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Frequency and phase estimation in time series with quasi periodic components

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In the memory of my father

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Abstract

A classical model in time series analysis is a stationary process superposed by one or several deterministic sinusoidal components. Different methods are applied to estimate the frequency (ω) of those components such as Least Squares Estimation and the maximization of the periodogram.

In many applications the assumption of a constant frequency is violated and we turn to a time dependent frequency function ($\omega(s)$). For example in the physics literature this is viewed as nonlinearity of the phase of a process. A way to estimate $\omega(s)$ is the local application of the above methods.

In this dissertation we study the maximum periodogram method on data segments as an estimator of $\omega(s)$ and subsequently a least squares technique for estimating the phase. We prove consistency and asymptotic normality in the context of "infill asymptotics", a concept that offers a meaningful asymptotic theory in cases of local estimations. Finally, we investigate an estimator based on a local linear approximation of the frequency function, prove its consistency and asymptotic normality in the "infill asymptotics" sense and show that it delivers better estimations than the ordinary periodogram. The theoretical results are also supported by some simulations.

Abstract (Deutsch)

Ein klassisches Model in der Zeitreihenanalyse ist ein von einer oder mehreren deterministischen sinusförmigen Komponenten überlagerter stationärer Prozess. Zur Schätzung der Frequenz (ω) dieser Komponenten wurden verschiedene Methoden entwickelt, wie die Maximierung des Periodograms und die Kleinste-Quadrate (KQ) Schätzung.

Bei vielen Anwendungen ist die Annahme einer konstanten Frequenz nicht erfüllt und es muss eine zeitabhängige Frequenzfunktion ($\omega(s)$) verwendet werden. In der Physikliteratur zum Beispiel spricht man in diesem Fall von einer nicht linearen Phase. Eine Art, $\omega(s)$ zu schätzen, ist die lokale Anwendung der oben genannten Methoden.

In der vorliegenden Dissertation untersuchen wir die Maximum Periodogram Methode angewandt auf Datensegmente als Schätzer von $\omega(s)$ und anschließend eine KQ Technik zur Schätzung der Phase. Wir beweisen die Konsistenz und asymptotische Normalität dieses Verfahrens im Rahmen der *Infill Asymptotics*, einem Konzept, das eine Asymptotik für lokale Schätzungen ermöglicht. Schließlich untersuchen wir einen auf einer lokalen linearen Approximation der Frequenzfunktion basierenden Schätzer, weisen seine Konsistenz und asymptotische Normalität im Rahmen der *Infill Asymptotics* nach und zeigen, dass er bessere Schätzungen als das übliche Periodogram liefert. Die theoretischen Ergebnisse werden durch Simulationen bestätigt.

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Chapter 1

Introduction

In many scientific areas, such as physics, biology, finance, etc. scientists have to deal with data being affected by phenomena that repeat themselves in time, i.e. data containing seasonal components. We mention temperature fluctuations, the beating of a heart and the consumption of electric energy in a house hold as examples from the above three fields. Mathematically, we could express our intuition on what a seasonality is through the following very general formula

$$Y(x) = f(x \mod T_0) + R_x, \quad x, T_0 \in \mathbb{S},$$

where $f(\cdot)$ is any function of x, T_0 is the period and $\mathbb{S}$ is the domain of x and can be time, space etc. From now on we assume $\mathbb{S}$ to be a discrete set. The term R_x contains all rest components that also affect the process but are not periodic, such as drift, noise, etc. By considering the three examples given above we can already distinguish between two main categories of seasonal components or periodicities: While for temperature fluctuations and energy consumption we can assume a fixed period of time after which the periodic component repeats itself (i.e. one day or one year), it is impossible to do so for the heart beat. The latter depends on the state of the body and is not constant in time. Such cases are the main focus of this study.

The easiest way to model periodic data is to use the simplest periodic function, the sinusoid. In the case of a fixed period the above formula becomes (in discrete time)

$$Y(t) = \gamma \cos(\omega t + \phi_0) + X_t, \quad t \in \mathbb{N}, \tag{1.1}$$

where γ is the amplitude, ω the frequency and ϕ_0 the starting phase of the oscillation. Throughout the hole study we assume that our processes are free of trend, as we can always mean-correct the observed data. In Figure 1.1 we show a cosine oscillation around zero with frequency $\omega = \pi/15$ and amplitude $\gamma = 1$ and the same oscillation with white noise from a $\mathscr{N}(0, 1)$ distribution.

There are many methods for estimating ω in the above model (for a review see [12]). We mention Maximum Likelihood methods, the Maximization of the Periodogram [8, 9, 4, 13],

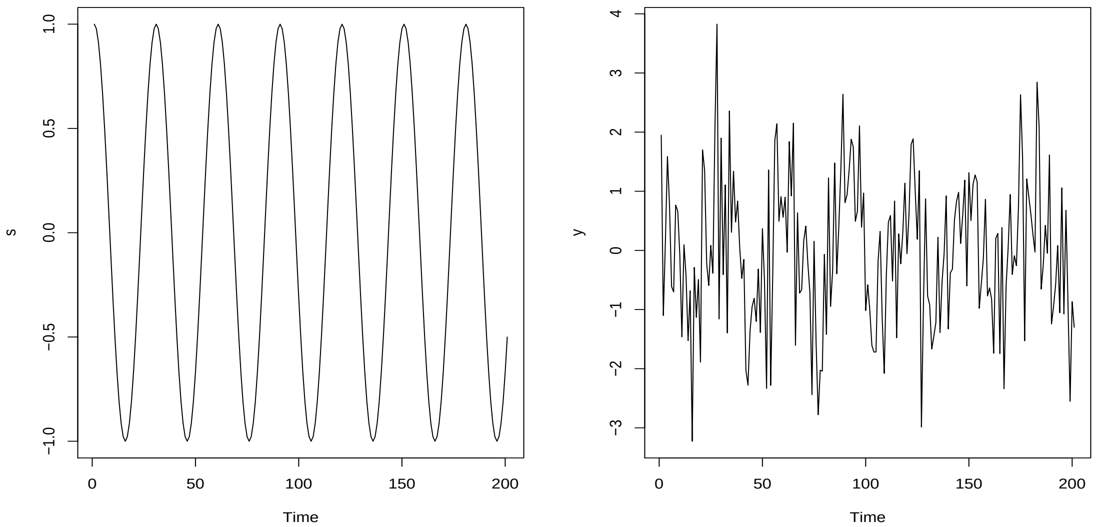

Figure 1.1: Cosine oscillation (s) and cosine oscillation with noise (y).

a variation of it called *the secondary analysis* [11, p.413] and Least Squares Estimation [18]. In particular, the estimate of ω through the maximization of the periodogram is

$$\begin{aligned}\widehat{\omega} &= \arg\sup_{\lambda} I_n(\lambda) \\ &= \arg\sup_{\lambda} \frac{1}{n}\left|\sum_{t=1}^{n} Y(t)\exp\{-i\lambda t\}\right|^2.\end{aligned}$$

Figure 1.2 shows the raw periodogram of the time series (y) from Figure 1.1 in logarithmic scale. The pick at the low frequencies indicates the periodicity of the data. Hannan proves in [9] consistency and asymptotic normality of this estimate in case of a stationary noise, indeed with asymptotic variance equal to the Cramér-Rao bound, which can be found in [12, p.62]. He uses there the following equivalent representation of model (1.1):

$$Y(t) = \alpha\cos(\omega t) + \beta\sin(\omega t) + X_t, \quad t \in \{1, ..., n\}. \tag{1.2}$$

After having estimated the frequency ω, the parameters α and β are also being estimated by fitting a linear model on $Y(t)$ with $\cos(\widehat{\omega}t)$ and $\sin(\widehat{\omega}t)$ as independent variables. Consistency and asymptotic normality of those estimates are also proved by means of the following central limit theorem:

$$\left(n^{1/2}(\hat{\alpha}_n - \alpha), n^{1/2}(\hat{\beta}_n - \beta), n^{3/2}(\widehat{\omega}_n - \omega)\right) \to \mathscr{N}(0, \Sigma),$$

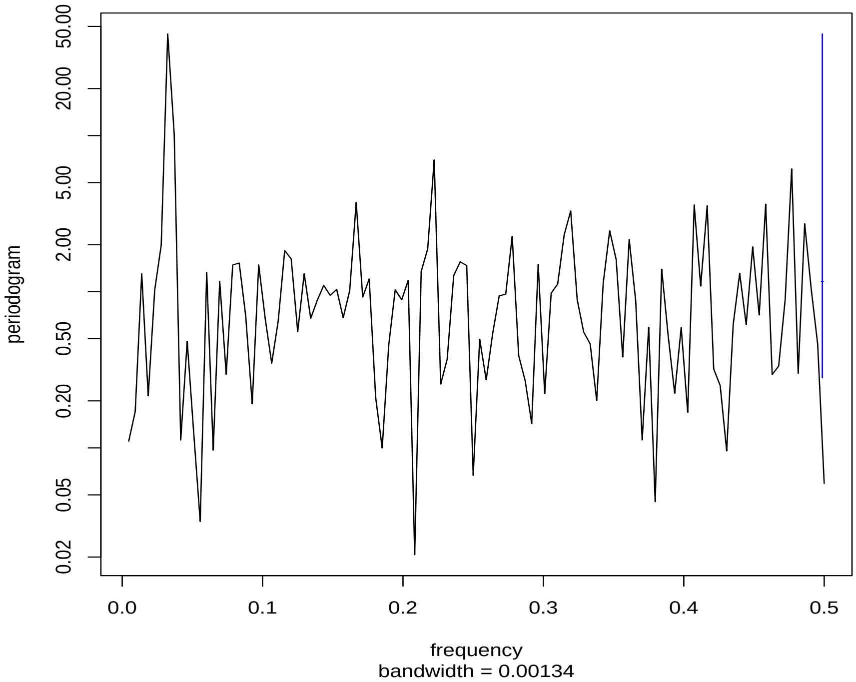

Figure 1.2: Raw periodogram of a noisy oscillation.

with asymptotic covariance matrix

$$\Sigma = 2\pi f(\omega) \begin{bmatrix} 1/2 & 0 & \beta/4 \\ 0 & 1/2 & -\alpha/4 \\ \beta/4 & -\alpha/4 & \frac{1}{6}(\alpha^2+\beta^2) \end{bmatrix}^{-1},$$

where $f(\cdot)$ is the spectral density of X_t. The main difference between Least Squares Estimation and the periodogram method is the cases $\omega = 0$ and $\omega = \pi$. While the maximization of the periodogram handles these cases without problems, one has to exclude them from the frequency domain to proceed to Least Squares Estimation. Apart from this, the two methods are asymptotically equivalent for white noise as it can be seen in [18], where the asymptotic distribution of the estimates is as above. Under Gaussianity there is equivalence also to Maximum Likelihood Estimation.

Some methods are proposed to make the Maximum of the Periodogram method computationally more efficient. They are based on maximizing the function only on the Fourier frequencies $2\pi\frac{j}{n}$ for $j = 1, ..., n$. As examples we mention the "Secondary analysis", a description of which can be found in [11, p.413], the zero padded Periodogram, methods based on Fourier coefficients and techniques based on filtering (c.f. [14]). The problem that arises by evaluating the periodogram only on the Fourier frequencies is that this does not constitute a fine enough grid (see [15]), so that the estimates are usually not optimal in terms of convergence rates.

Let us consider now the case of a time varying frequency or, "equivalently" of a non uniformly increasing phase. We use here the quotation marks because the instantaneous frequency is naturally defined as the first derivative of some phase function ϕ_t and this derivative does not have to exist. Though, this is a main assumption we do throughout the hole study; but since the domain is discrete, this is not very restricting. Following this natural definition of the frequency we consider the model

$$Y(t) = \gamma \cos\left(\int_0^t \widetilde{\omega}(s)\, ds + \phi_0\right) + X_t, \quad t \in \mathbb{N}. \tag{1.3}$$

In Figure 1.3 we have a realization of the above model for a time linear frequency function and the frequency function itself.

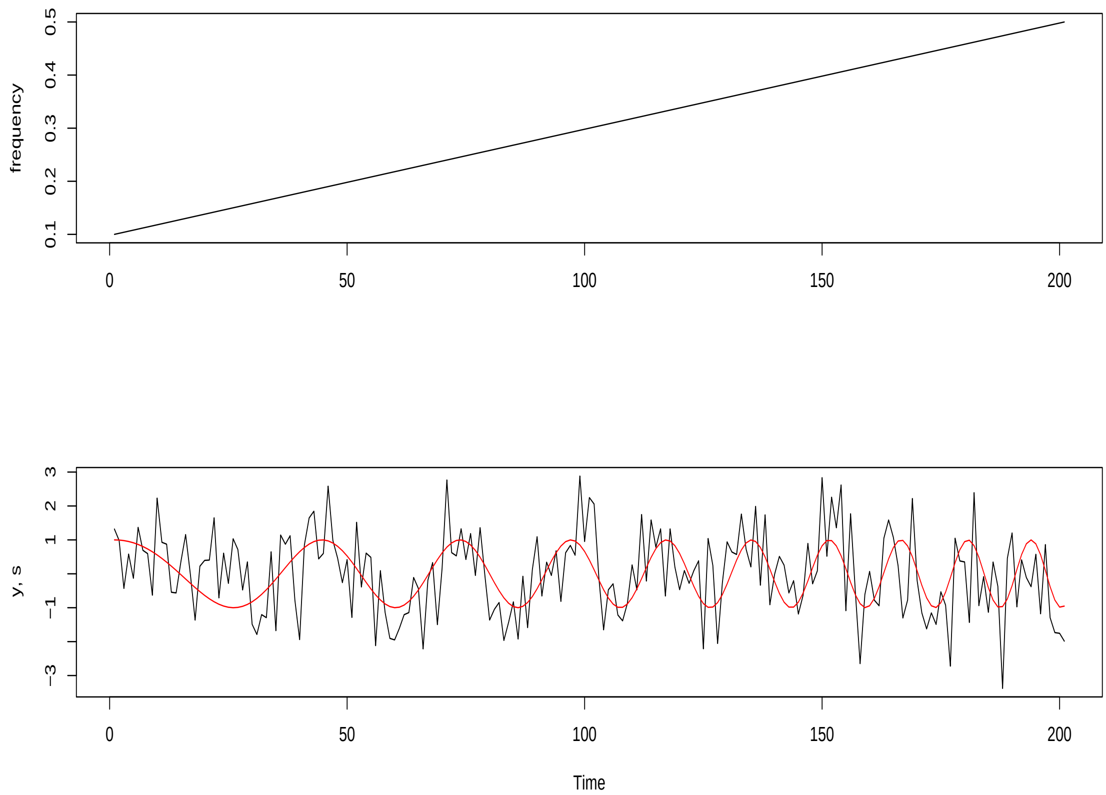

Figure 1.3: Oscillation with inhomogeneous phase increments.

Chapter 2

The periodogram is employed here segment wise to estimate the frequency locally, i.e. by using M_n of the n observations for estimating the function $\widetilde{\omega}(s)$ at each point $s \in \mathbb{N}$. The latent assumption is then that $\widetilde{\omega}(s)$ is locally smooth enough. How smooth it must be so that the estimate is consistent is one of the subjects of Chapter 2. In the same chapter we also prove consistency of $\hat{\alpha}_n$ and $\hat{\beta}_n$ in the alternative representation (1.2). All proofs are shown in the frame of "infill asymptotics", a concept that allows asymptotic theory when the parameter is estimated locally. The chapter concludes with a central limit theorem for the estimates, which in the case of the frequency estimate is derived by evaluating the asymptotic bias caused by the non constant frequency.

Chapter 3

In order to improve the estimates provided by the ordinary periodogram method, Katkovnik proposed the *Local Polynomial Periodogram* (LPP) [10], a method based on a local polynomial approximation of the phase function. Instead of the ordinary periodogram the function

$$\widetilde{J}_M(t_0; \lambda_0, \lambda_1, ...) := \frac{1}{2m+1} \left| \sum_{t=-m}^{m} Y(t+t_0) \exp\left\{ -i \left(\lambda_0 t + \lambda_1 \frac{t^2}{2!} + \lambda_2 \frac{t^2}{3!} + ... \right) \right\} \right|^2 ,$$

is used. In this chapter we focus on the case $\vec{\lambda} = (\lambda_0, \lambda_1)$ which we call the *modified* periodogram. Its maximization over λ_0 and λ_1 leads to estimates for $\widetilde{\omega}(t_0)$ and its derivative $\widetilde{\omega}'(t_0)$. The same least squares approach as before is then used for estimating the rest parameters. Again within the "infill asymptotics" frame we prove consistency and asymptotic normality. The theoretical results of the two estimators in question provide us with a straightforward comparison between them. To support these theoretical results, at the end of the two main chapters we present some simulation studies where we employ the two estimators described above and the very known Hilbert transform as a procedure for phase estimation (see [16]).

Chapter 2

The common periodogram

2.1 Introduction

A classical model in time series analysis is a stationary time series superposed by one or several deterministic sinusoidal components which in the case of one component takes the form

$$Y_t = \gamma \cos\Big(\omega\, t + \phi_0\Big) + X_t, \quad t \in \mathbb{N}.$$

Different methods have been used to estimate the parameter ω such as Least Squares Estimation (cf. [18]), the maximization of the periodogram (cf. [8, 9, 4, 13, 14]), the secondary analysis (cf. [11, p. 413]) and Maximum Likelihood methods. For an overview and additional methods see [12].

In many applications the frequency is not constant but time varying (cf. [12, p. 21 - 25], [1]). In particular in the physics literature (cf. [17, 7]) this is viewed as the estimation of the nonlinear phase of the process - i.e. instead of $\omega\, t + \phi_0$ the (possibly) nonlinear phase $\phi_t := \int_0^t \widetilde{\omega}(s)\, ds + \phi_0$ is estimated (in addition to the frequency curve $\widetilde{\omega}(s)$) in the model

$$Y_t = \gamma \cos\Big(\int_0^t \widetilde{\omega}(s)\, ds + \phi_0\Big) + X_t, \quad t \in \mathbb{N}. \tag{2.1}$$

where $\gamma > 0$ is the amplitude of the oscillation, $\widetilde{\omega}(s)$, $s \geq 0$ the frequency function and $\phi_0 \in [0, 2\pi)$ the starting phase of the oscillation. X_t is a stationary process. Also in the case with a time-varying frequency and a nonlinear phase several methods have been suggested (cf. [2, 12]) - many of them being a local variant of the existing methods. An example of this is the maximization of the periodogram on a local data segment.

We are not aware of any rigorous theoretical results on the estimators in the case of a time-varying frequency. In this chapter we shall do such a theoretical investigation for the first time by deriving the asymptotic properties of the maximum periodogram method on data segments as an estimate for $\widetilde{\omega}(s)$ and of a subsequent least squares technique for estimating the phase ϕ_t. A key problem addressed in this chapter is to find a proper framework for these investigations leading to the infill asymptotic approach in Section 2.2.1. We prove consistency and asymptotic normality for the estimates by generalizing previous results for constant frequencies to the time varying case. In Section 2.2.2 we derive consistency of the frequency estimator. In Section 2.2.3 we introduce the phase estimator and derive its consistency. In Section 2.3 we discuss the mean squared error and asymptotic normality for the estimators. Section 2.4 contains an empirical comparison of the estimator with the Hilbert transform. The proofs are postponed to the appendix.

2.2 Frequency and phase estimation for nonlinear phases

In this section we present the estimation procedure. At the beginning we introduce the infill asymptotics framework. We emphasize that this framework is only needed for the asymptotic results of the following sections. The method itself does not depend on this setup (and the rescaling used therein). This means that all methods can be easily formulated also without rescaling.

2.2.1 Rescaling and infill asymptotics

Asymptotic considerations as consistency and asymptotic normality are important tools in statistics for constructing tests, confidence intervals or to judge the quality of estimates. However, in the above setting of model (2.1) the simple asymptotics $t \to \infty$ is meaningless. From a theoretical point of view this is the same as in nonparametric regression or for non stationary time series. As a solution in the first example the setting $Y_{t,n} = m\big(\frac{t}{n}\big) + \varepsilon_t$ and in the second example e.g. the rescaled time varying AR-model $X_{t,n} = a\big(\frac{t}{n}\big) X_{t-1,n} + \sigma\big(\frac{t}{n}\big)\varepsilon_t$ (cf. [6]) are used where the curves of interest are rescaled to the unit interval. n then is assumed to tend to infinity leading to meaningful asymptotic results which can e.g. be used for the construction of approximate confidence intervals. We mention that the approach of infill asymptotics can usually not be interpreted in a physical sense.

The infill setting in the current situation is much more complicated: The naive setup $Y_{t,n} = \gamma \cos\big(\int_0^{t/n} \omega(u)\,du + \phi_0\big) + X_t$ does not make sense since the argument of the cos-function stays bounded and the signal does not oscillate at all. The correct solution is indicated by the substitution $s = un$

$$\int_0^t \widetilde{\omega}(s)\,ds = n \int_0^{t/n} \widetilde{\omega}(un)\,du =: n \int_0^{t/n} \omega(u)\,du \tag{2.2}$$

i.e. we use the model

$$Y_{t,n} = \gamma \cos\Big(n \int_0^{t/n} \omega(u)\,du + \phi_0\Big) + X_t =: S_{t,n} + X_t \tag{2.3}$$

with a fixed function $\omega(u) : [0,1] \to [0,\pi)$.

The additional factor n in (2.3) looks a bit strange and in fact it causes many technical problems in the derivations below (for example the remainder term in the Taylor expansion in Lemma 2.2 depends in a complicated way on n and is not getting small uniformly in t or t/n). Despite of this we are convinced that this is the correct approach for a meaningful asymptotics leading e.g. to approximate confidence intervals for $\omega(u_0)$ by using the central limit theorem.

The process X_t is assumed to be stationary. We assume that it satisfies the following assumption:

Assumption 2.1. *The observed process $Y_{t,n}$ satisfies (2.3) where the process $X(t)$ is strictly stationary with mean 0 and existing moments of all order and satisfies*

$$\sum_{u_1,...,u_{k-1}=-\infty}^{\infty} |c_k(u_1,...,u_{k-1})| < \infty,$$

where $c_k(.)$ is the k-th order cumulant.

Throughout this study we use the above setting and derive consistency, asymptotic normality and other results of the following estimator for $\omega(u_0)$ for $u_0 \in (0,1)$.

2.2.2 Frequency estimation based on the local periodogram

A good estimator of a constant unknown frequency is obtained via the maximization of the periodogram. In the present situation of a time-varying frequency we modify this estimator by considering the maximization of the periodogram in some neighborhood of the time point $u_0 \in (0,1)$: Let

$$J_M(u_0,\lambda) := \Big|\frac{1}{M(n)} \sum_{s=-m_\ell(n)}^{m_r(n)} Y_{n(u_0+\epsilon_n)+s,n} \exp(-i\lambda s)\Big|^2 \tag{2.4}$$

with $M(n) = m_l(n) + m_r(n) + 1$ being an increasing sequence of integers (usually the argument n is omitted). ϵ_n with $|\epsilon_n| < 1/n$ fills the "gap" between u_0 and the next t/n point, i.e. $\epsilon_n := \min_{t:\, t\geq nu_0}\{t/n - u_0\}$. It is usual, but not always necessary, that $m_l = m_r$. We define

$$\widehat{\omega}_n(u_0) = \arg\sup_{\lambda\in[0,\pi]} J_M(u_0,\lambda). \tag{2.5}$$

We now state consistency of the estimator $\widehat{\omega}(u_0)$ in the frame of the above infill asymptotics. The proofs are put into the appendix. The basic structure of the proof is similar to the proof of Hannan (cf. [9]) in the time-homogeneous case. In addition several problems occur due to the time varying situation and the complicated structure of the infill asymptotics.

Theorem 2.1. *Let Assumption 2.1 hold, $\omega(u)$ be Lipschitz continuous with values in $(0,\pi)$ and $u_0 \in (0,1)$. Then we have for the estimator defined in (2.5) with increasing sequences $m_l(n) = o(n^{1/2})$ and $m_r(n) = o(n^{1/2})$:*

$$\widehat{\omega}_n(u_0) \overset{n\to\infty}{\longrightarrow} \omega(u_0) \qquad a.s.$$

As in the classical case of a constant frequency we can even establish consistency with a rate. This result is needed for later proofs like the consistency of the phase estimator. It follows from a modification of the proof of Theorem 2.1.

Theorem 2.2. *Under the assumptions of Theorem 2.1 it holds that:*

$$M_n(\widehat{\omega}_n(u_0) - \omega(u_0)) \overset{n\to\infty}{\longrightarrow} 0, \qquad a.s. \tag{2.6}$$

2.2.3 Phase estimation based on a local regression

We now study estimation of the phase

$$\phi_n(u_0) := n \int_0^{u_0} \omega(u)\, du + \phi_0 = \int_0^{t_0} \widetilde{\omega}(s)\, ds + \phi_0 = \phi_{t_0}$$

which is the real occurring phase at time $u_0 = t_0/n$ (alternatively one could study estimation of $\int_0^{u_0} \omega(u)\, du$). In the physics literature the most usual way to estimate the phase ϕ_{t_0} is through the reconstruction of the analytic representation of a real signal Y_t (or $Y_{t,n}$ in the infill asymptotics context) using the Hilbert transform (cf. [16, 2]):

$$z_s(t) := Y_{t,n} + iH[Y_{t,n}],$$

where $H[\cdot]$ is the discrete time Hilbert transform operation. An immediate estimation of the phase then is

$$\hat{\phi}_t := \operatorname{arctan2}\left(H[Y_{t,n}], Y_{t,n}\right) = \operatorname{arctan2}\left(H[S_{t,n} + X_{t,n}], S_{t,n} + X_{t,n}\right), \tag{2.7}$$

where $S_{t,n}$ denotes the deterministic oscillating part of the process and $X_{t,n}$ is the noise term. This method has shown good behavior in simulations even with time-varying frequencies, but only when the signal/noise ratio was relatively large. In opposite cases it turned out to have greater MSE than the approach we describe below. Heuristically, this is easy to see because the estimator in (2.7) depends strongly on the realizations of the noise process at each time point.

In the following we suggest a method via a local regression based on the frequency estimator $\widehat{\omega}_n(u)$. We have

$$\begin{aligned} Y_{t,n} &= \gamma \cos\Big(n \int_0^{t/n} \omega(u)\, du + \phi_0\Big) + X_t \\ &= \gamma \cos\Big(n \int_{u_0}^{t/n} \omega(u)\, du + \phi_n(u_0)\Big) + X_t \\ &= \alpha_n(u_0) \cos\Big(n \int_{u_0}^{t/n} \omega(u)\, du\Big) + \beta_n(u_0) \sin\Big(n \int_{u_0}^{t/n} \omega(u)\, du\Big) + X_t \end{aligned} \tag{2.8}$$

with

$$\gamma^2 = \big(\alpha_n(u_0)\big)^2 + \big(\beta_n(u_0)\big)^2 \qquad \text{and} \qquad \phi_n(u_0) = -\operatorname{arctan2}\left(\beta_n(u_0), \alpha_n(u_0)\right). \tag{2.9}$$

Having estimated the frequency function $\widehat{\omega}_n(u_0)$ in (2.5) through the maximization of the periodogram we also want to estimate $\alpha_n(u_0)$ and $\beta_n(u_0)$ and finally the phase and amplitude of the process by using (2.9). Motivated by Hannan (cf. [9]) we do that by locally fitting the linear model

$$Y_{t,n} = \alpha \cos\Big(n \int_{u_0}^{t/n} \widehat{\omega}(u_0)\, du\Big) + \beta \sin\Big(n \int_{u_0}^{t/n} \widehat{\omega}(u_0)\, du\Big) + e_t \tag{2.10}$$

on our segment, i.e. for $t = n(u_0 + \epsilon_n) + s$ and $s \in \{-m_l, \ldots, m_r\}$.

We denote the local least squares-estimates by $\hat{\alpha}_n(u_0)$ and $\hat{\beta}_n(u_0)$. A formal definition of these estimates in terms of the formula $(X'X)^{-1}\, X'Y$ with X being the design-matrix is given in (A.22). For these estimates we can state

Lemma 2.1. *If the assumptions of Theorem 2.1 hold then we have for the estimator defined in (A.22):*

$$\begin{bmatrix} \hat{\alpha}_n(u_0) \\ \hat{\beta}_n(u_0) \end{bmatrix} - \begin{bmatrix} \alpha_n(u_0) \\ \beta_n(u_0) \end{bmatrix} \overset{n\to\infty}{\longrightarrow} 0 \qquad a.s.$$

where $m_l(n)$ and $m_r(n)$ are $o(n^{1/2})$ and increasing sequences.

Furthermore we set

$$\hat{\gamma} = [\{\hat{\alpha}_n(u_0)\}^2 + \{\hat{\beta}_n(u_0)\}^2]^{\frac{1}{2}} \qquad \text{and} \qquad \hat{\phi}_n(u_0) = -\operatorname{arctan2}\Big(\hat{\beta}_n(u_0), \hat{\alpha}_n(u_0)\Big). \tag{2.11}$$

Now we have

Theorem 2.3. *If the assumptions of Theorem 2.1 hold then $\hat{\gamma}$ converges to γ and $\hat{\phi}_n(u_0) - \phi_n(u_0)$ converges to 0 almost surely.*

2.3 The bias and asymptotic distributions

In this section we derive asymptotic normality and an approximate mean squared error for the local frequency estimate. The key trick for deriving these results is to approximate the original signal locally by one with constant frequency. Due to the complicated structure of the infill asymptotics the corresponding proofs turn out to be very involved.

2.3.1 The local signal approximation

Suppose we write the signal $Y_{t,n}$ from (2.3) as

$$Y_{t,N} = S_{t,n} + X_t \quad \text{with} \quad S_{t,n} = \gamma \cos\Big(n \int_{u_0}^{t/n} \omega(u)\, du + \phi_n(u_0)\Big). \tag{2.12}$$

The idea is to define a 'time homogeneous' approximation (with time-constant frequency) at each time point u_0 by

$$\widetilde{Y}_t(u_0) := \widetilde{S}_t(u_0) + X_t \quad \text{with} \quad \widetilde{S}_t(u_0) := \gamma \cos\Big(n\, \omega(u_0)\Big(\frac{t}{n} - u_0\Big) + \phi_n(u_0)\Big) \tag{2.13}$$

The two time series $Y_{t,N}$ and $\widetilde{Y}_t(u_0)$ coincide at time $t = u_0 n \in \mathbb{Z}$, i.e. $Y_{t,N} = \widetilde{Y}_t(t/n)$.

In the subsequent derivations we approximate the phase difference of the two signals by using the following Taylor-expansion.

Lemma 2.2. *Suppose $\omega(\cdot)$ is twice differentiable with Lipschitz continuous second derivative. Then we have*

$$\begin{aligned} n \int_{u_0}^{t/n} \omega(u)\,du - n\,\omega(u_0)\Big(\frac{t}{n} - u_0\Big) &= n \int_{u_0}^{t/n} \big(\omega(u) - \omega(u_0)\big)\,du \\ &= \frac{n}{2}\Big(\frac{t}{n} - u_0\Big)^2 \omega'(u_0) + \frac{n}{6}\Big(\frac{t}{n} - u_0\Big)^3 \omega''(u_0) + O\Big(n\Big(\frac{t}{n} - u_0\Big)^4\Big). \end{aligned}$$

The proof follows with a straightforward Taylor expansion.

As a consequence of Lemma 2.2 we have for the local times $t = n(u_0 + \epsilon_n) + s$ with $s \in \{-m_l, \ldots, m_r\}$ (see Remark A.2)

$$\begin{aligned} S_{n(u_0+\epsilon_n)+s,n} = \gamma \cos\Big[\frac{n}{2}\Big(\frac{s}{n}\Big)^2 \omega'(u_0) + \frac{n}{6}\Big(\frac{s}{n}\Big)^3 \omega''(u_0) + n\,\omega(u_0)\Big(\frac{s}{n} + \epsilon_n\Big) + \phi_n(u_0)\Big] \\ + O\Big(n\Big(\frac{s}{n}\Big)^4\Big) + O\Big(\frac{|s|+1}{n}\Big) \end{aligned}$$

while

$$\widetilde{S}_{n(u_0+\epsilon_n)+s,n}(u_0) = \gamma \cos\Big(n\,\omega(u_0)\Big(\frac{s}{n} + \epsilon_n\Big) + \phi_n(u_0)\Big).$$

This expansion is the basis for further bias calculations. The untypical form (with the additional factor n) is the source of many technical problems - c.f. Remark 2.5.

2.3.2 The bias and the mean squared error for the frequency estimator

The classical results on frequency estimation are results on estimating $\omega(u_0)$ from the series in (2.13) (which in this context is unobserved). However, instead of (2.13) we use the series in (2.3) which is non-homogeneous. This creates a bias which shall be investigated in the sequel.

Due to the technical problems in our calculations we restrict ourselves from now on to the case $m_r = m_\ell =: m$. Furthermore we omit n from $m(n), M(n)$ for simplicity. Beside the periodogram of the original series as defined in (2.4)

$$J_M(u_0, \lambda) := \Big|\frac{1}{M} \sum_{s=-m}^{m} Y_{n(u_0+\epsilon_n)+s,n} \exp(-i\lambda s)\Big|^2$$

we define the analogue for the approximate process by

$$\widetilde{J}_M(u_0, \lambda) := \Big|\frac{1}{M} \sum_{s=-m}^{m} \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0) \exp(-i\lambda s)\Big|^2 \tag{2.14}$$

with $M = 2m + 1$. Then the bias of the periodogram due to non-homogeneity is

$$B_M(u_0, \lambda) := J_M(u_0, \lambda) - \widetilde{J}_M(u_0, \lambda).$$

We now investigate how this bias transfers to a bias of the estimate $\widehat{\omega}_n(u_0)$. As usual the starting point of the proof of asymptotic normality is a Taylor expansion of the score function around $\omega(u_0)$. We have

$$J_M'\big(u_0, \widehat{\omega}_n(u_0)\big) - J_M'\big(u_0, \omega(u_0)\big) = \big(\widehat{\omega}_n(u_0) - \omega(u_0)\big)\, J_M''(u_0, \xi_n) \tag{2.15}$$

with $|\xi_n - \omega(u_0)| \leq |\widehat{\omega}_n(u_0) - \omega(u_0)|$. Since $J_M(u_0, \lambda)$ is periodic in λ the maximum at $\widehat{\omega}_n(u_0)$ is always a local one, i.e. $J_M'(u_0, \widehat{\omega}_n(u_0)) = 0$ leading to

$$-M^{-1/2}\, J_M'(u_0, \omega(u_0)) = \Big(\frac{1}{M^2}\, J_M''(u_0, \xi_n)\Big)\, M^{3/2}\big(\widehat{\omega}_n(u_0) - \omega(u_0)\big). \tag{2.16}$$

A key element in the proof then would be to show asymptotic normality of $J_M'\big(u_0, \omega(u_0)\big)$. Since the asymptotic properties of $\widetilde{J}_M'\big(u_0, \omega(u_0)\big)$ are well known from classical papers (see below) we only need to show that

$$M^{-1/2} B_M'(u_0, \omega(u_0)) = M^{-1/2} J_M'(u_0, \omega(u_0)) - M^{-1/2} \widetilde{J}_M'(u_0, \omega(u_0)) = o_p(1).$$

In this section we use the following assumptions:

Assumption 2.2. *The observed process $Y_{t,n}$ from (2.3) satisfies Assumption 2.1. $\omega : [0,1] \to [\delta, \pi)$ with some $\delta > 0$ is twice differentiable with Lipschitz continuous second derivative. The estimator $\widehat{\omega}_n(u_0)$ is defined by (2.5) with $0 < u_0 < 1$ and increasing sequences $m_l(n) = m_r(n) = m$ with $m = o(n^{1/2})$.*

Theorem 2.4. *Suppose Assumption 2.2 holds. Then we have with $\omega_0 := \omega(u_0)$*

$$\mathbf{E}\big(B_M'(u_0, \omega_0)\big) = \mathbf{E}\Big[J_M'(u_0, \omega_0) - \widetilde{J}_M'(u_0, \omega_0)\Big] = O_p\Big(\frac{m^2}{n}\Big)$$

and

$$\operatorname{var}\big(B_M'(u_0, \omega_0)\big) = \operatorname{var}\Big[J_M'(u_0, \omega_0) - \widetilde{J}_M'(u_0, \omega_0)\Big] = O_p\Big(\frac{m^5}{n^2}\Big).$$

As a consequence

$$MSE\,\Big(M^{-1/2} B_M'(u_0, \omega_0)\Big) = O_p\Big(\frac{m^4}{n^2}\Big)$$

which tends to zero iff $m << n^{1/2}$

Remark 2.5 (optimal MSE)**.** *We were not able to determine the optimal rate m_{opt} which minimizes the mean squared error. To indicate the reason we refer to the Taylor expansion (2.15) which implies*

$$\widehat{\omega}_n(u_0) - \omega(u_0) = J_M''(u_0, \xi_n)^{-1} J_M'(u_0, \omega_0).$$

In the proofs we were able to prove that $\frac{1}{m^2} J_M''(u_0, \xi_n) \xrightarrow{P} const$ in case $m << n^{1/2}$. On the other hand heuristic considerations and simulations indicate that $J_M''(u_0, \xi_n) << m^2$ for $m >> n^{1/2}$ leading to a smaller rate. For this reason we were not able to determine the optimal rate.

2.3.3 Asymptotic normality of the frequency estimator

To handle the second derivative in (2.16) we use again the approximation of $J_M(u_0, \lambda)$ by $\widetilde{J}_M(u_0, \lambda)$ from the last section:

Lemma 2.3. *Suppose Assumption 2.2 holds. Then we have*

$$J''_M(u_0, \xi_n) - \widetilde{J}''_M(u_0, \xi_n) = \begin{cases} o_p(m^2), & m << n^{1/2} \\ O_p(m^2), & \textit{otherwise} \end{cases} \tag{2.17}$$

for any stochastic sequence ξ_n *with* $|\xi_n - \omega(u_0)| \le |\widehat{\omega}_n(u_0) - \omega(u_0)|$.

As we mentioned before, the classical results on frequency estimation are results on estimating $\omega(u_0)$ from the series in (2.13) (which in this context is unobserved). In order to prove asymptotic normality for the series in (2.3) we use the results of Hannan (cf. [9]) who treats the case of constant frequency.

Theorem 2.6. *Suppose Assumption 2.2 holds. Then we have*

$$M(n)^{3/2}\left(\widehat{\omega}_n(u_0) - \omega(u_0)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, 24\, \frac{2\pi f\left(\omega(u_0)\right)}{\gamma^2}\Big)$$

where $f(\cdot)$ *is the spectral density of* X_t.

Proof. The Taylor expansion (2.16) yields

$$\begin{aligned} &- M^{-1/2}\left[\widetilde{J}'_M\big(u_0, \omega(u_0)\big) + B'_M\big(u_0, \omega(u_0)\big)\right] = \\ &\quad = \left[\frac{1}{M^2}\widetilde{J}''_M(u_0, \xi_n) + \frac{1}{M^2}\Big(J''_M(u_0, \xi_n) - \widetilde{J}''_M(u_0, \xi_n)\Big)\right] M^{3/2}\big(\widehat{\omega}_n(u_0) - \omega(u_0)\big) \end{aligned}$$

The assertion now follows from Theorem 2.4, Lemma 2.3 and the following classical results for time-homogeneous signals:

(i) $M^{-\frac{1}{2}}\, \widetilde{J}'_M\big(u_0, \omega(u_0)\big) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{\gamma^2}{24}\, 2\pi f\big(\omega(u_0)\big)\Big)$

(ii) $\frac{1}{M^2}\widetilde{J}''_M\big(u_0, \xi_n\big) \xrightarrow{P} -\frac{\gamma^2}{24}$

(c.f. [9, Theorem 2]; [4, Theorem 2.3]).

This establishes the result. □

2.3.4 Joined asymptotic distribution of the estimates

In this section we derive asymptotic normality of $\hat{\alpha}_n(u)$ and $\hat{\beta}_n(u)$ defined in (A.22) as well as their asymptotic covariance with $\widehat{\omega}_n(u)$. This should also serve as a tool for constructing e.g. confidence intervals for the quantities $\hat{\gamma}$ and $\hat{\phi}_n(u)$ defined in (2.11).

The starting point of the proof is a multidimensional Taylor expansion for the first derivatives of the square function

$$S_m(\alpha, \beta, \omega; u_0) := \sum_{s=-m}^{m} \left[Y_{n(u_0+\epsilon_n)+s,n} - \alpha\cos(\omega s) - \beta\sin(\omega s)\right]^2. \tag{2.18}$$

We set from now on $\epsilon_n = 0$. This is done only for reducing the complexity of the proofs. Nevertheless the results also hold if $\epsilon_n \neq 0$, although then we should use instead the function

$$\tilde{S}_m(\alpha, \beta, \omega; u_0) := \sum_{s=-m}^{m} \left[Y_{n(u_0+\epsilon_n)+s,n} - \alpha\cos(\omega s + \omega n\epsilon_n) - \beta\sin(\omega s + \omega n\epsilon_n)\right]^2.$$

Note that the estimating procedure in (A.22) is completely equivalent to minimizing $S_m(\alpha, \beta, \widehat{\omega}_n(u_0); u_0)$ over α and β. The variable ω that corresponds to the frequency of the process Y_t is also treated as a random variable, although it is not estimated through $S_m(\alpha, \beta, \omega; u_0)$ but pre-estimated through the maximization of the periodogram as described in previous sections. Now we have for some point sequence $\tilde{c}_n := (\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\omega}_n)$ between $(\hat{\alpha}_n(u_0), \hat{\beta}_n(u_0), \widehat{\omega}_n(u_0))$ and $(\alpha_n(u_0), \beta_n(u_0), \omega(u_0))$

$$\begin{aligned} -M^{-1/2}\frac{\partial S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha}\bigg|_{c_{n,0}} =& -M^{-1/2}\frac{\partial S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha}\bigg|_{\hat{c}_n} \\ &+ M^{-1}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha^2}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\alpha}_n(u_0) - \alpha_n(u_0)\Big) \\ &+ M^{-1}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha\partial\beta}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\beta}_n(u_0) - \beta_n(u_0)\Big) \\ &+ M^{-2}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha\partial\omega}\bigg|_{\tilde{c}_n} M^{3/2}\Big(\widehat{\omega}(u_0) - \omega_n(u_0)\Big) \end{aligned} \tag{2.19}$$

where $c_{n,0} = [\alpha_n(u_0), \beta_n(u_0), \omega(u_0)]$ and $\hat{c}_n = [\hat{\alpha}_n(u_0), \hat{\beta}_n(u_0), \widehat{\omega}_n(u_0)]$. Note that the first term on the right side of the equation is zero, as the function in question is maximized over α and β after it is evaluated on $\widehat{\omega}_n(u_0)$. On the other side we have

$$\begin{aligned} -M^{-1/2}\frac{\partial S_m(\alpha,\beta,\omega;u_0)}{\partial\beta}\bigg|_{c_{n,0}} =& -M^{-1/2}\frac{\partial S_m(\alpha,\beta,\omega;u_0)}{\partial\beta}\bigg|_{\hat{c}_n} \\ &+ M^{-1}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\beta^2}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\beta}_n(u_0) - \beta_n(u_0)\Big) \\ &+ M^{-1}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha\partial\beta}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\alpha}_n(u_0) - \alpha_n(u_0)\Big) \\ &+ M^{-2}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\beta\partial\omega}\bigg|_{\tilde{c}_n} M^{3/2}\Big(\widehat{\omega}(u_0) - \omega_n(u_0)\Big). \end{aligned} \tag{2.20}$$

In general we have to use a different $\tilde{c}_n$ sequence for each Taylor expansion, but since no ambiguity arises we use the same notation for avoiding unnecessary complexity.

Using (2.19) and (2.20) and combining with Theorem 2.6 we can show the following result about the joint asymptotic distribution of our estimates:

Theorem 2.7. *Suppose Assumption 2.2 holds and furthermore $m = o(n^{2/5})$. Then the vector*

$$\Big(M^{1/2}(\hat{\alpha}_n(u_0) - \alpha_n(u_0)), M^{1/2}(\hat{\beta}_n(u_0) - \beta_n(u_0)), M^{3/2}(\widehat{\omega}_n(u_0) - \omega(u_0))\Big)$$

is asymptotically normally distributed with zero mean and covariance matrix

$$2\pi f\big(\omega(u_0)\big)\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{24}{\gamma^2} \end{bmatrix}$$

where $f(\cdot)$ is the spectral density of X_t.

Comparing at this point the last result to the respective results in [9] and [18] one notes that the asymptotic covariance matrix is different in two ways: $\hat{\alpha}_n(u_0)$, $\hat{\beta}_n(u_0)$ and $\widehat{\omega}_n(u_0)$ are asymptotically independent and the variances of the two first do not depend any more on the true values α_n and β_n. As it is clear in the proof of the theorem (see Appendix A) this is due to the definition of $S_m(\alpha, \beta, \omega; u_0)$, which in the present is a summation from $-m$ to m and not from 1 to m like in the afore mentioned papers.

2.4 A simulation example

In the following we present a data simulation where the periodogram method are engaged to estimate the frequency and then the linear model described in Section 2.2.3 as well as the (approximated) Hilbert transform are applied to estimate the phase of the process. The data is generated by the model:

$$Y_{t,n} = 4\cos\underbrace{\left(\int_0^t \omega(s)dsds\right)}_{:=\phi(t)} + X_t, \quad 0 \le s < 200,$$

where $\omega(s) = 0.5 + 0.002s - 0.000005s^2$ (see Figure 2.1) and X_t is an AR(1) process with parameters $a = 0.8$ and $\sigma^2 = 1$. The simulation is repeated 300 times with the same deterministic part and different realizations of X_t.

First we estimate the frequency function by maximizing the periodogram using three data segments: $M = 25, 31$ and 41. In Figure 2.2 is shown the $MSE_\omega := \frac{1}{300}\sum_{j=1}^{300}(\widehat{\omega}_j(t) - \omega(t))^2$ in solid, dashed and dotted lines for $M = 25, 31$ and 41 respectively.

The reduction of the MSE as the data window grows can be explained heuristically as follows: the bias introduced in the segment by the signal inhomogeneity is still smaller than the reduction of the variance.

Furthermore we apply the two afore mentioned methods on the simulated data to estimate the phase process. In Figure 2.3 we show the mean squared error and the mean

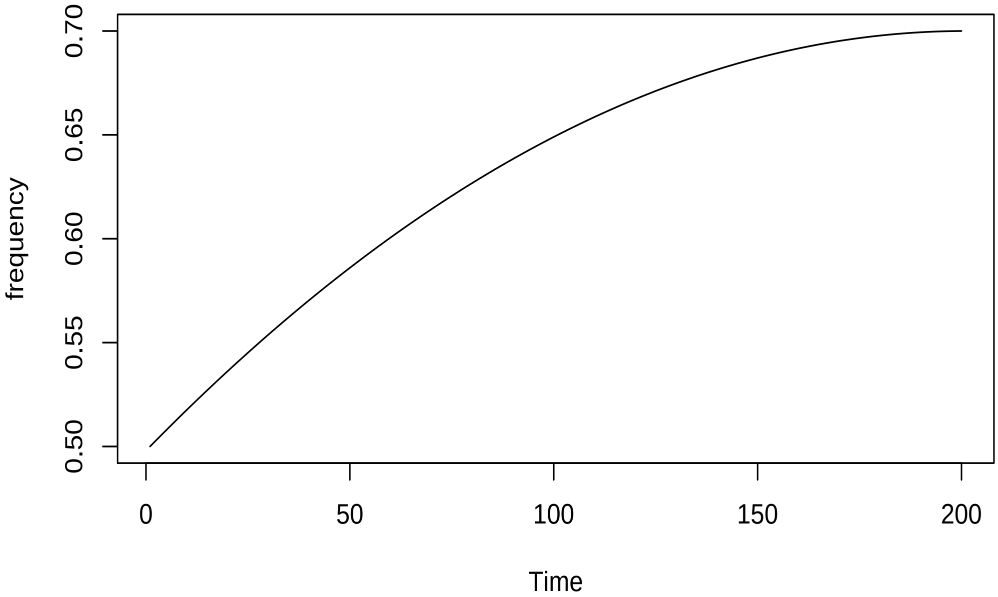

Figure 2.1: The frequency function.

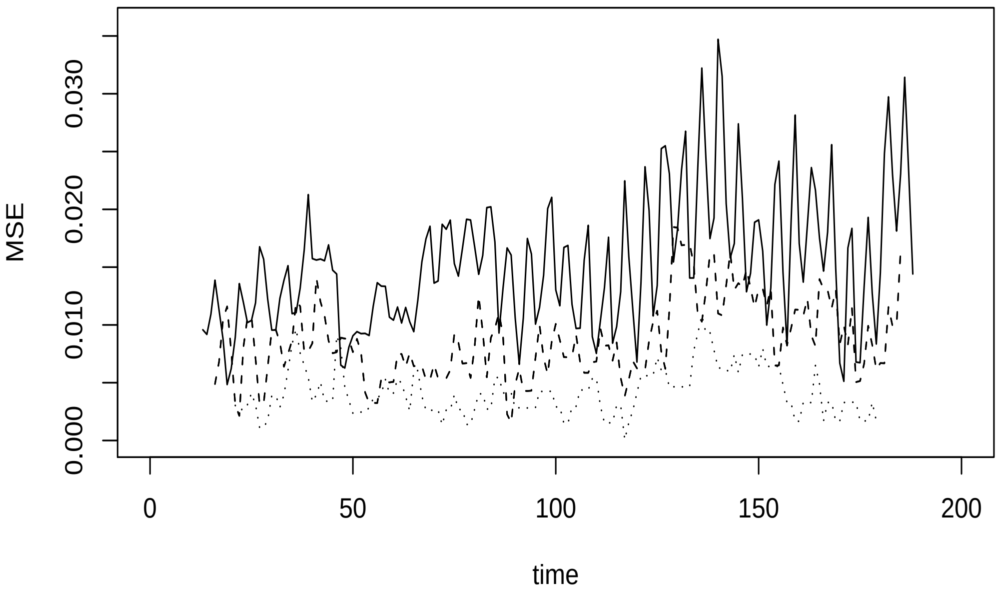

Figure 2.2: Frequency estimation MSE for $M = 25$ (solid), $M = 31$ (dashed) and $M = 41$ (dotted).

estimation error at each time point t for the two methods (solid line and cross is for the periodogram), computed as:

$$MEE_\phi(t) := \frac{1}{300}\sum_{j=1}^{300}(\hat{\phi}_j(t) - \phi(t)) \quad \text{and} \quad MSE_\phi(t) := \frac{1}{300}\sum_{j=1}^{300}(\hat{\phi}_j(t) - \phi(t))^2$$

respectively. In the case of the Hilbert transform the bias and the MSE do not seem to vary significantly for the different segment lengths. On the contrary, the estimates via the periodogram/linear model show greater bias and lower MSE the wider the time window is. Moreover we observe a radical reduce of the bias toward the right end of the data. This is due to the fact that there the frequency is almost constant and we are closer to the homogeneous case. In all cases the periodogram/linear model method seems to dominate the Hilbert transform in terms of MSE, while their bias is at comparable levels.

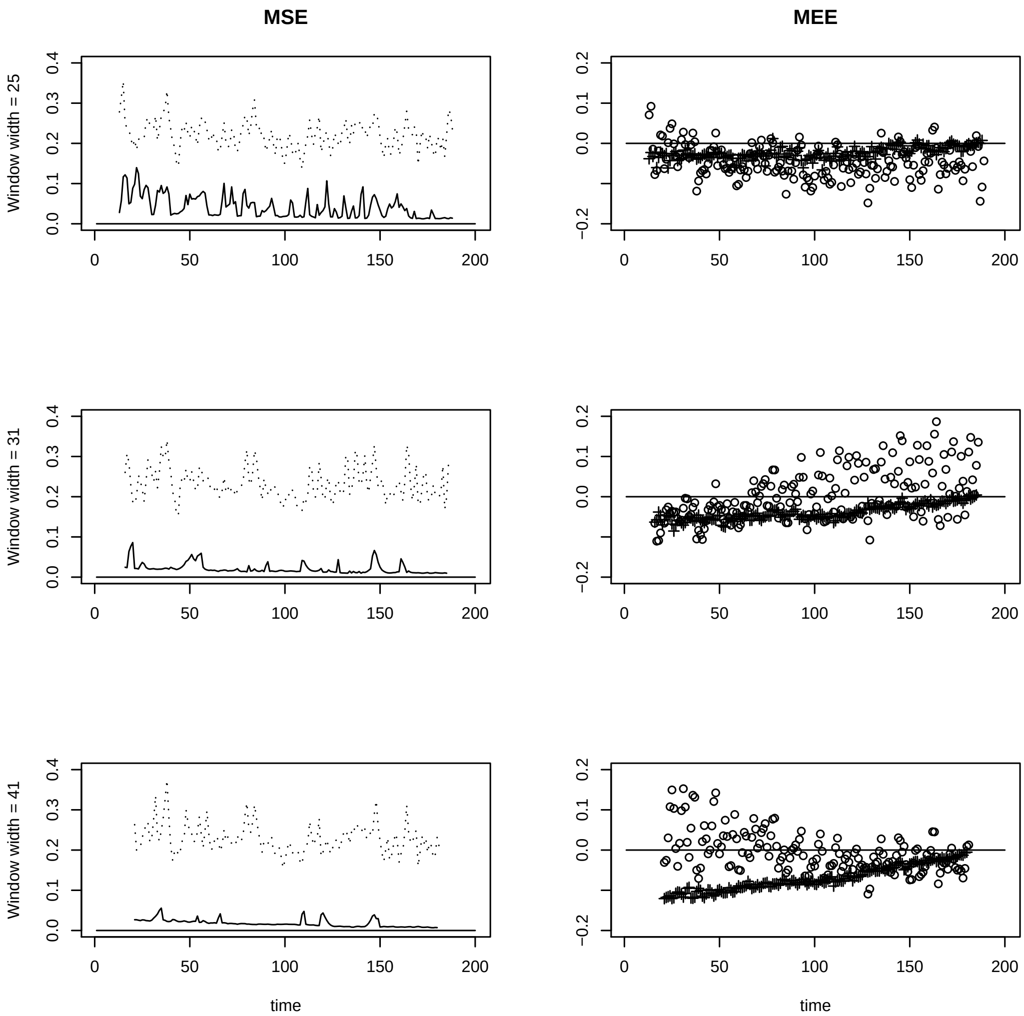

Figure 2.3: Phase estimation errors of Hilbert transform (dotted lines and circle) and Periodogram (first approach / solid lines and cross) methods.

A big improvement in the estimations can be achieved if in (2.10) we use $\widehat{\omega}(u)$ for all $u \in \{u_0 - m_l/n, ..., u_0 + m_r/n\}$ instead of the constant $\widehat{\omega}(u_0)$. Note that we do not have phase estimations for $2 \times (Window\, size)$ time points at the beginning and the end of the data set. In Figure 2.4 one can see a comparison between the two different approaches (solid line is for the old approach).

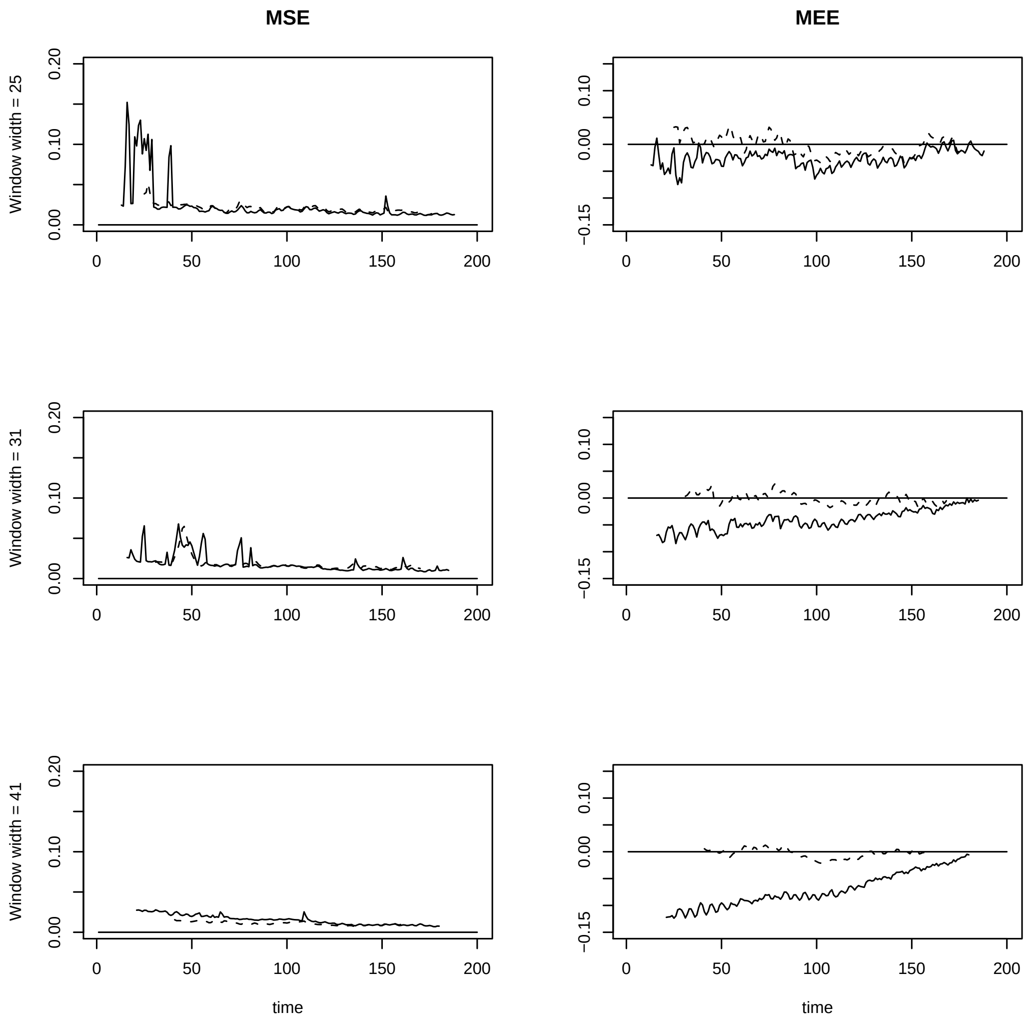

Figure 2.4: Phase estimation errors between the two approaches through linear models. Solid line is for the first and dotted line for the second approach.

Chapter 3

The modified periodogram

3.1 Introduction and motivation

As we did in the previous chapter, we generalize the classical model in time series analysis

$$Y_t = \gamma \cos(\omega t + \phi_0) + X_t, \qquad t = 1, ..., n \tag{3.1}$$

where X_t is a noise process, into the model

$$Y_t = \gamma \cos\left(\int_0^t \widetilde{\omega}(s)ds + \phi_0\right) + X_t, \qquad t = 1, ..., n \tag{3.2}$$

in order to include processes that are quasi periodic, i.e. they have time dependent frequency. We denote here the frequency function by $\widetilde{\omega}(s)$ because we use later the notation $\omega(s)$ for the rescaled function (see section 3.2.2). In general wherever we use the tilde sign it is for the same reason, if not stated otherwise. Methods that are engaged to estimate the frequency or the phase in the model (3.1) like Least Squares Estimation [18], the Hilbert transform [2, 16], the maximization of the periodogram [4, 9, 13, 14] or the "secondary analysis" [11, p. 413] are usually applied segment wise for local estimations in model (3.2) [12]. The latent assumption in these cases is that the frequency variates slowly in time and can be considered as *almost* constant within the respective segments. Heuristically, the less this assumption is satisfied, the more we have to reduce the length of the data segments to avoid additional bias. But with this we also reduce the efficiency of the estimator because of the noise term.

In the case of the maximization of the periodogram (see previous chapter), asymptotic bias is caused by the non constant frequency. To ensure good asymptotic properties, the asymptotic segment length needs to be bounded: $m(n) << n^{1/2}$. This limitation reduces the efficiency of the estimates. In [10] Katkovnik proposes the *Local Polynomial Periodogram* (LPP) by approximating the frequency (or phase) function by a polynomial. A similar method based on the same approximation is called the *Discrete Chirp Fourier Transform* [19] in the case of a time linear approximation. Indeed the same function as in

[10] is used (see below), but it is maximized over a finite set of frequencies, as the proposed method considers only frequencies and their first derivatives of the form $2\pi k/M$, where M is the sample size and $0 \leq k < M$.

To motivate the afore mentioned modification, the ordinary periodogram of a time series Y_t like in (3.1)

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^{n} Y_t \exp\{-i\lambda t\} \right|^2 \tag{3.3}$$

can be seen as some kind of "correlation" between the realization $y(1), ..., y(n)$ of Y_t and $\exp\{-i\lambda t\}$. The λ that maximizes this correlation is chosen as the estimate of the latent frequency. In the general case we have an arbitrary frequency function $\widetilde{\omega}(s)$ and not a constant one. It would be ideal if we knew its form up to some parameter $\vec{\lambda}$. In this case we could maximize the function

$$\widetilde{I}_n(\vec{\lambda}) = \frac{1}{n} \left| \sum_{t=1}^{n} Y_t \exp\left\{ i \int_0^t \widetilde{\omega}(s; \vec{\lambda}) ds \right\} \right|^2 \tag{3.4}$$

with respect to $\vec{\lambda}$ and this should provide us with an estimate of $\widetilde{\omega}(s)$. Since we don't know anything about this function we can instead work locally with its Taylor expansion around some observation time point t_0, where we want to estimate the function. First we have with $t = t_0 + \tau$

$$\exp\left\{ i \int_0^{t_0+\tau} \widetilde{\omega}(s) ds \right\} = \exp\left\{ i \int_0^{t_0} \widetilde{\omega}(s) ds \right\} \cdot \exp\left\{ i \int_{t_0}^{t_0+\tau} \widetilde{\omega}(s) ds \right\}$$

The first term of the right side is constant and drops out in (3.4). We expand $\widetilde{\omega}(s)$ in the second term:

$$\widetilde{\omega}(s) = \widetilde{\omega}(t_0) + (s - t_0)\widetilde{\omega}'(t_0) + O(...)$$

which integrated gives

$$\int_{t_0}^{t_0+\tau} \widetilde{\omega}(s) ds = \tau\widetilde{\omega}(t_0) + \frac{1}{2}\tau^2\widetilde{\omega}'(t_0) + O(...) \tag{3.5}$$

Plugging this into (3.4) we get

$$\widetilde{I}_n(t_0; \vec{\omega}) \approx \frac{1}{m_l + m_r + 1} \left| \sum_{\tau=-m_l}^{m_r} Y_{t_0+\tau} \exp\left\{ i \left[\tau\widetilde{\omega}(t_0) + \tau^2 \frac{\widetilde{\omega}'(t_0)}{2} \right] \right\} \right|^2 + O(...) \tag{3.6}$$

for proper values m_l and m_r depending on n. This motivates the definition of the *modified* periodogram:

$$\widetilde{J}_n(t_0; \lambda_0, \lambda_1) := \frac{1}{m_l + m_r + 1} \left| \sum_{\tau=-m_l}^{m_r} Y_{t_0+\tau} \exp\left\{ i \left[\tau\lambda_0 + \tau^2 \frac{\lambda_1}{2} \right] \right\} \right|^2. \tag{3.7}$$

For estimates of further derivatives (once assumed that they exist) of the frequency function, one should include more terms of the Taylor expansion and increase the dimension of the function to be maximized. Heuristically this should reduce the error term $O(...)$ in (3.5) and lead to better estimates. Note that already this Taylor expansion implies that the modified periodogram should be used locally, i.e. $m_l + m_r + 1 << n$. The above described method has some similarity to local polynomial fits in nonparametric regression.

In [10] an arbitrary high order expansion is used, but here we will restrict ourselves on the above function (3.7). Note that we use the same notation ($\widetilde{J}_n(\cdot)$ or $J_n(\cdot)$ in the rescaled time) for the modified periodogram, as for the local periodogram in the previous chapter. This is done on purpose to denote that also the common periodogram is a special case of the modified one if we take only one term of the Taylor expansion above. We could have included in the notation an index (i) implying the employed amount of Taylor terms (e.g. $\widetilde{J}_n^{(i)}(\cdot)$) but we avoided to do so to reduce the complexity in the notation.

The maximization of (3.7) over λ_0 and λ_1 delivers estimates for $\widetilde{\omega}(t_0)$ and $\widetilde{\omega}'(t_0)$. In [10], in the case of a time linear frequency function, they are shown to be strongly consistent with $\mathbf{E}\left(\widehat{\widetilde{\omega}}(t_0) - \widetilde{\omega}(t_0)\right)^2 = O(M^{-3})$ and $\mathbf{E}\left(\widehat{\widetilde{\omega}}'(t_0) - \widetilde{\omega}'(t_0)\right)^2 = O(M^{-5})$, where M is the sample size ([10], Proposition 2, Comment 2). Though, in this proposition, the following assumptions are made, in order to allow non polynomial frequency functions:

$$(a) \quad h^{s+1-p}|\omega_0^{(s)}| \text{ are small for all } s \geq p$$
$$(b) \quad h\frac{1}{(p+1)!}L_p \text{ is small,}$$

where h is the segment length for the data window (going to infinity), $(p-1) \in \mathbb{N}$ the order of polynomial approximation used (here $p-1=1$), $\omega_0^{(s)}$ the s^{th} derivative of the frequency function at t_0 and L_p some constant greater than zero. From the above it is obvious that both quantities in (a) and (b) can not remain small when h approaches infinity! Thus the assumptions can not be fulfilled and the proposition remains valid only for polynomial frequency functions. This is a common problem in the field of local estimation if the usual asymptotics is used and makes the use of the "infill asymptotics" concept inevitable. One further assumption (c) in the above paper is that the quantity

$$\frac{1}{(2m+1)^r}\sum_{t=-m}^{m} t^r \exp\left\{i\left(\omega_0 t + \frac{\omega_0'}{2}t^2\right)\right\}$$

must be bounded for $r = 0, 1$ and $m \to \infty$, where ω_0 and ω_0' are the real values of the frequency function at t_0 and its first derivative respectively. In different cases (that are easy to find, e.g. $\omega_0 = 0$ and $\omega_0' = 0.1\pi$) the convergence of the estimates can not be guaranteed, even for polynomial frequency functions. In this essay it turns out that in the frame of the "infill asymptotics" the assumptions (a), (b) and (c) are not necessary (see Section 3.4 and Lemma B.1 respectively). Finally, the results in [10] are proved for i.i.d. gaussian noise and in this study we generalize them to the case of an arbitrary stationary noise.

In the following we explicitly describe the use of the modified periodogram (i.e. LPP with two parameters) as an estimator of the frequency function. In particular in Section

3.2 we define the estimates for frequency ($\widetilde{\omega}(s)$), phase ($\phi_t := \int_0^t \widetilde{\omega}(s)ds + \phi_0$) and amplitude ($\gamma$). In Section 3.2.2 we set up again a meaningful asymptotic concept, the *infill asymptotics*, for investigating the asymptotic properties of the estimates, which is the subject of Sections 3.3 and 3.4. Finally, in Section 3.5 we apply the periodogram and the modified periodogram methods in a simulation. The proofs are postponed in Appendix B, where we also define the *modified* Fourier transform of a stationary process and derive its asymptotic distribution.

3.2 The estimates and *infill asymptotics*

In this section we introduce the estimates for the frequency function $\widetilde{\omega}(s)$, its first derivative and for the phase $\int_0^t \widetilde{\omega}(s)ds + \phi_0$ in (3.2). Furthermore we address the inefficiency of the normal asymptotics for our case and use the *infill asymptotics* concept to establish theoretical results for the estimates. We emphasize that the latter is only a means for the asymptotic investigation of the estimates and has no physical interpretation.

3.2.1 The frequency and phase estimates

We now define the estimator of the frequency function $\widetilde{\omega}(t)$ in (3.2) at a time point t_0:

$$\left(\widehat{\widetilde{\omega}}_n(t_0), \widehat{\widetilde{\omega}'}_n(t_0)\right) := \arg \sup_{\lambda_0, \lambda_1} \widetilde{J}_n(t_0; \lambda_0, \lambda_1), \tag{3.8}$$

where $\widetilde{\omega}_n'(t_0) := \frac{\partial}{\partial t}\widetilde{\omega}_n(t)$ at t_0 and $\widetilde{J}_n(t_0; \lambda_0, \lambda_1)$ is like in (3.7).

It is easy to see that the modified periodogram maintains the two basic properties of the usual periodogram, namely the periodicity and the symmetry around zero. Hence its values are repeated periodically. Just like with the normal periodogram, we don't need to maximize it in the hole $\mathbb{R}^2$. It is obvious that

$$\widetilde{J}_n(t_0; \lambda_0, \lambda_1) = \widetilde{J}_n(t_0; \lambda_0 + \kappa_0\pi, \lambda_1 + \kappa_1 2\pi)$$

for every argument (λ_0, λ_1) with κ_0, κ_1 integers and $\kappa_0 + \kappa_1 = 2K$, $K \in \mathbb{Z}$. This means that we can restrict ourselves in the subset $(\lambda_0, \lambda_1) \in [-\pi, \pi] \times [-2\pi, 2\pi]$. If now we take into account the symmetry around zero this set becomes $(\lambda_0, \lambda_1) \in [0, \pi] \times [-\pi, \pi]$ which could become even smaller if we considered cases like $\widetilde{J}_n(t_0; \pi, -\pi) = \widetilde{J}_n(t_0; 0, \pi)$, but we do not go into further details referring to such extrema-situations.
The model (3.2) can be written for any t_0 between 1 and n and $\tau = t - t_0$

$$\begin{aligned} Y_{t_0+\tau} = Y_t &= \gamma \cos\left(\int_0^{t_0} \widetilde{\omega}(s)ds + \int_{t_0}^{t_0+\tau} \widetilde{\omega}(s)ds + \phi_0\right) + X_t \\ &= \widetilde{\alpha}(t_0)\cos\left(\int_{t_0}^{t_0+\tau} \widetilde{\omega}(s)ds\right) + \widetilde{\beta}(t_0)\sin\left(\int_{t_0}^{t_0+\tau} \widetilde{\omega}(s)ds\right) + X_t \end{aligned} \tag{3.9}$$

with

$$\widetilde{\alpha}(t_0) = \gamma \cos\left(\int_0^{t_0} \widetilde{\omega}(s)ds + \phi_0\right) \quad \text{and} \quad \widetilde{\beta}(t_0) = -\gamma \sin\left(\int_0^{t_0} \widetilde{\omega}(s)ds + \phi_0\right).$$

Thus we see directly that

$$\gamma^2 = \{\widetilde{\alpha}(t_0)\}^2 + \{\widetilde{\beta}(t_0)\}^2 \qquad \text{and} \qquad \widetilde{\phi}_{t_0} := \int_0^t \widetilde{\omega}(s)ds + \phi_0 = -\operatorname{arctan2}\left(\widetilde{\beta}(t_0), \widetilde{\alpha}(t_0)\right) \tag{3.10}$$

Once we expressed the process Y_t from the "point of view" of a particular t_0 we proceed to the estimation of γ and $\widetilde{\phi}_{t_0}$ through estimating $\widetilde{\alpha}(t_0)$ and $\widetilde{\beta}(t_0)$ and plugging these into (3.10). Motivated by Hannan [9] that uses in the time homogeneous case a representation like in (3.9), we use, instead of the unknown $\widetilde{\omega}(s)$, the estimates (3.8) of the two first terms of the Taylor expansion in (3.5) and fit the following linear model

$$\begin{aligned}
Y_t &= \widetilde{\alpha}(t_0) \cos\left(\int_{t_0}^{t_0+\tau} \left[\widehat{\widetilde{\omega}}_n(t_0) + \widehat{\widetilde{\omega}'}_n(t_0)(s-t_0)\right] ds\right) \\
&\qquad + \widetilde{\beta}(t_0) \sin\left(\int_{t_0}^{t_0+\tau} \left[\widehat{\widetilde{\omega}}_n(t_0) + \widehat{\widetilde{\omega}'}_n(t_0)(s-t_0)\right] ds\right) + e_t \\
&= \widetilde{\alpha}(t_0) \cos\left(\tau\widehat{\widetilde{\omega}}_n(t_0) + \frac{1}{2}\tau^2\widehat{\widetilde{\omega}'}_n(t_0)\right) + \widetilde{\beta}(t_0) \sin\left(\tau\widehat{\widetilde{\omega}}_n(t_0) + \frac{1}{2}\tau^2\widehat{\widetilde{\omega}'}_n(t_0)\right) + e_t
\end{aligned} \tag{3.11}$$

which leads to least squares estimates $\widehat{\widetilde{\alpha}}(t_0)$ and $\widehat{\widetilde{\beta}}(t_0)$ for $\widetilde{\alpha}(t_0)$ and $\widetilde{\beta}(t_0)$ respectively. Finally we define

$$\hat{\gamma} := \sqrt{\left\{\widehat{\widetilde{\alpha}}(t_0)\right\}^2 + \left\{\widehat{\widetilde{\beta}}(t_0)\right\}^2} \qquad \text{and} \qquad \widehat{\widetilde{\phi}}_{t_0} := -\operatorname{arctan2}\left(\widehat{\widetilde{\beta}}(t_0), \widehat{\widetilde{\alpha}}(t_0)\right) \tag{3.12}$$

as estimates for amplitude and phase respectively.

3.2.2 Infill asymptotics

After defining the estimates, we want to investigate their asymptotic properties and in particular their consistency and asymptotic normality. However, in the above setting of model (3.2) the simple asymptotics $t \to \infty$ is, for the reasons explained in the previous chapter, meaningless. The solution is found again in the frame of infill asymptotics with the rescaling of time in the unit interval. We set

$$\int_0^t \widetilde{\omega}(s)\,ds = n\int_0^{t/n} \widetilde{\omega}(un)\,du =: n\int_0^{t/n} \omega(u)\,du$$

i.e. we use instead of (3.2) the model

$$Y_{t,n} = \gamma \cos\left(n\int_0^{t/n} \omega(u)\,du + \phi_0\right) + X_t$$

with a fixed function $\omega(u) : [0,1] \to [0,\pi)$ (see also Chapter 2). If we repeat now the steps (3.5) and (3.6) for the rescaled frequency function $\omega(s)$ we get the following definition of the periodogram in the "infill asymptotics" frame:

$$J_n(u_0;\lambda_0,\lambda_1) := \left| \frac{1}{M(n)} \sum_{s=-m_\ell(n)}^{m_r(n)} Y_{n(u_0+\epsilon_n)+s,n} \exp\left\{ -i\left(\lambda_0 s + \frac{\lambda_1}{2n}s^2\right)\right\}\right|^2 \tag{3.13}$$

with $M(n) = m_l(n) + m_r(n) + 1$ increasing sequences of integers (if not necessary we omit from now on the argument n). ϵ_n fills the "gap" between u_0 and the next t/n point, i.e. $\epsilon_n := \min_{t:\, t \geq nu_0}\{t/n - u_0\}$. Thus, we use m_l observations on the left side and $m_r + 1$ on the right side of u_0 to calculate the modified periodogram $J_n(u_0;\lambda_0,\lambda_1)$ that corresponds to u_0 (i.e. $-m_l \leq s \leq m_r$). It is usual, but not necessary, that $m_l = m_r$. The estimates are then defined like in (3.8) by maximizing (3.13) over λ_0, λ_1. Furthermore (3.9) takes the form (see also (2.8))

$$Y_{n(u_0+\epsilon_n)+s,n} =$$
$$\alpha_n(u_0)\cos\left(n\int_{u_0}^{u_0+\epsilon_n+s/n}\omega(s)ds\right) + \beta_n(u_0)\sin\left(n\int_{u_0}^{u_0+\epsilon_n+s/n}\omega(s)ds\right) + X_t \tag{3.14}$$

with $t = n(u_0+\epsilon_n)+s$ and

$$\alpha_n(u_0) = \gamma\cos\left(n\int_0^{u_0}\omega(s)ds + \phi_0\right) \quad \text{and} \quad \beta_n(u_0) = -\gamma\sin\left(n\int_0^{u_0}\omega(s)ds + \phi_0\right),$$

where s takes proper values in $\mathbb{Z}$ and ϵ_n is as above. Note that ϵ_n is of order $O(1/n)$ as this "gap" is becoming smaller and smaller for $n \to \infty$. Finally, (3.11) becomes

$$Y_{n(u_0+\epsilon_n)+s,n} =$$
$$\alpha_n(u_0)\cos\left(s\widehat{\omega}_n(u_0) + \frac{1}{2n}s^2\widehat{\omega'}_n(u_0)\right) + \beta_n(u_0)\sin\left(s\widehat{\omega}_n(u_0) + \frac{1}{2n}s^2\widehat{\omega'}_n(u_0)\right) + e_t, \tag{3.15}$$

which provides us with the infill asymptotics version of the least squares estimates for $\alpha_n(u_0)$ and $\beta_n(u_0)$ and we define

$$\hat{\gamma} := \sqrt{\{\hat{\alpha}_n(u_0)\}^2 + \left\{\hat{\beta}_n(u_0)\right\}^2} \quad \text{and} \quad \hat{\phi}_{n,u_0} := -\arctan\left(\hat{\beta}_n(u_0)/\hat{\alpha}_n(u_0)\right). \tag{3.16}$$

Note that the amplitude γ is not affected by the infill asymptotics frame, as it is assumed to be constant in time.

3.3 Consistency of the estimates

In this section we prove consistency of the two estimates described in Section 3.2.1 within the frame of infill asymptotics. For the consistency of $[\widehat{\omega}(u), \widehat{\omega}'(u)]$ at a certain $0 \leq u_0 \leq 1$ we also provide some kind of joined rate (see Theorem 3.2) that is needed for further proofs.

3.3.1 Consistency of the frequency function estimator

We define the frequency function estimator for a series like in (2.3):

$$[\widehat{\omega}_n(u_0), \widehat{\omega}'_n(u_0)] = \arg\sup_{\lambda_0,\lambda_1} J_n(u_0; \lambda_0, \lambda_1) \tag{3.17}$$

where $J_n(u_0; \lambda_0, \lambda_1)$ is the modified periodogram in (3.13).

Before we go on to the consistency in the infill asymptotics sense, we want to justify the quantity $n^p, 1/2 < p < 2/3$, that appears in this theorem and refers to the length sequence of the data segment to be used while applying the modified periodogram method. If we suppose that the frequency function $\omega(u), u \in [0, 1]$, is twice differentiable in some interval $u \in [u_0 - r, u_0 + r]$, we can use the Taylor theorem and get:

$$\omega(u) = \omega(u_0) + \omega'(u_0)(u - u_0) + R(u), \qquad \text{with } |R(u)| \leq M\frac{r^2}{2}, M < \infty. \tag{3.18}$$

The phase difference between u_0 and some $u = u_0 + s, |s| \leq r$ is:

$$\begin{aligned} \phi(u_0) - \phi(u) &= n\int_{u_0}^{u_0+s} \omega(x)dx \\ &= n\int_{u_0}^{u_0+s} \big[\omega(u_0) + \omega'(u_0)(x - u_0) + R(x)\big]\, dx. \end{aligned}$$

We focus now on the remaining term (for simplicity suppose $s > 0$):

$$\begin{aligned} \left| n\int_{u_0}^{u_0+s} R(x)dx \right| &\leq n\int_{u_0}^{u_0+s} |R(x)|\, dx \\ &\leq n\int_{u_0}^{u_0+s} M\frac{r^2}{2}dx \leq M\frac{nr^3}{2}. \end{aligned} \tag{3.19}$$

The quantity r is exactly the width of the time window and is defined, for every n, by the ratio m_n/n. This means that the above computed integral goes to zero for all $u \in [u_0 - r, u_0 + r]$ if we choose $m_n \sim n^p$ with $p < 2/3$. This -as it is clear from the consistency proof- eliminates asymptotically the effect of all Taylor coefficients of the $\omega(u)$-expansion that we do not wish to estimate, namely all derivatives greater than the first. Note that if the frequency function is locally linear $R(x)$ vanishes from some n_0 on and we can use every $p < 1$. On the other hand, if we chose $p < 1/2$, this would also eliminate the effect of the first derivative, and in this case we were not able to prove consistency of $\widehat{\omega}'_n(u_0)$. Thus, for applying the modified periodogram with two parameters, $1/2 < p < 2/3$ (or $1/2 < p < 1$ in case of polynomials of second degree) seems to be the proper choice.

Theorem 3.1. *Let $Y_{t,n}$ be as in (2.3), X_t be stationary with zero mean and satisfying (B.2) and $\omega(u) : [0, 1] \to [0, \pi)$ be twice differentiable with finite derivatives. Then we have for the estimates in (3.17), for every $u_0 \in [0, 1]$*

$$\lim_{n\to\infty} [\widehat{\omega}_n(u_0), \widehat{\omega}'_n(u_0)] = [\omega(u_0)\ (mod\ 2\pi), \omega'(u_0)] \qquad a.s.$$

where $m_l, m_r = O(n^p)$, $1/2 < p < 2/3$ and λ_1 is bounded. Moreover, if the frequency function is locally linear, the theorem holds for $1/2 < p < 1$.

3.3.2 Consistency of the phase estimator

Having estimated the frequency function in (2.3), (3.14) through the maximization of the modified periodogram we also want to estimate $\alpha_n(u_0)$ and $\beta_n(u_0)$ in (3.14) and finally the phase and amplitude of the process in (2.3). This is done as described in Section 3.2 by fitting the linear model in (3.15), with s running from $-m_l$ to m_r, $m_l, m_r = O(n^p)$, $1/2 < p < 2/3$. The estimated frequency function in a segment is the linear approximation $\widehat{\omega}_n(u) = \widehat{\omega}_n(u_0) + \widehat{\omega}'_n(u_0)(u - u_0)$. Before we prove consistency of $\hat{\alpha}_n(u_0)$ and $\hat{\beta}_n(u_0)$, we need the following result, which gives a first coarse rate of convergence of the frequency function estimates. It is obtained by a modification of the proof of Theorem 3.1.

Theorem 3.2. *Under the assumptions of Theorem 3.1 and for the estimates of the same theorem we have that*

$$\lim_{n\to\infty} \sup_{1\le t\le m} \left| (\widehat{\omega}_n(u_0) - \omega(u_0))t + \frac{\widehat{\omega}'_n(u_0) - \omega'(u_0)}{2n} t^2 \right| = 0, \quad a.s. \tag{3.20}$$

with $m = O(n^p)$, $1/2 < p < 2/3$ or $1/2 < p < 1$ in the case of a locally (around u_0) linear frequency function $\omega(u)$.

We can now state for the estimates of $\alpha_n(u_0)$ and $\beta_n(u_0)$ in (3.14), as they are described at the beginning of this section, the following lemma (consistency):

Lemma 3.1. *If assumptions of Theorem 3.1 hold then we have for the estimator derived by the linear model in (3.11) with*

$$\hat{\phi}_{s,n}^{(u_0)} = n \int_{u_0}^{u_0 + s/n + \epsilon_n} \Big[\widehat{\omega}(u_0) + \widehat{\omega}'(u_0)(u - u_0)\Big] du, \quad -m_l \le s \le m_r$$

where $m_l, m_r = O(n^p)$, p like in Theorem 3.2, $\widehat{\omega}(u_0), \widehat{\omega}'(u_0)$ are as in (3.17) and ϵ_n as in (3.14):

$$\begin{bmatrix} \hat{\alpha}_n(u_0) \\ \hat{\beta}_n(u_0) \end{bmatrix} - \begin{bmatrix} \alpha_n(u_0) \\ \beta_n(u_0) \end{bmatrix} \overset{n\to\infty}{\longrightarrow} 0 \qquad a.s.$$

If we use now the representation (2.3) of the process we can state the following theorem

Theorem 3.3. *Let assumptions of Theorem 3.1 hold. For every continuity point $u_0 \in [0, 1]$ for the same increasing sequences $m_l(n)$ and $m_r(n)$, $n \in \mathbb{N}$ of integers defined in Lemma 3.1 we have for the estimates:*

$$\hat{\gamma} = [\{\hat{\alpha}_n(u_0)\}^2 + \{\hat{\beta}_n(u_0)\}^2]^{\frac{1}{2}} \qquad and \qquad \hat{\phi}_n(u_0) = -\arctan2\left(\hat{\beta}_n(u_0), \hat{\alpha}_n(u_0)\right),$$

$$\hat{\gamma} \to \gamma \quad a.s.$$

and

$$\hat{\phi}_n(u_0) - \phi_n(u_0) \to 0 \quad a.s.$$

3.4 Mean squared error and asymptotic normality

In the present section we investigate the asymptotic Mean Squared Error of the frequency estimate depending on the segment length $m(n)$. This is made approximating the original signal by one with time linear frequency function and then evaluating the bias that results due to this approximation. The latter leads to an asymptotic normality theorem for the frequency function estimates.

3.4.1 The signal approximation

In order to prove asymptotic normality and determine the asymptotic MSE we define a signal approximation to the original one (compare also Chapter 2). The idea is again to prove the desired results for the approximation and then show that the bias terms caused by it converge to zero. We have (from now on we set $m_l = m_r = m$):

$$\check{Y}_{n(u_0+\epsilon_n)+s,n}(u_0) := \check{S}_{n(u_0+\epsilon_n)+s,n}(u_0) + X_{n(u_0+\epsilon_n)+s} \tag{3.21}$$

with

$$\check{S}_{n(u_0+\epsilon_n)+s,n} := \gamma \cos\left[\omega(u_0)s + \frac{\omega'(u_0)}{2}\frac{(s+n\epsilon_n)^2}{n} + \omega(u_0)n\epsilon_n + \phi_{u_0}\right]$$

and finally the modified periodogram of the approximation

$$\check{J}_M(u_0,\lambda) := \left|\frac{1}{M_n}\sum_{s=-m_n}^{m_n} \check{Y}_{n(u_0+\epsilon_n)+s,n}(u_0)\exp\left\{-i\left(\lambda_0 s + \frac{\lambda_1}{2}\frac{s^2}{n}\right)\right\}\right|^2 \tag{3.22}$$

where $M_n = 2m_n + 1$. Note that $\check{J}_M(u_0,\lambda)$ cannot be calculated from the original data. Now we can prove two lemmas on which the proof of the asymptotic normality is based.

Lemma 3.2. *Let $X(t)$ be stationary and satisfy (B.2). Furthermore let $\check{J}_M(u_0,\lambda)$ and $\check{Y}_{n(u_0+\epsilon_n)+s,n}(u_0)$ be like in (3.22, 3.21) with $m_n = O(n^p)$, $1/2 < p < 2/3$. Then*

$$-\begin{pmatrix} 1 & 0 \\ 0 & n/M_n \end{pmatrix} M_n^{-1/2}\,\nabla \check{J}_M(u_0;\omega(u_0),\omega'(u_0)) \xrightarrow{\mathfrak{D}} \mathcal{N}_2(\mathbf{0},\Sigma)$$

$$\textit{with}\quad \Sigma = \begin{pmatrix} 2\pi\frac{\gamma^2}{24} f_{XX}(\omega_0) & 0 \\ 0 & 2\pi\frac{\gamma^2}{240} f_{XX}(\omega_0) \end{pmatrix},$$

where $f_{XX}(\lambda)$ is the spectral density of X_t.

Note that the result of the theorem would hold for all $1/2 < p < 1$ if $\epsilon_n = 0$. This is because the term $\exp\{i\omega_0' s\epsilon_n\}$ in (B.25) would be equal to unity.

The next lemma refers to the second derivative of the modified periodogram. Note that it holds not only for the signal approximation:

Lemma 3.3. *Let $X(t)$ be stationary and satisfy (B.2). Furthermore let $m_n = O(n^p)$, $1/2 < p < 2/3$. If*

$$\lim_{n\to\infty} \sup_{1\le t\le m} \left| (\xi_{0,n} - \omega(u_0))t + \frac{\xi_{1,n} - \omega'(u_0)}{2n} t^2 \right| = 0. \tag{3.23}$$

then

$$\lim_{n\to\infty} M_n^{-2} \begin{pmatrix} \frac{\partial^2}{\partial\lambda_0^2} J_M(u_0;\lambda_0,\xi_{1,n})\Big|_{\lambda_0=\xi_{0,n}} \\ \frac{n^2}{M_n^2}\frac{\partial^2}{\partial\lambda_1^2} J_M(u_0;\xi_{0,n},\lambda_1)\Big|_{\lambda_1=\xi_{1,n}} \\ \frac{n}{M_n}\frac{\partial^2}{\partial\lambda_0\partial\lambda_1} J_M(u_0;\lambda_0,\lambda_1)\Big|_{\substack{\lambda_0=\xi_{0,n}\\ \lambda_1=\xi_{1,n}}} \end{pmatrix} = \begin{pmatrix} -\frac{\gamma^2}{24} \\ -\frac{1}{60}\frac{\gamma^2}{24} \\ 0 \end{pmatrix} \quad a.s.$$

Note again that:
(i) for time linear frequency functions (i.e. if we use the signal approximation in (3.22) instead of the real signal) the terms $O\left(\frac{m^3}{n^2}\right)$ disappear from all equations in the proof
(ii) the terms $\omega_0' s\epsilon_n$ vanish if we consider again estimating the frequency function only on observation points.
Under those two conditions the results of Lemma 3.3 hold for $1/2 < p < 1$.

Now we are ready to state the central limit theorem for the approximation (3.21).

Theorem 3.4. *Under the assumptions of Theorem 3.1 and (B.2) and if $\omega(u)$ is linear in time, then we have for the estimates*

$$(\widehat{\omega}_{0,n}, \widehat{\omega}'_{0,n}) = \arg\sup_{\lambda_0,\lambda_1} \check{J}_{M_n}(u_0;\lambda_0,\lambda_1)$$

with $M_n \sim \lfloor n^p \rfloor$, $1/2 < p < 2/3$:

$$\begin{pmatrix} 1 & 0 \\ 0 & M_n/n \end{pmatrix} M_n^{3/2} \begin{pmatrix} \widehat{\omega}_{0,n} - \omega_0 \\ \widehat{\omega}'_{0,n} - \omega_0' \end{pmatrix} \xrightarrow{\mathfrak{D}} \mathcal{N}_2(\mathbf{0}, \Sigma)$$

$$\text{with} \quad \Sigma = \begin{pmatrix} 2\pi\frac{24}{\gamma^2} f_{XX}(\omega_0) & 0 \\ 0 & 360\cdot 2\pi\frac{24}{\gamma^2} f_{XX}(\omega_0) \end{pmatrix}.$$

The result of the previous theorem holds for $1/2 < p < 1$ under the condition (ii) after Lemma 3.3.

3.4.2 MSE and asymptotic normality of the frequency estimate

Using the results from Section 2.3.1 we have

$$S_{n(u_0+\epsilon_n)+s,n} = \gamma\cos\big(a+c+d\big) + O\Big(n\Big(\frac{s}{n}\Big)^3\Big) + O\Big(\frac{|s|+1}{n}\Big) \tag{3.24}$$

$$\check{S}_{n(u_0+\epsilon_n)+s,n} = \gamma\cos\big(a+c+d\big) + O\Big(\frac{|s|+1}{n}\Big)$$

with

$$a := \frac{n}{2}\Big(\frac{s}{n}\Big)^2 \omega'(u_0), \quad c := \omega(u_0)\, s, \quad d := \phi_{u_0} + \omega(u_0)\, n\epsilon_n.$$

We have the results on estimating $\omega(u_0)$ and $\omega'(u_0)$ from the series in (3.21), which is an approximation of the series in (2.3). This creates a bias in the modified periodogram:

$$\check{B}_M(u_0, \lambda_0, \lambda_1) := J_M(u_0, \lambda_0, \lambda_1) - \check{J}_M(u_0, \lambda_0, \lambda_1).$$

where

$$J_M(u_0, \lambda_0, \lambda_1) := \Big|\frac{1}{M}\sum_{s=-m_\ell}^{m_r} Y_{n(u_0+\epsilon_n)+s,n} \exp\left\{-i\left(\lambda_0 s + \frac{\lambda_1}{2}\frac{s^2}{n}\right)\right\}\Big|^2$$

and

$$\check{J}_M(u_0, \lambda_0, \lambda_1) := \Big|\frac{1}{M}\sum_{s=-m_\ell}^{m_r} \check{Y}_{n(u_0+\epsilon_n)+s}(u_0) \exp\left\{-i\left(\lambda_0 s + \frac{\lambda_1}{2}\frac{s^2}{n}\right)\right\}\Big|^2$$

with $M = m_r + m_\ell + 1$ (we restrict ourselves to the case $m_r = m_\ell =: m$).

In the following we derive the asymptotic MSE and eventually the central limit theorem for $(\widehat{\omega}_n(u_0) - \omega(u_0))$ and $(\widehat{\omega}'_n(u_0) - \omega'(u_0))$. For this we will use the results on the approximation signal (3.21) evaluating the bias in the estimations that would be caused if we used the latter instead of the original signal.

MSE for $(\widehat{\omega}_n(u_0) - \omega(u_0))$:

Here all derivatives are with respect to λ_0. The starting point for both the derivation of the MSE and the asymptotic normality is the following Taylor expansion (derived from (B.37) and (B.38))

$$-\frac{1}{M^{1/2}}\frac{\partial J_M(u_0;\lambda_0,\omega'_0)}{\partial\lambda_0}\Big|_{\omega_0} = \left(\frac{1}{M^2}\frac{\partial^2 J_M(u_0;\lambda_0,\xi_{1,n})}{\partial\lambda_0^2}\Big|_{\xi_{0,n}} + o(1)\right) M^{3/2}(\widehat{\omega}_0 - \omega_0). \tag{3.25}$$

The following lemma refers to the bias caused to the first derivative due to the signal approximation. When this bias goes to zero the first derivative of the periodogram of the original signal will have the same asymptotic behavior as its approximation.

Lemma 3.4. *Under the assumptions of Theorem 3.1 we have for $n^{1/2} < m < n^{2/3}$*

$$MSE\left(\frac{1}{M^{1/2}}\frac{\partial\check{B}_M(u_0,\lambda_0,\omega'_0)}{\partial\lambda_0}\Big|_{\lambda_0=\omega_0}\right) = O\left(\frac{m^7}{n^4}\right).$$

Remark 3.5. *The MSE $\big(M^{-\frac{1}{2}}\check{B}'_M(u_0,\omega_0,\omega'_0)\big)$ tends to 0 if $m << n^{4/7}$, leading to a central limit theorem for $M^{-1/2}J'_M\big(u_0,\omega_0,\omega'_0\big)$ via the c.l.t. for $M^{-1/2}\check{J}'_M(u_0,\omega_0,\omega'_0)$.*

MSE for $(\widehat{\omega}'_n(u_0) - \omega'(u_0))$

Now the derivatives are with respect to λ_1. The starting point is the following Taylor expansion (compare to (B.38))

$$-\frac{n}{M^{3/2}}\frac{\partial J_M(u_0;\omega_0,\lambda_1)}{\partial\lambda_1}\Bigg|_{\omega_0'} = \left(\frac{n^2}{M^4}\frac{\partial^2 J_M(u_0;\xi_{0,n},\lambda_1)}{\partial\lambda_1^2}\Bigg|_{\xi_{1,n}} + o(1)\right)\frac{M^{5/2}}{n}(\widehat{\omega}_0' - \omega_0'). \tag{3.26}$$

Lemma 3.5. *Under the assumptions of Theorem 3.1 we have for $n^{1/2} < m < n^{2/3}$*

$$MSE\left(\frac{n}{M}\frac{1}{M^{1/2}}\frac{\partial\check{B}_M(u_0,\lambda_0,\lambda_1)}{\partial\lambda_1}\Big|_{\substack{\lambda_0=\omega_0\\ \lambda_1=\omega_0'}}\right) = O\left(\frac{m^7}{n^4}\right).$$

Remark 3.6. *The MSE $\left(\frac{n}{M}M^{-1/2}\check{B}_M'(u_0,\omega_0,\omega_0')\right)$ tends to 0 if $m << n^{4/7}$, leading to a central limit theorem for $\frac{n}{M}M^{-1/2}J_M'(u_0,\omega_0,\omega_0')$ via the c.l.t. for $\frac{n}{M}M^{-1/2}\check{J}_M'(u_0,\omega_0,\omega_0')$.*

Asymptotic normality

We can now state

Theorem 3.7. *Under the assumptions of Theorem 3.1 we have for the estimates*

$$(\widehat{\omega}_n(u_0), \widehat{\omega}_n'(u_0)) = \arg\sup_{\lambda_0,\lambda_1} J_{M_n}(u_0;\lambda_0,\lambda_1)$$

with $M_n \sim \lfloor n^p \rfloor$, $1/2 < p < 4/7$:

$$\begin{pmatrix} 1 & 0 \\ 0 & M_n/n \end{pmatrix} M_n^{3/2} \begin{pmatrix} \widehat{\omega}_n(u_0) - \omega_0 \\ \widehat{\omega}_n'(u_0) - \omega_0' \end{pmatrix} \xrightarrow{\mathfrak{D}} \mathcal{N}_2(\mathbf{0},\Sigma)$$

$$\text{with} \quad \Sigma = \begin{pmatrix} 2\pi\frac{24}{\gamma^2}f_{XX}(\omega_0) & 0 \\ 0 & 360\cdot 2\pi\frac{24}{\gamma^2}f_{XX}(\omega_0) \end{pmatrix}.$$

3.4.3 Joined asymptotic distribution of the estimates

In this section we derive asymptotic normality of $\hat{\alpha}_n(u)$ and $\hat{\beta}_n(u)$ of Lemma 3.1, as well as their asymptotic covariance with $\widehat{\omega}_n(u)$ and $\widehat{\omega}_n'(u)$. This should also serve as a tool for constructing e.g. confidence intervals for $\hat{\gamma}$ and $\hat{\phi}_n(u)$ defined in Theorem 3.3.

The starting point of the proof is a multidimensional Taylor expansion for the first derivatives of the square function

$$\bar{S}_m(\alpha,\beta,\lambda_0,\lambda_1;u_0) := \sum_{-m}^{m}\left[Y_{n(u_0+\epsilon_n)+s,n} - \alpha\cos\left(\lambda_0 t + \frac{\lambda_1}{2}\frac{t^2}{n}\right) - \beta\sin\left(\lambda_0 t + \frac{\lambda_1}{2}\frac{t^2}{n}\right)\right]^2 \tag{3.27}$$

We set from now on $\epsilon_n = 0$. This is done only for reducing the complexity of the proofs. Nevertheless the results also hold if $\epsilon_n \neq 0$, although then t should be replaced by $t+$

$n\epsilon_n$ in the above function. Note that the estimating procedure in (B.15) is completely equivalent to minimizing the function $\bar{S}_m(\alpha, \beta, \widehat{\omega}_n(u_0), \widehat{\omega}'_n(u_0); u_0)$ over α and β. The variables λ_0 and λ_1 that refer to the frequency function of the process Y_t are also treated as random variables, although they are not estimated through $\bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)$ but pre-estimated through the maximization of the modified periodogram as described in previous sections. Now we have for some point sequence $\tilde{c}_n := (\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\omega}_{0,n}, \tilde{\omega}_{1,n})$ between $\hat{c}_{n,0} := (\hat{\alpha}_n(u_0), \hat{\beta}_n(u_0), \widehat{\omega}_n(u_0), \widehat{\omega}'_n(u_0))$ and $c_n := (\alpha_n(u_0), \beta_n(u_0), \omega(u_0), \omega'(u_0))$

$$\begin{aligned} -M^{-1/2} \frac{\partial \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \alpha}\bigg|_{c_{n,0}} &= -M^{-1/2} \frac{\partial \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \alpha}\bigg|_{\hat{c}_n} \\ &+ M^{-1} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \alpha^2}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\alpha}_n(u_0) - \alpha_n(u_0)\Big) \\ &+ M^{-1} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \alpha \partial \beta}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\beta}_n(u_0) - \beta_n(u_0)\Big) \\ &+ M^{-2} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \alpha \partial \lambda_0}\bigg|_{\tilde{c}_n} M^{3/2}\Big(\widehat{\omega}(u_0) - \omega_n(u_0)\Big) \\ &+ nM^{-3} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \alpha \partial \lambda_1}\bigg|_{\tilde{c}_n} \frac{M^{5/2}}{n}\Big(\widehat{\omega}'(u_0) - \omega'_n(u_0)\Big) \end{aligned} \tag{3.28}$$

Note that the first term on the right side of the equation is zero, as the function in question is maximized over α and β AFTER it is evaluated on $\widehat{\omega}_n(u_0)$ and $\widehat{\omega}'_n(u_0)$. On the other side we have

$$\begin{aligned} -M^{-1/2} \frac{\partial \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \beta}\bigg|_{c_{n,0}} &= -M^{-1/2} \frac{\partial \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \beta}\bigg|_{\hat{c}_n} \\ &+ M^{-1} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \beta^2}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\beta}_n(u_0) - \beta_n(u_0)\Big) \\ &+ M^{-1} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \alpha \partial \beta}\bigg|_{\tilde{c}_n} M^{1/2}\Big(\hat{\alpha}_n(u_0) - \alpha_n(u_0)\Big) \\ &+ M^{-2} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \beta \partial \lambda_0}\bigg|_{\tilde{c}_n} M^{3/2}\Big(\widehat{\omega}(u_0) - \omega_n(u_0)\Big) \\ &+ nM^{-3} \frac{\partial^2 \bar{S}_m(\alpha, \beta, \lambda_0, \lambda_1; u_0)}{\partial \beta \partial \lambda_1}\bigg|_{\tilde{c}_n} \frac{M^{5/2}}{n}\Big(\widehat{\omega}'(u_0) - \omega'_n(u_0)\Big) \end{aligned} \tag{3.29}$$

In general we have to use a different $\tilde{c}_n$ sequence for each Taylor expansion, but since no ambiguity arises we use the same notation for avoiding unnecessary complexity.

Using (3.28) and (3.29) and combining with Theorem 3.7 we can show the following result about the joint asymptotic distribution of our estimates:

Theorem 3.8. *Suppose Assumptions of Theorem 3.7 hold and furthermore* $m = o(n^{4/7})$. *Then the vector*

$$M^{1/2}\left(\hat{\alpha}_n(u_0) - \alpha_n(u_0),\, \hat{\beta}_n(u_0) - \beta_n(u_0),\, M(\widehat{\omega}_n(u_0) - \omega(u_0)),\, \frac{M^2}{n}(\widehat{\omega}'_n(u_0) - \omega(u_0))\right)$$

is asymptotically normally distributed with zero mean and covariance matrix

$$2\pi f\big(\omega(u_0)\big)\begin{bmatrix} 2+60\frac{\beta_n^2}{\gamma^2} & 2-60\frac{\alpha_n\beta_n}{\gamma^2} & 0 & -720\frac{\beta_n}{\gamma^2} \\ 2-60\frac{\alpha_n\beta_n}{\gamma^2} & 2+60\frac{\alpha_n^2}{\gamma^2} & 0 & 720\frac{\alpha_n}{\gamma^2} \\ 0 & 0 & \frac{24}{\gamma^2} & 0 \\ -720\frac{\beta_n}{\gamma^2} & 720\frac{\alpha_n}{\gamma^2} & 0 & 360\frac{24}{\gamma^2} \end{bmatrix}$$

where $f(\cdot)$ *is the spectral density of* X_t.

We remind once more that -like all other asymptotic results- the asymptotic distribution of the estimates is derived within the infill asymptotics frame. Thus, any use of them, such as constructing confidence intervals, should be done taking into account this fact. Nevertheless, the result for $\widehat{\omega}_n(u_0)$ can be used in the usual manner, as it is asymptotically independent from the other estimates and its variance involves only $m(n)$ and not n itself.

3.5 A simulation

In the following we present a simulation study that consists of two parts. In the first we compare the modified periodogram frequency estimates to the ones from the ordinary periodogram. In the second we aim to test the performance of the modified periodogram method in the presence of a high noise component.

3.5.1 Comparison between ordinary and modified periodograms

We simulate data from three different models:

$$Y_t = \gamma\cos\Big(0.8\,t\Big) + X_t \tag{3.30}$$

$$Y_t = \gamma\cos\left(0.2\,t + \frac{0.02}{2}\,t^2\right) + X_t \tag{3.31}$$

$$Y_t = \gamma\cos\left(0.3\,t + \frac{0.02}{2}\,t^2 - \frac{0.0001}{3}\,t^3\right) + X_t \tag{3.32}$$

Model (3.30) is denoted by “constant”, (3.31) by “linear” and (3.31) by “quadratic” because of their corresponding frequency functions. X_t is in all cases an AR-process with parameters $\alpha = 0.8$ and $\sigma^2 = 1$. Note that the variance of the process X_t is $\sigma_X^2 = 1/0.36$. For all three models we simulate data for three different amplitudes ($\gamma = 2.5$, 3 and 4) and three different data segments ($2m+1 = 41$, 61 and 81). For each of these 27 combinations we get 300 realizations. t_0 is always the middle of each segment. The maximization of the modified periodogram and of the ordinary periodogram is done by evaluating them on a two dimensional grid on the space $(\lambda_0, \lambda_1) \in [0, \pi] \times [-\pi/20, \pi/20]$ and on a one dimensional grid on the space $\lambda \in [0, \pi]$. The maximization of λ_1 is restricted on a smaller area than its actual domain $[-\pi, \pi]$ only for making the procedure computationally faster. Figure 3.5 shows the evaluation of the modified periodogram function around its maximum-argument in one of the realizations of model (3.31).

Tables 3.1 and 3.2 show the mean squared error of the estimates $\widehat{\omega}(t_0)$ and $\widehat{\omega}'(t_0)$ for the modified periodogram and the periodogram methods. For $\widehat{\omega}'(t_0)$ in Table (3.1) the displayed values are the actual values multiplied by 10000. The real values of the frequency function and its derivative at t_0 are seen in the last column.

It can be seen that for constant frequency the ordinary periodogram method is better. This is clear since it is targeted for this case. Similarly, in the linear case the modified periodogram method is the better one. In the quadratic case the modified periodogram is better since it gives the better fit. We are convinced that this will hold the same for most other non constant frequencies, at least when they can be approximated by linear functions. To address this limitation we present in Table 3.3 the estimation MSE of the modified periodogram in a simulation using the frequency function $\widetilde{\omega}(t) = 0.3\,t + \frac{0.02}{2}\,t^2 - \frac{0.0002}{3}\,t^3$. Just by increasing (in absolute value) the quadratic coefficient of the frequency function by 0.0001 we see a significant reduction of the efficiency of the estimator. In the most extreme case ($n = 81$) the ordinary periodogram presents better results, which you can see in Table 3.4.

		Amplitude							
		2.5		3		4		Real values	
		$\widehat{\omega}(t_0)$	$\widehat{\omega}'(t_0)^*$	$\widehat{\omega}(t_0)$	$\widehat{\omega}'(t_0)^*$	$\widehat{\omega}(t_0)$	$\widehat{\omega}'(t_0)^*$	$\omega(t_0)$	$\omega'(t_0)$
	$n = 41$	0.1148	3.2734	0.0628	1.4001	0.0121	0.4503	0.80	0.000
constant	$n = 61$	0.0703	0.5974	0.0206	0.1347	0.0019	0.0070	0.80	0.000
	$n = 81$	0.0176	0.0558	0.0038	0.0119	$\approx$0.000	0.0008	0.80	0.000
	$n = 41$	0.0708	5.6435	0.0457	3.5175	0.0110	0.8303	0.60	0.020
linear	$n = 61$	0.0744	0.9343	0.0256	0.2296	0.0038	0.0432	0.80	0.020
	$n = 81$	0.0607	0.3767	0.0084	0.0404	$\approx$0.000	0.0009	1.00	0.020
	$n = 41$	0.0826	4.7787	0.0474	2.2585	0.0097	0.5669	0.66	0.016
quadratic	$n = 61$	0.0833	0.6544	0.0213	0.1239	0.0043	0.0265	0.81	0.014
	$n = 81$	0.0568	0.1496	0.0120	0.0278	0.0010	0.0007	0.94	0.012

* Displayed values are the actual values multiplied by 10000.

Table 3.1: Simulation MSE of $\widehat{\omega}(t_0)$ and $\widehat{\omega}'(t_0)$ for the modified periodogram.

To make more clear why this happens, in Figure 3.1 we have a visualization of the two different quadratic frequency functions used in the above simulations. While in the first simulation (solid line) the frequency can be well approximated by a linear function, in the second simulation (dashed line) the approximation is good enough only up to some time

		Amplitude			Real values	
		2.5	3	4		
		$\widehat{\omega}(t_0)$	$\widehat{\omega}(t_0)$	$\widehat{\omega}(t_0)$	$\omega(t_0)$	$\omega'(t_0)$
constant	$n = 41$	0.0759	0.0295	0.0022	0.80	0.000
	$n = 61$	0.0312	0.0043	≈0.000	0.80	0.000
	$n = 81$	0.0126	≈0.000	≈0.000	0.80	0.000
linear	$n = 41$	0.1311	0.1147	0.0773	0.60	0.020
	$n = 61$	0.3419	0.3015	0.2288	0.80	0.020
	$n = 81$	0.6618	0.6250	0.4928	1.00	0.020
quadratic	$n = 41$	0.1581	0.1211	0.0692	0.66	0.016
	$n = 61$	0.3192	0.2583	0.1472	0.81	0.014
	$n = 81$	0.4965	0.3795	0.2153	0.94	0.012

Table 3.2: Simulation MSE of $\widehat{\omega}(t_0)$ obtained by maximizing the ordinary periodogram.

		Amplitude						Real values	
		2.5		3		4			
		$\widehat{\omega}(t_0)$	$\widehat{\omega}'(t_0)^*$	$\widehat{\omega}(t_0)$	$\widehat{\omega}'(t_0)^*$	$\widehat{\omega}(t_0)$	$\widehat{\omega}'(t_0)^*$	$\omega(t_0)$	$\omega'(t_0)$
quadratic*	$n = 41$	0.0787	4.7263	0.0540	3.0715	0.0164	0.8481	0.62	0.012
	$n = 61$	0.0880	0.8772	0.0277	0.2994	0.0057	0.0201	0.72	0.008
	$n = 81$	0.1286	0.6553	0.0533	0.3059	0.0126	0.0559	0.78	0.004

* Displayed values are the actual values multiplied by 10000.

Table 3.3: Simulation MSE of $\widehat{\omega}(t_0)$ and $\widehat{\omega}'(t_0)$ for the modified periodogram (2).

		Amplitude			Real values	
		2.5	3	4		
		$\widehat{\omega}(t_0)$	$\widehat{\omega}(t_0)$	$\widehat{\omega}(t_0)$	$\omega(t_0)$	$\omega'(t_0)$
quadratic*	$n = 41$	0.1073	0.0672	0.0215	0.62	0.012
	$n = 61$	0.1667	0.0973	0.0313	0.72	0.008
	$n = 81$	0.0983	0.0367	0.0025	0.78	0.004

Table 3.4: Simulation MSE of $\widehat{\omega}(t_0)$ obtained by maximizing the ordinary periodogram (2).

point around $t = 50$. Hence the estimation results in Table 3.3 are good for $n = 41$ and $n = 61$, but not for $n = 81$. For the latter case, a constant approximation seems to be the more proper choice.

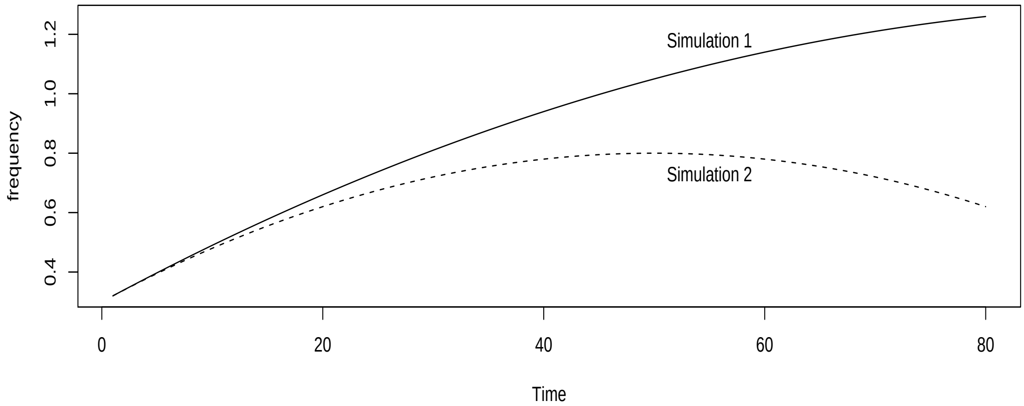

Figure 3.1: Quadratic frequency functions.

3.5.2 Performance of the modified periodogram in very noisy oscillations

In this section we aim to demonstrate the performance of the modified periodogram estimates in the presence of greater noise, in particular for the case where the noise variance is equal to the amplitude of the oscillation ($\sigma^2 = \gamma = 1$). We simulate 100 time series of length $2m + 1 = 201$ from the model

$$Y_t = \cos\left(0.1\,t + \frac{0.004}{2}\,t^2\right) + X_t, \qquad 0 \le t \le 200,$$

where X_t is white Gaussian noise with variance $\sigma^2 = 1$. In Figure 3.2 you can see one of the realizations of the above model, as well as the periodic component of the process (smooth line).

We apply the modified periodogram method on the hole segment for estimating the frequency function at $t_0 = 100$ ($\omega(t_0) = 0.5$) and its derivative ($\omega'(t_0) = 0.004$). The MSE for the first was 3.37×10^{-6} and for the second 5.8×10^{-9}. Figure 3.3 shows the histograms of the estimated values, which, as expected, are approximating the normal distribution.

Furthermore, we proceed to Least Squares Estimation to estimate $\widetilde{\alpha}(t_0) = 0.15425$ and $\widetilde{\beta}(t_0) = 0.98803$ in the alternative representation of the signal like in (3.9). The MSE of the estimations were 0.0295 and 0.0110 respectively. Note here the greater variance for $\widehat{\widetilde{\alpha}}(t_0)$ and compare to the result of Theorem 3.8 where its theoretical asymptotic variance is proportional to $\widetilde{\beta}^2(t_0)$ and vice versa. You can find the corresponding histograms for these estimations in Figure 3.4. Again the estimates seem to be normally distributed.

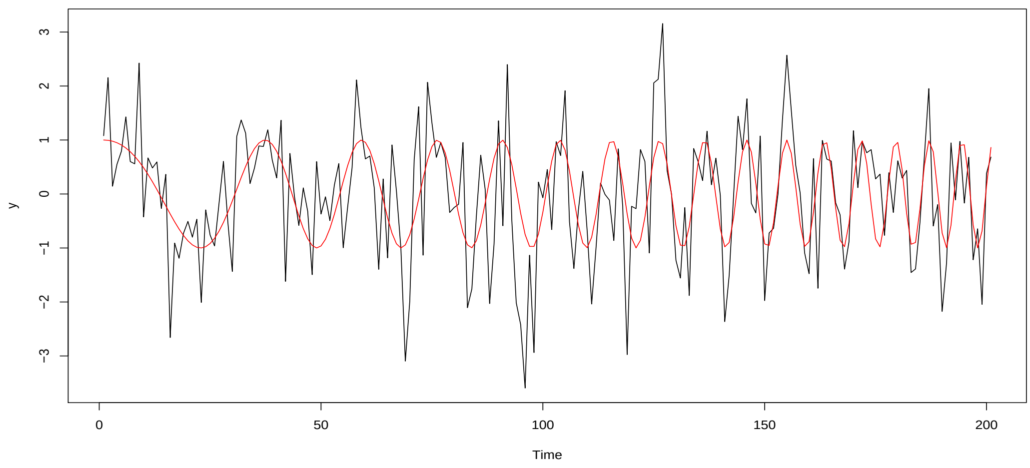

Figure 3.2: Noisy oscillation with "1 to 1" amplitude/noise ratio.

Finally, in Table 3.5 we present the empirical covariance matrix of the four estimates calculated by their 100 estimated values. Compare here the estimated variance of $\widehat{\widehat{\omega}}(t_0)$ (2.86×10^{-6}) with its theoretical asymptotic variance from Theorem 3.8 which was for the setting of this simulation $\text{var}(\widehat{\widehat{\omega}}(t_0)) = 24\sigma^2/M^3\gamma^2 = 2.96 \times 10^{-6}$. This is an indication that the theoretical result for $\widehat{\widehat{\omega}}(t_0)$ holds not only in the frame of infill asymptotics and can be used in the common way for constructing e.g. confidence intervals.

	$\widehat{\widehat{\alpha}}(t_0)$	$\widehat{\widehat{\beta}}(t_0)$	$\widehat{\widehat{\omega}}(t_0)$	$\widehat{\widehat{\omega}}'(t_0)$
$\widehat{\widehat{\alpha}}(t_0)$	2.61×10^{-2}	7.97×10^{-4}	3.64×10^{-6}	-8.66×10^{-6}
$\widehat{\widehat{\beta}}(t_0)$	7.97×10^{-4}	1.11×10^{-2}	-5.56×10^{-5}	5.14×10^{-7}
$\widehat{\widehat{\omega}}(t_0)$	3.64×10^{-6}	-5.56×10^{-5}	2.86×10^{-6}	1.12×10^{-8}
$\widehat{\widehat{\omega}}'(t_0)$	-8.66×10^{-6}	5.14×10^{-7}	1.12×10^{-8}	4.51×10^{-9}

Table 3.5: Empirical covariance matrix of the estimates.

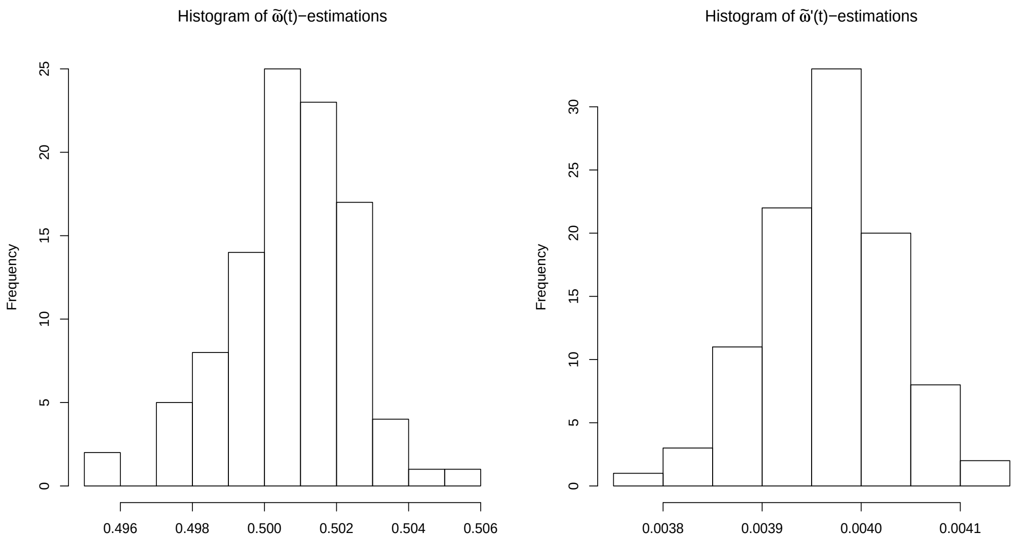

Figure 3.3: Histogram of estimated values for $\widetilde{\omega}(t_0)$ and $\widetilde{\omega}'(t_0)$.

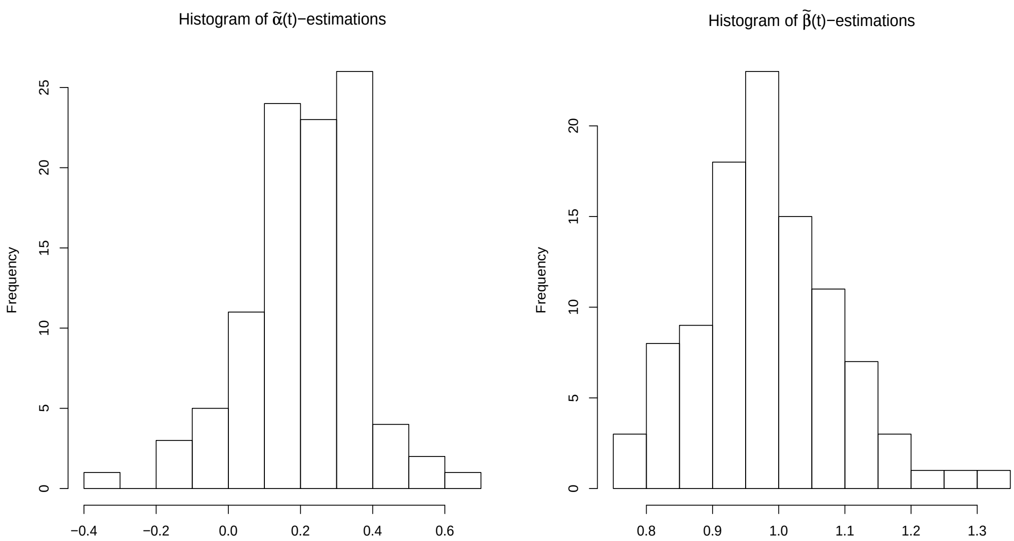

Figure 3.4: Histogram of estimated values for $\widetilde{\alpha}(t_0)$ and $\widetilde{\beta}(t_0)$.

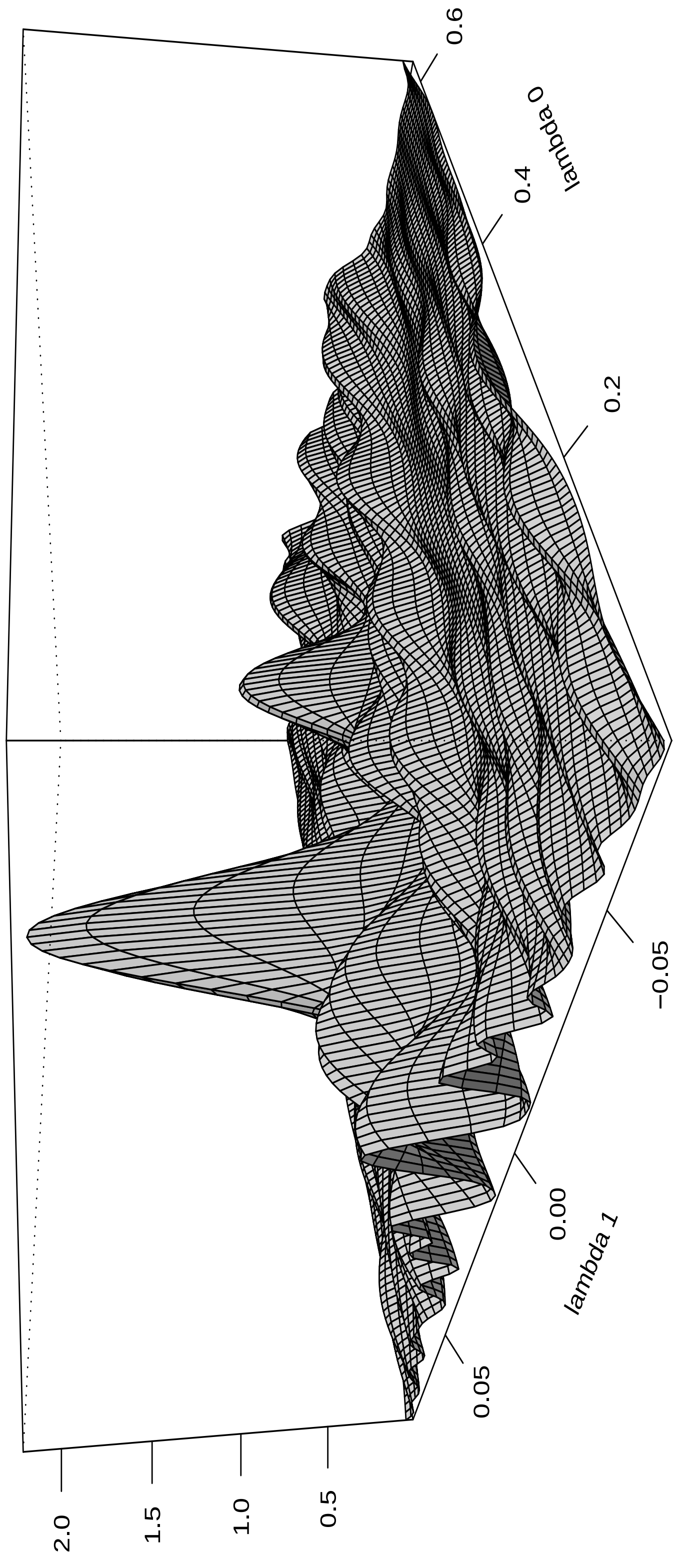

Figure 3.5: The modified periodogram function.

Chapter 4

Concluding remarks

Non parametric estimation of frequency in discretely observed noisy oscillations is widely reported in the literature due to its various applications. Most existing methods up to date deal with this problem assuming a - at least locally - constant frequency. In the introductory Chapter of the thesis we referred to some of these methods emphasizing on the maximizer of the periodogram.

As a result of the constancy assumption, estimates -and in particular the periodogram maximizer- are potentially affected by a systematic error induced by a possibly varying frequency. In this essay we have attempted to deal with this problem in two ways: first, we investigated the properties of the ordinary periodogram as an estimator of the instantaneous frequency, in order to find out to what extend this is justified. Second, we proposed a novel method based on a modification of the periodogram to reduce the bias caused by the (possible) non homogeneity of the frequency.

In particular, in Chapter 2 we determined an asymptotic upper bound for the segment length, under which the ordinary periodogram estimate is consistent and asymptotically normally distributed. This was achieved by evaluating the bias caused by the non constant frequency. Furthermore, using an equivalent representation of the basic model, we expressed its phase and amplitude as functions of the coefficients of the cosine and sine in this alternative representation, which we estimated through the least squares technique. For these two quantities we also proved consistency and asymptotic normality. We concluded with some simulations comparing this estimating procedure to the Hilbert transform as a phase estimator.

In Chapter 3 we investigated the local polynomial periodogram with two parameters, which we called the modified periodogram and whose maximization leads to an alternative estimation for the instantaneous frequency, as well as for its first derivative, given that this exists. Again using the equivalent representation of the signal we constructed estimates for the phase and amplitude. Like in the previous chapter we proved consistency and asymptotic normality for the estimates, which leaded to a straightforward theoretical comparison between the two periodogram based methods. The modified periodogram turned out to have better convergence rates due to the greater segment length (asymptotically) that can be used without it being affected by the frequency inhomogeneity. The theoretical results were also supported in a simulation, in which the two afore mentioned methods were applied.

All the proofs of the theorems are in the frame of *infill asymptotics* and they were postponed to the Appendix. There, we also define the *modified* Fourier transform of a stationary process (see Appendix B) and prove its asymptotic normality, which is then used in further proofs of theorems of Chapter 3.

Appendix A

Proofs of Chapter 2

A.1 Auxiliary results

Before proving consistency of our estimators we present two lemmas needed for this purpose:

Lemma A.1. *Under Assumption 2.1*

$$\lim_{M\to\infty} \sup_{\lambda} \left| \frac{1}{M} d_M^{(k)}(u_0,\lambda) \right| = 0, \quad a.s.$$

$$\text{where} \quad d_M^{(k)}(u_0,\lambda) := M^{-k} \sum_{s=-m}^{m} s^k \, X_s \, \exp(-is\lambda)$$

This is a standard result in time series analysis and can be found e.g. in [3, Theorem 4.5.4.]. Hannan (cf. [9]) also proves the same result under slightly different assumptions. Furthermore we have

Lemma A.2. *If $\omega(u) : [0,1] \to [0,\pi]$ is Lipschitz continuous at u_0, $\epsilon_n \le 1/n$ and $m_l, m_r = o(n^{1/2})$ then:*

$$\lim_{n\to\infty} \max_{m_l \le s \le m_r} \left| 1 - \exp\left\{ \pm\, in \int_{u_0}^{u_s} [\omega(u) - \omega(u_0)]\, du \right\} \right| = 0$$

with $u_s := u_0 + \epsilon_n + s/n$.

Proof. We have with some constant K uniformly in s

$$\left| n \int_{u_0}^{u_s} (\omega(u) - \omega(u_0))\, du \right| \le Kn \int_{u_0}^{u_s} |u - u_0|\, du \le \frac{K}{2} \frac{\max(m_\ell + 1, m_r + 1)^2}{n}.$$

Thus the lemma is established. □

For our proofs we also need some classical results on the discrete Fourier transform. We briefly summarize these results and adapt them to the situation of the present essay.

Lemma A.3 (The discrete Fourier transform)**.** *Suppose* $\big(X_t\big)_{t\in\mathbb{Z}}$ *is a stationary process with mean zero,* $\mathbf{E}X_t^2 < \infty$ *and continuous spectral density. Let*

$$d_M^{(k)}(u_0,\lambda) := \frac{1}{M^k}\exp(-i\lambda\, m)\sum_{s=-m}^{m} s^k\, X_{n(u_0+\epsilon_n)+s}\exp(-i\lambda\, s) \tag{A.1}$$
$$= \sum_{s=0}^{M-1}\Big(\frac{s}{M}-\frac{m}{M}\Big)^k X_{n(u_0+\epsilon_n)+s-m}\exp\big(-\,i\lambda s\big)$$

Then $\mathbf{E}d_M^{(k)}(u_0,\lambda)=0$ *and*

$$\mathrm{var}\big(\mathrm{Re}\, d_M^{(k)}(u_0,\lambda)\big) = O_p(M);$$
$$\mathrm{var}\big(\mathrm{Im}\, d_M^{(k)}(u_0,\lambda)\big) = O_p(M);$$
$$\mathrm{cov}\big(\mathrm{Re}\, d_M^{(k)}(u_0,\lambda), \mathrm{Im}\, d_M^{(k)}(u_0,\lambda)\big) = O_p(\log M).$$

Proof. The result is standard in time series analysis. It follows e.g. with straightforward calculations from [5, Theorem 1a]. Note that $\big(x-m/M\big)^k$ plays the role of a tapering function. □

Remark A.1. *We also need an upper bound for the (related) function*

$$H_M^{(k)}(\lambda) := \frac{1}{M^k}\exp(-i\lambda\, m)\sum_{s=-m}^{m} s^k\exp(-i\lambda\, s) \tag{A.2}$$
$$= \sum_{s=0}^{M-1}\Big(\frac{s}{M}-\frac{m}{M}\Big)^k \exp\big(-\,i\lambda s\big).$$

Let $L_M(\lambda)$ *be the periodic extension of*

$$L_M(\lambda) := \begin{cases} M, & |\lambda|\le 1/M \\ 1/|\lambda|, & 1/M\le|\lambda|\le\pi. \end{cases} \tag{A.3}$$

By using partial summation (cf. [5, (6)]) we obtain

$$|H_M^{(k)}(\lambda)| \le K L_N(\lambda). \tag{A.4}$$

In particular we obtain for $\lambda\neq 0$ $\;|H_M^{(k)}(\lambda)|\le K/|\lambda|$.

A.2 Proofs of theorems

A.2.1 Consistency of the frequency and phase estimator

Proof of Theorem 2.1.

Let $Y_{t,n}$ be as in (2.3), (2.8). We have with $u_s := u_0 + \epsilon_n + s/n$ (for simplicity we omit u_0 from $\alpha_n(u_0)$ and $\beta_n(u_0)$)

$$\begin{aligned}
&\frac{1}{M}\sum_{s=-m_l}^{m_r} Y_{n(u_0+\epsilon_n)+s,n}\exp\{-i\lambda s\}\\
=&\frac{1}{M}\sum_{-m_l}^{m_r}\left[\alpha_n\cos\left(n\int_{u_0}^{u_s}\omega(u)\,du\right)+\beta_n\sin\left(n\int_{u_0}^{u_s}\omega(u)\,du\right)+X_{n(u_0+\epsilon_n)+s}\right]\exp\{-i\lambda s\}\\
=&\quad\frac{\alpha_n}{2M}\sum_{s=-m_l}^{m_r}\left[\exp\left\{in\int_{u_0}^{u_s}\omega(u)\,du\right\}+\exp\left\{-in\int_{u_0}^{u_s}\omega(u)\,du\right\}\right]\exp\{-i\lambda s\}\\
&-\frac{i\,\beta_n}{2M}\sum_{s=-m_l}^{m_r}\left[\exp\left\{in\int_{u_0}^{u_s}\omega(u)\,du\right\}-\exp\left\{-in\int_{u_0}^{u_s}\omega(u)\,du\right\}\right]\exp\{-i\lambda s\}\\
&+\frac{1}{M}\sum_{s=-m_l}^{m_r}X_{n(u_0+\epsilon_n)+s}\exp\{-i\lambda s\}\\
=&\quad\frac{\alpha_n}{2M}B_n^+\sum_{s=-m_l}^{m_r}\exp\left\{in\int_{u_0}^{u_s}[\omega(u)-\omega(u_0)]\,du\right\}\exp\{is(\omega(u_0)-\lambda)\} &\text{(A.5)}\\
&+\frac{\alpha_n}{2M}B_n^-\sum_{s=-m_l}^{m_r}\exp\left\{-in\int_{u_0}^{u_s}[\omega(u)-\omega(u_0)]\,du\right\}\exp\{is(-\omega(u_0)-\lambda)\} &\text{(A.6)}\\
&-\frac{i\,\beta_n}{2M}B_n^+\sum_{s=-m_l}^{m_r}\exp\left\{in\int_{u_0}^{u_s}[\omega(u)-\omega(u_0)]\,du\right\}\exp\{is(\omega(u_0)-\lambda)\} &\text{(A.7)}\\
&+\frac{i\,\beta_n}{2M}B_n^-\sum_{s=-m_l}^{m_r}\exp\left\{-in\int_{u_0}^{u_s}[\omega(u)-\omega(u_0)]\,du\right\}\exp\{is(-\omega(u_0)-\lambda)\} &\text{(A.8)}\\
&+\frac{1}{M}\sum_{s=-m_l}^{m_r}X_{n(u_0+\epsilon_n)+s}\exp\{-i\lambda s\} &\text{(A.9)}
\end{aligned}$$

where $B_n^+ := \exp\{in\epsilon_n\omega(u_0)\}$ and $B_n^- := \exp\{-in\epsilon_n\omega(u_0)\}$. Furthermore

$$\begin{aligned}
(A.5) \;=\;& B_n^+\frac{\alpha_n}{2M}\sum_{s=-m_l}^{m_r}\exp\{is(\omega(u_0)-\lambda)\} &\text{(A.10)}\\
&-B_n^+\frac{\alpha_n}{2M}\sum_{s=-m_l}^{m_r}\exp\{is(\omega(u_0)-\lambda)\}\left(1-\exp\left\{in\int_{u_0}^{u_s}[\omega(u)-\omega(u_0)]\,du\right\}\right).
\end{aligned}$$

The term $\frac{1}{M}\sum\exp\{is(\omega(u_0)-\lambda)\}$ converges to zero if $\lambda \neq \omega(u_0)$, indeed uniformly for all $\lambda : |\lambda-\omega(u_0)| > \delta$ for any $\delta > 0$. On the other hand, if $\lambda = \omega(u_0)$ it converges to unity (one can easily check the validity of these two statements by writing the expression as $\frac{1}{M}\sum(\exp\{i(\omega(u_0)-\lambda)\})^s$ and finding the limit of the series for $m \to \infty$). For the second term in (A.10) we have

$$\left|B_n^+\frac{\alpha_n}{2M}\sum_s\exp\{is(\omega(u_0)-\lambda)\}\left(1-\exp\left\{in\int_{u_0}^{u_s}[\omega(u)-\omega(u_0)]\,du\right\}\right)\right| \qquad \text{(A.11)}$$

$$\leq \frac{\alpha_n}{2M} \sum_s \left| 1 - \exp\left\{ in \int_{u_0}^{u_s} [\omega(u) - \omega(u_0)]\, du \right\} \right|$$

which converges to zero because of Lemma (A.2). Altogether we obtain

$$\lim_{n\to\infty} (A.5) \sim \begin{cases} \frac{1}{2}\alpha_n B_n^+, & \lambda = \omega(u_0) \\ 0, & \lambda \neq \omega(u_0)^* \end{cases} \tag{A.12}$$

*uniformly for all $\lambda : |\lambda - \omega(u_0)| > \delta$ for any $\delta > 0$.

By following exactly the same steps we obtain

$$\lim_{n\to\infty} (A.7) \sim \begin{cases} -\frac{1}{2}i\beta_n B_n^+, & \lambda = \omega(u_0) \\ 0, & \lambda \neq \omega(u_0)^* \end{cases} \tag{A.13}$$

*uniformly for all $\lambda : |\lambda - \omega(u_0)| > \delta$ for any $\delta > 0$ and

$$\lim_{n\to\infty} (A.6) = \lim_{n\to\infty} (A.8) = 0 \tag{A.14}$$

uniformly for all $\lambda \in [0, \pi]$. Combining Lemma (A.1) with (A.12), (A.13) and (A.14) we receive:

$$\lim_{n\to\infty} J_M(u_0, \lambda) = \begin{cases} \frac{1}{4}(\alpha_n^2 + \beta_n^2), & \lambda = \omega(u_0) \\ 0, & \lambda \neq \omega(u_0) \end{cases} \quad a.s. \tag{A.15}$$

Note that $\alpha_n^2 + \beta_n^2 = \gamma^2$ is the amplitude of the periodic component, namely a constant. Furthermore, the second row of the right side of (A.15) holds uniformly for all $\lambda : |\lambda - \omega(u_0)| > \delta$ for any $\delta > 0$.

Now we prove the desired convergence by contradiction. Let's suppose that this is not the case. This means, there were some $\epsilon > 0$ for which the set $[0; \pi] \setminus (\omega(u_0) - \epsilon; \omega(u_0) + \epsilon)$ would contain infinitely many elements of the sequence $\widehat{\omega}_n(u_0)$. Consequently there were some limit point $\lambda' \neq \omega(u_0)$ (because $\widehat{\omega}_n(u_0)$ is bounded) and a subsequence converging to this very limit point. Along this subsequence $J_M(u_0, \widehat{\omega}_n(u_0))$ would converge to zero (because the convergence to zero of $J_M(u_0, \widehat{\omega}_n(u_0))$ is uniform for $|\lambda - \omega(u_0)| > \delta$ for any $\delta > 0$). But we have $J_M(u_0, \widehat{\omega}_n(u_0)) \geq J_M(u_0, \omega(u_0))$, as $\widehat{\omega}_n(u_0)$ is the value that maximizes the function every time and $J_M(u_0, \omega(u_0))$ converges to something greater than zero and thus we have a contradiction. □

Proof of Theorem 2.2.

We have with $X(s) := X_{n(u_0+\epsilon_n)+s}$ and $S(s) := S_{n(u_0+\epsilon_n)+s,n}$

$$
\begin{aligned}
J_M(u_0, \widehat{\omega}_n(u_0)) - J_M(u_0, \omega(u_0)) = & \\
= \quad & \Big|M_n^{-1}\sum_s S(s)e^{is\widehat{\omega}_n(u_0)}\Big|^2 - \Big|M_n^{-1}\sum_s S(s)e^{is\omega(u_0)}\Big|^2 \qquad \text{(A.16)}\\
& + \Big|M_n^{-1}\sum_s X(s)e^{is\widehat{\omega}_n(u_0)}\Big|^2 - \Big|M_n^{-1}\sum_s X(s)e^{is\omega(u_0)}\Big|^2 \qquad \text{(A.17)}\\
& + 2Re\Big[\Big(M_n^{-1}\sum_s X(s)e^{-is\widehat{\omega}_n(u_0)}\Big)\Big(M_n^{-1}\sum_s S(s)e^{is\widehat{\omega}_n(u_0)}\Big)\Big] \qquad \text{(A.18)}\\
& - 2Re\Big[\Big(M_n^{-1}\sum_s X(s)e^{-is\omega(u_0)}\Big)\Big(M_n^{-1}\sum_s S(s)e^{is\omega(u_0)}\Big)\Big] \qquad \text{(A.19)}
\end{aligned}
$$

where Re denotes the real part of the complex numbers. (A.17), (A.18) and (A.19) go to zero because of Lemma (A.1). If now we did the same as in (A.10) we could see that each one of the first two terms of (A.16) has a part that also goes to zero because of Lemma A.2. If we decompose these two terms we finally have

$$
\begin{aligned}
J_M(u_0, \widehat{\omega}_n(u_0)) - J_M(u_0, \omega(u_0)) = & \qquad \text{(A.20)}\\
= & \Big|M_n^{-1}\sum_s[\alpha_n\cos(s\omega(u_0)) + \beta_n\sin(s\omega(u_0))]e^{is\widehat{\omega}_n(u_0)}\Big|^2\\
& - \Big|M_n^{-1}\sum_s[\alpha_n\cos(s\omega(u_0)) + \beta_n\sin(s\omega(u_0))]e^{is\omega(u_0)}\Big|^2 + o_p(1)\\
= & \underbrace{\frac{1}{4}(\alpha_n^2+\beta_n^2)}_{\gamma^2/4}\Big\{\Big|M_n^{-1}\sum_s e^{is(\omega(u_0)-\widehat{\omega}_n(u_0))}\Big|^2 - 1\Big\} + o_p(1).
\end{aligned}
$$

This difference has to be non negative as $\widehat{\omega}_n(u_0)$ maximizes the function $J_M(u_0,\lambda)$. The first term on the right side is negative and as the positive $o_p(1)$ goes to zero it must also converge to zero. Thus $\Big|M_n^{-1}\sum_s e^{is(\omega(u_0)-\widehat{\omega}_n(u_0))}\Big|^2$ goes to unity. We show that this can only happen if $M_n(\widehat{\omega}_n(u_0)-\omega_n(u_0))$ converges to zero. Let us suppose that this is not the case. Then there must be an increasing subsequence $(n') \subset \mathbb{N}$ for which either the limit is some c with $0<|c|<\infty$ or we have divergence. We suppose for simplicity that the sum is from 0 to M_n as the proof is essentially the same with the summation being from $-m_l$ to m_r. In the case of divergence we have for $\delta_n = \widehat{\omega}_n(u_0) - \omega_n(u_0)$

$$
\Big|(M_{n'}+1)^{-1}\sum_{s=0}^{M_{n'}} e^{is\delta_{n'}}\Big|^2 = \Big|(M_{n'}+1)^{-1}\sum_{s=0}^{M_{g,n'}} e^{is\delta_{n'}} + (M_{n'}+1)^{-1}\sum_{s=M_{g,n'}+1}^{M_{n'}} e^{is\delta_{n'}}\Big|^2 \qquad \text{(A.21)}
$$

where $M_{g,n'}$ is the greatest integer for which $|\angle(e^{iM_{g,n'}\delta_{n'}})| + |\angle(e^{i(M_{g,n'}+1)\delta_{n'}})| = \delta_{n'}$ with the angles being considered always between $[-\pi,\pi]$. $e^{iM_{g,n'}\delta_{n'}}$ is namely the last vector before we "enter" the unity circle for the last time at each n'. This means that

$|1-e^{i(M_{g,n'}+1)\delta_{n'}}| \le |1-e^{i\delta_{n'}}|$ because $\delta_{n'}$ goes to zero. Now we have

$$\Big|(M_{n'}+1)^{-1}\sum_{s=0}^{M_{g,n'}} e^{is\delta_{n'}}\Big| = (M_{n'}+1)^{-1}\underbrace{\frac{|1-e^{i(M_{g,n'}+1)\delta_{n'}}|}{|1-e^{i\delta_{n'}}|}}_{\le 1} \stackrel{n\to\infty}{\longrightarrow} 0.$$

On the other hand

$$\Big|(M_{n'}+1)^{-1}\sum_{s=M_{g,n'}+1}^{M_{n'}} e^{is\delta_{n'}}\Big| \le (M_{n'}+1)^{-1}\sum_{s=M_{g,n'}+1}^{M_{n'}} |e^{is\delta_{n'}}| = \frac{M_{n'}-M_{g,n'}}{M_{n'}+1} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Thus (A.21) vanishes which is a contradiction. Now if $M_{n'}\delta_{n'}$ converges to some limit c we have

$$\Big|(M_{n'}+1)^{-1}\sum_{s=0}^{M_{n'}} e^{is\delta_{n'}}\Big|^2 = \Big|(M_{n'}+1)^{-1}\sum_{s=0}^{M_{n'}}[\cos(s\delta_{n'})+i\sin(s\delta_{n'})]\Big|^2$$

$$\stackrel{\delta_{n'}\to 0}{\longrightarrow} \Big|c^{-1}[\int_0^c \cos(u)\,du + i\int_0^c \sin(u)\,du]\Big|^2 = \frac{2}{c^2}[1-\cos(c)] = \frac{4}{c^2}\sin^2\frac{c}{2} < 1, \quad \text{for } c \neq 0.$$

Thus the convergence of $M_n(\widehat{\omega}_n(u_0)-\omega_n(u_0))$ to zero is established. □

Proof of Lemma 2.1.

The design matrix of the local regression in (2.10) is

$$X = \big(\cos(\hat{\phi}_s^{(u_0)}), \sin(\hat{\phi}_s^{(u_0)})\big)_{s=-m_\ell,\dots,m_r}$$

with $\hat{\phi}_s^{(u_0)} := n\int_{u_0}^{u_s}\widehat{\omega}(u)\,du$ where $u_s := u_0+\epsilon_n+s/n$. Thus the least squares estimate $(X'X)^{-1}X'Y$ is equal to

$$\begin{bmatrix} \hat{\alpha}_n(u_0) \\ \hat{\beta}_n(u_0) \end{bmatrix} = \frac{1}{|X'X|}\begin{bmatrix} \sum_s \sin^2(\hat{\phi}_s^{(u_0)}) & -\sum_s \cos(\hat{\phi}_s^{(u_0)})\sin(\hat{\phi}_s^{(u_0)}) \\ -\sum_s \cos(\hat{\phi}_s^{(u_0)})\sin(\hat{\phi}_s^{(u_0)}) & \sum_s \cos^2(\hat{\phi}_s^{(u_0)}) \end{bmatrix} \times$$
$$\times \begin{bmatrix} \sum_s \cos(\hat{\phi}_s^{(u_0)})\big(\alpha_n(u_0)\cos(\phi_s^{(u_0)})+\beta_n(u_0)\sin(\phi_s^{(u_0)})+X_s\big) \\ \sum_s \sin(\hat{\phi}_s^{(u_0)})\big(\alpha_n(u_0)\cos(\phi_s^{(u_0)})+\beta_n(u_0)\sin(\phi_s^{(u_0)})+X_s\big) \end{bmatrix}, \quad \text{(A.22)}$$

where $|X'X| = \Big[\sum_s \sin^2(\hat{\phi}_s^{(u_0)})\Big]\Big[\sum_s \cos^2(\hat{\phi}_s^{(u_0)})\Big] - \Big[\sum_s \cos(\hat{\phi}_s^{(u_0)})\sin(\hat{\phi}_s^{(u_0)})\Big]^2$ and $\phi_s^{(u_0)} = n\int_{u_0}^{u_s}\omega(u)\,du$ is the real phase function. We have (suppose for simplicity $\epsilon_n = 0$, as the approach is exactly the same if this is not the case)

$$\begin{aligned}\cos(\hat{\phi}_s^{(u_0)}) &= \frac{1}{2}\Big\{e^{i[s\omega_0+s(\widehat{\omega}_0-\omega_0)]} + e^{-i[s\omega_0+s(\widehat{\omega}_0-\omega_0)]}\Big\} \\ &= \frac{1}{2}\big\{e^{is\omega_0}[1+O(s(\widehat{\omega}_0-\omega_0))] + e^{-is\omega_0}[1+O(s(\widehat{\omega}_0-\omega_0))]\big\} \\ &= \cos(s\omega_0)+O(s(\widehat{\omega}_0-\omega_0))\end{aligned}$$

which because of Theorem 2.2 gives

$$\sum_s \cos^2(\hat{\phi}_s^{(u_0)}) = \sum_s \cos^2(s\omega_0) + o(M). \tag{A.23}$$

By using exactly the same arguments we can write

$$\sum_s \sin^2(\hat{\phi}_s^{(u_0)}) = \sum_s \sin^2(s\omega_0) + o(M). \tag{A.24}$$

and

$$\sum_s \sin(\hat{\phi}_s^{(u_0)}) \cos(\hat{\phi}_s^{(u_0)}) = o(M). \tag{A.25}$$

On the other hand, because $\omega(u)$ is Lipschitz continuous, we have

$$n \int_{u_0}^{u_0+s/n} |\omega(u) - \omega(u_0)| du \leq Ln \int_{u_0}^{u_0+s/n} |u - u_0| du = Ln \int_0^{s/n} |v| dv = O\left(\frac{s^2}{n}\right)$$

for some constant L and thus

$$\begin{aligned}
\cos(\phi_s^{(u_0)}) &= \frac{1}{2}\left\{ e^{in\int_{u_0}^{u_0+s/n}\omega(u)du} + e^{-in\int_{u_0}^{u_0+s/n}\omega(u)du} \right\} \\
&= \frac{1}{2}\left\{ e^{is\omega_0}\left[1 + O\left(\frac{s^2}{n}\right)\right] + e^{-is\omega_0}\left[1 + O\left(\frac{s^2}{n}\right)\right]\right\} \\
&= \cos(s\omega_0) + O\left(\frac{s^2}{n}\right),
\end{aligned}$$

which combined with the previous gives

$$\sum_{s=-m_l}^{m_r} \cos(\hat{\phi}_s^{(u_0)}) \cos(\phi_s^{(u_0)}) = \sum_s \cos^2(s\omega_0) + o_p(M), \tag{A.26}$$

because of Theorem 2.2 and the fact that $m_l, m_r = o(n^{1/2})$. Using exactly the same arguments we obtain

$$\sum_{s=-m_l}^{m_r} \sin(\hat{\phi}_s^{(u_0)}) \sin(\phi_s^{(u_0)}) = \sum_s \sin^2(s\omega_0) + o_p(M), \tag{A.27}$$

$$\sum_{s=-m_l}^{m_r} \cos(\hat{\phi}_s^{(u_0)}) \sin(\phi_s^{(u_0)}) = o_p(M), \tag{A.28}$$

$$\sum_{s=-m_l}^{m_r} \sin(\hat{\phi}_s^{(u_0)}) \cos(\phi_s^{(u_0)}) = o_p(M), \tag{A.29}$$

while Lemma (A.1) yields

$$\sum_{s=-m_l}^{m_r} \sin(\hat{\phi}_s^{(u_0)})X_s = o_p(M) = \sum_{s=-m_l}^{m_r} \cos(\hat{\phi}_s^{(u_0)})X_s. \tag{A.30}$$

Using (A.23), (A.24), (A.25), (A.26), (A.27), (A.28), (A.29), (A.30) we can express (A.22) as follows:

$$\begin{bmatrix} \hat{\alpha}_n(u_0) \\ \hat{\beta}_n(u_0) \end{bmatrix} = \frac{1}{|X'X|} \begin{bmatrix} \sum_s \sin^2(s\omega_{u_0}) + o_p(M) & o_p(M) \\ o_p(M) & \sum_s \cos^2(s\omega_{u_0}) + o_p(M) \end{bmatrix} \times$$

$$\times \begin{bmatrix} \alpha_n(u_0) \sum_s \cos^2(s\omega_{u_0}) + o_p(M) \\ \beta_n(u_0) \sum_s \sin^2(s\omega_{u_0}) + o_p(M) \end{bmatrix}, \tag{A.31}$$

where

$$|X'X| = \Big[\sum_s \sin^2(s\omega_{u_0})\Big]\Big[\sum_s \cos^2(s\omega_{u_0})\Big] + o_p(M^2).$$

It is now easy to see that (A.31) "converges" to $[\alpha_n(u_0), \beta_n(u_0)]'$ since $\alpha_n(u_0)$ and $\beta_n(u_0)$ are bounded and thus we have the desired result and the lemma is established. □

Proof of Theorem 2.3.
The assertion of the theorem follows immediately from Lemma 2.1. □

A.2.2 The bias and asymptotic normality

The main tool of this chapter is a Taylor expansion of the phase of the non stationary signal based on Lemma 2.2.

Remark A.2. *As a consequence of Lemma 2.2 we have*

$$S_{t,n}$$
$$= \gamma \cos\Big[\frac{n}{2}\Big(\frac{t}{n} - u_0\Big)^2 \omega'(u_0) + \frac{n}{6}\Big(\frac{t}{n} - u_0\Big)^3 \omega''(u_0) + O\Big(n\Big(\frac{t}{n} - u_0\Big)^4\Big) + n\,\omega(u_0)\Big(\frac{t}{n} - u_0\Big) + \phi_n(u_0)\Big]$$

and in local time with $s := t - n(u_0 + \epsilon_n)$, i.e. $\frac{t}{n} - u_0 = \frac{s}{n} + \epsilon_n$ with $|\epsilon_n| \leq \frac{1}{n}$

$$S_{n(u_0+\epsilon_n)+s,n}$$
$$= \gamma \cos\Big[\frac{n}{2}\Big(\frac{s}{n} + \epsilon_n\Big)^2 \omega'(u_0) + \frac{n}{6}\Big(\frac{s}{n} + \epsilon_n\Big)^3 \omega''(u_0) + O\Big(n\Big(\frac{s}{n} + \epsilon_n\Big)^4\Big) + n\,\omega(u_0)\Big(\frac{s}{n} + \epsilon_n\Big) + \phi_n(u_0)\Big].$$

We now remove the ϵ_n - terms. Since

$$\Big|n\Big(\frac{s}{n}\epsilon_n + \epsilon_n^2\Big) + n\Big(\big(\frac{s}{n}\big)^2\epsilon_n + \frac{s}{n}\epsilon_n^2 + \epsilon_n^3\Big) + n\Big(\big(\frac{s}{n}\big)^3\epsilon_n + \big(\frac{s}{n}\big)^2\epsilon_n^2 + \frac{s}{n}\epsilon_n^3 + \epsilon_n^4\Big)\Big| \leq 9\,\frac{|s|+1}{n}$$

we obtain

$$
\begin{aligned}
&S_{n(u_0+\epsilon_n)+s,n} \\
&= \gamma \cos\Big[\frac{n}{2}\Big(\frac{s}{n}\Big)^2\omega'(u_0) + \frac{n}{6}\Big(\frac{s}{n}\Big)^3\omega''(u_0) + O\Big(n\Big(\frac{s}{n}\Big)^4\Big) + O\Big(\frac{|s|+1}{n}\Big) \\
&\qquad\qquad + n\,\omega(u_0)\Big(\frac{s}{n}+\epsilon_n\Big) + \phi_n(u_0)\Big]
\end{aligned}
$$

The mean value-theorem now yields

$$
\begin{aligned}
&S_{n(u_0+\epsilon_n)+s,n} \\
&\quad= \gamma \cos\Big[\frac{n}{2}\Big(\frac{s}{n}\Big)^2\omega'(u_0) + \frac{n}{6}\Big(\frac{s}{n}\Big)^3\omega''(u_0) + \omega(u_0)s + \phi_n(u_0) + \omega(u_0)n\epsilon_n\Big] \\
&\qquad\quad + O\Big(n\Big(\frac{s}{n}\Big)^4\Big) + O\Big(\frac{|s|+1}{n}\Big) \\
&\quad= \gamma \cos\big(a+b+c+d\big) + O\Big(n\Big(\frac{s}{n}\Big)^4\Big) + O\Big(\frac{|s|+1}{n}\Big) \qquad \text{(A.32)}
\end{aligned}
$$

with

$$
a := \frac{n}{2}\Big(\frac{s}{n}\Big)^2\omega'(u_0), \quad b := \frac{n}{6}\Big(\frac{s}{n}\Big)^3\omega''(u_0), \quad c := \omega(u_0)\,s, \quad d := \phi_n(u_0) + \omega(u_0)\,n\epsilon_n. \qquad \text{(A.33)}
$$

Furthermore we have

$$
\widetilde{S}_{n(u_0+\epsilon_n)+s,n}(u_0) = \gamma \cos\Big(n\,\omega(u_0)\Big(\frac{s}{n}+\epsilon_n\Big) + \phi_n(u_0)\Big) = \gamma \cos\big(c+d\big). \qquad \text{(A.34)}
$$

In our calculations below we also use the lower order expansion

$$
S_{n(u_0+\epsilon_n)+s,n} = \gamma \cos\big(a+c+d\big) + O\Big(n\Big(\frac{s}{n}\Big)^3\Big) + O\Big(\frac{|s|+1}{n}\Big). \qquad \text{(A.35)}
$$

Proof of Theorem 2.4.
We have ('cc' means 'complex conjugate')

$$
\begin{aligned}
J_M'\big(u_0,\omega_0\big) = -i\,\Big(\frac{1}{M}\sum_{s=-m_\ell}^{m_r} s\,Y_{n(u_0+\epsilon_n)+s,n}\exp(-i\omega_0 s)\Big) \qquad \text{(A.36)} \\
\times\Big(\frac{1}{M}\sum_{t=-m_\ell}^{m_r} Y_{n(u_0+\epsilon_n)+t,n}\exp(i\omega_0 t)\Big) + \text{cc.}
\end{aligned}
$$

In order to estimate the difference $J_M'(u_0,\omega_0) - \widetilde{J}_M'(u_0,\omega_0)$ we need to replace in both summands the terms $Y_{n(u_0+\epsilon_n)+s,n}$ by $\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)$. We use the formula

$$
y_1y_2 - x_1x_2 = (y_1-x_1)x_2 + x_1(y_2-x_2) + (y_1-x_1)(y_2-x_2), \qquad \text{(A.37)}
$$

that is we have

$$
J_M'(u_0,\omega(u_0)) - \widetilde{J}_m'(u_0,\omega(u_0)) = -i\,\Big[(i)\times(iv) + (ii)\times(iii) + (i)\times(iii)\Big] + cc \qquad \text{(A.38)}
$$

with (cf. (A.32) - (A.35))

$$
\begin{aligned}
(i) &:= \frac{1}{M}\sum_{s=-m}^{m} s\,\big(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\big)\ \exp\big(-\,i\omega(u_0)s\big) \\
&= \frac{1}{M}\sum_{s=-m}^{m} s\,\Big(S_{n(u_0+\epsilon_n)+s,n} - \widetilde{S}_{n(u_0+\epsilon_n)+s,n}(u_0)\Big)\ \exp\big(-\,i\omega(u_0)s\big) \\
&= \frac{\gamma}{M}\sum_{s=-m}^{m} s\Big[\cos\big(a+b+c+d\big) - \cos\big(c+d\big) + O\Big(\frac{|s|^4}{n^3}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big]\ \exp\{-ic\} \\
&= \frac{\gamma}{2M}\sum_{s=-m}^{m} s\big[\exp\{i(a+b+c+d)\} - \exp\{i(c+d)\}\big]\ \exp\{-ic\} && \text{(A.39)} \\
&\quad + \frac{\gamma}{2M}\sum_{s=-m}^{m} s\big[\exp\{-i(a+b+c+d)\} - \exp\{-i(c+d)\}\big]\ \exp\{-ic\} && \text{(A.40)} \\
&\quad + O\Big(\frac{m^5}{n^3}\Big) + O\Big(\frac{m^2}{n}\Big) && \text{(A.41)}
\end{aligned}
$$

$$
(ii) := \frac{1}{M}\sum_{s=-m}^{m} s\,\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\ \exp\big(-\,i\omega(u_0)s\big)
$$

$$
\begin{aligned}
(iii) &:= \frac{1}{M}\sum_{s=-m}^{m} \big(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\big)\ \exp\big(i\omega(u_0)s\big) \\
&= \frac{\gamma}{M}\sum_{s=-m}^{m} \Big[\cos\big(a+c+d\big) - \cos\big(c+d\big) + O\Big(\frac{|s|^3}{n^2}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big]\ \exp\{ic\} \\
&= \frac{\gamma}{2M}\sum_{s=-m}^{m} \big[\exp\{-i(a+c+d)\} - \exp\{-i(c+d)\}\big]\ \exp\{ic\} && \text{(A.42)} \\
&\quad + \frac{\gamma}{2M}\sum_{s=-m}^{m} \big[\exp\{i(a+c+d)\} - \exp\{i(c+d)\}\big]\ \exp\{ic\} && \text{(A.43)} \\
&\quad + O\Big(\frac{m^3}{n^2}\Big) + O\Big(\frac{m}{n}\Big) && \text{(A.44)}
\end{aligned}
$$

$$
(iv) := \frac{1}{M}\sum_{s=-m}^{m} \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\ \exp\big(i\omega(u_0)s\big).
$$

We now construct upper bounds for these terms. The complexity with the following proof is that at different stages different techniques for deriving the upper bounds are needed. This is also the reason why we refrain from using Taylor-expansions (say for the cos-function) throughout the whole proof.

(i) We start with the term (A.39). We have

$$
\big[\exp\{i(a+b+c+d)\} - \exp\{i(c+d)\}\big]\exp\{-ic\}
$$

$$= \big[\exp\{i(a+b)\} - 1\big] \exp\{id\}$$
$$= \big[\cos(a+b) - 1\big] \exp\{id\} + i \sin(a+b) \exp\{id\}.$$

Since the sum in (A.39) is symmetric we use

$$\begin{aligned}\big[\cos(a+b)-1\big]-\big[\cos(a-b)-1\big] &= (\cos a \cos b - \sin a \sin b) - (\cos a \cos b + \sin a \sin b)\\ &= -2 \sin a \, \sin b\end{aligned}$$

and

$$\begin{aligned}\sin(a+b) - \sin(a-b) &= (\sin a \, \cos b + \cos a \, \sin b) - (\sin a \, \cos b - \cos a \, \sin b)\\ &= 2 \cos a \, \sin b\end{aligned}$$

and obtain

$$\begin{aligned}(A.39) &= -\frac{\gamma}{M}\sum_{s=1}^{m} s \, \sin a \, \sin b \, \exp\{id\} + i \frac{\gamma}{M}\sum_{s=1}^{m} s \, \cos a \, \sin b \, \exp\{id\}\\ &= i \frac{\gamma}{M}\sum_{s=1}^{m} s \, \sin b \, \exp\{ia\} \, \exp\{id\}\\ &= i \frac{\gamma}{M}\sum_{s=1}^{m} s \, \sin\Big\{\frac{n}{6}\Big(\frac{s}{n}\Big)^3 \omega''(u_0)\Big\} \, \exp\Big\{i\,\frac{n}{2}\Big(\frac{s}{n}\Big)^2 \omega'(u_0)\Big\} \, \exp\{id\}.\end{aligned}$$

Since

$$\begin{aligned}& i\,\frac{\gamma}{M}\sum_{s=1}^{m} s \, \sin\Big\{\frac{n}{6}\Big(\frac{s}{n}\Big)^3 \omega''\Big\} \, \exp\Big\{i\,\frac{n}{2}\Big(\frac{s}{n}\Big)^2 \omega'\Big\}\\ &= i\,\frac{\gamma}{M}\sum_{s=1}^{m} s \, \sin\Big\{\Big(\frac{s}{m}\Big)^3 \kappa_2\Big\} \, \exp\Big\{i\,\Big(\frac{s^2}{m^2}\Big)\kappa_1\Big\} \quad \text{with} \quad \kappa_1 = \frac{m^2}{2n}\,\omega', \ \kappa_2 = \frac{m^3}{6n^2}\,\omega''\\ &= i\,\frac{\gamma\, m}{2}\int_0^1 x \, \sin\big(x^3\kappa_2\big) \, \exp\big(i\,x^2\kappa_1\big)\, dx + O\left(\frac{m^2}{n}\right).\end{aligned}$$

The first term is equal to

$$\begin{aligned}& i\,\frac{\gamma\, m}{4\,\kappa_1}\int_0^{\kappa_1} \sin\Big(y^{3/2}\frac{\kappa_2}{\kappa_1^{3/2}}\Big) \, \exp(i\,y)\, dy \quad \text{with substitution} \;\; y = x^2\kappa_1\\ &= i\,\frac{\gamma\, m}{4\,\kappa_1}\int_0^{\kappa_1} \sin\Big(y^{3/2}\,\kappa_3\Big) \, \exp(i\,y)\, dy \quad \text{with } \kappa_3 = \kappa_2/\kappa_1^{3/2}\\ &= i\,\frac{\gamma\, m}{4\,\kappa_1}\Big[-\,i\,\sin\Big(y^{3/2}\,\kappa_3\Big)\exp(i\,y)\Big|_0^{\kappa_1} + i\,\frac{3}{2}\int_0^{\kappa_1} y^{1/2}\kappa_3\,\cos\Big(y^{3/2}\kappa_3\Big)\,\exp(i\,y)\,dy\Big]\\ &= \frac{\gamma\, m}{4\,\kappa_1}\,\sin\kappa_2\,\exp(i\,\kappa_1) + O\Big(\frac{m}{\kappa_1}\int_0^{\kappa_1} y^{1/2}\kappa_3\,dy\Big)\\ &= O\Big(\frac{m\,\kappa_2}{\kappa_1}\Big) = O\Big(\frac{m^2}{n}\Big)\end{aligned}$$

we obtain $(A.39) = O\big(\frac{m^2}{n}\big)$.

We now derive an upper bound for the term (A.40). With $M = 2m+1$ let

$$h_M(x) := x\Big[\exp\Big\{-i\,x^2\frac{M^2}{2\,n}\,\omega'(u_0)\Big\} - 1\Big].$$

By using partial summation (cf. Dahlhaus, 1983), (6)) we obtain with L_M defined as in (A.3) ($Var(h_M)$ denotes the variation of the function h_M)

$$\begin{aligned}
\big|(A.40)\big| &= \Big|\frac{\gamma}{2M}\sum_{s=-m}^{m} s\,\big[\exp\{-i(a+c+d)\} - \exp\{-i(c+d)\}\big]\,\exp\{-ic\}\Big| \\
&= \Big|\frac{\gamma}{2}\,\exp\{-id\}\sum_{s=-m}^{m} h_M\big(\frac{s}{M}\big)\,\exp\{-i\,2\,\omega(u_0)\,s\}\Big| \\
&\le \Big(Var(h_M) + \Big|h_M\big(\frac{m}{M}\big)\Big|\Big)\,L_M(2\,\omega(u_0)) \\
&\le \Big(\sup_{x\in[-1/2,1/2]}|\,h_M'(x)\,| + \Big|h_M\big(\frac{m}{M}\big)\Big|\Big)\,\frac{1}{|2\,\omega(u_0)|} \\
&\le 8\,\Big|\frac{M^2}{2\,n}\,\omega'(u_0)\Big|\,\frac{1}{|2\,\omega(u_0)|} = O\Big(\frac{m^2}{n}\Big)
\end{aligned}$$

for $\omega(u_0) \neq 0$. With the term $O\Big(\frac{m^5}{n^3}\Big)$ from (A.41) this leads in total to the upper bound

$$\big|(i)\big| = O\left(\frac{m^2}{n}\right), \qquad \text{for } m \le n^{1/2}.$$

(ii) We obtain with $\omega_0 = \omega(u_0)$ and $d_M^{(k)}(u_0,\lambda)$ and $H_M^{(k)}(\lambda)$ as in (A.1) and (A.2)

$$\begin{aligned}
&\frac{1}{M}\sum_{s=-m}^{m} s\,\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp(-i\omega_0 s) = \\
&\quad= \frac{1}{M}\sum_{s=-m}^{m} s\,\big(\widetilde{S}_{n(u_0+\epsilon_n)+s}(u_0) + X_{n(u_0+\epsilon_n)+s}\big)\,\exp(-i\omega_0 s) \\
&\quad= \frac{1}{M}\sum_{s=-m}^{m} s\,\frac{\gamma}{2}\Big[\exp\{i(c+d)\} + \exp\{-i(c+d)\}\Big]\,\exp\{-ic\} \\
&\qquad\qquad + \exp(i\omega_0 m)\,d_M^{(1)}(u_0,\omega_0) \\
&\quad= \frac{\gamma}{2}\,\exp\{id\}\,H_M^{(1)}(0) + \frac{\gamma}{2}\,\exp\{-id\}\,\exp(i\omega_0 m)\,H_M^{(1)}(2\omega_0) + \exp(i\omega_0 m)\,d_M^{(1)}(u_0,\omega_0) \\
&\quad= O(1) + \exp(i\omega_0 m)\,d_M^{(1)}(u_0,\omega_0).
\end{aligned}$$

(iii) Since

$$\cos x = 1 - \frac{x^2}{2} + O(x^4) \quad \text{and} \quad \sin x = x + O(|x|^3)$$

we have with $a = O\big(\frac{s^2}{n}\big)$

$$\begin{aligned}(A.42) &= \frac{\gamma}{2M} \sum_{s=-m}^{m} \big[\exp\{-i(a+c+d)\} - \exp\{-i(c+d)\}\big] \exp\{ic\} \\ &= \frac{\gamma}{2M} \sum_{s=-m}^{m} \Big(\big[\cos a - 1\big] - i\,\sin a\Big) \exp\{-id\} \\ &= \Big(- \frac{\gamma}{2M} \sum_{s=1}^{m} a^2 + O\Big(\frac{m^8}{n^4}\Big) - i\,2\,\frac{\gamma}{2M} \sum_{s=1}^{m} a - i\,O\Big(\frac{m^6}{n^3}\Big)\Big) \exp\{-id\} \\ &= O\Big(\frac{m^2}{n}\Big) \qquad \text{if } m^2 \le n.\end{aligned}$$

To derive an upper bound for (A.43) we set

$$h_M(x) := \Big[\exp\Big\{i\,x^2 \frac{M^2}{2\,n}\,\omega'(u_0)\Big\} - 1\Big].$$

with $M = 2m+1$. As above we obtain

$$\begin{aligned}\big|(A.43)\big| &= \Big|\frac{\gamma}{2M} \sum_{s=-m}^{m} \big[\exp\{i(a+c+d)\} - \exp\{i(c+d)\}\big]\ \exp\{ic\}\Big| \\ &= \Big|\frac{\gamma\,\exp\{id\}}{2M} \sum_{s=-m}^{m} h_M\big(\frac{s}{M}\big)\ \exp\{i\,2\,\omega(u_0)\,s\}\Big| = O_p\Big(\frac{m}{n}\Big)\end{aligned}$$

for $\omega(u_0) \neq 0$. With (A.44) this leads in total to the upper bound

$$\big|(iii)\big| = O_p\left(\frac{m^2}{n}\right), \qquad \text{for } m \le n^{1/2}.$$

(iv)

$$\begin{aligned}&\frac{1}{M} \sum_{t=-m}^{m} \widetilde{Y}_{n(u_0+\epsilon_n)+t}(u_0)\ \exp(i\omega_0 t) \\ &\quad = \frac{1}{M} \sum_{t=-m}^{m} \big(\widetilde{S}_{n(u_0+\epsilon_n)+t}(u_0) + X_{n(u_0+\epsilon_n)+t}\big)\ \exp(i\omega_0 t) \\ &\quad = \frac{\gamma}{2} \exp\{-id\} + \frac{\gamma}{2M} \exp\{id\}\ \exp(-i\omega_0 m)\, H_M^{(0)}(-2\omega_0) \\ &\qquad + \exp(-i\omega_0 m)\, \frac{1}{M}\, d_M^{(0)}(u_0, -\omega_0) \\ &\quad = \frac{\gamma}{2} \exp\{-id\} + O\Big(\frac{1}{m}\Big) + \exp(-i\omega_0 m)\, \frac{1}{M}\, d_M^{(0)}(u_0, -\omega_0).\end{aligned}$$

Since $\mathbf{E}\, d_m^{(0)}(u,\lambda) = \mathbf{E}\, d_m^{(1)}(u,\lambda) = 0$ for all u and λ we now obtain

$$\begin{aligned}\mathbf{E}\big(B'_M(u_0,\lambda)\big) &= \mathbf{E}\Big(- i\,\Big[(i)\times(iv) + (ii)\times(iii) + (i)\times(iii)\Big] + cc\Big) \\ &= O_p\left(\frac{m^2}{n}\right) \qquad \text{for } m \le n^2\end{aligned}$$

and with Lemma A.3 (note that (i) and (iii) are deterministic)

$$\begin{aligned}\operatorname{var}\big(B'_M(u_0,\lambda)\big) &= \operatorname{var}\Big(-\, i\,\Big[(i)\times(iv)+(ii)\times(iii)\Big]+cc\Big)\\ &= O_p\left(\frac{m^5}{n^2}\right) \qquad \text{for } m\le n^2.\end{aligned}$$

□

Proof of Lemma 2.3.
We have

$$\begin{aligned}&J''_M(u_0,\xi_n) = \qquad\qquad (A.45)\\ &\left\{\frac{1}{M}\sum_{s=-m_\ell}^{m_r} s\,Y_{n(u_0+\epsilon_n)+s,n}\,\exp(-i\xi_n s)\right\}\times\left\{\frac{1}{M}\sum_{s=-m_\ell}^{m_r} s\,Y_{n(u_0+\epsilon_n)+s,n}\,\exp(i\xi_n s)\right\}\\ &-\left\{\frac{1}{M}\sum_{s=-m_\ell}^{m_r} s^2\,Y_{n(u_0+\epsilon_n)+s,n}\,\exp(-i\xi_n s)\right\}\times\left\{\frac{1}{M}\sum_{s=-m_\ell}^{m_r} Y_{n(u_0+\epsilon_n)+s,n}\,\exp(i\xi_n s)\right\}+cc\end{aligned}$$

with $|\xi_n-\omega(u_0)|\le|\widehat{\omega}_n(u_0)-\omega(u_0)|$. Now we use again the formula (A.37) to estimate the difference $J''_M(u_0,\xi_n)-\widetilde{J}''_M(u_0,\xi_n)$. We have

$$\begin{aligned}J''_M(u_0,\xi_n)-\widetilde{J}''_M(u_0,\xi_n) &= [(I_1)\times(IV_1)+(II_1)\times(III_1)+(I_1)\times(III_1)]\\ &\quad-[(I_2)\times(IV_2)+(II_2)\times(III_2)+(I_2)\times(III_2)]+cc, \qquad (A.46)\end{aligned}$$

with:

$$\begin{aligned}(I_1):&=\frac{1}{M}\sum_{s=-m}^{m} s\left(Y_{n(u_0+\epsilon_n)+s,n}-\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right)\,\exp\left(-\,i\xi_n s\right)\\ &=(i)\\ &\quad+\frac{1}{M}\sum_{s=-m}^{m} s\left(Y_{n(u_0+\epsilon_n)+s,n}-\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right)\,\exp\left(-\,i\omega(u_0)s\right)\cdot\\ &\qquad\qquad\cdot\left[\exp\left\{-i[\xi_n-\omega(u_0)]s\right\}-1\right], \qquad (A.47)\end{aligned}$$

$$(II_1):=\frac{1}{M}\sum_{s=-m}^{m} s\,\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp\left(-\,i\xi_n s\right),$$

(III_1) is the complex conjugate of (I_1) and (IV_1) the complex conjugate of (II_1). On the other side:

$$\begin{aligned}(I_2):&=\frac{1}{M}\sum_{s=-m}^{m} s^2\left(Y_{n(u_0+\epsilon_n)+s,n}-\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right)\,\exp\left(-\,i\xi_n s\right)\\ &=\frac{1}{M}\sum_{s=-m}^{m} s^2\left(Y_{n(u_0+\epsilon_n)+s,n}-\widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right)\,\exp\left(-\,i\omega(u_0)s\right)\end{aligned}$$

$$+ \frac{1}{M} \sum_{s=-m}^{m} s^2 \left(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right) \exp\left(-i\omega(u_0)s\right) \cdot$$

$$\cdot \left[\exp\{-i[\xi_n - \omega(u_0)]s\} - 1\right],$$

$$= \frac{\gamma}{M} \sum_{s=-m}^{m} s^2 \Big[\cos\big(a+b+c+d\big) - \cos\big(c+d\big) + O\Big(\frac{|s|^4}{n^3}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big] \exp\{-ic\}$$

$$+ \frac{1}{M} \sum_{s=-m}^{m} s^2 \left(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right) \exp\left(-i\omega(u_0)s\right) \cdot$$

$$\cdot \left[\exp\{-i[\xi_n - \omega(u_0)]s\} - 1\right],$$

$$= \frac{\gamma}{M} \sum_{s=-m}^{m} s^2 \Big[\cos\big(a+b+c+d\big) - \cos\big(c+d\big)\Big] \exp\{-ic\} \tag{A.48}$$

$$+ O\Big(\frac{m^6}{n^3}\Big) + O\Big(\frac{m^3}{n}\Big) \tag{A.49}$$

$$+ \frac{1}{M} \sum_{s=-m}^{m} s^2 \left(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right) \exp\left(-i\omega(u_0)s\right) \cdot$$

$$\cdot \left[\exp\{-i[\xi_n - \omega(u_0)]s\} - 1\right], \tag{A.50}$$

(exactly as with (i))

$$(II_2) := \frac{1}{M} \sum_{s=-m}^{m} s^2\, \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0) \exp\left(-i\xi_n s\right)$$

$$(III_2) := \frac{1}{M} \sum_{s=-m}^{m} \left(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right) \exp\left(i\xi_n s\right)$$

$$= \frac{\gamma}{M} \sum_{s=-m}^{m} \Big[\cos\big(a+c+d\big) - \cos\big(c+d\big) + O\Big(\frac{|s|^3}{n^2}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big] \exp\{i\xi_n s\}$$

$$(IV_2) := \frac{1}{M} \sum_{s=-m}^{m} \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0) \exp\big(i\xi_n s\big).$$

We now construct upper bounds for these terms.

(I_1) We have for the term (A.47):

$$\Big|\frac{1}{M} \sum_{s=-m}^{m} s \left(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right) \exp\left(-i\omega(u_0)s\right) \cdot$$

$$\cdot \left[\exp\{-i[\xi_n - \omega(u_0)]s\} - 1\right]\Big|$$

$$\le \max_s \left|\exp\{-i[\xi_n - \omega(u_0)]s\} - 1\right| \sum_{s=-m}^{m} \left|Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right|,$$

which is $o_p(m)$ for $m << n^{1/2}$ (as $\max_s |\exp\{-i[\xi_n - \omega(u_0)]s\} - 1|$ goes in this case to zero because of Theorem 2.2) and $O(m)$ for otherwise. Altogether we obtain

$$(I_1) = \begin{cases} o_p(m), & m << n^{1/2} \\ O_p(m), & \text{otherwise} \end{cases} .$$

(II_1) We obtain with $\omega_0 = \omega(u_0)$:

$$\begin{aligned}
|(II_1)| &= \left| \frac{1}{M} \sum_{s=-m}^{m} s\, \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0) \exp(-i\xi_n s) \right| = \\
&= \left| \frac{1}{M} \sum_{s=-m}^{m} s \left(\widetilde{S}_{n(u_0+\epsilon_n)+s}(u_0) + X_{n(u_0+\epsilon_n)+s} \right) \exp(-i\xi_n s) \right| \\
&\leq \left| \frac{1}{M} \sum_{s=-m}^{m} s\, \frac{\gamma}{2} \Big[\exp\{i(c+d)\} + \exp\{-i(c+d)\} \Big] \exp\{-i\xi_n s\} \right| \\
&\qquad + \sup_{\lambda} \left| \frac{1}{M} \sum_{s=-m}^{m} s\, X_{n(u_0+\epsilon_n)+s} \exp(-i\lambda s) \right| \\
&= \frac{\gamma}{2} \exp\{id\}\, H_M^{(1)}(\omega_0 - \xi_n) + \frac{\gamma}{2} \exp\{-id\} \exp(i\omega_0 m)\, H_M^{(1)}(-\omega_0 - \xi_n) \\
&\qquad + \sup_{\lambda} \left| \frac{1}{M} \sum_{s=-m}^{m} s\, X_{n(u_0+\epsilon_n)+s} \exp(-i\lambda s) \right| \\
&= O(m) + O(1) + o_p(m) = O_p(m),
\end{aligned}$$

where the last line holds because of (A.4) and Lemma A.1.

(I_2) We have with (A.33)

$$\begin{aligned}
(A.48) &= \frac{\gamma}{M} \sum_{s=-m}^{m} s^2 \Big[\cos\big(a+b+c+d\big) - \cos\big(c+d\big) \Big] \exp\{-ic\} \\
&= \frac{\gamma}{M} \sum_{s=-m}^{m} s^2 \left[O\left(\frac{s^2}{n}\right) + O\left(\frac{s^3}{n^2}\right) \right] \exp\{-ic\} \\
&= O\left(\frac{m^4}{n}\right) + O\left(\frac{m^5}{n^2}\right) = O\left(\frac{m^4}{n}\right),
\end{aligned}$$

while for the term (A.50) it holds:

$$\left| \frac{1}{M} \sum_{s=-m}^{m} s^2 \left(Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0) \right) \exp\big(-i\omega(u_0)s\big) \cdot \left[\exp\left\{-i[\xi_n - \omega(u_0)]s\right\} - 1\right] \right|$$

$$\leq \max_s \left|\exp\left\{-i[\xi_n - \omega(u_0)]s\right\} - 1\right| \sum_{s=-m}^{m} |s| \left| Y_{n(u_0+\epsilon_n)+s,n} - \widetilde{Y}_{n(u_0+\epsilon_n)+s}(u_0) \right| = o(m^2),$$

for $m << n^{1/2}$. Altogether we obtain

$$(I_2) = \begin{cases} o(m^2), & m << n^{1/2} \\ O(m^2), & \text{otherwise} \end{cases} .$$

(II_2) We obtain exactly as with (II_1): $|(II_2)| = O_p(m^2)$.
(III_2) We have with $a = O\big(\frac{s^2}{n}\big)$ and the mean value theorem

$$\big|(III_2)\big| = \frac{\gamma}{M}\sum_{s=-m}^{m}\Big[O\big(\frac{s^2}{n}\big) + O\Big(\frac{|s|^3}{n^2}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big] = O\Big(\frac{m^2}{n}\Big).$$

(IV_2) We have

$$\begin{aligned}|(IV_2)| &\le \left|\frac{\gamma}{M}\sum_{s=-m}^{m}\cos(\dots)\exp\{-i\xi_n s\}\right| + \left|\frac{1}{M}\sum_{s=-m}^{m}X_{n(u_o+\epsilon_n)+s}\exp\{-i\xi_n s\}\right| \\ &= O(1) + O_p(1).\end{aligned}$$

Putting all these together we obtain

$$J''_M(u_0,\xi_n) - \widetilde{J}''_M(u_0,\xi_n) = \begin{cases} o_p(m^2), & m << n^{1/2} \\ O_p(m^2), & \text{otherwise} \end{cases}.$$

Thus the theorem is established. □

Proof of Theorem 2.7.

By making similar considerations like in Remark A.2 we can see that

$$Y_{nu_0+s,n} = \alpha_n(u_0)\cos\left(\omega(u_0)s\right) + \beta_n(u_0)\sin\left(\omega(u_0)s\right) + O\left(\frac{s^2}{n}\right) \tag{A.51}$$

We now start evaluating the terms in (2.19) and (2.20). Using (A.51) we have (for simplicity in the notation we omit u_0 and we set $\omega_0 := \omega(u_0)$)

$$\begin{aligned}M^{-1/2}\frac{\partial S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha}\bigg|_{c_{n,0}} &= -\frac{2}{M^{\frac12}}\sum_{-m}^{m}\left[Y_{nu_0+s,n} - \alpha_n\cos(\omega_0 s) - \beta_n\sin(\omega_0 s)\right]\cos(\omega_0 s) \\ &= -\frac{2}{M^{\frac12}}\sum_{-m}^{m}X_{nu_0+s,n}\cos(\omega_0 s) + O\left(\frac{m^{5/2}}{n}\right).\end{aligned} \tag{A.52}$$

Then we have

$$\begin{aligned}M^{-1}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha^2}\bigg|_{\tilde c_n} &= \frac{2}{M}\sum_{-m}^{m}\cos^2(\tilde\omega_n s) = \frac{2}{M}\sum_{-m}^{m}\cos^2[\omega_0 s + (\tilde\omega_n - \omega_0)s] \\ &= \frac{2}{M}\sum_{-m}^{m}[\cos^2(\omega_0 s) + O_p(m^{-1/2})] \\ &= 1 + o(m^{-1}) + O_p(m^{-1/2})\end{aligned} \tag{A.53}$$

because of Theorem 2.6. Furthermore

$$M^{-1}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha\partial\beta}\bigg|_{\tilde c_n} = M^{-1}\sum_{-m}^{m}\cos(\tilde\omega_n s)\sin(\tilde\omega_n s) = 0 \tag{A.54}$$

because $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ for all $x \in \mathbb{R}$. Finally, making similar considerations and using Lemma 2.1 we see that

$$M^{-2}\frac{\partial^2 S_m(\alpha,\beta,\omega;u_0)}{\partial\alpha\partial\omega}\bigg|_{\tilde{c}_n} = \frac{2}{M^2}\sum_{-m}^{m} s[Y_{nu_0+s,n} - \tilde{\alpha}_n\cos(\tilde{\omega}_n s) - \tilde{\beta}_n\sin(\tilde{\omega}_n s)]\sin(\tilde{\omega}_n s) \tag{A.55}$$

$$- \frac{2}{M^2}\sum_{-m}^{m} s[\tilde{\alpha}_n\sin(\tilde{\omega}_n s) - \tilde{\beta}_n\cos(\tilde{\omega}_n s)]\cos(\tilde{\omega}_n s)$$

$$= o_p(1)$$

for $m = o(n^{2/5})$. Thus, using (A.52), (A.53), (A.54), (A.55) and (2.19) and for $m = o(n^{2/5})$ we see that

$$M^{1/2}\Big(\alpha_n(u_0) - \hat{\alpha}_n(u_0)\Big) = -\frac{2}{M^{\frac{1}{2}}}\sum_{-m}^{m} X_{nu_0+s,n}\cos(\omega_0 s) + o_p(1). \tag{A.56}$$

Completely analogously we can show

$$M^{1/2}\Big(\beta_n(u_0) - \hat{\beta}_n(u_0)\Big) = -\frac{2}{M^{\frac{1}{2}}}\sum_{-m}^{m} X_{nu_0+s,n}\sin(\omega_0 s) + o_p(1). \tag{A.57}$$

Following Hannan ([9, p. 518]), we can analogously show that

$$M^{-1/2}J'_M(u_0,\omega_0) = \tag{A.58}$$
$$\beta_n\frac{1}{M^{3/2}}\sum_{-m}^{m} sX_{nu_0+s,n}\cos(\omega_0 s) - \alpha_n\frac{1}{M^{3/2}}\sum_{-m}^{m} sX_{nu_0+s,n}\sin(\omega_0 s) + o_p(1).$$

The difference between this last equation and the result of Hannan is due to the different summation, which in Hannan is from 1 to M. Summing from $-m$ to m makes some terms vanish. The assertion of the theorem now follows using (A.56), (A.57), (A.58), Theorem 2.4, Lemma 2.3 and the result in [3, Theorem 4.4.2.]. □

Appendix B

Proofs of Chapter 3

B.1 The *modified* Fourier transform

We introduce here the *modified* Fourier transform of a stationary time series as:

$$\mathbf{d}_X^{(m_n)}(P(\cdot)) = \left[\sum_t h_a(t/m_n) X_a(t) \exp\{-iP(t)\}\right] \tag{B.1}$$
$$= [d_a^{(m_n)}(P(\cdot))] \quad \text{for } a = 1, ..., r, \quad t = 1, ..., m_n$$

where m_n is an increasing integer function of n (where not needed we omit n from the notation), X_a is the a^{th} component of a stationary r vector-valued series, $P(t) = \lambda_0 t + \frac{\lambda_1}{n} t^2$ with $\lambda_0 \in [0, \pi]$ and λ_1 in a bounded space $\Lambda_1 \subset \mathbb{R}$ and $h_a(u)$ a taper function with $h_a(u) = u^k$ for $0 \leq u \leq 1$ and some natural k and vanishes for $|u| > 1$. Now we can prove

Theorem B.1. *Corresponds to Theorem 4.4.2 in [3].*
Let $\boldsymbol{X}(t)$, $t = 0, \pm 1, ...$ be an r vector-valued stationary time series satisfying

$$\sum_{u_1} \cdots \sum_{u_{k-1}} |c_{a_1 \ldots a_k}(u_1, ..., u_{k-1})| < \infty \tag{B.2}$$

with

$$c_{a_1 \ldots a_k}(u_1, ..., u_{k-1}) = \text{cum}\{X_{a_1}(t + u_1), ..., X_{a_1}(t + u_{k-1}), X_{a_k}(t)\}. \tag{B.3}$$

Let $P_j(t) = \lambda_0^{(j)} t + \frac{\lambda_1^{(j)}}{n} t^2$ for $0 \leq t \leq m_n$ where $n \in \mathbb{N}$, $m_n = O(n^p)$, $1/2 < p < 2/3$. Suppose $[\lambda_0^{(j)}, \lambda_1^{(j)}]' \neq [\lambda_0^{(k)}, \lambda_1^{(k)}]' \in (0, \pi) \times \mathbb{R}$ for $1 \leq j < k \leq J$. Let $d_a^{(m_n)}(P(\cdot))$ be like in (B.1). Then the $m_n^{-1/2} \boldsymbol{d}_X^{(m_n)}(P_j(\cdot))$, $j = 1, \ldots, J$ are asymptotically independent complex variates with asymptotic distribution $\mathcal{N}_r^C(\boldsymbol{0}, 2\pi[H_{ab}(0) f_{ab}(\lambda_0^{(j)})])$, where

$$H_{ab}(\lambda) = \int h_a^{(m)}(t) h_b^{(m)}(t) \exp\{-i\lambda t\} dt, \quad h_i^{(m)}(t) := h_i(t/m)$$
$$f_{ab}(\lambda) = (2\pi)^{-1} \sum_u \exp\{-i\lambda u\} c_{ab}(u).$$

Moreover, if $\boldsymbol{X}(t)$ has zero mean the above also holds for $1/2 < p < 1$.

In order to prove the theorem we proceed with a sequence of lemmas partly following the lemmas and theorems in [3, p.402-405] adapted to our case, i.e. using the *modified* Fourier transform instead of the usual one. The proofs involve a category of functions $h_a(u)$ that are bounded, of bounded variation and vanish for $|u| > 1$. For our purposes we only need a subset of it, namely $h_a(u) = u^n \mathbb{I}_{\{|u|\leq 1\}}$, for some $n \in \mathbb{N}$. In the proofs of Brillinger appears some integer T. For us this T is a function of n (i.e. m_n) with $m_n = O(n^p)$, for $1/2 < p < 1$. The index n though is mainly omitted for convenience.

Lemma B.1. *Let $\lambda_1 \neq 0$. For $\lambda_0 \neq 0$ (mod 2π) we have*

$$\sum_{t=1}^{m} \frac{t^k}{m^k} \exp\left\{i\left(\lambda_0 t + \tilde{\lambda}_0 \frac{s}{n} t + \lambda_1 \frac{t^2}{n}\right)\right\} = O\left(\frac{m^2}{n}\right)$$

uniformly for all $s : |s| \leq s_0$ for some $s_0 \in \mathbb{N}$ with $k \in \{1, 2, ...\}$ and $\tilde{\lambda}_0 \in \mathbb{R}$. Moreover, for arbitrary λ_0 we have

$$\lim_{n\to\infty} \frac{1}{m} \sum_{t=1}^{m} \frac{t^k}{m^k} \exp\left\{i\left(\lambda_0 t + \tilde{\lambda}_0 \frac{s}{n} t + \lambda_1 \frac{t^2}{n})\right)\right\} = 0.$$

Proof. Suppose $\lambda_0 \neq 0$. Let $h_m(x) := x^k \exp\left\{ix^2 \frac{m^2}{n}\lambda_1\right\}$. By using partial summation (cf. [5, (6)]) we get

$$\begin{aligned}
&\left|\sum_{t=1}^{m} \frac{t^k}{m^k} \exp\left\{i\left(\lambda_0 t + \tilde{\lambda}_0 \frac{s}{n} t + \lambda_1 \frac{t^2}{n}\right)\right\}\right| \\
&\quad = \left|\sum_{t=1}^{m} h_m\left(\frac{t}{m}\right) \exp\left\{i\left(\lambda_0 + \tilde{\lambda}_0 \frac{s}{n}\right) t\right\}\right| \leq \big(Var(h_m) + |h_m(1)|\big) L_m(\lambda_0 + \tilde{\lambda}_0 \frac{s}{n}) \\
&\quad \leq \big(\sup_{-1\leq x\leq 1} \left|h_m'(x)\right| + |h_m(1)|\big) \frac{1}{\lambda_0 + \tilde{\lambda}_0 \frac{s}{n}} \leq C \left|\frac{m^2}{n}\lambda_1\right| \frac{1}{\lambda_0 + \tilde{\lambda}_0 \frac{s}{n}}
\end{aligned}$$

for some constant C. Thus the expression in question is $O(\frac{m^2}{n})$, as $\tilde{\lambda}_0 \frac{s}{n}$ goes to zero uniformly for all s in question.

Now suppose $\lambda_0 = 0$ and $\lambda_1 \neq 0$.

$$\begin{aligned}
&\left|\frac{1}{m} \sum_{t=1}^{m} \frac{t^k}{m^k} \exp\left\{i\left(\tilde{\lambda}_0 \frac{s}{n} t + \lambda_1 \frac{t^2}{n}\right)\right\}\right| \\
&\leq \left|\frac{1}{m} \sum_{t=1}^{m} \frac{t^k}{m^k} \exp\left\{i\lambda_1 \frac{t^2}{n}\right\}\right| + \\
&\qquad + \frac{1}{m} \sum_{t=1}^{m} \left|\frac{t^k}{m^k} \exp\left\{i\lambda_1 \frac{t^2}{n}\right\}\right| \left|\left(\exp\left\{i\tilde{\lambda}_0 \frac{s}{n} t\right\} - 1\right)\right|
\end{aligned}$$

where the second term goes to zero uniformly for all $s : |s| \leq s_0$ for any $s_0 \in \mathbb{N}$ as

$\max_{s,t}\left|\left(\exp\left\{i\tilde{\lambda}_0 \frac{s}{n} t\right\}-1\right)\right|$ goes to zero. The square of the first term is

$$\left|\frac{1}{m}\sum_{t=1}^{m}\frac{t^k}{m^k}\exp\left\{i\lambda_1\frac{t^2}{n}\right\}\right|^2 =$$
$$\left[\frac{1}{m}\sum_{t=1}^{m}\frac{t^k}{m^k}\cos\left(\lambda_1\frac{t^2}{n}\right)\right]^2+\left[\frac{1}{m}\sum_{t=1}^{m}\frac{t^k}{m^k}\sin\left(\lambda_1\frac{t^2}{n}\right)\right]^2.$$

In the following we discuss the behavior of the first term of the right side, as the behavior of the second one is essentially the same. We have:

$$\frac{t^k}{m^k}\cos\left(\lambda_1\frac{t^2}{n}\right)\leq$$
$$\int_{t-1}^{t}\frac{x^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)dx+\sup_{x\in[t-1,t]}\frac{x^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)-\inf_{x\in[t-1,t]}\frac{x^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)$$

which, if we sum on both sides, gives:

$$\begin{aligned}\left|\frac{1}{m}\sum_{t=1}^{m}\frac{t^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)\right| &\leq \left|\frac{1}{m}\int_0^m\frac{x^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)dx\right|+\\ &+\left|\frac{1}{m}\sum_{t=1}^{m}\left[\sup_{x\in[t-1,t]}\frac{x^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)-\inf_{x\in[t-1,t]}\frac{x^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)\right]\right|\\ \leq\frac{1}{m}&\left[\left|\int_0^m\frac{x^k}{m^k}\cos\left(\lambda_1\frac{x^2}{n}\right)dx\right|+\sum_{t=1}^{m}\sup_{x,x'\in[t-1,t]}\left|\lambda_1\frac{x^2}{n}-\lambda_1\frac{x'^2}{n}\right|+\sum_{t=1}^{m}O\left(\frac{t^{k-1}}{m^k}\right)\right]\\ \leq\frac{1}{m^{k+1}}&\left|\int_0^m x^k\cos\left(\lambda_1\frac{x^2}{n}\right)dx\right|+\frac{1}{m}\sum_{t=1}^{m}\left[2\frac{\lambda_1}{n}t-\frac{\lambda_1}{n}\right]+O(m^{-1})\end{aligned}$$

The second term of the last equation is $O(m/n)$. The integral of the first term is for $k=0$

$$\int_0^m\cos\left(\lambda_1\frac{x^2}{n}\right)dx=\sqrt{\frac{\pi}{2\lambda_1}}n^{1/2}C\left(\sqrt{\frac{2\lambda_1}{\pi}}\frac{m}{n^{1/2}}\right)=O(n^{1/2})$$

where $C(u)$ is the Fresnel integral (bounded uniformly for all $u\in\mathbb{R}$). For $k\geq 1$ we have

$$\begin{aligned}\left|\int_0^m x^k\cos\left(\lambda_1\frac{x^2}{n}\right)dx\right| &= \left|\int_0^m\frac{nx^{k-1}}{2\lambda_1}\left[\sin\left(\lambda_1\frac{x^2}{n}\right)\right]'dx\right|\\ &=\left|\left[\frac{nx^{k-1}}{2\lambda_1}\sin\left(\lambda_1\frac{x^2}{n}\right)\right]_0^m-\int_0^m\frac{n(k-1)x^{k-2}}{2\lambda_1}\sin\left(\lambda_1\frac{x^2}{n}\right)dx\right|\\ &\leq\frac{nm^{k-1}}{2\lambda_1}+\frac{n(k-1)}{2\lambda_1}\int_0^m x^{k-2}dx=O(nm^{k-1})\end{aligned}$$

With that the lemma is established. □

Next we have

Lemma B.2. *Corresponds to Lemma P4.1 in [3]*
If $h_a^{(m)} = h_a(t/m)$ *for* $a = 1, ..., r$, *then*

$$\left|\sum_t h_{a_1}^{(m)}(t+u_1)\cdots h_{a_{k-1}}^{(m)}(t+u_{k-1})h_{a_k}^{(m)}(t)\exp\{-iP(t)\} - H_{a_1\cdots a_k}^{(m)}(P(\cdot))\right| \leq K\sum_1^{k-1}|u_j|$$

for any u_j, $j = 1, ..., k-1$ *and some finite* K *with*

$$H_{a_1\cdots a_k}^{(m)}(P(\cdot)) = \sum_t \left[\prod_1^k h_{a_j}^{(m)}(t)\right]\exp\{-iP(t)\}$$

with $a_1, ..., a_k = 1, ..., r$ *and* $P(t) = \lambda_0 t + \frac{\lambda_1}{n}t^2$ *for* $\lambda_0 \neq 0$ *and* $\lambda_1 \in \mathbb{R}$. *This holds uniformly for all such* $P(t)$.

Proof. Because of the definition of $H_{a_1\cdots a_k}^{(m)}(P(\cdot))$ and the fact that $|\exp\{-iP(t)\}| = 1$ the expression in question is

$$\leq \sum_t |h_{a_1}^{(m)}(t+u_1)\cdots h_{a_{k-1}}^{(m)}(t+u_{k-1}) - h_{a_1}^{(m)}(t)\cdots h_{a_{k-1}}^{(m)}(t)||h_{a_k}^{(m)}(t)|$$
$$\leq L\sum_{a=1}^{k-1}\sum_t |h_a^{(m)}(t+u_a) - h_a^{(m)}(t)|$$

for some finite L. Suppose for convenience $u_a > 0$. (The other cases are handled similarly.) The expression is now

$$\leq L\sum_{a=1}^{k-1}\sum_t\sum_{v=0}^{u_a-1}|h_a^{(m)}(t+v+1) - h_a^{(m)}(t+v)|$$
$$\leq M\sum_{a=1}^{k-1}\sum_{v=0}^{u_a-1}|\text{variation of } h_a| \leq K\sum_1^{k-1}|u_j|$$

for some finite M and K. □

Next we have

Lemma B.3. *Corresponds to Lemma P4.2 in [3]*

$$\textit{cumulant}\{d_{a_1}^{(m)}(P_1(\cdot)), ..., d_{a_k}^{(m)}(P_k(\cdot))\} =$$
$$\sum_{u_1=-S}^{S}\cdots\sum_{u_{k-1}=-S}^{S} H_{a_1\ldots a_k}^{(m)}(\tilde{P}_1(\cdot) + \cdots + \tilde{P}_{k-1}(\cdot) + P_k(\cdot))\times$$
$$\times\exp\{-i(P_1(u_1) + \cdots + P_{k-1}(u_{k-1}))\}c_{a_1\ldots a_k}(u_1, ..., u_{k-1}) + \epsilon_m$$

where $P_j(t) = \lambda_0^{(j)} t + \frac{\lambda_1^{(j)}}{n} t^2$, $\tilde{P}_j(t) = P_j(t) + 2\lambda_1^{(j)} \frac{u_j}{n} t$ *for* $j = 1, ..., k-1$, $S = 2(m-1)$ *and*

$$|\epsilon_m| \leq K \sum_{-S}^{S} \cdots \sum_{-S}^{S} (|u_1| + \cdots + |u_{k-1}|)|c_{a_1 \ldots a_k}(u_1, ..., u_{k-1})|$$

for some finite K.

Proof. The cumulant has the form

$$\sum_{t_1} \cdots \sum_{t_k} h_{a_1}^{(m)}(t_1) \cdots h_{a_k}^{(m)}(t_k) \exp\left\{-i \sum_1^k P_j(t_j)\right\} c_{a_1 \ldots a_k}(t_1 - t_k, ..., t_{k-1} - t_k)$$
$$= \sum_{u_1=-S}^{S} \cdots \sum_{u_{k-1}=-S}^{S} \exp\left\{-i \sum_1^{k-1} P_j(u_j)\right\} c_{a_1 \ldots a_k}(u_1, ..., u_{k-1}) \times$$
$$\times \sum_t h_{a_1}^{(m)}(t+u_1) \cdots h_{a_{k-1}}^{(m)}(t+u_{k-1}) h_{a_k}^{(m)}(t) \exp\left\{-i \sum_1^k \tilde{P}_j(t)\right\}$$

with $\tilde{P}_k(t) = P_k(t)$. The last equality results after changing the summation variables: $t = t_k$, $u_j = t_j - t$ for $j = 1, ..., k-1$. Using Lemma B.2 this equals

$$\sum_{u_1=-S}^{S} \cdots \sum_{u_{k-1}=-S}^{S} H_{a_1 \ldots a_k}^{(m)}(\tilde{P}_1(\cdot) + \cdots + \tilde{P}_{k-1}(\cdot) + P_k(\cdot)) \times$$
$$\times \exp\{-i(P_1(u_1) + \cdots + P_{k-1}(u_{k-1}))\} c_{a_1 \ldots a_k}(u_1, ..., u_{k-1}) + \epsilon_m$$

where ϵ_m has the indicated bound for some overall finite K, as Lemma B.2 holds uniformly for all parabolic functions $P(t)$. □

Lemma B.4. *Corresponds to Lemma P4.3 in [3].*
Under the condition (B.2), $\epsilon_m = o(m)$ *as* $m \to \infty$.

Proof.

$$m^{-1}|\epsilon_m| \leq K \sum_{-S}^{S} \cdots \sum_{-S}^{S} m^{-1}(|u_1| + \cdots + |u_{k-1}|)|c_{a_1 \ldots a_k}(u_1, ..., u_{k-1})|.$$

Now for any fixed $u_1, \ldots, u_{k-1} : m^{-1}(|u_1| + \cdots + |u_{k-1}|) \to 0$ as $m \to \infty$. Because of (B.2) we may now use the dominated convergence theorem to have the desired result. □

Now we can prove Theorem B.1:
We have

$$m^{-1/2} E d_a^{(m)}(P_j(\cdot)) = m^{-1/2} \sum_{t=0}^{m_n - 1} h_a(t/m) \exp\{-iP_j(t)\} EX_a(t)$$

which is either zero or goes to zero because of Lemma B.1. Then we have

$$cov\{m^{-1/2}d_a^{(m)}(P_j(\cdot)), m^{-1/2}d_b^{(m)}(P_k(\cdot))\} = m^{-1}cum\{d_a^{(m)}(P_j(\cdot)), \overline{d_b^{(m)}(P_k(\cdot))}\}$$
$$= m^{-1}\sum_{u=-S}^{S} H_{ab}^{(m)}(\tilde{P}_j(\cdot) - P_k(\cdot))\exp\{-iP_j(u)\}c_{ab}(u) + o(1)$$

where the last equation holds because of Lemmas B.3 and B.4. This tends to zero if $(\lambda_0^{(j)}, \lambda_1^{(j)}) \neq (\lambda_0^{(k)}, \lambda_1^{(k)})$ because of (B.2), Lemma B.1 and the dominated convergence theorem. If $(\lambda_0^{(j)}, \lambda_1^{(j)}) = (\lambda_0^{(k)}, \lambda_1^{(k)})$ it equals

$$m^{-1}\sum_{u=-S}^{S} H_{ab}^{(m)}\left(P_{0,j}(\cdot)\right)\exp\{-iP_j(u)\}c_{ab}(u) + o(1)$$
$$= m^{-1}\sum_{u=-\lfloor n^{\tilde{p}}\rfloor}^{\lfloor n^{\tilde{p}}\rfloor} H_{ab}^{(m)}\left(P_{0,j}(\cdot)\right)\exp\{-iP_j(u)\}c_{ab}(u) +$$
$$+ m^{-1}\sum_{\substack{|u|>\lfloor n^{\tilde{p}}\rfloor \\ -S\leq u\leq S}} H_{ab}^{(m)}\left(P_{0,j}(\cdot)\right)\exp\{-iP_j(u)\}c_{ab}(u) + o(1)$$

with $P_{0,j}(t) := \frac{\lambda_1^{(j)}}{n}ut$, for some positive $\tilde{p}$ such that $p + \tilde{p} < 1$. The second term goes to zero as $n \to \infty$ because of (B.2). The summary of the first term is equal to

$$\sum_{u=-\lfloor n^{\tilde{p}}\rfloor}^{\lfloor n^{\tilde{p}}\rfloor}\sum_{t=-m}^{m} h_a^{(m)}(t)h_b^{(m)}(t)\exp\left\{-\frac{\lambda_1^{(j)}}{n}ut\right\}\exp\left\{-i\left(\lambda_0^{(j)}u + \frac{\lambda_1^{(j)}}{n}u^2\right)\right\}c_{ab}(u)$$
$$= \sum_{u=-\lfloor n^{\tilde{p}}\rfloor}^{\lfloor n^{\tilde{p}}\rfloor}\sum_{t=-m}^{m} h_a^{(m)}(t)h_b^{(m)}(t)\exp\left\{-i\lambda_0^{(j)}u\right\}c_{ab}(u) +$$
$$\sum_{u=-\lfloor n^{\tilde{p}}\rfloor}^{\lfloor n^{\tilde{p}}\rfloor}\sum_{t=-m}^{m} h_a^{(m)}(t)h_b^{(m)}(t)\exp\left\{-i\lambda_0^{(j)}u\right\}\left(\exp\left\{-i\frac{\lambda_1^{(j)}}{n}(u^2+ut)\right\} - 1\right)c_{ab}(u).$$

The term $\exp\left\{-i\frac{\lambda_1^{(j)}}{n}(u^2+ut)\right\}$ goes to unity as $n \to \infty$ uniformly for all $-\lfloor n^{\tilde{p}}\rfloor \leq u \leq \lfloor n^{\tilde{p}}\rfloor$ and $-m \leq t \leq m$. Thus the second term is $o(m)$ because of (B.2), the fact that $h_a^{(m)}(t)$ is bounded and due to the theorem of the dominated convergence. Altogether we have for the case $(\lambda_0^{(j)}, \lambda_1^{(j)}) = (\lambda_0^{(k)}, \lambda_1^{(k)})$:

$$cov\{m^{-1/2}d_a^{(m)}(P(\cdot)), m^{-1/2}d_b^{(m)}(P(\cdot))\}$$
$$= \sum_{u=-\lfloor n^{\tilde{p}}\rfloor}^{\lfloor n^{\tilde{p}}\rfloor} m^{-1}\sum_{t=-m}^{m} h_a^{(m)}(t)h_b^{(m)}(t)\exp\left\{-i\lambda_0^{(j)}u\right\}c_{ab}(u) + o(1)$$
$$= m^{-1}H_{ab}^{(m)}(0)\sum_{u=-\lfloor n^{\tilde{p}}\rfloor}^{\lfloor n^{\tilde{p}}\rfloor}\exp\left\{-i\lambda_0^{(j)}u\right\}c_{ab}(u) + o(1)$$

Thus the second cumulant behaves in the manner required by the theorem. Finally it follows directly from Lemma B.3 that

$$\lim_{n\to\infty} m_n^{-k/2} \mathrm{cum}\{d_{a_1}^{(m)}(P_1(\cdot)), ..., d_{a_k}^{(m)}(P_k(\cdot))\} = 0$$

for $k \geq 3$ as $H_{a_1 \dots a_k}(.)$ is $O(m)$.

Putting the above results together, we see that the cumulants of the variates at issue, and the conjugates of those variates, tend to the cumulants of a normal distribution with the parameters indicated. Thus the theorem is established. □

B.2 Proofs of theorems

B.2.1 Consistency of the frequency estimator

Proof of Theorem 3.1

In order to prove Theorem 3.1 we need two lemmas, one referring to the stochastic and the other to the deterministic part of (2.3).

Lemma B.5. *Under the assumptions of Theorem B.1 and if* $\mathbf{E}X_t = 0$

$$\lim_{n\to\infty} \sup_{\lambda_0,\lambda_1} \left| r^{-1} \sum_{t=1}^{r} \frac{t^k}{r^k} X_t e^{i(\lambda_0 t + \frac{\lambda_1}{n} t^2)} \right| = 0, \qquad a.s.$$

for all $k \in \mathbb{N}$ *and for* $m = O(n^p)$ *where* $1/2 < p < 1$ *and* λ_1 *is bounded.*

Proof. We split the r summands into groups of K_n members, so that $K_n = o(n^{1-p})$ and K_n goes to infinity. Then, for $m_n = K_n^{-1} r$ and $R_n = r - \lfloor m_n \rfloor K_n$ we have:

$$\begin{aligned}
&\sup_{\lambda_0,\lambda_1} \left| r^{-1} \sum_{t=1}^{r} \frac{t^k}{r^k} X_t e^{i(\lambda_0 t + \frac{\lambda_1}{n} t^2)} \right| \qquad\qquad \text{(B.4)} \\
= &\sup_{\lambda_0,\lambda_1} \left| \frac{1}{m_n K_n} \left[\sum_{s=0}^{R_n} \frac{t^k}{r^k} X_s e^{i(\lambda_0 s + \frac{\lambda_1}{n} s^2)} + \sum_{j=1}^{\lfloor m_n \rfloor} \sum_{\substack{t=1+\\(j-1)K_n}}^{jK_n} \frac{t^k}{r^k} X_{t+R_n} e^{i(\lambda_0 t + \frac{\lambda_1}{n} t^2)} \right] \right| \\
\leq &\sup_{\lambda_0,\lambda_1} \left| \frac{1}{m_n K_n} \sum_{s=0}^{R_n} \frac{t^k}{r^k} X_s e^{i(\lambda_0 s + \frac{\lambda_1}{n} s^2)} \right| + \\
&\qquad\qquad \sup_{\lambda_0,\lambda_1} \left| m_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} K_n^{-1} \sum_t \frac{t^k}{r^k} X_{t+R_n} e^{i(\lambda_0 t + \frac{\lambda_1}{n} t^2)} \right| \\
\leq &\sup_{\lambda_0,\lambda_1} \left| m_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} K_n^{-1} \sum_t \frac{t^k}{r^k} X_{t+R_n} e^{i(\lambda_0 t + \frac{\lambda_1}{n} t^2)} \right| + \frac{1}{m_n K_n} \sum_{s=0}^{R_n} |X_s|,
\end{aligned}$$

where we set $X_0 = 0$. The second term of the right side of (B.4) converges to zero. To show that we write (note that $R_n \leq m_n + K_n$):

$$m_n^{-1} K_n^{-1} \sum_{s=0}^{R_n} |X_s| = \frac{m_n + K_n}{m_n K_n} \frac{1}{m_n + K_n} \sum_{s=0}^{R_n} |X_s|. \qquad \text{(B.5)}$$

Because $n^{-1}\sum_1^n |X_t|$ converges to $E|X_t| < \infty$ we have:

$$\forall \epsilon > 0 \quad \exists n_0' \in \mathbb{N}: \quad n^{-1}\sum_1^n |X_t| < E|X_t| + \epsilon \text{ (a.s.)}, \quad \forall n \geq n_0'.$$

Then it holds that:

$$\begin{aligned}\exists n_0 \in \mathbb{N} \quad &: \quad m_n + K_n \geq n_0', \quad \forall n \geq n_0 \\ &: \quad \frac{1}{m_n + K_n}\sum_{t=0}^{R_n}|X_t| \left(\leq \frac{1}{m_n + K_n}\sum_{t=0}^{m_n+K_n}|X_t|\right) < E|X_t| + \epsilon, \ \forall n \geq n_0.\end{aligned}$$

Combining this, (B.5) and the fact that $\frac{m_n+K_n}{m_nK_n}$ goes to zero we have the desired result. Now, we can write for all $j \in \{1, ..., \lfloor m_n \rfloor\}, t \in \{(j-1)K_n + 1, ..., jK_n\}$:

$$\begin{aligned}\frac{\lambda_1}{n}t^2 &= \frac{\lambda_1}{n}(t - jK_n + jK_n)^2 \\ &= \frac{\lambda_1}{n}(jK_n)^2 + \frac{\lambda_1}{n}(t - jK_n)^2 + 2\frac{\lambda_1}{n}(t - jK_n)jK_n \\ &= \frac{\lambda_1}{n}(jK_n)^2 + D_n(j,t),\end{aligned}$$

where $D_n(j,t) = 2\frac{\lambda_1}{n}(t-jK_n)jK_n + \frac{\lambda_1}{n}(t-jK_n)^2$ is $o(1)$ uniformly for all $j \in \{1, ..., \lfloor m_n \rfloor\}$, $t \in \{(j-1)K_n + 1, ..., jK_n\}$ because λ_1 is bounded, K_n is $o(n^{1-p})$, $m = K_n^{-1}r$, $t - jK_n$ is of the same order with K_n and $1/2 < p < 1$. For $t' := t + (j-1)K_n$ we can write the first term on the right side of (B.4) as:

$$\sup_{\lambda_0,\lambda_1}\left| m_n^{-1}\sum_{j=1}^{\lfloor m_n \rfloor} K_n^{-1}\sum_t \frac{t^k}{r^k}X_{t+R_n}e^{i(\lambda_0 t + \frac{\lambda_1}{n}t^2)}\right| = \tag{B.6}$$

$$\begin{aligned}&= \sup_{\lambda_0,\lambda_1}\left| m_n^{-1}\sum_{j=1}^{\lfloor m_n \rfloor} K_n^{-1}\sum_{t=1+(j-1)K_n}^{jK_n} \frac{t^k}{r^k}X_{t+R_n}e^{i(\lambda_0 t + \frac{\lambda_1}{n}j^2K_n^2 + D_n(j,t))}\right| \\ &\leq m_n^{-1}\sum_{j=1}^{\lfloor m_n \rfloor}\sup_{\lambda_0,\lambda_1}\left|e^{i(\lambda_0(j-1)K_n + \frac{\lambda_1}{n}j^2K_n^2)}\right| K_n^{-1}\left|\sum_{t=1}^{K_n}\frac{t'^k}{r^k}X_{t'+R_n}e^{i(\lambda_0 t + D_n(j,t'))}\right| \\ &= m_n^{-1}\sum_{j=1}^{\lfloor m_n \rfloor}\sup_{\lambda_0,\lambda_1} K_n^{-1}\left|\sum_{t=1}^{K_n}\frac{t'^k}{r^k}X_{t'+R_n}\left[e^{i(\lambda_0 t)} + e^{i(\lambda_0 t)}\left(e^{iD_n(j,t')} - 1\right)\right]\right| \\ &\leq m_n^{-1}\sum_{j=1}^{\lfloor m_n \rfloor}\sup_{\lambda_0,\lambda_1} K_n^{-1}\left|\sum_{t=1}^{K_n}\frac{t'^k}{r^k}X_{t'+R_n}e^{i(\lambda_0 t)}\right| \\ &\qquad + m_n^{-1}\sum_{j=1}^{\lfloor m_n \rfloor}\sup_{\lambda_0,\lambda_1} K_n^{-1}\left|\sum_{t=1}^{K_n}\frac{t'^k}{r^k}X_{t'+R_n}e^{i(\lambda_0 t)}\left(e^{iD_n(j,t')} - 1\right)\right| \\ &\leq m_n^{-1}\sum_{j=1}^{\lfloor m_n \rfloor}\sup_{\lambda_0,\lambda_1} K_n^{-1}\left|\sum_{t=1}^{K_n}\frac{t'^k}{r^k}X_{t'+R_n}e^{i(\lambda_0 t)}\right|\end{aligned}$$

$$+ m_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} \sup_{\substack{\lambda_0,\lambda_1,\\ j,t'}} K_n^{-1} \sum_{t=1}^{K_n} |X_{t'+R_n}| \left| e^{iD_n(j,t')} - 1 \right|$$

$$\le m_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} \sup_{\lambda_0,\lambda_1} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{t'^k}{r^k} X_{t'+R_n} e^{i(\lambda_0 t)} \right|$$

$$+ \sup_{\substack{\lambda_0,\lambda_1,\\ j,t'}} \left| e^{iD_n(j,t')} - 1 \right| m_n^{-1} K_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} \sum_{t=1}^{K_n} |X_{t'+R_n}|$$

$$\le m_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} \sup_{\lambda_0,\lambda_1} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{t'^k}{r^k} X_{t'+R_n} e^{i(\lambda_0 t)} \right| + \sup_{\substack{\lambda_0,\lambda_1,\\ j,t'}} \left| e^{iD_n(j,t')} - 1 \right| r^{-1} \sum_{t=1}^{r} |X_t|. \qquad \text{(B.7)}$$

The second one of these two terms goes to zero, as $D_n(j,t)$ goes to zero uniformly for all $j, t, \lambda_0, \lambda_1$ and $\frac{1}{r}\sum_t |X_t|$ converges to $E|X_t| < \infty$. For the first term we have the following:

$$\sup_{\lambda_0} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{t'^k}{r^k} X_{t'+R_n} e^{i(\lambda_0 t)} \right| = \sup_{\lambda_0} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{[(j-1)K_n + t]^k}{r^k} X_{t'+R_n} e^{i(\lambda_0 t)} \right|$$

$$\le \sup_{\lambda_0} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{[(j-1)K_n]^k}{r^k} X_{t'+R_n} e^{i(\lambda_0 t)} \right| +$$

$$\sup_{\lambda_0} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{\sum_{i=1}^{k} \binom{k}{i} t^i [(j-1)K_n]^{k-i}}{r^k} X_{t'+R_n} e^{i(\lambda_0 t)} \right|$$

The last term is

$$\le k \sup_{\lambda_0} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{t}{r} X_{t'+R_n} e^{i(\lambda_0 t)} \right| \le k \frac{K_n}{r} K_n^{-1} \sum_{t=1}^{K_n} |X_{t'+R_n}|$$

which goes to zero as $\mathbf{E}|X_t| < \infty$ and $K_n = o(n^{1-p})$, $r = O(n^p)$ with $p > 1/2$. Thus the dominating term in (B.7) is

$$m_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} \sup_{\lambda_0} K_n^{-1} \left| \sum_{t=1}^{K_n} \frac{[(j-1)K_n]^k}{r^k} X_{t'+R_n} e^{i(\lambda_0 t)} \right|$$

$$\le m_n^{-1} \sum_{j=1}^{\lfloor m_n \rfloor} \sup_{\lambda_0} K_n^{-1} \left| \sum_{t=1}^{K_n} X_{t'+R_n} e^{i(\lambda_0 t)} \right|.$$

Now let us for $i \ge 1$ consider the function

$$g_i^{(K)}(X) := \sup_{\lambda_0} K^{-1} \left| \sum_{t=1}^{K} X_{(i-1)K+t} e^{i(\lambda_0 t)} \right| : \mathbb{R}^K \to \mathbb{R}.$$

$g(X)$ is measurable and because X_t is ergodic we know from the ergodic theory that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{1}^{m} g_i^{(K)}(X) = \mathbf{E} g_i^{(K)}(X) \overset{K\to\infty}{\longrightarrow} 0,$$

where the last limit equation is a standard result in time series analysis (e.g.[3, p.98]). With that the lemma is established. □

Note that for $p < 1/2$ the term $\frac{\lambda_1}{n}t^2$ goes to zero uniformly for all t in question. Thus it is easy to prove that the lemma also holds for this case, but this is not needed for our purposes below.

Lemma B.6. *If β is bounded and $m_n = O(n^p)$ with $1/2 < p < 1$, then*

$$\lim_{n\to\infty}\left|\frac{1}{m}\sum_{t=1}^{m} e^{i(\alpha t+\frac{\beta}{n}t^2)}\right| = 0$$

uniformly for all $|\alpha| + |\beta| > \delta$, for every $\delta > 0$ (for $|\alpha|$ mod 2π).

Proof. We split the m summands into groups of K_n members, so that $K_n = o(n^{1-p})$ and K_n goes to infinity. Then, for $\tilde{m}_n = K_n^{-1}m$ we have:

$$\begin{aligned}
&\left|\frac{1}{m}\sum_{t=1}^{m} e^{i(\alpha t+\frac{\beta}{n}t^2)}\right| \\
= &\left|\frac{1}{m}\left[\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor}\sum_{t=(j-1)K_n+1}^{jK_n} e^{i(\alpha t+\frac{\beta}{n}t^2)} + \sum_{s=\lfloor\tilde{m}_n\rfloor K_n+1}^{\tilde{m}_n K_n} e^{i(\alpha s+\frac{\beta}{n}s^2)}\right]\right| \\
\leq &\left|\frac{1}{m}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor}\sum_{t=(j-1)K_n+1}^{jK_n} e^{i(\alpha t+\frac{\beta}{n}t^2)}\right| + \frac{1}{m}\sum_{s=\lfloor\tilde{m}_n\rfloor K_n+1}^{\tilde{m}_n K_n}\left|e^{i(\alpha s+\frac{\beta}{n}s^2)}\right| \\
\leq &\left|\frac{1}{m}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor}\sum_{t=(j-1)K_n+1}^{jK_n} e^{i(\alpha t+\frac{\beta}{n}t^2)}\right| + \frac{K_n}{m},
\end{aligned}$$

with K_n/m being $o(n^{1-2p})$ and therefor converging to zero. Now, we can write for all $j \in \{1, ..., \lfloor\tilde{m}_n\rfloor\}, t \in \{(j-1)K_n + 1, ..., jK_n\}$:

$$\begin{aligned}
\frac{\lambda_1}{n}t^2 &= \frac{\lambda_1}{n}(t - jK_n + jK_n)^2 \qquad \text{(B.8)}\\
&= \frac{\lambda_1}{n}(jK_n)^2 + \frac{\lambda_1}{n}(t - jK_n)^2 + 2\frac{\lambda_1}{n}(t - jK_n)jK_n \\
&= \frac{\lambda_1}{n}(jK_n)^2 + D_n(j, t),
\end{aligned}$$

where $D_n(j, t) = 2\frac{\lambda_1}{n}(t-jK_n)jK_n+\frac{\lambda_1}{n}(t-jK_n)^2$ is $o(1)$ uniformly for all $j \in \{1, ..., \lfloor\tilde{m}_n\rfloor\}$, $t \in \{(j-1)K_n + 1, ..., jK_n\}$ because λ_1 is bounded, K_n is $o(n^{1-p})$, $m = O(n^p)$, $t - jK_n$

is of the same order with K_n and $1/2 < p < 1$. Then

$$\begin{aligned}
&\left|\frac{1}{m}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor}\sum_{t=(j-1)K_n+1}^{jK_n} e^{i(\alpha t+\frac{\beta}{n}t^2)}\right| = \left|\frac{1}{m}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor}\sum_{t} e^{i(\alpha t+\frac{\beta}{n}j^2K_n^2+D_n(j,t))}\right| \\
\leq\ & \tilde{m}_n^{-1}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor}\left|e^{i(\alpha(j-1)K_n+\frac{\beta}{n}j^2K_n^2)}\right| K_n^{-1}\left|\sum_{t=1}^{K_n} e^{i(\alpha t+D_n(j,t'))}\right| \\
=\ & \tilde{m}_n^{-1}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor} K_n^{-1}\left|\sum_{t=1}^{K_n}\left[e^{i(\alpha t)}+e^{i(\alpha t)}\left(e^{iD_n(j,t')}-1\right)\right]\right| \\
\leq\ & \tilde{m}_n^{-1}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor} K_n^{-1}\left|\sum_{t=1}^{K_n} e^{i(\alpha t)}\right| + \tilde{m}_n^{-1}\sum_{j=1}^{\lfloor\tilde{m}_n\rfloor} K_n^{-1}\sum_{t=1}^{K_n}\left|e^{iD_n(j,t')}-1\right| \\
\leq\ & K_n^{-1}\left|\sum_{t=1}^{K_n} e^{i(\alpha t)}\right| + \max_{j,t}\left|e^{iD_n(j,t')}-1\right|,
\end{aligned}$$

where $t' = t+(j-1)K_n$. The second vanishes as n grows (see definition of $D_n(j,t)$ above), while the first one converges to zero uniformly for all $|\alpha| > \delta_\alpha \pmod{2\pi}$, for every $\delta_\alpha > 0$. Altogether we have:

$$\left|\frac{1}{m}\sum_{t=1}^{m} e^{i(\alpha t+\frac{\beta}{n}t^2)}\right| \text{ converges to zero uniformly for all } |\alpha| > \delta_\alpha \pmod{2\pi}, \text{ for every } \delta_\alpha > 0. \tag{B.9}$$

On the other hand, we can express the square of the same norm as follows:

$$\left|\frac{1}{m}\sum_{t=1}^{m} e^{i(\alpha t+\frac{\beta}{n}t^2)}\right|^2 = \left[\frac{1}{m}\sum_{t=1}^{m}\cos(\alpha t+\frac{\beta}{n}t^2)\right]^2 + \left[\frac{1}{m}\sum_{t=1}^{m}\sin(\alpha t+\frac{\beta}{n}t^2)\right]^2.$$

In the following we discuss the behavior of the first term of the right side, while the behavior of the second one is essentially the same. We have:

$$\begin{aligned}
&\cos(\alpha t+\frac{\beta}{n}t^2) \\
\leq\ & \int_{t-1}^{t}\cos(\alpha x+\frac{\beta}{n}x^2)dx + \sup_{x\in[t-1,t]}\cos(\alpha x+\frac{\beta}{n}x^2) - \inf_{x\in[t-1,t]}\cos(\alpha x+\frac{\beta}{n}x^2),
\end{aligned}$$

which, if we sum on both sides, gives:

$$\frac{1}{m}\sum_{t=1}^{m}\cos(\alpha t+\frac{\beta}{n}t^2) \leq \tag{B.10}$$

$$
\begin{aligned}
&\leq \frac{1}{m}\int_0^m \cos(\alpha x + \frac{\beta}{n}x^2)dx \\
&+ \frac{1}{m}\sum_{t=1}^{m}\left[\sup_{x\in[t-1,t]}\cos(\alpha x + \frac{\beta}{n}x^2) - \inf_{x\in[t-1,t]}\cos(\alpha x + \frac{\beta}{n}x^2)\right] \\
&\leq \frac{1}{m}\left[\int_0^m \cos(\alpha x + \frac{\beta}{n}x^2)dx + \sum_{t=1}^{m}\sup_{x,x'\in[t-1,t]}\left|\alpha x + \frac{\beta}{n}x^2 - \alpha x' - \frac{\beta}{n}x'^2\right|\right] \\
&\leq \frac{1}{m}\int_0^m \cos(\alpha x + \frac{\beta}{n}x^2)dx + \frac{1}{m}\sum_{t=1}^{m}\left[|\alpha| + 2\frac{|\beta|}{n}t - \frac{|\beta|}{n}\right] \\
&= \frac{1}{m}\int_0^m \cos(\alpha x + \frac{\beta}{n}x^2)dx + |\alpha| + o(n^{p-1})
\end{aligned}
$$

because $|\cos(x) - cos(y)| \leq |x-y|$ for all $x, y \in \mathbb{R}$. The integral on the right side of the last equation equals to (because $\cos(-x) = \cos(x)$ we can always assume that $\beta \geq 0$):

$$
\frac{1}{m}\int_0^m \cos(\alpha x + \frac{\beta}{n}x^2)dx \tag{B.11}
$$

$$
= \frac{1}{\sqrt{\beta}}\frac{\sqrt{n}}{m}\sqrt{\frac{\pi}{2}}\left[\cos\left(\frac{\alpha^2 n}{4\beta}\right)C\left(\frac{\alpha n + 2\beta x}{\sqrt{2\beta n\pi}}\right) + \sin\left(\frac{\alpha^2 n}{4\beta}\right)S\left(\frac{\alpha n + 2\beta x}{\sqrt{2\beta n\pi}}\right)\right]_0^m,
$$

where $C(x)$ and $S(x)$ are the Fresnel-C and Fresnel-S integrals, bounded for all $x \in \mathbb{R}$. This means that the above integral goes uniformly to zero for all $|\beta| > \delta_\beta$, for every $\delta_\beta > 0$. Putting together (B.10) and (B.11) we have for this second evaluation that:

$$
\text{For every } \epsilon_\beta \text{ there is } n_\beta \text{ with } \left|\frac{1}{\lfloor n^p \rfloor}\sum_{t=1}^{\lfloor n^p \rfloor} e^{i(\alpha t + \frac{\beta}{n}t^2)}\right| < |\alpha| + \epsilon_\beta \tag{B.12}
$$

$$
\text{for all } n \geq n_\beta \text{ and all } \beta : |\beta| > \delta_\beta, \text{ for every } \delta_\beta > 0.
$$

The combination of (B.9) and (B.12) establishes the lemma if we choose some $\delta_\alpha < \min\{\epsilon, \delta\}$, $\delta_\beta = \delta - \delta_\alpha$ and $\epsilon_\beta = \epsilon - \delta_\alpha$ with ϵ, δ being the quantities involved in the desired convergence of the lemma. □

Now we can prove the consistency theorem. We maximize the function in (3.17) divided by m, as this does not affect the maximizing arguments. We have:

$$
\left|m^{-1}\sum_{t=-m/2}^{m/2} Y_t e^{i(\lambda_0 t + \frac{\lambda_1}{2n}t^2)}\right|^2 \tag{B.13}
$$

$$
= \left|m^{-1}\sum_{t=-m/2}^{m/2}\left[\alpha_n^{(u_0)}\cos\left(\phi_n^{u_0}(t)\right) + \beta_n^{(u_0)}\sin\left(\phi_n^{u_0}(t)\right) + X(t)\right]e^{i(\lambda_0 t + \frac{\lambda_1}{2n}t^2)}\right|^2,
$$

where

$$\phi_n^{u_0}(t) = n\int_{u_0}^{u_0+\epsilon_n+t/n}\omega(x)dx \tag{B.14}$$
$$= t\omega(u_0) + t^2\frac{\omega'(u_0)}{2n} + n\epsilon_n\omega(u_0) + \frac{n\epsilon_n^2}{2}\omega'(u_0)$$
$$+ \underbrace{\epsilon_n t\omega'(u_0) + n\int_{u_0}^{u_0+\epsilon_n+t/n} R(x)dx}_{:=\tilde{R}_n^{u_0}(t)},$$

where $R(x)$ is defined in (3.18). Note that $\tilde{R}_n^{u_0}(t)$ vanishes for $n\to\infty$ uniformly for all $t\in\{-m/2:m/2\}$, $p<2/3$ because of (3.19) and the fact that $\epsilon_n\le 1/n$. In the case of a locally linear $\omega(u)$ it is from some n_0 on zero. The quantity $m^{-1}\sum_t X(t)e^{i(\lambda_0 t+\frac{\lambda_1}{2n}t^2)}$ goes to zero uniformly, for all $\lambda_0\in[0,\pi]$ and λ_1 bounded, according to Lemma B.5. In the following we describe the asymptotic behavior of $m^{-1}\sum_t\cos(\phi_n^{u_0}(t))\,e^{i(\lambda_0 t+\frac{\lambda_1}{2n}t^2)}$, as this of $m^{-1}\sum_t\sin(\phi_n^{u_0}(t))\,e^{i(\lambda_0 t+\frac{\lambda_1}{2n}t^2)}$ is exactly the same. We have:

$$\frac{1}{m}\sum_{t=-m/2}^{m/2}\cos(\phi_n^{u_0}(t))\,e^{i(\lambda_0 t+\frac{\lambda_1}{2n}t^2)}$$
$$= \frac{1}{2m}\sum_{t=-m/2}^{m/2}[e^{\phi_n^{u_0}(t)}+e^{-\phi_n^{u_0}(t)}]e^{i(\lambda_0 t+\frac{\lambda_1}{2n}t^2)} \qquad \text{and using (B.14):}$$
$$= \frac{\exp\left\{i\left(n\epsilon_n\omega(u_0)+\frac{n\epsilon_n^2}{2}\omega'(u_0)\right)\right\}}{2m}\sum_{t=-m/2}^{m/2} e^{i\tilde{R}_n^{u_0}(t)}e^{i\left[(\lambda_0+\omega(u_0))t+\frac{(\lambda_1+\omega'(u_0))}{2n}t^2\right]}$$
$$+ \frac{\exp\left\{-i\left(n\epsilon_n\omega(u_0)+\frac{n\epsilon_n^2}{2}\omega'(u_0)\right)\right\}}{2m}\sum_{t=-m/2}^{m/2} e^{-i\tilde{R}_n^{u_0}(t)}e^{-i\left[(\lambda_0-\omega(u_0))t+\frac{(\lambda_1-\omega'(u_0))}{2n}t^2\right]}$$
$$= \frac{\exp\left\{i\left(n\epsilon_n\omega(u_0)+\frac{n\epsilon_n^2}{2}\omega'(u_0)\right)\right\}}{2m}\sum_{t=-m/2}^{m/2} e^{i\left[(\lambda_0+\omega(u_0))t+\frac{(\lambda_1+\omega'(u_0))}{2n}t^2\right]}$$
$$+ \frac{\exp\left\{-i\left(n\epsilon_n\omega(u_0)+\frac{n\epsilon_n^2}{2}\omega'(u_0)\right)\right\}}{2m}\sum_{t=-m/2}^{m/2} e^{-i\left[(\lambda_0-\omega(u_0))t+\frac{(\lambda_1-\omega'(u_0))}{2n}t^2\right]}$$
$$+ \underbrace{o(n^{3p-2})}_{\to 0},$$

Now we distinguish two cases: If $\omega(u_0)>0$, according to Lemma B.6, this converges to $\frac{1}{2}\exp\left\{-i\left(n\epsilon_n\omega(u_0)+\frac{n\epsilon_n^2}{2}\omega'(u_0)\right)\right\}$ for $[\lambda_0,\lambda_1]=[\omega(u_0),\omega'(u_0)]$ and to zero uniformly for all $|\lambda_0-\omega(u_0)|+|\lambda_1-\omega'(u_0)|>\delta$ for every $\delta>0$. Altogether the right side of (B.13) converges to $\frac{\{\alpha_n^{u_0}\}^2+\{\beta_n^{u_0}\}^2}{4}=c>0$ for $[\lambda_0,\lambda_1]=[\omega(u_0),\omega'(u_0)]$ and to zero uniformly for all $|\lambda_0-\omega(u_0)|+|\lambda_1-\omega'(u_0)|>\delta$ for every $\delta>0$. If $\omega(u_0)=0$, then also $\omega'(u_0)=0$ because $\omega(u)\in[0,\pi]$ and it is differentiable. In this case the right side of (B.13) converges to $\{\alpha_n^{u_0}\}^2=\gamma^2>0$ for $[\lambda_0,\lambda_1]=[0,0]$ and to zero uniformly for all $|\lambda_0|+|\lambda_1|>\delta$

for every $\delta > 0$. The desired convergence follows now directly: If this were not the case, there would be a subsequence $[\widehat{\omega}_{\tilde{n}}(u_0), \widehat{\omega}'_{\tilde{n}}(u_0)]$ converging to some point $[x_0, y_0] \neq [\omega(u_0), \omega'(u_0)]$ or diverging. But $J_n(u_0; \widehat{\omega}_n(u_0), \widehat{\omega}'_n(u_0)) \geq J_n(u_0; \omega_n(u_0), \omega'_n(u_0)) \to c > 0$ while $J_n(u_0; \widehat{\omega}_{\tilde{n}}(u_0), \widehat{\omega}'_{\tilde{n}}(u_0)) \to 0$, which is a contradiction. □

B.2.2 Consistency of the phase estimator

Proof of Theorem 3.2

By using exactly the same arguments as in the proof of Theorem 2.2 we see that:

$$\begin{aligned} & J_n(u_0; \widehat{\omega}_n(u_0), \widehat{\omega}'_n(u_0)) - J_n(u_0; \omega(u_0), \omega'(u_0)) \\ = & \underbrace{\frac{1}{4}(\alpha_n^2 + \beta_n^2)}_{c} \left\{ \left| m^{-1} \sum_{t=1}^{m} e^{i[(\widehat{\omega}_n(u_0) - \omega(u_0))t + \frac{\widehat{\omega}'_n(u_0) - \omega'(u_0)}{2n} t^2]} \right|^2 - 1 \right\} + o(1). \end{aligned}$$

The difference has to be positive as the first arguments maximize the modified periodogram. The first part of the right side is less or equal to zero, thus it must converge to zero along with the positive $o(1)$ term. Thus the sum converges to 1. We will show that this is only possible if the statement of the theorem holds. Let us suppose that this is not the case. Then there must be a subsequence $\{n\} \subset \mathbb{N}$ for which either both extrema are bounded, or the limit is infinite, i.e. the maximum and/or the minimum of the function in question (without the norm) diverges/diverge.

We examine now the first case. We can write the polynomial function:

$$\begin{aligned} \tilde{f}_n(t) &= (\widehat{\omega}_n(u_0) - \omega(u_0))t + \frac{\widehat{\omega}'_n(u_0) - \omega'(u_0)}{2n} t^2, \qquad 1 \leq t \leq m \\ &= f_n(u) = a_n u + b_n u^2, \\ & \quad \text{for } a_n = (\widehat{\omega}_n(u_0) - \omega(u_0))m, \quad b_n = \frac{\widehat{\omega}'_n(u_0) - \omega'(u_0)}{2n} m^2, \quad u = \frac{t}{m}. \end{aligned}$$

Note that the number of u-points in some interval $(l_1, l_2) \subset [0, c]$ is of order $(l_2 - l_1)m$. Now we choose some $(l_1, l_2) \subset [0, c]$ and $(l_3, l_4) \subset [0, c]$ so that $0 < l_1 < l_2 < l_3 < l_4 < 2\pi$. Because of the boundedness of the extrema (see Lemma B.8), a_n and b_n have to be also bounded and along with them also the first derivative of $f_n(u)$. Thus, the Lebesgue measure of $\{u : u \in [0,1], f_n(u) \in (l_1, l_2)\}$ has a lower bound greater than zero, i.e. the number of $\frac{t}{m}$-points whose mapping is in (l_1, l_2) has a lower bound, let us say $L_{12}m$, with $0 < L_{12} < 1$. Let $L_{34}m > 0$ be the lower bound that refers to (l_3, l_4). Now we have:

$$\begin{aligned} & \left| m^{-1} \sum_{t=1}^{m} e^{i f_n(t)} \right| \\ = & \left| m^{-1} \left\{ \sum_{t: f_n(t) \in (l_1, l_2)} e^{i f_n(t)} + \sum_{t: f_n(t) \in (l_3, l_4)} e^{i f_n(t)} + \sum_{rest} e^{i f_n(t)} \right\} \right|, \end{aligned}$$

where in the two first sums of the right side we put only the first $L_{12}m$ and the first $L_{34}m$ t-points respectively that satisfy the condition. This last expression is:

$$\begin{aligned}
&\leq \frac{(1-L_{12}-L_{34})m}{m} \\
&+ m^{-1}\left\{\left|\sum_{t:f_n(t)\in(l_1,l_2)} e^{if_n(t)}\right| + \left|\sum_{t:f_n(t)\in(l_3,l_4)} e^{if_n(t)}\right|\right\} \\
&- m^{-2}\left\{2\left|\sum_{t:f_n(t)\in(l_1,l_2)} e^{if_n(t)}\right|\left|\sum_{t:f_n(t)\in(l_3,l_4)} e^{if_n(t)}\right|\cos(\theta)\right\},
\end{aligned}$$

where θ is the angle that form the arguments of these two norms and is strictly less than π, uniformly for all n because of the way l_1, l_2, l_3, l_4 are chosen! The norms themselves are of order $L_{12}m$ and $L_{34}m$ respectively. Thus, the last expression is asymptotically STRICTLY less than 1. And this is a contradiction.

Now we examine the case where at least one of the two extrema is diverging. For convenience suppose that only the sequence of the maximums diverges, as the proof is similar for the other cases. We choose all groups of $l_{1,k}, l_{2,k}, l_{3,k}, l_{4,k}$ with the properties: (i) $l_{j,k+1} = l_{j,k} + 2\pi$ for $k \in \mathbb{N}$ and $j \in \{1,2,3,4\}$ and (ii) $2k\pi < l_{1,k} < l_{2,k} < l_{3,k} < l_{4,k} < 2(k+1)\pi$. Note that the number of $f_n(t)$-points falling in $[2k\pi, 2(k+1)\pi]$ goes to infinity uniformly for all k such that $2(k+1)\pi \leq \sup_t f_n(t)$. By using similar arguments like in the previous case (see Lemma B.9) we can see that the number of t-points with $f_n(t) \in (l_{1,k}, l_{2,k})$ and the number of t-points with $f_n(t) \in (l_{3,k}, l_{4,k})$ for all $k : k \in \mathbb{N}$, $2(k+1)\pi \leq \sup_t f_n(t)$ is of order m. Thus we can go on exactly like in the previous case and attain the same contradiction. Thus the theorem is proved. □

Proof of Lemma 3.1

Before proving the consistency of $\alpha_n^{(u_0)}$ and $\beta_n^{(u_0)}$ in (3.14) we need the following lemma:

Lemma B.7. *For any* $(\kappa, \lambda) \neq (0(mod 2\pi), 0)$ *we have*

$$\lim_{n\to\infty} \frac{1}{m}\sum_{s=1}^{m} \cos^2\left(\kappa s + \frac{\lambda}{n}s^2\right) = \lim_{n\to\infty} \frac{1}{m}\sum_{s=1}^{m} \sin^2\left(\kappa s + \frac{\lambda}{n}s^2\right) = \frac{1}{2}$$

with $m = O(n^p)$, $0 < p < 1$.

We prove this statement only for the cosine-case, as the sine-case is essentially the same.

Proof. We have

$$\begin{aligned}
&\frac{1}{m}\sum_{s=1}^{m} \cos^2\left(\kappa s + \frac{\lambda}{n}s^2\right) \\
=&\frac{1}{4m}\sum_{s=1}^{m}\left[\exp\left\{i\left(\kappa s + \frac{\lambda}{n}s^2\right)\right\} + \exp\left\{-i\left(\kappa s + \frac{\lambda}{n}s^2\right)\right\}\right]^2
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{4m}\sum_{s=1}^{m}\left[\exp\left\{2i\left(\kappa s+\frac{\lambda}{n}s^2\right)\right\}+\exp\left\{-2i\left(\kappa s+\frac{\lambda}{n}s^2\right)\right\}+2\exp\{0\}\right]\\
=&\frac{1}{2}+\frac{1}{4m}\sum_{s=1}^{m}\left[\exp\left\{2i\left(\kappa s+\frac{\lambda}{n}s^2\right)\right\}+\exp\left\{-2i\left(\kappa s+\frac{\lambda}{n}s^2\right)\right\}\right]
\end{aligned}
$$

where the second part of the right side of the equation goes to zero following Lemma B.6. Thus this lemma is established. □

Now we prove the lemma. We have (suppose for simplicity $\epsilon_n = 0$, as the approach is exactly the same if this is not the case)

$$
\begin{aligned}
\begin{bmatrix} \hat{\alpha}_n^{(u_0)} \\ \hat{\beta}_n^{(u_0)} \end{bmatrix} &= \frac{1}{|X'X|}\begin{bmatrix} \sum_s \sin^2(\hat{\phi}_s^{(u_0)}) & -\sum_s \cos(\hat{\phi}_s^{(u_0)})\sin(\hat{\phi}_s^{(u_0)}) \\ -\sum_s \cos(\hat{\phi}_s^{(u_0)})\sin(\hat{\phi}_s^{(u_0)}) & \sum_s \cos^2(\hat{\phi}_s^{(u_0)}) \end{bmatrix}\times \\
&\times \begin{bmatrix} \sum_s \cos(\hat{\phi}_s^{(u_0)})(\alpha_n^{(u_0)}\cos(\phi_s^{(u_0)})+\beta_n^{(u_0)}\sin(\phi_s^{(u_0)})+X_s) \\ \sum_s \sin(\hat{\phi}_s^{(u_0)})(\alpha_n^{(u_0)}\cos(\phi_s^{(u_0)})+\beta_n^{(u_0)}\sin(\phi_s^{(u_0)})+X_s) \end{bmatrix}, \qquad \text{(B.15)}
\end{aligned}
$$

where $|X'X| = \left[\sum_s \sin^2(\hat{\phi}_s^{(u_0)})\right]\left[\sum_s \cos^2(\hat{\phi}_s^{(u_0)})\right] - \left[\sum_s \cos(\hat{\phi}_s^{(u_0)})\sin(\hat{\phi}_s^{(u_0)})\right]^2$, $\phi_s^{(u_0)}$ is the real phase function and $\hat{\phi}_s^{(u_0)} := s\widehat{\omega}_0 + \frac{s^2}{2n}\widehat{\omega}'_0$. We evaluate now all entries of the matrices in (B.15). We have (let $\omega_0 := \omega(u_0)$):

$$
\begin{aligned}
&\sum_{s=-m_l}^{m_r}\cos^2(\hat{\phi}_s^{(u_0)}) = \sum_{s=-m_l}^{m_r}\cos^2(s\widehat{\omega}_0+\frac{s^2}{2n}\widehat{\omega}'_0)\\
&=\sum_{s=-m_l}^{m_r}\cos^2(s\omega_0+\frac{s^2}{2n}\omega'_0+s(\widehat{\omega_0}-\omega_0)+\frac{s^2}{2n}(\widehat{\omega}'_0-\omega'_0))
\end{aligned}
$$

(and because of the mean value theorem)

$$
\begin{aligned}
&=\sum_{s=-m_l}^{m_r}\left[\cos(s\omega_0+\frac{s^2}{2n}\omega'_0)+O(s(\widehat{\omega_0}-\omega_0)+\frac{s^2}{2n}(\widehat{\omega}'_0-\omega'_0))\right]^2\\
&=\sum_{s=-m_l}^{m_r}\left[\cos^2(s\omega_0+\frac{s^2}{2n}\omega'_0)+O(s(\widehat{\omega_0}-\omega_0)+\frac{s^2}{2n}(\widehat{\omega}'_0-\omega'_0))\right]
\end{aligned}
$$

which because of Theorem 3.2 gives

$$
\sum_s \cos^2(\hat{\phi}_s^{(u_0)}) = \sum_s \cos^2(s\omega_0+\frac{s^2}{2n}\omega'_0)+o(M). \qquad \text{(B.16)}
$$

By using exactly the same arguments we can write

$$
\sum_s \sin^2(\hat{\phi}_s^{(u_0)}) = \sum_s \sin^2(s\omega_0+\frac{s^2}{2n}\omega'_0)+o(M). \qquad \text{(B.17)}
$$

and

$$
\sum_s \sin(\hat{\phi}_s^{(u_0)})\cos(\hat{\phi}_s^{(u_0)}) = o(M). \qquad \text{(B.18)}
$$

On the other hand we have

$$n\int_{u_0}^{u_0+s/n}\omega(u)du = n\int_{u_0}^{u_0+s/n}[\omega_0+(u-u_0)\omega_0'+\frac{1}{2}(u-u_0)^2\omega''(\xi)]du$$
$$= s\omega_0+\frac{s^2}{2n}\omega_0'+\frac{s^3}{6n^2}\omega''(\xi) = s\omega_0+\frac{s^2}{2n}\omega_0'+O\left(\frac{s^3}{n^2}\right), \quad |u_0-\xi|\le|u_0-u|$$

and thus

$$\sum_{s=-m_l}^{m_r}\cos(\hat{\phi}_s^{(u_0)})\cos(\phi_s^{(u_0)}) = \sum_{s=-m_l}^{m_r}\cos(s\widehat{\omega}_0+\frac{s^2}{2n}\widehat{\omega}_0')\cos\left(n\int_{u_0}^{u_0+s/n}\omega(u)du\right)$$
$$= \sum_{s=-m_l}^{m_r}\cos\left(s\omega_0+\frac{s^2}{2n}\omega_0'+O\left(s(\widehat{\omega}_0-\omega_0)+\frac{s^2}{2n}(\widehat{\omega}_0'-\omega_0')\right)\right)\cdot$$
$$\cos\left(s\omega_0+\frac{s^2}{2n}\omega_0'+O\left(\frac{s^3}{n^2}\right)\right)$$
$$= \sum_s\cos^2(s\omega_0+\frac{s^2}{2n}\omega_0')+o(M), \tag{B.19}$$

because of Theorem 3.2 and the fact that $m_l, m_r = o(n^{2/3})$. Using exactly the same arguments we get

$$\sum_{s=-m_l}^{m_r}\sin(\hat{\phi}_s^{(u_0)})\sin(\phi_s^{(u_0)}) = \sum_s\sin^2(s\omega_0+\frac{s^2}{2n}\omega_0')+o(M), \tag{B.20}$$

$$\sum_{s=-m_l}^{m_r}\cos(\hat{\phi}_s^{(u_0)})\sin(\phi_s^{(u_0)}) = o(M), \tag{B.21}$$

$$\sum_{s=-m_l}^{m_r}\sin(\hat{\phi}_s^{(u_0)})\cos(\phi_s^{(u_0)}) = o(M), \tag{B.22}$$

while Lemma B.5 yealds

$$\sum_{s=-m_l}^{m_r}\sin(\hat{\phi}_s^{(u_0)})X_s = o(M) = \sum_{s=-m_l}^{m_r}\cos(\hat{\phi}_s^{(u_0)})X_s. \tag{B.23}$$

Using (B.16), (B.17), (B.18), (B.19), (B.20), (B.21), (B.22), (B.23) we can express (B.15) as follows:

$$\begin{bmatrix}\hat{\alpha}_n^{(u_0)}\\ \hat{\beta}_n^{(u_0)}\end{bmatrix} =$$
$$\frac{1}{|X'X|}\begin{bmatrix}\sum_s\sin^2\left(s\omega_0+\frac{s^2}{2n}\omega_0'\right)+o(M) & o(M)\\ o(M) & \sum_s\cos^2\left(s\omega_0+\frac{s^2}{2n}\omega_0'\right)+o(M)\end{bmatrix}\times$$
$$\times\begin{bmatrix}\alpha_n^{(u_0)}\sum_s\cos^2\left(s\omega_0+\frac{s^2}{2n}\omega_0'\right)+o(M)\\ \beta_n^{(u_0)}\sum_s\sin^2\left(s\omega_0+\frac{s^2}{2n}\omega_0'\right)+o(M)\end{bmatrix}, \tag{B.24}$$

where

$$|X'X| = \left[\sum_s \sin^2\left(s\omega_0 + \frac{s^2}{2n}\omega_0'\right)\right]\left[\sum_s \cos^2\left(s\omega_0 + \frac{s^2}{2n}\omega_0'\right)\right] + o(M^2).$$

Because of Lemma B.7 and since $\alpha_n^{u_0}$ and $\beta_n^{u_0}$ are bounded it is now easy to see from (B.24) that $[\hat{\alpha}_n^{(u_0)}, \hat{\beta}_n^{(u_0)}] - [\alpha_n^{(u_0)}, \beta_n^{(u_0)}]$ converges to zero . Thus we have the desired result and the lemma is established. □

Proof of Theorem 3.3

The assertion follows immediately from Lemma 3.1. □

B.2.3 Asymptotic normality in the signal approximation (frequency)

Proof of Lemma 3.2

We have for $j = 0, 1$ with $\omega_0 := \omega(u_0)$, $\omega_0' := \omega'(u_0)$ and $\phi_0 := \phi_{u_0}$ (we omit "n" from M_n)

$$\begin{aligned}
&- M^{-1/2}\left(\frac{n}{M}\right)^j \frac{\partial}{\partial\lambda_j}\check{J}_M(u_0;\omega(u_0),\omega'(u_0)) \\
&= i\frac{1}{(j+1)!}M^{-1/2}\times \\
&\times \left[\frac{\gamma}{2}e^{i\left(\omega_0 n\epsilon_n + \omega_0'\frac{n\epsilon_n^2}{2} + \phi_0\right)}\sum_s\left(\frac{s}{M}\right)^{j+1}\exp\left\{i\left(\omega_0' s\epsilon_n\right)\right\}\right. && \text{(B.25)} \\
&\quad + \frac{\gamma}{2}e^{-i\left(\omega_0 n\epsilon_n + \omega_0'\frac{n\epsilon_n^2}{2} + \phi_0\right)}\sum_s\left(\frac{s}{M}\right)^{j+1}\exp\left\{-i\left(2\omega_0 s + \omega_0'\frac{s^2}{n} + \omega_0' s\epsilon_n\right)\right\} && \text{(B.26)} \\
&\quad \left. + \sum_s\left(\frac{s}{M}\right)^{j+1}X_{n(u_0+\epsilon_n)+s}\exp\left\{-i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\right]\times && \text{(B.27)} \\
&\times \left[\frac{\gamma}{2}e^{-i\left(\omega_0 n\epsilon_n + \omega_0'\frac{n\epsilon_n^2}{2} + \phi_0\right)}\sum_s\frac{1}{M}\exp\left\{-i\left(\omega_0' s\epsilon_n\right)\right\}\right. && \text{(B.28)} \\
&\quad + \frac{\gamma}{2}e^{i\left(\omega_0 n\epsilon_n + \omega_0'\frac{n\epsilon_n^2}{2} + \phi_0\right)}\sum_s\frac{1}{M}\exp\left\{i\left(2\omega_0 s + \omega_0'\frac{s^2}{n} + \omega_0' s\epsilon_n\right)\right\} && \text{(B.29)} \\
&\quad \left. + \sum_s\frac{1}{M}X_{n(u_0+\epsilon_n)+s}\exp\left\{i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\right] && \text{(B.30)} \\
&+ cc.
\end{aligned}$$

where all sums are from $-m_n$ to m_n and cc is the complex conjugate. Now because $|\epsilon_n| \leq 1/n$ and $\exp\{\pm i(\omega_0' s\epsilon_n)\} = 1 \pm i\omega_0' \exp\{i(\omega_0'\xi_s\epsilon_n)\} s\epsilon_n$ for some $0 \leq \xi_s \leq s$ we get

$$\begin{aligned}
(B.25) &= \frac{\gamma}{2}\exp\left\{i\left(\omega_0 n\epsilon_n + \omega_0'\frac{n\epsilon_n^2}{2} + \phi_0\right)\right\}\sum_s\left(\frac{s}{M}\right)^{j+1} + O\left(\frac{m_n^2}{n}\right) \\
(B.28) &= \frac{\gamma}{2}\exp\left\{-i\left(\omega_0 n\epsilon_n + \omega_0'\frac{n\epsilon_n^2}{2} + \phi_0\right)\right\} + O\left(\frac{m_n}{n}\right)
\end{aligned}$$

$$(B.26) = o(m_n^{1/2}) = m_n(B.29),$$

because of Lemma B.1 and the fact that $0 < \omega_0 < \pi$ and

$$(B.30) = O_p(m_n^{-1/2}), \quad \text{because of Theorem B.1.}$$

Putting all the previous together we have for $j = 0$

$$\begin{aligned}
&-\frac{\partial}{\partial\lambda_0} M^{-1/2} \check{J}_M(u_0;\omega(u_0),\omega'(u_0)) \qquad \text{(B.31)}\\
&= i\frac{\gamma}{2} e^{-i\left(\omega_0 n\epsilon_n + \omega_0' \frac{n\epsilon_n^2}{2} + \phi_0\right)} \sum_s \frac{s}{M^{\frac{3}{2}}} X_{n(u_0+\epsilon_n)+s} \exp\left\{-i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\\
&\quad + o(1) + cc
\end{aligned}$$

because (B.25) becomes $O\left(\frac{m_n^2}{n}\right)$ for $j = 0$ as $\sum_{-m}^{m} s = 0$. We know that if z_1 and z_2 are zero mean complex random variables then $Cov\{Re(z_1), Re(z_2)\} = \frac{1}{2}Re(E\{z_1 z_2\} + E\{z_1 \bar{z}_2\})$. Using Theorem B.1 we see that

$$\sum_s \frac{s}{M_n^{3/2}} X_{n(u_0+\epsilon_n)+s} \exp\left\{-i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\} \xrightarrow{\mathfrak{D}} \mathcal{N}^C\left(0, 2\pi\frac{1}{12} f_{XX}(\omega_0)\right)$$

where $f_{XX}(\omega_0)$ is as in Theorem B.1. Plugging this into (B.31) and by repeated use of Theorem B.1 we get

$$-\frac{\partial}{\partial\lambda_0} M^{-1/2} \check{J}_M(u_0;\omega(u_0),\omega'(u_0)) \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, 2\pi\frac{\gamma^2}{24} f_{XX}(\omega_0)\right). \qquad \text{(B.32)}$$

Similarly for $j = 1$ we have

$$\begin{aligned}
&-\frac{\partial}{\partial\lambda_1} \frac{n}{M^{3/2}} \check{J}_M(u_0;\omega(u_0),\omega'(u_0)) \qquad \text{(B.33)}\\
&= i\frac{\gamma}{4} e^{-i\left(\omega_0 n\epsilon_n + \omega_0' \frac{n\epsilon_n^2}{2} + \phi_0\right)} \sum_s \frac{s^2}{M^{\frac{5}{2}}} X_{n(u_0+\epsilon_n)+s} \exp\left\{-i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\\
&+ i\frac{\gamma}{4} e^{i\left(\omega_0 n\epsilon_n + \omega_0' \frac{n\epsilon_n^2}{2} + \phi_0\right)} \left(\sum_s \frac{s^2}{M^3}\right) \sum_s \frac{1}{M^{\frac{1}{2}}} X_{n(u_0+\epsilon_n)+s} \exp\left\{i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\\
&+ i\frac{\gamma^2}{8} \sum_s \frac{s^2}{M^{5/2}} + o(1) + cc.
\end{aligned}$$

The term $i\frac{\gamma^2}{8}\sum_s \frac{s^2}{M^{5/2}}$ vanishes with its complex conjugate. Using again Theorem B.1 we get

$$\sum_s \frac{s^2}{M_n^{5/2}} X_{n(u_0+\epsilon_n)+s} \exp\left\{-i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\} \xrightarrow{\mathfrak{D}} \mathcal{N}^C\left(0, 2\pi\frac{1}{80} f_{XX}(\omega_0)\right)$$

$$\sum_s \frac{1}{M_n^{1/2}} X_{n(u_0+\epsilon_n)+s} \exp\left\{i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\} \xrightarrow{\mathfrak{D}} \mathcal{N}^C\Big(0, 2\pi f_{XX}(\omega_0)\Big)$$

because $f_{XX}(\lambda) = f_{XX}(-\lambda)$ by its definition. Thus we get with help again of Theorem B.1

$$-\frac{\partial}{\partial\lambda_1}\frac{n}{M^{3/2}}\check{J}_M(u_0;\omega(u_0),\omega'(u_0)) \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, 2\pi\frac{\gamma^2}{24}\frac{1}{10}f_{XX}(\omega_0)\right). \tag{B.34}$$

As for the covariance between (B.32) and (B.34), Theorem B.1 implies that it goes to zero because the terms $H_{ab}^T(0)$ that appear contain summations of the form $\sum_{-m}^{m} s^k$ with $k = 1, 3$ which is zero. Thus the lemma is established. □

Proof of Lemma 3.3

Because of (3.19) with $r = m_n/n$ and $L_n = \omega_0 n\epsilon_n + \frac{\omega_0'}{2}n\epsilon_n^2 + \phi_0$ we have

$$\begin{aligned}
&\frac{1}{M^2}\frac{\partial^2}{\partial\lambda_0^2}J_{M_n}(u_0;\lambda_0,\lambda_1) = \\
&-\Bigg(e^{L_n}\frac{\gamma}{2}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\omega_0-\lambda_0)s + \frac{(\omega_0'-\lambda_1)s^2}{2n} + \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+e^{-L_n}\frac{\gamma}{2}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(-\omega_0-\lambda_0)s + \frac{(-\omega_0'-\lambda_1)s^2}{2n} - \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+\frac{\gamma}{2}\frac{1}{M^3}\sum_{s=-m}^{m} s^2 X_{n(u_0+\epsilon_n)+s}\exp\left\{-i\left[\lambda_0 s + \frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)\times \\
&\times\Bigg(e^{L_n}\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m} \exp\left\{i\left[(\omega_0+\lambda_0)s + \frac{(\omega_0'+\lambda_1)s^2}{2n} + \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+e^{-L_n}\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m} \exp\left\{i\left[(\lambda_0-\omega_0)s + \frac{(\lambda_1-\omega_0')s^2}{2n} - \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m} X_{n(u_0+\epsilon_n)+s}\exp\left\{i\left[\lambda_0 s + \frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg) \\
&+\Bigg(e^{L_n}\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(\omega_0-\lambda_0)s + \frac{(\omega_0'-\lambda_1)s^2}{2n} + \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+e^{-L_n}\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(-\omega_0-\lambda_0)s + \frac{(-\omega_0'-\lambda_1)s^2}{2n} - \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} sX_{n(u_0+\epsilon_n)+s}\exp\left\{-i\left[\lambda_0 s + \frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)\times \\
&\times\Bigg(e^{L_n}\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(\omega_0+\lambda_0)s + \frac{(\omega_0'+\lambda_1)s^2}{2n} + \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+e^{-L_n}\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(\lambda_0-\omega_0)s + \frac{(\lambda_1-\omega_0')s^2}{2n} - \omega_0' s\epsilon_n + O\left(\frac{m^3}{n^2}\right)\right]\right\} \\
&+\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} sX_{n(u_0+\epsilon_n)+s}\exp\left\{i\left[\lambda_0 s + \frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg) + cc.
\end{aligned}$$

Thus, because of Lemmas B.5 and B.1 and the fact that $\epsilon_n < 1/n$

$$\frac{1}{M^2}\frac{\partial^2}{\partial\lambda_0^2}J_M(u_0;\lambda_0,\xi_{1,n})\Big|_{\lambda_0=\xi_{0,n}} =$$
$$-\left(\frac{\gamma}{2}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\omega_0-\xi_{0,n})s+\frac{(\omega_0'-\xi_{1,n})s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)\times$$
$$\times\left(\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m}\exp\left\{i\left[(\xi_{0,n}-\omega_0)s+\frac{(\xi_{1,n}-\omega_0')s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)$$
$$+\left(\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(\omega_0-\xi_{0,n})s+\frac{(\omega_0'-\xi_{1,n})s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)\times$$
$$\times\left(\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(\xi_{0,n}-\omega_0)s+\frac{(\xi_{1,n}-\omega_0')s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)$$
$$+\,o(1)+cc.$$

The last two brackets and their complex conjugate vanish because of (3.23), $O(\frac{m^3}{n^2})=o(1)$ and the fact that $\sum_{-m}^{m} s = 0$. Altogether, again because of (3.23) and the fact that $O(\frac{m^3}{n^2})=o(1)$ we get

$$\frac{1}{M^2}\frac{\partial^2}{\partial\lambda_0^2}J_M(u_0;\lambda_0,\xi_{1,n})\Big|_{\lambda_0=\xi_{0,n}} = -\frac{\gamma^2}{24}+o(1). \tag{B.35}$$

On the other side we have

$$\frac{n^2}{M^4}\frac{\partial^2}{\partial\lambda_1^2}J_{M_n}(u_0;\lambda_0,\lambda_1) =$$
$$-\Bigg(e^{L_n}\frac{\gamma}{8}\frac{1}{M^5}\sum_{s=-m}^{m} s^4\exp\left\{i\left[(\omega_0-\lambda_0)s+\frac{(\omega_0'-\lambda_1)s^2}{2n}+\omega_0's\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{8}\frac{1}{M^5}\sum_{s=-m}^{m} s^4\exp\left\{i\left[(-\omega_0-\lambda_0)s+\frac{(-\omega_0'-\lambda_1)s^2}{2n}-\omega_0's\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+\frac{\gamma}{8}\frac{1}{M^5}\sum_{s=-m}^{m} s^4X_{n(u_0+\epsilon_n)+s}\exp\left\{-i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)\times$$
$$\times\Bigg(e^{L_n}\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m}\exp\left\{i\left[(\omega_0+\lambda_0)s+\frac{(\omega_0'+\lambda_1)s^2}{2n}+\omega_0's\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m}\exp\left\{i\left[(\lambda_0-\omega_0)s+\frac{(\lambda_1-\omega_0')s^2}{2n}-\omega_0's\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m} X_{n(u_0+\epsilon_n)+s}\exp\left\{i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)$$
$$+\Bigg(e^{L_n}\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\omega_0-\lambda_0)s+\frac{(\omega_0'-\lambda_1)s^2}{2n}+\omega_0's\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(-\omega_0-\lambda_0)s+\frac{(-\omega_0'-\lambda_1)s^2}{2n}-\omega_0's\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$

$$+\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2 X_{n(u_0+\epsilon_n)+s}\exp\left\{-i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)\times$$
$$\times\Bigg(e^{L_n}\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\omega_0+\lambda_0)s+\frac{(\omega_0'+\lambda_1)s^2}{2n}+\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\lambda_0-\omega_0)s+\frac{(\lambda_1-\omega_0')s^2}{2n}-\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2 X_{n(u_0+\epsilon_n)+s}\exp\left\{i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)+cc.$$

Now, using exactly the same arguments as before, we get

$$\frac{n^2}{M^4}\frac{\partial^2}{\partial\lambda_1^2}J_M(u_0;\xi_{0,n},\lambda_1)\Big|_{\lambda_1=\xi_{1,n}}=-\frac{1}{60}\frac{\gamma^2}{24}+o(1). \tag{B.36}$$

Finally

$$\frac{n}{M_n^3}\frac{\partial^2}{\partial\lambda_0\partial\lambda_1}J_M(u_0;\lambda_0,\lambda_1)=$$
$$-\Bigg(e^{L_n}\frac{\gamma}{8}\frac{1}{M^4}\sum_{s=-m}^{m} s^3\exp\left\{i\left[(\omega_0-\lambda_0)s+\frac{(\omega_0'-\lambda_1)s^2}{2n}+\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{8}\frac{1}{M^4}\sum_{s=-m}^{m} s^3\exp\left\{i\left[(-\omega_0-\lambda_0)s+\frac{(-\omega_0'-\lambda_1)s^2}{2n}-\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+\frac{\gamma}{8}\frac{1}{M^4}\sum_{s=-m}^{m} s^3 X_{n(u_0+\epsilon_n)+s}\exp\left\{-i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)\times$$
$$\times\Bigg(e^{L_n}\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m}\exp\left\{i\left[(\omega_0+\lambda_0)s+\frac{(\omega_0'+\lambda_1)s^2}{2n}+\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m}\exp\left\{i\left[(\lambda_0-\omega_0)s+\frac{(\lambda_1-\omega_0')s^2}{2n}-\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m} X_{n(u_0+\epsilon_n)+s}\exp\left\{i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)$$
$$+\Bigg(e^{L_n}\frac{\gamma}{4}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(\omega_0-\lambda_0)s+\frac{(\omega_0'-\lambda_1)s^2}{2n}+\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{4}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(-\omega_0-\lambda_0)s+\frac{(-\omega_0'-\lambda_1)s^2}{2n}-\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+\frac{\gamma}{4}\frac{1}{M^2}\sum_{s=-m}^{m} sX_{n(u_0+\epsilon_n)+s}\exp\left\{-i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)\times$$
$$\times\Bigg(e^{L_n}\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\omega_0+\lambda_0)s+\frac{(\omega_0'+\lambda_1)s^2}{2n}+\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$
$$+e^{-L_n}\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\lambda_0-\omega_0)s+\frac{(\lambda_1-\omega_0')s^2}{2n}-\omega_0' s\epsilon_n+O\left(\frac{m^3}{n^2}\right)\right]\right\}$$

$$+\frac{\gamma}{4}\frac{1}{M^3}\sum_{s=-m}^{m} s^2 X_{n(u_0+\epsilon_n)+s}\exp\left\{i\left[\lambda_0 s+\frac{\lambda_1 s^2}{2n}\right]\right\}\Bigg)+cc.$$

Again because of Lemmas B.5 and B.1 and the fact that $\epsilon_n < 1/n$ we have

$$\frac{1}{M^2}\frac{\partial^2}{\partial\lambda_0^2}J_M(u_0;\lambda_0,\lambda_1)\Big|_{\substack{\lambda_0=\xi_{0,n}\\ \lambda_1=\xi_{1,n}}} =$$

$$-\left(\frac{\gamma}{2}\frac{1}{M^4}\sum_{s=-m}^{m} s^3\exp\left\{i\left[(\omega_0-\xi_{0,n})s+\frac{(\omega_0'-\xi_{1,n})s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)\times$$

$$\times\left(\frac{\gamma}{2}\frac{1}{M}\sum_{s=-m}^{m}\exp\left\{i\left[(\xi_{0,n}-\omega_0)s+\frac{(\xi_{1,n}-\omega_0')s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)$$

$$+\left(\frac{\gamma}{2}\frac{1}{M^2}\sum_{s=-m}^{m} s\exp\left\{i\left[(\omega_0-\xi_{0,n})s+\frac{(\omega_0'-\xi_{1,n})s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)\times$$

$$\times\left(\frac{\gamma}{2}\frac{1}{M^3}\sum_{s=-m}^{m} s^2\exp\left\{i\left[(\xi_{0,n}-\omega_0)s+\frac{(\xi_{1,n}-\omega_0')s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right)$$

$$+o(1)+cc.$$

Now for $k=1,3$

$$\left|\frac{1}{M^{k+1}}\sum_{s=-m}^{m} s^k\exp\left\{i\left[(\omega_0-\xi_{0,n})s+\frac{(\omega_0'-\xi_{1,n})s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}\right|$$

$$\leq\left|\frac{1}{M^{k+1}}\sum_{s=-m}^{m} s^k\right|+$$

$$+\frac{1}{M^{k+1}}\sum_{s=-m}^{m}|s^k|\left|\exp\left\{i\left[(\omega_0-\xi_{0,n})s+\frac{(\omega_0'-\xi_{1,n})s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}-1\right|$$

$$=\sup_{-m\leq s\leq m}\left|\exp\left\{i\left[(\omega_0-\xi_{0,n})s+\frac{(\omega_0'-\xi_{1,n})s^2}{2n}+O\left(\frac{m^3}{n^2}\right)\right]\right\}-1\right|\frac{1}{M^{k+1}}\sum_{s=-m}^{m}|s^k|$$

where the last equality holds because $\sum_{s=-m}^{m} s^k = 0$. The last line goes to zero as the argument in the exp-function goes uniformly for all s to zero by assumption. Thus the lemma is proved. □

Proof of Theorem 3.4

The starting point of the proof of the asymptotic normality is the two following applications of the mean value theorem for the points (ω_0,ω_0') and $(\widehat{\omega}_{0,n},\widehat{\omega}_{0,n}')$. We have (we omit n from $\widehat{\omega}_{0,n},\widehat{\omega}_{0,n}'$)

$$\frac{1}{M^{1/2}}\frac{\partial\check{J}_M(u_0;\lambda_0,\widehat{\omega}_0')}{\partial\lambda_0}\Bigg|_{\lambda_0=\widehat{\omega}_0}-\frac{1}{M^{1/2}}\frac{\partial\check{J}_M(u_0;\lambda_0,\omega_0')}{\partial\lambda_0}\Bigg|_{\lambda_0=\omega_0}= \tag{B.37}$$

$$=\frac{1}{M^2}\frac{\partial^2\check{J}_M(u_0;\lambda_0,\xi_{1,n})}{\partial\lambda_0^2}\Bigg|_{\lambda_0=\xi_{0,n}} M^{3/2}(\widehat{\omega}_0-\omega_0)$$

$$+\frac{n}{M^3}\frac{\partial^2 \check{J}_M(u_0;\lambda_0,\lambda_1)}{\partial\lambda_0\partial\lambda_1}\Bigg|_{\substack{\lambda_0=\xi_{0,n}\\ \lambda_1=\xi_{1,n}}}\frac{M^{5/2}}{n}(\widehat{\omega}_0'-\omega_0')$$

$$\frac{n}{M^{3/2}}\frac{\partial \check{J}_M(u_0;\widehat{\omega}_0,\lambda_1)}{\partial\lambda_1}\Bigg|_{\lambda_1=\widehat{\omega}_0'}-\frac{n}{M^{3/2}}\frac{\partial \check{J}_M(u_0;\omega_0,\lambda_1)}{\partial\lambda_1}\Bigg|_{\lambda_1=\omega_0'}= \tag{B.38}$$

$$=\frac{n^2}{M^4}\frac{\partial^2 \check{J}_M(u_0;\tilde{\xi}_{0,n},\lambda_1)}{\partial\lambda_1^2}\Bigg|_{\lambda_1=\tilde{\xi}_{1,n}}\frac{M^{5/2}}{n}(\widehat{\omega}_0'-\omega_0')$$

$$+\frac{n}{M^3}\frac{\partial^2 \check{J}_M(u_0;\lambda_0,\lambda_1)}{\partial\lambda_1\partial\lambda_0}\Bigg|_{\substack{\lambda_0=\tilde{\xi}_{0,n}\\ \lambda_1=\tilde{\xi}_{1,n}}}M^{3/2}(\widehat{\omega}_0-\omega_0)$$

for some points $(\xi_{0,n},\xi_{1,n})$ and $(\tilde{\xi}_{0,n},\tilde{\xi}_{1,n})$ between (ω_0,ω_0') and $(\widehat{\omega}_0,\widehat{\omega}_0')$. Note that the first terms of the left side of both equations are zero, as $(\widehat{\omega}_0,\widehat{\omega}_0')$ maximizes the modified periodogram. If we solve (B.38) for $\frac{M_n^{5/2}}{n}(\widehat{\omega}_0'-\omega_0')$ and then plug it in (B.37) and make use of Lemma 3.3 and Theorem 3.2 we get

$$o(1)-\frac{1}{M^{1/2}}\frac{\partial \check{J}_M(u_0;\lambda_0,\omega_0')}{\partial\lambda_0}\Bigg|_{\omega_0}=\left(\frac{1}{M^2}\frac{\partial^2 \check{J}_M(u_0;\lambda_0,\xi_{1,n})}{\partial\lambda_0^2}\Bigg|_{\xi_{0,n}}+o(1)\right)M^{3/2}(\widehat{\omega}_0-\omega_0).$$

The result for $(\widehat{\omega}_0-\omega_0)$ follows now directly by use of Lemmas 3.2 and 3.3. The result for $(\widehat{\omega}_0'-\omega_0')$ is achieved by following the reverse procedure. The asymptotic independence also follows directly from Lemma 3.2.

B.2.4 MSE and asymptotic normality

Proof of Lemma 3.4

We have ('cc' means 'complex conjugate')

$$J_M'\Big(u_0,\omega_0,\omega_0'\Big)=-\,i\left(\frac{1}{M}\sum_{s=-m_\ell}^{m_r}s\,Y_{n(u_0+\epsilon_n)+s,n}\exp\left\{-i\left(\omega_0 s+\frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\right)\times \tag{B.39}$$
$$\times\left(\frac{1}{M}\sum_{s=-m_\ell}^{m_r}Y_{n(u_0+\epsilon_n)+s,n}\exp\left\{i\left(\omega_0 s+\frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\right)+cc$$

In order to estimate the difference $J_M'(u_0,\omega_0)-\check{J}_M'(u_0,\omega_0)$ we need to replace in both summands the terms $Y_{n(u_0+\epsilon_n)+s,n}$ by $\check{Y}_{n(u_0+\epsilon_n)+s}(u_0)$. We use the formula

$$y_1y_2-x_1x_2=(y_1-x_1)x_2+x_1(y_2-x_2)+(y_1-x_1)(y_2-x_2), \tag{B.40}$$

that is we have

$$J_M'(u_0,\omega_0,\omega_0')-\check{J}_m'(u_0,\omega_0,\omega_0')=-i\Big[(i)\times(iv)+(ii)\times(iii)+(i)\times(iii)\Big]+cc \tag{B.41}$$

with

$$S_{n(u_0+\epsilon_n)+s,n} = \gamma\,\cos\big(a+c+d\big) + O\Big(n\Big(\frac{s}{n}\Big)^3\Big) + O\Big(\frac{|s|+1}{n}\Big) \tag{B.42}$$
$$\check{S}_{n(u_0+\epsilon_n)+s,n} = \gamma\,\cos\big(a+c+d\big) + O\Big(\frac{|s|+1}{n}\Big)$$

with

$$a := \frac{n}{2}\Big(\frac{s}{n}\Big)^2\omega'(u_0), \quad c := \omega(u_0)\,s, \quad d := \phi_{u_0} + \omega(u_0)\,n\epsilon_n, \tag{B.43}$$

and

$$(i) := \frac{1}{M}\sum_{s=-m}^{m} s\,\big(Y_{n(u_0+\epsilon_n)+s,n} - \check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\big)\,\exp\Big\{-i\Big(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\Big)\Big\}$$
$$= \frac{1}{M}\sum_{s=-m}^{m} s\,\big(S_{n(u_0+\epsilon_n)+s,n} - \check{S}_{n(u_0+\epsilon_n)+s}(u_0)\big)\,\exp\Big\{-i\Big(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\Big)\Big\}$$
$$= \frac{\gamma}{M}\sum_{s=-m}^{m} s\Big[\cos\big(a+c+d\big) - \cos\big(a+c+d\big) + O\Big(\frac{|s|^3}{n^2}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big]\times$$
$$\times\exp\{-i(a+b)\}$$
$$= O\Big(\frac{m^4}{n^2}\Big) + O\Big(\frac{m^2}{n}\Big) = O\Big(\frac{m^4}{n^2}\Big) \tag{B.44}$$

$$(ii) := \frac{1}{M}\sum_{s=-m}^{m} s\,\check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp\Big\{-i\Big(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\Big)\Big\}$$

$$(iii) := \frac{1}{M}\sum_{s=-m}^{m} \big(Y_{n(u_0+\epsilon_n)+s,n} - \check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\big)\,\exp\Big\{-i\Big(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\Big)\Big\}$$
$$= \frac{\gamma}{M}\sum_{s=-m}^{m} \Big[\cos\big(a+c+d\big) - \cos\big(a+c+d\big) + O\Big(\frac{|s|^3}{n^2}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big]\times$$
$$\times\exp\{i(a+b)\}$$
$$= O\Big(\frac{m^3}{n^2}\Big) + O\Big(\frac{m}{n}\Big) = O\Big(\frac{m^3}{n^2}\Big) \tag{B.45}$$

$$(iv) := \frac{1}{M}\sum_{s=-m}^{m} \check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp\Big\{-i\Big(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\Big)\Big\}$$

where a, c, d are like in (B.43). We now construct upper bounds for these terms.

(ii) We obtain with $\omega_0 = \omega(u_0)$ and $\omega_0' = \omega'(u_0)$

$$\frac{1}{M}\sum_{s=-m}^{m} s\,\check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp\Big\{-i\Big(\omega_0 s + \frac{\omega_0' s^2}{2n}\Big)\Big\} =$$
$$= \frac{1}{M}\sum_{s=-m}^{m} s\,\big(\check{S}_{n(u_0+\epsilon_n)+s}(u_0) + X_{n(u_0+\epsilon_n)+s}\big)\,\exp\Big\{-i\Big(\omega_0 s + \frac{\omega_0' s^2}{2n}\Big)\Big\}$$

$$= \frac{1}{M}\sum_{s=-m}^{m} s\,\frac{\gamma}{2}\Big[\exp\{i(a+c+d)\} + \exp\{-i(a+c+d)\} + O\Big(\frac{|s|+1}{n}\Big)\Big]\exp\{-i(a+c)\}$$

$$+ \frac{1}{M}\sum_{s=-m}^{m} s\,X_{n(u_0+\epsilon_n)+s}\,\exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$

$$= \frac{1}{M}\sum_{s=-m}^{m} s\,X_{n(u_0+\epsilon_n)+s}\,\exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\} + O\left(\frac{m^2}{n}\right)$$

where the last line holds because of Lemma B.1.

(iv) We obtain with $\omega_0 = \omega(u_0)$ and $\omega_0' = \omega'(u_0)$

$$\frac{1}{M}\sum_{t=-m}^{m} \check{Y}_{n(u_0+\epsilon_n)+t}(u_0)\,\exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$

$$= \frac{1}{M}\sum_{t=-m}^{m} \big(\check{S}_{n(u_0+\epsilon_n)+t}(u_0) + X_{n(u_0+\epsilon_n)+t}\big)\,\exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$

$$= \frac{1}{M}\sum_{s=-m}^{m} \frac{\gamma}{2}\Big[\exp\{i(a+c+d)\} + \exp\{-i(a+c+d)\} + O\Big(\frac{|s|+1}{n}\Big)\Big]\exp\{i(a+c)\}$$

$$+ \frac{1}{M}\sum_{s=-m}^{m} X_{n(u_0+\epsilon_n)+s}\,\exp\left\{i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$

$$= \frac{1}{M}\sum_{s=-m}^{m} X_{n(u_0+\epsilon_n)+s}\,\exp\left\{i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\} + O(1)$$

again because of Lemma B.1.

Since $\mathbf{E}\,\check{d}_m^{(0)}(u,\lambda_0,\lambda_1) = \mathbf{E}\,\check{d}_m^{(1)}(u,\lambda_0,\lambda_1) = 0$ for all u and λ we now obtain

$$\mathbf{E}\big(\check{B}_M'(u_0,\omega_0,\omega_0')\big) = \mathbf{E}\Big(-i\,\Big[(i)\times(iv) + (ii)\times(iii) + (i)\times(iii)\Big] + cc = O\Big(\frac{m^4}{n^2}\Big)\Big)$$

and with Theorem B.1 (note that (i) and (iii) are deterministic)

$$\mathrm{var}\big(\check{B}_M'(u_0,\omega_0,\omega_0')\big) = \mathrm{var}\Big(-i\,\Big[(i)\times(iv) + (ii)\times(iii)\Big] + cc = O\Big(\frac{m^7}{n^4}\Big)\Big)$$

This means that $\mathrm{MSE}\big(M^{-1/2}\check{B}_M'(u_0,\omega_0,\omega_0')\big) = O\left(\frac{m^7}{n^4}\right)$, which goes to zero for $m << n^{4/7}$. □

Proof of Lemma 3.5

We have ('cc' means 'complex conjugate')

$$J_M'\Big(u_0,\omega_0,\omega_0'\Big) = -\frac{i}{2n}\left(\frac{1}{M}\sum_{s=-m_\ell}^{m_r} s^2\,Y_{n(u_0+\epsilon_n)+s,n}\,\exp\left\{-i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\right)\times \tag{B.46}$$

$$\times\left(\frac{1}{M}\sum_{s=-m_\ell}^{m_r} Y_{n(u_0+\epsilon_n)+s,n}\,\exp\left\{i\left(\omega_0 s + \frac{\omega_0'}{2}\frac{s^2}{n}\right)\right\}\right) + cc$$

In order to estimate the difference $J'_M(u_0,\omega_0) - \check{J}'_M(u_0,\omega_0)$ we need to replace in both summands the terms $Y_{n(u_0+\epsilon_n)+s,n}$ by $\check{Y}_{n(u_0+\epsilon_n)+s}(u_0)$. We use the formula (B.40), that is we have

$$J'_M(u_0,\omega_0,\omega'_0) - \check{J}'_m(u_0,\omega_0,\omega'_0) = -\frac{i}{2n}\Big[(I)\times(IV) + (II)\times(III) + (I)\times(III)\Big] + cc \tag{B.47}$$

with (cf. (B.42), (A.34) and (3.21))

$$\begin{aligned}
(I) &:= \frac{1}{M}\sum_{s=-m}^{m} s^2\left(Y_{n(u_0+\epsilon_n)+s,n} - \check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right)\,\exp\left\{-i\left(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\right)\right\}\\
&= \frac{1}{M}\sum_{s=-m}^{m} s^2\left(S_{n(u_0+\epsilon_n)+s,n} - \check{S}_{n(u_0+\epsilon_n)+s}(u_0)\right)\,\exp\left\{-i\left(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\right)\right\}\\
&= \frac{\gamma}{M}\sum_{s=-m}^{m} s^2\Big[\cos\left(a+c+d\right) - \cos\left(a+c+d\right) + O\Big(\frac{|s|^3}{n^2}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big]\times\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\exp\{-i(a+b)\}\\
&= O\Big(\frac{m^5}{n^2}\Big) + O\Big(\frac{m^3}{n}\Big) = O\Big(\frac{m^5}{n^2}\Big)
\end{aligned} \tag{B.48}$$

$$(II) := \frac{1}{M}\sum_{s=-m}^{m} s^2\,\check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp\left\{-i\left(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\right)\right\}$$

$$\begin{aligned}
(III) &:= \frac{1}{M}\sum_{s=-m}^{m}\left(Y_{n(u_0+\epsilon_n)+s,n} - \check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\right)\,\exp\left\{-i\left(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\right)\right\}\\
&= \frac{\gamma}{M}\sum_{s=-m}^{m}\Big[\cos\left(a+c+d\right) - \cos\left(a+c+d\right) + O\Big(\frac{|s|^3}{n^2}\Big) + O\Big(\frac{|s|+1}{n}\Big)\Big]\times\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\exp\{i(a+b)\}\\
&= O\Big(\frac{m^3}{n^2}\Big) + O\Big(\frac{m}{n}\Big) = O\Big(\frac{m^3}{n^2}\Big)
\end{aligned} \tag{B.49}$$

$$(IV) := \frac{1}{M}\sum_{s=-m}^{m}\check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp\left\{-i\left(\omega(u_0)s + \frac{\omega'(u_0)s^2}{2n}\right)\right\}$$

where a, c, d are like in (B.43). We now construct upper bounds for these terms.

(II) We obtain with $\omega_0 = \omega(u_0)$ and $\omega'_0 = \omega'(u_0)$

$$\begin{aligned}
&\frac{1}{M}\sum_{s=-m}^{m} s^2\,\check{Y}_{n(u_0+\epsilon_n)+s}(u_0)\,\exp\left\{-i\left(\omega_0 s + \frac{\omega'_0 s^2}{2n}\right)\right\} =\\
&\quad= \frac{1}{M}\sum_{s=-m}^{m} s^2\left(\check{S}_{n(u_0+\epsilon_n)+s}(u_0) + X_{n(u_0+\epsilon_n)+s}\right)\,\exp\left\{-i\left(\omega_0 s + \frac{\omega'_0 s^2}{2n}\right)\right\}
\end{aligned}$$

$$= \frac{1}{M} \sum_{s=-m}^{m} s^2 \frac{\gamma}{2} \Big[\exp\{i(a+c+d)\} + \exp\{-i(a+c+d)\} + O\Big(\frac{|s|+1}{n}\Big)\Big] \times$$
$$\times \exp\{-i(a+c)\}$$
$$+ \frac{1}{M} \sum_{s=-m}^{m} s^2 X_{n(u_0+\epsilon_n)+s} \exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$
$$= \frac{1}{M} \sum_{s=-m}^{m} s^2 X_{n(u_0+\epsilon_n)+s} \exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\} + O\left(\frac{m^3}{n}\right)$$

where the last line holds because of Lemma B.1.

(IV) We obtain with $\omega_0 = \omega(u_0)$ and $\omega_0' = \omega'(u_0)$

$$\frac{1}{M} \sum_{t=-m}^{m} \check{Y}_{n(u_0+\epsilon_n)+t}(u_0) \exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$
$$= \frac{1}{M} \sum_{t=-m}^{m} \big(\check{S}_{n(u_0+\epsilon_n)+t}(u_0) + X_{n(u_0+\epsilon_n)+t}\big) \exp\left\{-i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$
$$= \frac{1}{M} \sum_{s=-m}^{m} \frac{\gamma}{2} \Big[\exp\{i(a+c+d)\} + \exp\{-i(a+c+d)\} + O\Big(\frac{|s|+1}{n}\Big)\Big] \exp\{i(a+c)\}$$
$$+ \frac{1}{M} \sum_{s=-m}^{m} X_{n(u_0+\epsilon_n)+s} \exp\left\{i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\}$$
$$= \frac{1}{M} \sum_{s=-m}^{m} X_{n(u_0+\epsilon_n)+s} \exp\left\{i\left(\omega_0 s + \frac{\omega_0' s^2}{2n}\right)\right\} + O(1)$$

again because of Lemma B.1.

Since $\mathbf{E}\, \check{d}_m^{(0)}(u, \lambda_0, \lambda_1) = \mathbf{E}\, \check{d}_m^{(1)}(u, \lambda_0, \lambda_1) = 0$ for all u and λ we now obtain

$$\mathbf{E}\big(\check{B}_M'(u_0, \omega_0, \omega_0')\big) = \mathbf{E}\Big(-\frac{i}{2n} \Big[(I) \times (IV) + (II) \times (III) + (I) \times (III)\Big] + cc\Big)$$
$$= O\Big(\frac{m^5}{n^3}\Big)$$

and with Theorem B.1 (note that (I) and (III) are deterministic)

$$\mathrm{var}\big(\check{B}_M'(u_0, \omega_0, \omega_0')\big) = \mathrm{var}\Big(-\frac{i}{2n} \Big[(I) \times (IV) + (II) \times (III)\Big] + cc\Big)$$
$$= O\left(\frac{m^9}{n^6}\right).$$

This means that $\mathrm{MSE}\big(\frac{n}{M} M^{-1/2} \check{B}_M'(u_0, \omega_0, \omega_0')\big) = O\left(\frac{m^7}{n^4}\right)$, which goes to zero for $m << n^{4/7}$. □

Proof of Theorem 3.7

The proof follows directly from Theorem 3.4, Remark 3.5(i) and Remark 3.6(i) and the fact that Lemma 3.3 holds for any such $\xi_{0,n}, \xi_{1,n}$ with the required property. □

Proof of Theorem 3.8

Using (3.24) we can see that

$$Y_{nu_0+s,n} = \alpha_n(u_0)\cos\left(\omega(u_0)s + \frac{\omega'(u_0)}{2n}s^2\right) + \beta_n(u_0)\sin\left(\omega(u_0)s + \frac{\omega'(u_0)}{2n}s^2\right) + O\left(\frac{s^3}{n^2}\right) \tag{B.50}$$

We start evaluating the terms in (3.28) and (3.29). Using (B.50) we have (for simplicity in the notation we omit u_0 and we set $\omega_0 := \omega(u_0)$ and $\omega_0' := \omega'(u_0)$)

$$M^{-1/2}\frac{\partial \bar{S}_m(\alpha,\beta,\lambda_0,\lambda_1;u_0)}{\partial\alpha}\bigg|_{c_{n,0}} = -\frac{2}{M^{\frac{1}{2}}}\sum_{-m}^{m} X_{nu_0+s,n}\cos\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right) + O\left(\frac{m^{7/2}}{n^2}\right). \tag{B.51}$$

Then we have

$$\begin{aligned} M^{-1}\frac{\partial^2 \bar{S}_m(\alpha,\beta,\lambda_0,\lambda_1;u_0)}{\partial\alpha^2}\bigg|_{\tilde{c}_n} &= \frac{2}{M}\sum_{-m}^{m}\cos^2\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right) \qquad \text{(B.52)} \\ &= \frac{2}{M}\sum_{-m}^{m}\cos^2\left[\omega_0 s + \frac{\omega_0'}{2n}s^2 + (\tilde{\omega}_{0,n} - \omega_0)s + \frac{\tilde{\omega}_{1,n} - \omega_0'}{2n}s^2\right] \\ &= 1 + o_p(1) \end{aligned}$$

because of Theorem 3.7 and Lemma B.7. This is a crude evaluation but it suffices for our purposes. Furthermore

$$M^{-1}\frac{\partial^2 \bar{S}_m(\alpha,\beta,\lambda_0,\lambda_1;u_0)}{\partial\alpha\partial\beta}\bigg|_{\tilde{c}_n} = M^{-1}\sum_{-m}^{m}\cos\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right)\sin\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right) \tag{B.53}$$

$$= M^{-1}\sum_{-m}^{m}\frac{1}{2}\sin\left(2\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{n}s^2\right) = o(1)$$

because of Lemma B.6. Finally, making similar considerations and using additionally Lemmas 3.1 and B.1 we see that

$$\begin{aligned} &M^{-2}\frac{\partial^2 \bar{S}_m(\alpha,\beta,\lambda_0,\lambda_1;u_0)}{\partial\alpha\partial\lambda_0}\bigg|_{\tilde{c}_n} \qquad \text{(B.54)} \\ &= \frac{2}{M^2}\sum_{-m}^{m} s\left[Y_{nu_0+s,n} - \tilde{\alpha}_n\cos\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right) - \tilde{\beta}_n\sin\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right)\right]\times \\ &\qquad\times\sin\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right) \end{aligned}$$

$$-\frac{2}{M^2}\sum_{-m}^{m} s\left[\tilde{\alpha}_n \sin\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right) - \tilde{\beta}_n \cos\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right)\right]\cos\left(\tilde{\omega}_{0,n}s + \frac{\tilde{\omega}_{1,n}}{2n}s^2\right)$$

$$= \frac{2}{M^2}\sum_{-m}^{m} s\left[Y_{nu_0+s,n} - \alpha_n \cos\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right) - \beta_n \sin\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right)\right]\times$$

$$\times \sin\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right)$$

$$-\frac{2}{M^2}\sum_{-m}^{m} s\left[\alpha_n \sin\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right) - \beta_n \cos\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right)\right]\cos\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right) + o_p(1) \tag{B.55}$$

$$= \frac{2}{M^2}\sum_{-m}^{m} sX_{nu_0+s,n}\sin\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right) + O\left(\frac{m^3}{n^2}\right) + o(1) + o_p(1), \tag{B.56}$$

where the summary in (B.55) is $o(1)$ because of Lemma B.1. The summary in (B.56) also converges to zero (a.s.) because of Theorem B.1. In exactly the same way we can also show that

$$nM^{-3}\frac{\partial^2 \bar{S}_m(\alpha,\beta,\lambda_0,\lambda_1;u_0)}{\partial\alpha\partial\lambda_1}\bigg|_{\tilde{c}_n} = \frac{\beta_n}{12} + o_p(1) \tag{B.57}$$

Thus, using (B.51), (B.52), (B.53), (B.54), (B.57) and (3.28) and for $m = o(n^{4/7})$ we see that

$$M^{1/2}\Big(\alpha_n(u_0) - \hat{\alpha}_n(u_0)\Big) = -\frac{2}{M^{\frac{1}{2}}}\sum_{-m}^{m} X_{nu_0+s,n}\cos\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right) - \frac{\beta_n}{12}\frac{M^{5/2}}{n}\Big(\omega'(u_0) - \widehat{\omega}_n'(u_0)\Big) + o_p(1). \tag{B.58}$$

Completely analogously we can show

$$M^{1/2}\Big(\beta_n(u_0) - \hat{\beta}_n(u_0)\Big) = -\frac{2}{M^{\frac{1}{2}}}\sum_{-m}^{m} X_{nu_0+s,n}\sin\left(\omega_0 s + \frac{\omega_0'}{2n}s^2\right) + \frac{\alpha_n}{12}\frac{M^{5/2}}{n}\Big(\omega'(u_0) - \widehat{\omega}_n'(u_0)\Big) + o_p(1). \tag{B.59}$$

The theorem now follows using (B.31), (B.33), (B.35), (B.36), (B.37), (B.38), (B.58), (B.59) and Theorems B.1 and 3.7. □

B.2.5 A formal proof of Theorem 3.2

The next two lemmas are required for the proof of Theorem 3.2:

Lemma B.8. *Let*

$$f_n(u) = a_n u + b_n u^2, \quad 0 \le u \le 1 \tag{B.60}$$

be a sequence of quadratic functions with $a_n, b_n \in \mathbb{R}$, for which the least upper bound and the greatest lower bound are bounded uniformly for all n. Moreover let the least upper bound and/or the greatest lower bound converge to c_1 and/or c_2 respectively, not both equal to zero. Then for every

$$(l_1, l_2) \subset \big(0, c_1\big) \quad or \quad (l_1, l_2) \subset \big(c_2, 0\big), \quad l_2 > l_1,$$

depending on which limit exists, the Lebesgue measure of $f_n^{-1}([l_1, l_2])$ has a lower bound greater than zero.

Proof. Let $v_n = (a_n, b_n)'$ be a sequence in $\mathbb{R}^2$. We define the functions:

$$\rho_1(v_n) = \sqrt{a_n^2 + b_n^2}, \qquad \rho_2(v_n) = \sup_u |f_n(u)|, \quad 0 \le u \le 1.$$

The first function is the euclidean norm, while the second one also defines a norm (it is very easy to see that it satisfies all three conditions). According to the norm equivalence in finitely dimensioned spaces there is some $M < \infty$ for which: $\rho_1(v_n) \le M\rho_2(v_n)$. Since $\rho_2(v_n)$ is bounded per assumption, so is also $\rho_1(v_n)$. This makes a_n and b_n bounded and therewith also $f_n'(u)$. Now we take some sequence $(u_1^{(n)}, u_2^{(n)})$ with $\lim_n f_n(u_1^{(n)}) = l_1$, $\lim_n f_n(u_2^{(n)}) = l_2$ and $|u_1^{(n)} - u_2^{(n)}|$ is minimal. For this sequence we have:

$$\frac{|l_2 - l_1|}{|u_2^{(n)} - u_2^{(n)}|} \le \sup_{u,n} |f_n'(u)| \Rightarrow |u_2^{(n)} - u_2^{(n)}| \ge \frac{|l_2 - l_1|}{\sup_{u,n} |f_n'(u)|}$$

Since $|l_2 - l_1|$ is constant, $\sup_{u,n} |f_n'(u)|$ is bounded and thus the lemma is proved. □

Lemma B.9. *Let $f_{\tilde{n}}(u)$ be defined as in (B.60), $b_n \ge 0$, $\{\tilde{n}\} = \mathbb{N}$. Assume that the sequence of its lowest upper bounds or/and the sequence of its greatest lower bounds are not bounded. Then, for some subsequence $\{n\} \subseteq \{\tilde{n}\}$ there exists some $(u_m, u_M) \subset [0,1]$, $u_M > u_m$ with the Lebesgue measure $\lambda[f_n(u_m, u_M)]$ going to infinity and the following property: for every $l_1^{(n)}, l_2^{(n)}, L_1^{(n)}, L_2^{(n)}$ such that*

$$(l_1^{(n)}, l_2^{(n)}) \subset (L_1^{(n)}, L_2^{(n)}), \quad l_1^{(n)}, l_2^{(n)}, L_1^{(n)}, L_2^{(n)} \in f_n(u_m, u_M)$$

$$l_2^{(n)} - l_1^{(n)} = l < 2\pi, \quad L_2^{(n)} - L_1^{(n)} = 2\pi$$

the ratio of the Lebesgue measures $\frac{\lambda\left[f_n^{-1}([l_1^{(n)}, l_2^{(n)}])\right]}{\lambda\left[f_n^{-1}([L_1^{(n)}, L_2^{(n)}])\right]}$ converges to some $c > 0$ uniformly for all such $l_1^{(n)}, l_2^{(n)}, L_1^{(n)}, L_2^{(n)}$.

Proof. By using again the norm equivalence we can show exactly as in Lemma B.8 that, if $a_{\tilde{n}}$ and $b_{\tilde{n}}$ were bounded, the sequence of the extrema of $f_{\tilde{n}}(u)$ would also be bounded. This means that for some subsequence $\{n\}$ either a_n, or b_n or both of them diverge. Without loss of generality assume $b_n > 0$. The greatest lower bound of $f_n(u)$ for $u \in \mathbb{R}$ is attained for $u = -\frac{a_n}{2b_n}$. Now we distinguish three cases:

i. *Only b_n diverges:* We choose $(u_m, u_M) \subset (\delta, 1)$ for some δ slightly greater than 0. From some n_0, $-\frac{a_n}{2b_n}$ is smaller than δ which makes $f_n(u)$ monotone in $(\delta, 1)$. Moreover $f_n(1) - f_n(\delta) = a_n(1-\delta) + b_n(1-\delta^2) \to \infty$ for every such δ. Thus $\lambda[f_n(\delta, 1)]$ goes indeed to infinity. Let $u_1^{(n)}, u_2^{(n)}, U_1^{(n)}, U_2^{(n)} \in (\delta, 1)$ be sequences such that: $f_n(u_1^{(n)}) = l_1^{(n)}, f_n(u_2^{(n)}) = l_2^{(n)}, f_n(U_1^{(n)}) = L_1^{(n)}, f_n(U_2^{(n)}) = L_2^{(n)}$ with $l_1^{(n)}, l_2^{(n)}, L_1^{(n)}, L_2^{(n)}$ as described above. Now: $U_2^{(n)} = \frac{-a_n + \sqrt{a_n^2 + 4b_n(b_n(U_1^{(n)})^2 + a_n U_1^{(n)} + 2\pi)}}{2b_n}$. The first derivative of $f_n(u)$ is $f_n'(u) = a_n + 2ub_n$. We have for $U_1^{(n)} \leq u \leq U_2^{(n)}$:

$$\left[\frac{\sup f_n'(u)}{\inf f_n'(u)}\right]^2 = \left[\frac{f_n'(U_2^{(n)})}{f_n'(U_1^{(n)})}\right]^2 = \frac{a_n^2 + 4b_n(b_n(U_1^{(n)})^2 + a_n U_1^{(n)} + 2\pi)}{(a_n + 2b_n U_1^{(n)})^2} = \tag{B.61}$$

$$= \frac{(a_n + 2b_n U_1^{(n)})^2 + 8\pi b_n}{(a_n + 2b_n U_1^{(n)})^2} \overset{n\to\infty}{\longrightarrow} 1,$$

uniformly for all $l_1^{(n)}, l_2^{(n)}, L_1^{(n)}, L_2^{(n)}$ as described above. Furthermore, from n_0 on u lies in $(\delta, 1)$, so we can write:

$$\frac{\overbrace{l_2^{(n)} - l_1^{(n)}}^{=l}}{u_2^{(n)} - u_1^{(n)}} = f_n'(u_l^{(n)}) \quad \text{and} \quad \frac{\overbrace{L_2^{(n)} - L_1^{(n)}}^{=2\pi}}{U_2^{(n)} - U_1^{(n)}} = f_n'(u_L^{(n)})$$

for some $u_1^{(n)} \leq u_l^{(n)} \leq u_2^{(n)}$ and $U_1^{(n)} \leq u_L^{(n)} \leq U_2^{(n)}$. Dividing these two equations we receive:

$$\frac{u_2^{(n)} - u_1^{(n)}}{U_2^{(n)} - U_1^{(n)}} = \frac{\lambda\left[f_n^{-1}([l_1^{(n)}, l_2^{(n)}])\right]}{\lambda\left[f_n^{-1}([L_1^{(n)}, L_2^{(n)}])\right]} = \frac{l}{2\pi}\frac{f_n'(u_L)}{f_n'(u_l)},$$

which because of (B.61) converges to $l/2\pi$.

ii. *Only a_n diverges:* The proof is completely analogue with $(u_m, u_M) = (0, 1)$.

iii. *Both a_n and b_n diverge:* If b_n diverges faster than a_n the proof is exactly the same as in (i) whereas in the opposite case the same as in (ii). If they diverge with the same rate we choose the subsequence $\{n\}$ so that $-\frac{a_n}{2b_n} \leq 1/2$. If this is not possible then $-\frac{a_n}{2b_n} > 1/2$. We investigate only the first case, as both are handled in the same way. We choose $(u_m, u_M) = (1/2 + \delta, 1)$ for some δ slightly greater than 0. $f_n(u)$ is in this interval monotone and $f_n(1) - f_n(1/2 + \delta)$ goes to infinity. Furthermore (B.61) continues to hold. Thus the statement of the lemma also holds for this last case and the lemma is proved. □

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