GALOIS THEORY FOR ITERATIVE CONNECTIONS AND NONREDUCED GALOIS GROUPS

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ABSTRACT. This article presents a theory of *modules with iterative connection*. This theory is a generalisation of the theory of modules with connection in characteristic zero to modules over rings of arbitrary characteristic. We show that these modules with iterative connection (and also the modules with *integrable* iterative connection) form a Tannakian category, assuming some nice properties for the underlying ring, and we show how this generalises to modules over schemes. We also relate these notions to stratifications on modules, as introduced by A. Grothendieck (cf. [BO78]) in order to extend integrable (ordinary) connections to finite characteristic. Over smooth rings, we obtain an equivalence of stratifications and integrable iterative connections. Furthermore, over a regular ring in positive characteristic, we show that the category of modules with integrable iterative connection is also equivalent to the category of flat bundles as defined by D. Gieseker in [Gie75].

In the second part of this article, we set up a Picard-Vessiot theory for fields of solutions. For such a Picard-Vessiot extension, we obtain a Galois correspondence, which takes into account even nonreduced closed subgroup schemes of the Galois group scheme on one hand and inseparable intermediate extensions of the Picard-Vessiot extension on the other hand. Finally, we compare our Galois theory with the Galois theory for purely inseparable field extensions given by S. Chase in [Cha76].

1. Introduction

For characteristic zero, N. Katz described in [Kat87] a general setting of modules with connection to describe partial linear differential equations, and established a Galois theory from an abstract point of view: He showed that – under some assumptions on the ring – the category of modules with connection (and also that of modules with integrable connection) forms a neutral Tannakian category over the field of constants and neutral Tannakian categories are known to be equivalent to categories of finite dimensional representations of proalgebraic groups (see [DM89]). However, this theory works only in characteristic zero. This is mainly caused by the fact that in positive characteristic p, every p-th power of an element in a ring is differentially constant. A. Grothendieck gave a notion of stratifications (cf. [BO78]) which generalises the notion of integrable connections to arbitrary characteristic, and which turns out to be a "good" category. In positive characteristic, a theorem of Katz (see [Gie75]) shows that over smooth schemes, modules with stratifications are equivalent to flat bundles (or F-divided sheaves as they are called in [San07]), which enables Gieseker and Dos Santos to obtain further properties of the fundamental group scheme resp. the Tannakian group scheme.

In the first part of this article, we set up a theory over rings of arbitrary characteristic, which generalises the characteristic zero setting not only in the integrable case, but also in the non-integrable case, using so called iterative connections. The integrable version, however, (so called modules with integrable iterative connection) is again equivalent to flat bundles over a regular ring in positive characteristic (cf. Section 8).

For obtaining this theory, differentials will be replaced by a family of *higher differentials*, similar to the step from derivations to higher/iterative derivations in positive characteristic (see for example [Mat01] and [MvdP03]). In getting the right setting, the main idea is the following: For an algebra

Key words and phrases. Galois theory, Differential Galois theory, inseparable extensions, connections.

R over a perfect field K, regard an iterative derivation on R over K (or more generally, a higher derivation) not as a sequence of K-linear maps $\left(\partial^{(k)} : R \to R\right)_{k\in\mathbb{N}}$ (as it is done in [HS37], [Mat01] etc.) but as a homomorphism of K-algebras $\psi : R \to R[[T]]$ by summing up, in detail $\psi(r) := \sum_{k=0}^{\infty} \partial^{(k)}(r)T^k$ (ψ is often called the Taylor series), and moreover regard the ring of power series $R[[T]]$ as a completion of the graded R-algebra $R[T]$. This leads to the notion of "cgas" (completions of graded algebras; cf. Section 2), which allows to generalise the definition of a higher derivation and to obtain a universal object $\hat{\Omega}_{R/K}$ with a universal higher derivation $\mathrm{d}_R : R \to \hat{\Omega}_{R/K}$, replacing the module of differentials $\Omega_{R/K}$ used in the classical theory (cf. Theorem 3.10).

In Section 4, we introduce the definition of a higher connection on an R-module. Furthermore, we show that a finitely generated R-module that admits such a higher connection is locally free, if R is regular and a finitely generated K-algebra (Corollary 4.5). At least in positive characteristic, this is an improvement to the literature, since no integrability condition is needed. Although modules with higher connection might be interesting on their own, our main concern are modules with so called iterative connection and modules with integrable iterative connection (cf. Section 5), which are obtained by requiring additional properties on the higher connection. One of the main results of the first part is given in Section 6, namely

Theorem 6.10. *Let R be a regular ring over a perfect field K and the localisation of a finitely generated K-algebra, such that* $\mathrm{Spec}(R)$ *has a K-rational point. Then the categories* $\mathbf{HCon}(R/K)$, $\mathbf{ICon}(R/K)$ *and* $\mathbf{ICon_{int}}(R/K)$ *of R-modules with higher connection resp. iterative connection resp. integrable iterative connection are neutral Tannakian categories over K.*

The reason for considering iterative and integrable iterative connections becomes clear in the next two sections. In Section 7, we have a look at characteristic zero. Here we show that iterative connections on modules are in one-to-one correspondence to ordinary connections, if R is regular, and that the integrability conditions coincide via this correspondence. Hence the theory of modules with (integrable) iterative connection really is a generalisation of modules with (integrable) connection in characteristic zero.

Section 8 is dedicated to the case of positive characteristic. The main result here is the equivalence between the category $\mathbf{ICon_{int}}(R/K)$ and the category of Frobenius compatible projective systems (Fc-projective systems) over the ring R. (Again under the assumption that R is regular.) Essentially, Fc-projective systems over R can be identified with flat bundles over $\mathrm{Spec}(R)$ resp. F-divided sheaves on $\mathrm{Spec}(R)$. Using the equivalence above, we can deduce from Corollary 4.5 that for an Fc-projective system $\{M_i\}_{i\in\mathbb{N}}$, the R-module M_0 is locally free. This is a slight improvement of [San07], Lemma 6, where the underlying field K is supposed to be algebraically closed.

As mentioned in the beginning, stratifications on modules as introduced by A. Grothendieck (cf. [BO78]) also generalise the notion of integrable (ordinary) connections. At least if R is smooth over K and K is algebraically closed, in characteristic zero as well as in positive characteristic, we can deduce from our results that the category of stratified modules and the category $\mathbf{ICon_{int}}(R/K)$ are equivalent, using the equivalence between stratifications and integrable connections on modules in characteristic zero (cf. [BO78], Thm. 2.15) respectively a theorem of Katz on the equivalence of stratified modules and flat bundles in positive characteristic (cf. [Gie75], Thm. 1.3). However, there is no obvious direct correspondence between stratifications and integrable iterative connections, and it is an open question whether there is any correspondence at all, if R is not smooth.

We conclude the first part of this paper by outlining a generalisation of modules with higher connection to sheaves of modules with higher connection (resp. (integrable) iterative connection) over schemes in Section 9.

In the second part (Sections 10, 11 and 12), we consider solution rings and solution fields for modules with iterative connection, which we call pseudo Picard-Vessiot rings (PPV-rings) resp.

pseudo Picard-Vessiot fields (PPV-fields), following the notion of classical differential Galois theory. Indeed, the Picard-Vessiot theory given here is set up in a more general context (namely for so called *θ-fields*), so that it can be applied not only to modules with iterative connection, but also to the iterative differential modules as in [Mat01] and [MvdP03]. Given such a PPV-ring R over a θ-field F, we obtain a Galois group scheme $\mathcal{G} := \underline{\mathrm{Gal}}(R/F)$ defined over the constants C_F of F (cf. Prop. 10.9), and we show that $\mathrm{Spec}(R)$ is a $(\mathcal{G} \times_{C_F} F)$-torsor (cf. Cor. 10.11). The main theorem of this part is the Galois correspondence, namely

Theorem 11.5. (Galois correspondence)
Let R be a PPV-ring over some θ-field F, $E := \mathrm{Quot}(R)$ the quotient field of R and $\mathcal{G} := \underline{\mathrm{Gal}}(R/F)$ the Galois group scheme of R/F.

i) *There is an antiisomorphism of the lattices*

$$\mathfrak{H} := \{\mathcal{H} \mid \mathcal{H} \leq \mathcal{G} \text{ closed subgroup schemes of } \mathcal{G}\}$$

and

$$\mathfrak{M} := \{M \mid F \leq M \leq E \text{ intermediate } \theta\text{-fields}\}$$

given by $\Psi : \mathfrak{H} \to \mathfrak{M}, \mathcal{H} \mapsto E^{\mathcal{H}}$ and $\Phi : \mathfrak{M} \to \mathfrak{H}, M \mapsto \underline{\mathrm{Gal}}(RM/M)$.

ii) *If $\mathcal{H} \leq \mathcal{G}$ is normal, then $E^{\mathcal{H}} = \mathrm{Quot}(R^{\mathcal{H}})$ and $R^{\mathcal{H}}$ is a PPV-ring over F with Galois group scheme $\underline{\mathrm{Gal}}(R^{\mathcal{H}}/F) \cong \mathcal{G}/\mathcal{H}$.*

iii) *If $M \in \mathfrak{M}$ is stable under the action of $\mathcal{G}$, then $\mathcal{H} := \Phi(M)$ is a normal subgroup scheme of $\mathcal{G}$, M is a PPV-extension of F and $\underline{\mathrm{Gal}}(R \cap M/F) \cong \mathcal{G}/\mathcal{H}$.*

iv) *For $\mathcal{H} \in \mathfrak{H}$, the extension $E/E^{\mathcal{H}}$ is separable if and only if $\mathcal{H}$ is reduced.*

Here, $R^{\mathcal{H}}$ resp. $E^{\mathcal{H}}$ denote functorial invariants of R resp. E under the action of the group functor $\mathcal{H}$ (cf. Section 11).

Contrary to the Galois correspondence given by Matzat and van der Put in [MvdP03] in the iterative differential case, our correspondence takes into account not only reduced subgroup schemes and intermediate iterative differential fields over which E is separable[1], but even the nonreduced subgroup schemes and those intermediate fields over which E is inseparable. By part iv) of the theorem, in this general setting also the separability condition and the reducedness condition correspond to each other. Our Galois correspondence is quite similar to a Galois correspondence given by M. Takeuchi for so called C-ferential fields between intermediate C-ferential fields and closed subgroup schemes, although he uses a different definition of PV-extension. The relation to this correspondence is discussed in Remark 11.1. We conclude Section 11 by some examples to enlight our Galois correspondence (cf. Example 11.8).

In the last section, the Galois theory given here is compared with the Galois theory for purely inseparable field extensions given by S. Chase in [Cha76], who extended the theory of N. Jacobson (cf. [Jac64]) to Galois group schemes of arbitrary exponents.

2. Notation

Throughout this article, K denotes a perfect field, R and $\tilde{R}$ denote integral domains, which are finitely generated K-algebras (or localisations of finitely generated K-algebras) and $f : R \to \tilde{R}$ denotes a homomorphism of K-algebras. M will be a finitely generated R-module.

As mentioned in the introduction, we need the notion of "completions of graded algebras". So let $\bigoplus_{i=0}^{\infty} B_i$ be a graded R-algebra. Then the ideals $I_k := \bigoplus_{i=k}^{\infty} B_i$ form a filtration of the algebra and one obtains a completion of $\bigoplus_{i=0}^{\infty} B_i$ with respect to this filtration (cf. [Eis95], Ch. 7.1). As an R-module, this completion is isomorphic to $\prod_{i=0}^{\infty} B_i$.

[1] This separability condition is missing in [MvdP03], but has been added for example in [Ama07] and [Hei07].

Definition 2.1. (cgas) A commutative R-algebra B is called a **completion of a graded algebra**, or a **cga** for short, if B is the completion of a graded R-algebra $\bigoplus_{i=0}^{\infty} B_i$ in the above sense. We call B_i the **i-th homogeneous component** of B. B is called a **$\tilde{R}$-cga**, if B is a cga with $B_0 = \tilde{R}$. The augmentation map will be denoted by $\varepsilon : B \to B_0 = \tilde{R}$. More generally, the projection map to the i-th homogeneous component will be denoted by $\mathrm{pr}_i : B \to B_i$.

Example 2.2.
i) The ring of formal power series $R[[T]]$ is an R-cga, with i-th homogeneous component $R \cdot T^i$.
ii) The ring $\tilde{R}$ is a cga with $(\tilde{R})_0 = \tilde{R}$ and $(\tilde{R})_i = 0$ for $i > 0$. In particular, the ring R itself is the trivial R-cga with $(R)_i = 0$ for $i > 0$.

Remark 2.3.
i) Similar to the notation of a power series as an infinite sum, elements of a cga B are denoted by $\sum_{i=0}^{\infty} b_i$, where $b_i \in B_i$. This notation is also justified by the fact that indeed, $\sum_{i=0}^{\infty} b_i$ is the limit of the sequence of partial sums $(\sum_{i=0}^{n} b_i)_{n\in\mathbb{N}}$ in the given topology, or in other words, $\sum_{i=0}^{\infty} b_i$ is a convergent series.
ii) Since $\bigoplus_{k=0}^{\infty} B_k$ is dense in B, the continuous extension of a given homomorphism of graded R-algebras is unique. By a **homomorphism of cgas**, we will always mean a homomorphism that is induced by a graded homomorphism of the underlying graded algebras.
iii) For two cgas B and $\tilde{B}$, we define the **tensor product** $B \otimes \tilde{B}$, namely the cga with homogeneous components $(B \otimes \tilde{B})_k := \sum_{i+j=k} B_i \otimes_R \tilde{B}_j$.

We sometimes have to consider homomorphisms between cgas that aren't induced by homomorphisms of graded algebras. So let B and $\tilde{B}$ be cgas and let $g : B \to \tilde{B}$ be a continuous homomorphism of K-modules (or even K-algebras). Then we define "homogeneous components" $g^{(i)} : B \to \tilde{B}$ ($i \in \mathbb{Z}$) of g to be the continuous homomorphisms of K-modules given by

$$g^{(i)}|_{B_j} := \mathrm{pr}_{i+j} \circ g|_{B_j} : B_j \to \tilde{B}_{i+j}$$

for all $j \in \mathbb{N}$ (set $\tilde{B}_{i+j} := 0$ for $i + j < 0$). The $g^{(i)}$ uniquely determine g, because for all $b_j \in B_j$, $\sum\limits_{i=-j}^{\infty} g^{(i)}(b_j)$ converges to $g(b_j)$.

Such a continuous homomorphism of K-modules $g : B \to \tilde{B}$ is called **positive**, if $g^{(i)} = 0$ for $i < 0$, and we denote by $\mathrm{Hom}_K^+(B, \tilde{B})$ the set of positive continuous homomorphisms of K-modules from B to $\tilde{B}$. A short calculation shows that for cgas B and $\tilde{B}$, a continuous homomorphism $g : B \to \tilde{B}$ is a homomorphism of K-algebras if and only if the maps $g^{(k)}$ satisfy the property

$$\forall\, k \in \mathbb{N}, \forall\, r, s \in B : \quad g^{(k)}(rs) = \sum_{i+j=k} g^{(i)}(r) g^{(j)}(s).$$

Furthermore, the monoid $(K, \cdot)$ acts on the set $\mathrm{Hom}_K^+(B, \tilde{B})$ of positive continuous homomorphisms of K-modules by

$$(a.g)^{(i)} := a^i \cdot g^{(i)} \quad (i \geq 0)$$

for all $a \in K$, $g \in \mathrm{Hom}_K^+(B, \tilde{B})$. If g is a homomorphism of algebras, then $a.g$ also is a homomorphism of algebras. Moreover, for $g \in \mathrm{Hom}_K^+(B, \tilde{B})$, $h \in \mathrm{Hom}_K^+(\tilde{B}, \tilde{\tilde{B}})$ and $a \in K$, we have

$$a.(h \circ g) = a.h \circ a.g,$$

i. e. the action of K commutes with composition.

Definition 2.4. For a cga B, a **cgm** over B is the completion of a graded module over the graded algebra $\bigoplus_{k=0}^{\infty} B_k$, the completion taken by the topology induced from the grading. In the same manner as for cgas, we define homogeneous components of cgms, continuous homomorphisms between cgms and homogeneous components of those. There also is an action of the monoid $(K, \cdot)$ on the set of positive continuous homomorphisms between two given cgms.

Remark 2.5. Some special maps, that we will use here are the higher derivations on rings and modules (cf. Sections 3 and 4) – maps in $\mathrm{Hom}_K^+(R, B)$ resp. $\mathrm{Hom}_K^+(M, B \otimes_R M)$ –, the extension $\mathrm{d}_{\hat{\Omega}}$ of the universal derivation to the algebra of higher differentials – a map in $\mathrm{Hom}_K^+(\hat{\Omega}, \hat{\Omega})$ (cf. Section 3) –, the extensions of iterable higher derivations (cf. Section 10) as well as the extensions of higher connections on M to maps in $\mathrm{Hom}_K^+(\hat{\Omega} \otimes_R M, \hat{\Omega} \otimes_R M)$ (cf. Section 4) and the extensions of iterable higher derivations on M (cf. Section 10).

3. Higher Derivations and Higher Differentials

In this section we explain the notion of higher derivations on rings and modules. The definition used here is different from that introduced by Hasse and Schmidt in [HS37]. In fact, it is a generalisation, as we will show later on. This more general definition is necessary to define the algebra of higher differentials as a universal object.

Definition 3.1. Let B be a $\tilde{R}$-cga. (As mentioned earlier R, $\tilde{R}$ denote integral domains over K together with a homomorphism of K-algebras $f : R \to \tilde{R}$.) A **higher derivation** of R to B over K is a homomorphism of K-algebras $\psi : R \to B$ satisfying $\varepsilon \circ \psi = f : R \to B_0 = \tilde{R}$. The set of all higher derivations of R to B over K will be denoted by $\mathrm{HD}_K(R, B)$. In the special case of $B = R[[T]]$ (and $\tilde{R} = R$, $f = \mathrm{id}_R$) we set $\mathrm{HD}_K(R) := \mathrm{HD}_K(R, R[[T]])$.

Remark 3.2. i) Since a higher derivation $\psi \in \mathrm{HD}_K(R, B)$ can be regarded as a positive continuous homomorphism from the cga R to the cga B, the "homogeneous components" of ψ are denoted by $\psi^{(k)} : R \to B_k$ and for every $r \in R$, we then have $\psi(r) = \sum_{k=0}^{\infty} \psi^{(k)}(r)$. (The right hand side is a series that converges in the topology of B.)

ii) Let $\psi \in \mathrm{HD}_K(R)$. Then since $\varepsilon \circ \psi = \mathrm{id}_R$, the maps $\psi^{(k)} : R \to R \cdot T^k \cong R$ are homomorphisms of K-modules and satisfy the following properties:

$$\psi^{(0)} = \mathrm{id}_R \tag{1}$$

$$\forall\, k \in \mathbb{N}, \forall\, r, s \in R : \quad \psi^{(k)}(rs) = \sum_{i+j=k} \psi^{(i)}(r)\psi^{(j)}(s) \tag{2}$$

Furthermore, any sequence $\left(\partial^{(k)}\right)_{k \in \mathbb{N}}$ of K-module-homomorphisms $\partial^{(k)} : R \to R$ satisfying these two properties defines a higher derivation $\psi : R \to R[[T]]$ via $\psi(r) := \sum_{k=0}^{\infty} \partial^{(k)}(r) T^r$.

iii) As mentioned in the beginning, Hasse and Schmidt defined a higher derivation to be a sequence $\left(\partial^{(k)}\right)_{k \in \mathbb{N}}$ as above. Hence our definition of a higher derivation $\psi \in \mathrm{HD}_K(R)$ is equivalent to that of Hasse and Schmidt.

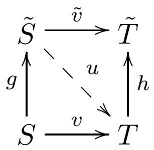
Example 3.3. i) If the characteristic of K is zero, then any K-derivation $\partial : R \to R$ (in the classical sense) gives rise to a higher derivation $\phi_\partial \in \mathrm{HD}_K(R)$ by

$$\phi_\partial(r) := \sum_{k=0}^{\infty} \frac{1}{k!} \partial^k(r) T^k$$

(see also Section 7).

ii) For a polynomial algebra $R = K[t_1, \dots, t_m]$, every higher derivation of R into some R-cga B is given by an m-tuple $(b_1, \dots, b_m)$ of elements of B satisfying $\varepsilon(b_j) = t_j$ for all $j = 1, \dots, m$. The higher derivations $\phi_{t_j} \in \mathrm{HD}_K(K[t_1, \dots, t_m])$ given by $\phi_{t_j}(t_i) = t_i$ for $i \neq j$ and $\phi_{t_j}(t_j) = t_j + T$ play an important role. In the classical context, $\phi_{t_j}^{(1)}$ is just the partial derivation with respect to t_j. We therefore call ϕ_{t_j} the **higher derivation with respect to t_j**. If $\tilde{R}$ is formally étale over $K[t_1, \dots, t_m]$ (see Def. 3.4 and Example 3.5), then the $\phi_{t_j} \in \mathrm{HD}_K(K[t_1, \dots, t_m])$ uniquely extend to higher derivations on $\tilde{R}$ by Proposition 3.7. These derivations will also be referred to as **higher derivation with respect to** t_j and will also be denoted by ϕ_{t_j}.

Definition 3.4. (cf. [Gro64], Def. 19.10.2)
Let $S, \tilde{S}$ be rings and $g : S \to \tilde{S}$ a homomorphism of rings. $\tilde{S}$ is called **formally étale** over S if, for each surjective homomorphism of rings $h : T \to \tilde{T}$ with nilpotent kernel, and all homomorphisms $v : S \to T$ and $\tilde{v} : \tilde{S} \to \tilde{T}$ satisfying $\tilde{v} \circ g = h \circ v$, there exists a unique homomorphism $u : \tilde{S} \to T$ such that $u \circ g = v$ and $h \circ u = \tilde{v}$, i. e., one obtains the following commutative diagram:



Example 3.5. i) As it is shown in [Gro64], Example 19.10.3(ii), localisations of a ring S are formally étale over S.
ii) Every finite separable extension of a field F is formally étale over F (cf. [Gro64], Ex. 19.10.3(iii)).

A more general example is the following:

Proposition 3.6. *Let S be a ring, let $\tilde{S} = S(y)$ be an extension of S with minimal polynomial $m(X) \in S[X]$ of y and such that $m'(y)$ is invertible in $\tilde{S}$, where $m'(X) := \frac{d}{dX}m(X)$. Then $\tilde{S}$ is formally étale over S.*

Proof. Let $h : T \to \tilde{T}$ be a surjective homomorphism with nilpotent kernel $I := \mathrm{Ker}(h)$ and let $v : S \to T$ and $\tilde{v} : \tilde{S} \to \tilde{T}$ be as in Definition 3.4. Since every lift u of v is given by the image of y in T, we have to show that there exists a unique element $z \in T$ with $h(z) = v(y) =: \tilde{z}$ and $m(z) = 0$. (By abuse of notation, we also write $m(X)$ for the polynomial $v(m)(X) \in T[X]$ and also for the polynomial $h(v(m))(X) \in \tilde{T}[X]$.) We will show by induction that for each $k \geq 1$, there exists a $z_k \in h^{-1}(\tilde{z})$ with $m(z_k) \in I^k$, and that z_k is unique modulo I^k with this property. Since I is nilpotent, this proves the claim by choosing k sufficiently large.

For $k = 1$, any preimage z_1 of $\tilde{z}$ works, since $I = \mathrm{Ker}(h)$. Now assume for given $k \geq 1$, there is a $z_k \in T$ satisfying $h(z_k) = \tilde{z}$ and $m(z_k) \in I^k$, which is unique modulo I^k. Since $m'(y)$ is invertible in $\tilde{S}$, we have $m'(\tilde{z}) \in \tilde{T}^\times$ and so $m'(z_k) \in T^\times$, by [Mats89], Ex. 1.1.

Using Taylor expansion, for $\zeta \in I^k$, we have $m(z_k + \zeta) \equiv m(z_k) + m'(z_k)\zeta \mod I^{k+1}$. Therefore $m(z_k + \zeta) \in I^{k+1}$ if and only if $\zeta \equiv -m'(z_k)^{-1}m(z_k) \mod I^{k+1}$. Since by hypothesis, $m(z_k) \in I^k$ and hence $-m'(z_k)^{-1}m(z_k) \in I^k$, the element $z_{k+1} := z_k - m'(z_k)^{-1}m(z_k) \in T$ satisfies $h(z_{k+1}) = \tilde{z}$ and $m(z_{k+1}) \in I^{k+1}$, and z_{k+1} is unique modulo I^{k+1} with these properties, since z_k was unique modulo I^k. □

We return to higher derivations (again using the notation given at the beginning of Section 2).

Proposition 3.7. *If $\tilde{R}$ is formally étale over R, then every higher derivation $\psi \in \mathrm{HD}_K(R, B)$ to a $\tilde{R}$-cga B can be uniquely extended to a higher derivation $\psi_e \in \mathrm{HD}_K(\tilde{R}, B)$.*

The proof is almost identical to H. Matsumura's proof for the case $B = \tilde{R}[[T]]$, so we will omit it here. (See [Mats89], Thm. 27.2; formally étale is called 0-etale there).

Definition 3.8. For $\psi \in \mathrm{HD}_K(R)$ we define a continuous endomorphism $\psi[[T]]$ on $R[[T]]$ by $\psi[[T]](\sum_{i=0}^{\infty} a_i T^i) := \sum_{i=0}^{\infty} \psi(a_i) T^i$. (In fact, $\psi[[T]]$ is an automorphism.) Using this we get a **multiplication** on $\mathrm{HD}_K(R)$ by

$$\psi_1 \cdot \psi_2 := \psi_1[[T]] \circ \psi_2 \in \mathrm{HD}_K(R) \tag{3}$$

for $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$. This defines a group structure on $\mathrm{HD}_K(R)$ (see [Mats89],§27).

The link to (ordinary) derivations is given by

Proposition 3.9. *For* $\mathrm{char}(K) = 0$*, the set* $\mathrm{Der}(R) := \{\psi^{(1)} \mid \psi \in \mathrm{HD}_K(R)\}$ *is the* R*-module of derivations on* R *(cf. Prop. 7.1).*

We now turn our attention to the universal higher derivation:

Theorem 3.10. *Up to isomorphism, there exists a unique* R*-cga* $\hat{\Omega}_{R/K}$ *(which we call the* ***algebra of higher differentials****) together with a higher derivation* $\mathrm{d}_R : R \to \hat{\Omega}_{R/K}$ *satisfying the following universal property:*
For each $\tilde{R}$*-cga* B *and each higher derivation* $\psi : R \to B$ *there exists a unique homomorphism of* $\tilde{R}$*-cgas* $\tilde{\psi} : \tilde{R} \otimes \hat{\Omega}_{R/K} \to B$ *with* $\tilde{\psi} \circ (1 \otimes \mathrm{d}_R) = \psi$*. In other words,* $\hat{\Omega}_{R/K}$ *represents the functor* $\mathrm{HD}_K(R, -)$*.*

Proof. We construct $\hat{\Omega}_{R/K}$. Uniqueness is given by the universal property.
Let $G = R[\mathrm{d}^{(k)}r \mid k \in \mathbb{N}_+, r \in R]$ be the polynomial algebra over R in the variables $\mathrm{d}^{(k)}r$ and let the degree of $\mathrm{d}^{(k)}r$ be k. Define $I \trianglelefteq G$ to be the ideal generated by the union of the sets

$$\{\mathrm{d}^{(k)}(r+s) - \mathrm{d}^{(k)}r - \mathrm{d}^{(k)}s \mid k \in \mathbb{N}_+; r, s \in R\},$$
$$\{\mathrm{d}^{(k)}a \mid k \in \mathbb{N}_+; a \in K\} \quad \text{and}$$
$$\{\mathrm{d}^{(k)}(rs) - \sum_{i=0}^{k} \mathrm{d}^{(i)}r \cdot \mathrm{d}^{(k-i)}s \mid k \in \mathbb{N}_+; r, s \in R\},$$

where we set $\mathrm{d}^{(0)}r = r$ for all $r \in R$. Therefore I is a homogeneous ideal and we define $\hat{\Omega}_{R/K}$ to be the completion of the graded algebra G/I. We also define the higher derivation $\mathrm{d}_R : R \to \hat{\Omega}_{R/K}$ by $\mathrm{d}_R(r) := \sum_{k=0}^{\infty} \mathrm{d}^{(k)}r$. (Here and in the following the residue class of $\mathrm{d}^{(k)}r \in G$ in $\hat{\Omega}_{R/K}$ will also be denoted by $\mathrm{d}^{(k)}r$.)
The universal property is seen as follows: Let $\psi : R \to B$ be a higher derivation. Then we define an R-algebra-homomorphism $g : G \to B$ by $g(\mathrm{d}^{(k)}r) := \psi^{(k)}(r)$ for all $k > 0$ and $r \in R$. The properties of a higher derivation imply that I lies in the kernel of g, and therefore g factors through $\overline{g} : G/I \to B$. Hence, we get a homomorphism of algebras $\hat{\Omega}_{R/K} \to B$ by extending $\overline{g}$ continuously and therefore a homomorphisms of $\tilde{R}$-cgas $\tilde{\psi} : \tilde{R} \otimes \hat{\Omega}_{R/K} \to B$. On the other hand, the condition $\tilde{\psi} \circ (1 \otimes \mathrm{d}_R) = \psi$ forces this choice of g and so $\tilde{\psi}$ is unique. □

Remark 3.11. In [Voj07], P. Vojta defined an R-algebra $HS^{\infty}_{R/K}$ that represents the functor $\tilde{R} \mapsto \mathrm{HD}_K(R, \tilde{R}[[T]])$ for any R-algebra $\tilde{R}$. Actually, $HS^{\infty}_{R/K}$ is a graded algebra, and by construction $\hat{\Omega}_{R/K}$ is just the completion of $HS^{\infty}_{R/K}$ (cf. the construction of $HS^{\infty}_{R/K}$ in [Voj07], Def. 1.3). Hence, some properties of $\hat{\Omega}_{R/K}$ can be deduced from the properties of $HS^{\infty}_{R/K}$ given in [Voj07].

Proposition 3.12. (a) *For every homomorphism of rings* $f : R \to \tilde{R}$ *there is a unique homomorphism of* $\tilde{R}$*-cgas* $Df : \tilde{R} \otimes \hat{\Omega}_{R/K} \to \hat{\Omega}_{\tilde{R}/K}$ *such that* $\mathrm{d}_{\tilde{R}} \circ f = Df \circ (1 \otimes \mathrm{d}_R)$*.*
(b) *If* $\tilde{R}$ *is formally étale over* R*, then* Df *is an isomorphism.*

Proof. Since $\mathrm{d}_{\tilde{R}} \circ f$ is a higher derivation on R, part (a) follows from the universal property of $\hat{\Omega}_{R/K}$. Part (b) follows from [Voj07], Thm. 3.6, where it is shown that the homomorphism of the underlying graded algebras is an isomorphism in this case. □

We consider three important examples.

Theorem 3.13. (a) *Let $R = K[t_1, \dots, t_m]$ be the polynomial ring in m variables. Then $\hat{\Omega}_{R/K}$ is the completion of the polynomial algebra $R[\mathrm{d}^{(i)}t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]$.*

(b) *Let $F/K(t_1, \dots, t_m)$ be a finite separable field extension. Then $\hat{\Omega}_{F/K}$ is the completion of the polynomial algebra $F[\mathrm{d}^{(i)}t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]$.*

(c) *Let $(R, \mathfrak{m})$ be a regular local ring of dimension m, let $t_1, \dots, t_m$ generate $\mathfrak{m}$ and assume that R is a localisation of a finitely generated K-algebra and that $R/\mathfrak{m}$ is a finite separable extension of K. Then $\hat{\Omega}_{R/K}$ is the completion of the polynomial algebra $R[\mathrm{d}^{(i)}t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]$.*

Remark 3.14. The completion of such a polynomial algebra will be denoted by $R[[\mathrm{d}^{(i)}t_j \mid i \in \mathbb{N}_+, j = 1, \dots, m]]$, although it is not really a ring of power series, because it contains infinite sums of different variables.

Proof. Part (a) is a direct consequence of [Voj07], Prop. 5.1. Part (b) then follows from Prop. 3.12(b), since by Example 3.5, $K(t_1, \dots, t_m)$ is formally étale over $K[t_1, \dots, t_m]$ and F is formally étale over $K(t_1, \dots, t_m)$.

It remains to prove (c): We will show that $(R/\mathfrak{m}) \otimes \hat{\Omega}_{R/K}$ is isomorphic to $(R/\mathfrak{m})[[\mathrm{d}^{(i)}t_j]]$. Then, since $\mathrm{Quot}(R) \otimes \hat{\Omega}_{R/K}$ is isomorphic to $\mathrm{Quot}(R)[[\mathrm{d}^{(i)}t_j]]$ (Prop. 3.12 and part (b)), by [Hart77], Ch.II, Lemma 8.9, it follows that $(\hat{\Omega}_{R/K})_k$ is a free R-module and that the residue classes of any basis of $(\hat{\Omega}_{R/K})_k$ form a basis of $(R/\mathfrak{m} \otimes \hat{\Omega}_{R/K})_k$. Hence we obtain $\hat{\Omega}_{R/K} = R[[\mathrm{d}^{(i)}t_j]]$.

First, let $\psi : R \to B$ be a higher derivation of R to an $R/\mathfrak{m}$-cga B. Then for all $k \in \mathbb{N}$ and $r_1, \dots, r_{k+1} \in \mathfrak{m}$, we have

$$\psi^{(k)}(r_1 \cdots r_{k+1}) = \sum_{i_1 + \cdots + i_{k+1} = k} \psi^{(i_1)}(r_1) \cdots \psi^{(i_{k+1})}(r_{k+1}) = 0,$$

since in each summand at least one $i_j = 0$, and so $\psi^{(i_1)}(r_1) \cdots \psi^{(i_{k+1})}(r_{k+1}) \in \mathfrak{m}B = 0$. Therefore $\psi^{(k)}$ (and $\psi^{(i)}$ for $i < k$) factors through $R/\mathfrak{m}^{k+1}$.

Next, since $R/\mathfrak{m}$ is a finite separable extension of K, there is an element $\bar{y} \in R/\mathfrak{m}$ which generates the extension $K \subset R/\mathfrak{m}$. Let $g(X) \in K[X]$ be the minimal polynomial of $\bar{y}$, then starting with an arbitrary representative $y \in R$ for $\bar{y}$, using the Newton approximation $y_{n+1} = y_n - g(y_n)g'(y_n)^{-1}$, we obtain an element $\tilde{y}_k \in R$ such that $g(\tilde{y}_k) \equiv 0 \mod \mathfrak{m}^{k+1}$ for given $k \in \mathbb{N}$. (Note that the Newton approximation is well defined and converges to a root of $g(X)$, due to the fact that $\overline{g(y)} = g(\overline{y}) = 0 \in R/\mathfrak{m}$ and $\overline{g'(y)} = g'(\overline{y}) \neq 0 \in R/\mathfrak{m}$, so $g(y) \in \mathfrak{m}$ and $g(y) \in R^\times$, as well as inductively for all $n \in \mathbb{N}$: $\bar{y}_{n+1} = \bar{y}_n = \bar{y} \in R/\mathfrak{m}$, $g(y_{n+1}) \in \mathfrak{m}$ and $g'(y_{n+1}) \in R^\times$.) This proves that for all $k \in \mathbb{N}$, the ring $R/\mathfrak{m}^{k+1}$ contains a subfield isomorphic to $R/\mathfrak{m}$.

Now by [Mats89], Theorem 14.4, the associated graded ring $\mathrm{gr}(R)$ of R is isomorphic to the polynomial ring $(R/\mathfrak{m})[t_1, \dots, t_m]$ and therefore we obtain $\mathrm{gr}(R/\mathfrak{m}^{k+1}) \cong (R/\mathfrak{m})[t_1, \dots, t_m]/\mathfrak{n}^{k+1}$, where $\mathfrak{n}$ is the ideal generated by $\{t_1, \dots, t_m\}$. Furthermore, since $R/\mathfrak{m}^{k+1}$ contains a subfield isomorphic to $R/\mathfrak{m}$, we see that the inclusion $\iota_k : (R/\mathfrak{m})[t_1, \dots, t_m]/\mathfrak{n}^{k+1} \to R/\mathfrak{m}^{k+1}$ (given by the inclusion $K[t_1, \dots, t_m]/\mathfrak{n}^{k+1} \subset R/\mathfrak{m}^{k+1}$ and $\bar{y} \mapsto \tilde{y}_k$) is an isomorphism.

Hence, every higher derivation $\psi_{\mathrm{gr}} : \mathrm{gr}(R) \to B$ into an $R/\mathfrak{m}$-cga B induces a higher derivation $\psi_R : R \to B$ on R by $\psi_R^{(k)} := \psi_{\mathrm{gr}}^{(k)} \circ \iota_k^{-1}$ $(k \in \mathbb{N})$ and vice versa. So $R/\mathfrak{m} \otimes_R \hat{\Omega}_{R/K} \cong R/\mathfrak{m} \otimes_{\mathrm{gr}(R)} \hat{\Omega}_{\mathrm{gr}(R)/K} = (R/\mathfrak{m})[[\mathrm{d}^{(i)}t_j]]$. □

Corollary 3.15. *Let R be a finitely generated K-algebra which is a regular ring, then the homogeneous components $(\hat{\Omega}_{R/K})_k$ $(k \in \mathbb{N})$ are projective R-modules of finite rank.*

Proof. For every maximal ideal $\mathfrak{m} \trianglelefteq R$, the localisation $R_\mathfrak{m}$ fulfills the conditions of Theorem 3.13(c). And so by Proposition 3.12, $R_\mathfrak{m} \otimes_R (\hat{\Omega}_{R/K})_k \cong (\hat{\Omega}_{R_\mathfrak{m}/K})_k$ is a free $R_\mathfrak{m}$-module of finite rank. Hence by [Eis95], Thm. A3.2, $(\hat{\Omega}_{R/K})_k$ is a projective R-module of finite rank. □

From now on we also write $\hat{\Omega}$ for $\hat{\Omega}_{R/K}$.

Theorem 3.16. *For each $a \in K$, there is a continuous homomorphism of K-algebras $a.\mathrm{d}_{\hat{\Omega}} : \hat{\Omega} \to \hat{\Omega}$ defined by*

$$a.\mathrm{d}_{\hat{\Omega}}\left(\mathrm{d}_R^{(i)} r\right) := \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}_R^{(i+j)} r$$

for all $i \in \mathbb{N}$ and $r \in R$. The homomorphisms $a.\mathrm{d}_{\hat{\Omega}}$ satisfy the following three conditions:

i) *$a.\mathrm{d}_{\hat{\Omega}}$ extends the higher derivation $a.\mathrm{d}_R : R \to \hat{\Omega}$.*
ii) *For all $a, b \in K$: $(a.\mathrm{d}_{\hat{\Omega}}) \circ (b.\mathrm{d}_{\hat{\Omega}}) = (a+b).\mathrm{d}_{\hat{\Omega}}$.*
iii) *$0.\mathrm{d}_{\hat{\Omega}} = \mathrm{id}_{\hat{\Omega}}$.*

For short, we will write $\mathrm{d}_{\hat{\Omega}}$ instead of $1.\mathrm{d}_{\hat{\Omega}}$ and $-\mathrm{d}_{\hat{\Omega}}$ instead of $-1.\mathrm{d}_{\hat{\Omega}}$.

Proof. It is not hard to check that $a.\mathrm{d}_{\hat{\Omega}}$ is well defined. Then the first and third statements are obvious and the second one is shown by an explicit calculation using some combinatorial identities (see [Rös07], Theorem 2.5 for details). □

Remark 3.17. i) By the second and the third property, we see that $a.\mathrm{d}_{\hat{\Omega}}$ is actually an automorphism of $\hat{\Omega}$ for all $a \in K$. The endomorphisms $a.\mathrm{d}_{\hat{\Omega}}$ play an important role in the iterative theory, as will be seen in Section 5.
ii) From the definition, we see that $a.\mathrm{d}_{\hat{\Omega}}$ is the image of $\mathrm{d}_{\hat{\Omega}}$ under the action of $a \in K$, as given in Section 2, thus the notation $a.\mathrm{d}_{\hat{\Omega}}$.

Proposition 3.18. *For all $i, j \in \mathbb{N}$ we have:*

$$\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)} = \binom{i+j}{i} \mathrm{d}_{\hat{\Omega}}^{(i+j)},$$

where $\mathrm{d}_{\hat{\Omega}}^{(i)}$ denotes the i-th homogeneous component of $\mathrm{d}_{\hat{\Omega}}$ (cf. Section 2).

Proof. For all $i, j \in \mathbb{N}$ and $\omega \in \hat{\Omega}$, the term $\left(\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)}\right)(\omega)$ is the coefficient of $a^i b^j$ in the expression $\left((a.\mathrm{d}_{\hat{\Omega}}) \circ (b.\mathrm{d}_{\hat{\Omega}})\right)(\omega)$. Furthermore by Theorem 3.16, $(a.\mathrm{d}_{\hat{\Omega}}) \circ (b.\mathrm{d}_{\hat{\Omega}}) = (a+b).\mathrm{d}_{\hat{\Omega}}$ and so $\left(\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)}\right)(\omega)$ is the coefficient of $a^i b^j$ in the expression $(a+b).\mathrm{d}_{\hat{\Omega}}(\omega) = \sum_{k=0}^{\infty} (a+b)^k \mathrm{d}_{\hat{\Omega}}^{(k)}(\omega)$, i.e., equals $\binom{i+j}{i} \mathrm{d}_{\hat{\Omega}}^{(i+j)}(\omega)$. (For a finite field K, one has to use the little trick explained in Remark 5.6 to justify this conclusion.) □

4. Higher Derivations on Modules and Higher Connections

In the following, M will denote a finitely generated R-module and B will be a $\tilde{R}$-cga.

Definition 4.1. Let $\psi : R \to B$ be a higher derivation of R to B over K. A **(higher) ψ-derivation** of M is an additive map $\Psi : M \to B \otimes_R M$ with $(\varepsilon \otimes \mathrm{id}_M) \circ \Psi = f \otimes \mathrm{id}_M$ and $\Psi(rm) = \psi(r)\Psi(m)$ for all $r \in R, m \in M$. The set of ψ-derivations of M is denoted by $\mathrm{HD}(M, \psi)$.
A **higher connection** on M is a d_R-derivation $\nabla \in \mathrm{HD}(M, \mathrm{d}_R)$. If ∇ is a higher connection on M, for any higher derivation $\psi \in \mathrm{HD}_K(R, B)$, we define the ψ-derivation ∇_ψ on M by

$$\nabla_\psi := (\tilde{\psi} \otimes \mathrm{id}_M) \circ \nabla : M \to \hat{\Omega}_{R/K} \otimes_R M \to B \otimes_R M$$

(cf. Remark 4.2(i)).
For all $a \in K$ we define an endomorphism $a._{\hat{\Omega}}\nabla : \hat{\Omega} \otimes_R M \to \hat{\Omega} \otimes_R M$ by

$$(a._{\hat{\Omega}}\nabla)(\omega \otimes x) := a.\mathrm{d}_{\hat{\Omega}}(\omega) \cdot (a.\nabla)(x)$$

for all $\omega \in \hat{\Omega}$ and $x \in M$, i.e. $a._{\hat{\Omega}}\nabla = (\mu_{\hat{\Omega}} \otimes \mathrm{id}_M) \circ (a.\mathrm{d}_{\hat{\Omega}} \otimes a.\nabla)$, where $\mu_{\hat{\Omega}}$ denotes the multiplication map in $\hat{\Omega}$.

Remark 4.2. i) For a given $\psi \in \mathrm{HD}_K(R, B)$, every homomorphism of $\tilde{R}$-cgas $g : B \to \tilde{B}$ induces a map $g_* : \mathrm{HD}(M, \psi) \to \mathrm{HD}(M, g \circ \psi), \Psi \mapsto (g \otimes \mathrm{id}_M) \circ \Psi$.
Using the action of K in this context, for every $\Psi \in \mathrm{HD}(M, \psi)$ and $a \in K$, we get an $(a.\psi)$-derivation $a.\Psi$.

ii) Let $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$ and $\Psi_i \in \mathrm{HD}(M, \psi_i)$ $(i = 1, 2)$. Then as in Definition 3.8, we have an automorphism $\Psi[[T]]$ of $R[[T]] \otimes M \cong M[[T]]$ and a product $\Psi_1 \cdot \Psi_2$, which is an element of $\mathrm{HD}(M, \psi_1\psi_2)$.

Our next aim is to show that for a regular ring R over a perfect field K every R-module that admits a higher connection is projective (or - in geometric terms - locally free). So we get the analogue of the well-known fact in characteristic zero that a coherent sheaf equipped with a holomorphic connection must be locally free, resp. the analogue of the corresponding fact in the not necessarily commutative situation given by Y. André in [And01], Cor. 2.5.2.2.

We first need the following lemma:

Lemma 4.3. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension m, let $t_1, \ldots, t_m$ generate $\mathfrak{m}$ and assume that R is a localisation of a finitely generated K-algebra and that $R/\mathfrak{m}$ is a finite separable extension of K. Let $\phi_{t_j} \in \mathrm{HD}_K(R)$ $(j = 1, \ldots, m)$ denote the higher derivations with respect to t_j (cf. Example 3.3 and Thm. 3.13(c)).*
Then for every $r \in R \setminus \{0\}$ there exist $k_1, \ldots, k_m \in \mathbb{N}$ such that

(1) $$\left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(r) \in R^\times,$$

(2) *for all $l_1, \ldots, l_m \in \mathbb{N}$ with $l_j \leq k_j$ $(j = 1, \ldots, m)$ and $l_i < k_i$ for some $i \in \{1, \ldots, m\}$,*

$$\left(\phi_{t_m}^{(l_m)} \circ \cdots \circ \phi_{t_1}^{(l_1)}\right)(r) \notin R^\times.$$

Proof. Let $r \in R \setminus \{0\}$. Choose $E \in \mathbb{N}$ such that $r \in \mathfrak{m}^E$ and $r \notin \mathfrak{m}^{E+1}$. Then r can (uniquely) be written as

$$r = \sum_{\substack{\boldsymbol{e}=(e_1,\ldots,e_m)\in\mathbb{N}^m \\ |\boldsymbol{e}|=E}} u_{\boldsymbol{e}} \boldsymbol{t}^{\boldsymbol{e}},$$

where $u_{\boldsymbol{e}} \in R$ and $u_{\boldsymbol{f}} \in R^\times$ for at least one $\boldsymbol{f} = (f_1, \ldots, f_m)$.
(We use the usual notation of multiindices: $|\boldsymbol{e}| = e_1 + \cdots + e_m$ and $\boldsymbol{t}^{\boldsymbol{e}} = t_1^{e_1} \cdots t_m^{e_m}$.) For arbitrary $\boldsymbol{l} = (l_1, \ldots, l_m) \in \mathbb{N}^m$ and $\boldsymbol{e} \in \mathbb{N}^m$ we have:

$$\left(\phi_{t_m}^{(l_m)} \circ \cdots \circ \phi_{t_1}^{(l_1)}\right)(\boldsymbol{t}^{\boldsymbol{e}}) = \binom{e_1}{l_1} \cdots \binom{e_m}{l_m} \boldsymbol{t}^{\boldsymbol{e}-\boldsymbol{l}} = \begin{cases} 0 & \text{if } l_i > e_i \text{ for some } i, \\ 1 & \text{if } l_j = e_j \text{ for all } j, \\ \in \mathfrak{m} & \text{if } |\boldsymbol{l}| < |\boldsymbol{e}|. \end{cases}$$

So if we choose $k_j = f_j$ $(j = 1, \ldots, m)$, we get

$$\begin{aligned} \left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(r) &= \sum_{|\boldsymbol{e}|=E} \left(\phi_{t_m}^{(k_m)} \circ \cdots \circ \phi_{t_1}^{(k_1)}\right)(u_{\boldsymbol{e}} \boldsymbol{t}^{\boldsymbol{e}}) \\ &= \sum_{|\boldsymbol{e}|=E} \sum_{\substack{0 \leq l_j \leq k_j \\ j=1,\ldots,m}} \left(\phi_{t_m}^{(k_m-l_m)} \circ \cdots \circ \phi_{t_1}^{(k_1-l_1)}\right)(u_{\boldsymbol{e}}) \left(\phi_{t_m}^{(l_m)} \circ \cdots \circ \phi_{t_1}^{(l_1)}\right)(\boldsymbol{t}^{\boldsymbol{e}}) \\ &\equiv u_{\boldsymbol{f}} \cdot 1 \mod \mathfrak{m}. \end{aligned}$$

So $\left(\phi_{t_m}^{(k_m)}\circ\cdots\circ\phi_{t_1}^{(k_1)}\right)(r)\in u_{\boldsymbol{f}}+\mathfrak{m}\subset R^\times$, and for all $\boldsymbol{l}\in\mathbb{N}^m$ with $l_j\leq k_j$ $(j=1,\dots,m)$ and $l_i<k_i$ for some i, we have $\left(\phi_{t_m}^{(l_m)}\circ\cdots\circ\phi_{t_1}^{(l_1)}\right)(r)\in\mathfrak{m}=R\setminus R^\times$, since $|\boldsymbol{l}|<E$. □

Theorem 4.4. *Let $(R,\mathfrak{m})$ be a regular local ring as in Lemma 4.3 and let M be a finitely generated R-module with a higher connection $\nabla\in\mathrm{HD}(M,\mathrm{d})$. Then M is a free R-module.*

Proof. Let $\{x_1,\dots,x_n\}$ be a minimal set of generators of M. Assume that $x_1,\dots,x_n$ are linearly dependent. Then there exists a nontrivial relation $\sum_{i=1}^n r_ix_i=0$, with $r_i\in R$. Choose $E\in\mathbb{N}$ such that $r_j\in\mathfrak{m}^E$ for all $j=1,\dots n$ and $r_i\notin\mathfrak{m}^{E+1}$ for at least one i. Without loss of generality, let $r_1\notin\mathfrak{m}^{E+1}$. Then choose $k_1,\dots,k_m\in\mathbb{N}$ for r_1 as given by the previous lemma. Then

$$\begin{aligned}0 &= \left(\nabla_{\phi_{t_m}}^{(k_m)}\circ\cdots\circ\nabla_{\phi_{t_1}}^{(k_1)}\right)\left(\sum_{i=1}^n r_ix_i\right)\\ &= \sum_{i=1}^n\sum_{\substack{0\leq l_j\leq k_j\\ j=1,\dots,m}}\left(\phi_{t_m}^{(l_m)}\circ\cdots\circ\phi_{t_1}^{(l_1)}\right)(r_i)\left(\nabla_{\phi_{t_m}}^{(k_m-l_m)}\circ\cdots\circ\nabla_{\phi_{t_1}}^{(k_1-l_1)}\right)(x_i)\\ &\equiv \sum_{i=1}^n\left(\phi_{t_m}^{(k_m)}\circ\cdots\circ\phi_{t_1}^{(k_1)}\right)(r_i)\cdot x_i \mod \mathfrak{m}M\end{aligned}$$

Since $\left(\phi_{t_m}^{(k_m)}\circ\cdots\circ\phi_{t_1}^{(k_1)}\right)(r_1)\in R^\times$, we get $x_1\in\langle x_2,\dots,x_n\rangle+\mathfrak{m}M$, so $M=\langle x_2,\dots,x_n\rangle+\mathfrak{m}M$ and therefore by Nakayama's lemma $M=\langle x_2,\dots,x_n\rangle$, in contradiction to the minimality of $\{x_1,\dots,x_n\}$. So $x_1,\dots,x_n$ is a basis for M and in particular M is a free R-module. □

Corollary 4.5. *Let K be a perfect field and let R be a finitely generated K-algebra which is a regular ring. Then every finitely generated R-module M with higher connection ∇ is a projective R-module.*

Proof. By the previous theorem, the localisations of M at every maximal ideal of R are free. Hence M is projective. □

5. Iterative Derivations and Iterative Connections

Definition 5.1. A higher derivation $\phi\in\mathrm{HD}_K(R)$ is called an **iterative derivation**, if

$$\forall\, i,j\in\mathbb{N}:\quad \phi^{(i)}\circ\phi^{(j)}=\binom{i+j}{i}\phi^{(i+j)}\ .$$

The set of iterative derivations on R is denoted by $\mathrm{ID}_K(R)$.
Let M be an R-module and $\phi\in\mathrm{ID}_K(R)$. A higher ϕ-derivation $\Phi\in\mathrm{HD}(M,\phi)$ is called an **iterative ϕ-derivation**, if

$$\forall\, i,j\in\mathbb{N}:\quad \Phi^{(i)}\circ\Phi^{(j)}=\binom{i+j}{i}\Phi^{(i+j)}\ .$$

The set of iterative ϕ-derivations is denoted by $\mathrm{ID}(M,\phi)$.

Example 5.2. If R is the polynomial ring $K[t_1,\dots,t_m]$ or a formally étale extension of that ring, the higher derivations ϕ_{t_j} with respect to t_j (cf. Example 3.3) are iterative derivations. (For $K[t_1,\dots,t_m]$ this is obvious and for extensions, it follows from Lemma 5.8.)

Remark 5.3. Note that there is no sense in defining an iterative derivation $\Phi\in\mathrm{HD}(M,\psi)$ for a non-iterative higher derivation $\psi\in\mathrm{HD}_K(R)$. This is seen by using the characterisation of the iterative derivations in Lemma 5.4 and Lemma 5.7: For all $a,b\in K^{\mathrm{sep}}$, $(a.\Phi)(b.\Phi)$ is an $(a.\psi)(b.\psi)$-derivation, whereas $(a+b).\Phi$ is an $(a+b).\psi$-derivation.

Lemma 5.4. (Characterisation of iterative derivations)
Let $\psi \in \mathrm{HD}_K(R)$ be a higher derivation. Then the following conditions are equivalent:

(i) *ψ is iterative,*
(ii) $\tilde{\psi} \circ \mathrm{d}_{\hat{\Omega}} = \psi[[T]] \circ \tilde{\psi}$, *(see Thm. 3.10 for the definition of $\tilde{\psi}$.)*
(iii) *For all $a \in K$: $\tilde{\psi} \circ (a.\mathrm{d}_{\hat{\Omega}}) = (a.\psi[[T]]) \circ \tilde{\psi}$.*

If K is an infinite field, then this is also equivalent to

(iv) *For all $a, b \in K$: $(a.\psi)(b.\psi) = (a+b).\psi$,*

whereas for arbitrary K the conditions (i)-(iii) *only imply condition* (iv).

Proof. For $a \in K$, $r \in R$ and $i \in \mathbb{N}$ we have:

$$\tilde{\psi} \circ (a.\mathrm{d}_{\hat{\Omega}})(\mathrm{d}^{(i)} r) = \tilde{\psi}\left(\sum_{j=0}^{\infty} a^j \binom{i+j}{j} \mathrm{d}^{(i+j)} r\right) = \sum_{j=0}^{\infty} a^j \binom{i+j}{j} \psi^{(i+j)}(r) T^{i+j}$$

and

$$(a.\psi[[T]]) \circ \tilde{\psi}(\mathrm{d}^{(i)} r) = a.\psi[[T]]\left(\psi^{(i)}(r) T^i\right) = \sum_{j=0}^{\infty} a^j \psi^{(j)}(\psi^{(i)}(r)) T^{i+j}.$$

So by comparing the coefficients of T^{i+j} one sees that condition (iii) is fulfilled if and only if $\tilde{\psi} \circ (a.\mathrm{d}_{\hat{\Omega}}) = (a.\psi[[T]]) \circ \tilde{\psi}$ holds for arbitrary $a \in K \setminus \{0\}$ (e.g. $a = 1$, i.e. condition (ii)). Moreover, this is fulfilled if and only if for all $i, j \in \mathbb{N}$ we have $\psi^{(j)} \circ \psi^{(i)} = \binom{i+j}{j} \psi^{(i+j)}$, i.e. ψ is iterative. Furthermore, we get for all $a, b \in K$:

$$((a.\psi)(b.\psi))^{(k)} = \sum_{i+j=k} (a.\psi)^{(i)} \circ (b.\psi)^{(j)} = \sum_{i+j=k} a^i b^j \psi^{(i)} \circ \psi^{(j)},$$

since $b \in K$, and

$$((a+b).\psi)^{(k)} = (a+b)^k \psi^{(k)} = \sum_{i+j=k} a^i b^j \binom{i+j}{i} \psi^{(i+j)}.$$

So if ψ is iterative, condition (iv) is fulfilled. If $\#K = \infty$, we obtain from condition (iv) that ψ is iterative by comparing the coefficients of a^i. □

If $\#K < \infty$, the following example shows that condition (iv) doesn't imply the others.

Example 5.5. Condition (iv) is in fact weaker if K is finite. If for example $K = \mathbb{F}_q$ and $R = \mathbb{F}_q[t]$, then $\psi \in \mathrm{HD}_K(R)$ defined by $\psi(t) = t + 1 \cdot T^{2q-1}$ is not iterative, since

$$(2q-1)\psi^{(2q-1)}(t) = 2q - 1 \neq 0 = \psi^{(2q-2)}\left(\psi^{(1)}(t)\right).$$

On the other hand, for all $a \in \mathbb{F}_q$ we have $a^{2q-1} = a$ and so

$$\begin{aligned}\big((a.\psi)(b.\psi)\big)^{(k)}(t) &= \sum_{i+j=k} a^i b^j \psi^{(i)}(\psi^{(j)}(t)) = a^k \psi^{(k)}(t) + a^{k-2q+1} b^{2q-1} \psi^{(k-2q+1)}(1) \\ &= \left\{\begin{array}{cc} t & k = 0 \\ a^{2q-1} + b^{2q-1} = (a+b)^{2q-1} & k = 2q-1 \\ 0 & \text{otherwise} \end{array}\right\} = \big((a+b).\psi\big)^{(k)}(t)\end{aligned}$$

for all $a, b \in K = \mathbb{F}_q$.

Remark 5.6. Condition (iv) is very useful for calculations – even if K is finite. If one has to show that some higher derivation $\psi \in \mathrm{HD}_K(R)$ is iterative, one can often use the following trick:
Let $\tilde{R} := K^{\mathrm{sep}} \otimes_{\bar{K}\cap R} R$ be the maximal separable extension of R by constants. Then by Proposition 3.7 the higher derivation ψ uniquely extends to a higher derivation $\psi_e \in \mathrm{HD}_K(\tilde{R}) = \mathrm{HD}_{K^{\mathrm{sep}}}(\tilde{R})$. Since $\#K^{\mathrm{sep}} = \infty$, we can use condition (iv) to show that ψ_e is iterative and therefore ψ is iterative. Whenever it will be shown that for all $a, b \in K^{\mathrm{sep}}$, $(a.\psi)(b.\psi) = (a+b).\psi$, this trick will be used, although we won't mention it explicitly.

Lemma 5.7. *Let $\phi \in \mathrm{ID}_K(R)$ be an iterative derivation and $\Psi \in \mathrm{HD}(M, \phi)$ be a ϕ-derivation. Then Ψ is iterative if and only if for all $a, b \in K^{\mathrm{sep}}$ the identity $(a.\Psi)(b.\Psi) = (a+b).\Psi$ holds.*

Proof. Analogous to the proof of Lemma 5.4. □

The next lemma states some structural properties of $\mathrm{ID}_K(R)$.

Lemma 5.8.
i) *If two iterative derivations $\phi_1, \phi_2 \in \mathrm{ID}_K(R)$ commute, i. e. they satisfy $\phi_1^{(i)} \circ \phi_2^{(j)} = \phi_2^{(j)} \circ \phi_1^{(i)}$ for all $i, j \in \mathbb{N}$, then $\phi_1\phi_2$ is again an iterative derivation.*
ii) *$\mathrm{ID}_K(R)$ is stable under the action of K.*
iii) *If $\tilde{R} \supset R$ is a ring extension such that every higher derivation on R uniquely extends to a higher derivation on $\tilde{R}$ (e. g. if $\tilde{R}$ is formally étale over R), then the extension $\phi_e \in \mathrm{HD}_K(\tilde{R})$ of an iterative derivation $\phi \in \mathrm{ID}_K(R)$ is again iterative.*

Proof. A short calculation shows that all occurring higher derivations satisfy condition (iv) of Lemma 5.4, hence are iterative. □

We have already seen that $\mathrm{d}_{\hat{\Omega}}$ satisfies the condition $\mathrm{d}_{\hat{\Omega}}^{(i)} \circ \mathrm{d}_{\hat{\Omega}}^{(j)} = \binom{i+j}{i}\mathrm{d}_{\hat{\Omega}}^{(i+j)}$ and that for iterative derivations $\phi \in \mathrm{ID}_K(R)$ we have the "same" condition $\phi^{(i)} \circ \phi^{(j)} = \binom{i+j}{i}\phi^{(i+j)}$. This motivates the following definition of an iterative connection.

Definition 5.9. A higher connection ∇ on M is called an **iterative connection** if the identity

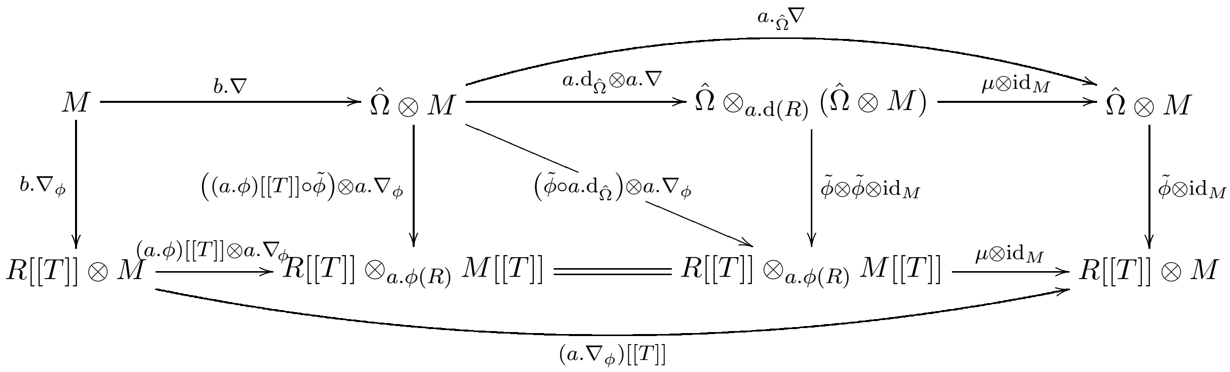
$$_{\hat{\Omega}}\nabla^{(i)} \circ {_{\hat{\Omega}}\nabla^{(j)}} = \binom{i+j}{i} {_{\hat{\Omega}}\nabla^{(i+j)}}$$

holds for all $i, j \in \mathbb{N}$. (As defined in Section 2 for the general case, $_{\hat{\Omega}}\nabla^{(i)}$ denotes the part of $_{\hat{\Omega}}\nabla$ that "increases degrees by i".)
An iterative connection ∇ on M is called an **integrable iterative connection** if for all higher derivations $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$ we have $\nabla_{\psi_1\psi_2} = \nabla_{\psi_1}\nabla_{\psi_2}$.

The notion of an integrable iterative connection is motivated by the correspondence to the integrable (ordinary) connections in characteristic zero (cf. Section 7).

Theorem 5.10. *Let ∇ be a higher connection on M. Then:*
i) *∇ is iterative if and only if for all $a, b \in K^{\mathrm{sep}}$: $a.{_{\hat{\Omega}}\nabla} \circ b.{_{\hat{\Omega}}\nabla} = (a+b).{_{\hat{\Omega}}\nabla}$ and if and only if for all $a, b \in K^{\mathrm{sep}}$: $a.{_{\hat{\Omega}}\nabla} \circ b.\nabla = (a+b).\nabla$.*
ii) *If ∇ is iterative, then for all iterative derivations $\phi \in \mathrm{ID}_K(R)$ the ϕ-derivation ∇_ϕ is again iterative. If R has enough iterative derivations, i. e. if for every nonzero $\omega \in \hat{\Omega}_{R/K}$ there exists an iterative derivation $\phi \in \mathrm{ID}_K(R)$ such that $\tilde{\phi}(\omega) \neq 0$, then the converse is also true (where $\tilde{\phi} : \hat{\Omega}_{R/K} \to R[[T]]$ is the unique homomorphism of cgas satisfying $\phi = \tilde{\phi} \circ \mathrm{d}$, cf. Thm. 3.10).*

Proof. The first statement is seen by a calculation similar to the one in Lemma 5.4. For proving the second part, let $\phi \in \mathrm{ID}_K(R)$ and consider the following diagram:



The square on the left commutes, since

$$b.\nabla_\phi = b.\left((\tilde{\phi}\otimes\mathrm{id}_M)\circ\nabla\right) = (\tilde{\phi}\otimes\mathrm{id}_M)\circ(b.\nabla).$$

The lower triangle commutes by Lemma 5.4, since ϕ is iterative. The upper triangle commutes, since $a.\nabla_\phi = (\tilde{\phi}\otimes\mathrm{id}_M)\circ(a.\nabla)$, and the square on the right commutes, since $\tilde{\phi}$ is a homomorphism of algebras. Furthermore the top of the diagram commutes by definition of $a.{}_{\hat{\Omega}}\nabla$ and the bottom commutes, since $a.\nabla_\phi$ is a $(a.\phi)$-derivation.
So the whole diagram commutes and we obtain

$$(\tilde{\phi}\otimes\mathrm{id}_M)\circ(a.{}_{\hat{\Omega}}\nabla)\circ(b.\nabla) = (a.\nabla_\phi)[[T]]\circ(b.\nabla_\phi) = (a.\nabla_\phi)(b.\nabla_\phi)$$

for all iterative derivations $\phi\in\mathrm{ID}_K(R)$.
If ∇ is iterative, we get

$$(a+b).\nabla_\phi = (\tilde{\phi}\otimes\mathrm{id}_M)\circ(a+b).\nabla = (\tilde{\phi}\otimes\mathrm{id}_M)\circ(a.{}_{\hat{\Omega}}\nabla)\circ(b.\nabla) = (a.\nabla_\phi)(b.\nabla_\phi)$$

by the first part of this theorem and so by Lemma 5.7, ∇_ϕ is iterative.
In turn, from the commuting diagram we see that if ∇_ϕ is iterative for an iterative derivation $\phi\in\mathrm{ID}_K(R)$, we get

$$(\tilde{\phi}\otimes\mathrm{id}_M)\circ(a.{}_{\hat{\Omega}}\nabla)\circ(b.\nabla) = (\tilde{\phi}\otimes\mathrm{id}_M)\circ(a+b).\nabla$$

for this ϕ. So if R has enough iterative derivations and ∇_ϕ is iterative for all $\phi\in\mathrm{ID}_K(R)$ we obtain $(a.{}_{\hat{\Omega}}\nabla)\circ(b.\nabla) = (a+b).\nabla$, i. e. ∇ is iterative. □

6. The Tannakian Category of Modules with Iterative Connection

In this section we show – assuming a slight restriction on the ring R – that the finitely generated projective modules (i.e. locally free of finite rank) with higher connection form an abelian category and that the modules with iterative resp. integrable iterative connection form full subcategories. Furthermore all these categories form tensor categories over K (in the sense of [Del90]); in fact they even form Tannakian categories.

Notation *From now on let R be a regular ring over K which is the localisation of a finitely generated K-algebra, such that K is algebraically closed in R. Also keep in mind that we assumed R to be an integral domain and K to be a perfect field.*
Furthermore, in the following a pair (M,∇) always denotes a finitely generated R-module M together with a higher connection $\nabla: M\to\hat{\Omega}\otimes_R M$, even if "finitely generated" is not mentioned. We recall the fact given in Corollary 4.5, namely that such a module is always projective.

Definition 6.1. Let (M_1,∇_1) and (M_2,∇_2) be R-modules with higher connection. Then we call $f\in\mathrm{Hom}_R(M_1,M_2)$ a **morphism of modules with higher connection**, or a morphism for short, if the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow{\scriptstyle \nabla_1} & & \downarrow{\scriptstyle \nabla_2} \\ \hat{\Omega}\otimes_R M_1 & \xrightarrow{\mathrm{id}_{\hat{\Omega}}\otimes f} & \hat{\Omega}\otimes_R M_2 \end{array}$$

commutes. The set of all morphisms $f \in \mathrm{Hom}_R(M_1, M_2)$ will be denoted by $\mathrm{Mor}\big((M_1,\nabla_1),(M_2,\nabla_2)\big)$. If the higher connections are clear from the context we will sometimes omit them.

Remark 6.2. It is clear that the set of modules with higher connection together with the sets of morphisms defined above forms a category. This category will be denoted by $\mathbf{HCon}(R/K)$. Furthermore the full subcategories of modules with iterative connection resp. integrable iterative connection will be denoted by $\mathbf{ICon}(R/K)$ resp. $\mathbf{ICon_{int}}(R/K)$ and by definition we have a chain of inclusions

$$\mathbf{HCon}(R/K) \supset \mathbf{ICon}(R/K) \supset \mathbf{ICon_{int}}(R/K).$$

As the objects of $\mathbf{HCon}(R/K)$ are modules with an extra structure and the morphisms are special homomorphisms, we have a forgetful functor $\boldsymbol{\omega} : \mathbf{HCon}(R/K) \to \mathbf{Mod}(R)$, which is faithful.

Definition 6.3. Let (M_1,∇_1) and (M_2,∇_2) be R-modules with higher connection. Then we define a higher connection $\nabla_\oplus$ on $(M_1 \oplus M_2)$ by

$$\nabla_\oplus : M_1 \oplus M_2 \xrightarrow{\nabla_1\oplus\nabla_2} \hat{\Omega}\otimes M_1 \oplus \hat{\Omega}\otimes M_2 \xrightarrow{\cong} \hat{\Omega}\otimes(M_1\oplus M_2)$$

and a higher connection $\nabla_\otimes$ on $M_1\otimes_R M_2$ by

$$\begin{aligned} \nabla_\otimes : M_1\otimes_R M_2 &\xrightarrow{\nabla_1\otimes\nabla_2} (\hat{\Omega}\otimes_R M_1)\otimes_{d(R)}(\hat{\Omega}\otimes_R M_2) \xrightarrow{\cong} \\ &\xrightarrow{\cong} (\hat{\Omega}\otimes_{d(R)}\hat{\Omega})\otimes_R(M_1\otimes_R M_2) \xrightarrow{\mu\otimes\mathrm{id}} \hat{\Omega}\otimes_R(M_1\otimes_R M_2). \end{aligned}$$

Furthermore we define a higher connection ∇_H on $\mathrm{Hom}_R(M_1,M_2)$ by the following:
For $f\in \mathrm{Hom}_R(M_1,M_2)$ the composition

$$1\otimes M_1 \hookrightarrow \hat{\Omega}\otimes_R M_1 \xrightarrow{({}_{\hat{\Omega}}\nabla_1)^{-1}} \hat{\Omega}\otimes_R M_1 \xrightarrow{\mathrm{id}_{\hat{\Omega}}\otimes f} \hat{\Omega}\otimes_R M_2 \xrightarrow{{}_{\hat{\Omega}}\nabla_2} \hat{\Omega}\otimes_R M_2$$

is an element of $\mathrm{Hom}_R(M_1, \hat{\Omega}\otimes_R M_2)$, and we have a natural isomorphism $\mathrm{Hom}_R(M_1,\hat{\Omega}\otimes_R M_2) \cong \hat{\Omega}\otimes_R \mathrm{Hom}_R(M_1,M_2)$. In this sense we define

$$\nabla_H(f) := {}_{\hat{\Omega}}\nabla_2 \circ (\mathrm{id}_{\hat{\Omega}}\otimes f)\circ \big({}_{\hat{\Omega}}\nabla_1\big)^{-1}|_{1\otimes M}.$$

Remark 6.4. (i) If ∇_1 is an iterative connection, the definition of ∇_H coincides with the definition given in [Rös07], because then we have $({}_{\hat{\Omega}}\nabla_1)^{-1}|_{1\otimes M_1} = -{}_{\hat{\Omega}}\nabla_1|_{1\otimes M_1} = -\nabla_1$.

(ii) As the referee pointed out, in the definition of the tensor product it might be possible to replace the twisting isomorphism $(\hat{\Omega}\otimes_R M_1)\otimes_{d(R)}(\hat{\Omega}\otimes_R M_2) \cong (\hat{\Omega}\otimes_{d(R)}\hat{\Omega})\otimes_R(M_1\otimes_R M_2)$ by a more general isomorphism in order to obtain a non-commutative theory. This might lead to an iterative variant of Y. André's framework for the theory of differential and difference equations (cf. [And01]) and also to a more abstract framework for the iterative q-difference theory of C. Hardouin (cf. [Hard08]).

Theorem 6.5. *The category* $\mathbf{HCon}(R/K)$ *is an abelian category and the categories* $\mathbf{ICon}(R/K)$ *and* $\mathbf{ICon_{int}}(R/K)$ *are abelian subcategories.*

Proof. For all $(M_1,\nabla_1),(M_2,\nabla_2)\in \mathbf{HCon}(R/K)$ the set of morphisms $\mathrm{Mor}(M_1,M_2)$ is a subgroup of $\mathrm{Hom}_R(M_1,M_2)$ and so it is an abelian group. Since $\mathbf{Mod}(R)$ is an abelian category, it is sufficient to show that kernels, direct sums and so on in the category $\mathbf{Mod}(R)$ can be equipped with a higher connection resp. (integrable) iterative connection and that all necessary homomorphisms (like the inclusion map of the kernel into the module) are morphisms.

The only point at which one has to be careful is the kernel: For a morphism $f \in \mathrm{Mor}(M_1, M_2)$, the higher connection ∇_1 induces a map $\nabla_1|_{\mathrm{Ker}(f)} : \mathrm{Ker}(f) \to \mathrm{Ker}(\mathrm{id}_{\hat{\Omega}} \otimes f)$ and one has to verify that $\mathrm{Ker}(\mathrm{id}_{\hat{\Omega}} \otimes f) = \hat{\Omega} \otimes \mathrm{Ker}(f)$. But this follows from the fact that $\hat{\Omega}_k$ is a projective R-module for all $k \in \mathbb{N}$ and hence $\hat{\Omega} \otimes -$ is an exact functor. (See [Rös07], Ch.4.1, for details.) □

Now we will show that these categories are tensor categories over K (we will use the definition of a tensor category over K given in [Del90]). By the previous theorem, they are all abelian, and by Corollary 4.5, all modules that arise are projective and the category $\mathbf{Proj\text{-}Mod}(R)$ of finitely generated projective R-modules is known to satisfy all properties of a tensor category apart from being an abelian category.

So we define

- the tensor product of (M_1, ∇_1) and (M_2, ∇_2) by
$$(M_1, \nabla_1) \otimes (M_2, \nabla_2) := (M_1 \otimes_R M_2, \nabla_\otimes)$$
(this tensor product is obviously associative and commutative),
- the unital object $\mathbf{1} := (R, \mathrm{d}_R)$ ($R \otimes_R M \to M, r \otimes m \mapsto rm$ is easily seen to be a morphism for all $(M, \nabla) \in \mathbf{HCon}(R/K)$),
- the dual object to (M, ∇) by
$$(M, \nabla)^* := (M^*, \nabla^*),$$
where $\nabla^*(f) := \mathrm{d}_{\hat{\Omega}} \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ ({}_{\hat{\Omega}}\nabla^{-1})|_{1 \otimes M} \in \mathrm{Hom}_R(M, \hat{\Omega})$ for $f \in M^* = \mathrm{Hom}_R(M, R)$ (here we used that $\hat{\Omega} \otimes_R R \cong \hat{\Omega}$ and $\mathrm{Hom}_R(M, \hat{\Omega}) \cong \mathrm{Hom}_R(M, R) \otimes \hat{\Omega}$. Cf. the definition of ∇_H in Definition 6.3),
- the internal hom object of (M_1, ∇_1) and (M_2, ∇_2) by
$$\underline{\mathrm{Hom}}\left((M_1, \nabla_1), (M_2, \nabla_2)\right) := \left(\mathrm{Hom}_R(M_1, M_2), \nabla_H\right).$$

Furthermore we recognize that every endomorphism in $\mathrm{End}(\mathbf{1})$ is given by the image of $1 \in R$, which has to be constant. Since all constants are algebraic over K and K is algebraically closed in R, $\mathrm{End}(\mathbf{1})$ is isomorphic to K.

Lemma 6.6. *For all $(M_1, \nabla_1), (M_2, \nabla_2) \in \mathbf{HCon}(R/K)$ the isomorphism of R-modules*
$$\iota_{M_1,M_2} : M_1^* \otimes_R M_2 \to \mathrm{Hom}_R(M_1, M_2), \quad f \otimes m \mapsto \{v \mapsto f(v) \cdot m\}$$
is a morphism (and therefore an isomorphism) in $\mathbf{HCon}(R/K)$.

Proof. For all $f \otimes m \in M_1^* \otimes_R M_2$ and for all $v \in M_1$, we have
$$\begin{aligned} \nabla_H(\iota_{M_1,M_2}(f \otimes m))(v) &= \Big({}_{\hat{\Omega}}\nabla_2 \circ \big(\mathrm{id}_{\hat{\Omega}} \otimes \iota_{M_1,M_2}(f \otimes m)\big) \circ ({}_{\hat{\Omega}}\nabla_1^{-1})\Big)(1 \otimes v) \\ &= {}_{\hat{\Omega}}\nabla_2\big((\mathrm{id}_{\hat{\Omega}} \otimes f)({}_{\hat{\Omega}}\nabla_1^{-1}(1 \otimes v)) \cdot (1 \otimes m)\big) \\ &= \mathrm{d}_{\hat{\Omega}}\big((\mathrm{id}_{\hat{\Omega}} \otimes f)({}_{\hat{\Omega}}\nabla_1^{-1}(1 \otimes v)\big) \cdot \nabla_2(m) \end{aligned}$$
and
$$\begin{aligned} (\mathrm{id}_{\hat{\Omega}} \otimes \iota_{M_1,M_2})(\nabla_\otimes(f \otimes m))(v) &= (\mathrm{id}_{\hat{\Omega}} \otimes \iota_{M_1,M_2})\Big(\big(\mathrm{d}_{\hat{\Omega}} \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ ({}_{\hat{\Omega}}\nabla_1^{-1}|_{1 \otimes M})\big) \otimes \nabla_2(m)\Big)(v) \\ &= \big(\mathrm{d}_{\hat{\Omega}} \circ (\mathrm{id}_{\hat{\Omega}} \otimes f) \circ ({}_{\hat{\Omega}}\nabla_1^{-1})\big)(1 \otimes v) \cdot \nabla_2(m). \end{aligned}$$
So $\nabla_H \circ \iota_{M_1,M_2} = (\mathrm{id}_{\hat{\Omega}} \otimes \iota_{M_1,M_2}) \circ \nabla_\otimes$, i. e. ι_{M_1,M_2} is a morphism. □

Lemma 6.7. *Let $(M, \nabla) \in \mathbf{HCon}(R/K)$, and let $\varepsilon_M : M \otimes M^* \to R$ and $\delta_M : R \to M^* \otimes M$ be the evaluation and coevaluation homomorphisms given in the definition of a tensor category, i. e., $\varepsilon_M(m \otimes f) = f(m)$ and $\delta_M(1) = \iota_{M,M}^{-1}(\mathrm{id}_M)$. Then ε_M and δ_M are morphisms in $\mathbf{HCon}(R/K)$.*

Proof. For $m\otimes f\in M\otimes M^*$, we have

$$\begin{aligned}
&(\mathrm{id}_{\hat\Omega}\otimes\varepsilon_M)(\nabla_\otimes(m\otimes f))\\
&=(\mathrm{id}_{\hat\Omega}\otimes\varepsilon_M)\Big((\mu\otimes\mathrm{id})\circ\big(\nabla(m)\otimes(\mathrm{d}_{\hat\Omega}\circ(\mathrm{id}_{\hat\Omega}\otimes f)\circ{}_{\hat\Omega}\nabla^{-1}|_{1\otimes M})\Big)\\
&=(\mu\otimes\mathrm{id})\Big(\big(\mathrm{id}_{\hat\Omega}\otimes\big(\mathrm{d}_{\hat\Omega}\circ(\mathrm{id}_{\hat\Omega}\otimes f)\circ{}_{\hat\Omega}\nabla^{-1}|_{1\otimes M}\big)\big)(\nabla(m))\Big)\\
&=\Big((\mu\otimes\mathrm{id})\circ(\mathrm{d}_{\hat\Omega}\otimes\mathrm{d}_{\hat\Omega})\circ(\mathrm{id}_{\hat\Omega\otimes\hat\Omega}\otimes f)\circ(\mathrm{d}_{\hat\Omega}^{-1}\otimes{}_{\hat\Omega}\nabla^{-1}|_{1\otimes M})\circ\nabla\Big)(m)\\
&=\Big(\mathrm{d}_{\hat\Omega}\circ(\mathrm{id}_{\hat\Omega}\otimes f)\circ({}_{\hat\Omega}\nabla^{-1})\circ\nabla\Big)(m)\\
&=\Big(\mathrm{d}_{\hat\Omega}\circ(\mathrm{id}_{\hat\Omega}\otimes f)\Big)(1\otimes m)\\
&=\mathrm{d}_R(f(m))=\mathrm{d}_R(\varepsilon_M(m\otimes f))
\end{aligned}$$

So ε_M is a morphism.

Since $\iota_{M,M}$ is an isomorphism in $\mathbf{HCon}(R/K)$, δ_M is a morphism if and only if $\iota_{M,M}\circ\delta_M$ is a morphism. Now

$$\nabla_H((\iota_{M,M}\circ\delta_M)(1))=\nabla_H(\mathrm{id}_M)={}_{\hat\Omega}\nabla\circ(\mathrm{id}_{\hat\Omega}\otimes\mathrm{id}_M)\circ{}_{\hat\Omega}\nabla^{-1}|_{1\otimes M}={}_{\hat\Omega}\nabla\circ{}_{\hat\Omega}\nabla^{-1}|_{1\otimes M}=1\otimes\mathrm{id}_M$$

and

$$(\mathrm{id}_{\hat\Omega}\otimes(\iota_{M,M}\circ\delta_M))(\mathrm{d}_R(1))=(\mathrm{id}_{\hat\Omega}\otimes(\iota_{M,M}\circ\delta_M))(1\otimes 1)=1\otimes\mathrm{id}_M,$$

so δ_M is a morphism. □

Theorem 6.8. $\mathbf{HCon}(R/K)$, $\mathbf{ICon}(R/K)$ *and* $\mathbf{ICon_{int}}(R/K)$ *are tensor categories over* K.

Proof. Since we have already shown that these categories are abelian, that $\mathbf{HCon}(R/K)$ is equipped with an associative and commutative tensor product, and that ε_M and δ_M are morphisms, we already know that $\mathbf{HCon}(R/K)$ is a tensor category. Hence, it only remains to show that the two subcategories are closed under tensor products and duals.

It is checked immediately that for $(M_1,\nabla_1),(M_2,\nabla_2)\in\mathbf{ICon}(R/K)$ the higher connection $\nabla_\otimes$ on $M_1\otimes M_2$ satisfies $(a._{\hat\Omega}\nabla_\otimes)\circ(b.\nabla_\otimes)=(a+b).\nabla_\otimes$ for all $a,b\in K^{\mathrm{sep}}$. One also checks easily that the higher connection ∇_1^* on M^* satisfies $\big(a._{\hat\Omega}\nabla_1^*\big)\circ(b.\nabla_1^*)=(a+b)._{\hat\Omega}\nabla_1^*$ for all $a,b\in K^{\mathrm{sep}}$, if ∇_1 is iterative. Hence $\mathbf{ICon}(R/K)$ is a tensor category.

Assuming that $(M_1,\nabla_1),(M_2,\nabla_2)\in\mathbf{ICon_{int}}(R/K)$, the integrability conditions for $\nabla_\otimes$ and ∇_1^* are obtained by a short calculation using the following lemma. □

Lemma 6.9. *Let* $(M_1,\nabla_1),(M_2,\nabla_2)\in\mathbf{HCon}(R/K)$ *and let* $\psi\in\mathrm{HD}_K(R)$*. Then we have:*

i) $(\nabla_\otimes)_\psi=(\mu\otimes\mathrm{id})\circ\big((\nabla_1)_\psi\otimes(\nabla_2)_\psi\big)$,

ii) *For all* $f\in\mathrm{Hom}_R(M_1,M_2)$*:*

$$\big((\nabla_H)_\psi\big)(f)\;=\;(\nabla_2)_\psi\,[[T]]\circ(\mathrm{id}_{R[[T]]}\otimes f)\circ\Big((\nabla_1)_\psi\,[[T]]\Big)^{-1}|_{1\otimes M}.$$

Proof. We have,

$$\begin{aligned}
(\nabla_\otimes)_\psi&=(\tilde\psi\otimes\mathrm{id})\circ(\mu\otimes\mathrm{id})\circ(\nabla_1\otimes\nabla_2)\\
&=(\mu\otimes\mathrm{id})\circ\big((\tilde\psi\otimes\mathrm{id}_{M_1})\otimes(\tilde\psi\otimes\mathrm{id}_{M_2})\big)\circ(\nabla_1\otimes\nabla_2)\\
&=(\mu\otimes\mathrm{id})\circ\big((\nabla_1)_\psi\otimes(\nabla_2)_\psi\big),
\end{aligned}$$

which shows the first part. For the proof of the second part, consider the following diagram:

$$\begin{array}{ccccccc}
\hat{\Omega} \otimes_R M_1 & \xleftarrow[\cong]{{}_{\hat{\Omega}}\nabla_1} & \hat{\Omega} \otimes_R M_1 & \xrightarrow{\mathrm{id}\otimes f} & \hat{\Omega} \otimes_R M_2 & \xrightarrow[\cong]{{}_{\hat{\Omega}}\nabla_2} & \hat{\Omega} \otimes_R M_2 \\
\Big\downarrow{\scriptstyle \tilde{\psi}\otimes \mathrm{id}_{M_1}} & & \Big\uparrow & & & & \Big\downarrow{\scriptstyle \tilde{\psi}\otimes \mathrm{id}_{M_2}} \\
 & & 1 \otimes M_1 & & & & \\
 & & \Big\downarrow & & & & \\
R[[T]] \otimes_R M_1 & \xleftarrow[\cong]{(\nabla_1)_\psi[[T]]} & R[[T]] \otimes_R M_1 & \xrightarrow{\mathrm{id}\otimes f} & R[[T]] \otimes_R M_2 & \xrightarrow[\cong]{(\nabla_2)_\psi[[T]]} & R[[T]] \otimes_R M_2
\end{array}$$

It is sufficient to show that both maps from the upper left corner of the diagram to the lower right corner are equal. Both parts of the diagram (starting at $1 \otimes M$ in the middle) commute by definition of $(\nabla_1)_\psi$ resp. $(\nabla_2)_\psi$. Furthermore, $\hat{\Omega} \otimes M_1$ is generated as an $\hat{\Omega}$-module by elements in $1 \otimes M_1$ and since ${}_{\hat{\Omega}}\nabla_1$ is an isomorphism, $\hat{\Omega} \otimes M_1$ is also generated as an $\hat{\Omega}$-module by elements of the form ${}_{\hat{\Omega}}\nabla_1(1 \otimes m)$ for $m \in M_1$. The equality of the maps then follows from the $\hat{\Omega}$-linearity of the upper row and the $R[[T]]$-linearity of the lower row. $\square$

Theorem 6.10. *The categories* $\mathbf{HCon}(R/K)$*,* $\mathbf{ICon}(R/K)$ *and* $\mathbf{ICon_{int}}(R/K)$ *are Tannakian categories with the forgetful functor* $\boldsymbol{\omega} : \mathbf{HCon}(R/K) \to \mathbf{Mod}(R)$ *(restricted to the respective category) as fibre functor. If moreover* R *has a* K*-rational point, i. e., there exists a maximal ideal* $\mathfrak{m} \trianglelefteq R$ *with* $K \cong R/\mathfrak{m}$*, then these categories are neutral Tannakian categories with fibre functor* $\boldsymbol{\omega}_K : \mathbf{HCon}(R/K) \xrightarrow{\boldsymbol{\omega}} \mathbf{Mod}(R) \xrightarrow{\otimes_R R/\mathfrak{m}} \mathbf{Vect}(K)$*.*

Proof. By construction, the functor $\boldsymbol{\omega}$ is a fibre functor and so the tensor categories $\mathbf{HCon}(R/K)$, $\mathbf{ICon}(R/K)$ and $\mathbf{ICon_{int}}(R/K)$ are Tannakian categories. If R has a K-rational point, by [Del90].2.8, $\boldsymbol{\omega}_K$ is a fibre functor. This proves the second part. $\square$

By Tannakian duality, every neutral Tannakian category $\mathcal{T}$ over K with fibre functor $\boldsymbol{\omega}_K$ is equivalent to the category of finite dimensional representations of a group scheme defined over K, called the Tannaka group scheme (or fundamental group scheme) $\Pi(\mathcal{T}, \boldsymbol{\omega}_K)$ of $\mathcal{T}$. Furthermore, this group scheme is the projective limit of all quotients $G_V := \Pi(\ll V \gg), \boldsymbol{\omega}_K|_{\ll V \gg})$, where V ranges over all objects of $\mathcal{T}$ and $\ll V \gg$ denotes the smallest Tannakian subcategory of $\mathcal{T}$ that contains V.

Using the Picard-Vessiot theory in the second part of this paper, we obtain the following proposition:

Proposition 6.11. *Let* R *have a* K*-rational point and let* K *be algebraically closed. Then the Tannaka group schemes* $\Pi(\mathbf{ICon}(R/K), \boldsymbol{\omega}_K)$ *and* $\Pi(\mathbf{ICon_{int}}(R/K), \boldsymbol{\omega}_K)$ *are reduced group schemes.*

Proof. If $\mathrm{char}(K) = 0$, then by general theory all group schemes are reduced, and nothing has to be shown. So assume $\mathrm{char}(K) = p > 0$.

Using the equivalence of categories given in Thm. 8.7 and the identification in Rem. 8.3, we obtain from [San07], 2.3.1 that the Tannaka group scheme associated to $\mathbf{ICon_{int}}(R/K)$ is reduced, even a perfect group scheme. (The reducedness also follows from the reducedness of $\Pi(\mathbf{ICon}(R/K), \boldsymbol{\omega}_K)$, since $\Pi(\mathbf{ICon_{int}}(R/K), \boldsymbol{\omega}_K)$ is a quotient.)

Since $\Pi(\mathbf{ICon}(R/K), \boldsymbol{\omega}_K)$ is the projective limit of all $G_{(M,\nabla)} := \Pi(\ll (M, \nabla) \gg, \boldsymbol{\omega}_K|_{\ll (M,\nabla) \gg})$, it suffices to show that all $G_{(M,\nabla)}$ are reduced. Let F be the quotient field of R and E/F be a PPV-extension for $F \otimes_R M$ as defined in Section 10, which exists since K is algebraically closed (cf. Remark 10.6). Then $G_{(M,\nabla)}$ is isomorphic to the Galois group scheme $\underline{\mathrm{Gal}}(E/F)$ (cf. Rem. 10.13). By Prop. 8.1, we have $\mathrm{Ker}(\mathrm{d}_F^{(1)}) = F^p$, and hence by Cor. 11.7, $\underline{\mathrm{Gal}}(E/F) \cong G_{(M,\nabla)}$ is reduced. $\square$

Remark 6.12. One might ask whether the inclusions in the chain of categories $\mathbf{HCon}(R/K) \supset \mathbf{ICon}(R/K) \supset \mathbf{ICon_{int}}(R/K)$ are strict.

Clearly, $\mathbf{HCon}(R/K) \neq \mathbf{ICon}(R/K)$, because if for example M is a free R-module of dimension 1 with basis $b_1 \in M$, every $\omega = \sum_{j=0}^{\infty} \omega_j \in \hat{\Omega}_{R/K}$ with $\omega_0 = 1$ defines a higher connection $\nabla : M \to \hat{\Omega}_{R/K} \otimes_R M, b_1 \mapsto \omega \otimes b_1$, but in general this higher connection is not iterative, because if ∇ is iterative, ω satisfies the condition

$$0 = \left(-_{\hat{\Omega}}\nabla \circ \nabla\right)^{(2)}(b_1) = (2\omega_2 - \omega_1^2 + \mathrm{d}_{\hat{\Omega}}^{(1)}(\omega_1)) \otimes b_1.$$

(The only exception is the case when R is algebraic over K, because in this case $\hat{\Omega}_{R/K} = R$ and hence all categories above are equivalent to $\mathbf{Mod}(R)$).

The inclusion $\mathbf{ICon}(R/K) \supset \mathbf{ICon_{int}}(R/K)$ is also strict in general, because in the next section we will see that in characteristic zero, the category $\mathbf{ICon}(R/K)$ is equivalent to the category of modules with (ordinary) connection over R and $\mathbf{ICon_{int}}(R/K)$ is equivalent to the category of modules with integrable connection over R, and it is known that those two categories are different if for example $R = K(t_1, t_2)$. However, it is also known that every (ordinary) connection is integrable if $\mathrm{char}(K) = 0$ and R is an algebraic function field in one variable over K. In Section 8, we will see that also $\mathbf{ICon}(R/K) = \mathbf{ICon_{int}}(R/K)$, if R is an algebraic function field in one variable over K and $\mathrm{char}(K) = p$.

7. Characteristic Zero

For $\mathrm{char}(K) = 0$, in general one gets the usual constructions of derivations, differentials and connections by restricting to the terms of degree 1. On the other hand these constructions can be uniquely extended to iterative derivations and iterative connections. Moreover integrable connections, i. e. connections which preserve commutators of derivations, correspond to integrable iterative connections. This will be proven in this section.

So throughout this section, K is a field of characteristic zero and R is an integral domain which is a regular ring and the localisation of a finitely generated K-algebra. Furthermore we assume that R has a maximal ideal $\mathfrak{m} \trianglelefteq R$, such that $R/\mathfrak{m}$ is a finite extension of K. M denotes a finitely generated R-module.

Proposition 7.1.

i) *The map*

$$\mathrm{Der}(R/K) \longrightarrow \mathrm{ID}_K(R), \partial \mapsto \phi_\partial,$$

given by

$$\phi_\partial(r) := \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n(r) T^n$$

for all $r \in R$, is a bijection and the inverse map is given by $\phi \mapsto \phi^{(1)}$.

For a given derivation ∂ on R and the corresponding iterative derivation ϕ_∂ the map $\mathrm{I} : \mathrm{Der}_R(M) \to \mathrm{ID}(M, \phi_\partial), \partial_M \mapsto \Phi_{\partial_M}$ given by

$$\Phi_{\partial_M}(m) := \sum_{n=0}^{\infty} \frac{1}{n!} \partial_M^n(m) T^n,$$

for all $m \in M$, is a bijection and the inverse map is given by $\Phi \mapsto \Phi^{(1)}$.

ii) *The R-module $(\hat{\Omega}_{R/K})_1$ is canonically isomorphic to the module of (ordinary) differentials $\Omega_{R/K}$ and $\mathrm{d}^{(1)} : R \to (\hat{\Omega}_{R/K})_1 \cong \Omega_{R/K}$ is the universal derivation.*

iii) *For every iterative connection ∇ on M, the map $\nabla^{(1)} : M \to (\hat{\Omega}_{R/K})_1 \otimes M \cong \Omega_{R/K} \otimes M$ is a connection on M and every connection $\nabla^{(1)}$ on M uniquely extends to an iterative connection on M. Furthermore, ∇ is an integrable iterative connection if and only if $\nabla^{(1)}$ is an integrable connection.*

Proof. i) Let $\partial \in \mathrm{Der}(R/K)$ be a derivation. Then for all $i,j \in \mathbb{N}$: $\frac{1}{i!}\partial^i \circ \frac{1}{j!}\partial^j = \binom{i+j}{i}\frac{1}{(i+j)!}\partial^{i+j}$. So ϕ_∂ is an iterative derivation. On the other hand, for every iterative derivation ϕ, one obtains $\phi^{(k)} = \frac{1}{k!}(\phi^{(1)})^k$ for all $k \in \mathbb{N}$ by applying the formula $\phi^{(i)} = \frac{1}{i}\phi^{(1)} \circ \phi^{(i-1)}$ inductively. Finally by Remark 3.2, for all $r,s \in R$ we have $\phi^{(1)}(rs) = r\phi^{(1)}(s) + \phi^{(1)}(r)s$, i.e. $\phi^{(1)} \in \mathrm{Der}(R/K)$.
The bijection $\mathrm{I} : \mathrm{Der}_R(M) \to \mathrm{ID}(M, \phi_\partial)$ is shown analogously.
ii) The construction of $(\hat{\Omega}_{R/K})_1$ given in the proof of Theorem 3.10 is the same as the usual construction of $\Omega_{R/K}$ (e.g. in [Hart77], Ch. II.8).
iii) The bijection of the iterative connections and the ordinary connections is shown analogous to the first part. So we only prove the equivalence of the integrability conditions. Let $\partial_1, \partial_2 \in \mathrm{Der}(R/K)$ be derivations, let $\phi_i := \phi_{\partial_i}$ $(i = 1,2)$ be the corresponding iterative derivations, and let ∇ be an iterative connection on M. By an explicit calculation one gets

$$\left(\phi_1\phi_2\phi_1^{-1}\phi_2^{-1}\right)^{(1)} = 0 \quad \text{and} \quad \left(\phi_1\phi_2\phi_1^{-1}\phi_2^{-1}\right)^{(2)} = \partial_1 \circ \partial_2 - \partial_2 \circ \partial_1 = [\partial_1, \partial_2]\,.$$

From this and the iterativity condition of ∇, one computes that

$$\nabla^{(2)}_{\phi_1\phi_2\phi_1^{-1}\phi_2^{-1}} = \left(\nabla^{(1)}\right)_{[\partial_1,\partial_2]}.$$

(The last expression means that we apply the ordinary connection $\nabla^{(1)}$ to the derivation $[\partial_1, \partial_2]$.) On the other hand, by quite the same calculation as before one obtains

$$\left(\nabla_{\phi_1}\nabla_{\phi_2}\nabla_{\phi_1}^{-1}\nabla_{\phi_2}^{-1}\right)^{(2)} = \nabla^{(1)}_{\phi_1} \circ \nabla^{(1)}_{\phi_2} - \nabla^{(1)}_{\phi_2} \circ \nabla^{(1)}_{\phi_1} = \left[\left(\nabla^{(1)}\right)_{\partial_1}, \left(\nabla^{(1)}\right)_{\partial_2}\right].$$

So if ∇ is an integrable iterative connection, then $\nabla^{(2)}_{\phi_1\phi_2\phi_1^{-1}\phi_2^{-1}} = \left(\nabla_{\phi_1}\nabla_{\phi_2}\nabla_{\phi_1}^{-1}\nabla_{\phi_2}^{-1}\right)^{(2)}$ and hence $\nabla^{(1)}$ is an integrable connection.

For the converse, consider the complete local ring $\hat{R}_\mathfrak{m}$. We first note that for an arbitrary R-cga B, every higher derivation $\psi \in \mathrm{HD}_K(R, B)$ can be extended canonically to a higher derivation $\psi_e \in \mathrm{HD}_K(\hat{R}_\mathfrak{m}, \hat{R}_\mathfrak{m} \otimes B)$ in the following way: Every homogeneous component $\psi^{(k)}$ $(k \in \mathbb{N})$ can uniquely be extended to the localisation $R_\mathfrak{m}$ (see Prop. 3.7). This extension is continuous with respect to the $\mathfrak{m}$-adic topology, since for all $i \in \mathbb{N}$, $\psi^{(k)}(\mathfrak{m}^i) \subseteq \mathfrak{m}^{i-k}(R_\mathfrak{m} \otimes B_k)$. So $\psi^{(k)}$ can be uniquely extended to a map $\psi_e^{(k)} : \hat{R}_\mathfrak{m} \to \hat{R}_\mathfrak{m} \otimes B_k$, which is continuous with respect to the $\mathfrak{m}$-adic topology. One easily verifies that indeed $\psi_e := \sum_{k=0}^\infty \psi_e^{(k)} : \hat{R}_\mathfrak{m} \to \hat{R}_\mathfrak{m} \otimes B$ is a higher derivation.
Since the extension is canonical, we obtain the identities $(\mathrm{id}_{\hat{R}_\mathfrak{m}} \otimes \widetilde{\psi}) \circ \mathrm{d}_{R,e} = \psi_e$ ($\mathrm{d}_{R,e}$ denotes the extension of the universal derivation d_R) and $(\psi_1\psi_2)_e = \psi_{1,e}\psi_{2,e}$ for all $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$.
Now let ∇ be an iterative connection such that $\nabla^{(1)}$ is an integrable connection. By [Kat70], Prop. 8.9, the $\hat{R}_\mathfrak{m}$-module $\hat{R}_\mathfrak{m} \otimes_R M$ is a trivial differential module, i.e., there is an $\hat{R}_\mathfrak{m}$-basis $\boldsymbol{b} = (b_1, \dots, b_n)$ of $\hat{R}_\mathfrak{m} \otimes_R M$ such that $\nabla^{(1)}(\boldsymbol{b}) = 0$, where $\nabla^{(1)}$ is extended to $\hat{R}_\mathfrak{m} \otimes_R M$ in the same manner as the higher derivations. Since ∇ is iterative, this implies $\nabla^{(k)}(\boldsymbol{b}) = 0$ for all $k > 0$. Hence for all $\psi_1, \psi_2 \in \mathrm{HD}_K(R)$ and all vectors $\boldsymbol{y} \in \hat{R}_\mathfrak{m}^n$, s.t. $\sum y_i b_i \in M$, we have:

$$\nabla_{\psi_1\psi_2}(\sum y_i b_i) = (\widetilde{\psi_1\psi_2} \otimes \mathrm{id})\left(\sum \mathrm{d}_{R,e}(y_i)\nabla(b_i)\right) = (\widetilde{\psi_1\psi_2} \otimes \mathrm{id})\left(\sum \mathrm{d}_{R,e}(y_i) \cdot b_i\right) = \sum (\psi_1\psi_2)_e(y_i) \cdot b_i,$$

and

$$\nabla_{\psi_1}\nabla_{\psi_2}\left(\sum y_i b_i\right) = \sum \psi_{1,e}[[T]](\psi_{2,e}(y_i)) \cdot b_i = \sum (\psi_1\psi_2)_e(y_i) \cdot b_i.$$

Hence $\nabla_{\psi_1\psi_2} = \nabla_{\psi_1}\nabla_{\psi_2}$, i.e., ∇ is an integrable iterative connection. □

As a consequence of the previous proposition, we obtain

Theorem 7.2. *Under the assumptions given above, the category* $\mathbf{ICon}(R/K)$ *(resp.* $\mathbf{ICon_{int}}(R/K)$*) of finitely generated* R*-modules with iterative connection (resp. integrable iterative connection) and the category of finitely generated* R*-modules with connection (resp. integrable connection) are equivalent.*

We end this section with a comparison of integrable iterative connections and stratifications (cf. [BO78], Def. 2.10): From the previous theorem and the fact that for a smooth ring in characteristic zero a stratification is equivalent to an integrable connection (cf. [BO78], Thm 2.15 for a sketch of the proof), we deduce the following corollary. In the next section, we will see that the corollary also holds if the characteristic of K is not zero (cf. Cor. 8.8).

Corollary 7.3. *Let* R *be smooth over* K*, then the category* $\mathbf{ICon_{int}}(R/K)$ *is equivalent to the category of stratified modules over* R*.*

8. Positive Characteristic

In this section, we consider the case that K has positive characteristic p. Contrary to characteristic zero, iterative derivations and iterative connections are not longer determined by the term of degree 1. Moreover, not every derivation $\partial \in \mathrm{Der}(R/K)$ can be extended to an iterative derivation ϕ with $\phi^{(1)} = \partial$, because the conditions on an iterative derivation imply $(\phi^{(1)})^p = p! \cdot \phi^{(p)} = 0$, i. e., at least ∂ has to be nilpotent.

But there are some other structural properties: The main result is that every module with integrable iterative connection gives rise to a projective system and vice versa, similar to the correspondence of projective systems and iterative differential modules over function fields given in [Mat01], Ch. 2.2. In fact, when R is an algebraic function field in one variable, the projective systems defined here are equal to those defined by Matzat and van der Put, which shows that in this case the categories $\mathbf{ICon}(R/K)$, $\mathbf{ICon_{int}}(R/K)$, $\mathbf{Proj}_R$ and $\mathbf{ID}_R$ are equivalent. (Here $\mathbf{Proj}_R$ denotes the category of projective systems over R and $\mathbf{ID}_R$ denotes the category of ID-modules as given in [Mat01]).

So in this section, let K be a perfect field of characteristic $p > 0$ and let R be an integral domain which is a regular ring and the localisation of a finitely generated K-algebra. Furthermore let $t_1, \dots, t_m$ denote a separable transcendence basis for R, i. e. $\mathrm{Quot}(R)$ is a separable algebraic extension of the rational function field $K(t_1, \dots, t_m)$.[2]

R has a natural sequence of K-subrings $(R_l)_{l\in\mathbb{N}}$ given by $R_l := R^{p^l}$. The following proposition gives a characterisation of this sequence by the higher differential:

Proposition 8.1. (Frobenius Compatibility) *For all* $l \in \mathbb{N}$*:*

$$R_l = \bigcap_{0<j<p^l} \mathrm{Ker}(\mathrm{d}_R^{(j)}).$$

Proof. Since d_R is a homomorphism of algebras, $\mathrm{d}_R(R_l) = \mathrm{d}_R(R^{p^l}) \subset (\hat{\Omega}_{R/K})^{p^l}$ and therefore $\mathrm{d}_R^{(j)}(r) = 0$ $(0 < j < p^l)$ for all $r \in R_l$. The other inclusion is shown inductively: The case $l = 0$ is trivial. Now let $r \in R$ satisfy $\mathrm{d}_R^{(j)}(r) = 0$ for $0 < j < p^l$. By induction hypothesis $r \in R_{l-1}$. So there exists $s \in R$ with $s^{p^{l-1}} = r$. If $s \notin R^p$, then s is a separating element of R and we can find separating variables $s = s_1, s_2, \dots, s_m$ for R, i. e. $\mathrm{Quot}(R)/K(s_1, \dots, s_m)$ is a finite separable extension. By applying Prop. 3.12 and Theorem 3.13(b), we see that $\mathrm{d}_R^{(1)}(s) \neq 0$. And so

$$0 \neq \left(\mathrm{d}_R^{(1)}(s)\right)^{p^{l-1}} = \mathrm{d}_R^{(p^{l-1})}\left(s^{p^{l-1}}\right) = \mathrm{d}_R^{(p^{l-1})}(r),$$

[2] This includes the case $m = 0$, although in this case everything becomes trivial.

which is a contradiction. So $s \in R^p$ and $r \in R_l$. □

In the case of R being an algebraic function field in one variable, it was shown by F. K. Schmidt (see [HS37], Thm. 10 and 15) that for an iterative derivation $\phi \in \mathrm{ID}_K(R)$ satisfying $\phi^{(1)} \neq 0$, we have $R^{p^l} = \bigcap_{0<j<p^l} \mathrm{Ker}(\phi^{(j)})$. So in this case we obtain the same sequence of subfields, when "only" looking at an iterative derivation instead of the universal derivation.

Definition 8.2. A **Frobenius compatible projective system over** R (or an **Fc-projective system over** R for short) is a sequence $(M_l, \varphi_l)_{l\in\mathbb{N}}$ with the following properties:

i) For all $l \in \mathbb{N}$, M_l is a finitely generated R_l-module.
ii) $\varphi_l : M_{l+1} \hookrightarrow M_l$ is a monomorphism of R_{l+1}-modules that extends to an isomorphism $\mathrm{id}_{R_l} \otimes \varphi_l : R_l \otimes_{R_{l+1}} M_{l+1} \to M_l$.

A **morphism** $\alpha : (M_l, \varphi_l) \to (M'_l, \varphi'_l)$ **of Fc-projective systems over** R is a sequence $\alpha = (\alpha_l)_{l\in\mathbb{N}}$ of homomorphisms of modules $\alpha_l : M_l \to M'_l$ satisfying $\varphi'_l \circ \alpha_{l+1} = \alpha_l \circ \varphi_l$.

Remark 8.3. An Fc-projective system $(M_l, \varphi_l)_{l\in\mathbb{N}}$ over R is nothing else than a flat bundle on $\mathrm{Spec}(R)$ (cf. [Gie75], Def. 1.1) resp. an F-divided sheaf on $\mathrm{Spec}(R)$ (cf. [San07], Def. 4), if one identifies $R_l = R^{p^l}$ with R via the Frobenius homomorphism $\mathbf{F}_l : R \to R_l, x \mapsto x^{p^l}$. Then all M_l can be regarded as R-modules and the maps $\mathrm{id}_{R_l} \otimes \varphi_l : R_l \otimes_{R_{l+1}} M_{l+1} \to M_l$ become R-linear isomorphisms $\sigma_l : \mathbf{F}_1^*(M_{l+1}) \to M_l$ from the Frobenius pullback of M_{l+1} to M_l, i. e. $(M_l, \sigma_l)_{l\in\mathbb{N}}$ is a flat bundle on $\mathrm{Spec}(R)$.

Proposition 8.4. *Every Fc-projective system $(M_l, \varphi_l)_{l\in\mathbb{N}}$ over R defines an integrable iterative connection ∇ on $M := M_0$ satisfying*

$$\bigcap_{0<j<p^l} \mathrm{Ker}(\nabla^{(j)}) = (\varphi_0 \circ \cdots \circ \varphi_{l-1})(M_l).$$

For a morphism $(\alpha_l)_{l\in\mathbb{N}} : (M_l, \varphi_l) \to (M'_l, \varphi'_l)$ of Fc-projective systems over R, the homomorphism $\alpha_0 : M = M_0 \to M' = M'_0$ then is a morphism of modules with higher connection.

Proof. The proof is similar to the construction of a stratification related to a flat bundle in the proof of [Gie75], Thm. 1.3.

In order to define $\nabla^{(k)}$, choose $l \in \mathbb{N}$ such that $p^l > k$ and let $\chi_l : R \otimes_{R_l} M_l \xrightarrow{\cong} M$ be the composition of the isomorphisms $\mathrm{id}_R \otimes \varphi_j : R \otimes_{R_{j+1}} M_{j+1} \to R \otimes_{R_j} M_j$ $(0 \leq j < l)$. Then we define $\nabla^{(k)}$ to be the composition:

$$\nabla^{(k)} : M \xrightarrow{\chi_l^{-1}} R \otimes_{R_l} M_l \xrightarrow{\mathrm{d}_R^{(k)} \otimes \mathrm{id}_{M_l}} \left(\hat{\Omega}_{R/K}\right)_k \otimes_{R_l} M_l \xrightarrow{\mathrm{id}_{\hat{\Omega}} \otimes \chi_l} \left(\hat{\Omega}_{R/K}\right)_k \otimes_R M.$$

This is well defined, because $\mathrm{d}_R^{(k)}$ is R_l-linear, and is also independent of the chosen l. The definition also shows that the necessary conditions for ∇ being an integrable iterative connection are fulfilled modulo degrees $\geq p^l$, since d_R is an integrable iterative connection. As l can be chosen arbitrarily large, ∇ fulfills all conditions for being an integrable iterative connection.

It remains to show that $\bigcap_{0<j<p^l} \mathrm{Ker}(\nabla^{(j)}) = (\varphi_0 \circ \cdots \circ \varphi_{l-1})(M_l)$. Since we have just constructed an iterative connection on M, by Corollary 4.5, M is projective and by the same argument, all M_l are projective. Hence $\mathrm{Ker}(\mathrm{d}_R^{(j)} \otimes \mathrm{id}_{M_l}) = \mathrm{Ker}(\mathrm{d}_R^{(j)}) \otimes_{R_l} M_l$ for all $j < p^l$ and so

$$\bigcap_{0<j<p^l} \mathrm{Ker}(\nabla^{(j)}) = \chi_l(R_l \otimes_{R_l} M_l) = (\varphi_0 \circ \cdots \circ \varphi_{l-1})(M_l).$$

Finally, let $(\alpha_l)_{l\in\mathbb{N}} : (M_l, \varphi_l) \to (M'_l, \varphi'_l)$ be a morphism of Fc-projective systems over R. We have to show that $\nabla' \circ \alpha_0 = (\mathrm{id}_{\hat\Omega} \otimes \alpha_0) \circ \nabla$, or equivalently, that for all $k \in \mathbb{N}$

$$\nabla'^{(k)} \circ \alpha_0 = (\mathrm{id}_{\hat\Omega} \otimes \alpha_0) \circ \nabla^{(k)}.$$

But the last condition is easily seen to hold by choosing $l \in \mathbb{N}$ such that $p^l > k$ and by using the definition of the iterative connections above. □

In what follows, we will show that the converse is also true, i. e. that a module with integrable iterative connection gives rise to an Fc-projective system over R. For this, we consider the quotient field $F := \mathrm{Quot}(R)$ of R and a monomial ordering on $\hat\Omega_{F/K} = F[[\mathrm{d}^{(i)}t_j]]$, namely the lexicographical order, where the variables are ordered by $\mathrm{d}^{(i_1)}t_{j_1} > \mathrm{d}^{(i_2)}t_{j_2}$ if $i_1 > i_2$ or if $i_1 = i_2$ and $j_1 > j_2$. The leading term of $\omega \in \hat\Omega_{F/K}$ (if it exists) is then denoted by $\mathrm{LT}(\omega)$.

Lemma 8.5. *Let $\omega \in \hat\Omega_{F/K}$ be homogeneous of degree p^l and $\omega \notin F\hat\Omega^{p^l}_{F/K}$. Let $\mathrm{d}^{(i_0)}t_{j_0}$ be the highest variable with the property that there exist $e_0 \in \mathbb{N}$, $p \nmid e_0$ and a monomial $\omega' \in \hat\Omega_{F/K}$ such that $(\mathrm{d}^{(i_0)}t_{j_0})^{e_0}\omega'$ is a monomial term of ω. Let e_0 and ω' be chosen such that $(\mathrm{d}^{(i_0)}t_{j_0})^{e_0}\omega'$ is maximal among those monomials. Then for every $k \le p^l(p-1)$, we have:*

$$\mathrm{LT}(\mathrm{d}^{(k)}_{\hat\Omega}(\omega)) \le e_0 \mathrm{d}^{(i_0+p^l(p-1))}t_{j_0} \cdot (\mathrm{d}^{(i_0)}t_{j_0})^{e_0-1}\omega',$$

with equality if and only if $k = p^l(p-1)$ and $i_0 < p^l$.

Proof. For $i \in \mathbb{N}$, $j \in \{1, \dots, m\}$, $e \in \mathbb{N}_+$ and $k \in \mathbb{N}$, we have

$$\mathrm{d}^{(k)}_{\hat\Omega}\left((\mathrm{d}^{(i)}t_j)^e\right) = \sum_{k_1+\cdots+k_e=k} \binom{i+k_1}{i} \cdots \binom{i+k_e}{i} \mathrm{d}^{(i+k_1)}t_j \cdots \mathrm{d}^{(i+k_e)}t_j.$$

So

$$\begin{aligned}
\mathrm{LT}\left(\mathrm{d}^{(k)}_{\hat\Omega}\left((\mathrm{d}^{(i)}t_j)^e\right)\right) &= e \cdot \binom{i+k}{i} \mathrm{d}^{(i+k)}t_j (\mathrm{d}^{(i)}t_j)^{e-1} \quad \text{if } e\binom{i+k}{i} \neq 0 \in \mathbb{F}_p \\
\mathrm{d}^{(k)}_{\hat\Omega}\left((\mathrm{d}^{(i)}t_j)^e\right) &= 0 \qquad \text{if } p \mid e \text{ and } p \nmid k \quad \text{and} \\
\mathrm{d}^{(k)}_{\hat\Omega}\left((\mathrm{d}^{(i)}t_j)^e\right) &= \left(\mathrm{d}^{(\frac{k}{p})}_{\hat\Omega}\left((\mathrm{d}^{(i)}t_j)^{\frac{e}{p}}\right)\right)^p \quad \text{if } p \mid e \text{ and } p \mid k.
\end{aligned}$$

So for $k \le p^l(p-1)$, a variable $\mathrm{d}^{(i)}t_j \neq \mathrm{d}^{(i_0)}t_{j_0}$ occurring in ω gives a contribution to $\mathrm{d}^{(k)}_{\hat\Omega}(\omega)$ of variables

(i) less than $\mathrm{d}^{(i_0+k)}t_{j_0}$ if it occurs in a power not divided by p and
(ii) less than $\mathrm{d}^{(i+\frac{k}{p})}t_j$ otherwise.

In the second case, $i \le p^{l-1}$, since ω is of degree p^l, and so $i + \frac{k}{p} \le p^{l-1} + p^{l-1}(p-1) = p^l$. So $\mathrm{d}^{(i+\frac{k}{p})}t_j < \mathrm{d}^{(i_0+p^l)}t_{j_0}$. Therefore the highest variable that may occur is $\mathrm{d}^{(i_0+k)}t_{j_0}$ (or $\mathrm{d}^{(i_0+p^l)}t_{j_0}$ if $k < p^l$) and $\mathrm{d}^{(i_0+p^l(p-1))}t_{j_0}$ occurs if and only if $k = p^l(p-1)$ and $\binom{i_0+p^l(p-1)}{i_0} \neq 0 \in \mathbb{F}_p$, i. e. $i_0 \neq p^l$. The highest corresponding monomial then is

$$e_0 \mathrm{d}^{(i_0+p^l(p-1))}t_{j_0} \cdot (\mathrm{d}^{(i_0)}t_{j_0})^{e_0-1}\omega'.$$ □

Proposition 8.6. *Every R-module M with integrable iterative connection ∇ defines an Fc-projective system (M_l, φ_l) over R, where $M_l := \bigcap_{0<j<p^l} \mathrm{Ker}(\nabla^{(j)})$ and $\varphi_l : M_{l+1} \to M_l$ is the inclusion map, and a morphism $f : (M, \nabla) \to (M', \nabla')$ of modules with higher connection defines a morphism $\alpha : (M_l, \varphi_l) \to (M'_l, \varphi'_l)$ of Fc-projective systems over R by $\alpha_l := f|_{M_l}$.*

Proof. Since ∇ is an integrable iterative connection on M, $\nabla^{(1)}$ is an integrable connection on M (cf. proof of Prop. 7.1,iii)), and is of p-curvature zero. Now let $M_1 := \mathrm{Ker}\left(\nabla^{(1)}\right)$ ($= \bigcap_{0<j<p^1} \mathrm{Ker}(\nabla^{(j)})$, since ∇ is iterative). Then by Cartier's Theorem on the p-curvature (cf. [Kat70], Thm. 5.1), M_1 is an R_1-module and $R \otimes_{R_1} M_1 \to M$ is an isomorphism of R-modules.

Next, we will show that $\nabla(M_1) \subset (\hat{\Omega}_{R/K})^p \otimes_{R_1} M_1$. Since $(\hat{\Omega}_{R/K})^p$ is isomorphic to $\hat{\Omega}_{R_1/K}$ as an algebra by the map $\left(\mathrm{d}^{(i)}x\right)^p \mapsto \mathrm{d}^{(i)}(x^p)$, this means that essentially $\nabla|_{M_1}$ is an integrable iterative connection on the R_1-module M_1. It then follows inductively that $R_l \otimes_{R_{l+1}} M_{l+1} \xrightarrow{\cong} M_l$ and that, essentially, $\nabla|_{M_{l+1}}$ is an integrable iterative connection on the R_{l+1}-module M_{l+1}.

Since M_1 and $\hat{\Omega}_{R/K}$ are locally free, and hence localisation is injective, it suffices to show the statement for the quotient field $F := \mathrm{Quot}(R)$ of R. For simplicity, we again write M and M_1 for what should be $F \otimes_R M$ and $F_1 \otimes_{R_1} M_1$:

Since ∇ is iterative, we only have to show that $\nabla^{(p^l)}(M_1) \subset (\hat{\Omega}_{F/K})^p \otimes_{F_1} M_1$ for all $l \geq 1$. So fix an F_1-basis $\boldsymbol{b} = (b_1, \ldots, b_n)$ of M_1 (written as a row) and let $A_l \in \mathrm{Mat}_n(\hat{\Omega}_{p^l})$ with $\nabla^{(p^l)}(\boldsymbol{b}) = \boldsymbol{b}A_l$.[3] From $0 = {}_{\hat{\Omega}}\nabla^{(p^l)}(\nabla^{(1)}(\boldsymbol{b})) = {}_{\hat{\Omega}}\nabla^{(1)}(\nabla^{(p^l)}(\boldsymbol{b})) = \boldsymbol{b}\mathrm{d}^{(1)}_{\hat{\Omega}}(A_l)$ we conclude $\mathrm{d}^{(1)}_{\hat{\Omega}}(A_l) = 0$. Assume there is an entry $\omega \in \hat{\Omega}_{p^l} \subset F[\mathrm{d}^{(i)}t_j \mid i = 1, \ldots, p^l, j = 1, \ldots, m]$ of A_l with $LT(\omega) = r\mathrm{d}^{(p^l)}t_j$ (for some $r \in F$ and $j \in \{1, \ldots, m\}$). Since $\mathrm{d}^{(1)}_{\hat{\Omega}}(r\mathrm{d}^{(p^l)}t_j) = \mathrm{d}^{(1)}(r)\mathrm{d}^{(p^l)}t_j + r\mathrm{d}^{(p^l+1)}t_j$, and since for all other monomials of ω, the image under $\mathrm{d}^{(1)}_{\hat{\Omega}}$ doesn't contain the variable $\mathrm{d}^{(p^l+1)}t_j$, we obtain $\mathrm{d}^{(1)}_{\hat{\Omega}}(\omega) \neq 0$, a contradiction. So $\omega \in F[\mathrm{d}^{(i)}t_j \mid i = 1, \ldots, p^l - 1, j = 1, \ldots, m]$.

Furthermore, since ∇ is iterative, ${}_{\hat{\Omega}}\nabla^{(p^l(p-1))} \circ \nabla^{(p^l)} = \binom{p^{l+1}}{p^l}\nabla^{(p^{l+1})} = 0$, and therefore

$$0 = {}_{\hat{\Omega}}\nabla^{(p^l(p-1))}(\boldsymbol{b}A_l) = \boldsymbol{b} \cdot \mathrm{d}^{(p^l(p-1))}_{\hat{\Omega}}(A_l) + \sum_{k=0}^{p^l(p-1)-1} \nabla^{(p^l(p-1)-k)}(\boldsymbol{b}) \cdot \mathrm{d}^{(k)}_{\hat{\Omega}}(A_l).$$

If $A_l \notin \mathrm{Mat}_n(F\hat{\Omega}^p)$, then by the previous lemma, $\mathrm{d}^{(p^l(p-1))}(A_l)$ has an entry with leading term $e_0\mathrm{d}^{(i_0+p^l(p-1))}t_{j_0}\left(\mathrm{d}^{(i_0)}t_{j_0}\right)^{e_0-1} \cdot \omega'$ for some $\omega' \in \hat{\Omega}$, $i_0 \leq p^l$ and $j_0 \in \{1, \ldots, m\}$, and the variables occurring in $\mathrm{d}^{(k)}_{\hat{\Omega}}(A_l)$ $(k < p^l(p-1) - 1)$ are less than $\mathrm{d}^{(i_0+p^l(p-1))}t_{j_0}$. Moreover, those occurring in $\nabla^{(p^l(p-1)-k)}(\boldsymbol{b})$ are even less than or equal to $\mathrm{d}^{(p^l(p-1))}t_m$. So we would have ${}_{\hat{\Omega}}\nabla^{(p^l(p-1))}(\boldsymbol{b}A_l) \neq 0$. Therefore $A_l \in \mathrm{Mat}_n(F\hat{\Omega}^p)$.

At last, since $\mathrm{d}^{(1)}_{\hat{\Omega}}(A_l) = 0$, in fact $A_l \in \mathrm{Mat}_n(\hat{\Omega}^p)$, which completes the proof. □

Theorem 8.7. *The category* $\mathbf{Proj}_R$ *of Fc-projective systems over* R *and the category* $\mathbf{ICon_{int}}(R/K)$ *are equivalent. Furthermore, if* R *is an algebraic function field in one variable over* K *and* $\phi \in \mathrm{ID}_K(R)$ *with* $\phi^{(1)} \neq 0$, *then they are also equivalent to the category* $\mathbf{ID}_R$ *of iterative differential modules over* (R, ϕ) *(cf.* [Mat01]*, Ch. 2 and* [MvdP03]*, Ch. 2) and to the category* $\mathbf{ICon}(R/K)$.

Proof. The first statement follows immediately from the previous two propositions, since the given maps are functors that are inverses to each other. The proof of Proposition 8.6 shows that the integrability condition is not necessary when R is an algebraic function field in one variable. So $\mathbf{ICon}(R/K)$ is equivalent to $\mathbf{Proj}_R$ in this case. Furthermore, Matzat and van der Put showed in [Mat01], Thm. 2.8, resp. [MvdP03], Prop. 5.1, that $\mathbf{ID}_R$ is also equivalent to $\mathbf{Proj}_R$. □

Corollary 8.8. *If* K *is algebraically closed and* R *is smooth over* K, *then the category* $\mathbf{ICon_{int}}(R/K)$ *is equivalent to the category of stratified modules over* R.

[3] For simplicity we use vector notations: $\boldsymbol{b}A_l$ denotes the row vector with j-th component $\sum_{i=1}^n (A_l)_{ij}b_i$, and ∇ and $\mathrm{d}_{\hat{\Omega}}$ are always applied to the components of a vector or a matrix. Also we abbreviate $\hat{\Omega}_{F/K}$ by $\hat{\Omega}$.

Proof. By the previous theorem, the category $\mathbf{ICon_{int}}(R/K)$ is equivalent to the category $\mathbf{Proj}_R$ of Fc-projective systems over R. Furthermore under the given assumptions, $\mathbf{Proj}_R$ is equivalent to the category of stratified modules over R, by [Gie75], Thm. 1.3. So the statement follows. □

In the previous section, we have seen that the same corollary holds for $\mathrm{char}(K) = 0$ (cf. Cor. 7.3). However, there is still no proof of this equivalence that works in arbitrary characteristic. Furthermore, it is an open question whether stratifications and integrable iterative connections are equivalent or even related, when R is not smooth over K.

9. Higher Connections on Schemes

Next, we outline a generalisation of modules with iterative connections to modules over schemes. Throughout this section, let K be a perfect field, let X be a nonsingular, geometrically integral K-scheme which is separated and of finite type over K, and let $\mathcal{O}_X$ denote the structure sheaf of X.

Definition 9.1. We define the **sheaf of higher differentials on** X, denoted by $\hat{\Omega}_{X/K}$, to be the sheaf associated to the presheaf given by

$$U \mapsto \hat{\Omega}_{\mathcal{O}_X(U)/K}$$

for each open subset $U \subseteq X$ and by the restriction maps

$$D(\rho_V^U) : \hat{\Omega}_{\mathcal{O}_X(U)/K} \to \hat{\Omega}_{\mathcal{O}_X(V)/K}$$

for all open subsets $V \subseteq U \subseteq X$, as defined in Proposition 3.12, where $\rho_V^U : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is the restriction map of $\mathcal{O}_X$.

Remark 9.2. By Proposition 3.12, for all open subsets $V \subseteq U \subseteq X$, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\mathrm{d}_{\mathcal{O}_X(U)}} & \hat{\Omega}_{\mathcal{O}_X(U)/K} \\ \downarrow{\scriptstyle \rho_V^U} & & \downarrow{\scriptstyle D(\rho_V^U)} \\ \mathcal{O}_X(V) & \xrightarrow{\mathrm{d}_{\mathcal{O}_X(V)}} & \hat{\Omega}_{\mathcal{O}_X(V)/K} \end{array}$$

commutes and so the collection of maps $\mathrm{d}_{\mathcal{O}_X(U)}$ induces a morphism of sheaves of K-algebras $\mathrm{d}_X : \mathcal{O}_X \to \hat{\Omega}_{X/K}$.

Proposition 9.3. *If X is an affine scheme, then the presheaf $U \mapsto \hat{\Omega}_{\mathcal{O}_X(U)/K}$ is in fact a sheaf.*

Proof. The given presheaf is a sheaf if and only if for all open subsets $U \subseteq X$ and all open coverings $\bigcup_{i\in I} U_i = U$, the sequence

$$0 \to \hat{\Omega}_{\mathcal{O}_X(U)/K} \to \prod_{i\in I} \hat{\Omega}_{\mathcal{O}_X(U_i)/K} \to \prod_{i,j\in I} \hat{\Omega}_{\mathcal{O}_X(U_i\cap U_j)/K}$$

is exact. Since this is a sequence of cgas, it suffices to show that the sequence is exact in each homogeneous component.
For every open subset $V \subseteq U$, $\mathcal{O}_X(V)$ is a localisation of $\mathcal{O}_X(U)$ and so by Proposition 3.12, $\hat{\Omega}_{\mathcal{O}_X(V)/K} \cong \mathcal{O}_X(V) \otimes \hat{\Omega}_{\mathcal{O}_X(U)/K}$. By Corollary 3.15, the homogeneous components $(\hat{\Omega}_{\mathcal{O}_X(U)/K})_k$ ($k \in \mathbb{N}$) are projective $\mathcal{O}_X(U)$-modules and therefore tensoring with $(\hat{\Omega}_{\mathcal{O}_X(U)/K})_k$ is exact. So the sequence above is exact in each homogeneous component, if the sequence

$$0 \to \mathcal{O}_X(U) \to \prod_{i\in I} \mathcal{O}_X(U_i) \to \prod_{i,j\in I} \mathcal{O}_X(U_i \cap U_j)$$

is exact. But this is true since $\mathcal{O}_X$ is itself a sheaf. □

As an immediate consequence of this proposition, we have the following corollary:

Corollary 9.4. *For every affine open subset $U \subseteq X$, we have $\hat{\Omega}_{X/K}(U) = \hat{\Omega}_{\mathcal{O}_X(U)/K}$.*

Definition 9.5. Let M be a coherent $\mathcal{O}_X$-module. A **higher connection on** M is a morphism of sheaves $\nabla : M \to \hat{\Omega}_{X/K} \otimes_{\mathcal{O}_X} M$ which locally (i. e. on affine open subsets) is a higher connection in the sense of Section 4. The higher connection ∇ is called **iterative resp. integrable iterative** if ∇ locally is an iterative resp. integrable iterative connection.

Remark 9.6. i) By Corollary 4.5, every coherent $\mathcal{O}_X$-module M that admits a higher connection $\nabla : M \to \hat{\Omega}_{X/K} \otimes_{\mathcal{O}_X} M$ is locally free and of finite rank.

ii) Following the notion of modules with higher connection over rings, the categories of coherent $\mathcal{O}_X$-modules with higher connection, with iterative connection and with integrable iterative connection will be denoted by $\mathbf{HCon}(X/K)$, $\mathbf{ICon}(X/K)$ resp. $\mathbf{ICon_{int}}(X/K)$. By standard methods of algebraic geometry, one obtains that again $\mathbf{HCon}(X/K)$, $\mathbf{ICon}(X/K)$ and $\mathbf{ICon_{int}}(X/K)$ are tensor categories over K and that they are Tannakian categories. And if X has a K-rational point, they are in fact neutral Tannakian categories over K.

10. Picard-Vessiot theory

In Section 6, we showed that the category of modules with higher connection $\mathbf{HCon}(R/K)$ with fibre functor $\boldsymbol{\omega}_K : \mathbf{HCon}(R/K) \to \mathbf{Vect}(K)$ is a neutral Tannakian category over K and that $\mathbf{ICon}(R/K)$ and $\mathbf{ICon_{int}}(R/K)$ are Tannakian subcategories. By Tannaka duality, this means that $\mathbf{HCon}(R/K)$ is equivalent to the category of finite dimensional representations of a certain group scheme and that $\mathbf{ICon}(R/K)$ and $\mathbf{ICon_{int}}(R/K)$ are equivalent to the category of finite dimensional representations of quotients of this group scheme. (In positive characteristic, the group scheme associated to $\mathbf{ICon_{int}}(R/K)$ is isomorphic to the fundamental group scheme for F-divided sheaves $\Pi^{Fdiv}(\mathrm{Spec}(R), \boldsymbol{\omega}_K)$ in [San07], by Thm. 8.7 and Rem. 8.3.) Furthermore, for every module with higher (or iterative or integrable iterative) connection (M, ∇), one obtains the Tannakian Galois group $G_{(M,\nabla)}$, which is the group scheme corresponding to the smallest Tannakian subcategory that contains (M, ∇). In this section, we obtain these Galois group schemes for modules with iterative connection from another point of view, namely as automorphisms of solution rings (so called pseudo Picard-Vessiot rings, or PPV-rings for short). The fact that the automorphism group scheme of a PPV-ring of (M, ∇) is isomorphic to the Tannakian Galois group scheme $G_{(M,\nabla)}$ can be shown in the same manner as in [vdPS03], Thm. 2.33 for differential modules, or as in [Pap08], Sections 3.5 – 4.5, for t-motives, and is sketched in Remark 10.13 at the end of this section.

Some of the constructions and proofs given here will be quite similar to those of T. Dyckerhoff in [Dyc08], who used Galois group schemes for obtaining a differential Galois theory in characteristic zero over non algebraically closed fields of constants. However, we have to deal with an additional phenomenon occurring in positive characteristic, namely inseparability of the extensions and nonreduced group schemes.

Since the Picard-Vessiot theory we provide does not only work for modules with iterative connections, but for a large class of higher derivations, we make the following definition.

Definition 10.1. Let F be a K-algebra and let $\tilde{\Omega}$ be an F-cga. A higher derivation $\theta : F \to \tilde{\Omega}$ will be called **iterable** if the following hold:

(i) For all $k \in \mathbb{N}$ the homogeneous component $\tilde{\Omega}_k$ is generated by $\{\theta^{(k)}(r) \mid r \in F\}$.

(ii) θ can be extended to a continuous endomorphism $\theta_{\tilde{\Omega}} : \tilde{\Omega} \to \tilde{\Omega}$, satisfying the iteration rule $\theta_{\tilde{\Omega}}^{(i)} \circ \theta_{\tilde{\Omega}}^{(j)} = \binom{i+j}{i} \theta_{\tilde{\Omega}}^{(i+j)}$ $(i, j \in \mathbb{N})$, or equivalently satisfying $(a.\theta_{\tilde{\Omega}}) \circ (b.\theta_{\tilde{\Omega}}) = (a+b).\theta_{\tilde{\Omega}}$ for all $a, b \in K^{\mathrm{sep}}$.

Let θ be iterable, let M be an F-module and Θ a higher θ-derivation on M. As for higher connections we can define an endomorphism $\Theta_{\tilde{\Omega}}$ on $\tilde{\Omega}\otimes_F M$ by

$$\Theta_{\tilde{\Omega}}(\omega\otimes x) := \theta_{\tilde{\Omega}}(\omega)\cdot\Theta(x)$$

for all $\omega\in\tilde{\Omega}$ and $x\in M$. The θ-derivation Θ is called **iterable** if $\Theta_{\tilde{\Omega}}$ satisfies the iteration rule $\Theta_{\tilde{\Omega}}^{(i)}\circ\Theta_{\tilde{\Omega}}^{(j)} = \binom{i+j}{i}\Theta_{\tilde{\Omega}}^{(i+j)}$ $(i,j\in\mathbb{N})$.

Example 10.2. The universal derivation $\mathrm{d}_F : F\to\hat{\Omega}_{F/K}$ is an iterable higher derivation with extension $\mathrm{d}_{\hat{\Omega}}$. Other examples are appropriate extensions of the universal derivation to extensions of F (e.g. to PPV-rings R over F for some iterable higher differential equation; cf. Def. 10.5), or the canonical extension of $\mathrm{d}_{K(t)}$ to $K((t))$ (i. e. a higher derivation $\theta : K((t))\to K((t))\otimes_{K(t)}\hat{\Omega}_{K(t)/K}$). Further examples are iterative derivations $\phi : F\to F[[T]]$ with $\phi^{(1)}\neq 0$ (the additional assumption is only necessary to fulfill condition (i)), and also m-variate iterative derivations $\phi : F\to F[[T_1,\dots,T_m]]$ defined by F. Heiderich in his Diplomarbeit (cf. [Hei07]).

Let $\theta_F : F\to\tilde{\Omega}$ be an iterable higher derivation and let L/F be a finite field extension. If L/F is separable, then θ_F extends uniquely to a higher derivation $\theta_L : L\to L\otimes\tilde{\Omega}$ (cf. Prop. 3.7 and Ex. 3.5), which therefore is also iterable. If L/F is not separable, there may not exists an extension of θ_F to L. However, if $\tilde{\Omega}$ has no nilpotent elements, there exists at most one extension, which then is iterable. This relies on the fact that for some $k\geq 0$, L^{p^k} lies in a separable algebraic extension $\tilde{F}$ of F, and hence $\theta_L(s)^{p^k} = \theta_{\tilde{F}}(s^{p^k})$ determines $\theta_L(s)$ uniquely for all $s\in L$.

Remark 10.3. If a higher derivation $\theta : F\to\tilde{\Omega}$ is iterable, the extension $\theta_{\tilde{\Omega}}$ is unique, since $\tilde{\Omega}_k$ is generated by $\theta^{(k)}(F)$ for all k. Furthermore, the iteration rule implies that $\theta_{\tilde{\Omega}}$ is an automorphism of $\tilde{\Omega}$.

From now on, we fix an arbitrary field K, a field F containing K, an F-cga $\tilde{\Omega}$ having no zero-divisors, and an iterable higher derivation $\theta : F\to\tilde{\Omega}$, such that $K=\{t\in F\mid\theta(t)=t\}$.

We introduce some notation.

Definition 10.4. A **θ-ring** is an F-algebra R together with an iterable higher derivation $\theta_R : R\to R\otimes\tilde{\Omega}$ that extends θ. The pair (R,θ_R) is called a **θ-field**, if R is a field. The set $C_R := \{r\in R\mid\theta_R(r)=r\otimes 1\}$ is called the **ring of constants** of (R,θ_R). An ideal $I\trianglelefteq R$ is called a **θ-ideal** if for all $k\in\mathbb{N}$, $\theta_R^{(k)}(I)\subseteq I\otimes\tilde{\Omega}_k$; R is **θ-simple** if R has no proper nontrivial θ-ideals. Localisations of θ-rings are again θ-rings by $\theta(\frac{r}{s}) := \theta(r)\theta(s)^{-1}$ (as for iterative derivations one easily shows that these extensions are again iterable). The tensor product $R\otimes_F\tilde{R}$ of two θ-rings R and $\tilde{R}$ is a θ-ring by

$$\theta_{R\otimes\tilde{R}}^{(k)}(r\otimes\tilde{r}) := \sum_{i+j=k}\theta_R^{(i)}(r)\cdot\theta_{\tilde{R}}^{(j)}(\tilde{r})\in R\otimes\tilde{R}\otimes\tilde{\Omega},$$

for all $k\geq 0$, $r\in R$ and $\tilde{r}\in\tilde{R}$. A homomorphism of θ-rings $f : R\to\tilde{R}$ is called a **θ-homomorphism** if $\theta_{\tilde{R}}\circ f = (f\otimes\mathrm{id}_{\tilde{\Omega}})\circ\theta_R$. The set of all θ-homomorphisms is denoted by $\mathrm{Hom}^\theta(R,\tilde{R})$. Furthermore for θ-rings $R\geq\tilde{R}\geq F$, the set of θ-automorphisms of R that leave the elements of $\tilde{R}$ fixed, is denoted by $\mathrm{Aut}^\theta(R/\tilde{R})$.
Given a θ-ring R and a K-algebra L, the tensor product $R\otimes_K L$ can be given the structure of a θ-ring by $\theta_{R\otimes_K L}(r\otimes a) = \theta_R(r)\otimes a$ $(r\in R,\ a\in L)$. We say that θ_R is **extended trivially** to $R\otimes_K L$.

Let $A = \sum_{k=0}^{\infty} A_k \in \mathrm{GL}_n(\tilde{\Omega})$ with $A_0 = \mathbf{1}_n$ (identity matrix) and for all $k, l \in \mathbb{N}$, $\binom{k+l}{l} A_{k+l} = \sum_{i+j=l} \theta^{(i)}(A_k) \cdot A_j \in \mathrm{Mat}(n \times n), \tilde{\Omega}_{k+l})$. Then an equation

$$(*) \qquad \theta(\boldsymbol{y}) = A\boldsymbol{y},$$

where $\boldsymbol{y}$ is a vector of indeterminates, is called an **iterable higher differential equation**. Notice that the condition on the matrices A_k is the same as to say that for an F-vector space M with basis $\boldsymbol{b} = (b_1, \dots, b_n)$, the θ-derivation Θ defined by $\Theta(\boldsymbol{b}) = \boldsymbol{b}A^{-1}$ is iterable, and a vector $\boldsymbol{x} \in F^n$ is a solution of the equation (*), if and only if $\boldsymbol{bx} \in M$ is constant, i. e. satisfies $\Theta(\boldsymbol{bx}) = \boldsymbol{bx}$.

Definition 10.5. A θ-ring (R, θ_R) is called a **pseudo Picard-Vessiot ring** (PPV-ring) for $\theta(\boldsymbol{y}) = A\boldsymbol{y}$, if the following holds:

i) R is θ-simple.
ii) There is a fundamental solution matrix $Y \in \mathrm{GL}_n(R)$, i. e. an invertible matrix satisfying $\theta_R(Y) = AY$.
iii) As an F-algebra, R is generated by the coefficients of Y and by $\det(Y)^{-1}$.
iv) $C_R = C_F = K$.

The quotient field $E = \mathrm{Quot}(R)$ is called a **pseudo PV-field** for the equation (*).

Remark 10.6. Analogous to [Mat01], Prop. 3.2, resp. [MvdP03], Lemma 3.2, one shows that R is an integral domain, so the quotient field E exists. Furthermore, for every θ-simple θ-ring which is finitely generated as an F-algebra, its constants are algebraic over K. Hence if K is algebraically closed, a PPV-ring for the equation (*) is given by $R = S/P$, where $S := F[X_{ij}, \det(X)^{-1} \mid i, j = 1, \dots, n]$ is a θ-ring by $\theta_S(X) := AX$ and $P \trianglelefteq S$ is a maximal θ-ideal. Hence in this case PPV-rings always exist and – by a similar proof as for [Mat01], Thm. 3.4 – are unique up to θ-isomorphisms.

For a PPV-ring R/F we define the functor

$$\underline{\mathrm{Aut}}^{\theta}(R/F) : (\mathsf{Algebras}/K) \to (\mathsf{Groups}), L \mapsto \mathrm{Aut}^{\theta}(R_L/F_L)$$

where $F_L := F \otimes_K L$, $R_L := R \otimes_K L$ and θ resp. θ_R is extended trivially to F_L resp. R_L.

We will show that the functor $\underline{\mathrm{Aut}}^{\theta}(R/F)$ is representable by a K-algebra of finite type and hence is an affine group scheme of finite type over K.

Lemma 10.7. *Let R be a θ-simple θ-ring with $C_R = K$, let L be a finitely generated K-algebra and $R_L := R \otimes_K L$, with θ-structure trivially extended from R. Then there is a bijection*

$$\begin{array}{rcl} \mathcal{I}(L) & \longleftrightarrow & \mathcal{I}^{\theta}(R_L) \\ I & \longmapsto & R_L(1 \otimes_K I) = R \otimes_K I \\ J \cap (1 \otimes_K L) & \longleftarrow & J \end{array}$$

between the ideals of L and the θ-ideals of R_L.

Proof. Obviously, both maps are well defined, so we only have to show that they are inverses to each other.

(i) We need to show that for $I \in \mathcal{I}(L)$, we have $(R \otimes_K I) \cap (1 \otimes_K L) = I$.

It is clear that I is contained in the left side. For the other inclusion, let $\{e_i \mid i \in \tilde{N}\}$ be a K-basis of I. Then $(R \otimes_K I)$ is a free R-module with the same basis and an element $f = \sum_{i \in \tilde{N}} r_i \otimes e_i \in (R \otimes_K I)$ is constant, if and only if all r_i are constant, i. e. if $f \in I$.

(ii) We need to show that for $J \in \mathcal{I}_{\theta}(R_L)$, we have $R \otimes_K (J \cap (1 \otimes_K L)) = J$.

It is clear that the left side is contained in J, since J is an ideal. For the other inclusion, let $\{e_i \mid i \in N\}$ be a K-basis of L, where N denotes an index set. Then $\{e_i \mid i \in N\}$ also is a basis for the free R-module R_L.

For any subset $N_0 \subseteq N$ and $i_0 \in N_0$, let $\mathfrak{A}_{N_0,i_0} \trianglelefteq R$ denote the ideal of all $r \in R$ such that there exists an element $g = \sum_{j\in N_0} s_j \otimes e_j \in J$ with $s_{i_0} = r$. We will show that $\mathfrak{A}_{N_0,i_0}$ is a θ-ideal of R and so by θ-simplicity of R is equal to (0) or to R:
Let $r \in \mathfrak{A}_{N_0,i_0}$, $g = \sum_{j\in N_0} s_j \otimes e_j \in J$ with $s_{i_0} = r$ and $k \in \mathbb{N}$. We have to show that $\theta_R^{(k)}(r) \in \mathfrak{A}_{N_0,i_0} \otimes \tilde{\Omega}_k$. So let $\{\omega_\alpha\}$ be an F-basis of $\tilde{\Omega}_k$ and let $g_\alpha \in R_L$ such that $\theta^{(k)}(g) = \sum_\alpha g_\alpha \otimes \omega_\alpha$. Since J is a θ-ideal, we have $g_\alpha \in J$. On the other hand, let $\theta^{(k)}(s_j) = \sum_\alpha s_{\alpha,j} \otimes \omega_\alpha$ for some $s_{\alpha,j} \in R$, then

$$\theta^{(k)}(g) = \sum_{j\in N_0} \theta^{(k)}(s_j) \otimes e_j = \sum_{j\in N_0} \sum_\alpha s_{\alpha,j} \otimes e_j \otimes \omega_\alpha.$$

So $g_\alpha = \sum_{j\in N_0} s_{\alpha,j} \otimes e_j$ and therefore $s_{\alpha,i_0} \in \mathfrak{A}_{N_0,i_0}$. Hence,

$$\theta_R^{(k)}(r) = \theta_R^{(k)}(s_{i_0}) = \sum_\alpha s_{\alpha,i_0} \otimes \omega_\alpha \in \mathfrak{A}_{N_0,i_0} \otimes \tilde{\Omega}_k.$$

Now, let $N_0 \subset N$ be a subset, which is minimal for the property that $\mathfrak{A}_{N_0,i_0} \neq (0)$ for at least one index $i_0 \in N_0$ (minimal in the lattice of subsets). So there exists $f = \sum_{j\in N_0} r_j \otimes e_j \in J$ with $r_{i_0} = 1$ and by minimality of N_0, for all $k > 0$ we obtain $\theta^{(k)}(f) = \sum_{j\in N_0, j\neq i_0} \theta^{(k)}(r_j) \otimes e_j = 0$. Hence $f \in J \cap (1 \otimes_K L)$.
Now let $g = \sum_{j\in N} s_j \otimes e_j \in J$ be an arbitrary element and denote by N_1 the set of indices j where $s_j \neq 0$. By definition, for all $i \in N_1$, $\mathfrak{A}_{N_1,i} \neq (0)$. Hence there is $N_0 \subseteq N_1$ minimal as above, $i_0 \in N_0$ and $f = \sum_{j\in N_0} r_j \otimes e_j \in J \cap (1 \otimes_K L)$ with $r_{i_0} = 1$. By induction on the magnitude of N_1, we may assume that $g - s_{i_0} f \in R \otimes_K (J \cap (1 \otimes_K L)) \subset J$. So $g = (g - s_{i_0} f) + s_{i_0} f \in R \otimes_K (J \cap (1 \otimes_K L))$ and hence $R \otimes_K (J \cap (1 \otimes_K L)) = J$. □

Proposition 10.8. *Let R be a PPV-ring for the equation (*) and let $T \geq F$ be a θ-simple θ-ring with $C_T = K$ such that there exists a fundamental solution matrix $Y \in \mathrm{GL}_n(T)$. Then there exists a finitely generated K-algebra U and a T-linear θ-isomorphism*

$$\gamma_T : T \otimes_F R \to T \otimes_K U,$$

where (again) the θ-structure is extended trivially to $T \otimes_K U$.
(Actually U is isomorphic to the ring of constants of $T \otimes_F R$.)

Proof. R is obtained as a quotient of $F[X, X^{-1}]$ with θ-structure given by $\theta(X) = AX$ (for short we write $F[X, X^{-1}]$ instead of $F[X_{ij}, \det(X)^{-1}]$) by a maximal θ-ideal $P \trianglelefteq F[X, X^{-1}]$. Let $L := K[Z, Z^{-1}] = K[\mathrm{GL}_n]$. We then define a T-linear homomorphism

$$\gamma_T : T \otimes_F F[X, X^{-1}] \to T \otimes_K K[Z, Z^{-1}]$$

by $X_{ij} \mapsto \sum_{k=1}^n Y_{ik} \otimes Z_{kj}$ (or $X \mapsto Y \otimes Z$ for short). γ_T is indeed an isomorphism and – if we extend the θ-structure trivially to $T \otimes_K K[Z, Z^{-1}]$ – γ_T is a θ-isomorphism.
By the previous lemma, the θ-ideal $\gamma_T(T \otimes P)$ is equal to $T \otimes I$ for an ideal $I \trianglelefteq K[Z, Z^{-1}]$. So for $U := K[Z, Z^{-1}]/I$, γ_T induces a θ-isomorphism

$$\gamma_T : T \otimes_F R \to T \otimes_K U.$$

□

Proposition 10.9. *Let R be a PPV-ring over F. Then the group functor $\underline{\mathrm{Aut}}^\theta(R/F)$ is represented by the finitely generated K-algebra $U = C_{R\otimes_F R}$, i. e. $\underline{\mathrm{Aut}}^\theta(R/F)$ is an affine group scheme of finite type over K, which we call the* **Galois group scheme** *$\underline{\mathrm{Gal}}(R/F)$ of R over F or also the Galois group scheme $\underline{\mathrm{Gal}}(E/F)$ of $E := \mathrm{Quot}(R)$ over F.*

Proof. First we show that for every K-algebra L any F_L-linear θ-homomorphism $f : R_L \to R_L$ is an isomorphism: The kernel of such a homomorphism f is a θ-ideal of R_L and so by Lemma 10.7, it is generated by constants, i. e. elements in $1 \otimes L$. But f is L-linear and so $\mathrm{Ker}(f) = \{0\}$. If $X \in \mathrm{GL}_n(R)$ is a fundamental matrix, then $f(X) \in \mathrm{GL}_n(R_L)$ is also a fundamental matrix and so there is a matrix $D \in \mathrm{GL}_n(C_{R_L}) = \mathrm{GL}_n(L)$ such that $X = f(X)D = f(XD)$. Hence $X_{ij}, \det(X)^{-1} \in \mathrm{Im}(f)$ $(i, j = 1, \ldots, n)$, and since R is generated over F by the X_{ij} and by $\det(X)^{-1}$, the homomorphism f is also surjective.

Using the isomorphism $\gamma := \gamma_R$ of Prop. 10.8, for a K-algebra L, we obtain a chain of isomorphisms:

$$\begin{aligned}
\mathrm{Aut}^\theta(R_L/F_L) &= \mathrm{Hom}^\theta_{F_L}(R_L, R_L) = \mathrm{Hom}^\theta_{F_L}(F_L \otimes_F R, R_L) \\
&\cong \mathrm{Hom}^\theta_R(R \otimes_F R, R_L) \\
&\cong \mathrm{Hom}^\theta_R(R \otimes_K U, R_L) \\
&\cong \mathrm{Hom}^\theta_K(U, R_L) \\
&\cong \mathrm{Hom}_K(U, L)
\end{aligned}$$

So U is representing the functor $\underline{\mathrm{Aut}}^\theta(R/F)$. □

Remark 10.10. A careful look at the isomorphisms in the previous proof shows that the universal object $\mathrm{id}_U \in \mathrm{Hom}_K(U, U)$ corresponds to the θ-automorphism $\rho \otimes \mathrm{id}_U : R \otimes_K U \to R \otimes_K U$, where $\rho = \gamma_R \circ (1 \otimes \mathrm{id}_R) : R \to R \otimes_F R \to R \otimes_K U$. Furthermore we obtain that the action of $g \in \underline{\mathrm{Aut}}^\theta(R/F)(L) = \mathrm{Hom}_K(U, L)$ on $r \in R$ is given by

$$g.r = (\mathrm{id}_R \otimes g)\big(\gamma_R(1 \otimes r)\big) \in R \otimes_K L.$$

Corollary 10.11. *Let R be a PPV-ring over F and $\mathcal{G} := \underline{\mathrm{Gal}}(R/F)$ the Galois group scheme of R. Then* $\mathrm{Spec}(R)$ *is a $\mathcal{G}_F$-torsor.*

Proof. The isomorphism $\gamma = \gamma_R$ of Proposition 10.8 determines an isomorphism of schemes

$$\mathrm{Spec}(\gamma) : \mathrm{Spec}(R) \times_F \mathcal{G}_F = \mathrm{Spec}(R) \times_K \mathcal{G} \to \mathrm{Spec}(R) \times_F \mathrm{Spec}(R).$$

By the previous remark and R-linearity of γ, the composition of $\mathrm{Spec}(\gamma)$ with the second projection is the morphism which describes the action of $\mathcal{G}_F$ on $\mathrm{Spec}(R)$, and the composition of $\mathrm{Spec}(\gamma)$ with the first projection is equal to the first projection $\mathrm{Spec}(R) \times_F \mathcal{G}_F \to \mathrm{Spec}(R)$. In other words, $\mathrm{Spec}(R)$ is a $\mathcal{G}_F$-torsor. □

The next proposition shows that being a torsor indicates a θ-simple θ-ring to be a PPV-ring.

Proposition 10.12. *Let R/F be a θ-simple θ-ring with constants $C_R = K$. Further let $\mathcal{G} \leq \mathrm{GL}_{n,K}$ be an affine group scheme over K and assume that* $\mathrm{Spec}(R)$ *is a $\mathcal{G}_F$-torsor such that the corresponding isomorphism $\gamma : R \otimes_F R \to R \otimes_K K[\mathcal{G}]$ is a θ-isomorphism. Then R is a PPV-ring over F.*

Proof. Since $\mathrm{Spec}(R)$ is a $\mathcal{G}_F$-torsor, the fibration $\mathrm{Spec}(R) \times^{\mathcal{G}_F} \mathrm{GL}_{n,F}$ is a $\mathrm{GL}_{n,F}$-torsor. (The scheme $\mathrm{Spec}(R) \times^{\mathcal{G}_F} \mathrm{GL}_{n,F}$ is obtained as the quotient of the direct product by the action of $\mathcal{G}_F$ given by $(x, h).g := (xg, g^{-1}h)$, and is a right $\mathrm{GL}_{n,F}$-scheme by the action on the second factor.) By Hilbert 90, every $\mathrm{GL}_{n,F}$-torsor is trivial, i. e. we have a $\mathrm{GL}_{n,F}$-equivariant isomorphism $\mathrm{Spec}(R) \times^{\mathcal{G}_F} \mathrm{GL}_{n,F} \to \mathrm{GL}_{n,F}$. Then the closed embedding $\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(R) \times^{\mathcal{G}_F} \mathrm{GL}_{n,F} \to \mathrm{GL}_{n,F}$ induces an epimorphism $F[X, X^{-1}] \to R$ which is $\mathcal{G}_F$-equivariant. Let the image of X be denoted by Y. We then obtain that the action of $\mathcal{G}$ on Y is given by $Y \mapsto Y \cdot g$ for any L-valued point $g \in \mathcal{G}(L) \subset \mathrm{GL}_n(L)$. Since by assumption for every K-algebra L, the action of $\mathcal{G}(L)$ commutes with θ, the matrix $\theta(Y)Y^{-1}$ is $\mathcal{G}$-invariant. So $\theta(Y)Y^{-1} =: A \in \mathrm{GL}_n(\tilde{\Omega})$, and Y is a fundamental solution matrix for the equation $\theta(\boldsymbol{y}) = A\boldsymbol{y}$. Hence R is a PPV-ring. □

Remark 10.13. As indicated in the beginning of this section, the Tannakian Galois group scheme $G_{(M,\nabla)}$ of a module with iterative connection and the Galois group scheme of a PPV-extension for (M,∇) are isomorphic. We now sketch this isomorphism.

So let K be a perfect field, and let S be a regular integral domain which is the localisation of a finitely generated K-algebra, and such that there is a maximal ideal $\mathfrak{m} \trianglelefteq S$ with $S/\mathfrak{m} \cong K$. Furthermore, let $(M,\nabla) \in \mathbf{ICon}(S/K)$, and let $F := \mathrm{Quot}(S)$ denote the quotient field of S and $\theta := \mathrm{d}_F : F \to \hat{\Omega}_{F/K}$ the universal derivation of F. Since M is a locally free module (cf. Cor. 4.5), by [Hart77], Ch.II, Lemma 8.9, there exists a basis $\boldsymbol{b} := (b_1,\dots,b_n)$ of the F-vector space $F \otimes_S M$ with $b_i \in M$ $(i=1,\dots,n)$, and such that the residue classes in $M/\mathfrak{m}M$ form a K-basis of $M/\mathfrak{m}M$. We assume that there exists a PPV-ring R for the corresponding iterable differential equation $\nabla(\boldsymbol{bx}) = \boldsymbol{bx}$ with fundamental solution matrix $Y \in \mathrm{GL}_n(R)$.

For obtaining the correspondence, we fix an isomorphism of R-modules $\varphi : R \otimes_S M \to R \otimes_S M$ given by $\varphi(\boldsymbol{b}) = \boldsymbol{b}Y$. The correspondence is then given as follows:

For any K-algebra L (with trivial θ-structure), an element $\sigma \in G_{(M,\nabla)}(L)$ is determined by $\sigma_M \in \mathrm{GL}(L \otimes_K \boldsymbol{\omega}_K(M))$ which can be identified with a matrix $D_\sigma \in \mathrm{GL}_n(L)$ by $\sigma_M(\boldsymbol{b}) = \boldsymbol{b}D_\sigma$. So σ_M induces an $(R\otimes_K L)$-linear automorphism $\tilde{\sigma}_M$ of $(R \otimes_K L) \otimes_S M$ by $\boldsymbol{b} \mapsto \boldsymbol{b}D_\sigma$ and we obtain a θ-isomorphism $\hat{\sigma} := \varphi \circ \tilde{\sigma}_M \circ \varphi^{-1}$ of $(R \otimes_K L) \otimes_S M$ mapping the constant basis $\boldsymbol{b}Y$ to the constant basis $\boldsymbol{b}YD_\sigma$. One shows that this induces a θ-isomorphism of $R \otimes_K L$ over $F \otimes_K L$ given by $Y \mapsto YD_\sigma$, i.e. an element of $\underline{\mathrm{Gal}}(R/F)(L)$.

On the other hand, every θ-isomorphism of $R \otimes_K L$ over $F \otimes_K L$ is given by $Y \mapsto YD$ for some $D \in \mathrm{GL}_n(L)$ and by reversing the steps above, one obtains an element $\sigma_M \in \mathrm{GL}(L \otimes_K \boldsymbol{\omega}_K(M))$, and one shows that indeed σ_M defines an element $\sigma \in G_{(M,\nabla)}(L)$.

11. Galois correspondence

In this section, we prove a Galois correspondence between all intermediate θ-fields of a PPV-extension E/F and all closed subgroup schemes of the Galois group scheme $\underline{\mathrm{Gal}}(E/F)$. This includes θ-fields over which E is inseparable and nonreduced subgroup schemes, and hence is an improvement of the correspondence given by Matzat and van der Put (cf. [MvdP03], Thm. 3.5), which only considers reduced subgroup schemes and intermediate fields over which E is separable. (However, this separability condition is missing in their statement, but has been added for example in [Ama07], Thm. 2.5, and in [Hei07], Thm. 6.5.2.)

Remark 11.1. One should also mention the work of M. Takeuchi (cf. [Tak89]) on a Picard-Vessiot theory of so called C-ferential fields (a huge class of fields with extra structure to which the iterative differential fields and the θ-fields defined below belong). But Takeuchi used a definition of a PV-extension, that differs from ours and the “usual” one. The main difference is that instead of requiring the existence of a fundamental solution matrix he imposed a condition which is equivalent to an isomorphism $R \otimes_F R \cong R \otimes_K C_{R\otimes_F R}$. (Here F denotes a C-ferential field, R a PV-ring over F, $K = C_F = C_R$ the field of constants of F and R, and $C_{R\otimes_F R}$ the constants of $R \otimes_F R$; cf. [Tak89], Def. 2.3). Showing that this isomorphism also exists by our definition was the statement of Proposition 10.8. In fact, Proposition 10.8 and Proposition 10.12 imply that both definitions coincide in the case of θ-fields. Our Galois correspondence is quite the same as the one given by Takeuchi (cf. [Tak89], Thm 2.10), but we give maps in both directions (Takeuchi only constructed the subgroup scheme corresponding to an intermediate field) and also include the correspondence of the separability condition and the reducedness condition (separability and reducedness are not mentioned at all in Takeuchi's work).

In order to provide the Galois correspondence for PPV-extensions, we need a functorial definition of invariants. Let S be a K-algebra and $\mathcal{H}/K$ be a subgroup functor of the functor $\underline{\mathrm{Aut}}(S/K)$, i. e. for every K-algebra L, the set $\mathcal{H}(L)$ is a group acting on S_L and this action is functorial in L. An element $s \in S$ is then called **invariant** if for all L, the element $s \otimes 1 \in S_L$ is invariant under $\mathcal{H}(L)$. The ring of invariants is denoted by $S^{\mathcal{H}}$. (In [Jan03], I.2.10 the invariant elements are called "fixed points".) Let $E = \mathrm{Quot}(S)$ be the localisation of S by all non zero divisors. We call an element $e = \frac{r}{s} \in E$ **invariant** under $\mathcal{H}$, if for all K-algebras L and all $h \in \mathcal{H}(L)$,

$$h.(r \otimes 1) \cdot (s \otimes 1) = (r \otimes 1) \cdot h.(s \otimes 1) \in S \otimes_K L.$$

The ring of invariants of E is denoted by $E^{\mathcal{H}}$. One can easily verify that this definition of an invariant element $e \in E$ is independent of the chosen representation $\frac{r}{s}$.

Remark 11.2. One has to take care that in general the group functor $\underline{\mathrm{Aut}}(S/K)$ is not a subgroup functor of $\underline{\mathrm{Aut}}(E/K)$, because not every automorphism $S \otimes_K L \to S \otimes_K L$ can be extended to an automorphism $E \otimes_K L \to E \otimes_K L$. Hence a subgroup functor $\mathcal{H}$ of $\underline{\mathrm{Aut}}(S/K)$ does not have to be a subgroup functor of $\underline{\mathrm{Aut}}(E/K)$. That is why we use this more complicated definition of the invariants $E^{\mathcal{H}}$.

In the following, let R be a PPV-ring over F, $E = \mathrm{Quot}(R)$ its quotient field and $\mathcal{G} = \underline{\mathrm{Gal}}(R/F)$ the Galois group scheme of R over F.

Lemma 11.3. *Let $\mathcal{H} \le \mathcal{G}$ be a closed subgroup scheme and let $\pi^{\mathcal{G}}_{\mathcal{H}} : K[\mathcal{G}] \to K[\mathcal{H}]$ denote the epimorphism corresponding to the inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$. Then an element $\frac{r}{s} \in E$ is invariant under the action of $\mathcal{H}$ if and only if $r \otimes s - s \otimes r$ is in the kernel of the map*

$$(\mathrm{id}_R \otimes \pi^{\mathcal{G}}_{\mathcal{H}}) \circ \gamma : R \otimes_F R \to R \otimes_K K[\mathcal{H}].$$

Proof. An element $\frac{r}{s} \in E$ is invariant under the action of $\mathcal{H}$ if and only if it is invariant under the universal element in $\mathcal{H}$, namely $\pi^{\mathcal{G}}_{\mathcal{H}} \in \mathcal{G}(K[\mathcal{H}])$. By Remark 10.10 and R-linearity of γ, we have

$$(\mathrm{id}_R \otimes \pi^{\mathcal{G}}_{\mathcal{H}})\big(\gamma(r \otimes s)\big) = (r \otimes 1) \cdot \pi^{\mathcal{G}}_{\mathcal{H}}.(s \otimes 1) \in R \otimes_K K[\mathcal{H}].$$

Hence $r \otimes s - s \otimes r$ is in the considered kernel if and only if $\frac{r}{s}$ is invariant under $\mathcal{H}$. □

Theorem 11.4. *For every closed subgroup scheme $\mathcal{H} \le \mathcal{G}$, the ring $E^{\mathcal{H}}$ is a θ-field. Furthermore we have $E^{\mathcal{H}} = F$ if and only if $\mathcal{H} = \mathcal{G}$.*

Proof. By the previous lemma, it is obvious that $E^{\mathcal{H}}$ is a field. Next let $\frac{r}{s} \in E^{\mathcal{H}}$. Then for all $k \in \mathbb{N}$, we have

$$\begin{aligned} & \theta^{(k)}(r \otimes s - s \otimes r) \cdot (s^k \otimes s^k) \\ = & \sum_{i_1+i_2+i_3=k} \theta^{(i_1)}(\frac{r}{s}) s^k \theta^{(i_2)}(s) \otimes \theta^{(i_3)}(s) s^k - \theta^{(i_2)}(s) s^k \otimes \theta^{(i_1)}(\frac{r}{s}) s^k \theta^{(i_3)}(s) \\ = & \sum_{i_1+i_2+i_3=k} \left(\theta^{(i_2)}(s) \otimes \theta^{(i_3)}(s)\right) \left(\theta^{(i_1)}(\frac{r}{s}) s^k \otimes s^k - s^k \otimes \theta^{(i_1)}(\frac{r}{s}) s^k\right) \\ = & \sum_{i+j=k} \theta^{(i)}(s \otimes s) \left(\theta^{(j)}(\frac{r}{s}) s^k \otimes s^k - s^k \otimes \theta^{(j)}(\frac{r}{s}) s^k\right). \end{aligned}$$

The left hand side lies in $\mathrm{Ker}((\mathrm{id}_R \otimes \pi^{\mathcal{G}}_{\mathcal{H}}) \circ \gamma) \otimes \tilde{\Omega}_k$, since the kernel is a θ-ideal. So by induction, we obtain $(s \otimes s)\left(\theta^{(k)}(\frac{r}{s}) s^k \otimes s^k - s^k \otimes \theta^{(k)}(\frac{r}{s}) s^k\right) \in \mathrm{Ker}((\mathrm{id}_R \otimes \pi^{\mathcal{G}}_{\mathcal{H}}) \circ \gamma) \otimes \tilde{\Omega}_k$ and hence $\theta^{(k)}(\frac{r}{s}) \in E^{\mathcal{H}} \otimes \tilde{\Omega}_k$.

For the second statement: If $\mathcal{H} = \mathcal{G}$, then $\pi^{\mathcal{G}}_{\mathcal{H}} = \mathrm{id}_{K[\mathcal{G}]}$ and the considered kernel is trivial. Hence $r \otimes s = s \otimes r \in R \otimes_F R$ for all $\frac{r}{s} \in E^{\mathcal{G}}$. So $r = c \cdot s$ for an appropriate element $c \in F$, i.e. $\frac{r}{s} = c \in F$.

Assume $\mathcal{H} \lneq \mathcal{G}$. Since $\mathcal{X} = \mathrm{Spec}(R)$ is a $\mathcal{G}_F$-torsor, the quotient scheme $\mathcal{X}/\mathcal{G}_F$ is equal to $\mathrm{Spec}(F)$, in particular it is a scheme, and since $\mathcal{G}_F$ and $\mathcal{H}_F$ are affine, $\mathcal{G}_F/\mathcal{H}_F$ also is a scheme. So by

[Jan03],I.5.16.(1), $\mathcal{X}/\mathcal{H}_F \cong \mathcal{X} \times^{\mathcal{G}_F} (\mathcal{G}_F/\mathcal{H}_F)$ is a scheme, too. Let $\overline{\mathcal{U}} \subset \mathcal{X}/\mathcal{H}_F$ be an arbitrary affine open subset and $\mathcal{U} = \mathrm{pr}^{-1}(\overline{\mathcal{U}}) \subset \mathcal{X}$ its inverse image, where $\mathrm{pr} : \mathcal{X} \to \mathcal{X}/\mathcal{H}_F$ denotes the canonical projection. Then we get a monomorphism $\mathrm{pr}_* : \mathcal{O}_{\mathcal{X}/\mathcal{H}_F}(\overline{\mathcal{U}}) \to \mathcal{O}_{\mathcal{X}}(\mathcal{U})$, whose image is $\mathcal{O}_{\mathcal{X}}(\mathcal{U})^{\mathcal{H}}$. If $E^{\mathcal{H}} = F$, then also $\mathcal{O}_{\mathcal{X}}(\mathcal{U})^{\mathcal{H}} = F$. So for every open affine subset $\overline{\mathcal{U}} \subset \mathcal{X}/\mathcal{H}_F$, we would have $\mathcal{O}_{\mathcal{X}/\mathcal{H}_F}(\overline{\mathcal{U}}) = F$, i. e. $\overline{\mathcal{U}} \cong \mathrm{Spec}(F)$ is a single point. Hence $\mathcal{X}/\mathcal{H}_F = \mathrm{Spec}(F)$ which contradicts the assumption $\mathcal{H} \lneq \mathcal{G}$. □

Theorem 11.5. (Galois correspondence)

i) *There is an antiisomorphism of the lattices*

$$\mathfrak{H} := \{\mathcal{H} \mid \mathcal{H} \leq \mathcal{G} \text{ closed subgroup schemes of } \mathcal{G}\}$$

and

$$\mathfrak{M} := \{M \mid F \leq M \leq E \text{ intermediate } \theta\text{-fields}\}$$

given by $\Psi : \mathfrak{H} \to \mathfrak{M}, \mathcal{H} \mapsto E^{\mathcal{H}}$ *and* $\Phi : \mathfrak{M} \to \mathfrak{H}, M \mapsto \underline{\mathrm{Gal}}(RM/M)$.

ii) *If* $\mathcal{H} \leq \mathcal{G}$ *is normal, then* $E^{\mathcal{H}} = \mathrm{Quot}(R^{\mathcal{H}})$ *and* $R^{\mathcal{H}}$ *is a PPV-ring over* F *with Galois group scheme* $\underline{\mathrm{Gal}}(R^{\mathcal{H}}/F) \cong \mathcal{G}/\mathcal{H}$.

iii) *If* $M \in \mathfrak{M}$ *is stable under the action of* $\mathcal{G}$, *then* $\mathcal{H} := \Phi(M)$ *is a normal subgroup scheme of* $\mathcal{G}$, M *is a PPV-extension of* F *and* $\underline{\mathrm{Gal}}(R \cap M/F) \cong \mathcal{G}/\mathcal{H}$.

iv) *For* $\mathcal{H} \in \mathfrak{H}$, *the extension* $E/E^{\mathcal{H}}$ *is separable if and only if* $\mathcal{H}$ *is reduced.*

Proof. i) Let $M \in \mathfrak{M}$ be an intermediate θ-field. Then the composite $RM \subseteq E$ of R and M is a PPV-ring over M. Furthermore, the canonical θ-epimorphism $RM \otimes_F R \to RM \otimes_M RM$ gives rise to a θ-epimorphism

$$RM \otimes_K K[\mathcal{G}] \xrightarrow{\gamma_{RM}^{-1}} RM \otimes_F R \to RM \otimes_M RM.$$

By Lemma 10.7, the kernel of this epimorphism is given by $RM \otimes_K I$ for an ideal $I \trianglelefteq K[\mathcal{G}]$. Let $\mathcal{H}$ denote the closed subscheme of $\mathcal{G}$ defined by I, then γ_{RM} induces an isomorphism

$$RM \otimes_M RM \xrightarrow{\cong} RM \otimes_K K[\mathcal{H}].$$

By construction, this isomorphism is the isomorphism γ for the base field M. Hence the subscheme $\mathcal{H}$ equals the Galois group scheme $\underline{\mathrm{Gal}}(RM/M)$. So $\underline{\mathrm{Gal}}(RM/M)$ is indeed a closed subgroup scheme of $\mathcal{G}$.

From Theorem 11.4 – applied to the extension E/M – we see that $E^{\underline{\mathrm{Gal}}(RM/M)} = M$, so $\Psi \circ \Phi = \mathrm{id}_{\mathfrak{M}}$. On the other hand, for given $\mathcal{H} \in \mathfrak{H}$ and $M := E^{\mathcal{H}}$, we obtain a θ-epimorphism $RM \otimes_M RM \to RM \otimes_K K[\mathcal{H}]$ induced from γ_{RM}. This gives $\mathcal{H}$ as a closed subgroup scheme of $\underline{\mathrm{Gal}}(RM/M)$. But $(\mathrm{Quot}(RM))^{\mathcal{H}} = E^{\mathcal{H}} = M$, and so by Theorem 11.4, we have $\mathcal{H} = \underline{\mathrm{Gal}}(RM/M)$. Hence $\Phi \circ \Psi = \mathrm{id}_{\mathfrak{H}}$.

ii) Let $\mathcal{H} \leq \mathcal{G}$ be normal. The isomorphism γ is $\mathcal{H}$-equivariant (by the action of $\mathcal{H}$ on the right factor) and hence we get a θ-isomorphism

$$R \otimes_F R^{\mathcal{H}} \cong R \otimes_K K[\mathcal{G}]^{\mathcal{H}}.$$

Since $\mathcal{H}$ is normal, $\mathcal{G}/\mathcal{H}$ is an affine group scheme with $K[\mathcal{G}/\mathcal{H}] \cong K[\mathcal{G}]^{\mathcal{H}}$ (cf. [DG70],III,§3, Thm. 5.6 and 5.8). Again by taking invariants (this time $\mathcal{H}$ acting on the first factor) the isomorphism above restricts to an isomorphism

$$R^{\mathcal{H}} \otimes_F R^{\mathcal{H}} \cong R^{\mathcal{H}} \otimes_K K[\mathcal{G}/\mathcal{H}].$$

$R^{\mathcal{H}}$ is θ-simple, because for every θ-ideal $P \trianglelefteq R^{\mathcal{H}}$, the ideal $P \cdot R \trianglelefteq R$ is a θ-ideal, hence equals (0) or R, and so $P = (P \cdot R)^{\mathcal{H}}$ is (0) or $R^{\mathcal{H}}$. Since $F \leq R^{\mathcal{H}} \leq R$, we also have $C_{R^{\mathcal{H}}} = K$. So by Proposition 10.12, $R^{\mathcal{H}}$ is a PPV-ring over F with Galois group scheme $\mathcal{G}/\mathcal{H}$. It remains to show that $E^{\mathcal{H}} = \mathrm{Quot}(R^{\mathcal{H}})$:

Let $\tilde{F} := \mathrm{Quot}(R^{\mathcal{H}})$ and $\tilde{\mathcal{G}} := \underline{\mathrm{Gal}}(E/\tilde{F})$. Then $\mathcal{H}$ is a normal subgroup of $\tilde{\mathcal{G}}$ and by the previous,

$(R\cdot\tilde{F})^{\mathcal{H}}$ is a $\left(\tilde{\mathcal{G}}/\mathcal{H}\right)_{\tilde{F}}$-torsor. But $(R\cdot\tilde{F})^{\mathcal{H}} = R^{\mathcal{H}}\cdot\tilde{F} = \tilde{F}$, so $\tilde{\mathcal{G}} = \mathcal{H}$, and hence $E^{\mathcal{H}} = E^{\tilde{\mathcal{G}}} = \tilde{F} = \mathrm{Quot}(R^{\mathcal{H}})$.
iii) It suffices to show that $\mathcal{H}$ is normal in $\mathcal{G}$. The rest then follows from ii). Let L be a K-algebra and let $h\in\mathcal{H}(L)$ and $g\in\mathcal{G}(L)$. Then for all $r\in R\cap M$, we have

$$ghg^{-1}.(r\otimes 1) = gh.(g^{-1}.(r\otimes 1)) = g.(g^{-1}.(r\otimes 1)) = (r\otimes 1),$$

since $g^{-1}.(r\otimes 1)\in(R\cap M)\otimes_K L$ by $\mathcal{G}$-stability of M. So $ghg^{-1}\in\mathcal{H}(L)$, and therefore $\mathcal{H}$ is normal in $\mathcal{G}$.
iv) Without loss of generality let $\mathcal{H}=\mathcal{G}$. If $\mathcal{G}$ is reduced, then $F = E^{\mathcal{G}} = E^{\mathcal{G}(\bar{K})}$ and hence $E\otimes_K\bar{K}/F\otimes_K\bar{K}$ is separable, and so E/F is separable. On the other hand, if $\mathcal{G}$ is not reduced, let $\mathcal{G}_{red}\leq\mathcal{G}$ denote the closed reduced subgroup scheme given by the nilradical ideal. Since $\mathcal{G}_{red}$ is normal in $\mathcal{G}$, by the second statement $\tilde{F} := \mathrm{Quot}(R^{\mathcal{G}_{red}})$ is a PPV-extension of F with Galois group scheme $\underline{\mathrm{Gal}}(\tilde{F}/F) = \mathcal{G}/\mathcal{G}_{red}$ which is an infinitesimal group scheme. Let $ev: K[\mathcal{G}/\mathcal{G}_{red}]\to K$ denote the evaluation map corresponding to the neutral element of the group, then for any $\frac{r}{s}\in\tilde{F}$, we have

$$(\mathrm{id}\otimes ev)(\gamma(r\otimes s - s\otimes r)) = 0.$$

Since $\mathcal{G}/\mathcal{G}_{red}$ is infinitesimal, the kernel of ev is the nilradical, and hence there is some $k\in\mathbb{N}$ such that $(r\otimes s - s\otimes r)^{p^k} = 0$, where $p=\mathrm{char}(F)$. Hence $r^{p^k}\otimes s^{p^k} = s^{p^k}\otimes r^{p^k}\in\tilde{F}\otimes_F\tilde{F}$ which means that $\frac{r^{p^k}}{s^{p^k}}\in F$. So $\tilde{F}/F$ is purely inseparable, and hence E/F has also been inseparable. □

Corollary 11.6. *Let E/F be a PPV-extension with Galois group scheme $\mathcal{G}$. Then E/F is a purely inseparable extension if and only if $\mathcal{G}$ is an infinitesimal group scheme.*

Proof. In the proof of Theorem 11.5,iv) we have already shown, that E/F is purely inseparable if $\mathcal{G}$ is infinitesimal. On the other hand, if E/F is purely inseparable, then $\mathcal{G}(\bar{K}) = \mathrm{Aut}^\theta(E\otimes_K\bar{K}/F\otimes_K\bar{K})$ is the trivial group, since $E\otimes_K\bar{K}/F\otimes_K\bar{K}$ also is a purely inseparable extension. Hence $\mathcal{G}$ is infinitesimal. □

Corollary 11.7. *Let $p := \mathrm{char}(F) > 0$. If $\mathrm{Ker}(\theta_F^{(1)}) = F^p$, then all PPV-extensions E/F are separable, and the corresponding Galois group schemes are reduced.*

Proof. By Thm. 11.5,iv), the separability of a PPV-extension E/F is equivalent to the reducedness of $\underline{\mathrm{Gal}}(E/F)$. Assume, there exists an inseparable PPV-extension E/F. Then since $\underline{\mathrm{Gal}}(E/F)$ is not reduced, there is a purely inseparable intermediate extension $\tilde{F}/F$ (cf. the proof of Thm. 11.5,iv)). So there is an element $s\in\tilde{F}\setminus F$ such that $s^p\in F$ (and $s^p\notin F^p$). But $\theta_F^{(1)}(s^p) = ps^{p-1}\theta_{\tilde{F}}^{(1)}(s) = 0$ in contradiction to the hypothesis $\mathrm{Ker}(\theta_F^{(1)}) = F^p$. □

Example 11.8. We consider some examples for subfields of $(K((t)),\theta)$, where $\theta := \phi_t\in\mathrm{ID}_K(K((t)))$ is the iterative derivation with respect to t (cf. Example 3.3) and K denotes a field of characteristic $p>0$. For simplicity we assume that K is algebraically closed.

i) Let $F = K(t)\subseteq K((t))$. Then the IDE given by

$$\theta^{(p^l)}(y) = a_l t^{-p^l} y \quad (a_l\in K)$$

has a PPV-ring $R = F[s,s^{-1}]$ and PPV-field $E = F(s)$, where s is a solution of the IDE. This implies that $\underline{\mathrm{Gal}}(E/F)$ is a subgroup of $\mathbb{G}_m$. If the a_l are chosen appropriately then we have $\underline{\mathrm{Gal}}(E/F) = \mathbb{G}_m$ (cf. [Mat01], Thm. 3.13, resp. [MvdP03], Section 4) and s is transcendental over F. Furthermore the isomorphism $\gamma: R\otimes_F R\to R\otimes_K K[\mathbb{G}_m]$ is given by $1\otimes s\mapsto s\otimes x$ ($K[\mathbb{G}_m] =: K[x,x^{-1}]$).
All closed subgroup schemes of $\mathbb{G}_m$ are given by the ideals $(x^k-1)\trianglelefteq K[x,x^{-1}]$ for $k\in\mathbb{N}$ (the

so called subgroups μ_k of k-th roots of unity) and the corresponding intermediate θ-fields are $E^{\mu_k} = \mathrm{Quot}(R^{\mu_k}) = F(s^k)$. Hence, there are also intermediate θ-fields over which E is inseparable, namely for all $k > 0$ that are divisible by p.

ii) Let $F \subseteq K((t))$ be the subfield generated over K by t, $s_1 := \prod_{l=0}^{\infty}\left(1 + t^{a_l p^l}\right)$ and $s_2 := \prod_{l=0}^{\infty}\left(1 + t^{b_l p^l}\right)$, where $a_l, b_l \in \{0, 1, \ldots, p-1\}$ are chosen such that t, s_1 and s_2 are algebraically independent. Consider the IDE

$$\theta^{(p^l)}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{l+1}\left(1 + t^{a_{l+1}p^{l+1}}\right)^{-1} & 0 \\ 0 & b_{l+1}\left(1 + t^{b_{l+1}p^{l+1}}\right)^{-1} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (l \in \mathbb{N}).$$

A solution of this IDE is given by $\binom{r_1}{r_2} \in K((t))^2$ with $r_1^p = (1 + t^{a_0})^{-1} \cdot s_1$ and $r_2^p = (1 + t^{b_0})^{-1} \cdot s_2$. Hence the corresponding PPV-ring is $R = F[r_1, r_2]$ and the Galois group scheme – a priori a subgroup of $\mathbb{G}_m \times \mathbb{G}_m$ – is equal to $\mu_p \times \mu_p$. The action of the Galois group scheme on R is given by the homomorphism $\rho : R \to R \otimes_K K[\mu_p \times \mu_p] \cong R \otimes_K K[x_1, x_2]/(x_1^p - 1, x_2^p - 1)$, which maps r_i to $r_i \otimes x_i$ $(i = 1, 2)$. Since the nontrivial subgroups of $\mu_p \times \mu_p$ are given by the ideals $(x_1^k x_2 - 1) \trianglelefteq K[x_1, x_2]/(x_1^p - 1, x_2^p - 1)$ $(k \in \{0, 1, \ldots, p-1\})$ and $(x_1 - 1) \trianglelefteq K[x_1, x_2]/(x_1^p - 1, x_2^p - 1)$, there are exactly $p+1$ intermediate θ-fields unequal to E and F, namely $F(r_1^k r_2)$ resp. $F(r_1)$.

So in this case, although E/F has infinitely many intermediate fields, there are only finitely many intermediate θ-fields.

iii) Let $F \subseteq K((t))$ be the subfield generated over K by $t, s_1 := \sum_{l=0}^{\infty} a_l t^{p^l}$ and $s_2 := \sum_{l=0}^{\infty} b_l t^{p^l}$, where $a_l, b_l \in \mathbb{F}_p$ are chosen such that t, s_1, s_2 are algebraically independent. In this case we also have a purely inseparable PPV-extension of degree p^2, namely $E = F(r_1, r_2) \subseteq K((t))$ with $r_1^p = s_1 - a_0 t$, $r_2^p = s_2 - b_0 t$. r_1 is a solution of the IDE

$$\theta^{(p^l)}\begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a_{l+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix} \quad (l \in \mathbb{N}),$$

and r_2 a solution of the IDE with a_{l+1} replaced by b_{l+1}. Hence the Galois group scheme – a subgroup scheme of $\mathbb{G}_a \times \mathbb{G}_a$ – is equal to $\alpha_p \times \alpha_p$ (where α_p denotes the Frobenius kernel inside $\mathbb{G}_a$).

In this case, there are infinitely many intermediate θ-fields, since $\alpha_p \times \alpha_p$ has infinitely many subgroups which are given by the ideals $(ay_1 + by_2) \trianglelefteq K[y_1, y_2]/(y_1^p, y_2^p) := K[\alpha_p \times \alpha_p]$ $(a, b \in K)$.

The action is given by $\rho : R \to R \otimes_K K[y_1, y_2]/(y_1^p, y_2^p)$ with $\rho(r_i) = r_i \otimes 1 + 1 \otimes y_i$ $(i = 1, 2)$. So the corresponding intermediate θ-fields are $F(ar_1 + br_2)$, $a, b \in K$.

Comparing this example with the one before, we see that – even for finite extensions – the Galois group scheme depends on the iterative derivation. This is contrary to finite separable PPV-extensions, where the Galois group is already determined by the extension of fields itself (cf. [Mat01], Thm. 1.15).

12. Finite inseparable extensions

In this section we compare our results for finite purely inseparable PPV-extensions with the Galois theory for purely inseparable field extensions given by Chase in [Cha76].

So let us first give a brief overview on some results in [Cha76]: Let E/F be a purely inseparable field extension. Then the group functor

$$G_t(E/F) : (\mathsf{TruncAlg}/F) \to (\mathsf{Groups}), L \mapsto \mathrm{Aut}(E \otimes_F L/L)$$

from the category of truncated F-algebras (i.e. algebras of the form $F[t_1,\dots,t_r]/(t_1^{n_1},\dots,t_r^{n_r})$) to the category of groups is representable by a truncated F-algebra U. If the extension E/F is modular (i.e. for all $i\in\mathbb{N}$, E^{p^i} and F are linearly disjoint over $E^{p^i}\cap F$), then $E^{G_t(E/F)}=F$ and $\dim_F(U)=[E:F]^{[E:F]}$. In this case, there is a Galois correspondence between the intermediate fields $F\leq M\leq E$, s.t. E/M is modular and certain closed subgroup schemes of $G_t(E/F)$, given in the usual way by taking fixed fields respectively subgroups fixing the given intermediate field. Furthermore, he showed that a purely inseparable field extension E/F is modular if and only if there exists a truncated group scheme $\mathcal{G}$ (i.e. an affine group scheme represented by a truncated F-algebra) which acts on E/F, s.t. $\mathrm{Spec}(E)$ is a $\mathcal{G}$-torsor. Given such a group scheme $\mathcal{G}$, then $G_t(E/F)\cong\mathcal{G}(E\otimes_F -)$ as truncated group schemes over F. However, although the group scheme $G_t(E/F)$ is unique, there might be several such group schemes $\mathcal{G}$.

Return now to the case that E/F is a purely inseparable PPV-extension and $\mathcal{G}:=\underline{\mathrm{Gal}}(E/F)$. By Prop. 11.6, $\mathcal{G}$ is infinitesimal and since K is perfect, $K[\mathcal{G}]$ is a truncated K-algebra (cf. [DG70], III, §3, Cor. 6.3) and so $F[\mathcal{G}]$ is a truncated F-algebra. As shown in Corollary 10.11, E is a $\mathcal{G}_F$-torsor.

By the statements above, we obtain that E/F is a modular field extension and that $G_t(E/F)$ equals $\mathcal{G}_F(E\otimes_F -)$. So we can regain the truncated Galois group scheme $G_t(E/F)$ from our Galois group scheme $\underline{\mathrm{Gal}}(E/F)$.

However, starting with $G_t(E/F)$, the iterable higher derivation leads to a natural choice for a group scheme $\mathcal{G}_F\leq G_t(E/F)$ over which E is a torsor (namely $\underline{\mathrm{Gal}}(E/F)_F$) and also gives a natural description of the intermediate fields corresponding to the closed subgroup schemes of $\mathcal{G}$. For instance, in Example 11.8,ii)+ iii), $F=K(t,s_1,s_2)$ is the rational function field in three variables and E/F is a purely inseparable field extension of degree p^2 and exponent 1. Hence in both examples, we have the same (abstract) field extension. But in one case the iterable higher derivation leads to the Galois group scheme $\underline{\mathrm{Gal}}(E/F)=\alpha_p\times\alpha_p$ and in the other case to $\underline{\mathrm{Gal}}(E/F)=\mu_p\times\mu_p$. Other iterable higher derivations would also lead to different Galois group schemes. The truncated Galois group scheme $G_t(E/F)$ only gives a bound on which Galois group schemes $\underline{\mathrm{Gal}}(E/F)$ may occur, because every one of them will be a closed subgroup scheme of $G_t(E/F)$.

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