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Abstract

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*** Author's Covering Letter**

11.08.2009

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Journal of Differential Equations
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Dear Editors,

I hereby submit to you the manuscript of my Article *Existence of Solution for a Model of Film Condensation and Crystallization* for publication in *Journal of Differential Equations*

Best regards,

Martin Heida

Main File of Manuscript

Existence of Solution for a Model of Film Condensation and Crystallization

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Abstract

A model for vapor transport with condensation and evaporation on a solid-air interface is set up. It consists of a convection-diffusion equation describing vapor transport, an ordinary equation describing condensation and a Stefan-type equation on with convection describing energy transport. The proof of existence of a solution is based on a method used by J.F. Rodriguez in several publications on the convective Stefan problem. The new part in this system is a lower-dimensional Stefan problem on the air-solid interface that describes possible freezing of the condensed water. The Model described in this article could also be applied to crystalization problems.

Key words: Stefan Problem, Convection, Condensation, Dissolution, Crystallization

1. Introduction

The model analyzed in this paper rises up in modeling processes like condensation or crystallization on a solid surface. In the first case, we assume vapor transported by an air velocity field to condensate at the boundary of a solid material due to over saturation in the gaseous phase. In the second case it may be assumed that some material transported by water crystallizes at the surface of solid matter. In both cases, for simplicity the temperature field is assumed to be continuous across the interface. The condensation or crystallization film is assumed to be very thin and therefore is assumed to be two-dimensional.

Thus, the domain $\Omega \subset \mathbb{R}^3$ in which all these processes take place can be divided into three parts of interest: The solid domain Ω_1, in which only heat transport has to be considered, the air/water domain Ω_2 in which all the dynamical processes take place and the interface Γ. The model equations for vapor transport and condensation read as follows:

$$\begin{aligned} \partial_t c - \operatorname{div}(K\nabla c) + \operatorname{div}(w\,c) &= 0 \quad \text{on } \Omega_2 && (1)\\ (w\,c - K\nabla c)\nu_2 &= j \quad \text{on } \partial\Omega_2 && (2)\\ \partial c_\Gamma &= j \quad \text{on } \Gamma_{as} && (3) \end{aligned}$$

whereas the equations for heat transport are given by

$$\begin{aligned} \partial_t u(\vartheta) - \operatorname{div}(K_1\nabla\vartheta) &= 0 \quad \text{on } \Omega_1 && (4)\\ \partial_t u(\vartheta) - \operatorname{div}(K_2\nabla\vartheta) + \operatorname{div}\,((w\,c - K\nabla c)m_c(\vartheta)) + \operatorname{div}\,(w\,m_2(\vartheta)) &= 0 \quad \text{on } \Omega_2 && (5)\\ \partial_t u(\vartheta) - ((w\,c - K\nabla c)m_c(\vartheta) - w\,m_2(\vartheta) - K_2\nabla\vartheta + K_1\nabla\vartheta) &= 0 \quad \text{on } \Gamma_{as} && (6) \end{aligned}$$

In the first set of equations, the variable c denotes the vapor concentration in the air phase, c_Γ the amount of condensed water, w is the air velocity and fulfills $\operatorname{div} w = 0$, i.e. the air phase is assumed to be incompressible. Furthermore, K is some positive constant and j is the condensation rate on the boundary. In the second set of equations, u denotes the inner energy density of the system which is assumed to be a function of the temperature ϑ as well as of c and c_Γ. In particular, $u(\vartheta) = m_2(\vartheta) + c\, m_c(\vartheta)$ on Ω_2, $u(\vartheta) = m_1(\vartheta)$ on Ω_1 and $u(\vartheta) = c_\Gamma\, m_{c_\Gamma}(\vartheta)$ on Γ_{as}. In this context, K_1 and K_2 are positive constants and m_i, i as an arbitrary index, are strongly monotone increasing functions with $m_i(0) = 0$. As an additional degree of freedom, the functions m_{c_Γ} and m_1 need not to be continuous but may have jumps.

Evidently, this problem is closely related to the Stefan problem and we will shortly summarize what has been done in this direction: The abstract Stefan problem usually is expressed by $\partial_t a(\vartheta) - \operatorname{div}(\nabla b(\vartheta)) = 0$, $\partial_t a(\vartheta) - b(\Delta\vartheta) = 0$, $a(\partial_t\vartheta) - \operatorname{div}(\nabla b(\vartheta)) = 0$ or similar equations. A broad overview over different types of Stefan problems and the corresponding literature can be found in the book of Visintin [14]. Alt and Luckhaus [1] treated the problem

$$\partial_t a(\vartheta) - \operatorname{div}(a(\vartheta, \nabla\vartheta)) = f(\vartheta)$$

in great generality. In chapter 4 of [15], Rodriguez developed a method to treat the Stefan problem with a convective term and nonlinear Neumann boundary conditions. He successfully applied this method in [9, 10] together with Urbano to a Stefan-convection problem coupled with Stokes and Darcy flow fields. Di Benedetto and O'Leary [4] and Blanchard and Porretta [2] considered a (nonlinear) convection-diffusion problem coupled with the Stefan equation for energy. Further work on Stefan-convection Problems can be found e.g. in [16, 17, 12]. This list is not surely not complete but rather reflects the author's reading.

There are several papers dealing with crystallization coupled with reaction kinetics on the boundary [5, 13, and references therein] but not involving heat transfer. Some numerical scheme for such problems can be found in [3]. In contrast to present article, they deal with a set valued condensation term of the form $\partial_t c_\Gamma \in j(c, \vartheta, c_\Gamma)$. Such an approach would come up with even more difficulties than the present one and may be overcome by some further approximation or by some new techniques.

The challenge of the presented model lies in its condensation boundary condition coupled with a Stefan problem on the same interface. To the authors knowledge no Stefan problem coupled with an additional lower dimensional Stefan problem has been treated in literature analytically. The analysis of the system is based on the method of Rodriguez [15, 9, 10] together with an approximation ansatz and Schauders Fixed Point Theorem. The reason for this approach will be given in section 4.

This article is organized as follows: in section 2 some basic tools that will be necessary for the analysis of the problem will be introduced. In section 3 the weak formulation of the problem will be derived and in section 4 a more easy to solve approximated problem will be formulated. In 5 to 7 the approximated problem is solved by solving convenient decoupled equations and combining them using Schauders fixed point theorem. In the last section, it will be shown that the approximations can be dropped and a solution to the initial problem is obtained.

2. Mathematical tools

We will start by constructing some important Hilbert spaces and citing or proving some results on embedding properties which are valid for these spaces.

Theorem 2.1. *[8]*

Let $B_0 \subset B \subset B_1$, three Banach spaces such that B_0 and B_1 are reflexive. Suppose also that the injection $B_0 \hookrightarrow B$ is compact and define

$$W = \left\{ v \mid v \in L^{p_0}(a, b; B_0),\ \ \frac{\partial v}{\partial t} \in L^{p_1}(a, b; B_1) \right\}$$

with $1 < p_0, p_1 < +\infty$. Then W is a Banach space with respect to the norm of the graph defined by

$$||u||_W = ||u||_{L^{p_0}(a,b;B_0)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^{p_1}(a,b;B_1)}$$

and the injection $W \hookrightarrow L^{p_0}(a,b;B)$ is compact.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1, $0 < T < \infty$, $Q := (0,T) \times \Omega$, $\Sigma := (0,T) \times \partial\Omega$ and $B_0 := H^1(\Omega)$, it is possible to choose $B := H^s(\Omega)$, $0 \le s < 1$ and therefore also $B = L^2(Q)$ or $B = L^2(\Sigma)$.*

Lemma 2.3. *Assume $\Omega, K \subset \mathbb{R}^3$ are open and bounded with $\Omega \subset\subset K$. There exists a continuous operator $T : H^1(\Omega) \to H^1_0(K)$ such that $Tu(x) = u(x)$ for all $x \in \Omega$ and $\Delta Tu \equiv 0$ on $K\backslash\Omega$.*

This is easily proved by solving the corresponding partial differential equation.

Lemma 2.4. *[7]*

Assume $\Omega \subset \mathbb{R}^3$ is open, bounded and the boundary has bounded first and second order derivatives. Then there exists $C > 0$ such that

$$||u||_{W^{2,2}(\Omega)} \le C\,||\Delta u||_{L^2(\Omega)} \qquad \forall u \in W^{2,2}(\Omega) \cap W^1_0(\Omega)$$

This lemma gives rise to the assumption, that we could expect some similar result for $u \in W^{2,2}(\Omega)$. However, we would need at least $H^{\frac{3}{2},2}(\partial\Omega)$ estimates of the boundary values in order to proof $||u||_{W^{2,2}(\Omega)} \le C\,(||\Delta u||_{L^2(\Omega)} + ||u||_{H^{\frac{3}{2},2}(\partial\Omega)})$. But we can state the following

Lemma 2.5. *Let Ω be an open bounded $C^{0,1}$-domain in $\mathbb{R}^n$. Define*

$$\begin{aligned} ||u||_{W^{1,2}_{\Delta,\partial}(\Omega)} &:= ||u||_{H^1(\Omega)} + ||\Delta u||_{L^2(\Omega)} + ||\partial_\nu u||_{L^2(\partial\Omega)} \quad \forall u \in W^{2,2}(\Omega) \\ W^{1,2}_{\Delta,\partial}(\Omega) &:= \overline{W^{2,2}(\Omega)}^{||\bullet||_{W^{1,2}_{\Delta,\partial}(\Omega)}} \end{aligned}$$

Then $W^{1,2}_{\Delta,\partial}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and the embedding is compact.

Proof.
For any weak converging series $u_n \rightharpoonup u$ in $W^{1,2}_{\Delta,\partial}(\Omega)$ and $w_n \rightharpoonup w$ in $W^{1,2}(\Omega)$ calculate

$$\begin{aligned} \int_\Omega (u_n w_n + \nabla u_n \nabla w_n) &= \int_\Omega u_n w_n - \int_\Omega w_n \Delta u_n + \int_{\partial\Omega} \partial_\nu u_n w_n \\ &\to \int_\Omega (uw - w\Delta u) + \int_{\partial\Omega} \partial_\nu u\, w = \int_\Omega (uw + \nabla u \nabla w) \end{aligned}$$

Where the limit follows from the strong convergence of $w_n \to w$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$ and the weak convergence of $(u_n, \Delta u_n, \partial_\nu u_n) \rightharpoonup (u, \Delta u, \partial_\nu u)$ in $L^2(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega)$.

□

Furthermore, we define the following space according to Temam [11]

$$\begin{aligned} ||u||_{E(\Omega)} &:= \left(\int_\Omega (u^2 + (\operatorname{div} u)^2)\right)^{\frac{1}{2}} \quad \forall \phi \in H^1(\Omega)^n \\ E(\Omega) &:= \overline{H^1(\Omega)^n}^{||\cdot||_{E(\Omega)}} \end{aligned}$$

Lemma 2.6. *(Temam: [11, Theorem 1.2.])*

Let Ω be an open bounded set of class C^2. Then there exists a linear continuous operator $\gamma_\nu \in \mathcal{L}(E(\Omega), H^{-\frac{1}{2}}(\partial\Omega))$ such that

$$\gamma_\nu u = u\nu \text{ for every } u \in \mathcal{D}(\overline{\Omega})$$

The following generalized Stokes formula is true for all $u \in E(\Omega)$ and $w \in H^1(\Omega)$:

$$\int_\Omega u\nabla w + \int_\Omega w div\, u = \int_\Gamma (\gamma_\nu u)\, w$$

Lemma 2.7. *Let Ω be an open, bounded set with $C^{1,1}$-boundary. For any sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\Omega)^n$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)^n$ holds $u_n\nu \rightharpoonup u\nu$ weakly in $H^{\frac{1}{2}}(\partial\Omega)$, where ν is the outer normal vector of $\partial\Omega$. For $(u_n)_{n\in\mathbb{N}} \subset E(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $E(\Omega)$ and $u_n \to u$ strongly in $L^2(\Omega)^n$ holds $u_n\nu \to u\nu$ strongly in $H^{-1/2}(\partial\Omega)$.*

Proof.

For any $\widetilde{w} \in H^{\frac{1}{2}}(\partial\Omega)$, choose a function $w \in H^1(\Omega)$ such that $w\Big|_{\partial\Omega} \equiv \widetilde{w}$ and $\Delta w = 0$ in Ω and calculate:

$$\begin{aligned}
\lim_{n\to\infty} \int_{\partial\Omega} u_n w\, d\sigma &= \lim_{n\to\infty} \left(\int_\Omega u_n \nabla w\, dx + \int_\Omega w \operatorname{div} u_n\, dx \right) \\
&= \int_\Omega u\nabla w\, dx + \int_\Omega w \operatorname{div} u\, dx = \int_{\partial\Omega} uw\, d\sigma
\end{aligned}$$

in the second case do the same calculation for a weakly convergent sequence $\widetilde{w}_n \rightharpoonup \widetilde{w}$ in $H^{1/2}(\partial\Omega)$. □

Remark that a similar result holds also in the case of $u_n \rightharpoonup u$ in $L^2(0,T; E(\Omega))$, $u_n \to u$ in $L^2(0,T; L^2(\Omega)^n)$ and sequences $w_n \rightharpoonup w$ in $L^2(0,T; H^1(\Omega))$ with $w_n \to w$ in $L^2(0,T; L^2(\Omega))$.

We will also need some results about sequences of bounded functions and Lipschitz continuous functions evaluated with bounded L^2-functions. Additionally, as we are dealing with set valued functions and inclusions, we need some results about the limit behavior of such sequences.

Lemma 2.8. *For a measure space (Ω, σ, μ) with finite measure μassume that for a converging sequence of functions $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mu)$ with $u = \lim_{n\to\infty} u_n$ holds $-\infty < c_1 \leq u_n \leq c_2 < +\infty$ μ-almost surely in Ω for all n. Then $c_1 \leq u \leq c_2$ μ- almost surely.*

Proof.

Define $a_n := \max\{c_1, \min\{c_2, u_n\}\}$ with $a_n \to u$ in $L^2(\Omega, \mu)$. It holds for a sub sequence that $a_n(x) \to u(x)$ point wise for μ-almost every x. Therefore $c_1 \leq u \leq c_2$ μ-almost surely.

□

Lemma 2.9. *[14]*

For a finite measure space (Ω, μ) and a maximal monotone graph $m : \mathbb{R} \to \mathbb{R}$ such that $|m(x)| \leq C(1+|x|)$ for some constant C, let $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mu)$ be a converging sequence with limit u such that $m(u_n) \rightharpoonup m^*$ weakly in $L^2(\Omega, \mu)$. Then $m^* \in m(u)$.

Lemma 2.10. *Assume $\gamma_1, \gamma_2 : X \to \mathbb{R}$ are two Lipschitz continuous mappings on a normed space X with Lipschitz constants C_1, C_2 and that $\|\gamma_1(x)\| + \|\gamma_2(x)\| \leq C_3$ for all $x \in X$. Then $x \mapsto \gamma_1(x)\gamma_2(x)$ is Lipschitz continuous with Lipschitz constant $C_3\,(C_1 + C_2)$.*

For (Ω, σ, μ) a measure space and a Lipschitz continuous function $j : \mathbb{R}^m \to \mathbb{R}$ which is monotone in any argument consider sequences $(\alpha^i_j)_{j\in\mathbb{N}} \subset L^2(\Omega)$ such that $\alpha^i_j \to \alpha^i$ in $L^2(\Omega)$ for $i = 1, \ldots, m$ and $|\alpha^j_i| < a^i < \infty$ a.s.. Then $j(\alpha^1_j, \ldots, \alpha^m_j) \to j(\alpha^1, \ldots, \alpha^m)$ in $L^2(\Omega)$.

Proof.
For $m = 2$ find $\left|j(\alpha^1_j, \alpha^2_j) - j(\alpha^1, \alpha^2)\right| \leq \left|j(\alpha^1_j, \alpha^2_j) - j(\alpha^1_j, \alpha^2)\right| + \left|j(\alpha^1_j, \alpha^2) - j(\alpha^1, \alpha^2)\right|$. □

We finally cite the following version of Schauder's Fixed Point Theorem:

Theorem 2.11. *(Schauder's second fixed point theorem)[18]*
Suppose that

1. *X is a reflexive, separable Banach space*
2. *The map $T : M \subset X \to M$ is weakly sequentially continuous, i.e., if $x_n \rightharpoonup x$ as $n \to \infty$, then also $T(x_n) \to T(x)$*
3. *The set M is nonempty, closed, bounded and convex*

Then T has a fixed point.

3. Formulating the Mathematical Problem

In the following, $\int_\Gamma \phi$ with Γ being a $(n-1)$-dimensional Manifold denotes the integral of ϕ with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}(\cdot \cap \Gamma)$ on Γ.

$L^2(\Omega \cup \Gamma_{as} \cup \partial\Omega)$ denotes the space of all square-integrable functions with respect to the measure $\mu(A) := \mathcal{L}(A \cap \Omega) + \mathcal{H}^{n-1}(A \cap (\Gamma_{as} \cup \partial\Omega))$ which is a sum of the Lebesgue measure on Ω and the Hausdorff measure on $\Gamma_{as} \cup \partial\Omega$. $\int_{\Omega\cup\Gamma_{as}\cup\partial\Omega}$ has to be understood in this context.

Suppose we were given $\Omega \subset \mathbb{R}^n$ bounded and open with C^2-boundary. Suppose $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_{as}$ with Ω_1, Ω_2 being open sets with piecewise $C^{2,1}$-boundary and $\Omega_1 \cap \Omega_2 \cap \Gamma_{as} = \emptyset$. In Particular, $\Gamma_{as} = \partial\Omega_1 \cap \partial\Omega_2$ and is a piecewise smooth $(n-1)$-dimensional manifold. For the rest of this section, we define ν_2 as the outer normal vector of Ω_2.

The transport and condensation equations read as follows:

$$\left.\begin{aligned} \partial_t c + \operatorname{div} j &= 0 \qquad \text{in } \Omega_2 \\ c &\equiv 0 \qquad \text{in } \Omega_1 \\ j := j_w &= -K\nabla c + w\,c \\ j\,\nu_2 &= j_{w,\Gamma}(c, \vartheta, c_\Gamma) \qquad \text{in } \Gamma_{as} \\ j\Big|_{\partial\Omega\cap\partial\Omega_2} \nu_2 &= j_{w,\partial\Omega}(c, \vartheta) \end{aligned}\right\} \tag{7}$$

$$\partial_t c_\Gamma = j_{w,\Gamma}(c, \vartheta, c_\Gamma) \quad \text{in } \Gamma_{as} \tag{8}$$

whereas the energy transport can be described by the following set of equations:

$$
\begin{aligned}
\partial_t u + \operatorname{div} j_u &= 0 \qquad \text{in } \Omega_2 \cup \Omega_1 && (9)\\
j_u &= \begin{cases} j_u^2 := -K_2 \nabla\vartheta + j\, m_c(\vartheta) + w\, m(\vartheta) & \text{in} \quad \Omega_2 \\ j_u^1 := -K_1 \nabla\vartheta & \text{in} \quad \Omega_1 \end{cases} \\
\partial_t u - (j_u^2 - j_u^1)\nu_2 &= 0 \qquad \text{on } \Gamma_{as} \\
\partial_t u + g(\vartheta) &= j_u \nu_\Omega \qquad \text{on } \partial\Omega \\
u_1(x,t,\vartheta) &= u\big|_{\Omega_1}(x,t,\vartheta) \in m_1(\vartheta) && (10)\\
u_2(x,t,\vartheta) &= u\big|_{\Omega_2}(x,t,\vartheta) \in m_2(\vartheta) + c(x,t)\, m_c(\vartheta) && (11)\\
u_\Gamma(x,t,\vartheta) &= u\big|_{\Gamma}(x,t,\vartheta) \in c_\Gamma(x,t) m_{c_\Gamma}(\vartheta) && (12)\\
u_{\partial\Omega}(x,t,\vartheta) &= u\big|_{\partial\Omega}(x,t,\vartheta) = 0 && (13)
\end{aligned}
$$

Assume that

$$
j_{w,\partial\Omega_2}(c,\vartheta,c_\Gamma) = j_0(c,\vartheta)\, j_1(c_\Gamma)
$$

where $j_0(c,\vartheta) = \widetilde{j_0}(c - c_0(\vartheta))$ with c_0, $\widetilde{j_0}$, j_1 and $\frac{j_1(\bullet)}{\bullet}$ are Lipschitz continuous monotone increasing functions (with Lipschitz-constants C_0, $\widetilde{J_0}$, J_1, $J_{1,*}$), $c_0 \geq 0$ and c_0 and $\widetilde{j_0}$ being bounded by some constants $C_{0,\max}$ and $\widetilde{J_0}_{\max}$ and $\widetilde{j_0}(0) = j_1(0) = 0$ with Lipschitz-constant J_1. Furthermore c_0 may be strongly continuous. Evidently, for every $-\infty < \vartheta_{\text{krit}} < \vartheta^{\text{krit}} < +\infty$, there are $-\infty < c_{\text{krit}} < c^{\text{krit}} < +\infty$ such that

$$
j_0(c,\vartheta) \begin{cases} \leq 0 & \text{if } (c,\vartheta) \in (-\infty, c_{\text{krit}}] \times [\vartheta_{\text{krit}}, \vartheta^{\text{krit}}] \cup [c_{\text{krit}}, c^{\text{krit}}] \times [\vartheta^{\text{krit}}, \infty) \\ \geq 0 & \text{if } (c,\vartheta) \in [c^{\text{krit}}, \infty) \times [\vartheta_{\text{krit}}, \vartheta^{\text{krit}}] \cup [c_{\text{krit}}, c^{\text{krit}}] \times (-\infty, \vartheta_{\text{krit}}] \end{cases}
$$

On $[c_{\text{krit}}, c^{\text{krit}}] \times [\vartheta_{\text{krit}}, \vartheta^{\text{krit}}]$ j_0 is bounded by some constant J_0 and $|\partial_t c_\Gamma| < J_0 J_1$ a.s..

Furthermore, $H^*(\Omega)$ denotes the dual space of $H^1(\Omega)$ and $\psi \in L^2(0,T;H^*(\Omega))$ means:

$$
\int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \psi\phi \leq C\|\phi\|_{H^1(\Omega)} \qquad \forall\phi \in H^1(\Omega)
$$

In connection with this definition, we call $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$, $H^{-2}(\Omega)$ the dual space of $H_0^1(\Omega) \cap H^2(\Omega)$ and $H^{-1}(\Omega_2)$ and $H^{-2}(\Omega_2)$ respectively.

Testing the equations (7) and (9) with some $\phi \in C^1(\overline{\Omega})$, partial integration and inserting the boundary conditions yields

$$
\begin{aligned}
\int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \partial_t u(\vartheta)\,\phi - \int_0^T \int_\Omega j_u \nabla\phi + \int_0^T \int_{\partial\Omega} g(\vartheta)\phi &= 0 && (14)\\
\int_0^T \int_{\Omega_2} \partial_t c\,\phi + \int_0^T \int_{\Omega_2} (K\nabla c - w\,c)\,\nabla\phi + \int_0^T \int_{\partial\Omega_2} j_{w,\partial\Omega_2}(c,\vartheta,c_\Gamma) &= 0 && (15)
\end{aligned}
$$

while equality (8) should hold in the sense of $L^2(\Gamma_{as})$

$$
\partial_t c_\Gamma = j_{w,\Gamma}(c,\vartheta,c_\Gamma) \quad \text{in } \; L^2(\Gamma_{as}) \tag{16}
$$

Inserting the explicit form of j_u in equation (14) and partial integration of the convective term in space yields a second possible formulation of the problem:

$$\int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \partial_t u(\vartheta)\,\phi + \int_0^T \int_\Omega \nabla\vartheta\nabla\phi + \int_0^T \int_{\Omega_2} \nabla\left(j\, m_c(\vartheta) + w\, m_2(\vartheta)\right)\,\phi$$
$$- \int_0^T \int_{\Gamma_{as}} j_{w,\Gamma}(c,\vartheta,c_\Gamma)\, m_c(\vartheta)\phi + \int_0^T \int_{\partial\Omega} \left(g(\vartheta) - j_{w,\partial\Omega}(c,\vartheta)\, m_c(\vartheta)\right)\phi = 0 \quad (17)$$

This will be the basic ansatz to show L^∞ estimates on ϑ.

Problem 3.1. *Assume* $\widetilde{c_0} \in H^1(\Omega_2)$, $\widetilde{\vartheta_0} \in H^1(\Omega)$, $\widetilde{c_{\Gamma 0}} \in L^\infty(\Gamma_{as})$, $\widetilde{u_0} \in L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega)$ *with constants* $+\infty > \vartheta^{krit} > \vartheta_{krit} > -\infty$, $c_{krit} := c_0^{-1}(\vartheta_{krit})$, $c^{krit} := c_0^{-1}(\vartheta^{krit})$, $\vartheta^{krit} \geq \widetilde{\vartheta_0} \geq \vartheta_{krit}$, $c^{krit} \geq \widetilde{c_0} \geq c_{krit} \geq 0$ *almost surely,* $\widetilde{c_{\Gamma 0}} \geq 0$ *almost surely and* $(\widetilde{c_0}, \widetilde{\vartheta_0}, \widetilde{c_{\Gamma,0}}, \widetilde{u_0})$ *satisfying equations* (10)- (13) *almost surely. Furthermore,* $g = g(\vartheta, x, t)$ *is assumed to be bounded and Lipschitz in* ϑ *with a Lipschitz constant independent on* (x, t), $g(\vartheta, x, t) \geq 0$ *for* $\vartheta > \vartheta^{krit}$ *and* $g(\vartheta, x, t) \leq 0$ *for* $\vartheta < \vartheta_{krit}$ *for all* $(x, t) \in \partial\Omega \times (0, T)$.

Find $c \in L^2(0, T; H^1(\Omega_2)) \cap H^1(0, T; H^{-1}(\Omega_2))$, $\vartheta \in L^2(0, T; H^1(\Omega))$, $\vartheta \in H^1(0, T; H^{-1}(\Omega_2))$, $\vartheta \in H^1(0, T; L^2(A))\ \forall A \subset\subset \Omega_1$, $c_\Gamma \in H^1(0, T; L^\infty(\Gamma_{as}))$, $u \in L^2(0, T; L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega)) \cap L^\infty((0, T)\times\Omega)$, $\partial_t u \in H^*(\Omega)$ *such that* u *satisfies equations* (10) - (13) *and* $(c, \vartheta, c_\Gamma, u)$ *satisfies* (14) - (16) *with* $c(0) = \widetilde{c_0}$, $\vartheta(0) = \widetilde{\vartheta_0}$, $c_\Gamma(0) = \widetilde{c_{\Gamma 0}}$ *and* $u(0) = \widetilde{u_0}$ *and the essential boundedness conditions* $\vartheta^{krit} > \vartheta \geq \vartheta_{krit}$, $c^{krit} \geq c \geq c_{krit}$.

4. An Approximated Problem

The energy equation on Ω_2 is the most difficult in the system of equations. It seems evident, that the coefficients have to be smoothed out if one wants to overcome problems in terms like $\partial_t(c\, m_c(\vartheta))$ or $\operatorname{div}(j\, m_c(\vartheta))$ which is not in L^2 or even in H^{-1} as long as it is not known that ϑ is essentially bounded. However, if we smooth out the coefficients in the energy equations by a Dirac sequence and take the limit, weak convergence of j in L^2 would not be enough to get sufficient convergence behavior of the boundary conditions. To this aim, an other approach is introduced giving more regularity of the vapor concentration c.

The first step is to modify the system describing vapor transport by changing the boundary condition into

$$j\nu_2 = j_{w,\Gamma}(c, \vartheta, c_\Gamma) + \delta\partial_t c$$

this will lead to an L^2-estimate on $\partial_t c$. The limit $\delta \to 0$ seems very delicate in this context and it will turn out to be the last step in the approximation procedure. The crucial point is, that the introduction of this term does not change the type of the equation, i.e. the equation remains of parabolic type.

We assume that $c \in H^1(\Omega_2)$ and choose an extension of c on $(-1, T + 1) \times K$ with some balls K and $\widetilde{K}$ satisfying $\Omega \subset\subset \widetilde{K} \subset\subset K$, by extending it harmonically on K according to Lemma 2.3 such that $c|_{\partial\widetilde{K}} \equiv c_{\mathrm{krit}}$ and $c|_{\partial K} \equiv 0$. By the weak maximum principle, $c^{\mathrm{krit}} \geq c \geq c_{\mathrm{krit}}$ still holds in $\widetilde{K}$ almost surely and we extent this function on $(0, T)$ constant on $(-1, T + 1)$. Smoothing with $\eta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ such that $\eta \geq 0$, $\|\eta\|_{L^1} = 1$ and $\operatorname{supp}(\eta * \chi_{(0,T)\times\Omega}) \subset (-0.5, T + 0.5) \times \widetilde{K}$ yields two functions:

$$c^* \ := \ (\eta * c) \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n) \quad (18)$$
$$j_1 \ := \ (\eta * (-K\nabla c + wc)) \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n) \quad (19)$$

such that c^* still satisfies $c^{\text{krit}} \geq c^* \geq c_{\text{krit}}$ on $(0,T) \times \Omega_2$ almost surely. Now solve:

$$\partial_t c^* + \operatorname{div} j_1 - \Delta\psi = 0 \qquad \psi \in H_0^1(\Omega_2) \tag{20}$$

Proposition 4.1. *For the solution of equation* (20) *holds* $\psi \in C^\infty(\Omega_2) \cap H^{2,2}(\Omega_2)$ *and*

$$\|\psi\|_{L^2(0,T;H^{2,2}(\Omega_2))} \leq C(\Omega_2)\left(\|\partial_t c^*\|_{L^2(0,T;L^2(K))} + \|div\, j_1\|_{L^2(0,T;L^2(K))}\right)$$

For $\Omega_2 \in C^{2+m}$, *we get*

$$\|\psi\|_{L^2(0,T;H^{m,2}(\Omega_2))} \leq C(\Omega_2)\left(\|\partial_t c^*\|_{L^2(0,T;L^2(K))} + \|div\, j_1\|_{L^2(0,T;L^2(K))}\right)$$

Proof.
The existence of ψ and the $C^\infty(\Omega)$-regularity follows from standard results (see [6]), the $H^{2,2}(K)$-estimate from Lemma 2.4 as well as the higher estimates from [6]. □

Note that the mapping $c \mapsto j^* := -\nabla\psi + j_1$ as a mapping $L^2(0,T;H^1(\Omega_2)) \cap H^1(0,T;L^2(\Omega_2)) \to L^2(0,T;H^1(\Omega_2)^n)$ is continuous.

The equations for (u,ϑ) are changed the following way:

$$j_u = \begin{cases} -K_2\nabla\vartheta + j^* m_c(\vartheta) + w\, m_2(\vartheta) & \text{in} \quad \Omega_2 \\ -K_1\nabla\vartheta & \text{in} \quad \Omega_1 \end{cases} \tag{21}$$

$$u_1(x,t,\vartheta) = u\big|_{\Omega_1}(x,t,\vartheta) = m_1(\vartheta) \tag{22}$$

$$u_2(x,t,\vartheta) = u\big|_{\Omega_2}(x,t,\vartheta) = m_2(\vartheta) + c^*(x,t)\, m_c(\vartheta) \tag{23}$$

$$u_\Gamma(x,t,\vartheta) = u\big|_{\Gamma}(x,t,\vartheta) = m_\Gamma(\vartheta) + c_\Gamma(x,t) m_{c_\Gamma}(\vartheta) \tag{24}$$

$$u_{\partial\Omega}(x,t,\vartheta) = u\big|_{\partial\Omega}(x,t,\vartheta) = m_{\partial\Omega}(\vartheta) \tag{25}$$

Remark that we introduced the functions m_Γ and $m_{\partial\Omega}$ which will be needed to get an L^2-control of $\partial_t\vartheta$. It will be one of the last steps of the considerations below to get rid of these two terms.

Problem 4.1. *Keep the assumptions on* $j_{w,\partial\Omega_2}(c,\vartheta,c_\Gamma)$, $\widetilde{c_0}$, $\widetilde{\vartheta_0}$, $\widetilde{c_{\Gamma 0}}$ *and* $\widetilde{u_0}$ *as in Problem 3.1 but with* $(\widetilde{u_0},\widetilde{\vartheta_0})$ *now satisfying equations* (22)-(25)

Find $(c,\vartheta,c_\Gamma) \in Y$, $u \in L^2(0,T;L^2(\Omega \cup \Gamma_{as} \cup \partial\Omega)) \cap L^\infty((0,T)\times\Omega)$, $\partial_t u \in H^*(\Omega)$ *such that equations* (22)- (25) *are satisfied and*

$$\begin{aligned}
0 &= \int_0^T \int_{\Omega_2} \partial_t c\phi + \int_0^T \int_{\partial\Omega_2} \delta\partial_t c\phi - \int_0^T \int_{\Omega_2} j_w \nabla\phi + \int_0^T \int_{\partial\Omega_2} j_{w,\partial\Omega_2}(c,\vartheta,c_\Gamma)\phi \\
0 &= c_\Gamma - \int_0^T j_{w,\Gamma_{as}}(c,\vartheta,c_\Gamma) \\
0 &= \int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \partial_t u(\vartheta)\,\phi + \int_0^T \int_{\Omega_2} \nabla\left(j^*\, m_c(\vartheta) + w\, m(\vartheta)\right)\,\phi \\
&\quad + \int_0^T \int_\Omega \nabla\vartheta\nabla\phi - \int_0^T \int_{\Gamma_{as}} j_{w,\Gamma}(c,\vartheta,c_\Gamma)\,(m_c(\vartheta) - m_{c_\Gamma}(\vartheta))\phi \\
&\quad + \int_0^T \int_{\partial\Omega} (g(\vartheta) - j_{w,\partial\Omega}(c,\vartheta)\, m_c(\vartheta))\,\phi - \int_0^T \int_{\Gamma_{as}} \partial_t c_\Gamma\, m_{c_\Gamma}(\vartheta)\phi
\end{aligned}$$

with $c(0) = \widetilde{c_0}$, $\vartheta(0) = \widetilde{\vartheta_0}$, $c_\Gamma(0) = \widetilde{c_{\Gamma 0}}$ *and* $u(0) = \widetilde{u_0}$.

The hope is, that the sequence of solutions to this problem will converge to a solution of the original problem if η is replaced by a Dirac-sequence and the additional terms tend to zero in a reasonable sense. Remark that due to the reformulation with the additional boundary terms, one of the critical questions will be whether $j^* \to j_{w,\Gamma}(c, \vartheta, c_\Gamma)$ in a reasonable space can be shown.

The strategy to obtain a solution of the approximated system looks as follows: in section 7, the system above will be decomposed into three appropriate equations. In each of the equations, two variables will be considered as parameters and the last one is the free variable. Then, the equations will be extended by some terms depending on the solutions and the parameters. These terms will cancel out in case the parameters and the solutions coincide. The system is then solved using Schauder's second Fixed Point Theorem. The three types of equations which are analyzed in the following two sections have to be understood as general form of the equations arising from the decomposition in section 7.

5. The Transport and the Condensation Equation

It is easy to see that

$$\begin{aligned} |c_\Gamma(t)| &\leq \exp(J_0 J_1 t)\, c_\Gamma(0) \qquad \text{and} \\ \partial_t(c_{\Gamma\,1} - c_{\Gamma\,2}) &\leq J_0(c_{\Gamma\,1} - c_{\Gamma\,2}) + |(j_0(c_1, \vartheta_1) - j_0(c_2, \vartheta_2))|\, c_{\Gamma\,2} \end{aligned} \tag{26}$$

or after multiplying with $\Delta c_\Gamma := (c_{\Gamma\,1} - c_{\Gamma\,2})$:

$$\frac{1}{2}\frac{d}{dt}|\Delta c_\Gamma|^2 \leq C(J_0, J_1, T, c_\Gamma(0))\left(|\Delta c_\Gamma|^2 + |j_0(c_1, \vartheta_1) - j_0(c_2, \vartheta_2)|^2\right)$$

which yields by Gronwall's lemma after integrating in space $||c_{\Gamma\,1} - c_{\Gamma\,2}||_{L^\infty(0,T;L^2(\Gamma_{as}))} \to 0$ as $||(j_0(c_1, \vartheta_1) - j_0(c_2, \vartheta_2))||_{L^2(0,T;L^2(\Gamma_{as}))} \to 0$. Therefore by the explicit form of $\partial_t c_\Gamma$: $||c_{\Gamma\,1} - c_{\Gamma\,2}||_{H^1(0,T;L^2(\Gamma_{as}))} \to 0$. Note that $c_\Gamma(T) > 0$ whenever $c_\Gamma(0) > 0$.

Proposition 5.1. *For every $(c, \vartheta) \in L^2(0, T; H^1(\Omega_2))^2$ with $c_{krit} \leq c \leq c^{krit}$ and $\vartheta_{krit} \leq \vartheta \leq \vartheta^{krit}$ a.s. on Γ_{as} there is a unique solution $c_\Gamma \in H^1(0, T; L^2(\Gamma_{as}))$ to*

$$\partial_t c_\Gamma = j_{w,\partial\Omega_2}(c, \vartheta, c_\Gamma)$$

with $||c_\Gamma||_{H^1(0,T;L^2(\Gamma_{as}))} \leq C$ with C independent on (c, ϑ). c_Γ depends Lipschitz continuously on c, $\vartheta \in L^2(0, T; L^2(\Gamma_{as}))$.

For the Transport equation, the following theorem can be obtained

Theorem 5.2. *For $\widetilde{c_0} \in H^1(\Omega_2)$ with $c^{krit} \geq \widetilde{c_0} \geq c_{krit}$ almost surely and $j : [0, T] \times \partial\Omega_2 \times \mathbb{R}$ such that j is uniformly Lipschitz continuous in the last variable with $j(t, x, c) \geq 0$ for $c \geq c^{krit}$ and $j(t, x, c) \leq 0$ for $c \leq c^{krit}$ there is a unique solution $c \in H^1((0, T) \times \Omega_2) \cap H^1(0, T; L^2(\partial\Omega_2)$ with $c(t, x) \geq 0$ almost surely in space and time to the problem*

$$\int_0^T \int_{\Omega_2} \partial_t c\phi + \int_0^T \int_{\partial\Omega_2} \delta\partial_t c\phi - \int_0^T \int_{\Omega_2} j_w \nabla\phi + \int_0^T \int_{\partial\Omega_2} j(c)\phi = 0 \tag{27}$$

$\forall \phi \in H^1((0, T) \times \Omega_2)$ with $c(0, \cdot) = \widetilde{c_0}$ and an estimate

$$||c||_{L^2(0,T;H^1(\Omega_2))} + ||\partial_t c||_{L^2(0,T;H^*(\Omega_2))} \leq C(T, \Omega_2, \widetilde{c_0})$$

where the constant C does not depend on ϑ or c_Γ and

$$\|\partial_t c\|_{L^2(0,T;L^2(\Omega_2))} + \|\Delta c\|_{L^2(0,T;L^2(\Omega_2))} \leq C(T, \Omega_2, \widetilde{c_0}, \delta)$$

Furthermore, $c^{krit} \geq c \geq c_{krit}$ almost surely.

Proof.

Take an complete orthonormal system $(v_n)_{n\in\mathbb{N}}$ of $H^1(\Omega_2)$ and define $H_m(\Omega_2) := \operatorname{span}(v_1, \ldots, v_m)$ for every $m \in \mathbb{N}$. We show that there is $c_m \in H_m(\Omega_2)$ such that

$$\int_0^T \int_{\Omega_2} \partial_t c_m \phi + \int_0^T \int_{\partial\Omega_2} \delta \partial_t c_m \phi - \int_0^T \int_{\Omega_2} j_w \nabla \phi + \int_0^T \int_{\partial\Omega_2} j(c_m)\phi = 0$$

for every $\phi \in H_m(\Omega_2)$. To this aim, set

$$c_m := \sum_{i=1}^m \xi_i(t) v_i$$

Inserting this ansatz yields a system of equations

$$A\xi'(t) = F(\xi)$$

with $\xi_i(0) = \int_{\Omega_2 \cup \partial\Omega_2} \widetilde{c_0} v_i$, Lipschitz continuous function F and

$$A = (a_{i,j}) = (B(v_i, v_j)) := \int_{\Omega_2} v_i v_j + \delta \int_{\partial\Omega_2} v_i v_j$$

Since B is a strongly positive bi linear form on $H^1(\Omega_2)$, $A \in \mathbb{R}^{n\times n}$ is invertible and there exists a unique ξ satisfying the ODE above.

By standard arguments, testing with c_m yields:

$$\frac{d}{dt}\left(\int_{\Omega_2} c_m^2 + \delta \int_{\partial\Omega_2} c_m^2\right) + \int_{\Omega_2} |\nabla c_m|^2 \leq \int_{\partial\Omega_2} |j| \cdot |c_m| + \int_{\Omega_2} |w|\,|c_m|\,|\nabla c_m|$$

and by a simple calculation

$$\sup_{t\in[0,T]} \|c_m(t)\|_{L^2(\Omega_2 \cup \partial\Omega_2)} + \|\nabla c_m\|_{L^2(0,T;L^2(\Omega_2))} < C$$

Testing with $\partial_t c$ yields

$$\int_{\Omega_2} (\partial_t c_m)^2 + \delta \int_{\partial\Omega_2} (\partial c_m)^2 + \frac{d}{dt} \int |\nabla c_m|^2 \leq \int_{\partial\Omega_2} |j| \cdot |\partial_t c_m| + \int_{\Omega_2} |w|\,|\partial_t c_m|\,|\nabla c_m|$$

which is again easy to handle and yields

$$\|\partial_t c_m\|_{L^2(0,T;L^2(\Omega_2 \cup \partial\Omega_2)} + \sup_{t\in[0,T]} \|\nabla c_m(t)\|_{L^2(\Omega_2)} < C$$

Passage to the limit yields a solution to the problem satisfying the claimed regularities on $\|c\|_{L^2(0,T;H^1(\Omega_2))} + \|\partial_t c\|_{L^2(0,T;H^*(\Omega_2))}$ and on $\|\partial_t c\|_{L^2(0,T;L^2(\Omega_2 \cup \partial\Omega_2))} + \|\Delta c\|_{L^2(0,T;L^2(\Omega_2))}$. Remark that the last estimate stems from the fact that $\partial_t c - \Delta c + v\nabla c = 0$ a.s.. For two solutions c_1, c_2 with identical initial values, it can be checked quickly that

$$\frac{d}{dt}\left(\int_{\Omega_2}(c_1-c_2)^2 + \delta\int_{\partial\Omega_2}(c_1-c_2)^2\right) + \int_{\Omega_2}|\nabla(c_1-c_2)|^2$$
$$\leq \int_{\partial\Omega_2} J_0 J_1 C_0 (c_1-c_2)^2 + \int_{\Omega_2}|w|\,|c_1-c_2|\,|\nabla(c_1-c_2)|$$

which yields uniqueness of c by application of Gronwall's inequality.

To check the independence of the estimates' constants form the parameters c_Γ and ϑ, just remember that according to the definitions in 5 we calculate $|j_0(c_1,\vartheta) - j_0(c_2,\vartheta)| \leq J_0|c_1 - c_0(\vartheta) - c_2 + c_0(\vartheta)|$ to see that the Lipschitz constant of $j_{w,\partial\Omega_2}$ is independent on the choice of ϑ and by equation (26) also c_Γ. □

6. The Energy equation

Problem 6.1. *For given $u_0 \in L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega)$ and $\vartheta_0 \in H^1(\Omega)$ with $u_0(x) \in u(x,0,\vartheta_0)$ where u is defined as below and $\vartheta^{krit} > \vartheta_0 > \vartheta_{krit}$ almost surely, find $u \in L^2(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega))$, $\partial_t u \in L^2(0,T;H^*(\Omega))$ $\vartheta \in H^1(0,T;L^2(\Omega,\mathcal{L}+\mu_\Gamma)) \cap L^2(0,T;H^1(\Omega))$ such that:*

$$\int_0^T\int_\Omega \partial_t u(\vartheta)\phi + \int_0^T\int_\Omega \nabla\vartheta\nabla\phi + \int_0^T\int_{\Omega_2}\nabla\left(j^* m_c(\vartheta) + w\,m_2(\vartheta)\right)\phi + \dots$$
$$\dots - \int_0^T\int_{\Gamma_{as}}\partial_t c_\Gamma\, m_{c_\Gamma}(\vartheta) + \int_0^T\int_{\Gamma_{as}\cup\partial\Omega}(\partial_t u(\vartheta) + g(x,t,\vartheta))\phi = 0 \quad (28)$$

where j_u is given by equation (21) and (22)-(25) are satisfied. Suppose $\vartheta(0) = \vartheta_0$, $c^* \in C^{2,1}([0,T]\times\overline{\Omega_2})$, $j^* \in C^{2,1}([0,T]\times\overline{\Omega_2})^n$ such that $\partial_t c^* + div\, j^* = 0$. Furthermore $w \in L^\infty(0,T;\Omega_2)^n$, $div\, w = 0$, $v\big|_{\partial\Omega_2} \equiv 0$ and $c_\Gamma \in H^1(0,T;L^2(\Gamma_{as})) \cap L^\infty((0,T)\times\Gamma_{as})$. $g = g(\vartheta,x,t)$ is assumed to be monotone increasing and Lipschitz in ϑ with a Lipschitz constant independent on (x,t), $g(\vartheta,x,t) \geq 0$ for $\vartheta > \vartheta^{krit}$ and $g(\vartheta,x,t) \leq 0$ for $\vartheta < \vartheta_{krit}$.

Finally, m_1, m_2, m_c, m_{c_Γ},$m_{\partial\Omega}$ and m_Γ are maximal monotone graphs with $0 \in m_i(0)$, $m_i(s) = b_i(s) + \alpha_i H_i(s)$, $b_i \in C^1(\mathbb{R})$, $\alpha_i \in \mathbb{R}^+$, $\infty > b^* \geq b_i'(s)\,\forall i$, $b_i'(s) \geq b_* > 0$ for $i = 1,2,\Gamma$, $\alpha_2 = \alpha_c = 0$ and

$$H_i(s) := \begin{cases} 0 & s < h_i \\ [0,1] & s = h_i \\ 1 & s > h_i \end{cases}$$

with h_i being arbitrary positive constants.

Theorem 6.1. *Assume that m_1, m_2, m_c, m_{c_Γ}, $m_{\partial\Omega}$ and m_Γ above are in $C^{1,1}(\mathbb{R})$, strongly monotone with bounded derivatives, i.e.*

$$0 < b_* < m_1', m_2', m_\Gamma', m_c', m_{c_\Gamma}', m_{\partial\Omega}' < b^* < \infty$$

, and $c^* \geq 0$ as well as $c_\Gamma \geq 0$. Then Problem 6.1 has a unique solution which satisfies the estimate:

$$\|\vartheta\|_{L^\infty(0,T;H^1(\Omega))} + \|\vartheta\|_{H^1(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega))} \quad\leq\quad C$$

with $C = C(T, b_*, b^*, \|c_\Gamma\|_{H^1(0,T;L^\infty(\Gamma_{as}))}, \|j^* + w\|_{L^\infty((0,T)\times\Omega_2)}, \vartheta(0))$ only depending on these constants and $\vartheta_{krit} \leq \vartheta \leq \vartheta^{krit}$ almost surely.

Proof.
1. Step: Galerkin Approximation

Take an orthonormal Basis $(v_i)_{i\in\mathbb{N}}$ of $H^1(\Omega)$ and define $H_m(\Omega) := \operatorname{span}(v_1,\dots,v_m)$ for every $m\in\mathbb{N}$. The first step is to show that there is a solution $\vartheta_m \in C^1([0,T];H_m(\Omega))$ to the system

$$\begin{aligned}0 = \int_0^T \int_\Omega \partial_t u(\vartheta_m) v_i \, d(\mathcal{L}+\mathcal{H}) - \int_0^T \int_\Omega j_u(\vartheta_m)\nabla v_i \\ + \int_0^T \int_{\Gamma_{as}\cup\partial\Omega} (\partial_t u(\vartheta_m) + g(\vartheta_m) - \partial_t c_\Gamma \, m_{c_\Gamma}(\vartheta_m)) v_i \quad \text{for } i = 1,\dots,m\end{aligned} \tag{29}$$

with $\vartheta_m(0) = \sum_{i=1}^m \langle\vartheta_0, v_i\rangle v_i$. To this aim use the ansatz:

$$\vartheta_m = \sum_{i=1}^m \xi_i(t) v_i$$

and get a system of equations:

$$\sum_{j=1}^m \left(\int_\Omega (m'_{1,2}(\vartheta_m) + c^* \, m'_c(\vartheta_m))\xi'_j(t) v_i v_j + \int_\Gamma (c_\Gamma m'_{c_\Gamma}(\vartheta_m) + m'_\Gamma(\vartheta_m))\xi'_j(t) v_i v_j \right) = F(\xi,t) \quad i = 1,\dots,m$$

which can be written in the form:

$$A(\xi,t)\,\xi'(t) = F(\xi,t) \tag{30}$$

with the corresponding initial conditions. Here, A is a linear mapping $\mathbb{R}^m \to \mathbb{R}^m$ and F is Lipschitz-continuous in ξ. There is a locally unique Solution on $(0,t_0)$ with $t_0 \le T$, if the matrix inverse $A(\xi,t)^{-1}$ exists and is Lipschitz continuous in ξ since

$$A(\xi_1,t)^{-1}F(\xi_1,t) - A(\xi_2,t)^{-1}F(\xi_2,t) = \left(A(\xi_1,t)^{-1} - A(\xi_2,t)^{-1}\right) F(\xi_1,t) + A(\xi_2,t)^{-1}\left(F(\xi_1) - F(\xi_2)\right) \tag{31}$$

It holds $|F(\xi)| < C|\xi|$ by definition and the relation $|\xi|^2 \le C(T)$ will follow from equation (37). We therefore assume for the moment $t_0 = T$.

Now, A^{-1} exists, since H_m is isomorphic to $\mathbb{R}^m$ and

$$B(\phi,\psi) := \int_\Omega (m'(\vartheta_m) + c^* \, m'_c(\vartheta_m))\phi\,\psi + \int_\Gamma (c_\Gamma m'_{c_\Gamma}(\vartheta_m) + m'_\Gamma(\vartheta_m))\phi\,\psi$$

is a continuous bi linear form with $B^*\|\phi\|^2_{H_m} \ge B(\phi,\phi) \ge B_*\|\phi\|^2_{H_m}$ with B^* and B_* being constants independent on m and ϑ_m. By the Lax-Milgram-Lemma, the Matrix representing B is invertible and from above we see that this matrix is A. A depends Lipschitz-continuous on ξ and $A \mapsto A^{-1}$ is locally Lipschitz-continuous with a Lipschitz constant depending on $\|A\|$.

2. Step: A Priory Estimates and Limit

Define

$$a(x,t,\vartheta) := \int_0^\vartheta u(x,t,s)ds \tag{32}$$

for arbitrary ϑ, the following simple equation holds:

$$\int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \partial_t u(\vartheta)\vartheta = \int_\Omega \Big((u(\vartheta(t))\vartheta(t) - a(\vartheta(t))) \Big|_0^T \Big) + \int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} a_t(\vartheta) \tag{33}$$

Since u takes the abstract form $u(\vartheta) = m(\vartheta)$ one gets with $M(\vartheta) := \int_0^\vartheta m(s)ds$:

$$\frac{1}{2} b^* \vartheta^2 \geq u(\vartheta)\vartheta - a(\vartheta) = m(\vartheta)\vartheta - M(\vartheta) = \int_0^\vartheta m'(s)s\,ds \geq \frac{1}{2} b_* \vartheta^2$$

furthermore, one gets for the last term in equation (33):

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} a_t(\vartheta) = \int_0^T \left(\int_\Omega \partial_t c^* M_c(\vartheta) + \int_{\Gamma_{as}} \partial_t c_\Gamma M_{c_\Gamma}(\vartheta) \right) \tag{34}$$

In order to get a priory estimates, take ϑ_m as a valid test function:

$$0 = \int_0^T \int_\Omega \partial_t u(\vartheta_m)\vartheta_m + \int_0^T \int_\Omega |\nabla\vartheta_m| - \int_0^T \int_{\Omega_2} (j^* m_c(\vartheta_m) + w\, m_2(\vartheta_m))\nabla\vartheta_m$$
$$+ \int_0^T \int_{\Gamma_{as}\cup\partial\Omega} (\partial_t u(\vartheta_m)\vartheta_m - \partial_t c_\Gamma\, m_{c_\Gamma}(\vartheta_m)\vartheta_m + g(\vartheta_m)\vartheta_m)$$

We need to estimate the third term on the right hand side:

$$\begin{aligned}
\int_0^T \int_{\Omega_2} \operatorname{div}(j^* m_c(\vartheta_m) + w\, m_2(\vartheta_m))\vartheta_m &= -\int_0^T \int_{\Omega_2} (j^* m_c(\vartheta_m) + w\, m_2(\vartheta_m))\nabla\vartheta_m + \int_0^T \int_{\partial\Omega_2} j^* \nu_2 m_c(\vartheta_m)\vartheta_m \\
&= -\int_0^T \int_{\Omega_2} (j^* \nabla M_c(\vartheta_m) + w\nabla M_2(\vartheta_m)) + \int_0^T \int_{\partial\Omega_2} j^* \cdot \nu_2 m_c(\vartheta_m)\vartheta_m \\
&= -\int_0^T \int_{\partial\Omega_2} j^* \cdot \nu_2 \,(M_c(\vartheta) - m_c(\vartheta_m)\vartheta_m) + \int_0^T \int_{\Omega_2} \operatorname{div} j^* M_c(\vartheta_m)
\end{aligned} \tag{35}$$

We see from equations (35) and (34) that the sum of terms including $\partial_t c^*$ and $\operatorname{div} j^*$ vanish and get by $c^* \geq 0$:

$$\begin{aligned}
\frac{1}{2} b_* \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} \vartheta_m^2 + \int_0^T \int_\Omega |\nabla\vartheta_m|^2 < \int_0^T \int_{\Gamma_{as}} \partial_t c_\Gamma (m_{c_\Gamma}(\vartheta_m)\vartheta_m - M_{c_\Gamma}(\vartheta_m)) + C \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \vartheta_m(0)^2 \\
+ \int_0^T \int_{\partial\Omega_2} |j^*|\,(M_c(\vartheta_m) + m_c(\vartheta_m)\vartheta_m)
\end{aligned} \tag{36}$$

Since $M_{c_\Gamma}(\vartheta) \leq \frac{1}{2} b^* \vartheta^2$, $M_c(\vartheta) \leq \frac{1}{2} b^* \vartheta^2$, Gronwall's lemma applied to this inequality yields:

$$\int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \vartheta_m^2(T) + \int_0^T \int_\Omega |\nabla\vartheta_m|^2 \leq C(b_*, b^*, \vartheta(0), c_\Gamma)\,(1 + C\,T \exp(C\,T)) \tag{37}$$

and this yields together with equation (31) by $|\xi|^2 \leq C(1 + T\exp(CT))$ the global existence and uniqueness in $(0, T)$. In a second step, we choose $\partial_t\vartheta_m$ as a test function and make use of

$$\begin{aligned}
\partial_t u_1(\vartheta_m) &= m_1'(\vartheta_m)\partial_t\vartheta_m \\
\partial_t u_2(\vartheta_m) &= m_2'(\vartheta_m)\partial_t\vartheta_m + \partial_t c^*\, m_c(\vartheta_m) + c^* m_c'(\vartheta_m)\partial_t\vartheta_m \\
\partial_t u_{\Gamma_{as}}(\vartheta_m) &= m_\Gamma'(\vartheta_m)\partial_t\vartheta_m + \partial_t c_\Gamma\, m_{c_\Gamma}(\vartheta_m) + c_\Gamma m_{c_\Gamma}'(\vartheta_m)\partial_t\vartheta_m \\
\partial_t u_{\partial\Omega}(\vartheta_m) &= m_{\partial\Omega}'(\vartheta_m)\partial_t\vartheta_m
\end{aligned}$$

The convective term turns into:

$$+\int_{\Omega_2} \operatorname{div}\,(j^* m_c(\vartheta_m) + w\, m_2(\vartheta_m))\partial_t\vartheta_m = \int_{\Omega_2} (\operatorname{div}\, j^* m_c(\vartheta_m))\,\partial_t\vartheta_m + \int_{\Omega_2} (j^* m_c'(\vartheta_m) + w\, m_2'(\vartheta_m))\nabla\vartheta_m\partial_t\vartheta_m$$

So one obtains:

$$\begin{aligned}
&\int_{\Omega_2} (m_2'(\vartheta_m) + c^* m_c'(\vartheta_m))(\partial_t\vartheta_m)^2 + \int_{\Omega_1} m_1'(\vartheta_m)(\partial_t\vartheta_m)^2 + \int_{\Gamma_{as}} m_\Gamma'(\vartheta_m)(\partial_t\vartheta_m)^2 \\
&\quad + \int_{\Gamma_{as}} c_\Gamma m_{c_\Gamma}'(\vartheta_m)(\partial_t\vartheta_m)^2 + \int_{\partial\Omega} m_{\partial\Omega}'(\vartheta_m)(\partial_t\vartheta_m)^2 + \frac{d}{dt}\int_\Omega (\nabla\vartheta_m)^2 \\
&\quad \leq \|j^* + w\|_\infty b^* \|\nabla\vartheta_m\|_{L^2(\Omega_2)} \|\partial_t\vartheta_m\|_{L^2(\Omega_2)} + C\|g\|_\infty \|\partial_t\vartheta_m\|_{L^2(\partial\Omega_2)} \\
&\quad + C\|j^*\|_{L^\infty(\partial\Omega)} \|m_c(\vartheta_m)\|_{L^2(\partial\Omega)} \|\partial_t\vartheta_m\|_{L^2(\partial\Omega)}
\end{aligned} \tag{38}$$

where $m_c(\vartheta)\vartheta$ and $m_{c_\Gamma}(\vartheta)\vartheta$ were estimated by $b^*\vartheta^2$, $m_2', m_1', m_\Gamma', m_{\partial\Omega} > b_*$, $c^* \geq 0$ and $m_\Gamma', m_c' \geq 0$. Gronwall's lemma yields:

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (\partial_t\vartheta_m)^2 + \sup_{0\leq t\leq T} \int_\Omega (\nabla\vartheta_m)^2 \leq C(\vartheta(0), T, j^*, w, c_\Gamma, b_*, b^*) \tag{39}$$

Choosing any test function $\phi \in H^1(\Omega)$ finally yields:

$$\|\partial_t u(\vartheta_m)\|_{L^2(0,T;H^*)} \leq C(\|(j^* + w)\|_\infty)\left(\|\nabla\vartheta_m\|_{L^2(\Omega)} + C\|\vartheta_m\|_{L^2(\partial\Omega\cup\Gamma_{as})}\right) \tag{40}$$

For ϑ_m and u we get the following convergences for a sub sequence:

$$\begin{aligned}
\vartheta_m &\rightharpoonup \vartheta && \text{in } L^2(0,T;H^1(\Omega)) \\
\partial_t\vartheta_m &\rightharpoonup \vartheta_t && \text{in } L^2(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega)) \\
\partial_t u(\vartheta_m) &\rightharpoonup u_t && \text{in } L^2(0,T;(H^*))
\end{aligned}$$

Which yields by Lions Theorem

$$\vartheta_m \to \vartheta \quad \text{in } L^2(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega)) \quad \partial_t\vartheta = \vartheta_t$$

Since u depends by some Lipschitz-continuous terms on ϑ,

$$u(\vartheta_m) \to u(\vartheta) \text{ in}$$

strongly in $L^2(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega))$ and

$$\iint u_t\phi \xleftarrow{m\to\infty} \iint \partial_t u(\vartheta_m)\phi = -\iint u(\vartheta_m)\partial_t\phi \xrightarrow{m\to\infty} -\iint u(\vartheta)\partial_t\phi$$

for suitable test functions with zero boundary values in time. Since $u(\vartheta) \in H^1(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega))$, we have $u_t = \partial_t u(\vartheta)$

Using all the above convergences , equation (28) holds for (u, ϑ).

3. Step: Essential Boundedness

The basic idea to show the essential boundedness of ϑ, is testing the equation with $(\vartheta - \vartheta^{\text{krit}})^+$, where $x^+ = \chi_{\mathbb{R}^+}(x)\,x$. In this context, it can first be observed, that

$$\int_{\partial\Omega_2} g(x,t,\vartheta)(\vartheta - \vartheta^{\text{krit}})^+ \geq 0$$

by definition of g. Therefore, the term will be neglected in the calculations below.

Following Rodriguez [9, 10] define $M := \vartheta^{\text{krit}}$ and calculate

$$\beta_{M,i}(r) := \begin{cases} \int_0^r (m_i^{-1}(s) - M)^+ dr & \text{if } r > m_i(M) \\ 0 & \text{if } r \leq m_i(M) \end{cases}$$

Remark, that $\beta_{M,i}(m_i(\vartheta(0))) = 0$ for all i. Choose $(\vartheta - M)^+$ as a valid test function and calculate:

$$\begin{aligned}\int_0^T \int_{\Omega_2} \partial_t u_2(\vartheta)(\vartheta - M)^+ &= \int_0^T \int_{\Omega_2} (\partial_t \beta_{M,2}(m_2(\vartheta)) + \partial_t c^* m_c(\vartheta)(\vartheta - M)^+ + c^* \partial_t \beta_{M,c}(m_c(\vartheta))) \\ &= \int_0^T \int_{\Omega_2} \partial_t c^* \,(m_c(\vartheta)(\vartheta - M)^+ - \beta_{M,c}(m_c(\vartheta))) + \int_{\Omega_2} (c^*(T)\beta_{M,c}(\vartheta(T)) + \beta_{M,2}(m_2(\vartheta(T))))\end{aligned}$$

$$\int_0^T \int_{\Omega_1} \partial_t u_1(\vartheta)(\vartheta - M)^+ = \int_0^T \int_{\Omega_1} \partial_t \beta_{M,1}(m_1(\vartheta))$$

$$\begin{aligned}\int_0^T \int_{\Gamma_{as}} (\partial_t u_\Gamma(\vartheta) - \partial_t c_\Gamma m_{c_\Gamma}(\vartheta))(\vartheta - M)^+ &= \int_0^T \int_{\Gamma_{as}} c_\Gamma \partial_t \beta_{M,c_\Gamma}(m_{c_\Gamma}(\vartheta)) + \int_0^T \int_{\Gamma_{as}} \partial_t \beta_{M,\Gamma}(m_\Gamma(\vartheta)) \\ &= \int_{\Gamma_{as}} c_\Gamma \beta_{M,c_\Gamma}(m_{c_\Gamma}(\vartheta))\Big|_0^T - \int_0^T \int_{\Gamma_{as}} (\partial_t c_\Gamma \beta_{M,c_\Gamma}(m_{c_\Gamma}(\vartheta)) - \partial_t \beta_{M,\Gamma}(m_\Gamma(\vartheta)))\end{aligned}$$

$$\int_0^T \int_{\partial\Omega} \partial_t u_{\partial\Omega}(\vartheta)(\vartheta - M)^+ = \int_0^T \int_{\partial\Omega} \partial_t \beta_{M,\partial\Omega}(m_{\partial\Omega}(\vartheta))$$

$$\begin{aligned}\int_0^T \int_{\Omega_2} \operatorname{div}(j^* m_c(\vartheta))(\vartheta - M)^+ &= \int_0^T \int_{\Omega_2} (\operatorname{div} j^* m_c(\vartheta)(\vartheta - M)^+ + j^* \nabla \beta_{M,c}(m_c(\vartheta))) \\ &= \int_0^T \int_{\Omega_2} (\operatorname{div} j^* m_c(\vartheta)(\vartheta - M)^+ - \operatorname{div} j^* \,\beta_{M,c}(m_c(\vartheta))) + \int_0^T \int_{\Gamma_{as} \cup \partial\Omega} j^* \beta_{M,c}(m_c(\vartheta))\end{aligned}$$

From the transformation theorem for Integrals follows

$$\begin{aligned} 0 \leq \beta_{M,i}(m_i(\vartheta)) &= \;= \int_0^{m_i(\vartheta)} (m_i^{-1}(s) - M)^+ ds \int_0^\vartheta (m_i^{-1}(m_i(s)) - M)^+ m_i'(s)\, ds \\ &= \begin{cases} \int_0^\vartheta (s - M)^+ m_i'(s)\, ds & \text{if } \vartheta > M \\ 0 & \text{if } \vartheta \leq M \end{cases}\end{aligned}$$

and from the boundedness of the derivatives of the m_i' follows the existence of $C_{i,j}$ for each couple i, j such that

$$\beta_{M,i}(m_i(\vartheta)) \leq C_{i,j} \beta_{M,j}(m_j(\vartheta))$$

therefore, the estimate on $\beta_{M,i}(m_i(\vartheta))$ reads as follows:

$$\int_{\Omega_2}\beta_{M,2}(m_2(\vartheta(t)))+\int_{\Omega_1}\beta_{M,1}(m_1(\vartheta(t)))+\int_{\Gamma_{as}}\beta_{M,\Gamma}(m_\Gamma(\vartheta(t)))+\int_{\partial\Omega}\beta_{M,\partial\Omega}(m_{\partial\Omega}(\vartheta(t)))$$
$$\leq C\int_0^t\left(\int_{\Omega_2}\beta_{M,2}(m_2(\vartheta))+\int_{\Omega_1}\beta_{M,1}(m_1(\vartheta))+\int_{\Gamma_{as}}\beta_{M,\Gamma}(m_\Gamma(\vartheta))+\int_{\partial\Omega}\beta_{M,\partial\Omega}(m_{\partial\Omega}(\vartheta))\right)$$

for every $t\in[0,T]$ and therefore the expression on the left hand side is zero for all t. This implies $\vartheta\leq M$. In the same way it can be shown that $\vartheta\geq\vartheta_{\text{krit}}$.

4. Step: Uniqueness

Assume the test function ϕ having $\Delta\phi\in H^*(\Omega)$ and $\partial_{tt}\phi\in H^*(\Omega)$ and $\phi(T)\equiv 0$. Take two solutions (u_1,ϑ_1) and (u_2,ϑ_2) with $(u_1(0),\vartheta_1(0))=(u_2(0),\vartheta_2(0))$ of the problem:

$$\int_0^T\int_\Omega(\vartheta_1-\vartheta_2)(-\alpha\partial_t\phi-\Delta\phi-\beta\nabla\phi)+\int_0^T\int_{\Gamma_{as}\cup\partial\Omega}(\vartheta_1-\vartheta_2)(-\alpha\partial_t\phi+[\nabla\phi]+\delta\phi)$$
$$=\int_\Omega(u_1(0)-u_2(0))\phi(0)=0$$

with $\alpha=\frac{(u_1-u_2)}{(\vartheta_1-\vartheta_2)}$ on Ω and $\Gamma\cup\partial\Omega$ respectively,

$$\begin{aligned}\beta &:= \chi_{\Omega_2}\cdot\left(j^*(m_c(\vartheta_1)-m_c(\vartheta_2))+w(m_2(\vartheta_1)-m_2(\vartheta_2))\right)/(\vartheta_1-\vartheta_2)\\ \delta &:= \beta\nu_2\big|_{\partial\Omega_2}+\left((g(\vartheta_1)-g(\vartheta_2))-c_\Gamma(m_{c_\Gamma}(\vartheta_1)-m_{c_\Gamma}(\vartheta_2))\right)/(\vartheta_1-\vartheta_2)\end{aligned}$$

and $[\nabla\phi]:=(\nabla\phi|_{\Omega_2}-\nabla\phi|_{\Omega_1})\,\nu_2$ is a term which enters due to jumps of $\nabla\phi$ at the interface Γ_{as}. Note, that $|\alpha|\geq b_*$.

We insert a solution to the problem:

$$\begin{aligned}\varepsilon\partial_t^2\phi-\alpha\partial_t\phi-\Delta\phi-\beta\nabla\phi &= (\vartheta_1-\vartheta_2)\quad\text{in }\Omega &(41)\\ \varepsilon\partial_t^2\phi-\alpha\partial_t\phi+\delta\phi &= -[\nabla\phi]+(\vartheta_1-\vartheta_2)\quad\text{on }\Gamma_{as}\cup\partial\Omega &(42)\\ \phi(T) &\equiv 0 &(43)\\ \partial_t\phi(T) &\equiv 0 &(44)\end{aligned}$$

with the a priory estimate (45) from Lemma 6.2 to obtain for $u_1(0)=u_2(0)$:

$$\int_0^T\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}}(\vartheta_1-\vartheta_2)^2\leq\left|\int_0^T\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}}(\vartheta_1-\vartheta_2)\varepsilon\partial_t^2\phi\right|$$
$$\leq\left|\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}}(\vartheta_1-\vartheta_2)\varepsilon\partial_t\phi\Big|_0^T\right|+\varepsilon\left|\int_0^T\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}}\partial_t(\vartheta_1-\vartheta_2)\partial_t\phi\right|$$

where the first term on the right hand side of the last inequality vanishes due to the fact that $\vartheta_1(0)=\vartheta_2(0)$and $\partial_t\phi(T)=0$. By the boundedness of $\partial_t(\vartheta_1-\vartheta_2)$and $\partial_t\phi$ in $L^2(0,T;L^2(\Omega\cup\partial\Omega\cup\Gamma_{as}))$ we get

$$\int_0^T\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}}(\vartheta_1-\vartheta_2)^2\leq 0$$

□

Lemma 6.2. *Let $\alpha \in L^\infty(\Omega \cup \Gamma_{as} \cup \partial\Omega)$, $\beta \in L^\infty(\Omega)$, $\delta \in L^\infty(\Gamma_{as} \cup \partial\Omega)$ with $\alpha \geq \alpha_* > 0$.For every $f \in L^2(\Omega \cup \partial\Omega \cup \Gamma_{as}, \mathcal{L}+\mathcal{H})$ there exists a solution $\phi \in H^1(0,T; L^2(\Omega \cup \partial\Omega \cup \Gamma_{as})) \cap L^2(0,T; H^1(\Omega))$ with $\Delta\phi, \partial_t^2\phi \in L^2(0,T; H^*(\Omega))$ to the problem*

$$\begin{aligned} \varepsilon\partial_t^2\phi - \alpha\partial_t\phi - \Delta\phi - \beta\nabla\phi &= f \quad \text{in } \Omega \\ \varepsilon\partial_t^2\phi - \alpha\partial_t\phi + \delta\phi &= -[\nabla\phi] + f \quad \text{on } \Gamma_{as} \cup \partial\Omega \\ \phi(T) &\equiv 0 \\ \partial_t\phi(T) &\equiv 0 \end{aligned}$$

where $[\nabla\phi] := (\nabla\phi|_{\Omega_2} - \nabla\phi|_{\Omega_1})\, \nu_2$, that satisfies the following a priory estimates:

$$\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (\partial_t\phi_m)^2 + \int_\Omega (\nabla\phi_m)^2 + \int_{\Gamma_{as}\cup\partial\Omega} (\phi_m)^2 \leq C\left(||f||^2_{L^2(\Omega)}\right) \tag{45}$$

where the constant C depends only on Ω, α_*, β, δ.

Note that an important condition in this lemma is the strict positivity of α.

Proof. Perform the transformation $t \rightsquigarrow -t$. Using the Galerkin approximation with spaces $H_m(\Omega)$ as in the previous proof, we get the existence of a solution $\phi_m \in H^2(0,T; H_m(\Omega))$ to the problem

$$\begin{aligned} \int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} f\,\psi &= \int_0^T \int_\Omega \left(\varepsilon\partial_t^2\phi_m\,\psi + \alpha\partial_t\phi_m\,\psi + \nabla\phi_m\nabla\psi - \beta\nabla\phi_m\,\psi\right) \\ &\quad + \int_0^T \int_{\Gamma_{as}\cup\partial\Omega} \left(\varepsilon\partial_t^2\phi_m + \alpha\partial_t\phi_m + \delta\phi_m\right)\psi \qquad \forall\psi \in H^1(0,T; H_m(\Omega)) \end{aligned}$$

Choosing $\partial_t\phi_m$ as a test function, the following estimates can be obtained:

$$\begin{aligned} \frac{d}{dt}\left(\int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \varepsilon(\partial_t\phi_m)^2 + \int_\Omega (\nabla\phi_m)^2 + \int_{\Gamma_{as}\cup\partial\Omega} (\phi_m)^2\right) + \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} \alpha(\partial_t\phi_m)^2 \\ \leq \left(||f||_{L^2(\Omega)} + ||\beta||_{L^\infty(\Omega)}||\nabla\phi_m||_{L^2(\Omega)}\right)||\partial_t\phi_m||_{L^2(\Omega)} \\ + (||\delta||_{L^\infty(\Gamma_{as}\cup\partial\Omega)} + 1)||\phi_m||_{L^2(\Gamma_{as}\cup\partial\Omega)}||\partial_t\phi_m||_{L^2(\Gamma_{as}\cup\partial\Omega)} \end{aligned}$$

since α is bounded from below $0 < \alpha_0 \leq \alpha$, it can be seen by absorbing the terms including $\partial_t\phi_m$ on the right hand side, that:

$$\begin{aligned} \frac{d}{dt}\left(\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} \varepsilon(\partial_t\phi_m)^2 + \int_\Omega (\nabla\phi_m)^2 + \int_{\Gamma_{as}\cup\partial\Omega} (\phi_m)^2\right) + \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (\partial_t\phi_m)^2 \\ \leq C\left(||f||^2_{L^2(\Omega)} + ||\nabla\phi_m||^2_{L^2(\Omega)} + ||\phi_m||_{L^2(\Gamma_{as}\cup\partial\Omega)} + \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} \varepsilon(\partial_t\phi_m)^2\right) \end{aligned}$$

and therefore

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (\partial_t\phi_m)^2 + \varepsilon\int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (\partial_t\phi_m)^2 + \int_\Omega (\nabla\phi_m)^2 + \int_{\Gamma_{as}\cup\partial\Omega} (\phi_m)^2 \leq C\left(||f||^2_{L^2(\Omega)}\right)$$

The H^*-estimates on $\partial_t^2\phi_m$ and $\Delta\phi_m$ are trivial. An estimate on $||\phi_m||_{L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega)}$ can be obtained by $\phi_m(x,T) = \phi_m(x,0) + \int_0^T \partial_t\phi(x,s)ds$ and Jensen's inequality.

□

Lemma 6.3. *Let $A \subset\subset (0,T) \times \Omega_1$, $H^1_\infty(\Omega) := H^1(\Omega) \cap L^\infty(\Omega)$ with $\|\cdot\|_{H^1_\infty} := \|\cdot\|_{H^1} + \|\cdot\|_{L^\infty}$, $H^1_{\infty,0}(\Omega) := H^1_0(\Omega) \cap L^\infty(\Omega)$ and H^{1*}_∞, $H^{1*}_{\infty,0}$ be the corresponding dual spaces. The solution ϑ from Theorem 6.1 satisfies the estimates:*

$$
\begin{aligned}
\|\nabla\vartheta\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{1*}_\infty(\Omega))} + \|\partial_t\vartheta\|_{L^2(0,T;H^{1*}_{\infty,0}(\Omega_2))} &\leq C \\
\|\partial_t\vartheta\|_{L^2(0,T;L^2(A))} &\leq C_A
\end{aligned}
$$

where C only depends on $b_*, b^*, \vartheta(0), c_\Gamma, T, \|j^*\|_{L^2(0,T;L^2(\Omega_2))}, \vartheta^{krit}, \vartheta_{krit}$ and C_A additionally depends on A.

Proof.
Using the result above and testing the problem with $\phi \in L^2(0,T;H^1_\infty(\Omega))$ yields

$$
\left|\int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \partial_t u\,\phi\right| \leq \left(\|\nabla\vartheta\|_{L^2(0,T;L^2(\Omega))} + C\|j^*\|_{L^2(0,T;L^2(\Omega))}\right)\|\nabla\phi\|_{L^2(0,T;L^2(\Omega))} + \int_0^T\int_{\partial\Omega_2} j^* m_c(\vartheta)\phi
$$

since $\vartheta, \phi \in L^2(0,T;H^1_\infty(\Omega))$ and m_c is differentiable follows $m_c(\vartheta)\phi \in L^2(0,T;H^1_\infty(\Omega))$ and

$$
\left|\int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \partial_t u\,\phi\right| \leq C\left(\|\vartheta\|_{L^2(0,T;H^1(\Omega))}, \|j^*\|_{L^2(0,T;L^2(\Omega))}, \vartheta^{\mathrm{krit}}, \vartheta_{\mathrm{krit}}\right)\|\phi\|_{L^2(0,T;H^1_\infty(\Omega))}
$$

Testing again with ϑ and using the fact that $m_c(\vartheta)\vartheta \in H^1_\infty(\Omega)$ equations (33) to (37) yield the estimate

$$
\int_0^T\int_\Omega |\nabla\vartheta|^2 \leq C\left(b_*, b^*, \vartheta(0), c_\Gamma, T, \|j^*\|_{L^2(0,T;L^2(\Omega_2))}, \vartheta^{\mathrm{krit}}, \vartheta_{\mathrm{krit}}\right)
$$

To get the estimates on $\partial_t\vartheta$, choose any $\phi \in H^1_0(\Omega_2) \cap H^1_\infty(\Omega)$ and test the equation with $\psi := \phi(c^* m_c'(\vartheta) + m_2'(\vartheta))^{-1}$ with $\|\psi\|_{L^2(0,T;H^1(\Omega))} \leq C\,\|\phi\|_{L^2(0,T;H^1(\Omega))}\|\vartheta\|_{L^2(0,T;H^1(\Omega))}$ to calculate:

$$
\begin{aligned}
\int_0^T\int_{\Omega_2} \partial_t\vartheta\phi &\leq \left|\int_0^T\int_{\Omega_2} \partial_t u\,\psi\right| + \left|\int_0^T\int_{\Omega_2} \partial_t c^* m_c(\vartheta)\psi\right| \\
&\leq C(\|\partial_t c^*\|_{L^2(0,T;H^*(\Omega_2))})\,\|\phi\|_{H^1_\infty(\Omega_2)}
\end{aligned}
$$

For any set $A \subset\subset (0,T) \times \Omega_1$ take some $\phi \in C_0^\infty((0,1)\times\Omega_1)$ with $\phi \equiv 1$ on A and $\phi \geq 0$ to calculate:

$$
\begin{aligned}
b_*\int_A (\partial_t\vartheta)^2 &\leq \int_0^T\int_{\Omega_1} m_1'(\vartheta)(\partial_t\vartheta)^2\phi^2 \leq \left|\int_0^T\int_{\Omega_1} |\nabla\vartheta|^2\phi\partial_t\phi\right| \\
&\leq C_A\|\nabla\vartheta\|_{L^2(0,T;L^2(\Omega_1))}
\end{aligned}
$$

□

The next theorem gives unique existence for the general class of coefficients.

Theorem 6.4. *Problem 6.1 has a unique solution which satisfies the estimate:*

$$
\|\vartheta\|_{L^\infty(0,T;H^1(\Omega))} + \|\vartheta\|_{H^1(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega))} \leq C
$$

with $C = C(T, b_*, b^*, \|c_\Gamma\|_{H^1(0,T;L^\infty(\Gamma_{as}))}, \|j^* + w\|_{L^\infty((0,T)\times\Omega_2)}, \vartheta(0))$ only depending on these constants and $\vartheta_{krit} \leq \vartheta \leq \vartheta^{krit}$ almost surely.

Proof.

The maximal monotone graphs m_i ($i = 1, 2, \Gamma, c, c_\Gamma, \partial\Omega$) can be approximated by monotone functions m_i^ε satisfying the conditions of Theorem 6.1 such that the corresponding functions $u^\varepsilon(\vartheta)$ converge uniformly in ϑ. Remark that we can choose the m_i^ε such that $0 < b_* < {m_1^\varepsilon}' < {m_2^\varepsilon}' < {m_c^\varepsilon}' < {m_{c_\Gamma}^\varepsilon}' < {m_{\Gamma_{as}}^\varepsilon}' < {m_{\partial\Omega}^\varepsilon}'$ independent on the approximation and taking a look at equation (36) it is evident that the constants in (37) can be chosen independent on m_i^ε. The same holds for the inequalities (38) and (39) and the boundedness of $\|\partial_t u^\varepsilon(\vartheta^\varepsilon)\|_{L^2(0,T;H^*(\Omega))}$ in equation (40).

As $\varepsilon \to 0$ the following convergences hold for a sub sequence of the solutions $(u^\varepsilon, \vartheta^\varepsilon)$:

$$\begin{array}{rcll} \vartheta^\varepsilon & \rightharpoonup & \vartheta & \text{in } L^2(0,T;H^1(\Omega)) \\ \partial_t\vartheta^\varepsilon & \rightharpoonup & \vartheta_t & \text{in } L^2(0,T;L^2(\Omega \cup \Gamma_{as} \cup \partial\Omega)) \\ \partial_t u^\varepsilon(\vartheta^\varepsilon) & \rightharpoonup & u_t & \text{in } L^2(0,T;(H^*(\Omega)) \end{array} \tag{46}$$

Which yields by Lions Theorem

$$\vartheta^\varepsilon \to \vartheta \text{in } L^2(0,T;L^2(\Omega) \cap L^2(\Gamma_{as})) \quad \partial_t\vartheta = \vartheta_t \tag{47}$$

So far, one can argue the same way as for Theorem 6.1. The weak convergence $u^\varepsilon(\vartheta^\varepsilon) \rightharpoonup u(\vartheta)$ in $L^2(\Omega\cup\partial\Omega\cup\Gamma_{as})$ can be shown by the boundedness $|u^\varepsilon(\vartheta^\varepsilon)| \le c(1 + |\vartheta^\varepsilon|)$ which shows $u^\varepsilon(\vartheta^\varepsilon) \rightharpoonup u^*$ in $L^2(\Omega \cup \partial\Omega \cup \Gamma_{as})$. The relation $u^* \in u(\vartheta)$ follows from the fact, that for any $\phi \in H^1(\Omega)$ with $\zeta := u(\phi)$ holds:

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (u^\varepsilon(\vartheta^\varepsilon) - \zeta)\left(\vartheta^\varepsilon - (u^\varepsilon)^{-1}(\zeta)\right) \ge 0$$

and by the convergences (46), (47) and the uniform convergence of $(u^\varepsilon)^{-1}(\cdot)$ on the interval $(\vartheta_{\text{krit}}, \vartheta^{\text{krit}})$, the limit inequality reads:

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (u^* - \zeta)\left(\vartheta - u^{-1}(\zeta)\right) \ge 0$$

like in [15] Chapter 4 Proposition 4.1 substituting $\zeta = u^* + \lambda\xi$ for some $\lambda \in \mathbb{R}$ and $\xi \in L^\infty(\Omega \cup \partial\Omega \cup \Gamma_{as})$ yields:

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} \xi(\vartheta - u^{-1}(u^*)) = 0 \qquad \forall \xi \in L^\infty(\Omega \cup \partial\Omega \cup \Gamma_{as})$$

and we have:

$$\iint u_t\phi \xleftarrow{m\to\infty} \iint \partial_t u^\varepsilon(\vartheta^\varepsilon)\phi = -\iint u^\varepsilon(\vartheta^\varepsilon)\partial_t\phi \xrightarrow{m\to\infty} -\iint u(\vartheta)\partial_t\phi$$

for suitable test functions and therefore $u_t = \partial_t u(\vartheta)$.

The uniqueness follows again by considering the equation:

$$\begin{aligned} \int_0^T \int_\Omega (\vartheta_1 - \vartheta_2)(-\alpha_m\partial_t\phi - \Delta\phi - \beta\nabla\phi) + \int_0^T \int_{\Gamma_{as}\cup\partial\Omega} (\vartheta_1 - \vartheta_2)(-\alpha_m\partial_t\phi + [\nabla\phi] + \delta\phi) \\ = \int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} ((u_1 - u_2) - \alpha_m(\vartheta_1 - \vartheta_2))\,\partial_t\phi \end{aligned}$$

where $\alpha_m \in L^\infty(\Omega \cup \partial\Omega \cup \Gamma_{as})$ with $|\alpha_m| \leq m$ and $\alpha_m(\vartheta_1 - \vartheta_2) \to (u_1 - u_2)$ in $L^2(0,T; L^2(\Omega \cup \partial\Omega \cup \Gamma_{as}))$. ϕ is a solution to

$$\begin{aligned}
\varepsilon\partial_t^2\phi - \alpha_m\partial_t\phi - \Delta\phi - \beta\nabla\phi &= (\vartheta_1 - \vartheta_2) \quad \text{in}\Omega \\
\varepsilon\partial_t^2\phi - \alpha_m\partial_t\phi + \delta\phi &= -[\nabla\phi] + (\vartheta_1 - \vartheta_2) \quad \text{on}\Gamma_{as} \cup \partial\Omega \\
\phi(T) &\equiv 0 \\
\partial_t\phi(T) &\equiv 0
\end{aligned}$$

Since $||\partial_t\phi||_{L^2(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega))} < C$ with C independent on m, it follows that

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (\vartheta_1 - \vartheta_2)^2 \leq \int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} ((u_1 - u_2) - \alpha_m(\vartheta_1 - \vartheta_2))\, \partial_t\phi + \varepsilon \int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} \partial_t^2\phi(\vartheta_1 - \vartheta_2)$$

which implies by the convergence of the first term in m and the convergence of the second term in ε

$$\int_0^T \int_{\Omega\cup\partial\Omega\cup\Gamma_{as}} (\vartheta_1 - \vartheta_2)^2 = 0$$

The result $u_1 = u_2$ follows from the uniqueness of the evolution operator. The boundedness can be shown the same way as for Theorem 6.1. Alternatively, ϑ inherits the boundedness from the ϑ^ε by Lemma 2.8.

□

7. Solving the approximated system

The existence of a solution to the original coupled system can be shown by an application of Schauder's fixed point theorem in Y:

$$\begin{aligned}
Y &:= V_1 \times V_2 \times V_3 && (48) \\
V_1 &:= H^1((0,T) \times \Omega_2) \cap L^2(0,T; W^{1,2}_{\Delta,\partial}(\Omega_2)) && (49) \\
V_2 &:= L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega \cup \Gamma_{as} \cup \partial\Omega)) && (50) \\
V_3 &:= H^1(0,T; L^2(\Gamma_{as})) && (51)
\end{aligned}$$

Theorem 7.1. *There exists at least one solution $(c, \vartheta, c_\Gamma) \in Y$ and a corresponding function $u \in L^2(0,T; L^2(\Omega \cup \Gamma_{as} \cup \partial\Omega))$ to the coupled problem 4.1.*

Proof.
Define the set

$$K: \;=\; \left\{ (c,\vartheta,c_\Gamma) \in Y \;\middle|\; \begin{array}{l} c_{\text{krit}} \leq c \leq c^{\text{krit}},\ \vartheta_{\text{krit}} \leq \vartheta \leq \vartheta^{\text{krit}},\ ||c_\Gamma||_{V_3} \leq C_{c_\Gamma},\ ||c||_{V_1} \leq C_c,\ ||\vartheta||_{V_2} \leq C_\vartheta, \\ c(0,\cdot) = \widetilde{c_0},\ \vartheta(0,\cdot) = \widetilde{\vartheta_0},\ c_\Gamma(0,\cdot) = \widetilde{c_{\Gamma,0}},\ ||\partial_t c_\Gamma||_\infty \leq J_0 J_1,\ c_\Gamma \geq 0\ a.s. \end{array} \right\}$$

Where C_{c_Γ}, C_c, C_ϑ are the constants from Proposition 5.1, Theorem 5.2 and Theorem 6.4. Then, $K \subset Y$ is a nonempty, closed, bounded and convex subset.

Consider the following map from K onto itself: For a given triple $(c_1, \vartheta_1, c_{\Gamma,1}) \in K$ calculate the unique solution to

$$\partial_t c_\Gamma = j_{w,\partial\Omega_2}(c_1, \vartheta_1, c_\Gamma), \qquad c_\Gamma(0) = \widetilde{c_{\Gamma 0}}$$

according to Proposition 5.1. Define

$$j_a(c, \vartheta, c_\Gamma) := j_{w,\Gamma}(c, \vartheta, c_\Gamma)\, c_\Gamma^{-1} = j_0(c, \vartheta)\, \frac{j_1(c_\Gamma)}{c_\Gamma}$$

Due to the assumptions on j_1, j_a is a bounded locally Lipschitz continuous function on $\mathbb{R}^3$.

Calculate c as the unique solution to the problem

$$\int_0^T \int_{\Omega_2} \partial_t c\phi + \delta \int_0^T \int_{\partial\Omega_2} \partial_t c\phi - \int_0^T \int_{\Omega_2} j_w \nabla\phi + \int_0^T \int_{\partial\Omega_2\cap\partial\Omega} j_{w,\partial\Omega}(\vartheta, c)\phi + \int_0^T \int_{\Gamma_{as}} j_a(c, \vartheta_1, c_\Gamma) c_{\Gamma,1}\phi = 0 \tag{52}$$

for all $\phi \in H^1((0,T)\times\Omega_2)$ with $c(0) = \widetilde{c_0}$ by Theorem 5.2.

Finally, find ϑ as the solution to the problem

$$\int_0^T \int_\Omega \partial_t u(\vartheta)\phi + \int_0^T \int_\Omega \nabla\vartheta\nabla\phi + \int_0^T \int_{\Omega_2} \nabla\left(j^* m_c(\vartheta) + v\, m_2(\vartheta)\right)\phi$$
$$+ \int_0^T \int_{\Gamma_{as}\cup\partial\Omega} (\partial_t u(\vartheta) + \widetilde{g}(x,t,\vartheta))\phi - \int_0^T \int_{\Gamma_{as}} \partial_t c_{\Gamma,1} m_{c_\Gamma}(\vartheta)\phi = 0$$

where c^* and j^* are calculated from c_1 and

$$\widetilde{g}(x,t,\vartheta) := \begin{cases} j_0(x,t,\vartheta) m_c(\vartheta) + g(x,t,\vartheta) & \text{on } \partial\Omega\cap\partial\Omega_2 \\ g(x,t,\vartheta) & \text{on } \partial\Omega\backslash\partial\Omega_2 \\ -j_a(c_1, \vartheta, c_\Gamma) c_{\Gamma,1}(x,t)(m_c(\vartheta) - m_{c_\Gamma}(\vartheta)) & \text{on } \Gamma_{as} \end{cases} \tag{53}$$

Remember, that $m_c(\vartheta) \geq m_{c_\Gamma}(\vartheta)$ for all ϑ. If it is assumed in (53) that $m_c(\vartheta) = const$ and $m_{c_\Gamma}(\vartheta) = const$ on $\mathbb{R}\backslash[\vartheta_{\text{krit}}, \vartheta^{\text{krit}}]$, $\widetilde{g}$ is Lipschitz continuous according to Lemma 2.10 and by Theorem 6.4 there is a unique solution to the problem above that satisfies $\vartheta^{\text{krit}} \geq \vartheta \geq \vartheta_{\text{krit}}$. Then, due to the upper and lower essential bound on ϑ, the assumption on m_c and m_{c_Γ} can be made w.l.o.g..

It only remains to show that the mapping $(c_1, \vartheta_1, c_{\Gamma,1}) \mapsto (c, \vartheta, c_\Gamma)$ is weakly sequentially continuous on K. Schauder's second Fixed Point Theorem 2.11 will then yield the existence of a solution to problem 4.1 by the simple remark that $\partial_t c_\Gamma = j_{w,\Gamma}(c, \vartheta, c_\Gamma) = j_a(c, \vartheta, c_\Gamma) c_\Gamma$.

To prove the weakly sequentially continuity of $(c_1, \vartheta_1, c_{\Gamma,1}) \mapsto (c, \vartheta, c_\Gamma)$ take any sequence $(c_1^n, \vartheta_1^n, c_{\Gamma,1}^n) \in Y$ such that $(c_1^n, \vartheta_1^n, c_{\Gamma,1}^n) \rightharpoonup (c_1, \vartheta_1, c_{\Gamma,1})$ weakly in Y. Evidently, $c_1^n \to c_1$ in $L^2(0,T;H^1(\Omega))$ and $\vartheta_1^n \to \vartheta_1$ strongly in $L^2(0,T;H^s(\Omega_2))$ for $s \in (0,1)$. Therefore $c_1^n \to c_1$ strongly in $L^2(0,T;L^2(\partial\Omega_2))$ and $\vartheta_1^n \to \vartheta_1$ strongly in $L^2(0,T;L^2(\Gamma_{as}\cup\partial\Omega))$.

First observe that Proposition 5.1 states the strong convergence of $c_\Gamma^n \to c_\Gamma$ in V_3.

The Lipschitz-continuity of j_a yields $j_a(c_1^n, \vartheta_1^n, c_\Gamma^n) \to j_a(c_1, \vartheta_1, c_\Gamma)$ strongly in $L^2(0,T;L^2(\Gamma_{as}))$. It follows

$$\int_0^T \int_{\Gamma_{as}} j_a(c^n, \vartheta_1^n, c_\Gamma^n) c_{\Gamma,1}^n \phi \to \int_0^T \int_{\Gamma_{as}} j_a(c, \vartheta_1, c_\Gamma) c_{\Gamma,1}\phi \quad \forall\phi \in C^1(\overline{(0,T)\times\Omega_2}) \cap H^1((0,T)\times\Omega_2)$$

and the limit c is the unique solution to (52).

For a sub sequence of the associated solutions holds $\vartheta^n \rightharpoonup \vartheta$ weakly in V_2. Therefore, strong convergence of $j_a(c_1^n, \vartheta^n, c_\Gamma^n) \to j_a(c_1, \vartheta, c_\Gamma)$, $m_c(\vartheta^n) \to m_c(\vartheta)$ and due to boundedness also of $j_a(c_1^n, \vartheta^n, c_\Gamma^n)m_c(\vartheta^n) \to j_a(c_1, \vartheta, c_\Gamma)m_c(\vartheta)$ in $L^2(0, T; L^2(\Gamma_{as}))$ are obtained. By the weak convergence of $c_{\Gamma,1}^n \rightharpoonup c_{\Gamma,1}$ in $L^2(0, T; L^2(\Gamma_{as}))$ follows

$$\int_0^T \int_{\Gamma_{as}} j_a(c_1^n, \vartheta^n, c_\Gamma^n)c_{\Gamma,1}^n\, m_c(\vartheta^n)\phi \quad\to\quad \int_0^T \int_{\Gamma_{as}} j_a(c_1, \vartheta, c_\Gamma)c_{\Gamma,1}\, m_c(\vartheta)\phi \quad \forall\phi \in C^\infty(\overline{\Omega})$$
$$\int_0^T \int_{\Gamma_{as}} j_a(c_1^n, \vartheta^n, c_\Gamma^n)c_{\Gamma,1}^n\, m_{c_\Gamma}(\vartheta^n)\phi \quad\to\quad \int_0^T \int_{\Gamma_{as}} j_a(c_1, \vartheta, c_\Gamma)c_{\Gamma,1}\, m_{c_\Gamma}(\vartheta)\phi \quad \forall\phi \in C^\infty(\overline{\Omega})$$

and since the right hand side can be considered as a linear functional on $\phi \in H^1(\Omega)$, the Banach-Steinhaus Theorem yields convergence of the latter limit for all such ϕ. Since $c_1^n \to c_1$ strongly in $L^2(0, T; H^1(\Omega))$ and the extension operator is continuous as well as the folding operator, we get $j^{*,n} \to j^*$ strongly in $L^2(0, T; H^1(\Omega_2))$. The weak convergence of $c_{\Gamma,1}^n \rightharpoonup c_{\Gamma,1}$ in $H^1(0, T; L^2(\Gamma_{as}))$ finally yields

$$\int_0^T \int_{\Gamma_{as}} \partial_t c_{\Gamma,1}^n\, m_{c_\Gamma}(\vartheta^n)\phi \to \int_0^T \int_{\Gamma_{as}} \partial_t c_{\Gamma,1}\, m_{c_\Gamma}(\vartheta)\phi \quad \forall\phi \in C^\infty(\overline{\Omega})$$

Since $u^n \in L^2(0, T; L^2(\Omega \cup \Gamma_{as} \cup \partial\Omega))$ is bounded by $(c_1^n, \vartheta^n, c_{\Gamma,1}^n)$, $u^n \rightharpoonup u^*$ weakly in $L^2(0, T; L^2(\Omega \cup \Gamma_{as} \cup \partial\Omega))$. It's easy to see that $m_2(\vartheta^n) \to m_2(\vartheta)$ and $m_c(\vartheta^n) \to m_c(\vartheta)$ in $L^2((0, T) \times \Omega_2)$ as well as $m_\Gamma(\vartheta^n) \to m_\Gamma(\vartheta)$ in $L^2((0, T) \times \Gamma_{as})$ and $m_{\partial\Omega}(\vartheta^n) \to m_{\partial\Omega}(\vartheta)$ in $L^2((0, T) \times \partial\Omega)$. By Lemma 2.9, $m_1(\vartheta^n) \ni u_1^n \rightharpoonup u_1 \in m_1(\vartheta)$ weakly in $L^2(0, T; L^2(\Omega_1))$.

To see that $u_\Gamma^n \rightharpoonup u_\Gamma \in m_\Gamma(\vartheta) + c_\Gamma m_{c_\Gamma}(\vartheta)$ in $L^2((0, T) \times \Gamma_{as})$, remember that for every function $\xi \in L^\infty(0, T; L^\infty(\Gamma_{as}))$ holds

$$\int_0^T \int_{\Gamma_{as}} (u_\Gamma^n - c_\Gamma^n\xi - m_\Gamma(m_{c_\Gamma}^{-1}(\xi)))\,(\vartheta^n - m_{c_\Gamma}^{-1}(\xi)) \geq 0 \tag{54}$$

and therefore also

$$\int_0^T \int_{\Gamma_{as}} (u_\Gamma - c_\Gamma\xi - m_\Gamma(m_{c_\Gamma}^{-1}(\xi)))\,(\vartheta - m_{c_\Gamma}^{-1}(\xi)) \geq 0$$

by the strong convergence of ϑ^n and the weak convergence of u_Γ^n and c_Γ^n. It follows from monotonicity, that $u_\Gamma \in m_\Gamma(\vartheta) + c_\Gamma m_{c_\Gamma}(\vartheta)$.

The last convergences show that

$$\lim_{n\to\infty} \int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} u^n\phi = \int_0^T \int_{\Omega\cup\Gamma_{as}\cup\partial\Omega} u\phi \qquad \forall\phi \in C^2([0, T] \times \overline{\Omega})$$

with $u^* = u \in u(\vartheta)$ and by the boundedness of u for all $\phi \in L^2(0, T; H^1(\Omega))$. It is easy to see that $\partial_t u^n \rightharpoonup \partial_t u$ in $L^2(0, T; H^*(\Omega))$ and therefore the limit function ϑ is identical with the unique solution to the heat transfer problem with parameters $(c_1, c_{\Gamma,1})$. □

Since the relation $\partial_t c_\Gamma = j_{w,\Gamma}(c,\vartheta,c_\Gamma) = j_a(c,\vartheta,c_\Gamma)c_\Gamma$ holds, the problem takes the following form:

$$\begin{aligned}
0 &= \int_0^T\int_{\Omega_2}\partial_t c\phi + \int_0^T\int_{\partial\Omega_2}\delta\partial_t c\phi - \int_0^T\int_{\Omega_2} j_w\nabla\phi + \int_0^T\int_{\partial\Omega_2} j_{w,\partial\Omega_2}(c,\vartheta,c_\Gamma)\phi\\
0 &= c_\Gamma - \int_0^T j_{w,\Gamma_{as}}(c,\vartheta,c_\Gamma)\\
0 &= \int_0^T\int_{\Omega\cup\Gamma_{as}\cup\partial\Omega}\partial_t u(\vartheta)\,\phi + \int_0^T\int_\Omega\nabla\vartheta\nabla\phi + \int_0^T\int_{\Omega_2}\nabla\left(j^*\,m_c(\vartheta) + w\,m(\vartheta)\right)\,\phi\\
&\quad - \int_0^T\int_{\Gamma_{as}} j_{w,\Gamma}(c,\vartheta,c_\Gamma)\,m_c(\vartheta)\phi + \int_0^T\int_{\partial\Omega}\left(g(\vartheta) - j_{w,\partial\Omega}(c,\vartheta)\,m_c(\vartheta)\right)\phi
\end{aligned}$$

8. Solving the original Problem

Theorem 8.1. *There is at least one solution to problem 3.1.*

Proof.

To show existence of a solution of the original system, some uniform estimates on (c,ϑ,c_Γ) independent on the approximation $c^* = c*\eta$, m_Γ and $m_{\partial\Omega}$ are needed. The estimates on c and c_Γ from Theorem 5.2 and Proposition 5.1 only depend on ϑ^{krit} and ϑ_{krit} so there is only need for some new estimates on ϑ. To this aim write the heat transport equation in the following way:

$$\int_0^T\int_{\Omega\cup\Gamma_{as}\cup\partial\Omega}\partial_t u(\vartheta)\phi + \int_0^T\int_\Omega\nabla\vartheta\nabla\phi - \int_0^T\int_{\Omega_2}\left(j^* m_c(\vartheta) + w\,m_2(\vartheta)\right)\nabla\phi$$
$$+ \int_0^T\int_{\partial\Omega}\widetilde{g}(x,t,\vartheta)\phi + \int_0^T\int_{\partial\Omega_2} j^* m_c(\vartheta)\phi = 0$$

$$\text{with } \widetilde{g} := \begin{cases} g(\vartheta) - j_{w,\partial\Omega}(c,\vartheta)m_c(\vartheta) & \text{on } \partial\Omega\cap\partial\Omega_2\\ g(\vartheta) & \text{on } \partial\Omega\backslash\partial\Omega_2\\ -j_{w,\Gamma}(c,\vartheta,c_\Gamma)m_c(\vartheta) & \text{on } \Gamma_{as}\end{cases}$$

We remember the estimates from Lemma 6.3:

$$\begin{aligned}
\|\nabla\vartheta\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{1*}_\infty(\Omega))} + \|\partial_t\vartheta\|_{L^2(0,T;H^{1*}_{\infty,0}(\Omega_2))} &\le C\\
\|\partial_t\vartheta\|_{L^2(0,T;L^2(A))} &\le C_A
\end{aligned}$$

where C and C_A do not depend on our approximation but C_A depends on $A\subset\subset(0,T)\times\Omega_1$.

First Limit Problem with fixed δ, m_Γ and $m_{\partial\Omega}$

Assume

$$\begin{aligned}
c^{*,\varepsilon} &:= (\eta^\varepsilon * c^\varepsilon)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)\\
j_1^\varepsilon &:= (\eta^\varepsilon * (-K\nabla c^\varepsilon + wc^\varepsilon))\in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)
\end{aligned}$$

with η^ε being the standard mollifier such that $\eta^\varepsilon \xrightarrow{\varepsilon\to 0} \delta_0$ in sense of distribution. Let ψ^ε and $j^{*,\varepsilon}$ be the corresponding sequences according to (20). For every ε, there is a solution $(c^\varepsilon, c^\varepsilon_\Gamma, \vartheta^\varepsilon, u^\varepsilon)$ and we obtain from the estimates above

$$\begin{array}{rcl}
\vartheta^\varepsilon & \rightharpoonup & \vartheta \quad \text{in } L^2(0,T;H^1(\Omega)) \\
\partial_t\vartheta^\varepsilon & \rightharpoonup & \vartheta_t \quad \text{in } L^2(0,T;H^{1*}_{\infty,0}(\Omega_2)) \\
\partial_t\vartheta^\varepsilon & \rightharpoonup & \vartheta_t \quad \text{in } L^2(0,T;L^2(A)) \text{ for } \quad A\subset\subset(0,T)\times\Omega_1 \\
\partial_t u^\varepsilon(\vartheta^\varepsilon) & \rightharpoonup & u_t \quad \text{in } L^2(0,T;H^{1*}_{\infty}(\Omega)) \\
c^\varepsilon & \rightharpoonup & c \quad \text{in } H^1(0,T;H^1(\Omega_2)) \cap L^2(0,T;W^{1,2}_{\Delta,\partial}(\Omega_2))
\end{array}$$

Which yields by Lions Theorem

$$\begin{array}{rcl}
\vartheta^\varepsilon & \to & \vartheta \text{ in } L^2(0,T;H^1(\Omega)) \quad \partial_t\vartheta = \vartheta_t \\
c^\varepsilon & \to & c \text{ in } L^2(0,T;H^1(\Omega_2)) \quad c^\varepsilon \to c \text{ in } L^2(0,T;L^2(\partial\Omega_2))
\end{array}$$

In the same way as for Proposition 5.1 follows

$$\begin{array}{rcl}
c^\varepsilon_\Gamma & \rightharpoonup & c_\Gamma \text{ in } V_3 \\
c^\varepsilon_\Gamma & \to & c_\Gamma \text{ in } L^2(0,T;L^2(\Gamma_{as}))
\end{array}$$

Therefore also $j_w(c^\varepsilon, \vartheta^\varepsilon, c^\varepsilon_\Gamma) = j_0(c^\varepsilon - c_0(\vartheta^\varepsilon))c^\varepsilon_\Gamma$ converges weakly in $L^2(0,T;L^2(\Gamma_{as}))$ to $j_w(c,\vartheta,c_\Gamma)$. From these convergences follows immediately:

$$\int_0^T\int_{\Omega_2}\partial_t c\phi + \delta\int_0^T\int_{\partial\Omega_2}\partial_t c\phi + \int_0^T\int_{\Omega_2}(\nabla c - v\,c)\nabla\phi + \int_0^T\int_{\partial\Omega_2} j_w(\theta_v,\vartheta,c_\Gamma)\phi = 0 \tag{55}$$

$$\partial_t c_\Gamma = j_w(c,\vartheta,c_\Gamma) \tag{56}$$

for the limit functions.

Similar to the approximated problem, it can be seen that $u \in u(\vartheta)$. This is evident for $u_1, u_2, u_{\partial\Omega}$ and for u_Γ remember inequality (54):

$$\int_0^T\int_{\Gamma_{as}}(u^\varepsilon_\Gamma - c^\varepsilon_\Gamma\xi - m_\Gamma(m^{-1}_{c_\Gamma}(\xi)))\,(\vartheta^\varepsilon - m^{-1}_{c_\Gamma}(\xi)) \geq 0$$

to obtain $u_\Gamma \in c_\Gamma m_{c_\Gamma}(\vartheta) + m_\Gamma(\vartheta)$.

The strong convergence $c^\varepsilon \to c$ in $L^2(0,T;H^1(\Omega_2))$ yields $j^\varepsilon_w \to j_w$ strongly in $L^2(0,T;L^2(\Omega_2))^n$ as well as $j^\varepsilon_1 \to j_w$ strongly in $L^2(0,T;L^2(\Omega_2))^n$. Since $\partial_t c^{*,\varepsilon} \rightharpoonup \partial_t c$ weakly in $L^2(0,T;L^2(\Omega_2))$, $j^\varepsilon_1 \rightharpoonup j_w$ weakly in $E(\Omega_2)$ and $\psi^\varepsilon \rightharpoonup \psi$ for some ψ weakly in $L^2(0,T;H^1_0(\Omega_2)\cap H^2(\Omega_2))$ with $\Delta\psi = 0$, we conclude from Lemma 2.4 that $\psi = 0$ and from Lemma 2.7 that $\nabla\psi^\varepsilon\nu_2 \rightharpoonup 0$ in $L^2(0,T;H^{\frac{1}{2}}(\partial\Omega_2)$. Due to the argumentation after Lemma 2.7 follows

$$\int_0^T\int_{\partial\Omega_2} j^{*,\varepsilon}\phi^\varepsilon \to \int_0^T\int_{\partial\Omega_2}(j^{\Gamma_{as}}_w(\theta_v,\vartheta,c_\Gamma) + \delta\partial_t c)\phi$$

for all $\phi^\varepsilon \rightharpoonup \phi$ weakly in $L^2(0,T;H^1(\Omega_2))$ with $\phi^\varepsilon \to \phi$ strongly in $L^2(0,T;L^2(\Omega\cup\Gamma_{as}\cup\partial\Omega))$. We therefore finally get as a limit equation for $(u^\varepsilon,\vartheta^\varepsilon)$:

$$\int_0^T\int_{\Omega\cup\Gamma_{as}\cup\partial\Omega}\partial_t u(\vartheta)\phi+\int_0^T\int_\Omega\nabla\vartheta\nabla\phi-\int_0^T\int_{\Omega_2} j_w m_c(\vartheta)\nabla\phi$$
$$+\int_0^T\int_{\partial\Omega} g(\vartheta)\phi+\int_0^T\int_{\partial\Omega_2}\delta\partial_t c\, m_c(\vartheta)\phi=0$$

For all $\phi\in L^2(0,T;H^1(\Omega)\cap C^\infty(\overline{\Omega}))$ and by the regularity of the terms for all $\phi\in L^2(0,T;H^1(\Omega))$

Second Limit Problem with fixed δ

Fixing δ and choosing a sequence of functions $(m_\Gamma^n)_{n\in\mathbb{N}}$, $(m_{\partial\Omega}^n)_{n\in\mathbb{N}}$ with $m_\Gamma^n \xrightarrow{n\to\infty} 0$, $m_{\partial\Omega}^n \xrightarrow{n\to\infty} 0$ uniformly on $[\vartheta_{\text{krit}},\vartheta^{\text{krit}}]$ for the corresponding sequence of functions (ϑ^n,u^n) holds:

$$\begin{aligned}
\vartheta^n &\rightharpoonup \vartheta && \text{in } L^2(0,T;H^1(\Omega))\\
\partial_t\vartheta^n &\rightharpoonup \vartheta_t && \text{in } L^2(0,T;H^{1*}_{\infty,0}(\Omega_2))\\
\partial_t\vartheta^n &\rightharpoonup \vartheta_t && \text{in } L^2(0,T;L^2(A)) \text{ for } A\subset\subset(0,T)\times\Omega_1\\
\partial_t u^n(\vartheta^n) &\rightharpoonup u_t && \text{in } L^2(0,T;H^{1*}_\infty(\Omega))\\
c^n &\rightharpoonup c && \text{in } H^1(0,T;H^1(\Omega_2))\cap L^2(0,T;H^2(\Omega_2))
\end{aligned}$$

Which yields by Lions Theorem

$$\begin{aligned}
\vartheta^n &\to \vartheta \text{ in } L^2(0,T;H^s(\Omega)) \quad \partial_t\vartheta=\vartheta_t\\
c^n &\to c \text{ in } L^2(0,T;H^1(\Omega_2)) \quad c^n\to c \text{ in } L^2(0,T;L^2(\partial\Omega_2))
\end{aligned}$$

for some $s\in(\frac{1}{2},1)$ by the local strong convergence of the sequence. Furthermore

We have $c_\Gamma^n \rightharpoonup c_\Gamma$ in V_3 and $c_\Gamma^n \to c_\Gamma$ in $L^2(0,T;L^2(\Gamma_{as}))$ and therefore also $j_w(c^n,\vartheta^n,c_\Gamma^n)=j_0(c^n-c_0(\vartheta^n))c_\Gamma^n$ converges weakly in $L^2(0,T;L^2(\Gamma_{as}))$ to $j_w(c,\vartheta,c_\Gamma)$. From these convergences follows immediately (55) and (56) for the limit functions.

Similar to the approximated problem, it can be seen that $u\in u(\vartheta)$. This is evident for $u_1,u_2,u_{\partial\Omega}$ (actually, $u_{\partial\Omega_2}\to 0$). For u_Γ remember inequality (54):

$$\int_0^T\int_{\Gamma_{as}}(u_\Gamma^n-c_\Gamma^n\xi-m_\Gamma(m_{c_\Gamma}^{-1}(\xi)))\,(\vartheta^n-m_{c_\Gamma}^{-1}(\xi))\geq 0$$

and use $m_\Gamma\to 0$ to obtain $u_\Gamma\in c_\Gamma m_{c_\Gamma}(\vartheta)$.

We therefore finally get as a limit equation for (u_n,ϑ_n):

$$\int_0^T\int_{\Omega\cup\Gamma_{as}\cup\partial\Omega}\partial_t u(\vartheta)\phi+\int_0^T\int_\Omega\nabla\vartheta\nabla\phi-\int_0^T\int_{\Omega_2} j_w m_c(\vartheta)\nabla\phi$$
$$+\int_0^T\int_{\partial\Omega} g(\vartheta)\phi+\int_0^T\int_{\partial\Omega_2}\delta\partial_t c\, m_c(\vartheta)\phi=0$$

For all $\phi\in L^2(0,T;H^1(\Omega)\cap C^\infty(\overline{\Omega}))$ and by the regularity of the terms for all $\phi\in L^2(0,T;H^1(\Omega))$.

The Limit $\delta \to 0$

In order to get rid of the δ-terms, choose a sequence $\delta \to 0$ and test the equation

$$\int_0^T \int_{\Omega_2} (\partial_t c\phi + (K\nabla c - w)\nabla\phi) + \int_0^T \int_{\partial\Omega_2} (\delta\partial_t c + j(c, \vartheta, c_\Gamma))\phi = 0$$

with $\phi \in H_0^1(\Omega_2)$ to obtain $\|\partial_t c\|_{L^2(0,T;H^{-1}(\Omega_2))} \leq C$ with C independent on δ. Furthermore, use the estimates from Proposition 5.1, Theorem 5.2 and Lemma 6.3 to obtain sequences

$$\begin{array}{rcl}
\vartheta^\delta & \rightharpoonup & \vartheta \quad \text{in } L^2(0,T;H^1(\Omega)) \\
\partial_t\vartheta^\delta & \rightharpoonup & \vartheta_t \quad \text{in } L^2(0,T;H^{1*}_{\infty,0}(\Omega_2)) \\
\partial_t\vartheta^\delta & \rightharpoonup & \vartheta_t \quad \text{in } L^2(0,T;L^2(A)) \text{ for } \mathrm{A} \subset\subset (0,\mathrm{T}) \times \Omega_1 \\
u^\delta & \rightharpoonup & u \quad \text{in } L^2(0,T;L^2(\Omega \cup \Gamma_{as})) \\
\partial_t u^\delta(\vartheta^\delta) & \rightharpoonup & u_t \quad \text{in } L^2(0,T;H^{1*}_\infty(\Omega)) \\
c^\delta & \rightharpoonup & c \quad \text{in } H^1(0,T;H^{-1}(\Omega_2)) \cap L^2(0,T;H^1(\Omega_2)) \\
c^\delta_\Gamma & \rightharpoonup & c_\Gamma \text{ in } V_3
\end{array}$$

which implies

$$\begin{array}{rcl}
\vartheta^\delta & \to & \vartheta \text{ in } L^2(0,T;H^s(\Omega)) \quad \partial_t\vartheta = \vartheta_t \\
c^\delta & \to & c \text{ in } L^2(0,T;H^s(\Omega_2)) \text{ for } s \in (\frac{1}{2}, 1) \\
c^\delta_\Gamma & \to & c_\Gamma \text{ in } L^2(0,T;L^2(\Gamma_{as}))
\end{array}$$

For any $\phi \in H^1(0,T;L^2(\partial\Omega_2))$ with $\phi(T) \equiv 0$, a short calculation yields by the boundedness of c

$$\begin{aligned}
\int_0^T \int_{\partial\Omega_2} \delta\partial_t c^\delta \phi &= \delta \int_{\partial\Omega_2} c^\delta(0)\phi(0) - \delta \int_0^T \int_{\partial\Omega_2} c^\delta \partial_t\phi \\
&\to 0 \quad \text{as } \delta \to 0
\end{aligned}$$

and the other convergences are evident from the above calculations. Therefore, there exists at least one solution to the problem with the claimed regularity. □

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