $|e(\mathbf{k})| < \epsilon$, contains a subintegral

$$Q(\epsilon) = \sup_{\mathbf{q}} \max_{v_i=\pm 1} \int d\theta_1 d\theta_2 \; 1(|e(v_1\mathbf{p}(0,\theta_1) + v_2\mathbf{p}(0,\theta_2) + \mathbf{q})| \leq \epsilon), \quad (5.5)$$

where $1(\cdot)$ equals one if the condition holds, zero otherwise, $\mathbf{p}(\rho,\theta)$ denotes a parametrization of a neighborhood of the Fermi surface, with $\rho = 0$ corresponds to the Fermi surface itself. Then there is a constant $Q_V > 0$ dependent on the curvature of the Fermi surface, such that

$$Q(\epsilon) \leq Q_V |\epsilon| |\log \epsilon|. \quad (5.6)$$

If G is overlapping, a volume improvement factor which depends on the inverse of the curvature of the Fermi surface can be extracted to compensate for the bad factor from the derivative. But for dilute Fermi gas the Fermi surface is so small, hence all the volume improvement effects from overlapping are absent.

5.1 Arch expansion

We start to study the self-energy. The self-energy is defined through its Feynman graph expansion. In this approach, $\Sigma(x,y)$ is restricted to the contributions to the two-point connected subgraphs which are 1-particle-irreducible in the channel $x-y$, that is, in which x and y cannot be disconnected by the deletion of a single line. Note that, for convenience, we shall simply write in the following 1-PI to mean "1-particle-irreducibility in the channel $x-y$".

From the previous chapter we see that the tree expansion ensures only the connectivity of the graphs but not the 1-point-irreducibility. In principle we would have to expand out all the remaining determinant to know which contributions are 1-PI or not. This seems not constructive, since the number of terms after expanding the complete determinant proliferates, which gives a factorial $p!$. In this section we will introduce the arch expansion [DR00b], which allows us to select 1-PI graphs from the tree expansion by expanding out explicitly some additional lines from the determinant. It shows that 1-PI can be extracted by expanding some loops out of the determinant without generating any factorials in the bound.

We explain in detail the arch expansion for an expression of a tree expansion which has a expression of the type,

$$\mathcal{A}_T = \prod_{l\in T} C(Y_l, Y_l') \det G_T. \tag{5.7}$$

We consider a graph with p vertices, equipped with its spanning tree T. We distinguish in T the unique path connecting the two external vertices x and y through T, denoted by $\mathcal{P}(x,y)$. If $\mathcal{P}(x,y)$ has length zero, namely the two external lines are hooked to the same vertex $x=y$, we have a generalized tadpole, which is automatically 1-PI. No additional expansion is needed. Otherwise there are $n-2 \le p-2$ vertices in the path $\mathcal{P}(x,y)$ joining $x=x_1$ and $y=x_n$. The intermediate $n-2$ vertices are numbered by $x_2,\dots,x_{n-1}$. The remaining vertices are divided into n disjoint subsets V_i, $i=1,\dots,n$. By definition, a vertex belongs to V_i if and only if the unique path in T joining the vertex to the external vertex x_1 met x_i but not pass though x_{i+1}. See Figure 5.2.

A complete expansion of the determinant $\det G_T$ costs $p!$. We just want to select 1-PI graphs with respect to the path $\mathcal{P}(x,y)$. To do this, among all the possible contraction schemes contained in the determinant, we select the contractions which have a contraction between an element of V_1 and some vertex between $V_{k_1} = \bigcup_{i=2}^{k_1} V_i$, with $k_1 > 1$. This is done by a Taylor

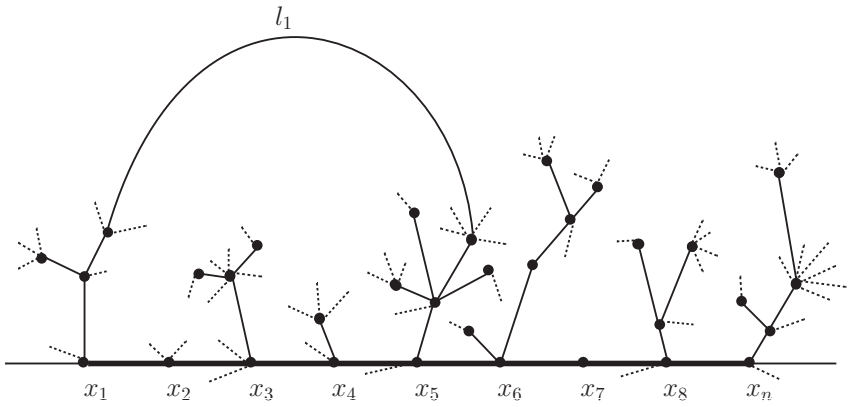

Figure 5.2: Example for a spanning tree. The path $\mathcal{P}(x, y)$ joining $x = x_1$ and $y = x_n$ is drawn in bold line. The external fields are represented by thin lines, the fields in the determinant by dotted half-lines. We have added to this tree an explicit line l_1 joining the subset V_1 to V_{k_1}, with $k_1 = 5$. Thus this graph is 1-particle-irreducible in the channel $x_1 - x_5$.

expansion step with an interpolation parameter $0 \leq s_1 \leq 1$. Defining

$$C_{ij}(s_1) = \begin{cases} s_1 C_{ij} & \text{if} \quad i \in V_1, j \notin V_1, \\ C_{ij} & \text{otherwise.} \end{cases} \tag{5.8}$$

the expansion of the determinant gives

$$\det G_T = \det G_T(s_1) \mid_{s_1=0} + \int_0^1 ds_1 \frac{d}{ds_1} \det G_T(s_1). \tag{5.9}$$

The first term $s_1 = 0$ means that the graph is 1-PR (by cutting the line joining x_1 and x_2 as no further line connects V_1 to its complement). Otherwise one loop connecting a vertex of V_1 to a vertex in V_{k_1} is extracted from the determinant. At this stage, the graph obtained is 1-PI in the channel $x_1 - x_{k_1}$. If $k_1 = n$ we obtain a 1-PI graph in the channel $x_1 - x_n$. Otherwise we repeat this procedure between $\bigcup_{i=1}^{k_1} V_i$ and its non-empty complements by expanding $\det G_T(s_1)$ with a new interpolation parameter $0 \leq s_2 \leq 1$, we set

$$C_{ij}(s_1, s_2) = \begin{cases} s_2 C_{ij}(s_1) & \text{if} \quad i \in \bigcup_{i=1}^{k_1} V_i, j \notin \bigcup_{i=1}^{k_1} V_i, \\ C_{ij}(s_1) & \text{otherwise.} \end{cases} \tag{5.10}$$

we have

$$\det G_T(s_1) = \det G_T(s_1, s_2) \mid_{s_2=0} + \int_0^1 ds_2 \frac{d}{ds_2} \det G_T(s_1, s_2). \tag{5.11}$$

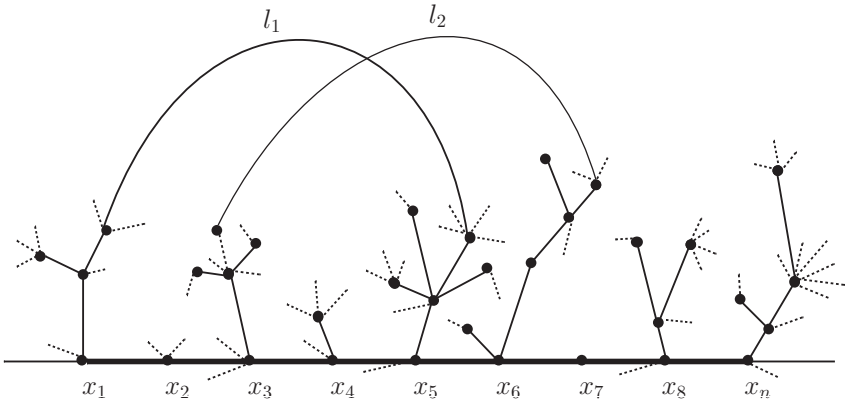

Figure 5.3: The tree by two line from the arch expansion.

The first term means that the graph is 1-PR by cutting the line joining x_{k_1} and x_{k_1+1}. The second term extracts a loop line to link $\bigcup_{i=1}^{k_1} V_i$ with its complement. See Figure 5.3. Note that

$$\begin{aligned} & \int_0^1 ds_2 \frac{d}{ds_2} \det G_T(s_1, s_2) \\ = & \int_0^1 ds_2 \frac{\partial}{\partial s_2} C_{i_2 j_2}(s_1, s_2) \frac{\partial}{\partial C_{i_2 j_2}(s_1, s_2)} \det G_T(s_1, s_2), \end{aligned} \tag{5.12}$$

with $i_2 \in \bigcup_{i=1}^{k_1} V_i$, $j_2 \in V_{k_2}$, $k_2 > k_1$ and

$$\frac{\partial}{\partial s_2} C_{i_2 j_2}(s_1, s_2) = \begin{cases} C_{i_2 j_2}, & \text{if} \quad i_2 \in \bigcup_{i=2}^{k_1} V_i, \\ s_1 C_{i_2 j_2}, & \text{if} \quad i_2 \in V_1. \end{cases} \tag{5.13}$$

In most n steps we shall reach the end vertex y and we have a 1-PI graph in the channel $x - y$. If the procedure stops at q-th step, we call any final set of q archs derived in this way as a q-arch system. We can express the 1-PI part of the determinant as

$$\begin{aligned} & \det G_{T,1PI} \\ = & \sum_{q-arch, q \le n} \left(\prod_{r=1}^{q} \int_0^1 ds_r \right) \left(\prod_{r=1}^{q} C(s_1, \ldots, s_{r-1}) \right) \frac{\partial^q}{\prod_{r=1}^{q} \partial C_r} \det G_T(\mathbf{s}), \end{aligned}$$

where

$$\sum_{q-arch, q \le n} = \sum_{1 < k_1 < \cdots < k_q = n} \sum_{i_1 \in V_1, j_1 \in V_{k_1}} \sum_{i_2 \in \bigcup_{i=1}^{k_1} V_i, j_2 \in V_{k_2}} \cdots \sum_{i_q \in \bigcup_{i=1}^{k_q - 1} V_i, j_q \in V_{k_q} = V_n}$$

and

$$\frac{\partial^q}{\prod_{r=1}^q \partial C_r} \det G_T(\mathbf{s}) = \frac{\partial^q \det G_T(s_1,\ldots,s_q)}{\partial C_{i_1 j_1}\cdots\partial C_{i_q j_q}}. \tag{5.14}$$

For the derived propagators we have an expression as

$$\left(\prod_{r=1}^{q} C(s_1,\ldots,s_{r-1})\right) = \prod_{l=1}^{q} C_{i_l j_l}(s_1,\ldots,s_l) \prod_{l=1}^{q} s_l^{q_l}, \tag{5.15}$$

with the integer $q_l \geq 0$. This expansion ensures that the presence of the s_i parameters does not destroy the positivity of the remaining determinant. The Gram's bound is the same, with all these parameters being set to 1.

Now we want to prove that the sum over all possible arch systems are bounded by (constant)^p. Before stating the lemma we introduce some new notations. We call $l(V)$ the number of loop fields hooked to the vertices in V, W_i the set of vertices from where the line l_i may start from, so that

$$W_i = V_1 \cup V_2 \cup \cdots \cup V_{k_{i-1}}, \tag{5.16}$$

and r_i the number of loop fields where l_i may contract without crossing more than one arch, in other words, it is the number of fields in W_i but not in W_{i-1},

$$r_i = l(W_i \backslash W_{i-1}). \tag{5.17}$$

Note that $r_1 = l(V_1)$ and $r_2 = l(V_1 \cap V_2 \cap \cdots \cap V_{k_1} \setminus V_1)$. The following lemma counts the number of 1-PI graphs.

Lemma 13 (Lemma 9 in [DR00b])**.** *The sum over all possible arch systems that connect p points in a such way to obtain a 1-PI block is bounded by*

$$\begin{aligned} &\sum_{n=0}^{p}\sum_{q=1}^{n} \sum_{1<k_1<k_2<k_q=n} \sum_{j_r\in V_r}^{r=1,\ldots,q} \int_0^1 \prod_{r=1}^{q} ds_r \sum_{i_r\in W_r}^{r=1,\ldots,q} a(s_1,\ldots,s_q) \\ \leq\ & 8^p 6^{\sum_{i=1}^p (m_i+\bar{m}_i)} \end{aligned}$$

where $a(s_1,\ldots,s_q) = \prod_{r=1}^q s_r^{q_r}$*, with* $q_r = 0$ *or* 1*.*

Proof. The proof is analogous to the one in [DR00b], but we generalize the number of fields hooked to a vertex by $m_i + \bar{m}_i$. We reproduce the proof here for completeness. We have

$$\sum_{i_r\in W_r, r=1,\ldots,q} a(s_1,\ldots,s_q) \leq \prod_{r=1}^{q} a_r(s_1,\ldots,s_{r-1}), \tag{5.18}$$

where a_r is defined inductively by $a_1 = r_1$ and

$$a_r(s_1, \ldots, s_{r-1}) = r_r + s_{r-1} a_{r-1}(s_1, \ldots, s_{r-2}). \tag{5.19}$$

To see this we remark that we have r_i choices to choose i_i if it does not hook to W_{i-1}. If it does hook to W_{i-1}, we have a_{i-1} choices, but we have a factor s_{i-1}. The s-integrals implies the following inequality,

$$\int_0^1 \prod_{r=1}^{q} ds_r \sum_{i_r \in W_r, r=1,\ldots,q} a(s_1, \ldots, s_q) \leq e^{\sum_{i=1}^{q} r_i}. \tag{5.20}$$

(5.20) can be shown inductively in the following way

$$\begin{aligned}
& \int_0^1 \prod_{r=1}^{q} ds_r \sum_{i_r \in W_r, r=1,\ldots,q} a(s_1, \ldots, s_q) \leq \int_0^1 \prod_{r=1}^{q} ds_r \prod_{i=1}^{q} a_i(s_1, \ldots, s_{i-1}) \\
\leq & \int_0^1 \prod_{r=1}^{q-2} ds_r \prod_{i=1}^{q-1} a_i(s_1, \ldots, s_{i-1}) \int_0^1 ds_{q-1}\, a_q(s_1, \ldots, s_{q-1}) \\
\leq & \int_0^1 \prod_{r=1}^{q-3} ds_r \prod_{i=1}^{q-2} a_i(s_1, \ldots, s_{i-1}) \int_0^1 ds_{q-2}\, e^{a_{q-1}+r_q} \\
\leq & \int_0^1 \prod_{r=1}^{q-3} ds_r \prod_{i=1}^{q-3} a_i(s_1, \ldots, s_{i-1})\, e^{a_{q-2}+r_{q-1}+r_q} \\
\leq & \; e^{\sum_{i=1}^{q} r_i}.
\end{aligned} \tag{5.21}$$

where we have inductively used the inequality

$$\int_0^1 (as+b) ds \leq \int_0^1 e^{as+b} ds \leq \frac{1}{a} e^{a+b}, \quad \text{for } a > 0, b > 0. \tag{5.22}$$

Now, as $r_i = l(W_i \backslash W_{i-1})$,

$$\sum_{i=1}^{q} r_i \leq \sum_{i=1}^{n} l(V_i) < \sum_{i=1}^{p} (m_i + \bar{m}_i), \tag{5.23}$$

since $\sum_{i=1}^{p}(m_i+\bar{m}_i)$ is the total number of fields in the amputated two-point function. Moreover,

$$\sum_{j_r \in V_r, r=1,\ldots,q} 1 = \sum_{i=1}^{n} l(V_{k_i}) < \sum_{i=1}^{p} (m_i + \bar{m}_i) < 2^{\sum_{i=1}^{p}(m_i+\bar{m}_i)}, \tag{5.24}$$

the summation $\sum_{1<k_1<k_2<k_q=n} 1$ corresponds to the number of partitions of $\{1, \ldots, n\}$ into q intervals, hence can be bounded by $2^n \leq 2^p$. Putting all

together it is easy to check that

$$\begin{aligned} & \sum_{n=0}^{p}\sum_{q=1}^{n}\sum_{1<k_1<k_2<k_q=n}\sum_{j_r\in V_r, r=1,\dots,q} e^{\sum_{i=1}^{q} r_i} \leq \frac{p(p+1)}{2}\, 2^p (2e)^{\sum_{i=1}^{p}(m_i+\bar{m}_i)} \\ \leq\ & 8^p 6^{\sum_{i=1}^{p}(m_i+\bar{m}_i)}. \end{aligned} \tag{5.25}$$

This completes the proof. □

5.2 Upper bounds on the self-energy

Everything is now prepared for estimating the bounds. In this section we evaluate the upper bounds on the self-energy and its derivatives by combining the tree with the arch formula for the two-point vertex function. We do not repeat all details but concentrate on what is new with respect to the usual tree expansion. We consider the flow of the self-energy, which has the following expression:

$$\begin{aligned} & \left|\Sigma^{(j-1)} - \Sigma^{(j)}\right| \\ \leq\ & \sum_{p\geq 2}\frac{1}{p!}\sum_{m_1,\dots,m_p}\sum_{\bar{m}_1,\dots,\bar{m}_p}\prod_{q=1}^{p}\left[\int d^{\bar{m}_q}\underline{X}'_q d^{m_q}\underline{X}_q v^{(j)}_{\bar{m}_q,m_q}(\underline{X}'_q,\underline{X}_q)\right] \\ \cdot\ & \sum_{T\in\mathcal{T}_p}\left[\prod_{\{q,q'\}\in T}\left|C_j\left((q,X),(q',X')\right)\right|\right]\sum_{\theta_1,\dots,\theta_p}\sum_{\bar{\theta}_1,\dots,\bar{\theta}_p}\prod_{q=1}^{p}\binom{m_q}{\theta_q}\theta_q!\binom{\bar{m}_q}{\bar{\theta}_q}\bar{\theta}_q! \\ \cdot\ & \sum_{\substack{a_1,\dots,a_p>0\\ \bar{a}_1,\dots,\bar{a}_p>0}}\prod_{q=1}^{p}\binom{m_q-\theta_q}{a_q}\binom{\bar{m}_q-\bar{\theta}_q}{\bar{a}_q}\delta_{\sum_{q=1}^{p}(m_q-\theta_q-a_q+\bar{m}_q-\bar{\theta}_q-\bar{a}_q)=2} \\ \cdot\ & \sum_{n=0}^{p}\sum_{q=1}^{n}\sum_{q-arch, q\leq n}\left(\prod_{r=1}^{q}\int_0^1 ds_r\right)\left(\prod_{r=1}^{q}C(s_1,\dots,s_{r-1})\right) \\ \cdot\ & \frac{\partial^q}{\prod_{r=1}^{q}\partial C_r}\left(\det G^{D}_{T,\theta,a_q}+\det G^{C_{\leq j-1}}_{T,\theta,a_q}+\det G^{C_{\leq j},qq}_{T,\theta,a_q}\right), \end{aligned} \tag{5.26}$$

where the arch expansions have been performed for all two-point functions which appeared in the tree expansion. Applying the Lemma 13, which counts the number of 1-PI graphs, and using the technique used to show the

analytic properties of the effective action, we obtain

$$\begin{aligned}
&\left|\Sigma^{(j-1)} - \Sigma^{(j)}\right|_{1,\infty} \\
\leq\ & \sum_{p\geq 2} \frac{1}{p!}\, \omega_{C_j}^{p-1} \gamma_{C_j}^{2(p-1)} \sum_{m_1,\ldots,m_p} \sum_{\bar{m}_1,\ldots,\bar{m}_p} \prod_{q=1}^{p} |v^{(j)}_{\bar{m}_q,m_q}|_{1,\infty} \\
& \cdot \sum_{T\in\mathcal{T}_p} \sum_{\theta_1,\ldots,\theta_p} \sum_{\bar{\theta}_1,\ldots,\bar{\theta}_p} \prod_{q=1}^{p} \binom{m_q}{\theta_q} \theta_q! \binom{\bar{m}_q}{\bar{\theta}_q} \bar{\theta}_q! \sum_{\substack{a_1,\ldots,a_p>0 \\ \bar{a}_1,\ldots,\bar{a}_p>0}} \prod_{q=1}^{p} \binom{m_q-\theta_q}{a_q} \binom{\bar{m}_q-\bar{\theta}_q}{\bar{a}_q} \\
& \cdot 8^p \prod_{q=1}^{p} (6\cdot 3\gamma_{C_j})^{\bar{a}_q+a_q} \delta_{\sum_{q=1}^{p} \bar{m}_q-\bar{\theta}_q-\bar{a}_q+m_q-\theta_q-a_q=2} \\
\leq\ & \omega_{C_j}^{-1} (18\gamma_{C_j})^{-2} \sum_{p\geq 2} \left(8\, \omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j''}\right)^p
\end{aligned} \tag{5.27}$$

with $h_j'' = (2 + 2\cdot 3\cdot 6)\gamma_{C_j} = 38\gamma_{C_j}$. We can chose suitable α, such that

$$8\, \omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j''} \leq \omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} \tag{5.28}$$

with $h_j = \alpha\gamma_{C_j}$, (5.27) reduces to

$$\left|\Sigma^{(j-1)} - \Sigma^{(j)}\right|_{1,\infty} \leq \omega_{C_j}^{-1} (18\gamma_{C_j})^{-2} \sum_{p\geq 2} \left(\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j}\right)^p . \tag{5.29}$$

According to the Lemma 12 the right hand side of (5.29) reduces to

$$\left|\Sigma^{(j-1)} - \Sigma^{(j)}\right|_{1,\infty} \leq \begin{cases} C_{s>} M^j |g_0|^2 & \text{if} \quad g_0 > 0, \\ C_{s<} M^j |g_j| & \text{if} \quad g_0 < 0. \end{cases} \tag{5.30}$$

with two positive constants $C_{s>}$ and $C_{s<}$. Using (5.30), due to the decay factor M^j, the total self-energy can be bounded by

$$\left|\Sigma^{(J-1)}\right|_{1,\infty} \leq \sum_{i=J-1}^{0} \left|\Sigma^{(i)} - \Sigma^{(i+1)}\right|_{1,\infty} \leq \begin{cases} C'_{s>} |g_0|^2 & \text{if} \quad g_0 > 0, \\ C'_{s<} |g_0| & \text{if} \quad g_0 < 0. \end{cases}$$

with two positive constants $C'_{s>}$ and $C'_{s<}$.

The derivatives correspond to the multiplication by a factor $x_\alpha - y_\alpha$ in position space, for $\alpha = 0, 1, 2$, therefore

$$\left|\partial_{k_0}^{n_0} \partial_{k_1}^{n_1} \partial_{k_2}^{n_2} \Sigma(k)\right| \leq \left| \prod_{\alpha=0}^{2} (x_\alpha - y_\alpha)^{n_\alpha} \Sigma(x,y) \right|_{1,\infty} . \tag{5.31}$$

we denote $n = n_0 + n_1 + n_2$. We perform the calculation as usual, the only difference is the change of the decay bound, where we put the multiplication of the distance. By using

$$\alpha_{C_i}^{\nu} \leq c_\alpha M^{-i(1+n_0+\frac{n_1+n_2}{2})}, \tag{5.32}$$

the derivatives of the self-energy have the bounds

$$|\partial_{k_0}^{n_0}\partial_{k_1}^{n_1}\partial_{k_2}^{n_2}\Sigma(k)| \leq \sum_{i=J-1}^{0} M^{(1-n_0-\frac{n_1+n_2}{2})i} \begin{cases} C'_{s>}|g_0|^2 & \text{if} \quad g_0 > 0, \\ C'_{s<}|g_j| & \text{if} \quad g_0 < 0. \end{cases} \tag{5.33}$$

Theorem 5 follows directly from (5.33).

5.3 Lower bounds on the self-energy

In this section we prove that a certain second derivative of the self-energy at a particular value of the external momentum is not uniformly bounded in the analytic domain of the initial coupling constant.

Let us consider, in Fourier space, the amplitude of the graph represented in Figure 5, with an incoming momentum $k = (k_0, \mathbf{k})$, denoted as $\Sigma_2(k_0, \mathbf{k})$, can be written as

$$\Sigma_2 (k_0, \mathbf{k}) = \int_{-\beta/2}^{\beta/2} d\tau \int_{\Lambda} d\mathbf{x}\; e^{-ik_0\tau - ik\cdot\mathbf{x}} C\left(\tau, \mathbf{x}\right)^2 C(-\tau, -\mathbf{x}). \tag{5.34}$$

More precisely, we shall consider the second frequency derivative of this quantity, which up to a global inessential minus sign is

$$\partial_{k_0}^2 \Sigma_2 (k_0, \mathbf{k}) = \int_{-\beta/2}^{\beta/2} d\tau\; \tau^2 \int_{\Lambda} d\mathbf{x}\; e^{-ik_0\tau - i\mathbf{k}\cdot\mathbf{x}}\, C\left(\tau, \mathbf{x}\right)^2 C(-\tau, -\mathbf{x}). \tag{5.35}$$

We state now our lemma:

Lemma 14. *There exits some positive constant C, such that for a temperature small enough,*

$$|\partial_{k_0}^2 \Sigma_2 (k_0, 0)\,| \geq C\beta \tag{5.36}$$

Proof. Let $f_\beta(e)$ be the Fermi-distribution, The propagator $C(\tau, x)$ is

$$C\left(\tau, \mathbf{x}\right) = \int_{\Lambda^*} d^2\mathbf{k}\; e^{i\mathbf{k}\cdot\mathbf{x}}\, C\left(\tau, e_{\mathbf{k}}\right) \tag{5.37}$$

with

$$C(\tau, e) = -1_{\tau>0} e^{-\tau e} f_\beta(-e) + 1_{\tau\leq 0} e^{-\tau e} f_\beta(e) \tag{5.38}$$

then the quantity we are going to study is explicitly written as

$$\partial_{k_0}^2 \Sigma_2 (k_0, 0) = \int_{-\beta/2}^{\beta/2} d\tau\; \tau^2 e^{-ik_0\tau} F(\tau) \tag{5.39}$$

with

$$\begin{aligned} F(\tau) &= \int_{\Lambda^*} d^2\mathbf{p} \int_{\Lambda^*} d^2\mathbf{q} \left(e^{-\tau e_{\mathbf{p}}} e^{-\tau e_{\mathbf{q}}} e^{\tau e_{\mathbf{p+q}}} f_\beta(-e_{\mathbf{p}}) f_\beta(-e_{\mathbf{q}}) f_\beta(e_{\mathbf{p+q}}) 1_{\tau>0} \right. \\ &\quad \left. - e^{-\tau e_{\mathbf{p}}} e^{-\tau e_{\mathbf{q}}} e^{\tau e_{\mathbf{p+q}}} f_\beta(e_{\mathbf{p}}) f_\beta(e_{\mathbf{q}}) f_\beta(-e_{\mathbf{p+q}}) 1_{\tau\leq 0} \right) \\ &= \int_{\Lambda^*} d^2\mathbf{p} \int_{\Lambda^*} d^2\mathbf{q}\, f_\beta(-e_{\mathbf{p}}) f_\beta(-e_{\mathbf{q}}) f_\beta(-e_{\mathbf{p+q}}) \\ &\quad \cdot \left(e^{-\tau e_{\mathbf{p}}} e^{-\tau e_{\mathbf{q}}} e^{-(\beta-\tau) e_{\mathbf{p+q}}} 1_{\tau>0} - e^{-(\tau+\beta) e_{\mathbf{p}}} e^{-(\tau+\beta) e_{\mathbf{q}}} e^{\tau e_{\mathbf{p+q}}} 1_{\tau\leq 0} \right) \end{aligned}$$

Inserting $e(\mathbf{p}) = \mathbf{p}^2 - \frac{\nu}{\beta}$, we have

$$F(\tau) = \int_{\Lambda^*} d^2\mathbf{p} \int_{\Lambda^*} d^2\mathbf{q} \left(G_+ e^{-2(\beta-\tau)(\mathbf{p}\cdot\mathbf{q})} 1_{\tau>0} - G_- e^{2\tau(\mathbf{p}\cdot\mathbf{q})} 1_{\tau\leq 0} \right), \quad (5.40)$$

where

$$G_+ = e^{(\beta+\tau)\frac{\nu}{\beta}} f_\beta(-e_p) f_\beta(-e_q) f_\beta(-e_{p+q}), \quad (5.41)$$

for $0 < \tau \leq \beta/2$, and

$$G_- = e^{(2\beta+\tau)\frac{\nu}{\beta}} f_\beta(-e_p) f_\beta(-e_q) f_\beta(-e_{p+q}), \quad (5.42)$$

for $-\beta/2 \leq \tau \leq 0$. We observe that both G_+ and G_- have constant upper and lower bounds, which are of order one for $0 < \nu \leq 1$. Thus we have

$$F(\tau) \geq C_L \int_{\Lambda^*} d^2\mathbf{p} \int_{\Lambda^*} d^2\mathbf{q} \left(e^{-2(\beta-\tau)(\mathbf{p}\cdot\mathbf{q})} 1_{\tau>0} - e^{2\tau(\mathbf{p}\cdot\mathbf{q})} 1_{\tau\leq 0} \right) \quad (5.43)$$

where C_L is a suitable constant. Let us now consider the momentum integral. In polar coordinates the $\tau > 0$ part of $F(\tau)$ can be rewritten as

$$\int dr_p \int dr_q\, r_p r_q\, e^{-\beta(r_p^2+r_q^2)} \int_0^{2\pi} d\phi\, e^{-2(\beta-\tau) r_p r_q \cos\phi} 1_{\tau>0}, \quad (5.44)$$

we do the ϕ integral

$$\int_0^{2\pi} d\phi\, e^{-2(\beta-\tau) r_p r_q \cos\phi} = \sum_{n=0} \frac{(-2(\beta-\tau) r_p r_q)^n}{n!} \int_0^{2\pi} d\phi\, \cos^n \phi, \quad (5.45)$$

using the identities

$$\int_0^{2\pi} d\phi\, \cos^{2n} \phi = \frac{2\pi}{2^{2n}} \binom{2n}{n}, \qquad \int_0^{2\pi} d\phi\, \cos^{2n+1} \phi = 0, \quad (5.46)$$

(5.44) becomes

$$\sum_{n=0}^{\infty} \frac{2\pi}{2n!} \binom{2n}{n} \int dr_p \int dr_q\, r_p^{2n+1} r_q^{2n+1}\, e^{-\beta(r_p^2+r_q^2)} (\beta-\tau)^{2n} 1_{\tau>0}. \quad (5.47)$$

To integrate out r_p and r_q, we rename $E_p = r_p^2, E_q = r_q^2$,

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{2n!}\binom{2n}{n}\left(\int_0^{E_0} dE_p\; E_p^n e^{-\beta E_p}\right)^2 (\beta-\tau)^{2n}1_{\tau>0}, \tag{5.48}$$

since

$$\int dx\; x^n e^{ax} = e^{ax}\left(\sum_{k=0}^{n}\frac{(-1)^k k!\binom{n}{k}}{a^{k+1}}x^{n-k}\right), \tag{5.49}$$

the energy integration gives

$$\int_0^{E_0} dE_p\; E_p^n e^{-\beta E_p} = e^{-\beta E_0}\left(\sum_{k=0}^{n}\frac{-k!\binom{n}{k}}{\beta^{k+1}}E_0^{n-k}\right) + \frac{n!}{\beta^{n+1}}, \tag{5.50}$$

then the momentum integral gives

$$\frac{\pi}{2\beta^2}\sum_{n=0}^{\infty}\frac{1}{\beta^{2n}}\left(1-e^{-\beta E_0}\sum_{k=0}^{n}\frac{(\beta E_0)^k}{k!}\right)^2(\beta-\tau)^{2n}1_{\tau>0}. \tag{5.51}$$

Similarly, the $\tau<0$ part of $F(\tau)$ gives

$$\frac{\pi}{2\beta^2}\sum_{n=0}^{\infty}\frac{1}{\beta^{2n}}\left(1-e^{-\beta E_0}\sum_{k=0}^{n}\frac{(\beta E_0)^k}{k!}\right)^2\tau^{2n}1_{\tau>0}. \tag{5.52}$$

By estimating upper and lower bounds on the term in the bracket, we have

$$\left(1-e^{-\beta E_0}\sum_{k=0}^{n}\frac{(\beta E_0)^k}{k!}\right)^2 \approx c^{2n}, \tag{5.53}$$

with $c<1$. Putting all together we obtain

$$F(\tau) \geq \frac{C_L\pi}{2\beta^2}\sum_{n=0}^{\infty}\frac{c^{2n}}{\beta^{2n}}\left((\beta-\tau)^{2n}1_{\tau>0} - \tau^{2n}1_{\tau>0}\right). \tag{5.54}$$

Denoting $x=\frac{\tau}{\beta}$, $k_0=\frac{\pi}{\beta}$, the second derivative of the self energy is given by

$$\left|\partial_{k_0}^2\Sigma_2\left(\pi/\beta,0\right)\right| \;\geq\; \frac{C_L\pi}{2}|H|\beta \tag{5.55}$$

with

$$|H| = \left|\int_{-1/2}^{1/2} dx\; x^2 e^{-i\pi x}\left(\frac{1}{1-c_1^2(1-x)^2}1_{x>0} - \frac{1}{1-c_1^2x^2}1_{x\leq 0}\right)\right| \tag{5.56}$$

a numerical calculation shows that $|H|$ is given by a constant. The prefactor C_L is of order one, so that $\left|\partial_{k_0}^2\Sigma_2\left(k_0,0\right)\right| > C\beta$. □

In general we have the lower bound for the n-th derivatives of the self-energy with respect to the frequency,

$$\left|\partial_{k_0}^{(n)}\Sigma_2(k_0,0)\right| \geq C\beta^{n-1}. \tag{5.57}$$

5.4 The large order behavior

In this last section of the self-energy we consider the problem of the large order behavior of the second frequency derivatives. We denote $\Sigma_{\geq 3}(k)$ as the perturbation series starting with contributions of order three in $|g|$. We study the ratio

$$\left| \frac{\partial^2_{k_0} \Sigma_{\geq 3}(k_0, \mathbf{k})}{\partial^2_{k_0} \Sigma_2(k_0, 0)} \right|. \tag{5.58}$$

The previous analysis shows that

$$|\partial^2_{k_0} \Sigma_{\geq 3}(k)| \leq \begin{cases} C\beta|g|^3, & \text{if} \quad g > 0, \\ C\beta(\log \beta)^3|g|^3, & \text{if} \quad g < 0, \end{cases} \tag{5.59}$$

where C is a constant. Together with the lower bound (5.36) the ratio (5.58) diverges as $(\log \beta)^2$ for $g < 0$. The bound (5.59) is not optimal. A more careful analysis would spare these two additional logarithms and prove that (5.58) can be bounded indeed by a small constant.

Let us now consider the flow of the four-point vertex function, the tree expansion gives

$$|v_{2,2}^{(j-1)} - v_{2,2}^{(j)}|_{1,\infty} \leq C_4' \sum_{p \geq 2} (\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j})^p, \tag{5.60}$$

where $C_4' = c_\omega^{-1}(3c_\gamma)^{-4}$. Now we define inductively a sequence of new variables $\{\lambda_1, \lambda_0, \ldots, \lambda_{J-1}\}$, with $\lambda_1 = |g|$, and

$$\lambda_{j-1} - \lambda_j = \sum_{p \geq 2} (\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j})^p, \tag{5.61}$$

for all $J \leq j \leq 0$. It is obvious that λ_{j-1} is well defined only if $\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} < 1/2$ holds. With this definition, (5.60) and (5.29), the flow of four-point function and the self-energy can be expressed in term of λ_j as

$$|v_{2,2}^{(j-1)} - v_{2,2}^{(j)}|_{1,\infty} \leq C_4'(\lambda_{j-1} - \lambda_j), \tag{5.62}$$

and

$$|\Sigma^{(j-1)} - \Sigma^{(j)}|_{1,\infty} \leq C_\sigma M^j (\lambda_{j-1} - \lambda_j) \tag{5.63}$$

with $C_\sigma = c_\omega^{-1}(18c_\gamma)^{-2}$. The analytic properties of λ_j are proved in the following lemma:

Lemma 15. *There exist positive constants C'_λ and $F > 1$, such that for $\lambda_1 < \frac{C'_\lambda}{\log \beta}$,*

$$\omega_{C_j} \|\mathcal{V}^j\|_{h_j} \leq F\lambda_j. \tag{5.64}$$

holds for all $J \leq j \leq 0$,

Proof. The proof is very similar to the one for the convergence problem. It will be done by induction. The case $j = 0$ is trivial. We assume the lemma holds for $j < 0$, then for scale $j-1$, (5.61) implies that λ_{j-1} is well defined, we decompose $\mathcal{V}^{(j-1)}$ in the following way:

$$\mathcal{V}^{(j-1)} = \mathcal{V}_2^{(j-1)} + \sum_{i=j-1}^{0} \mathcal{P}_{2,2}(\mathcal{V}^{(i)} - \mathcal{V}^{(i+1)}) + \tilde{R}(\mathcal{V}^{(j)}) \tag{5.65}$$

where the rest term is given by

$$\tilde{R}(\mathcal{V}^{(j)}) = \sum_{i=j-1}^{0} \mathcal{P}_{\geq 3,\geq 3}\left(W_{\geq 2}^{(i)}(\mathcal{V}^{(i+1)})\right). \tag{5.66}$$

The second term in (5.65) can be bounded by

$$\omega_{C_{j-1}} h_{j-1}^4 \sum_{i=j-1}^{0} |\mathcal{P}_{2,2}(\mathcal{V}^{(i)} - \mathcal{V}^{(i+1)})|_{1,\infty} \leq C_\lambda \sum_{i=j-1}^{0} (\lambda_i - \lambda_{i+1}) \leq C_\lambda \lambda_{j-1}.$$

with $C_\lambda = (\alpha/3)^4$. The rest term $\tilde{R}$ has a bound $\frac{\tilde{K}}{2M}$ as usual. Under an assumption that $|v_{1,1}^{(j)}|_{1,\infty} \leq C_{2\lambda} M^j \lambda_1$, which could be shown like Lemma 12, we obtain

$$\omega_{C_{j-1}} \|\mathcal{V}^{j-1}\|_{h_{j-1}} \leq C_{2\lambda} C_2 \lambda_1 + C_\lambda \lambda_{j-1} + \frac{\tilde{K}}{M} \frac{1}{2}. \tag{5.67}$$

Since $\lambda_j > \lambda_{j+1}$ for all $J - 1 \leq j \leq 0$, hence it is possible to find a constant $F > 1$, such that $\omega_{C_{j-1}} \|\mathcal{V}^{j-1}\|_{h_{j-1}} \leq F\lambda_{j-1}$ holds. □

By the definition of λ_j (5.61) and the Lemma 15 we have following flow equation of λ_j,

$$\lambda_{j-1} - \lambda_j \leq \sum_{p \geq 2} (F\lambda_j)^p, \tag{5.68}$$

with $F > 1$, which implies

Lemma 16. *If λ_j are well defined, and satisfy (5.68), for all $J \leq j \leq 0$. Then for $\lambda_1 < (F^2(|J|+2))^{-1}$, λ_j can be represented as a formal power series in terms of λ_1 ,*

$$\lambda_j \leq \lambda_1 + \sum_{p\geq 2} a_{p,j}\lambda_1^p. \tag{5.69}$$

with $a_{p,j} \geq 0$ are the expansion coefficients, which have upper bounds given by

$$|a_{p,j}| \leq \left(F^2(|j|+1)\right)^{p-1}. \tag{5.70}$$

Proof. The proof is induction on the scale j. The case $j = 0$ is trivial, since $F > 1$. We assume that (5.70) holds for $j < 0$, inserting

$$\lambda_j \leq \lambda_1 + \sum_{p\geq 2} a_{p,j}\lambda_1^p \tag{5.71}$$

into (5.68) we have

$$\begin{aligned}
\lambda_{j-1} &\leq \lambda_1 + \sum_{p\geq 2}\left[\left(F^2(|j|+1)\right)^{p-1}\lambda_1^p + \left(F\sum_{l\geq 1}\left(F^2(|j|+1)\right)^{l-1}\lambda_1^l\right)^p\right] \\
&\leq \lambda_1 + \sum_{p\geq 2}\Bigg[\left(F^2(|j|+1)\right)^{p-1}\lambda_1^p \\
&\qquad + (F(|j|+1))^{-p}\sum_{k=p}^{\infty}\binom{k-1}{k-p}\left(F^2(|j|+1)\lambda_1\right)^k\Bigg],
\end{aligned} \tag{5.72}$$

where the identity

$$\left(\sum_{l=1}^{\infty} x^l\right)^p = \sum_{k=p}^{\infty}\binom{k-1}{k-p}x^k, \tag{5.73}$$

with $|x| < 1$ has been used. We prove (5.73):

$$\begin{aligned}
&\left(\sum_{l=1}^{\infty} x^l\right)^p = x^p(1-x)^{-p} \\
=\ & x^p\left(1 + px + \cdots + \frac{(-p)(-p-1)\cdots(-p-n+1)}{n!}(-1)^n x^n + \cdots\right) \\
=\ & \sum_{n=0}^{\infty}\frac{p(p+1)\cdots(p+n-1)}{n!}x^{p+n},
\end{aligned} \tag{5.74}$$

setting $k = p + n$,

$$\begin{aligned}\left(\sum_{l=1}^{\infty} x^l\right)^p &= \sum_{k=p}^{\infty} \frac{p(p+1)\cdots(k-1)}{n!} x^k = \sum_{k=p}^{\infty} \frac{(k-1)!}{(k-p)!(p-1)!} x^k \\ &= \sum_{k=p}^{\infty} \binom{k-1}{k-p} x^k. \end{aligned} \tag{5.75}$$

From (5.72) the expansion coefficient for λ_{j-1} is given by

$$\begin{aligned} a_{p,j-1} &= \left(F^2(|j|+1)\right)^{p-1} + \sum_{l=2}^{p} \left(F(|j|+1)\right)^{-l} \binom{p-1}{p-l} \left(F^2(|j|+1)\right)^p \\ &= F^{2(p-1)} \left((|j|+1)^{p-1} + \sum_{l=2}^{p} \binom{p-1}{p-l} F^{2-l} (|j|+1)^{p-l} \right) \end{aligned} \tag{5.76}$$

for $F > 1$, it follows

$$\begin{aligned} a_{p,j-1} &\leq F^{2(p-1)} \left((|j|+1)^{p-1} + \sum_{l=2}^{p} \binom{p-1}{p-l} (|j|+1)^{p-l} \right) \\ &\leq \left(F^2\left(|j|+2\right)\right)^{p-1}. \end{aligned} \tag{5.77}$$

this completes the proof. □

Inserting (5.70) into (5.68), the flow equation for λ_j can be written in terms of λ_1 as

$$\begin{aligned} \lambda_{j-1} - \lambda_j &\leq \sum_{p\geq 2}^{\infty} (F\lambda_j)^p \leq \sum_{p\geq 2}^{\infty} \left(F \sum_{k\geq 1}^{\infty} \left(F^2(|j|+1)\right)^{k-1} \lambda_1^k \right)^p \\ &\leq \sum_{p\geq 2}^{\infty} \left(F|j|\right)^{-p} \sum_{k=p}^{\infty} \binom{k-1}{k-p} \left(F^2(|j|+1)\lambda_1\right)^k \\ &\leq \sum_{p\geq 2}^{\infty} F^{2(p-1)} \sum_{k=2}^{p} \binom{p-1}{p-k} F^{2-k} (|j|+1)^{p-k} \lambda_1^p \\ &\leq \sum_{p\geq 2}^{\infty} F^{2(p-1)} \left((|j|+2)^{p-1} - (|j|+1)^{p-1} \right) \lambda_1^p, \end{aligned} \tag{5.78}$$

since

$$\begin{aligned} & (|j|+2)^{p-1} - (|j|+1)^{p-1} \\ = \; & (|j|+2)^{p-2} \left(1 + \left((|j|+1) - (|j|+1) \left(\frac{|j|+1}{|j|+2} \right)^{p-2} \right) \right) \\ \leq \; & 2(|j|+2)^{p-2}, \end{aligned} \tag{5.79}$$

we arrive at

$$\lambda_{j-1} - \lambda_j \leq 2F^2\lambda_1^2 + 2\sum_{p\geq 3}^{\infty} F^{2(p-1)}(|j|+2)^{p-2}\lambda_1^p. \tag{5.80}$$

Combining (5.63) and (5.80), we get

$$\begin{aligned} \left|\partial_{k_0}^2 \Sigma_{\geq 3}(k)\right| &\leq 2\sum_{i=J-1}^{1} M^{(1-n)i} \sum_{p\geq 3}^{\infty} F^{2(p-1)}(|j|+2)^{p-2}\lambda_1^p \\ &\leq C\beta \log\beta |g|^3 \end{aligned} \tag{5.81}$$

where C is a constant. Finally together with the lower bound estimated on the second order contribution (5.58) becomes

$$\left|\frac{\partial_{k_0}^2 \Sigma_{\geq 3}(k_0, \mathbf{k})}{\partial_{k_0}^2 \Sigma_2(k_0, 0)}\right| \leq C' \log\beta |g|. \tag{5.82}$$

Thus it is possible to choose a constant C, for $|g|\log\beta < C$, (5.58) can be bounded by a small number.

Chapter 6

Response Functions

Observables of a fermionic system are given by polynomials in the fields, which can be calculated from the connected Green's functions. In this chapter we turn to an important class of observables that are the correlations of fermionic bilinears. It is convenient to calculate the correlation decay by using the tree expansion developed in the previous chapter. We derive and discuss them now.

We first make some discussions on the fermionic bilinear. The spatial and spin structure of bilnears is determined by the functions $b(x, Y, Y')$. A charge invariant bilinear is of the form

$$\mathcal{B}(x) = \int dY' \int dY \; \bar{\psi}(Y') \; b(x, Y', Y) \; \psi(Y). \tag{6.1}$$

For instance, $b(x, Y', Y) = \delta(y, y')\delta_{\sigma,\sigma'}$ corresponds to the charge density. Similarly, a charge non-invariant bilinear is of the form

$$\mathcal{B}(x) = \int dY' \int dY \left(\psi(Y')b(x, Y', Y)\psi(Y) + \bar{\psi}(Y')b(x, Y', Y)\bar{\psi}(Y)\right). \tag{6.2}$$

For translation invariant systems, we can consider a momentum representation. The most well-known example involving the charge non-invariant bilinear is the Cooper pairs

$$\mathcal{B}(k) = \int dp \left[\bar{\psi}_\sigma(p)\Delta_{\sigma,\sigma'}(p, k)\bar{\psi}_{\sigma'}(-p + k) + \psi_\sigma(p)\Delta_{\sigma,\sigma'}(p, k)\psi_{\sigma'}(-p + k)\right]$$

for $k = 0$, $\Delta_{\sigma,\sigma'}(p, k) = \Delta(p)\delta_{\sigma,-\sigma'}$ is the gap function, this is the singlet Cooper pairing.

Now let us introduce an external scale field J, which is not the integration variable, so it can be regarded as a functional parameter. We couple $\mathcal{B}(x)$ to this source field J, and define

$$(J, \mathcal{B}) = \int dx \; J(x)\mathcal{B}(x). \tag{6.3}$$

To generate response functions of the bilinears, it is convenient to add the term (6.3) to the action, we define

$$S(J) = \log \int d\mu_C(\psi)\; e^{\mathcal{V}(\psi)+(J,\mathcal{B})}. \tag{6.4}$$

with $\mathcal{V}(\psi)$ given by $-V(\psi)+\mathcal{K}(\psi)$. Using $S(J)$, the response function of $\mathcal{B}$ is defined as

$$R(x,y) = \langle \mathcal{B}(x)\,;\mathcal{B}(y)\rangle = \left[\frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(y)}\; S(J)\right]_{J=0}, \tag{6.5}$$

where the semicolon in $\langle\cdot;\cdot\rangle$ indicates truncated expectation:

$$\langle A\,;B\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle. \tag{6.6}$$

(6.5) can be calculated by using the techniques introduced to show the regularity of the effective action and the self-energy in the previous chapters. As the first step we decompose the propagators and fields as usual,

$$C(k) = C_<(k) + C_>(k) = \sum_{j=J}^{0} C_j(k) + C_>(k) \tag{6.7}$$

and $\psi = \sum_{j=J}^{0}\psi^{(j)} + \psi^{(1)}$. The Grassmann integral implies

$$S(J) = \log \int d\mu_{C_<}(\psi^{(<)}) \int d\mu_{C_>}(\psi^{(1)})\; e^{\mathcal{V}(\psi)+(J,\mathcal{B})}. \tag{6.8}$$

Performing the integration over $\psi^{(1)}$ fields, we get up to an inessential constant

$$S(J) = S^{(0)}(J) + \log \int d\mu_{C_{\le 0}}(\psi^{(\le 0)})\; e^{\mathcal{V}^{(0)}(\psi^{(\le 0)})+\mathcal{B}^{(0)}(\psi^{(\le 0)},J)}, \tag{6.9}$$

where $\mathcal{V}^{(0)}(\psi^{(\le 0)})$ is the effective potential on scale 0, $\mathcal{B}^{(0)}(\psi^{(\le 0)},J)$ collects the terms depending both on $\psi^{(\le 0)}$ and J, $S^{(0)}(J)$ depends only on J generated by the ultraviolet integration. It is not difficult to verify that $\mathcal{V}^{(0)}$ has the form

$$\mathcal{V}^{(0)}(\psi^{(\le 0)}) = \sum_{m,\bar m\ge 1} \int d^{\bar m}\underline{X}' d^{m}\underline{X}\; v^{(0)}_{\bar m,m}(\underline{X}',\underline{X})\bar\psi^{(\le 0)\bar m}(\underline{X}')\psi^{(\le 0)m}(\underline{X}),$$

and the integral kernels satisfy

$$\sup_{j\in \bar m+m} \int \prod_{i\ne j}^{\bar m+m} dX_i\; |v^{(0)}_{\bar m,m}\left(\underline{X}',\underline{X}\right)| \le C|g|^{\,max\{1,m-1\}} \tag{6.10}$$

with some constant $C > 0$. The function $\mathcal{B}^{(0)}(\psi^{(\leq 0)}, J)$ has a very similar bound and representation, i.e.

$$\begin{aligned}\mathcal{B}^{(0)}(\psi^{(\leq 0)}, J) &= \sum_{m,\bar{m}\geq 1} \sum_{n\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X} \int d^n \underline{y}\; b^{(0)}_{\bar{m},m,n,}(\underline{X}', \underline{X}, \underline{y}) \\ &\cdot \bar{\psi}^{(\leq 0)\bar{m}}(\underline{X}')\psi^{(\leq 0)m}(\underline{X}) J^n(\underline{y}), \end{aligned} \tag{6.11}$$

The kernels $b^{(0)}_{\bar{m},m,n}(\underline{X}', \underline{X}, \underline{y})$ are analytic in g ,they admit bounds analogous to (6.10).

Now we proceed to compute the infrared contribution to the response function. The integration over infrared modes is performed in an iterative way as usual. After integrating out the degrees of freedom on scales $0, -1, \ldots, j+1$, $j < 0$,

$$S(J) = S^{(\geq j)}(J) + \log \int d\mu_{C_j}(\psi^{(\leq j)})\; e^{\mathcal{V}^{(j)}(\psi^{(\leq j)}) + \mathcal{B}^{(\leq j)}(\psi^{(\leq j)}, J)}, \tag{6.12}$$

where $S^{(\geq j)}(J) = \sum_{i\geq j} S^{(i)}(J)$, with

$$S^{(i)}(J) = \log \int d\mu_{C_{i+1}}(\psi^{(i+1)})\; e^{\mathcal{V}^{(i+1)}(\psi^{(i+1)}) + \mathcal{B}^{(i+1)}(\psi^{(i+1)}, J)}. \tag{6.13}$$

is the contribution coming from the ith scale. In previous chapters we have shown that the polynomials $\mathcal{V}^{(i)}(\psi^{(i)})$ and $\mathcal{B}^{(i)}(\psi^{(i)}, J)$ are well defined, if the constraints on g, $|g|^2 \log \beta < C$ or $|g| \log \beta < C$ are satisfied. Moreover, they can be written as

$$\mathcal{V}^{(i)}(\psi^{(i)}) = \sum_{m,\bar{m}\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X}\; v^{(i)}_{\bar{m},m}\left(\underline{X}', \underline{X}\right) \bar{\psi}^{(i)\bar{m}}\left(\underline{X}'\right) \psi^{(i)m}\left(\underline{X}\right),$$

and

$$\begin{aligned}\mathcal{B}^{(i)}(\psi^{(i)}, J) &= \sum_{m,\bar{m}\geq 1} \sum_{n\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X} \int d^n \underline{y}\; b^{(j)}_{\bar{m},m,n}\left(\underline{X}', \underline{X}, \underline{y}\right) \\ &\cdot \bar{\psi}^{(j)\bar{m}}\left(\underline{X}'\right) \psi^{(j)m}\left(\underline{X}\right) J^n(\underline{y}). \end{aligned}$$

By (6.12) and (6.13), the response function can be reexpressed by a sum over the derivatives of $S^i(J)$ with respect to J fields

$$R(x,y) = \sum_{i=J-1}^{0} \left[\frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)}\; S^{(i)}(J)\right]_{J=0}. \tag{6.14}$$

Let us now consider the contribution to $R(x,y)$ coming from the ith scale explicitly. Because the derivative is evaluated at $J = 0$, we observe that

only terms with one or two J fields contribute to (6.14), we have

$$\begin{aligned}&\left[\frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(y)}\,S^{(i)}(J)\right]_{J=0}\\ =\;&\left[\frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(y)}\log\int d\mu_{C_{i+1}}(\psi^{(i+1)})\,e^{Q^{(i+1)}}\right]_{J=0},\end{aligned}\tag{6.15}$$

where

$$Q^{(i+1)}=\mathcal{V}^{(i+1)}(\psi^{(i+1)})+\mathcal{B}_1^{(i+1)}(\psi^{(i+1)},J)+\mathcal{B}_2^{(i+1)}(\psi^{(i+1)},J)\tag{6.16}$$

with

$$\begin{aligned}\mathcal{B}_l^{(i+1)}(\psi^{(i+1)},J)\;=\;&\sum_{m,\bar{m}\geq 1}\int d^{\bar{m}}\underline{X}'d^{m}\underline{X}\int\prod_{q=1}^{l}dy_q\;b^{(i+1)}_{\bar{m},m,l}(\underline{X}',\underline{X},\underline{y})\\ &\cdot\bar{\psi}^{(i+1)\bar{m}}\left(\underline{X}'\right)\psi^{(i+1)m}\left(\underline{X}\right)J^{l}(\underline{y}).\end{aligned}$$

$l=1,2$. Before we continue, it is useful to estimate the bound on $\mathcal{B}_l^{(i+1)}(\psi^{(i+1)},J)$. We define the norm on $\mathcal{B}_l^{(i+1)}(\psi^{(i+1)},J)$ as

$$\|\mathcal{B}_l^{(i+1)}(\psi^{(i+1)},J)\|_{h_{i+1}}=\sum_{\bar{m},m\geq 1}|b^{(i+1)}_{\bar{m},m,l}|_{1,\infty}h_{i+1}^{\bar{m}+m}\tag{6.17}$$

with $h_{i+1}=\alpha\gamma_{C_{i+1}}$. $|b^{(i+1)}_{\bar{m},m,l}|_{1,\infty}$ is defined as

$$|b^{(i+1)}_{\bar{m},m,l}|_{1,\infty}=\max_j\sup_{X_j\text{ or }y_j}\int\prod_{q=1,q\neq j}^{\bar{m}+m}dX_q\int\prod_{q=\bar{m}+m,q\neq j}^{\bar{m}+m+l}dy_q\;|b^{(i+1)}_{\bar{m},m,l}(\underline{X}',\underline{X},\underline{y})|.$$

According to the results of previous chapters we have shown that for $|g|\log\beta<C$, $\omega_{C_{i+1}}\|\mathcal{B}^{(i+1)}\|_{h_{i+1}}\leq C|g_{i+1}|$, and $|g_i|\leq c|i||g|$, then we have

$$\|\mathcal{B}_l^{(i+1)}(\psi^{(i+1)},J)\|_{h_{i+1}}\leq cM^{i+1}|g|\leq cM^{i+1}\tag{6.18}$$

with c as a constant.

We turn back to calculate (6.15). We expand the exponential function and select terms with one or two J fields, we have

$$\left[\frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(y)}\,S^{(i)}(J)\right]_{J=0}=R_1^{(i)}(x,y)+R_2^{(i)}(x,y)\tag{6.19}$$

with

$$R_1^{(i)}(x,y)=\sum_{p=0}^{\infty}\frac{1}{2p!}\left[\frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(y)}\log\int d\mu_{C_{i+1}}(\psi^{(i+1)})\prod_{q=1}^{p}\mathcal{V}_q^{(i+1)}\prod_{q=1}^{2}\mathcal{B}_{1,q}^{(i+1)}\right]$$

and

$$R_2^{(i)}(x,y) = \sum_{p=0}^{\infty} \frac{1}{p!} \left[\frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \log \int d\mu_{C_{i+1}}(\psi^{(i+1)}) \prod_{q=1}^{p} \mathcal{V}_q^{(i+1)} \cdot \mathcal{B}_2^{(i+1)} \right].$$

Let us first consider $R_1^{(i)}(x,y)$. In the perturbation theory the Grassmann Gaussian integral reduces to

$$\begin{aligned}
& R_1^{(i)}(x,y) \\
= & \sum_{p\geq 0} \frac{(-1)^{a_p}}{2p!} \sum_{m_1,\ldots,m_p} \sum_{\bar{m}_1,\ldots,\bar{m}_p} \sum_{n_1,n_2} \sum_{\bar{n}_1,\bar{n}_2} \prod_{q=1}^{p} \int d^{\bar{m}_q} \underline{U}'_q d^{m_q} \underline{U}_q \\
\cdot & \prod_{q=1}^{2} \int d^{\bar{n}_q} \underline{V}'_q d^{n_q} \underline{V}_q \int dw_1 \int dw_2 \; \delta(w_1,x)\delta(w_2,y) \\
\cdot & \prod_{q=1}^{p} v^{(i+1)}_{\bar{m}_q,m_q}(\underline{U}'_q,\underline{U}_q) \prod_{q=1}^{2} b^{(i+1)}_{\bar{n}_q,n_q,1}(\underline{V}'_q,\underline{V}_q,w_q) \\
\cdot & \left\langle \bar{\psi}^{(i+1)\bar{m}_1}\left(\underline{U}'_1\right) \psi^{(i+1)m_1}\left(\underline{U}_1\right); \cdots ; \bar{\psi}^{(i+1)\bar{m}_p}\left(\underline{U}'_p\right) \psi^{(i+1)m_p}\left(\underline{U}_p\right); \right. \\
& \left. \bar{\psi}^{(i+1)\bar{n}_1}\left(\underline{V}'_1\right) \psi^{(i+1)n_1}\left(\underline{V}_1\right); \bar{\psi}^{(i+1)\bar{n}_2}\left(\underline{V}'_2\right) \psi^{(i+1)n_2}\left(\underline{V}_2\right) \right\rangle_c \qquad (6.20)
\end{aligned}$$

where $a_p = 0, 1$, $\underline{U}', \underline{U}, \underline{V}', \underline{V} \in \Gamma$ and $w', w \in \Lambda'$. We denote also $Z = U \cup V$ and $Z' = U' \cup V'$. The fermionic expectation $\langle \cdot \rangle_c$ is defined as

$$\begin{aligned}
& \left\langle \bar{\psi}^{(i+1)\bar{m}_1}\left(\underline{U}'_1\right) \psi^{(i+1)m_1}\left(\underline{U}_1\right); \cdots ; \bar{\psi}^{(i+1)\bar{m}_p}\left(\underline{U}'_p\right) \psi^{(i+1)m_p}\left(\underline{U}_p\right); \right. \\
& \left. \bar{\psi}^{(i+1)\bar{n}_1}\left(\underline{V}'_1\right) \psi^{(i+1)n_1}\left(\underline{V}_1\right); \bar{\psi}^{(i+1)\bar{n}_2}\left(\underline{V}'_2\right) \psi^{(i+1)n_2}\left(\underline{V}_2\right) \right\rangle_c \\
= & \int d\mu_{C_{i+1}}(\psi^{(i+1)}) \prod_{q=1}^{p} \bar{\psi}^{(i+1)\bar{m}_q}\left(\underline{U}'_q\right) \psi^{(i+1)m_q}\left(\underline{U}_q\right) \prod_{q=1}^{2} \bar{\psi}^{(i+1)\bar{n}_q}\left(\underline{V}'_q\right) \psi^{(i+1)n_q}\left(\underline{V}_q\right)
\end{aligned}$$

which can be evaluated by using the tree formula,

$$\begin{aligned}
& \sum_{T\in\mathcal{T}_{p+2}} \left[\prod_{\{q,q'\}\in T} \left| C_{i+1}(Z_q, Z'_{q'}) \right| \right] \sum_{\theta_1,\ldots,\theta_p} \sum_{\bar{\theta}_1,\ldots,\bar{\theta}_p} \left[\prod_{q=1}^{p} \binom{m_q}{\theta_q} \theta_q! \binom{\bar{m}_q}{\bar{\theta}_q} \bar{\theta}_q! \right] \\
\cdot & \sum_{\theta'_1,\theta'_2} \sum_{\bar{\theta}'_1,\bar{\theta}'_2} \left[\prod_{q=1}^{p} \binom{n_q}{\theta'_q} \theta'_q! \binom{\bar{n}_q}{\bar{\theta}'_q} \bar{\theta}'_q! \right] \cdot \det G^{C_{i+1}}_{T,\theta,H(T,\theta,\mathbf{s})}. \qquad (6.21)
\end{aligned}$$

The trees consist of $p+2$ vertices. $G^{C_{i+1}}_{T,\theta,H(T,\theta,\mathbf{s})}$ is a $\sum_{q=1}^{p}(\bar{m}_q + m_q) + \sum_{q=1}^{2}(\bar{n}_q + n_q)$ by $\sum_{q=1}^{p}(\bar{m}_q + m_q) + \sum_{q=1}^{2}(\bar{n}_q + n_q)$ matrix, whose entries are C_{i+1}. Applying the Gram estimate for the determinant and summing

over the trees and θs as in previous chapters the fermionic expectation is bounded by

$$\leq \; p!\; \gamma_{C_{i+!}}^{-2(p+1)} \prod_{\{q,q'\}\in T} \left|C_{i+1}(Z_q, Z'_{q'})\right| \left(3\gamma_{C_{i+1}}\right)^{\sum_{q=1}^{p} \bar{m}_q + m_q + \sum_{q=1}^{2} \bar{n}_q + n_q}$$

In order to take into account the decay between x and y, we can extract a decay factor

$$\rho_{i+1}(x,y) = \frac{C_N}{1 + (M^{i+1}|\tau_x - \tau_y| + M^{\frac{i+1}{2}}|x_1 - y_1| + M^{\frac{i+1}{2}}|x_2 - y_2|)^N} \quad (6.22)$$

with C_N as a constant and $N > 1$, from the product of the propagators in the spanning tree. Summing over $\bar{m}, m$ and $\bar{n}, n$, we obtain finally

$$|R_1^{(i)}(x,y)| \leq \frac{1}{2}\, |\rho_{i+1}(x,y)|\, \|\mathcal{B}_1^{(i+1)}\|_{h_{i+1}}^2 \sum_{p\geq 0} (\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}})^p. \quad (6.23)$$

By (6.18), and $\omega_{C_{i+1}} \|\mathcal{V}^{(i+1)}\|_{h_{i+1}} \leq 1/2$, (6.23) reduces to

$$|R_1^{(i)}(x,y)| \leq |\rho_{i+1}(x,y)| \|\mathcal{B}_1^{(i+1)}\|_{h_{i+1}}^2 \leq c_1 |\rho_{i+1}(x,y)| M^{2(i+1)}, \quad (6.24)$$

with c_1 as a constant.

Now we turn our attention to $R_2^{(i)}(x,y)$. By an expansion in the effective potential $\mathcal{V}^{i+1}$, we write

$$\begin{aligned}
& R_2^{(i)}(x,y) \\
= \; & \sum_{p\geq 0} \frac{(-1)^{a_p}}{p!} \sum_{m_1,\ldots,m_p} \sum_{\bar{m}_1,\ldots,\bar{m}_p} \sum_{n,\bar{n}} \prod_{q=1}^{p} \int d^{\bar{m}_q} \underline{U}'_q d^{m_q} \underline{U}_q \prod_{q=1}^{p} v_{\bar{m}_q, m_q}^{(i+1)} (\underline{U}'_q, \underline{U}_q) \\
\cdot \; & \int d^{\bar{n}} \underline{V}' d^n \underline{V} \int dw_1 \int dw_2\; \delta(w_1, x)\delta(w_2, y)\; b_{\bar{n},n,2}^{(i+1)} (\underline{V}'_q, \underline{V}_q, w_1, w_2) \\
\cdot \; & \left\langle \bar{\psi}^{(i+1)\bar{m}_1} \left(\underline{U}'_1\right) \psi^{(i+1)m_1} \left(\underline{U}_1\right); \cdots; \bar{\psi}^{(i+1)\bar{m}_p} \left(\underline{U}'_p\right) \psi^{(i+1)m_p} \left(\underline{U}_p\right); \right. \\
& \left. \bar{\psi}^{(i+1)\bar{n}} \left(\underline{V}'\right) \psi^{(i+1)n} \left(\underline{V}\right) \right\rangle_c
\end{aligned}$$

The fermionic expectation is bounded by

$$\leq (p-1)!\, \gamma_{C_{i+1}}^{-2p} \prod_{\{q,q'\}\in T} \left|C_{i+1}(Z_q, Z'_{q'})\right| \left(3\gamma_{C_{i+1}}\right)^{\sum_{q=1}^{p} \bar{m}_q + m_q + \bar{n} + n} \quad (6.25)$$

$|R_2^{(i)}(x,y)|$ can be bounded in a way analogous to (6.23), but with the difference that we get a gain factor $c_V M^{2(i+1)}$, with c_V as a constant. This is because the two J fields hook to the same vertex, since two derivatives cost two volume integral, but the norm excludes only one volume integrals, we

has one more factor $c_V M^{2(i+1)}$ from the remaining volume integrals. Due to the same reason, the decay factor is extracted from the determinant, not from the spanning tree. We have

$$\begin{aligned} |R_2^{(i)}(x,y)| &\leq c_V\ |\rho_{i+1}(x,y)|\ M^{2(i+1)}\ \|\mathcal{B}_2^{(i+1)}\|_{h_{i+1}}\ \sum_{p\geq 0}(\omega_{C_{i+1}}\|\mathcal{V}^{(i+1)}\|_{h_{i+1}})^p \\ &\leq 2c_V\ |\rho_{i+1}(x,y)|\ M^{2(i+1)}\ \|\mathcal{B}_2^{(i+1)}\|_{h_{i+1}}. \end{aligned} \tag{6.26}$$

By (6.18), $\|\mathcal{B}_2^{(i+1)}\|_{h_{i+1}} \leq cM^{i+1}|g|$, but this bound is not good enough. A better bound should be found. We consider the integral

$$\mathcal{B}_2^{(i)}(\psi^{(\leq i)}, J) = \mathcal{P}_{2,J}\left(\int d\mu_{C_{i+1}}(\psi^{(i+1)}) e^{\mathcal{V}^{(i+1)}(\psi^{(i+1)})+\mathcal{B}^{(i+1)}(\psi^{(i+1)},J)}\right)$$

where $\mathcal{P}_{2,J}$ is a project operator, which selects terms with two J fields. Expanding in the effective potential we have

$$\mathcal{B}_2^{(i)}(\psi^{(\leq i)}, J) = \mathcal{B}_{2,1}^{(i)}(\psi^{(\leq i)}, J) + \mathcal{B}_{2,2}^{(i)}(\psi^{(\leq i)}, J) \tag{6.27}$$

with

$$\mathcal{B}_{2,1}^{(i)}(\psi^{(\leq i)}, J) = \sum_{p=0}^{\infty}\frac{1}{2p!}\log\int d\mu_{C_{i+1}}(\psi^{(\leq i+1)})\prod_{q=1}^{p}\mathcal{V}_q^{(i+1)}\ \prod_{q=1}^{2}\mathcal{B}_{1,q}^{(i+1)} \tag{6.28}$$

and

$$\mathcal{B}_{2,2}^{(i)}(\psi^{(\leq i)}, J) = \sum_{p=0}^{\infty}\frac{1}{p!}\log\int d\mu_{C_{i+1}}(\psi^{(\leq i+1)})\prod_{q=1}^{p}\mathcal{V}_q^{(i+1)}\cdot\mathcal{B}_2^{(i+1)}. \tag{6.29}$$

We see that these can be bounded as

$$\|\mathcal{B}_{2,1}^{(i)}(\psi^{(\leq i)}, J)\|_{h_i} \leq \omega_{C_{i+1}}\|\mathcal{B}_1^{(i+1)}\|_{h'_{i+1}}^2\sum_{p\geq 0}(\omega_{C_{i+1}}\|\mathcal{V}^{(i+1)}\|_{h'_{i+1}})^p \tag{6.30}$$

and

$$\|\mathcal{B}_{2,2}^{(i)}(\psi^{(\leq i)}, J)\|_{h_i} \leq \|\mathcal{B}_2^{(i+1)}\|_{h'_{i+1}}\sum_{p\geq 0}(\omega_{C_{i+1}}\|\mathcal{V}^{(i+1)}\|_{h'_{i+1}})^p. \tag{6.31}$$

with $h_i = \alpha\gamma_{C_i}$, $h'_{i+1} = \left(5+\frac{\alpha}{\sqrt{M}}\right)\gamma_{C_{i+1}}$. Since we have shown that

$$\|\mathcal{B}_2^{(i+1)}\|_{h'_{i+1}} \leq \tilde{A}^2\|\mathcal{B}_2^{(i+1)}\|_{h_{i+1}}, \tag{6.32}$$

recall that $\tilde{A} = \left(\frac{1}{\sqrt{M}}+\frac{5}{\alpha}\right)^2$ and $\tilde{A}^2 < 1$. By (6.18), (6.27), (6.31) and $\omega_{C_{i+1}}\|\mathcal{V}^{(i+1)}\|_{h_{i+1}} < \frac{1}{2}$, we have

$$\|\mathcal{B}_2^{(i)}\|_{h_i} \leq \tilde{A}^4\omega_{C_{i+1}}\|\mathcal{B}_1^{(i+1)}\|_{h_{i+1}}^2 + \tilde{A}^2\|\mathcal{B}_2^{(i+1)}\|_{h_{i+1}}. \tag{6.33}$$

Iterating this procedure till the initial scale, we get

$$\begin{aligned} \|\mathcal{B}_2^{(i)}\|_{h_i} &\leq \sum_{l=i+1} (\tilde{A}^2)^{l-i+1} \omega_{C_l} \|\mathcal{B}_1^{(l)}\|_{h_l}^2 \\ &\leq c|g|^2 \end{aligned} \tag{6.34}$$

with c as a constant. With this, we have

$$|R_2^{(i)}(x,y)| \leq c_2 |\rho_{i+1}(x,y)| M^{2(i+1)}, \tag{6.35}$$

where c_2 is a constant. Putting (6.18) and (6.35) together we have

$$|R^{(i)}(x,y)| \leq c_3 |\rho_{i+1}(x,y)| M^{2(i+1)} \tag{6.36}$$

with $c_3 = c_1 + c_2$ is a suitable constant. Now we can bound the response function,

$$\begin{aligned} R(x,y) &= \sum_{i=J-1}^{0} R^{(i)}(x,y) \leq c_3 \sum_{i=J-1}^{0} |\rho_{i+1}(x,y)| M^{2(i+1)} \\ &\leq \frac{c_3}{|x-y|^2} \sum_{i=J} \frac{(M^i|x-y|)^2}{1 + (M^i|\tau_x - \tau_y| + M^{\frac{i}{2}}|x_1 - y_1| + M^{\frac{i}{2}}|x_2 - y_2|)^N} \\ &\leq \frac{c_3}{|x-y|^2} \frac{1}{1 + (M^J|x-y|)^{N-2}} \sum_{i=J}^{0} M^{(J-i)(N-2)}, \end{aligned} \tag{6.37}$$

for $N > 2$, the summation over i converges. We obtain finally

$$|R(x,y)| \leq \frac{C}{|x-y|^2} \tag{6.38}$$

with C as a suitable constant.

Chapter 7

Bilayer Graphene

In this chapter we consider the Hubbard model on the Bernal stacked honeycomb bilayer at half filling and weak coupling. In the simplest form with the nearest neighbor hoping only, the tight-binding approximation gives rise to a band structure with two bands touching quadratically at the Fermi level near two non-equivalent points in the Brillouin zone. This can be regarded as the limit case of the dilute Fermi gas model with a zero radius Fermi circle, except that they are two branches. Therefore similar regularity properties of the effective action like the one of dilute Fermi gas are expected. When the next nearest hoping is take into account, the low energy spectrum becomes anisotropic. The Fermi surface breaks into four Dirac points, close to which, the dispersion relation vanishes linearly. Using the same method in manifesting the convergence series we prove that for a small enough initial coupling, the convergence radius is temperature independent, which implies that the noninteracting system is stable toward infinitesimal coupling, and there are no truly weak coupling instabilities when this trigonal warping is presented. In the last part of this chapter, we will discuss possible physical instabilities and some presented theoretical and experimental results will be introduced.

7.1 Functional integral representation

As for dilute Fermi gas, let us now rewrite the bilayer graphene model in a Grassmann integral representation. We introduce a time spacing $\epsilon_\tau > 0$, the inverse temperature β, such that $n_\tau = \frac{\beta}{2\epsilon_\tau}$ is a large number. Let n_τ be even, and $\mathcal{T} = \{\tau = n\epsilon_\tau : n \in \mathbb{Z}, -\frac{n_\tau}{2} \leq n \leq \frac{n_\tau}{2}\}$, we denote $\Lambda' = \mathcal{T} \times \Lambda$, $X = (\tau, \mathbf{x}, \sigma, \varepsilon, \rho) = (x, \sigma, \varepsilon, \rho) \in \Gamma' = \Lambda' \times \{-1, 1\} \times \{a, b\} \times \{u, l\}$, where $\varepsilon = a, b$ labels the sublattices, $\rho = u, l$ labels the top and bottom layer,

respectively. Let $\mathcal{A}$ be the Grassmann algebra generated by $\psi(X)$ and $\bar{\psi}(X)$, we denote also $\psi(X) = \psi_{\sigma,\varepsilon,\rho}(x)$. The elements of $\mathcal{A}$ are polynomials in Grassmann even subalgebra

$$\mathcal{V}(\psi) = \sum_{m,\bar{m}\geq 0} \int_{\Gamma'} d^{\bar{m}}\underline{X}' d^m \underline{X}\; v_{\bar{m},m}(\underline{X}', \underline{X}) \bar{\psi}^{\bar{m}}(\underline{X}) \psi^m(\underline{X}), \tag{7.1}$$

where we abbreviated

$$\int_{\Gamma'} dX F(X) = \sum_{\sigma=\pm 1} \sum_{\varepsilon=a,b} \sum_{\rho=u,l} \epsilon_\tau \sum_{\tau\in\mathcal{T}} \int_\Lambda d^2\mathbf{x}\; F(\tau, \mathbf{x}, \sigma, \varepsilon, \rho). \tag{7.2}$$

The Fourier transformation of $\psi(X)$ and $\bar{\psi}(X)$ are

$$\psi_{\sigma,\varepsilon,\rho}(k) = \int_{\Lambda'} dx\; e^{-ikx} \psi_{\sigma,\varepsilon,\rho}(x), \quad \bar{\psi}_{\sigma,\varepsilon,\rho}(k) = \int_{\Lambda'} dx\; e^{-ikx} \bar{\psi}_{\sigma,\varepsilon,\rho}(x) \tag{7.3}$$

for $k = (k_0, \mathbf{k})$, $kx = k_0\tau + \mathbf{k}\cdot\mathbf{x}$. The momentum k is in $\mathcal{B}'_L = \mathcal{M}_{k_0,n_\tau} \times \mathcal{B}_L$, with $\mathcal{B}_L$ being the first Brillouin zone. The inverse transformation is $\psi_{\sigma,\varepsilon,\rho}(x) = \int_{\mathcal{B}'_L} dk\; e^{ikx} \psi_{\sigma,\varepsilon,\rho}(x)$, where we abbreviated

$$\int_{\mathcal{B}'_L} dk\; F(k) = \frac{1}{\beta} \sum_{k_0\in\mathcal{M}_{k_0,n_\tau}} \int_{\mathbf{k}\in\mathcal{B}_L} d^2\mathbf{k}\; F(k_0, \mathbf{k}). \tag{7.4}$$

To simplify our analysis we consider the model in the limit $n_\tau \to \infty$, where the time variable becomes continuous, and we treat infinite dimensional Grassmann integrals. The thermodynamical limits $L \to \infty$ will be taken at the end of the analysis.

We are interested in the generator of the connected, amputated Green's functions,

$$W(\mathcal{V})(\phi) = \log \int d\mu_C(\psi)\; e^{-\mathcal{V}(\psi+\phi)}, \tag{7.5}$$

where ϕ is another set of Grassmann fields. The Grassmann Gaussian measure $d\mu_C(\psi)$ is defined as

$$d\mu_C(\psi) = \mathcal{N} \prod_{\sigma,\varepsilon,\rho,k\in\Lambda'} d\bar{\psi}_{\sigma,\varepsilon,\rho}(k) d\psi_{\sigma,\varepsilon,\rho}(k) e^{-\sum_\sigma \int_{\mathcal{B}'_L} dk\; \bar{\Psi}_\sigma(k) C^{-1}(k) \Psi_\sigma(k)}$$

with $\mathcal{N}$ as a normalization constant, such that $\int d\mu_C(\psi) = 1$. $\bar{\Psi}_\sigma(k) = \big(\bar{\psi}_{\sigma,a,l}(k), \bar{\psi}_{\sigma,b,u}(k), \bar{\psi}_{\sigma,a,u}(k), \bar{\psi}_{\sigma,b,l}(k)\big)$. The matrix $C(k)^{-1} = (ik_0 - H_0(\mathbf{k}))$ is invertible, thus the propagator $C(k) = (ik_0 - H_0(\mathbf{k}))^{-1}$ exists. Moreover, $d\mu_C(\psi)$ is characterized by its covariance

$$\int d\mu_C(\psi)\; \bar{\psi}_{\sigma',\varepsilon',\rho'}(k')\; \psi_{\sigma,\varepsilon,\rho}(k) = \delta_{\sigma'\sigma} \left[(ik_0 - H_0(\mathbf{k}))^{-1}\right]_{\varepsilon'\rho',\varepsilon\rho}. \tag{7.6}$$

Due to the particle-hole symmetry at half filling the Fermi surface stays fixed as interactions being turned on, so that renormalization of the Fermi surface becomes unnecessary. In other words, the counterterm vanishes. Hence the initial effective potential $\mathcal{V}(\psi)$ is easily given by the on-site interaction

$$\mathcal{V}(\psi) = g \sum_{\epsilon=a,b} \sum_{\rho=u,l} \int_{\mathbf{x}\in\Lambda} d\mathbf{x}\ \bar{\psi}_{\uparrow,\varepsilon,\rho}(\mathbf{x})\bar{\psi}_{\downarrow,\varepsilon,\rho}(\mathbf{x})\psi_{\uparrow,\varepsilon,\rho}(\mathbf{x})\psi_{\downarrow,\varepsilon,\rho}(\mathbf{x}). \tag{7.7}$$

with g being the coupling constant, which can be either positive or negative.

By (2.2), we have two high energy bands with $|E(\mathbf{k})| > 0$ for all $\mathbf{k} \in \mathcal{B}_L$ and two degenerate low energy bands touching at $E(\mathbf{k}) = 0$. In the weak coupling limit, since only low energy modes are important to determine the behavior of the system at low temperature, a formulation of an effective low energy model that contains only the relevant bands near the Fermi level is required. For this purpose we need to project out the bands which originate from the two high energy bands associated with the fermionic fields $\psi_{\sigma,a,u}$ and $\psi_{\sigma,b,l}$.

We start from the Hamiltonian $H_0(\mathbf{k})$, four 2×2 blocks can be identified:

$$H_{11} = -\begin{pmatrix} 0 & \gamma_3 A(\mathbf{k}) \\ \gamma_3 A^*(\mathbf{k}) & 0 \end{pmatrix}, \qquad H_{22} = -\begin{pmatrix} 0 & \gamma_1 \\ \gamma_1 & 0 \end{pmatrix} \tag{7.8}$$

and

$$H_{12} = H_{21} = -\begin{pmatrix} 0 & \gamma_0 A^*(\mathbf{k}) \\ \gamma_3 A^*(\mathbf{k}) & 0 \end{pmatrix}. \tag{7.9}$$

To simplify our notations we define a bilinear form

$$(\bar{\psi}, F\psi) = \sum_{\sigma} \int_{\mathcal{B}'_L} dk\ \bar{\psi}(k)F(k)\psi(k). \tag{7.10}$$

Moreover, we denote $C_H = (ik_0 - H_{22})^{-1}$, which is well defined and without singular points. Let $\Psi_{H,\sigma} = (\psi_{\sigma,a,u}(k), \psi_{\sigma,b,l}(k))$, $\Psi_{L,\sigma} = (\psi_{\sigma,a,l}(k), \psi_{\sigma,b,u}(k))$, $D\bar{\Psi}_H D\Psi_H = \prod_{\sigma,k} d\bar{\psi}_{\sigma,a,u}(k)d\psi_{\sigma,a,u}(k)d\bar{\psi}_{\sigma,b,l}(k)d\psi_{\sigma,b,l}(k)$, and

$$\begin{aligned} \mathcal{V}_H(\Psi_H) \;=\; & -g\Big[\int_{\mathbf{x}\in\Lambda} d\mathbf{x}\ \bar{\psi}_{\uparrow,a,u}(\mathbf{x})\bar{\psi}_{\downarrow,a,u}(\mathbf{x})\psi_{\uparrow,a,u}(\mathbf{x})\psi_{\downarrow,a,u}(\mathbf{x}) \\ & + \int_{\mathbf{x}\in\Lambda} d\mathbf{x}\ \bar{\psi}_{\uparrow,b,l}(\mathbf{x})\bar{\psi}_{\downarrow,b,l}(\mathbf{x})\psi_{\uparrow,b,l}(\mathbf{x})\psi_{\downarrow,b,l}(\mathbf{x})\Big] \end{aligned} \tag{7.11}$$

is the interaction relates to the fields $\psi_{a,u}$ and $\psi_{b,l}$. The integration over

high energy modes gives

$$\begin{aligned} & \int D\bar{\Psi}_H D\Psi_H \; e^{-(\bar{\Psi}_H, C_H^{-1}\Psi_H)+(\bar{\Psi}_L H_{12}, \Psi_H\rangle+(\bar{\Psi}_H, H_{21}\Psi_L)+\mathcal{V}_H(\Psi_H+\phi)} \\ = \;& e^{(\bar{\Psi}_L, H_{12}C_H H_{21}\Psi_L)} \\ & \cdot \int D\bar{\Psi}_H D\Psi_H \; e^{-(\bar{\Psi}_H - C_H^T H_{12}\bar{\Psi}_L, C_H^{-1}(\Psi_H - C_H H_{21}\Psi_L))+\mathcal{V}_H(\Psi_H+\phi)}. \end{aligned} \tag{7.12}$$

Using the identity

$$\int \prod_{k=1}^{L} d\bar{\psi}_k \psi_k \; f(\bar{\psi} + A\bar{\eta}, \psi + B\eta) = \int \prod_{k=1}^{L} d\bar{\psi}_k \psi_k \; f(\bar{\psi}, \psi), \tag{7.13}$$

the integral over Ψ_H fields in (7.12) gives

$$e^{\mathcal{V}_h(C_H H_{21}\Psi_L+\phi)} = \int D\bar{\Psi}_H D\Psi_H \; e^{-(\bar{\Psi_H}, C_H^{-1}\Psi_H)+\mathcal{V}_H(\Psi_H + C_H H_{21}\Psi_L+\phi)}. \tag{7.14}$$

The integral above is up to the normalization the exponential of the effective action coming from the high energy levels for the theory. It can be seen as the correction to the initial interaction for the low energy fields. Using the method introduced in the previous chapters, and due to the well defined propagator C_H, it is easy to check that $\mathcal{V}_h(C_H H_{21}\Psi_L + \phi)$ is a well defined object, and has the form

$$\mathcal{V}_h(\psi) = \sum_{m,\bar{m}\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X} \; v_{h,\bar{m},m}\left(\underline{X}', \underline{X}\right) \bar{\psi}^{\bar{m}}\left(\underline{X}'\right) \psi^m\left(\underline{X}\right), \tag{7.15}$$

and it satisfies that for a small enough $|g|$, $\|\mathcal{V}_h\|_h \leq 2|g|$, with $h > 0$.

On the next step we move the the exponential prefactor $e^{(\bar{\Psi}_L, H_{12}C_H H_{21}\Psi_L)}$ in (7.12) into the new propagator for the low energy fields

$$C(k) = (ik_0 - H_{11} - H_{12}C_H H_{21})^{-1}, \tag{7.16}$$

where

$$\begin{aligned} H_{12}C_H H_{21} &= \begin{pmatrix} 0 & \gamma_0 A^*(\mathbf{k}) \\ \gamma_0 A(\mathbf{k}) & 0 \end{pmatrix} \begin{pmatrix} ik_0 & \gamma_1 \\ \gamma_1 & ik_0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \gamma_0 A^*(\mathbf{k}) \\ \gamma_0 A(\mathbf{k}) & 0 \end{pmatrix} \\ &= -\frac{1}{k_0^2 + \gamma_1^2} \begin{pmatrix} ik_0\gamma_0^2 |A(\mathbf{k})|^2 & -\gamma_1\gamma_0^2 A^{*2}(\mathbf{k}) \\ -\gamma_1\gamma_0^2 A^2(\mathbf{k}) & ik_0\gamma_0^2 |A(\mathbf{k})|^2 \end{pmatrix}. \end{aligned} \tag{7.17}$$

Since we want to get an effective low energy model, and we are interested in the modes near the Fermi surface, so we can set $k_0 = 0$ in (7.17). The

term arising from the corrections are irrelevant in a perturbation sense. An elementary calculation shows that

$$C(k) = \begin{pmatrix} C_s(k) & -C_t(k)e^{-i2\varphi_A(\mathbf{k})} \\ -C_t(k)e^{i2\varphi_A(\mathbf{k})} & C_s(k) \end{pmatrix}, \tag{7.18}$$

with

$$C_{s,t}(k) = \frac{1}{2}\left(\frac{1}{ik_0 - e(\mathbf{k})} \pm \frac{1}{ik_0 + e(\mathbf{k})}\right), \tag{7.19}$$

where $e(\mathbf{k}) = -\gamma_3|A(\mathbf{k})| + \frac{\gamma_0^2}{\gamma_1}|A(\mathbf{k})|^2$, and $\varphi_A(\mathbf{k})$ is the phase factor of $A(\mathbf{k})$.

Before we continue, we note that the lattice model has an intrinsic ultraviolet cutoff in the $\mathbf{k}$ variables, which scales similarly to a lattice spacing. Now let us redefine the linear functional $d\mu_C(\psi)$ and the effective action $W(\mathcal{V})$ after integrating out the high energy modes. To simplify our notation we rename $\psi_{\sigma,1}(k) = \psi_{\sigma,a,l}(k)$ and $\psi_{\sigma,2}(k) = \psi_{\sigma,b,u}(k)$. We define the linear functional

$$d\mu_C(\psi) = \mathcal{N}_L \prod_{\sigma,\rho,k\in\Lambda'} d\bar{\psi}_{\sigma,\rho}(k) d\psi_{\sigma,\rho}(k)\; e^{-\sum_\sigma \sum_{\rho,\rho'} \int_{\mathcal{B}'_L} dk\; \bar{\psi}_{\sigma,\rho}(k) C^{-1}_{k_0 c}(k)_{\rho\rho'} \psi_{\sigma,\rho'}(k)}, \tag{7.20}$$

where $\mathcal{N}_L$ is a normalization constant, $\int d\mu_C(\psi) = 1$. The reduced effective action for the low energy modes is

$$W(\mathcal{V})(\phi) = \log \frac{1}{Z} \int d\mu_C(\psi)\; e^{\mathcal{V}(\psi+\phi)} \tag{7.21}$$

where Z is a constant such that $W(\mathcal{V})(0) = 0$, with the initial effective potential $\mathcal{V}(\psi)$

$$\mathcal{V}(\psi) = -g \sum_{\rho=1,2} \int_{\mathbf{x}\in\Lambda} d\mathbf{x}\; \bar{\psi}_{\uparrow,\rho}(\mathbf{x}) \bar{\psi}_{\downarrow,\rho}(\mathbf{x}) \psi_{\uparrow,\rho}(\mathbf{x}) \psi_{\downarrow,\rho}(\mathbf{x}) + \mathcal{V}_h(\psi). \tag{7.22}$$

The regularity of (7.21) will be studied in perturbation theory for two cases, depending on whether the trigonal warping vanishes or not.

If the trigonal warping vanishes, $\gamma_3 = 0$, $e(\mathbf{k}) = \frac{\gamma_0^2}{\gamma_1}|A(\mathbf{k})|^2$. By (7.18) the propagator $C_{k_0 c}(k)$ becomes singular at $k_0 = 0$ and $\mathbf{k} = \mathbf{k}_F^{\pm}$, with

$$\mathbf{k}_F^{\pm} = \left(\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}}\right). \tag{7.23}$$

These two points are called Fermi points, and denoted as $K^{(+)}$ and $K^{(-)}$, respectively. approaching to these Fermi points, the dispersion relation vanishes quadratically,

$$e(\mathbf{k}' + \mathbf{k}_F^{\pm}) = c_1|\mathbf{k}'|^2 + R(\mathbf{k}') \tag{7.24}$$

with $c_1 = \frac{9\gamma_0^2}{4\gamma_1}$, and $\mathbf{k}' = \mathbf{k} - \mathbf{k}_F^{\pm}$. Since the second derivatives of $e(\mathbf{k})$ with respect to $\mathbf{k}$ is bounded for all $\mathbf{k} \in \mathcal{B}_L$, so that the rest term can be bounded by

$$|R(\mathbf{k}')| \leq \frac{9\sqrt{2}\gamma_0^2}{2\gamma_1}|\mathbf{k}'|^3. \tag{7.25}$$

In this sense, in the low energy limit, the model resembles the two dimensional dilute Fermi gas model.

When we take into account the trigonal warping term $\gamma_3 \neq 0$, the Fermi point $K^{(+)}$ or $K^{(-)}$ point splits into four equivalent points. Thus instead of two Fermi points we have now eight ones. Around these points, due to the linear term in $A(\mathbf{k})$ of $e(\mathbf{k})$, within the low energy regime the system has a linear spectrum,

$$e(\mathbf{k}' + \mathbf{k}) = c_1'|\mathbf{k}'| + R(\mathbf{k}'), \tag{7.26}$$

with $c_1' = \frac{3\gamma_0}{\gamma_1}$ and $R(\mathbf{k}') \leq c|\mathbf{k}'|^2$. For the valley $K^{(+)}$, we denote $\tan\phi = \frac{k_1'}{k_2'}$, then one of the Fermi points stays at $K^{(+)}$ with $|\mathbf{k}'| = 0$, the other three stay at $|\mathbf{k}'| = \frac{\gamma_3\gamma_1}{\gamma_0^2}$, and $\phi = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, whereas for $K^{(-)}$, one stays at $K^{(-)}$ with $|\mathbf{k}'| = 0$, the other three stay at $|\mathbf{k}'| = \frac{\gamma_3\gamma_1}{\gamma_0^2}$, and $\phi = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

7.2 Renormalization group analysis

In this section we illustrate the multiscale integration of the fermionic functional integral of interest. The analysis is very similar to the dilute Fermi gas case, hence we do not repeat all details but give only what is new with respect to the dilute Fermi gas.

7.2.1 Without trigonal warping

Let us first consider the situation with $\gamma_3 = 0$. The first step in the calculation of the effective action is to integrate out the ultraviolet degrees of freedom corresponding to the large value of k. To do this, we decompose the propagator $C(k)$ into a sum of two propagators supported in the ultraviolet and infrared regions, respectively. The regions of large or small k_0 are defined in term of the smooth support functions $h(x)$. We define $h(x) \in \mathbb{C}^\infty(\mathbb{R}, [0,1])$,

$$h(x) = \begin{cases} 1, & \text{for} \quad x < \epsilon_0, \\ 0, & \text{for} \quad x \geq 2\epsilon_0, \end{cases} \tag{7.27}$$

$h'(x) < 0$ for all $x \in (\epsilon_0, 2\epsilon_0)$. The constant ϵ_0 is chosen so that the supports of $h\left(\left|ik_0 - c_1|\mathbf{k} - \mathbf{k}_F^{(+)}|^2\right|\right)$ and $h\left(\left|ik_0 - c_1|\mathbf{k} - \mathbf{k}_F^{(-)}|^2\right|\right)$ are disjoint. To satisfy this condition, it is enough that $2\epsilon_0 < c_1 \frac{4\pi^2}{27}$. We define

$$f_>(k) = 1 - h\left(\left|ik_0 - c_1|\mathbf{k} - \mathbf{k}_F^{(+)}|^2\right|\right) - h\left(\left|ik_0 - c_1|\mathbf{k} - \mathbf{k}_F^{(-)}|^2\right|\right) \quad (7.28)$$

and $f_<(k) = 1 - f_>(k)$, so that we can rewrite $C(k)$ as:

$$C(k) = f_<(k)C(k) + f_>(k)C(k) = C_<(k) + C_>(k). \quad (7.29)$$

With these decomposition we introduce two independent sets of Grassmann fields $\psi^{(<)}_{\sigma,\rho}(k)$ and $\psi^{(>)}_{\sigma,\rho}(k)$, with $\sigma = \pm$, $\rho = 1, 2$, and the Grassmann integration defined by

$$\int d\mu_{C_>}(\psi^{(>)})\ \bar{\psi}^{(>)}_{\sigma,\rho}(k)\psi^{(>)}_{\sigma',\rho'}(k') = \delta_{\sigma\sigma'}\delta_{kk'}C_>(k)_{\rho\rho'}, \quad (7.30)$$

$$\int d\mu_{C_<}(\psi^{(<)})\ \bar{\psi}^{(<)}_{\sigma,\rho}(k)\psi^{(<)}_{\sigma',\rho'}(k') = \delta_{\sigma\sigma'}\delta_{kk'}C_<(k)_{\rho\rho'}. \quad (7.31)$$

where $d\mu_{C_>}(\psi^{(>)})$ and $d\mu_{C_<}(\psi^{(<)})$ admit an explicit representation analogous to (7.20), with $C(k)$ replaced by $C_<$ or $C_>$, $\psi(k)$ replaced by $\psi^{(<)}(k)$ or $\psi^{(>)}(k)$, and the sum over k are restricted to the values in the supports of $f_<(k)$ and $f_>(k)$, respectively. The additional property of the Grassmann integration implies

$$\begin{aligned} W(\mathcal{V})(\phi) &= \log \frac{1}{Z} \int d\mu_{C_<}(\psi^{(<)}) \int d\mu_{C_>}(\psi^{(>)})\ e^{\mathcal{V}(\psi+\phi)} \\ &= \log \frac{1}{Z_0} \int d\mu_{C_<}(\psi^{(<)})\ e^{\mathcal{V}^{(<)}(\psi^{(<)}+\phi)} \end{aligned} \quad (7.32)$$

where

$$\mathcal{V}^{(<)}(\psi^{(<)} + \phi) = \log \int d\mu_{C_>}(\psi^{(>)})\ e^{\mathcal{V}(\psi+\phi)} \quad (7.33)$$

is the ultraviolet part of the integration. To do this integral, we have to know the asymptotic properties of the ultraviolet propagator. Let us first redefine $X = (x, \sigma, \rho)$, with $\rho = 1, 2$ labeling two fields, and the following lemma holds:

Lemma 17. *There is a constant $\tilde{g}$, such that for $|g| < \tilde{g}$, $\mathcal{V}^{(<)}(\psi)$ is analytic in g, and it satisfies the following bound*

$$\|\mathcal{V}^{(<)}\|_h \leq c_h|g|, \quad (7.34)$$

where h and c_h are positive constants.

The proof of this lemma is the same as the Lemma 10. We do not repeat it here. With this lemma, $\mathcal{V}^{(<)}(\psi)$ has the form

$$\mathcal{V}^{(<)}(\psi^{(<)}) = \sum_{m,\bar{m}\geq 1} \int d^{\bar{m}}\underline{X}' d^{m}\underline{X}\; v^{(<)}_{\bar{m},m}(\underline{X}',\underline{X})\bar{\psi}^{(<)\bar{m}}\left(\underline{X}'\right)\psi^{(<)m}\left(\underline{X}\right).$$

and its kernel functions satisfy

$$|v^{(<)}_{\bar{m},m}(\underline{X}',\underline{X})|_{1,\infty} \leq C|g|^{\max\{1,m-1\}}, \tag{7.35}$$

with C as a constant. We note that the bound is independent of the temperature.

We are now left with dealing the infrared integration. We proceed in an iterative fashion. Because of the two non-equivalent Fermi points, it is convenient to decompose the infrared propagator as

$$C_<(X,X') = \sum_{\alpha=\pm} e^{ik_F^{(\alpha)}(x-x')}C^{(<)}_{\alpha}(X,X'), \tag{7.36}$$

where $k_F^{(\alpha)} = (0,\mathbf{k}_F^{(\alpha)})$, $\alpha = \pm$ refers to the two Fermi points and

$$C^{(<)}_{\alpha}(X,X') = \delta_{\sigma\sigma'}\int_{k'\in\mathcal{B}_L'^{(\alpha)}} dk'\; e^{ik'(x-x')}h\left(\left|ik_0 - c_1|\mathbf{k}'|^2\right|\right)C(k'+k_F^{(\alpha)}) \tag{7.37}$$

where $k' = k - k_F^{(\alpha)} = (k_0,\mathbf{k}') \in \mathcal{B}_L'^{(\alpha)} = \mathcal{M}^{(<)}_{k_0}\times\mathcal{B}_L^{(\alpha)}$, with $\mathcal{B}_L^{(\alpha)} = \{\mathbf{k}' = \frac{n_1}{L}b_1 + \frac{n_2}{L}b_2 - \mathbf{k}_F^{(\alpha)},\; 0\leq n_1,n_2\leq L-1\}$. Correspondingly we express $\psi^{(<)}(X)$ as a sum of independent Grassmann fields:

$$\psi^{(<)}(X) = \sum_{\alpha=\pm} e^{ik_F^{(\alpha)}x}\psi^{(<)}_{\alpha}(X). \tag{7.38}$$

The Fourier transform of $\psi^{(<)}_{\sigma,\rho,\alpha}(x)$ is

$$\psi^{(<)}_{\sigma,\alpha,\rho}(k) = \int dx\; e^{-i(k-k_F^{(\alpha)})x}\psi^{(<)}_{\sigma,\alpha,\rho}(x). \tag{7.39}$$

where ρ labels the two fields, α labels the two valleys, respectively. We redefine $X = (x,\sigma,\rho,\alpha) \in \Lambda'\times\{-1,1\}\times\{1,2\}\times\{+,-\}$, and

$$\int dX F(X) = \sum_{\sigma}\sum_{\rho=1,2}\sum_{\alpha=\pm}\int d\tau\int d^2\mathbf{x}\; F(\tau,\mathbf{x},\sigma,\rho,\alpha). \tag{7.40}$$

By (7.37) and (7.38), we rename $\psi^{(\leq 0)} = \psi^{(<)} + \phi$, $\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \mathcal{V}^{(<)}(\psi^{(<)}+\phi)$, thus the effective action can be rewritten in the form

$$W(\mathcal{V})(\phi) = \log\frac{1}{Z_0}\int d\mu_{C_<}(\psi^{(<)})e^{\mathcal{V}^{(0)}(\psi^{(\leq 0)})}, \tag{7.41}$$

where $\mathcal{V}^{(0)}(\psi^{(\leq 0)})$ is

$$\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \sum_{m,\bar{m}\geq 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X}\; v^{(0)}_{\bar{m},m}(\underline{X}',\underline{X})\; \bar{\psi}^{(\leq 0)\bar{m}}(\underline{X}')\psi^{(\leq 0)m}(\underline{X}).$$

The kernel $v^{(0)}_{\bar{m},m}(\underline{X}',\underline{X})$ has a representation in term of its Fourier transform,

$$\begin{aligned} v^{(0)}_{\bar{m},m}(\underline{X}',\underline{X}) \;&= \int \prod_{i=1}^{m+\bar{m}} dk'_j\; e^{i\sum_{j=1}^{m+\bar{m}}(-1)^j k_j x_j}\, \delta\left(\sum_{j=1}^{m+\bar{m}} (-1)^j (k_F^{(\alpha_j)} + k'_j)\right) \\ &\quad \cdot\, \hat{v}^{(0)}_{\bar{m},m}(k'_1,\ldots,k'_m,k'_{m+1},\ldots,k'_{m+\bar{m}}), \end{aligned} \tag{7.42}$$

with

$$\hat{v}^{(0)}_{\bar{m},m}(k'_1,\ldots,k'_{m+\bar{m}}) = \hat{v}^{(0)}_{\bar{m},m}(k'_1 + k_F^{(\alpha_1)},\ldots,k'_{m+\bar{m}} + k_F^{(\alpha_{\bar{m}+m})}). \tag{7.43}$$

Moreover, the Grassmann Gaussian measure $d\mu_{C_<}(\psi^{(<)})$ is defined as

$$d\mu_{C_<}(\psi^{(<)}) = \mathcal{N} \int \prod_k \prod_{\sigma,\rho,\alpha} d\bar{\psi}^{(<)}_{\sigma,\rho,\bar{\alpha}}(k) d\psi^{(<)}_{\sigma,\rho,\alpha}(k)\; e^{\mathcal{A}(\bar{\psi}^{(<)}_{\sigma,\bar{\rho},\bar{\alpha}},\psi^{(<)}_{\sigma,\rho,\bar{\alpha}})} \tag{7.44}$$

with

$$\mathcal{A}(\bar{\psi}^{(<)}_{\sigma,\bar{\rho},\bar{\alpha}},\psi^{(<)}_{\sigma,\rho,\bar{\alpha}}) = -\sum_{\alpha,\bar{\rho},\rho}\sum_{\sigma}\int_{k'\in\mathcal{B}'(\alpha)} dk' \bar{\psi}^{(<)}_{\sigma,\bar{\rho},\alpha}(k') C^{(<)}_\alpha(k')^{-1} \psi^{(<)}_{\sigma,\rho,\alpha}(k') \tag{7.45}$$

$\mathcal{N}$ is a constant, so that $\int d\mu_{C_<}(\psi^{(<)}) = 1$. The propagator in (7.45) is given by

$$C^{(<)}_\alpha(k') = h(\left|ik_0 - c_1|\mathbf{k}'|^2\right|) C(k' + k_F^{(\alpha)}). \tag{7.46}$$

Note that in (7.45) the α index of the ψ fields is the same, since the terms with different α's vanish, which can be easily checked. As a consequence there are no umklapp processes contributing to the infrared effective potential.

To do the infrared integration we have to scale $C^{(<)}_\alpha(k')$ by introducing the scales $j = 0, -1, -2, \ldots$. Then for all $j \leq 0$ the jth scale function is defined as

$$f^{(\alpha)}_j(k') = h(M^{-j}\left|ik_0 - c_1|\mathbf{k}'|^2\right|) - h(M^{-j+1}\left|ik_0 - c_1|\mathbf{k}'|^2\right|),$$

and obeys

$$h(\left|ik_0 - c_1|\mathbf{k}'|^2\right|) = \sum_{j=-\infty}^{0} f^{(\alpha)}_j(k'). \tag{7.47}$$

By construction the support of $f_j^{(\alpha)}(k')$ is contained in

$$\{k' \in \mathcal{B}'^{(\alpha)} |\ M^{j-1}\epsilon_0 \le |ik_0 - c_1|\mathbf{k}'|^2| \le 2M^j\epsilon_0\} \tag{7.48}$$

and is identically one on

$$\{k' \in \mathcal{B}'^{(\alpha)} |\ 2M^{j-1}\epsilon_0 \le |ik_0 - c_1|\mathbf{k}'|^2| \le M^j\epsilon_0\}. \tag{7.49}$$

At finite temperature $|k_0| > \frac{\pi}{\beta}$, $\forall k_0 \in \mathcal{M}_{k_0}$, we define the infrared cutoff scales

$$J = \max\{j \le 0, M^{j-1}\epsilon_0 < \frac{\pi}{\beta}\}. \tag{7.50}$$

Now by (7.47), (7.50), we have the identity

$$C_\alpha^{(<)}(k') \quad = \quad \sum_{j=J}^{0} C_\alpha^{(j)}(k') \tag{7.51}$$

with $C_\alpha^{(j)}(k') = f_j^{(\alpha)}(k')C(k' + k_F^{(\alpha)})$, and the Fourier transform of $C_\alpha^{(j)}(k')$ is

$$C_\alpha^{(j)}(X, X') = \delta_{\sigma\sigma'} \int_{k' \in \mathcal{B}_L'^{(\alpha)}} dk'\ e^{ik'(x-x')} C_\alpha^{(j)}(k'). \tag{7.52}$$

Correspondingly, by (7.36) we have

$$C_j(X, X') = \sum_{\alpha=\pm} e^{ik_F^{(\alpha)}(x-x')} C_\alpha^{(j)}(X, X'). \tag{7.53}$$

Similarly we decompose the Grassmann fields as $\psi^{(<)} = \sum_{j=J}^{0} \psi^{(j)}$. Subsequently we will successively integrate out $\psi^{(j)}$, which generates a sequence of effective potential $\mathcal{V}^{(0)} \dots \mathcal{V}^{(J-1)}$. We prove inductively in the perturbation theory that for all $j < 0$, the integration

$$\mathcal{V}^{(j-1)}(\psi^{(\le j-1)}) =: \log \int d\mu_{C_j}(\psi^{(j)}) e^{:\mathcal{V}^{(j)}(\psi^{(\le j)}):_{C_{\le j}}} :_{-C_{\le j-1}} . \tag{7.54}$$

is well defined, if $\mathcal{V}^{(j)}(\psi^{(\le j)}$ being well defined effective potential at scale j, and has the form

$$\mathcal{V}^{(j)}(\psi^{(\le j)}) = \sum_{m,\bar{m}\ge 1} \int d^{\bar{m}}\underline{X}' d^m \underline{X}\ v_{\bar{m},m}^{(j)}\left(\underline{X}', \underline{X}\right) \bar{\psi}^{(\le j)\bar{m}}\left(\underline{X}'\right) \psi^{(\le j)m}\left(\underline{X}\right).$$

Proceeding as in the previous chapter we have to estimate the Gram and decay bound of the propagator at first. It is easy to verify that the single scale propagator has a Gram bound

$$c_\gamma' M^{j/2} \le \gamma_{C_j} \le c_\gamma M^{j/2} \tag{7.55}$$

with c'_γ, c_γ two suitable constants, and a decay bound

$$\alpha_{C_j} \leq c_\alpha M^{-j}. \tag{7.56}$$

c_α is a constant. Both bounds are same as the bounds of the propagator in dilute Fermi gas. Similar results like the Lemma 10 are expected. By (7.55) and (7.56), using the tree expansion, we summarize our result in the following lemma:

Lemma 18. *Suppose that there exist positive constants $\tilde{g}$, C_2, C_g and suitable M and α, such that the conditions*

$$|v^{(j)}_{1,1}|_{1,\infty} \leq C_2 M^j |g| \tag{7.57}$$

for $J \leq j \leq 0$, $|g| < \tilde{g}$, $|g| \leq \frac{C_g}{\log \beta}$ are satisfied, $h_j = \alpha\gamma_{C_j}$, then for all $J \leq j \leq 0$,

$$\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j} < \frac{1}{2} \tag{7.58}$$

holds.

The proofs of this lemma and the assumption on the two-point function are similar as the proof for dilute Fermi gas (Lemma 10 and Lemma 12). We do not repeat them here.

Theorem 2 follows directly from this Lemma. We conclude that the perturbation expansion in g for $W(\mathcal{V})$ converges if g is small enough and the temperature is higher than an exponential small one.

7.2.2 With trigonal warping

Let us now consider the case with trigonal warping, $\gamma_3 \neq 0$. This next nearest neighbor coupling may have important effects in the low energy limit. It splits the parabolic degeneracy into four Dirac points, close to which the dispersion relation vanishes linearly.

The integration of $W(\mathcal{V})$ will be implemented in an analogous way as the previous section by replacing the smooth support function by

$$f_j(k) = h(M^{-j}|ik_0 - c'_1|\mathbf{k}'||) - h(M^{-j+1}|ik_0 - c'_1|\mathbf{k}'||). \tag{7.59}$$

Due to the linear dispersion relation it can be shown that the scaled infrared propagator has a better Gram bound

$$c_\gamma M^j \leq \gamma_{C_j} \leq c'_\gamma M^j, \tag{7.60}$$

where c_γ and c'_γ are positive constants. However the decay bound remains the same

$$\alpha_{C_j} \leq c_\alpha M^{-j} \tag{7.61}$$

with $c_\alpha > 0$. With these bounds, we first consider a naive power counting for the Green's functions. We found that only the two legged vertices are relevant in the renormalization flow. All others are irrelevant. This makes the further analysis much easier than the previous one. We consider the integral

$$\mathcal{V}^{(j-1)}(\psi^{(\leq j-1)}) =: \log \int d\mu_{C_j}(\psi^{(j)}) e^{:\mathcal{V}^{(j)}(\psi^{(\leq j)}):_{C_{\leq j}}} :_{-C_{\leq j-1}} . \tag{7.62}$$

In perturbation theory, we decompose $\mathcal{V}^{(j-1)}$ in the following way:

$$\mathcal{V}^{(j-1)} = \mathcal{V}_2^{(j-1)} + \mathcal{P}_{\geq 2,\geq 2}(\mathcal{V}^{(j-1)} - \mathcal{V}^{(j)}) + \mathcal{P}_{\geq 2,\geq 2}(\mathcal{V}^{(j)}) \tag{7.63}$$

where $\mathcal{V}_2^{(j-1)} = \mathcal{P}_{1,1}(\mathcal{V}^{(j-1)})$ is the two fields parts of the effective potential at scale $j-1$. We try to bound

$$\begin{aligned} \omega_{C_{j-1}} \|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \quad \leq \quad & \omega_{C_{j-1}} \|\mathcal{V}_2^{(j-1)}\|_{h_{j-1}} + \omega_{C_{j-1}} \|\mathcal{P}_{\geq 2,\geq,2}(\mathcal{V}^{(j)})\|_{h_{j-1}} \\ & + \omega_{C_{j-1}} \|\mathcal{P}_{\geq 2,\geq,2}(\mathcal{V}^{(j-1)} - \mathcal{V}^{(j-1)})\|_{h_{j-1}}. \end{aligned} \tag{7.64}$$

The second term on the right hand side of (7.64) has a trivial bound

$$\begin{aligned} & \omega_{C_{j-1}} \|\mathcal{P}_{\geq 2,\geq 2}(\mathcal{V}^{(j)})\|_{h_{j-1}} \\ \leq \quad & \omega_{C_j} \left(\frac{\omega_{C_{j-1}}}{\omega_{C_j}}\right) \sum_{\bar{m},m\geq 2} |v^{(j)}_{\bar{m},m}|_{1,\infty} h_j^{\bar{m}+m} \left(\frac{h_{j-1}}{h_j}\right)^{\bar{m}+m} \\ \leq \quad & \frac{1}{M}\, \omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j}. \end{aligned} \tag{7.65}$$

The last term in (7.64) can be bounded by using the usual tree expansion

$$\|\mathcal{P}_{\geq 2,\geq 2}(\mathcal{V}^{(j-1)} - \mathcal{V}^{(j)})\|_{h_{j-1}} \leq \sum_{p\geq 2} \omega_{C_j}^{p-1} \|\mathcal{V}^{(j)}\|^p_{h'_j}, \tag{7.66}$$

with $h'_j = \left(5 + \frac{\alpha}{M}\right) \gamma_{C_j}$. Denoting $\tilde{A} = \left(\frac{1}{M} + \frac{5}{\alpha}\right)$, and assuming $\tilde{A} < 1$, following inequalities hold,

$$\sum_{m,\bar{m}\geq 1} |v^{(j)}_{\bar{m},m}|_{1,\infty} h_j'^{\bar{m}+m} \leq \tilde{A}^2 \sum_{m,\bar{m}\geq 1} |v^{(j)}_{\bar{m},m}|_{1,\infty} h_j^{\bar{m}+m}. \tag{7.67}$$

Putting (7.67) into (7.66),

$$\omega_{C_{j-1}} \|\mathcal{V}^{(j-1)} - \mathcal{V}^{(j)}\|_{h_{j-1}} \leq \frac{K}{M} \sum_{p\geq 2} (\omega_{C_j} \|\mathcal{V}^{(j)}\|_{h_j})^p. \tag{7.68}$$

where $K = M^4\tilde{A}^4$. We will choose M and α, so that $K/M < 1$ holds. Putting all together, we get

$$\begin{aligned} & \omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \\ \leq \; & \omega_{C_{j-1}}\|\mathcal{V}_2^{(j-1)}\|_{h_{j-1}} + \frac{K}{M}\sum_{p\geq 2}(\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j})^p + \frac{1}{M}\,\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j}. \end{aligned} \tag{7.69}$$

We show the following lemma:

Lemma 19. *Suppose that there exist positive constants $\tilde{g}$, C_2, C_g and suitable M and α, such that the condition*

$$|v_{1,1}^{(j)}|_{1,\infty} \leq C_2 M^j |g| \tag{7.70}$$

for $J \leq j \leq 0$, $|g| < \tilde{g}$ holds, $h_j = \alpha\gamma_{C_j}$, then for all $J \leq j \leq 0$,

$$\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} < \frac{1}{2} \tag{7.71}$$

holds.

Proof. The proof is easy by induction on j. When $j = 0$, it is trivial. Let $j \leq -1$ and the statement holds for all $j' \geq j$, we have

$$\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq \left(\frac{K+1}{M}\right)\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} + C_2'|g_0|, \tag{7.72}$$

thus for $|g_0| \leq \frac{1}{2}\frac{M-K-1}{C_2'M} \leq \tilde{g}$, we have $\omega_{C_{j-1}}\|\mathcal{V}^{(j-1)}\|_{h_{j-1}} \leq \frac{1}{2}$. This proves the lemma. □

As a corollary of above Lemma we find that

$$\omega_{C_j}\|\mathcal{V}^{(j)}\|_{h_j} \leq C_{g0}|g|. \tag{7.73}$$

for a suitable constant C_{g0}. The assumption on the two points function in the lemma can be shown by imposing a zero renormalization mass condition. Thus the perturbation expansion of the generating functional $\mathcal{W}(\mathcal{V})$ converges for $|g| \leq \tilde{g}$, which is temperature independent.

7.3 Symmetry breaking states

In this section we would like to discuss the possible physical instability in short. Fixing an initial coupling constant we let the temperature be a scale for the energy and let the temperature go down, the renormalization group

analysis used in this work brings out three energy regimes. In the first regime the running coupling functions remain small and the perturbation expansion of the generating functional in g converges. This is the normal order phase, where the symmetry breaking gaps remain small. The symmetry breaking takes place in the second regime. The symmetry breaking gaps rapidly start to grow and finally saturate at a scale. The running coupling function is no longer small. In the third regime the system can be described by Goldstone Bosons.

Actually we would like to construct a model which shows a spontaneous symmetry breaking and a phase transition as the running coupling constant grows. An idea to do this is to use the source term which can be written as

$$\sum_{\mu,\nu,\omega} \Delta_{\mu\nu\omega}\ \bar{\Psi}(x) Q_{\mu\nu\omega} \Psi(x) \tag{7.74}$$

where $\bar{\Psi}(x)$ is an eight-component ferimonic fields,

$$\bar{\Psi} = \left(\bar{\psi}_{\uparrow,1,+}, \bar{\psi}_{\uparrow,2,+}, \bar{\psi}_{\downarrow,1,+}, \bar{\psi}_{l,\downarrow,2,+}, \bar{\psi}_{u,\uparrow,1,-}, \bar{\psi}_{\uparrow,2,-}, \bar{\psi}_{\downarrow,1,-}, \bar{\psi}_{\downarrow,2,-}\right). \tag{7.75}$$

The order parameter is given by

$$Q_{\mu\nu\omega} = \tau_\mu \otimes \sigma_\nu \otimes \eta_\omega, \tag{7.76}$$

where $\mu, \nu, \omega = 0, 1, 2, 3$, and τ_μ, σ_ν, η_ω are Pauli matrices and zero denotes the unit matrices, acting on layer, spin, valley induces, respectively. $\Delta_{\mu\nu\omega}$ are the gap functions. For bilayer graphene in low energy regime, we consider

$$C_\Delta(\phi, \bar{\phi}) = \log \frac{1}{Z_\Delta} \int d\mu_C(\psi, \bar{\psi})\ e^{-\mathcal{V}_\Delta + (\bar{\phi}, \psi) + (\bar{\psi}, \phi)} \tag{7.77}$$

with

$$\mathcal{V}_\Delta(\psi, \bar{\psi}) = \mathcal{V}(\psi, \bar{\psi}) + \int dk\ \bar{\Psi}(k) \delta e_k \Psi(k) - \int dx \sum_{\mu,\nu,\omega} \Delta_{\mu\nu\omega}\ \bar{\Psi}(x) Q_{\mu\nu\omega} \Psi(x)$$

at half filling $\delta e(k) = 0$. In order to get a constructive model we have to show the existence of the limit

$$C(\phi, \bar{\phi}) = \lim_{\Delta \to 0} C_\Delta(\phi, \bar{\phi}) \tag{7.78}$$

If this limit can be proven nonpertubatively, we are success to construct a model which shows a spontaneous symmetry breaking. This is a difficult task. We will not cover this problem in this work.

Instead we would like to discuss the properties of some instabilities. The symmetry breaking states can be divided into two classes, based on whether

a gap emergent between the two low energy bands. A gapped state breaks generally the inversion symmetry, the order parameter has a form $\tau_3 \otimes R$, where $R = \sigma_\nu \otimes \eta_\omega$, with an arbitrary choice of the spin and valley matrices. The most important breaking symmetry states in this class are ferroelectric state (FE), layer-antiferromagnetic state (AF), anomalous quantum Hall insulator (AQH) and spin quantum Hall insulator (SQH). Their properties are summarized in table 7.1.

	$\tau_3 \otimes R$	Broken symmetry	Properties
FE	$1 \otimes 1$	inversion (Z_2)	layer pesudospin polarized, spontaneous charge transform between layers,
AF	$\sigma_3 \otimes 1$	inversion (Z_2), time reversal ($\mathcal{T}$), spin rotation ($SU(2)$)	opposite spin polarization on top and bottom layers, zero Hall conductivity.
AQH	$1 \otimes \eta_3$	inversion (Z_2), time reversal ($\mathcal{T}$), valley (Z_2)	Hall conductivity $\sigma = \frac{4e^2}{h}$, persists to zero magnetic field.
SQH	$\sigma_3 \otimes \eta_3$	inversion (Z_2), spin rotation ($SU(2)$), valley (Z_2)	zero Hall conductivity, finite Hall conductivity for each spin.

Table 7.1: Gapped breaking symmetry states.

In contrast with the gapped states there are instabilities which lead to gapless states. e.g. the nematic states. The order parameters have the form $\tau_{1,2} \otimes R$. The two Fermi points $K^{\pm}$ split to two non-equivalent Dirac points.

There have been a number of theoretical investigations for the symmetry breaking states of bilayer graphene by using both mean field and renormalization group approaches. At first sight the results were controversial.

We first introduce the results coming from the mean field approach. There is a result about the instability of a quadratic band crossing point two dimensional fermionic system in [SYFK09]. Using a variational wave function technique [NNPG06] argued for a ferromagnetic phase for long-range interactions and for short-range interactions layer-antiferromagnetic phase was argued by susceptibility calculations. [MBPM08] predicted a layer pesudospin magnet states (AF, QAH) by calculating a chiral two-dimensional electron system (C2DES) Hartree-Fock Hamiltonian. Similar results fol-

lowed by [JZM11], where a lattice Hartree-Fock model was investigated, they found that states with a quantized valley Hall effect were lowest in energy (FE). Both papers considered a Coulomb interaction. In [NL10] a mean field in combination with random phase approximation to second order was used, they showed that by a dynamically screened Coulombs interaction gapped states were favored. e.g. FE, AF, AQH. All had the same energy because of the $SU(4)$ invariant of the Hamiltonian under spin-valley rotation.

There are also a number of papers that employ the renormalization group approach introduced in [Sha94] to investigate the symmetry breaking phases. In [VY10, Vaf10] they introduced a symmetry breaking source term in the action, and calculated numerically how the symmetry breaking term diverged, as they approach the energy scale where the running coupling function diverges. They argue for an existence of a nematic state for extremely long-range Coulomb interaction and a layer-antiferromagnetic state for an on-site interaction. A similar paper [LATF10] came to the same result that a nematic state is favored by Coulomb interactions. In [ZMPM10] an inversion symmetry breaking state with short-range interactions was estimated by a perturbation renormalization group calculation.

All the results indicated that the dominant instability may depend on the profile of the interaction. In [TV12] they extended the renormalization group analysis of bilayer graphene with different, but finite interactions from extremely short to extremely long ranges. They found that the system enters a gapped antiferromagnetic phase for short-range interactions and a gapless nematic phase for long-range interactions. The application of functional renormalization group technique [SUH11] presented a phase diagram of the possible ground states.

It should be note that in [MKAF11] it was shown that the mechanical deformations of the crystal may lead to a Lifshitz transition, the parabolic bands splits into two Dirac cores. Thus strain has the same effect as the presence of a nematic order.

Experimentally, [MFW$^+$10, WAF$^+$10] found evidence for an anomalous quantum Hall phase by a compressibility measurement. A recent experiment [MEMK$^+$11] estimated a nematic phase by measuring a temperature dependence of the width and the amplitude of the conductivity minimum, and the cyclotron gaps as a function of the applied magnetic field for different filling factors. They found that the lowest Landau level is eight-fold degenerate, which implies a nematic state. The controversial results are due to the different experimental setup.

Chapter 8

Conclusion

In the present thesis we described constructive renormalization group approach to study systematically the properties of interacting non-relativistic Fermi systems in two or three dimensions. In particular we focused on rigorous mathematical constructions of two models for Fermions: A dilute Fermi gas model in continuum two dimensions with short-range interaction and a Hubbard model for bilayer graphene at half-filling with on-site interaction. The constructions were based on checking whether given models satisfy Salmhofer's criterion for Fermi liquid behavior or not.

For dilute Fermi gas model, to prove that renormalized perturbation theory in the coupling constant g converges, we employed multiscale analysis and discrete renormalization group techniques. The renormalization group map was implemented by using a tree expansion, which allowed to package the perturbation expansion in terms of trees rather than Feynman graphs, and give right scaling properties for the convergence radius of the expansion. The convergence radius was proved to depend on the sign of g. By a detailed analysis of the flow equation of the running coupling constant, we found that for an attractive interaction the Green's functions converge to an analytic function if g satisfies $\{(g, \beta) : |g| \log \beta < \text{const}\}$. On the other sides, for a repulsive interaction the analytic region was replaced by $\{(g, \beta) : g^2 \log \beta < \text{const}\}$. In this case the running coupling constant decreased during the renormalization flow, and the theory was infrared asymptotically free. The restrictions on g removed possible instabilities and drove the systems from their normal phases into symmetry breaking phases.

The regularity of the self-energy was used to distinguish Fermi liquids from other possible states of the fermion system. Verifying its regularity we supplemented the same procedure but with a number of modifications to get the desired bounds. We followed the arch-expansion to generate 1-PI graphs

from the fermion determinants in the tree expansion for the self-energy. This additional expansion ensures that not too many terms were generated and the expansion could be controlled constructively. The self-energy was proved to be $\mathbb{C}^1$ uniformly in β, but not $\mathbb{C}^2$ uniformly. By estimating a lower and an upper bound on the second derivatives of the self-energy with respect to the frequency, it grew like β for large β. However, we found that the high order corrections to this quantity were much smaller than its second order contribution.

We extended our method to evaluate the correlation decay of fermionic bilinears. We don't found anomalous decay exponents in the analytic domain of the theory.

The Hubbard model for bilayer graphene was constructed in the similar way. We have seen that the next nearest interlayer hopping γ_3, which called trigonal warping played a role in our constructions. We conclude that for $|g| < C_3$, the perturbation expansion in g converges, even the temperature goes to zero, in other words, the system was stable under weak interaction. For $C_3 < |g| < C_1$, it converges only for temperature $\beta^{-1} > e^{-\frac{C_2}{|g|}}$.

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