A nonlinear structured population model: Global existence and structural stability of measure-valued solutions

Piotr Gwiazda

Institute of Applied Mathematics and Mechanics, University of Warsaw, Warszawa 02-097, Poland
E-mail: pgwiazda@mimuw.edu.pl

Thomas Lorenz

Institute of Applied Mathematics, University of Heidelberg, 69120 Heidelberg, Germany
E-mail: thomas.lorenz@iwr.uni-heidelberg.de

Anna Marciniak-Czochra

Institute of Applied Mathematics, University of Heidelberg, 69120 Heidelberg, Germany
E-mail: anna.marciniak@iwr.uni-heidelberg.de

Abstract

This paper is devoted to the study of the global existence and structural stability of measure-valued solutions to a nonlinear structured population model given in the form of a nonlocal first-order hyperbolic problem on $\mathbb{R}^+$. In distinction to previous studies, where the L^1 norm was used, we apply the flat metric, similar to the Wasserstein W_1 distance. We argue that stability using this metric, in addition to mathematical advantages, is consistent with intuitive understanding of empirical data. Structural stability and the uniqueness of the weak solutions are shown under the assumption about the Lipschitz continuity of the kinetic functions. The stability result is based on the duality formula and the Gronwall-type argument. Using a framework of mutational equations, existence of solutions to the equations of the model is also shown under weaker assumptions, i.e., without assuming Lipschitz continuity of the kinetic functions.

Key words: structural stability, Radon measures, population dynamics, flat metric, structured population model

1 Introduction

Models describing the time evolution of physiologically structured populations have been extensively studied for many years [18,25,28]. The dynamics of such populations have been modeled by partial differential equations of transport type, as described for example in [28] and [18] and in the references therein, by integral and functional-integral equations (e.g. [3]), and more recently by constructing the next-state operators, which define a semigroup [9].

This paper focuses on measure-valued solutions to a nonlocal first-order hyperbolic problem on $\mathbb{R}^+$ describing a physiologically structured population. We find continuity assumptions about the kinetic functions sufficient for global existence and for structural stability of distributional solutions whose values are finite Radon measures on $\mathbb{R}^+$. These results can be extended to systems of more than one species.

1.1 A joint framework for continuous and discrete distributions: Radon measures

Global existence and stability of structured population models were established for solutions defined in Banach space L^1 [14,28]. In this case it was possible to prove strong continuity and structural stability of solutions. However, it is often necessary to describe populations in which the initial distribution of the individuals is concentrated with respect to the structure, i.e., it is not absolutely continuous with respect to the Lebesgue measure. In these cases it is relevant to consider initial data in the space of Radon measures as proposed in [18]. The set of finite Radon measures on the Euclidean space is defined as the dual space of all real-valued continuous functions vanishing at infinity. For linear age-dependent population dynamics, a qualitative theory using semigroup methods and spectral analysis has been laid out in [11]. The follow-up work [9] is devoted to nonlinear models. Some analytical results concerning the existence of solutions are given in [10]. All the results there concerning continuous dependence of solutions on time and initial state are based on the weak$*$ topology of Radon measures. Moreover, there exist simple counter-examples indicating that continuous dependence, either with respect to time or to initial state, generally cannot be expected in the strong topology. As concluded in [10] "*structural stability, in the sense of continuous dependence with respect to the modeling ingredients, is still to be established*". This is important in the context of numerical approximation and experimental data. Structural stability of the model solutions is essential for calibration of the model.

We have therefore been motivated to study the problem of structural stability of the solutions to the equations of structured population dynamics. Our new approach is based on a theory of nonlinear semigroups in metric spaces, instead of weak$*$ semigroups on Banach spaces. The framework of the Wasserstein metric in the spaces of probability measures (see for example [2]) is important for the analysis of transport equations. Since the nonlinear structured population model is not conservative, we cannot expect the solutions to be probability measures, even if the initial data are. Therefore, instead of the Wasserstein metric, we apply a version of the flat metric, defined in Section 2. The flat metric corresponds to the dual norm of $W^{1,\infty}(\mathbb{R}^+)$. It metrizes both weak* and narrow topologies on each tight subset of Radon measures with uniformly bounded total variation.

The approach we apply helps to alleviate one difficulty of the classical approach, which is the inconsistency of the L^1 norm with empirical data. In particular, even if we assume that a real population has a distribution absolutely continuous with respect to the Lebesgue measure, in which case the distribution density exists, the data from experiments provide information on the number of individuals in some range of the state variable (age, size, etc.). Discrete data in the observational series approximate the integrals of the density over some intervals of the considered quantity, and not the density itself. In other words, we may record the series $\{a_n\}_{n=1}^{\infty} = \left\{\int_{[nh,(n+1)h[} d\mu\right\}_{n=1}^{\infty}$, where h is the size of the considered interval of the state variable. Therefore, having two functions with equal values of the integrals over even short intervals does not imply that the L^1 norm of the difference of these quantities is small. For example assume that $\{a_n\}_{n=1}^{\infty}$ is given and define

$$A = \{\mu \in \mathcal{M}^+(\mathbb{R}^+) \mid \int\limits_{[nh,(n+1)h[} d\mu = a_n, n = 1, 2, \ldots\}.$$

The diameter of the set A, $\mathrm{diam}_{\|\cdot\|_{\mathcal{M}}}(A) = \mathrm{diam}_{\|\cdot\|_{L^1}}(A \cap L^1) = 2\sum_{n=1}^{\infty} a_n$, does not depend on h and therefore, even if h is small, we cannot conclude that the distance between different possible initial data is small. On the contrary, for the flat metric ϱ, which we define later on, $\mathrm{diam}_{\varrho}(A) \leq h\sum_{n=1}^{\infty} a_n$. This suggests that considering L^1 stability for equations describing biological processes, or any processes basing on the data of empirical type, may not be an optimal approach.

1.2 A nonlinear model of a physiologically structured population

We recall the structured population model considered in [28,14] for solution $u(\cdot, t) \in L^1(\mathbb{R}^+)$,

$$\begin{cases} \partial_t\, u(x,t) \;+\; \partial_x\left(F_2(u(\cdot,t),\, x, t)\, u(x,t)\right) = F_3(u(\cdot,t), x, t)\, u(x,t), & \text{in } \mathbb{R}^+ \times [0,T], \\ \qquad F_2(u(\cdot,t),\, 0, t)\, u(0,t) \;= \displaystyle\int_{\mathbb{R}^+} F_1(u(\cdot,t),\, x, t)\, u(x,t)dx & \text{in }]0,T], \\ \qquad u(x,0) = u_0(x), & \text{in } \mathbb{R}^+. \end{cases}$$

Here x denotes the state of an individual (for example the size, level of neoplastic transformation, stage of differentiation) and $u(x,t)$ the density of individuals being in state $x \in \mathbb{R}^+$ at time t. By $F_3(u, x, t)$ we denote a function describing the individual's rate of evolution, such as the growth or death rate. $F_2(u, x, t)$ describes the rate of the dynamics of the structure, i.e., the dynamics of the transformation of the individual state. The boundary term describes the influx of new individuals to state $x = 0$. Finally, u_0 denotes the initial population density.

In this paper, we investigate the existence of measure-valued solutions $\mu_t \in \mathcal{M}(\mathbb{R}^+)$ to the nonlinear model

$$\begin{cases} \partial_t\, \mu_t \;+\; \partial_x\left(F_2(\mu_t, t)\, \mu_t\right) \;=\; F_3(\mu_t, t)\, \mu_t, & \text{in } \mathbb{R}^+ \times [0,T] \\ \qquad F_2(\mu_t, t)(0)\, \mu_t(0) \;=\; \displaystyle\int_{\mathbb{R}^+} F_1(\mu_t, t)(x)\;\; d\mu_t(x), & \text{in }]0,T] \\ \qquad \mu_0 \;=\; \nu_0, \end{cases} \tag{1}$$

and their dependence on the initial measure $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ and on three coefficient functions $F_1, F_2, F_3 : \mathcal{M}(\mathbb{R}^+) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^+)$.

Problem (1) is interpreted in a weak sense. Accordingly, the desired solutions are narrowly continuous (see Definition 2.1) curves $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+) = (C_0^0(\mathbb{R}^+))'$ satisfying the problem in a weak sense, i.e., in duality with all test functions in $C^1(\mathbb{R}^+ \times [0,T]) \cap W^{1,\infty}(\mathbb{R}^+ \times [0,T])$. The additional assumption $F_1(\cdot) \geq 0$ guarantees that positivity of initial measure ν_0 is preserved by the solution μ_t constructed here. This feature is essential for modeling population dynamics.

In this paper we focus on two aspects:
(i) Under very general conditions on $F_1, F_2, F_3 : \mathcal{M}(\mathbb{R}^+) \times [0,T] \longrightarrow W^{1,\infty}(\mathbb{R}^+)$ we prove the existence of measure-valued solutions $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+)$, which are narrowly continuous with respect to time. The main result concerning existence is formulated in Theorem 4.3 and Corollary 4.4.
(ii) The method used for constructing these solutions is stable with respect to perturbations of coefficients and initial measure. Therefore, assuming continuity of the functions F_1, F_2 and F_3 with respect to time and Lipschitz continuity in the flat metric with respect to measure μ, we prove the existence of solutions, which are structurally stable, i.e., which exhibit Lipschitz dependence on the initial data and model parameters. This second main result is formulated in Theorem 4.5 and Theorem 4.6.

1.3 Comparison with earlier results

Model (1) is a generic formulation of a nonlinear single-species model with a one-dimensional structure. The model was considered by Diekmann and Getto in reference [10] in a case where the functions F_i depend on the population density via weighted integrals $\int \gamma_i(x) d\mu_t$. Diekmann and Getto proved the global existence of solutions and their continuous dependence on time and initial state in the weak* topology of $\mathcal{M}(\mathbb{R}^+)$. The results were formulated under the assumptions of Lipschitz continuity of functions F_1, F_2 and F_3 and the global Lipschitz property of the output function γ_i. To solve a nonlinear problem Diekmann and Getto applied the so-called method of interaction variables. The method consists of replacing the dependence on the measure μ incorporated in F_1, F_2 and F_3 by input $I(t)$ at time t, and splitting the nonlinear problem (1) into a linear problem coupled to a fix-point problem. This leads to a linear problem depending on the parameter function $I(\cdot)$, which can be solved by extending the concept of semigroup. The feedback law relates the parameter function $I(\cdot)$ to the solution and thus provides a fix-point problem equivalent to the original nonlinear problem. Appropriate assumptions on the coefficients allow application of the contraction principle.

In this paper, the analysis of model (1) is based on the estimates obtained for the linear problem,

$$\begin{cases} \partial_t \, \mu_t \;+\; \partial_x \, (b \, \mu_t) = c \, \mu_t, & \text{in } \mathbb{R}^+ \times [0,T], \\ b(0) \, \mu_t(0) = \displaystyle\int_{\mathbb{R}^+} a \, d\mu_t, & \text{in }]0,T], \\ \mu_0 = \nu_0. & \end{cases} \tag{2}$$

where $a(\cdot)$, $b(\cdot)$, $c(\cdot) \in W^{1,\infty}(\mathbb{R}^+)$ and $b(0) > 0$.

The key estimates are obtained using the concepts of the duality theory applied for transport equation in ref. [12]. Recently, these ideas were further developed by Perthame and co-workers in the context of long-time asymptotics of linear structured population models [21]. In the present paper the smooth solution to a dual partial differential equation provides an integral representation of a measure-valued solution $\mu : [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^+)$ to equation (2). In particular, this solution exists and depends continuously on the initial measure ν_0 and on the coefficients $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$.

To prove structural stability we apply concepts similar to those used by Bianchini and Colombo [6]. However, instead of the framework of quasidifferential equations [20], we apply a similar concept: the mutational equations introduced by Aubin [4,5] and generalised by Lorenz [15,16]. In comparison to the approach of Diekmann and co-workers [9,10], the connection with the nonlinear problem (1) is not based on the contraction principle, but on compactness. This allows us to prove existence of weak solutions to the nonlinear population model (1) without assuming Lipschitz dependence of the coefficients $F_1(\cdot, t)$, $F_2(\cdot, t)$, $F_3(\cdot, t)$, but only continuity. In addition, assuming Lipschitz continuity of the model coefficients $F_1(\cdot, t)$, $F_2(\cdot, t)$, $F_3(\cdot, t)$ guarantees the uniqueness of the weak solution.

1.4 Coupling with other dynamical systems using mutational equations

In a similar way to the Peano Theorem for ordinary differential equations, the existence result holds for systems of mutational equations. Therefore, the framework of mutational equations allows extending the results obtained in the present paper to multispecies models and to more involved structured population models consisting of systems of structured equations coupled with ordinary differential equations controlling the dynamics of the structure (through the coupling in F_2 function), or regulating processes on the level of the population (through the coupling in F_3 and F_1 functions). This latter type of models seems to be of special interest in the context of cell populations, whose dynamics is controlled by the intracellular signaling pathways, density of cell membrane receptors or other processes, which take place on the level of single cells.

1.5 Structure of the paper

We conclude this section with a brief description of the plan of the paper. In the remainder of this section we introduce the notations which are used throughout the paper. In Section 2 we define the flat metric on Radon measures and specify the relations to weak* and narrow convergence in tight subsets. Section 3 is devoted to proving *a priori* estimates for solutions to the linear structured population model (2). First we solve the dual problem and find estimates for its solutions in the $W^{1,\infty}$ space. Applying the duality formula, we obtain estimates for the solution to the original linear problem in the space of Radon measures with flat metric. In Section 4, by applying the framework of mutational equations and using the estimates obtained in Section 3, we show existence and structural stability of solutions to the nonlinear model. The existence result is formulated in Theorem 4.3 and Corollary 4.4 with the stability result being given in Theorem 4.5. The uniqueness of the weak solutions is shown in Theorem 4.6.
At the end of this paper, Appendix A provides proofs of the lemmas formulated in Section 3. Appendix B is a self-contained overview of mutational equations in metric spaces.

1.6 Notation

Throughout this paper we use the following notations:
$\mathbb{R}^+ = [0, +\infty[$ and $C_c^0(\mathbb{R}^+)$ is the space of continuous functions $\mathbb{R}^+ \longrightarrow \mathbb{R}$ with compact

support, and $C_0^0(\mathbb{R}^+)$ is its closure with respect to the supremum norm. $C_c^0(\mathbb{R}^+, \mathbb{R}^+)$ denotes the subset of functions $\varphi \in C_c^0(\mathbb{R}^+)$ with $\varphi \geq 0$ and correspondingly, $C_0^0(\mathbb{R}^+, \mathbb{R}^+)$ denotes its closure.
Furthermore, $\mathcal{M}(\mathbb{R}^+)$ consists of all finite real–valued Radon measures on $\mathbb{R}^+$. As a consequence of the Riesz Theorem, it is the dual space of $C_0^0(\mathbb{R}^+)$ and, the total variation of a Radon measure is equal to the dual norm [1, Remark 1.57].
$\mathcal{M}^+(\mathbb{R}^+)$ denotes the set of all nonnegative finite Radon measures on $\mathbb{R}^+$, i.e., $\mathcal{M}^+(\mathbb{R}^+) := \{\mu \in \mathcal{M}(\mathbb{R}^+) \mid \mu(\cdot) \geq 0\}$.

2 The flat metric on finite Radon measures

In this section, we specify a suitable metric ρ in the space $\mathcal{M}(\mathbb{R}^+)$ of finite Radon measures on $\mathbb{R}^+$. First, we recall the definition of the narrow convergence in the space of Radon measures (cf. [2]).

Definition 2.1 *A sequence $(\mu^n)_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}^+)$ converges narrowly to $\mu \in \mathcal{M}(\mathbb{R}^+)$ as $n \to \infty$ if*

$$\lim_{n\to\infty} \int_{\mathbb{R}^+} \varphi \; d\mu^n \;=\; \int_{\mathbb{R}^+} \varphi \; d\mu$$

for every bounded and continuous function $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}$.

Definition 2.2 *A sequence $(\mu^n)_{n\in\mathbb{N}}$ is tight if*

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \int_{[M,\infty[} 1 d|\mu^n| = 0.$$

Remark 2.3 For any tight sequence $(\mu^n)_{n\in\mathbb{N}}$ weak$*$ convergence in the space $\mathcal{M}(\mathbb{R}^+)$ is equivalent to the narrow convergence.

Remark 2.4 A sequence $(\mu^n)_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^+)$ is tight if and only if there exists a nondecreasing nonnegative function $\phi \in C^0(\mathbb{R}^+)$ with $\lim_{x\to\infty} \phi(x) = \infty$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^+} \phi \; d|\mu^n| < \infty,$$

or equivalently

$$\sup_{n\in\mathbb{N}} \sup_{\substack{\psi \in C^0(\mathbb{R}^+): \\ |\psi| \leq \phi}} \int_{\mathbb{R}^+} \psi \; d\mu^n < \infty.$$

The topology of narrow convergence on $\mathcal{M}(\mathbb{R}^+)$ is metrizable on tight subsets with uniformly bounded total variation. It is induced by a norm, which is usually called the flat norm and has been established for differential forms in geometric measure theory [13, 4.1.12]. Less abstract setting in terms of differential operators can be found in [19, § 3]. For the analysis of the nonlinear population model, we apply the corresponding metric ρ.

Definition 2.5 *The flat metric* $\rho : \mathcal{M}(\mathbb{R}^+) \times \mathcal{M}(\mathbb{R}^+) \longrightarrow \mathbb{R}^+$ *is defined by*

$$\rho(\mu,\,\nu) := \sup\ \Big\{ \int_{\mathbb{R}^+} \psi\ d(\mu-\nu)\ \Big|\ \psi \in C^1(\mathbb{R}^+),\ \ \|\psi\|_\infty \le 1,\ \|\partial_x \psi\|_\infty \le 1 \Big\}.$$

Note that unlike the positive cone $\mathcal{M}^+(\mathbb{R}^+)$, the linear space $\mathcal{M}(\mathbb{R}^+)$ of all signed Radon measures is not complete with respect to ρ.

Remark 2.6 (i) For any $\lambda > 0$ and $\mu, \nu \in \mathcal{M}(\mathbb{R}^+)$, the flat metric $\rho(\mu,\nu)$ can be characterised by continuously differentiable test functions with a uniform bound λ,

$$\rho(\mu,\,\nu) = \sup\ \Big\{ \tfrac{1}{\lambda} \int_{\mathbb{R}^+} \varphi\ d(\mu-\nu)\ \Big|\ \varphi \in C^1(\mathbb{R}^+),\ \ \|\varphi\|_\infty \le \lambda,\ \|\partial_x \varphi\|_\infty \le \lambda \Big\}.$$

(ii) Every element of $W^{1,\infty}(\mathbb{R}^+)$ can be approximated on an arbitrary compact set by elements of $C_c^\infty(\mathbb{R}^+) \subset C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ with respect to supremum norm. Hence for all $\lambda > 0$

$$\begin{aligned}
\rho(\mu,\,\nu) &= \sup \Big\{ \tfrac{1}{\lambda} \int_{\mathbb{R}^+} \varphi\ d(\mu-\nu)\ \Big|\ \varphi \in C_c^\infty(\mathbb{R}^+), \quad \|\varphi\|_\infty \le \lambda,\ \|\partial_x \varphi\|_\infty \le \lambda \Big\} \\
&= \sup \Big\{ \tfrac{1}{\lambda} \int_{\mathbb{R}^+} \varphi\ d(\mu-\nu)\ \Big|\ \varphi \in W^{1,\infty}(\mathbb{R}^+),\ \ \|\varphi\|_\infty \le \lambda,\ \|\partial_x \varphi\|_\infty \le \lambda \Big\} \\
&= \|\mu - \nu\|_{(W^{1,\infty})^*}.
\end{aligned}$$

The above representation of ρ proves to be very useful for the analysis of the linear population model as test functions are transformed along characteristics.

The following Theorem 2.7 summarizes the main properties of ρ. In particular, it specifies the relationship between the topologies of the flat metric ρ, weak* convergence and narrow convergence for tight subsets of $\mathcal{M}(\mathbb{R}^+)$. The following proof is similar to [19, § 3, Theorem], except that the space of Radon measures considered here includes measures defined on the noncompact set $\mathbb{R}^+$.

Theorem 2.7 (*i*) *For any tight sequence* $(\mu^n)_{n\in\mathbb{N}}$ *and* μ *in* $\mathcal{M}(\mathbb{R}^+)$, *the following equivalence holds*

$$\mu^n \longrightarrow \mu \quad \textit{weak*} \quad \textit{for} \ \ n \longrightarrow \infty \qquad \Longleftrightarrow \qquad \begin{cases} \lim\limits_{n\to\infty} \ \rho(\mu^n, \mu) = 0 \\ \sup\limits_{n\in\mathbb{N}} \ |\mu^n|(\mathbb{R}^+) < \infty. \end{cases}$$

(*ii*) *For any threshold* $r > 0$, *the set* $\{\mu \in \mathcal{M}(\mathbb{R}^+) \mid |\mu|(\mathbb{R}^+) \le r\}$ *endowed with* $\rho(\cdot,\cdot)$ *constitutes a complete separable metric space.*
(*iii*) *For any threshold* $r > 0$, *the set* $\mathcal{K} \subset \{\mu \in \mathcal{M}(\mathbb{R}^+) \mid |\mu|(\mathbb{R}^+) \le r\}$ *is relatively compact with respect to the flat metric* ρ *if the set* $\mathcal{K}$ *is tight* (*i.e., every sequence* $(\mu^n)_{n\in\mathbb{N}}$ *in* $\mathcal{K}$ *is tight*).

Proof. (i) ($\Leftarrow$) Assume that a sequence $(\mu^n)_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^+)$ with $\sup\limits_{n\in\mathbb{N}} |\mu^n|(\mathbb{R}^+) < \infty$ converges to μ with respect to ρ, i.e., $\lim\limits_{n\to\infty} \rho(\mu^n,\mu) = 0$. Using Remark 2.6 (ii) and the fact that $W^{1,\infty}(\mathbb{R}^+) \cap C_0^0(\mathbb{R}^+)$ is dense in $(C_0^0(\mathbb{R}^+), \|\cdot\|_\infty)$, we conclude that the sequence $(\mu^n)_{n\in\mathbb{N}}$ converges also weak$*$ in $\mathcal{M}(\mathbb{R}^+) = (C_0^0(\mathbb{R}^+))'$. Indeed,

$$|\int\limits_{\mathbb{R}^+} (\varphi - \varphi^k) d\mu^n - \int\limits_{\mathbb{R}^+} (\varphi - \varphi^k) d\mu| \le (\sup_{n\in\mathbb{N}} \|\mu^n\|_{\mathcal{M}} + \|\mu\|_{\mathcal{M}}) \|\varphi - \varphi^k\|_\infty.$$

Choosing $\varphi^k \in W^{1,\infty}$ with $\|\varphi - \varphi^k\| \le \frac{1}{k}$, we obtain

$$|\int\limits_{\mathbb{R}^+} \varphi d\mu^n - \int\limits_{\mathbb{R}^+} \varphi d\mu| \le |\int\limits_{\mathbb{R}^+} (\varphi - \varphi^k) d\mu^n - \int\limits_{\mathbb{R}^+} (\varphi - \varphi^k) d\mu| + |\int\limits_{\mathbb{R}^+} \varphi^k d\mu^n - \int\limits_{\mathbb{R}^+} \varphi^k d\mu|$$

and the last term converges to zero, as n tends to ∞.
($\Rightarrow$) Let $(\mu^n)_{n\in\mathbb{N}}$ converge weak* to $\mu \in \mathcal{M}(\mathbb{R}^+)$. Then, $\sup\limits_{n\in\mathbb{N}} |\mu^n|(\mathbb{R}^+) < \infty$ due to the Banach-Steinhaus Theorem. Using the definition of ρ, we obtain

$$\rho(\mu^n,\mu) \ \le \ \sup\Big\{ \int\limits_{[0,a]} \varphi \, d(\mu^n - \mu) \,\Big|\, \|\varphi\|_\infty \le 1, \ \|\partial_x \varphi\|_\infty \le 1 \Big\} + \int\limits_{[a,\infty[} 1 \, d(|\mu^n| + |\mu|).$$

The tightness condition yields $\sup\limits_n |\mu^n|([a,\infty[) + |\mu|([a,\infty[) \longrightarrow 0$ for $a \to \infty$. In particular, for every $\varepsilon > 0$, there exists $a \in \mathbb{R}^+$ such that $\sup\limits_n |\mu^n|([a,\infty[) + |\mu|([a,\infty[) \le \frac{\varepsilon}{3}$. The set

$$K_{a+1} := \{\varphi \in W^{1,\infty}(\mathbb{R}^+) \mid \ \|\varphi\|_\infty \le 1, \ \|\partial_x\varphi\|_\infty \le 1 \ \text{ and } \ \operatorname{supp}\varphi \subset [0,\, a{+}1]\,\}$$

is compact in $(C_0^0(\mathbb{R}^+), \|\cdot\|_\infty)$ according to the Arzela-Ascoli Theorem. Therefore, there exists a finite set of Lipschitz functions $\{\varphi_i\}_{i=1}^{k_\varepsilon}$ in K_{a+1} such that

$$\Big(\sup_{n\in\mathbb{N}} |\mu^n|(\mathbb{R}^+) + |\mu|(\mathbb{R}^+)\Big) \sup_{\varphi \in K_{a+1}} \Big\{ \inf_{i\in\{1\ldots k_\varepsilon\}} \|\varphi - \varphi_i\|_\infty \Big\} \le \tfrac{\varepsilon}{3}$$

and thus,

$$\rho(\mu^n,\mu) \ \le \ \max_{i\in\{1\ldots k_\varepsilon\}} \int\limits_{[0,a]} \varphi_i \, d(\mu^n - \mu) \ + \ \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3}.$$

Due to the weak* convergence of $(\mu^n)_{n\in\mathbb{N}}$, there exists $m_\varepsilon \in \mathbb{N}$ such that

$$\sup_{i\,\in\,\{1,\ldots,k_\varepsilon\}} \Big| \int\limits_{[0,a]} \varphi_i \ d(\mu^n - \mu) \Big| \ \le \ \tfrac{\varepsilon}{3}$$

for every $n \ge m_\varepsilon$. This implies $\rho(\mu^n,\mu) < \varepsilon$ for every $n \ge m_\varepsilon$ and thus, $\lim\limits_{n\to\infty} \rho(\mu^n,\mu) = 0$.
(ii) The subset $\{\mu \in \mathcal{M}(\mathbb{R}^+) \mid |\mu|(\mathbb{R}^+) \le r\}$ (with arbitrary $r > 0$) is complete with respect to weak* convergence since $\mathcal{M}(\mathbb{R}^+)$ is the dual space of $C_0^0(\mathbb{R}^+)$ (see e.g. [1,23]). Consequently the first part of the proof (i) implies its completeness with respect to the flat metric ρ.
(iii) This statement results from property (i) and sequential version of the Banach-Alaoglu Theorem. □

Remark 2.8 Note that Remark 2.3 yields the same conclusions as in Theorem 2.7 (i) replacing weak* convergence with narrow convergence.

3 The linear population model

In this section we consider a linear structured population model,

$$
\begin{cases}
\partial_t \mu_t + \partial_x (b\, \mu_t) = c\, \mu_t, & \text{in } \mathbb{R}^+ \times [0,T], \\
b(0)\, \mu_t(0) = \int\limits_{\mathbb{R}^+} a\, d\mu_t, & \text{in }]0,T], \\
\mu_0 = \nu_0,
\end{cases}
\tag{3}
$$

where $a, b, c : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are bounded and Lipschitz continuous functions with $b(\cdot) \in C^1(\mathbb{R}^+)$, $b(0) > 0$, and $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ is given initial data.

Formal integration by parts motivates the following definition of a weak solution to problem (3).

Definition 3.1 *$\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+)$, $t \longmapsto \mu_t$ is called a* weak solution *to problem (3) if μ is narrowly continuous with respect to time and, for all $\varphi \in C^1(\mathbb{R}^+ \times [0,T]) \cap W^{1,\infty}(\mathbb{R}^+ \times [0,T])$,*

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^+} \varphi(x,T) d\mu_T(x) - \int\limits_{\mathbb{R}^+} \varphi(x,0) d\nu_0(x) \\
&= \int\limits_0^T \int\limits_{\mathbb{R}^+} \partial_t \varphi(x,t) d\mu_t(x) dt + \int\limits_0^T \int\limits_{\mathbb{R}^+} \Big(\partial_x \varphi(x,t) b(x) + \varphi(x,t) c(x) \Big) d\mu_t(x) dt \\
&\qquad + \int\limits_0^T \varphi(0,t) \int\limits_{\mathbb{R}^+} a(x) d\mu_t(x) dt.
\end{aligned}
\tag{4}
$$

The key point of this section is an implicit characterization of the solution to the linear problem (4) by an integral equation exploiting the notion of characteristics. This solution is derived for any initial finite Radon measure $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ and coefficient $b(\cdot) \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ with $b(0) > 0$. Motivated by the application to population dynamics, we then specify a sufficient condition on $a(\cdot)$ for preserving nonnegativity of measures, namely $a(\cdot) \geq 0$. The corresponding solution map can easily be extended to less regular coefficients $b(\cdot) \in W^{1,\infty}(\mathbb{R}^+)$, which we prove in Corollary 3.10.

Definition 3.2 *$X_b : [0,T] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is said to be induced by the flow along b, if for any initial point $x_0 \in \mathbb{R}^+$, the curve $X_b(\cdot, x_0) : [0,T] \longrightarrow \mathbb{R}^+$ is the continuously differentiable solution to the Cauchy problem*

$$
\begin{cases}
\dfrac{d}{dt} x(t) = b(x(t)), & \textit{in } [0,T], \\
x(0) = x_0 \in \mathbb{R}^+.
\end{cases}
\tag{5}
$$

The assumptions $b \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $b(0) > 0$ and Gronwall's Lemma imply continuous differentiability of solutions to ordinary differential equations with respect to parameters and initial data [26].

Lemma 3.3 $X_b : [0,T] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is continuously differentiable with*

(*i*) $\|\partial_x X_b(t,\cdot)\|_\infty \le e^{\|\partial_x b\|_\infty\, t}$,

(*ii*) $\mathrm{Lip}\;\; \partial_x\, X_b(\cdot,x) \le \|\partial_x b\|_\infty\; e^{\|\partial_x b\|_\infty\, T}$,

(*iii*) $\|X_b(t,\cdot) - X_{\widetilde{b}}(t,\cdot)\|_\infty \le \|b-\widetilde{b}\|_\infty\; t\; e^{\|\partial_x \widetilde{b}\|_\infty\, t}$ *for any* $\widetilde{b} \in W^{1,\infty}(\mathbb{R}^+),\ \widetilde{b}(0) > 0$.

For every weak solution $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+)$, integration by parts provides a characterization using a dual problem in the form of a partial differential equation.

Definition 3.4 *Let* $\psi \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$. *We call* $\varphi_{t,\psi} \in C^1(\mathbb{R}^+ \times [0,t])$ *the solution to the dual problem related to* $\psi(\cdot)$ *and* t *if it satisfies*

$$\begin{cases} \partial_\tau\, \varphi_{t,\psi} + b(x)\,\partial_x\, \varphi_{t,\psi} + c(x)\,\varphi_{t,\psi} + a(x)\,\varphi_{t,\psi}(0,\tau) \;=\; 0, \;\; in\;\; \mathbb{R}^+ \times [0,t], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \varphi_{t,\psi}(\cdot,t) \;=\; \psi, \;\; in\;\; \mathbb{R}^+. \end{cases} \tag{6}$$

The formulation of the dual problem is particularly useful as tool for proving existence of weak solutions. Knowing the solution to the dual problem, the solution to the linear problem (3) is given by the integral formula explicitly stated in Lemma 3.6. In the following lemma we collect the properties of a solution to the dual problem. Its proof is deferred to Appendix A.

Lemma 3.5 *Let* $a,b,c \in W^{1,\infty}(\mathbb{R}^+)$ *and* $b \in C^1(\mathbb{R}^+),\ b(0) > 0$. *For any function* $\psi \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ *and time* $t \in\,]0,T]$, *the solution* $\varphi := \varphi_{t,\psi}$ *to the related dual problem (6) is unique and its equivalent characterization is given by the integral equation*

$$\begin{aligned} \varphi(x,\tau) = {} & \psi\left(X_b(t-\tau,x)\right)\; e^{\int_\tau^t c(X_b(r-\tau,x))\;dr} \\ & + \int_\tau^t a\left(X_b(s-\tau,x)\right)\;\; \varphi(0,s)\; e^{\int_\tau^s c(X_b(r-\tau,x))\;dr}\;\; ds. \end{aligned} \tag{7}$$

Moreover, for any $t > 0$ *and* $\psi \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ *fixed, the following statements hold*

(*i*) $\varphi(0,\cdot) : [0,t] \longrightarrow \mathbb{R}$ *is a bounded and continuously differentiable solution to the following inhomogeneous Volterra equation of second type*

$$\begin{aligned} \varphi(0,\tau) = {} & \psi\left(X_b(t-\tau,0)\right)\; e^{\int_\tau^t c(X_b(r-\tau,0))\;dr} \\ & + \int_\tau^t a\left(X_b(s-\tau,0)\right)\;\; \varphi(0,s)\; e^{\int_\tau^s c(X_b(r-\tau,0))\;dr}\;\; ds, \end{aligned} \tag{8}$$

with $\quad \|\varphi(0,\cdot)\|_\infty \le \sup\limits_{z \,\le\, \|b\|_\infty\, t} |\psi(z)|\;(1 + \|a\|_\infty\, t)\; e^{(\|a\|_\infty + \|c\|_\infty)\,t}$,

$$\|\partial_\tau \varphi(0,\cdot)\|_\infty \le \mathrm{const}(\|a\|_{W^{1,\infty}},\ \|b\|_\infty,\ \|c\|_{W^{1,\infty}})\ \max\{\|\psi\|_\infty, \|\partial_x\psi\|_\infty\}\; e^{2\,(\|a\|_\infty + \|c\|_\infty)\,t}\;(1+t).$$

(ii) $\varphi(x,\cdot):[0,t]\longrightarrow\mathbb{R}$ *is continuously differentiable for each* $x\in\mathbb{R}^+$ *with*

$$\|\partial_\tau\varphi(x,\cdot)\|_\infty \le \mathrm{const}(\|a\|_{W^{1,\infty}},\ \|b\|_\infty,\ \|c\|_{W^{1,\infty}})\ \max\{\|\psi\|_\infty, \|\partial_x\psi\|_\infty\}$$
$$e^{2\,(\|a\|_\infty+\|c\|_\infty)\,t}\ (1+t).$$

(iii) $\varphi(\cdot,\tau):\mathbb{R}^+\longrightarrow\mathbb{R}$ *is continuously differentiable for every* $\tau\in[0,t]$ *and satisfies*

$$\|\varphi(\cdot,\tau)\|_\infty \le \|\psi\|_\infty\, e^{2\,(\|a\|_\infty+\|c\|_\infty)\,t},$$
$$\|\partial_x\varphi(\cdot,\tau)\|_\infty \le \max\{\|\partial_x\psi\|_\infty,1\}\ e^{\max\{\|\psi\|_\infty,1\}\,3(\|a\|_{W^{1,\infty}}+\|\partial_x b\|_\infty+\|c\|_{W^{1,\infty}})\,t}.$$

(iv) For every $t>0$ *and* $\psi\in C^1(\mathbb{R}^+)\cap W^{1,\infty}(\mathbb{R}^+)$ *there exists a continuously differentiable solution* $\varphi:\mathbb{R}^+\times[0,t]\longrightarrow\mathbb{R}$ *to integral equation (7). It is unique and has the regularity properties stated in parts (ii) and (iii).*

Using the solution to the dual problem, we establish below some properties of the measure-valued solution to equation (4).

Lemma 3.6 *Let* $\varphi_{t,\psi}\in C^1(\mathbb{R}^+\times[0,t])$ *denote the solution to the dual problem (6) (or equivalently, the integral equation (7)) for any* $t>0$ *and* $\psi\in C^1(\mathbb{R}^+)\cap W^{1,\infty}(\mathbb{R}^+)$*. For any measure* $\mu_0\in\mathcal{M}(\mathbb{R}^+)$*, let* $\mu:[0,T]\longrightarrow\mathcal{M}(\mathbb{R}^+)$*,* $t\longmapsto\mu_t$*, be given by*

$$\int_{\mathbb{R}^+}\psi(x)\,d\mu_t(x)=\int_{\mathbb{R}^+}\varphi_{t,\psi}(x,0)\,d\mu_0(x). \tag{9}$$

Then

(i) μ *satisfies the following form of the semigroup property for every* $0\le s\le t\le T$ *and* $\psi\in C^1(\mathbb{R}^+)\cap W^{1,\infty}(\mathbb{R}^+)$*:*

$$\int_{\mathbb{R}^+}\psi(x)\,d\mu_t(x)=\int_{\mathbb{R}^+}\varphi_{t,\psi}(x,s)\,d\mu_s(x). \tag{10}$$

(ii) $t\longmapsto\int_{\mathbb{R}^+}\psi\,d\mu_t$ *is Lipschitz continuous for every* $\psi\in C^1(\mathbb{R}^+)\cap W^{1,\infty}(\mathbb{R}^+)$ *with a Lipschitz constant bounded by*

$$C\ =\ const(\|a\|_{W^{1,\infty}},\|b\|_\infty,\|c\|_{W^{1,\infty}},T)\ \|\psi\|_{W^{1,\infty}}\ |\mu_0|(\mathbb{R}^+).$$

Furthermore, $|\mu_t|(\mathbb{R}^+)\le e^{2\,(\|a\|_\infty+\|c\|_\infty)\,t}\ |\mu_0|(\mathbb{R}^+)$*.*

(iii) μ *is a weak solution to the linear problem (3) (in the sense of Definition 3.1).*

(iv) For any $\phi\in C^0(\mathbb{R}^+)$ *such that* $\operatorname{supp}\phi\subset[\|b\|_\infty t,\infty[$*, the following estimate holds with* $\widetilde{\phi}(x):=\sup_{z\le x}\phi(z)$ *:*

$$\int_{\mathbb{R}^+}\widetilde{\phi}(x+\|b\|_\infty t)\,d|\mu_0|(x)\ge e^{-\|c\|_\infty\,t}\int_{\mathbb{R}^+}\phi(x)\,d\mu_t(x).$$

Proof. (i) Choose arbitrary $0\le s<t\le T$ and $\psi\in C^1(\mathbb{R}^+)\cap W^{1,\infty}(\mathbb{R}^+)$. Let $\xi\in C^1(\mathbb{R}^+\times[0,s])$ denote a solution to the semilinear partial differential equation

$$\begin{aligned}\partial_\tau\xi+b(x)\,\partial_x\,\xi\ +\ c(x)\,\xi+a(x)\,\xi(0,\tau)&=0 &&\text{in }\ \mathbb{R}^+\times[0,s],\\ \xi(\cdot,s)&=\varphi_{t,\psi}(\cdot,s) &&\text{in }\ \mathbb{R}^+,\end{aligned}$$

or (as an equivalent formulation) to the integral equation

$$\xi(x,\tau) \;=\; \varphi_{t,\psi}\left(X_b(s-\tau,x),\; s\right) e^{\int_\tau^s c(X_b(r-\tau,x))\,dr} + \int_\tau^s a\left(X_b(\sigma-\tau,x)\right)\,\xi(0,\sigma)\, e^{\int_\tau^\sigma c(X_b(r-\tau,x))\,dr} d\sigma \quad \text{for } (x,\tau)\in\mathbb{R}^+\times[0,s].$$

According to Lemma 3.5 (iv), such a solution exists and is unique since $\varphi_{t,\psi}(\cdot,s)$ is continuously differentiable and bounded in $W^{1,\infty}(\mathbb{R}^+)$. Thus, $\xi \equiv \varphi_{t,\psi}(\cdot,\cdot)|_{\mathbb{R}^+\times[0,s]}$ and, using the duality formula (9), we conclude that

$$\int_{\mathbb{R}^+} \psi(x)\, d\mu_t(x) \;=\; \int_{\mathbb{R}^+} \varphi_{t,\psi}(x,0)\, d\mu_0(x) \;=\; \int_{\mathbb{R}^+} \xi(x,0)\, d\mu_0(x) \;=\; \int_{\mathbb{R}^+} \varphi_{t,\psi}(x,s)\, d\mu_s(x).$$

(ii) The total variation of μ_t can be characterized as a supremum [1, Proposition 1.47]. Therefore, due to Lemma 3.5 (iii),

$$\begin{aligned}
|\mu_t|(\mathbb{R}^+) \;&=\; \sup\Big\{ \int_{\mathbb{R}^+} u(x)\; d\mu_t(x) \Big|\; u\in C_c^0(\mathbb{R}^+),\; \|u\|_\infty \le 1\Big\}\\
&=\; \sup\Big\{ \int_{\mathbb{R}^+} u(x)\; d\mu_t(x) \Big|\; u\in C_c^1(\mathbb{R}^+),\; \|u\|_\infty \le 1\Big\}\\
&\overset{(9)}{=}\; \sup\Big\{ \int_{\mathbb{R}^+} \varphi_{t,u}(x,0)\; d\mu_0(x) \Big|\; u\in C_c^1(\mathbb{R}^+),\; \|u\|_\infty \le 1\Big\}\\
&\le\; \sup\Big\{ \|\varphi_{t,u}(\cdot,0)\|_\infty |\mu_0|(\mathbb{R}^+) \Big|\; u\in C_c^1(\mathbb{R}^+),\; \|u\|_\infty \le 1\Big\}\\
&\le\; e^{2\,(\|a\|_\infty+\|c\|_\infty)\,t}\; |\mu_0|(\mathbb{R}^+).
\end{aligned}$$

Choosing arbitrary $0\le s<t\le T$ and $\psi\in W^{1,\infty}(\mathbb{R}^+)\cap C^1(\mathbb{R}^+)$, we obtain

$$\begin{aligned}
\Big|\int_{\mathbb{R}^+} \psi\, d\mu_t \;-\; \int_{\mathbb{R}^+} \psi\, d\mu_s\Big| &= \Big|\int_{\mathbb{R}^+} \varphi_{t,\psi}(x,s)\;\, d\mu_s(x) \;\;-\; \int_{\mathbb{R}^+} \varphi_{t,\psi}(x,t)\;\, d\mu_s(x)\Big|\\
&\le\; \int_{\mathbb{R}^+} \Big|\varphi_{t,\psi}(x,s) \;-\; \varphi_{t,\psi}(x,t)\Big|\; d|\mu_s|(x)\\
&\le\; (t-s)\;\|\partial_\tau\, \varphi_{t,\psi}\|_\infty \quad |\mu_s|(\mathbb{R}^+)\;\;.
\end{aligned}$$

Lemma 3.5 (ii) implies Lipschitz continuity due to $\psi\in W^{1,\infty}(\mathbb{R}^+)$.

(iii) For arbitrary $\psi\in W^{1,\infty}(\mathbb{R}^+)\cap C^1(\mathbb{R}^+)$ and $t\in\,]0,T]$, we first prove

$$\lim_{h\downarrow 0}\;\; \frac{1}{h}\Big(\int_{\mathbb{R}^+} \psi\, d\mu_t \;-\; \int_{\mathbb{R}^+} \psi\, d\mu_{t-h}\Big) \;=\; \int_{\mathbb{R}^+} \Big(b\,\partial_x\psi \;+\; c\,\psi \;+\; a\,\psi(0)\Big)\, d\mu_t. \tag{11}$$

Indeed, applying (i) allows us to calculate

$$\frac{1}{h}\,\Big(\int_{\mathbb{R}^+} \psi\, d\mu_t \;-\; \int_{\mathbb{R}^+} \psi\, d\mu_{t-h}\Big) = \int_{\mathbb{R}^+} \frac{1}{h}\,\big(\varphi_{t,\psi}(x,\,t-h) \;-\; \psi(x)\big)\, d\mu_{t-h}(x).$$

Note that on compact sets $\frac{1}{h}(\varphi_{t,\psi}(x,t-h)-\psi(x))$ converges to $\partial_t\varphi_{t,\psi}(x,t)$ in the supremum norm as $h\downarrow 0$. Then, applying a truncation function,

$$T_M(x) = \begin{cases} 1 & \text{for } x \in [0, M[, \\ M+1-x & \text{for } x \in]M, M+1[, \\ 0 & \text{for } x \in [M+1, \infty[, \end{cases}$$

defined for arbitrary $M > 0$, we obtain

$$\int_{\mathbb{R}^+} \frac{1}{h} \left(\varphi_{t,\psi}(x, t-h) - \psi(x)\right) d\mu_{t-h}(x) \; = \int_{\mathbb{R}^+} \frac{1}{h} \left(\varphi_{t,\psi}(x, t-h) - \psi(x)\right) T_M(x) d\mu_{t-h}(x)$$
$$+ \int_{\mathbb{R}^+} \frac{1}{h} \left(\varphi_{t,\psi}(x,\, t-h) - \psi(x)\right) (1 - T_M(x))\, d\mu_{t-h}(x).$$

We calculate that

$$\int_{\mathbb{R}^+} \frac{\varphi_{t,\psi}(x,\, t-h) - \psi(x)}{h}\; T_M(x)\; d\mu_{t-h}(x)$$
$$= \int_{\mathbb{R}^+} [\frac{\varphi_{t,\psi}(x,\, t-h) - \psi(x)}{h} - \partial_t \varphi_{t,\psi}(x,t)] T_M(x) d\mu_{t-h}(x) + \int_{\mathbb{R}^+} \partial_t \varphi_{t,\psi}(x,t) T_M(x) d\mu_{t-h}(x).$$

Since the first integral converges to zero and μ_{t-h} converges weak* to μ_t, we obtain

$$\lim_{h \downarrow 0} \int_{\mathbb{R}^+} \frac{\varphi_{t,\psi}(x,\, t-h) \; - \; \psi(x)}{h}\; T_M(x)\; d\mu_{t-h}(x) = \int_{\mathbb{R}^+} \; \partial_t \varphi_{t,\psi}(x,t)\; T_M(x)\; d\mu_t(x).$$

The estimate in (iv) implies that the family of measures μ_t, $t \in [0, T]$ is tight and

$$\left\| \frac{1}{h} (\varphi_{t,\psi}(\cdot, t-h) - \psi(\cdot)) \right\|_\infty \leq \text{const}(\|a\|_{W^{1,\infty}},\; \|b\|_\infty,\; \|c\|_{W^{1,\infty}},\; T)\; \|\psi\|_{W^{1,\infty}},$$

and therefore,

$$\lim_{M \to 0} \sup_{h > 0} \int_{\mathbb{R}^+} \frac{\varphi_{t,\psi}(x,\, t-h) \; - \; \psi(x)}{h}\; (1 - T_M(x))\; d\mu_{t-h}(x) = 0.$$

Finally, equation (11) follows directly from the dual equation (6).

To show that μ is a weak solution to the linear problem (3), we define an auxiliary function $\zeta \,:\, [0, T] \times [0, T] \,\to\, \mathbb{R}$, $\zeta(s, t) \,=\, \int_{\mathbb{R}^+} \varphi(x, t) d\mu_s(x)$. For an arbitrary test function $\varphi \in C^1(\mathbb{R}^+ \times [0, T]) \cap W^{1,\infty}(\mathbb{R}^+ \times [0, T])$, the following statements hold:

$$\frac{\partial}{\partial s}\, \zeta(s, t) \; = \int_{\mathbb{R}^+} \Big(b\; \partial_x\, \varphi(\cdot, t) + c\, \varphi(\cdot, t) + a\; \varphi(0, t)\Big)\; d\mu_s \;\text{ is in } C^0([0, T] \times [0, T]),$$

and

$$\frac{\partial}{\partial t}\,\zeta(s,t) \;=\; \int_{\mathbb{R}^+} \partial_t\varphi(x,t)\;\, d\mu_s(x) \;\text{ is in } C^0([0,T]\times[0,T]).$$

Hence $\zeta(\cdot,\cdot) \in C^1([0,T]\times[0,T])$ and

$$\frac{d}{d\,t}\,\zeta(t,t) = \left(\frac{\partial}{\partial\,t_1}\,\zeta(t_1,t_2) + \frac{\partial}{\partial\,t_2}\,\zeta(t_1,t_2)\right)\Big|_{t_1=t_2=t},$$

which yields that $[0,T] \longrightarrow \mathbb{R},\; t \longmapsto \zeta(t,t)$ is continuously differentiable with

$$\frac{d}{dt}\zeta(t,t) = \int_{\mathbb{R}^+} \Big(b\;\partial_x\,\varphi(\cdot,t) + c\;\varphi(\cdot,t) + a\;\varphi(0,t)\Big)d\mu_t + \int_{\mathbb{R}^+} \partial_t\varphi(\cdot,t)d\mu_t.$$

(iv) supp $\phi \subset [t\,\|b\|_\infty, \infty[$ implies $\|\varphi_{t,\phi}(0,\cdot)\|_\infty = 0$ due to Lemma 3.5 (i). Hence the integral equation (7) for $\varphi_{t,\phi}$ simplifies to

$$\varphi_{t,\phi}(x,\tau) = \phi\,(X_b(t-\tau,x))\;\; e^{\int_\tau^t c(X_b(r-\tau,x))dr}$$

for all $x \in \mathbb{R}^+$ and $\tau \in [0,t]$. Finally, we conclude

$$\begin{aligned}
e^{\|c\|_\infty\,t} \int_{\mathbb{R}^+} \widetilde{\phi}(x + t\,\|b\|_\infty)d|\mu_0|(x) &\geq \int_{\mathbb{R}^+} \widetilde{\phi}(X_b(t,x))\;\, e^{\int_0^t\, c(X_b(r,x))dr}d|\mu_0|(x)\\
&\geq \int_{\mathbb{R}^+} \phi(X_b(t,x))\;\, e^{\int_0^t c(X_b(r,x))dr}d\mu_0(x)\\
&= \int_{\mathbb{R}^+} \varphi_{t,\phi}(x,0)d\mu_0(x) = \int_{\mathbb{R}^+} \phi(x)d\mu_t(x).
\end{aligned}$$

□

We can also exploit the preceding properties to demonstrate nonnegativity preservation of finite Radon measures.

Corollary 3.7 *Under the additional hypothesis that $a(\cdot) \geq 0$, the weak solution μ : $[0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+)$ presented in Lemma 3.6 is a nonnegative Radon measure for every nonnegative initial measure μ_0.*

Proof. The construction of μ_t using equation (9) implies that nonnegativity of measures is preserved if we can ensure that

$$\psi(\cdot) \geq 0 \Longrightarrow \quad \varphi_{t,\psi}(\cdot,0) \geq 0.$$

Setting $x = 0$ in the integral characterization (7) of $\varphi_{t,\psi}$ leads to the Volterra equation (8) for $\varphi_{t,\psi}(0,\cdot)$. Supposing $\psi(\cdot) \geq 0$ implies

$$\begin{aligned}
\varphi_{t,\psi}(0,\tau) &= \psi\,(X_b(t-\tau,0))\,e^{\int_\tau^t c(X_b(r-\tau,0))dr} + \int_\tau^t a\,(X_b(s-\tau,0))\,\varphi_{t,\psi}(0,s)e^{\int_\tau^s c(X_b(r-\tau,0))dr}ds\\
&\geq \int_\tau^t a\,(X_b(s-\tau,0))\;\,\varphi_{t,\psi}(0,s)\;e^{\int_\tau^s c(X_b(r-\tau,0))dr}ds.
\end{aligned}$$

Therefore, for $a(\cdot) \geq 0$,

$$
\begin{aligned}
|\varphi_{t,\psi}(0,\tau)|_- &\leq \Big| \int_\tau^t a\left(X_b(s-\tau,0)\right) \varphi_{t,\psi}(0,s)\, e^{\int_\tau^s c(X_b(r-\tau,0))dr} ds \Big|_- \\
&\leq \int_\tau^t a\left(X_b(s-\tau,0)\right)\ |\varphi_{t,\psi}(0,s)|_-\ e^{\int_\tau^s c(X_b(r-\tau,0))\,dr}\ ds,
\end{aligned}
$$

where $|\xi|_- = -\min\{0,\xi\}$. Using the Gronwall Lemma we conclude that

$$\psi(\cdot) \geq 0 \Longrightarrow \quad \varphi_{t,\psi}(0,\tau) \geq 0, \quad \text{for every } \tau \in [0,\mathrm{t}].$$

□

The preceding lemmas provide more information than the existence of solutions.
Using the construction of Lemma 3.6, we obtain a continuous solution map for the linear problem (3). Furthermore, as we will be prove by means of the following lemma, these solutions depend continuously on the coefficients $a(\cdot)$, $b(\cdot)$, $c(\cdot)$. The proof of the lemma is deferred to Appendix A.

Lemma 3.8 *Suppose $a,\ \widetilde{a},\ c,\ \widetilde{c} \in W^{1,\infty}(\mathbb{R}^+)$, $b,\ \widetilde{b} \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ with $b(0) > 0$ and $\widetilde{b}(0) > 0$. For $t \in]0,1]$, $\lambda \in [0,1]$ and a function $\psi \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ fixed, let $\varphi^\lambda \in C^0(\mathbb{R}^+ \times [0,t])$ satisfy the integral equation*

$$
\begin{aligned}
\varphi^\lambda(x,\tau) = \psi\Big|_{\left(\lambda\ X_b(t-\tau,x) + (1-\lambda)\ X_{\widetilde{b}}(t-\tau,x)\right)}\ & e^{\int_\tau^t \left(\lambda\ c(X_b(r-\tau,x)) + (1-\lambda)\ \widetilde{c}(X_{\widetilde{b}}(r-\tau,x))\right)dr} \\
+ \int_\tau^t \Big(\lambda\ a\left(X_b(s-\tau,x)\right) &+ (1-\lambda)\ \widetilde{a}\left(X_{\widetilde{b}}(s-\tau,x)\right)\Big)\ \varphi^\lambda(0,s) \\
& \cdot e^{\int_\tau^s \left(\lambda\ c(X_b(r-\tau,x)) + (1-\lambda)\ \widetilde{c}(X_{\widetilde{b}}(r-\tau,x))\right)dr} ds. \qquad (12)
\end{aligned}
$$

Then, $\lambda \longmapsto \varphi^\lambda(x,\tau)$ is continuously differentiable for every $x \in \mathbb{R}^+$ and $\tau \in [0,t]$ and there is a constant $C = C(\|a\|_{W^{1,\infty}},\ \|\widetilde{a}\|_{W^{1,\infty}},\ \|b\|_{W^{1,\infty}},\ \|\widetilde{b}\|_{W^{1,\infty}},\ \|c\|_{W^{1,\infty}},\ \|\widetilde{c}\|_{W^{1,\infty}})$ such that
$\left|\frac{\partial}{\partial\lambda}\varphi^\lambda(x,\tau)\right| \leq C \max\{\|\psi\|_\infty, \|\partial_x\psi\|_\infty, 1\}(\|a-\widetilde{a}\|_\infty + \|b-\widetilde{b}\|_\infty + \|c-\widetilde{c}\|_\infty)(t-\tau)e^{C(t-\tau)}$.

The following proposition summarizes the properties of the solutions to linear problem (3), i.e., the existence of a Lipschitz semigroup of solutions, which is acting on the metric space $(\mathcal{M}(\mathbb{R}^+), \rho)$.

Proposition 3.9 *Let $a(\cdot), c(\cdot) \in W^{1,\infty}(\mathbb{R}^+)$ and $b(\cdot) \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ satisfy $b(0) > 0$. The weak solutions to the linear problem (3), characterized in Lemma 3.6, induce a map $\vartheta_{a,b,c} : [0,1] \times \mathcal{M}(\mathbb{R}^+) \longrightarrow \mathcal{M}(\mathbb{R}^+)$, $(t,\mu_0) \longmapsto \mu_t$ satisfying the following conditions for any $\mu_0, \nu_0 \in \mathcal{M}(\mathbb{R}^+)$, $t, h \in [0,1]$, $\widetilde{a}, \widetilde{c} \in W^{1,\infty}(\mathbb{R}^+)$, $\widetilde{b} \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$ with $t+h \leq 1$, $\widetilde{b}(0) > 0$:*

(i) $\vartheta_{a,b,c}(0,\cdot) = \mathrm{Id}_{\mathcal{M}(\mathbb{R}^+)}$,

(ii) $\vartheta_{a,b,c}(h,\ \vartheta_{a,b,c}(t,\mu_0)) = \vartheta_{a,b,c}(t+h,\mu_0)$,

(iii) $|\vartheta_{a,b,c}(h,\mu_0)|(\mathbb{R}^+) \le |\mu_0|(\mathbb{R}^+)\ e^{2\,(\|a\|_\infty + \|c\|_\infty)\,h}$,

(iv) $\rho\left(\vartheta_{a,b,c}(t,\mu_0),\ \vartheta_{a,b,c}(t{+}h,\mu_0)\right) \le\ h\ \mathrm{const}(\|a\|_{W^{1,\infty}}, \|b\|_\infty, \|c\|_{W^{1,\infty}})\ |\mu_0|(\mathbb{R}^+)$,

(v) $\rho\left(\vartheta_{a,b,c}(h,\mu_0),\ \vartheta_{a,b,c}(h,\nu_0)\right) \le\ \rho(\mu_0,\nu_0)\ e^{3\,(\|a\|_{W^{1,\infty}} + \|\partial_x b\|_\infty + \|c\|_{W^{1,\infty}})\,h}$,

(vi) $\rho\left(\vartheta_{a,b,c}(h,\mu_0),\ \vartheta_{\widetilde{a},\widetilde{b},\widetilde{c}}(h,\mu_0)\right) \le\ h\ (\|a-\widetilde{a}\|_\infty + \|b-\widetilde{b}\|_\infty + \|c-\widetilde{c}\|_\infty)\ \widehat{C}\ |\mu_0|(\mathbb{R}^+)$,
with a constant $\widehat{C} = \widehat{C}(\|a\|_{W^{1,\infty}},\ \|\widetilde{a}\|_{W^{1,\infty}},\ \|b\|_{W^{1,\infty}},\ \|\widetilde{b}\|_{W^{1,\infty}},\ \|c\|_{W^{1,\infty}},\ \|\widetilde{c}\|_{W^{1,\infty}})$,

(vii) If additionally $a(\cdot) \ge 0$, *then* $\vartheta_{a,b,c}([0,1], \mathcal{M}^+(\mathbb{R}^+)) \subset \mathcal{M}^+(\mathbb{R}^+)$.

Proof. (i) This is a consequence of equation (9) in Lemma 3.6.

(ii) This results from equation (10) in Lemma 3.6 (i), which can be written in the form

$$\int_{\mathbb{R}^+} \psi(x) d\mu_{t+h}(x) = \int_{\mathbb{R}^+} \varphi_{t+h,\,\psi}(x,t) d\mu_t(x) = \int_{\mathbb{R}^+} \varphi_{h,\,\psi}(x,0) d\mu_t(x).$$

for every $\psi \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$. Indeed, $\varphi_{t+h,\,\psi}(\cdot,t) \equiv \varphi_{h,\,\psi}(\cdot,0)$ results from the partial differential equation (6) characterizing $\varphi_{h,\,\psi}$, because all its coefficients are autonomous.

(iii) This has already been verified in Lemma 3.6 (ii).

(iv) This results directly from Lemma 3.6 (ii) and the definition of $\rho(\cdot,\cdot)$:

$$\begin{aligned}
&\rho\left(\vartheta_{a,b,c}(t,\mu_0), \vartheta_{a,b,c}(t+h,\mu_0)\right) = \\
&= \sup\ \Big\{ \int_{\mathbb{R}^+} \psi d(\vartheta_{a,b,c}(t+h,\mu_0) - \vartheta_{a,b,c}(t,\mu_0))\ \Big|\ \psi \in C^1(\mathbb{R}^+),\ \|\psi\|_\infty \le 1,\ \|\partial_x \psi\|_\infty \le 1 \Big\} \\
&\le h\ \mathrm{const}(\|a\|_{W^{1,\infty}},\ \|b\|_\infty,\ \|c\|_{W^{1,\infty}})\ |\mu_0|(\mathbb{R}^+)
\end{aligned}$$

(v) Choose $\psi \in C^1(\mathbb{R}^+)$ with $\|\psi\|_\infty \le 1$ and $\|\partial_x \psi\|_\infty \le 1$. Employing the notation of Lemma 3.6, we obtain

$$\int_{\mathbb{R}^+} \psi d\Big(\vartheta_{a,b,c}(h,\mu_0)\ -\ \vartheta_{a,b,c}(h,\nu_0)\Big)\ =\ \int_{\mathbb{R}^+} \varphi_{h,\psi}(x,0) d\left(\mu_0 - \nu_0\right)(x),$$

and, due to Lemma 3.5 (iii), $x \longmapsto \varphi_{h,\psi}(x,t)$ is continuously differentiable with

$$\begin{aligned}
\|\varphi_{h,\psi}(\cdot,t)\|_\infty &\le e^{2\,(\|a\|_\infty + \|c\|_\infty)\,h}, \\
\|\partial_x\, \varphi_{h,\psi}(\cdot,t)\|_\infty &\le e^{3\,(\|a\|_{W^{1,\infty}} + \|\partial_x b\|_\infty + \|c\|_{W^{1,\infty}})\,h}.
\end{aligned}$$

Therefore, Remark 2.6 concerning the metric $\rho(\cdot,\cdot)$ implies

$$\begin{aligned}
\int_{\mathbb{R}^+} \varphi_{h,\psi}(\cdot,0)\ d\left(\mu_0 - \nu_0\right) &\le \rho(\mu_0,\nu_0)\ \max\Big\{ e^{2\,(\|a\|_\infty + \|c\|_\infty)h}, e^{3\,(\|a\|_{W^{1,\infty}} + \|\partial_x b\|_\infty + \|c\|_{W^{1,\infty}})h} \Big\} \\
&\le \rho(\mu_0,\nu_0)\ e^{3\,(\|a\|_{W^{1,\infty}} + \|\partial_x b\|_\infty + \|c\|_{W^{1,\infty}})h}
\end{aligned}$$

and thus,

$$\rho\left(\vartheta_{a,b,c}(h,\mu_0),\ \vartheta_{a,b,c}(h,\nu_0)\right)\ \le\ \rho(\mu_0,\nu_0)\, e^{3\,(\|a\|_{W^{1,\infty}}+\|\partial_x b\|_\infty+\|c\|_{W^{1,\infty}})\,h}.$$

(vi) This is based on the estimate in Lemma 3.8 and therefore it uses notation $\varphi^\lambda(\cdot,\cdot)$ for an arbitrarily chosen function $\psi\in C^1(\mathbb{R}^+)$ with $\|\psi\|_\infty\le 1,\ \|\partial_x\psi\|_\infty\le 1$ (see equation (12)). Indeed, Lemma 3.6 implies that for every $\mu_0\in\mathcal{M}(\mathbb{R}^+)$ and $t\in[0,1]$

$$\begin{aligned}\int_{\mathbb{R}^+}\psi d\left(\vartheta_{a,b,c}(t,\mu_0)-\vartheta_{\widetilde{a},\widetilde{b},\widetilde{c}}(t,\mu_0)\right) &= \int_{\mathbb{R}^+}\left(\varphi^1(x,0)-\varphi^0(x,0)\right)d\mu_0(x)\\ &= \int_{\mathbb{R}^+}\int_0^1\frac{\partial}{\partial\lambda}\varphi^\lambda(x,0)\,d\lambda\, d\mu_0(x).\end{aligned}$$

Lemma 3.8 guarantees that for every $x\in\mathbb{R}^+$

$$\left|\tfrac{\partial}{\partial\lambda}\,\varphi^\lambda(x,0)\right|\ \le\ C\left(\|a-\widetilde{a}\|_\infty+\|b-\widetilde{b}\|_\infty+\|c-\widetilde{c}\|_\infty\right)t\,e^{C\,t},$$

with a constant $C=C(\|a\|_{W^{1,\infty}},\ \|\widetilde{a}\|_{W^{1,\infty}},\ \|b\|_{W^{1,\infty}},\ \|\widetilde{b}\|_{W^{1,\infty}},\ \|c\|_{W^{1,\infty}},\ \|\widetilde{c}\|_{W^{1,\infty}})$.

Therefore,

$$\int_{\mathbb{R}^+}\psi d\left(\vartheta_{a,b,c}(t,\mu_0)-\vartheta_{\widetilde{a},\widetilde{b},\widetilde{c}}(t,\mu_0)\right)\le C(\|a-\widetilde{a}\|_\infty+\|b-\widetilde{b}\|_\infty+\|c-\widetilde{c}\|_\infty)\,t\,e^{Ct}|\mu_0|(\mathbb{R}^+).$$

(vii) If additionally $a(\cdot)\ge 0$, then nonnegative initial measures lead to solutions with values in $\mathcal{M}^+(\mathbb{R}^+)$ according to Corollary 3.7. □

Corollary 3.10 *For any functions $a(\cdot),b(\cdot),c(\cdot)\in W^{1,\infty}(\mathbb{R}^+)$ satisfying $b(0)>0$, a map $\vartheta_{a,b,c}:[0,1]\times\mathcal{M}(\mathbb{R}^+)\longrightarrow\mathcal{M}(\mathbb{R}^+)$ can be constructed in such a way that $\vartheta_{a,b,c}(\cdot,\mu_0)$ is a weak solution to the linear problem (3) for each $\mu_0\in\mathcal{M}(\mathbb{R}^+)$ and the statements (i)–(vii) of Proposition 3.9 hold for all $\mu_0,\nu_0\in\mathcal{M}(\mathbb{R}^+)$, $t,h\in[0,1]$, $\widetilde{a},\widetilde{b},\widetilde{c}\in W^{1,\infty}(\mathbb{R}^+)$ with $t+h\le 1$, $\widetilde{b}(0)>0$.*

Proof. The solution map $\vartheta_{a,b,c}:[0,1]\times\mathcal{M}(\mathbb{R}^+)\longrightarrow\mathcal{M}(\mathbb{R}^+)$ is continuous with respect to the coefficients $(a(\cdot),b(\cdot),c(\cdot))$. In particular, Proposition 3.9 (vi) indicates that the distance between two solutions to the problem with the same initial data but a different coefficient $b(\cdot)$ can be estimated by the L^∞ norm of the difference in the values of b. Therefore, we can extend our obtained results to the problems with coefficients $b(\cdot)\in W^{1,\infty}(\mathbb{R}^+)$ that are not continuously differentiable. Indeed, $C^1(\mathbb{R}^+)\cap W^{1,\infty}(\mathbb{R}^+)$ is dense in $W^{1,\infty}(\mathbb{R}^+)$ with respect to the supremum norm and therefore, any $b(\cdot)\in W^{1,\infty}(\mathbb{R}^+)$ can be approximated by a sequence $b^n(\cdot)$ converging to $b(\cdot)$ in $L^\infty(\mathbb{R}^+)$. Since from Theorem 2.7 (ii) the space of measures $\{\mu\in\mathcal{M}(\mathbb{R}^+):\ |\mu|(\mathbb{R}^+)\le r\}$ (with arbitrary $r>0$) is complete with respect to the metric ρ and a sequence of the solutions $\vartheta_{a,b^n,c}(t,\mu_0)$ is bounded, then the Cauchy sequence $\vartheta_{a,b^n,c}(t,\mu_0)$ has a limit $\vartheta_{a,b,c}(t,\mu_0)$ with $b=\lim_{n\to\infty}b^n$. As a consequence, we can extend Proposition 3.9 to coefficients $b(\cdot)\in W^{1,\infty}(\mathbb{R}^+)$ with $b(0)>0$. □

Proposition 3.11 (*Euler compactness*) *Choose the initial measure $\mu_0\in\mathcal{M}(\mathbb{R}^+)$, time $T\in\,]0,\infty[$ and bound $M>0$ arbitrarily. Let $\mathcal{N}=\mathcal{N}(\mu_0,T,M)$ denote the set of all* *measure-valued functions* $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+)$ *constructed in the following piecewise way: For any finite equidistant partition* $0 = t_0 < t_1 < \ldots < t_n = T$ *of* $[0,T]$ *and* n *tuples* $\{(a_j^n, b_j^n, c_j^n)\}_{j=1}^n \subset W^{1,\infty}(\mathbb{R}^+)^3$ *with* $b_j^n(0) > 0$, $\|a_j^n\|_{W^{1,\infty}} + \|b_j^n\|_{W^{1,\infty}} + \|c_j^n\|_{W^{1,\infty}} \leq M$ *for each* $j = 1 \ldots n$

define $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+)$ *by*

$$\mu(0) := \mu_0, \qquad \mu(t) := \vartheta_{a_j^n, b_j^n, c_j^n}(t - t_j,\ \mu(t_{j-1})) \quad \text{for } t \in]t_{j-1}, t_j],\ j = 1 \ldots n.$$

Then the union of all images $\{\mu(t) \mid \mu \in \mathcal{N},\ t \in [0,T]\} \subset \mathcal{M}(\mathbb{R}^+)$ *is tight and relatively compact in the metric space* $(\mathcal{M}(\mathbb{R}^+), \rho)$.

Proof. Note that the constant sequence with all elements equal to $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$ is compact. Therefore, $\{\mu_0\}$ is tight due to Theorem 2.7 (ii) and, Remark 2.4 provides a nondecreasing nonnegative continuous function ϕ_0 with $\lim_{x\to\infty} \phi_0(x) = \infty$ such that

$$\int_{\mathbb{R}^+} \phi_0(x) d|\mu_0| < \infty.$$

Setting $\bar{x} := M\,T \geq \sup_{j\in\{1\ldots n\}} \|b_j^n\|_\infty T$, let us define the nondecreasing nonnegative function $\phi_T \in C^0(\mathbb{R}^+)$ as

$$\phi_T(x) := \begin{cases} 0, & \text{for} \quad x \leq \bar{x}, \\ \phi_0(x - \bar{x}), & \text{for} \quad x > \bar{x}. \end{cases}$$

Considering now any measure-valued function $\mu(\cdot) \in \mathcal{N}$, Lemma 3.6 (iv) implies a uniform integral bound for any function $\psi_T \in C^0(\mathbb{R}^+)$ satisfying $|\psi_T| \leq \phi_T$ and for each time $t \in [0,T]$

$$\int_{\mathbb{R}^+} \psi_T \, d\mu(t) \ \leq\ e^{\|c\|_\infty T} \int_{\mathbb{R}^+} \phi_T \, d|\mu_0| \ \leq\ e^{\|c\|_\infty T} \int_{\mathbb{R}^+} \phi_0 \, d|\mu_0| \ <\ \infty.$$

Therefore, the set of all values $\{\mu(t) |\ \mu \in \mathcal{N}, t \in [0,T]\} \subset \mathcal{M}(\mathbb{R}^+)$ is tight based on Remark 2.4. Furthermore, all total variations $|\mu(t)|(\mathbb{R}^+)$ are uniformly bounded, i.e.,

$$\sup_{\substack{\mu \in \mathcal{N} \\ t \in [0,T]}} |\mu(t)|(\mathbb{R}^+) < \infty,$$

as a consequence of Proposition 3.9 (iii) and Corollary 3.10, and the piecewise construction of each $\mu(\cdot) \in \mathcal{N}$. Hence by Theorem 2.7 (iii) the assertion follows. □

4 Solution to the nonlinear population model

The preceding section revealed some properties of measure-valued solutions to the linear problem (3) (in the distributional sense of equation (4)). Now we consider a nonlinear problem

$$
\begin{cases}
\partial_t \mu_t \;+\; \partial_x \left(F_2(\mu_t, t)\, \mu_t\right) \;=\; F_3(\mu_t, t)\, \mu_t & \text{in } \mathbb{R}^+ \times [0,T] \\
F_2(\mu_t, t)(0)\, \mu_t(0) \;=\; \int\limits_{\mathbb{R}^+} F_1(\mu_t, t)(x)\, d\mu_t(x) & \text{in }]0,T] \\
\mu_0 \;=\; \nu_0
\end{cases}
\tag{13}
$$

with $F : \mathcal{M}(\mathbb{R}^+) \times [0,T] \longrightarrow \{(a,b,c) \in W^{1,\infty}(\mathbb{R}^+)^3 \,|b(0) > 0\}$ and $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ given. By definition, $\mu : [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^+), t \longmapsto \mu_t$, is regarded as a weak solution to the nonlinear problem (13) if it is narrowly continuous with respect to time and, for all test function $\varphi \in C^1(\mathbb{R}^+ \times [0,T]) \cap W^{1,\infty}(\mathbb{R}^+ \times [0,T])$, it satisfies,

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^+} \varphi(x,T)\, d\mu_T(x) - \int\limits_{\mathbb{R}^+} \varphi(x,0)\, d\nu_0(x) \;= \\
&\int\limits_0^T \int\limits_{\mathbb{R}^+} \partial_t \varphi(x,t)\, d\mu_t(x)\, dt + \int\limits_0^T \int\limits_{\mathbb{R}^+} \Big(\partial_x \varphi(x,t) F_2(\mu_t,t)(x) + \varphi(x,t) F_3(\mu_t,t)(x)\Big)\, d\mu_t(x)\, dt \\
&+ \int\limits_0^T \varphi(0,t) \int\limits_{\mathbb{R}^+} F_1(\mu_t,t)(x)\, d\mu_t(x)\, dt.
\end{aligned}
$$

To solve the nonlinear problem (13), we successively freeze the coefficients on an equidistant grid of $[0,T]$ and then investigate the constructed approximations for a vanishing grid size. The procedure is analogous to the Euler method of solving the ordinary differential equation. To describe the properties of the limit of the approximations we apply the framework of mutational equations. A self-contained overview of this approach is presented in Appendix B.

We will consider the so-called transitions on a given metric space. Each transition ϑ indicates the state $\vartheta(t,x)$ that the initial point x reaches at time $t \in [0,1]$. If ϑ satisfies appropriate continuity conditions with respect to both arguments, then the Euler method and a suitable form of sequential compactness provide the tools for solving the initial value problems, formulated in the terms of the mutational equations. In Definition B.1, these conditions on transitions are specified. Corollary 3.10 implies that the solutions $\vartheta_{a,b,c} : [0,1] \times \mathcal{M}(\mathbb{R}^+) \longrightarrow \mathcal{M}(\mathbb{R}^+)$ to the linear problem (3) induce transitions on $\mathcal{M}(\mathbb{R}^+)$ equipped with the flat metric and the total variation.

Corollary 4.1 *For any functions* $a, b, c \in W^{1,\infty}(\mathbb{R}^+)$ *with* $b(0) > 0$, *the map* $\vartheta_{a,b,c}$: $[0,1] \times \mathcal{M}(\mathbb{R}^+) \longrightarrow \mathcal{M}(\mathbb{R}^+), (t, \mu_0) \longmapsto \mu_t$ *defined by the solutions to the linear problem (3) is a transition on* $(\mathcal{M}(\mathbb{R}^+), \rho, |\cdot|)$ *in the sense of Definition B.1 with*

$$
\begin{array}{llcl}
(i) & \alpha(\vartheta_{a,b,c}\,;\ r) & := & 3\left(\|a\|_{W^{1,\infty}} + \|\partial_x b\|_\infty + \|c\|_{W^{1,\infty}}\right),\\
(ii) & \beta(\vartheta_{a,b,c}\,;\ r) & := & \mathrm{const}(\|a\|_{W^{1,\infty}}, \|b\|_\infty, \|c\|_{W^{1,\infty}})\,r,\\
(iii) & \zeta(\vartheta_{a,b,c}) & := & 2\left(\|a\|_\infty + \|c\|_\infty\right),\\
(iv) & D(\vartheta_{a,b,c},\ \vartheta_{\widetilde{a},\widetilde{b},\widetilde{c}};\ r) & \leq & \left(\|a-\widetilde{a}\|_\infty + \|b-\widetilde{b}\|_\infty + \|c-\widetilde{c}\|_\infty\right) \widehat{C}\,r,
\end{array}
$$

with a constant $\widehat{C} = \widehat{C}(\|a\|_{W^{1,\infty}},\ \|\widetilde{a}\|_{W^{1,\infty}},\ \|b\|_{W^{1,\infty}},\ \|\widetilde{b}\|_{W^{1,\infty}},\ \|c\|_{W^{1,\infty}},\ \|\widetilde{c}\|_{W^{1,\infty}})$.

If, in addition, $a(\cdot) \geq 0$*, then the Radon measure* $\vartheta_{a,b,c}(t,\mu_0)$ *is nonnegative for every* $\mu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ *and* $t \in [0,1]$.

The set of all transitions defined as in Corollary 4.1 is denoted $\Theta(\mathcal{M}(\mathbb{R}^+), \rho, |\cdot|)$. Proposition 3.11 states that the tuple $(\mathcal{M}(\mathbb{R}^+), \rho, \Theta(\mathcal{M}(\mathbb{R}^+), \rho, |\cdot|))$ satisfies the conditions of Euler compactness given in Definition B.9.

Remark 4.2 Please note that $\alpha(\vartheta_{a,b,c}; r)$ does not depend on r and that $\zeta(\vartheta_{a,b,c}) \leq \alpha(\vartheta_{a,b,c}; r)$. This estimate for $\zeta(\vartheta_{a,b,c})$ will be used in the conditions of Euler compactness (Def. B.9).

Exploiting the abstract framework of mutational equations, we obtain that continuity of F implies the existence of a mutational solution $\mu : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^+)$ and that $\mu(\cdot)$ is a narrowly continuous weak solution to the nonlinear problem (13).

Theorem 4.3 (Existence)
Suppose that $F : \mathcal{M}(\mathbb{R}^+) \times [0,T] \longrightarrow \{(a,b,c) \in W^{1,\infty}(\mathbb{R}^+)^3 \mid b(0) > 0\}$ *satisfies*

(i) $\sup_{t\in[0,T]} \sup_{\nu\in\mathcal{M}(\mathbb{R}^+)} \|F(\nu,t)\|_{W^{1,\infty}} < \infty$, *and*

(ii) $F : (\mathcal{M}(\mathbb{R}^+), \rho) \times [0,T] \longrightarrow (W^{1,\infty}(\mathbb{R}^+)^3, \|\cdot\|_\infty)$ *is continuous.*

Then, for any initial measure $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$*, there exists a Lipschitz continuous solution* $\mu : [0,T[\longrightarrow (\mathcal{M}(\mathbb{R}^+), \rho)$ *to the mutational equation* $\overset{\circ}{\mu}_t \ni F(\mu_t, t)$ *in* $[0,T[$ *with* $\mu(0) = \nu_0$*, i.e.,*

(a) $\limsup_{h\downarrow 0} \frac{1}{h}\, \rho\left(\vartheta_{F_1(\mu_t,t),\, F_2(\mu_t,t),\, F_3(\mu_t,t)}(h,\mu_t),\, \mu_{t+h}\right) = 0$ *for every* $t \in [0,T[$, *and*

(b) $\sup_{0\leq t<T} |\mu_t|(\mathbb{R}^+) < \infty$.

Moreover, every solution $\mu : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^+)$ *to this mutational equation is a weak solution to the model (13).*
If, in addition, $\nu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ *and* $F_1(\nu,t)(\cdot) \geq 0$ *for every* $\nu \in \mathcal{M}^+(\mathbb{R}^+)$, $t \in [0,T]$*, then the nonlinear population model (13) has a weak solution with values in* $\mathcal{M}^+(\mathbb{R}^+)$.

Proof. We identify $F(\mu,t)$ with the corresponding transition on $(\mathcal{M}(\mathbb{R}^+),\ \rho,\ |\cdot|)$

$$\vartheta_{F_1(\mu,t),\, F_2(\mu,t),\, F_3(\mu,t)} : [0,1] \times \mathcal{M}(\mathbb{R}^+) \longrightarrow \mathcal{M}(\mathbb{R}^+).$$

Proposition B.10 guarantees the existence of a Lipschitz continuous solution $\mu : [0,T[\longrightarrow (\mathcal{M}(\mathbb{R}^+), \rho)$, $t \longmapsto \mu_t$ to the mutational equation $\overset{\circ}{\mu}_t \ni F(\mu_t, t)$ with $\mu_0 = \nu_0$. In particular, $\mu : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^+)$ is narrowly continuous due to Theorem 2.7 (i) and Remark 2.3.

We still have to verify that μ is a weak solution to the nonlinear population model (13). Choose any $\psi(x,t) = \psi_1(t)\psi_2(x)$, where $\psi_1 \in C^\infty([0,T])$ and $\psi_2 \in C_c^\infty(\mathbb{R}^+)$. Then, $\Psi(t) = \int_{\mathbb{R}^+} \psi(x,t)\, d\mu_t(x)$ is Lipschitz continuous, since

$$\begin{aligned}
\Big|\Psi(t) - \Psi(s)\Big| \;&=\; \Big|\int_{\mathbb{R}^+} \psi(x,t)\, d\mu_t - \int_{\mathbb{R}^+} \psi(x,s)\, d\mu_s\Big| \\
\leq \Big|\psi_1(t)\Big|\Big|\int_{\mathbb{R}^+} \psi_2(x) d\mu_t &- \int_{\mathbb{R}^+} \psi_2(x) d\mu_s\Big| + \Big|\psi_1(t) - \psi_1(s)\Big|\Big|\int_{\mathbb{R}^+} \psi_2(x) d\mu_s\Big|
\end{aligned}$$

and

$$\begin{aligned}
\Big|\int_{\mathbb{R}^+} \psi_2 d\mu_t - \int_{\mathbb{R}^+} \psi_2 d\mu_s\Big| \;&\leq\; \max\{\|\psi_2\|_\infty,\, \|\partial_x\psi_2\|_\infty\}\; \rho(\mu_t, \mu_s) \\
&\leq\; \max\{\|\psi_2\|_\infty,\, \|\partial_x\psi_2\|_\infty\}\; L\; |t-s|.
\end{aligned}$$

Choosing $t \in [0,T[$ as a point of differentiability of Ψ, we obtain for $h \in]0,1]$

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^+} \psi_2 d\left(\mu_{t+h} - \vartheta_{F(\mu_t,t)}(h,\mu_t)\right)\Big| \\
&\leq\; \max\{1,\, \|\psi_2\|_\infty,\, \|\partial_x\psi_2\|_\infty\}\; \rho\left(\mu_{t+h},\; \vartheta_{F(\mu_t,t)}(h,\mu_t)\right) \;=\; o(h).
\end{aligned}$$

Therefore, equation (4) implies

$$\begin{aligned}
\Psi'(t) = \psi_1'(t) \int_{\mathbb{R}^+} \psi_2(x)\, d\mu_t + \psi_1(t)\; \lim_{h\downarrow 0} \frac{1}{h} \int_0^h \int_{\mathbb{R}^+} \Big(\psi_2(0) F_1(\mu_t,t)(x) + \partial_x\psi_2(x) F_2(\mu_t,t)(x) \\
+ \psi_2(x) F_3(\mu_t,t)(x)\Big)\, d\vartheta_{F(\mu_t,t)}(s,\mu_t)(x)\;\; ds.
\end{aligned}$$

Remark 2.6 and Proposition 3.9 (iv) yield for any $s \in]0,1]$

$$\begin{aligned}
&\int_{\mathbb{R}^+} \Big(\psi_2(0) F_1(\mu_t,t) + \partial_x\psi_2 F_2(\mu_t,t) + \psi_2 F_3(\mu_t,t)\Big) d\left(\vartheta_{F(\mu_t,t)}(s,\mu_t) \;-\; \mu_t\right) \\
&\leq \mathrm{const}(M, \|\psi\|_{W^{1,\infty}})\; \rho\left(\vartheta_{F(\mu_t,t)}(s,\mu_t),\;\; \mu_t\right) \\
&\leq \mathrm{const}(M, \|\psi\|_{W^{1,\infty}})\; \mathrm{const}(M, \sup_\tau\, |\mu_\tau|(\mathbb{R}^+))\; s
\end{aligned}$$

with a constant $M := \sup_{t\in[0,T]} \sup_{\nu\in\mathcal{M}(\mathbb{R}^+)} \|F(\nu,t)\|_{W^{1,\infty}} < \infty$. Therefore, we obtain

$$\Psi'(t) = \int_{\mathbb{R}^+} \Big(\partial_t\psi(x,t) + \psi(0,t) F_1(\mu_t,t)(x) + \partial_x\psi(x,t) F_2(\mu_t,t)(x) + \psi(x,t) F_3(\mu_t,t)(x)\Big) d\mu_t(x).$$

Finally, integrating with respect to time, we conclude that

$$\int_{\mathbb{R}^+} \psi(x,t) d\mu_t - \int_{\mathbb{R}^+} \psi(x,t) d\nu_0$$

$$= \int_0^t \int_{\mathbb{R}^+} \Big(\partial_t \psi(x,t) + \psi(0,t) F_1(\mu_t, t) + \partial_x \psi(x,t) F_2(\mu_t,t) + \psi(x,t) F_3(\mu_t,t) \Big) d\mu_s ds$$

for $t \in [0,T]$ and $\psi = \psi_1(t)\psi_2(x)$, where $\psi_1 \in C^\infty[0,T]$ and $\psi_2 \in C_c^\infty(\mathbb{R}^+)$.

The linear hull of functions $\psi(x,t) = \psi_1(t)\psi_2(x)$ is dense in $C_c^1([0,T] \times \mathbb{R}^+)$. Tightness of the family μ_t, $t \in [0,T]$ provides that the set of test functions in weak formulation $C_c^1([0,T] \times \mathbb{R}^+)$ can be replaced with $C^1(\mathbb{R}^+ \times [0,T]) \cap W^{1,\infty}(\mathbb{R}^+ \times [0,T])$. Therefore, we conclude that μ_t is a weak solution to the nonlinear problem (13).

To conclude, we prove nonnegativity preservation of the Radon measures μ_t. Suppose that $F_1(\nu,t) \in W^{1,\infty}(\mathbb{R}^+)$ is nonnegative for every $\nu \in \mathcal{M}^+(\mathbb{R}^+)$, $t \in [0,T]$. Then the piecewise Euler approximations used in Proposition B.10 have values in $\mathcal{M}^+(\mathbb{R}^+)$ due to Corollary 4.1. As $\mathcal{M}^+(\mathbb{R}^+)$ is closed in $(\mathcal{M}(\mathbb{R}^+), \rho)$, all values of the resulting solution μ of $\overset{\circ}{\mu}_t \ni F(\mu_t, t)$ are also contained in $\mathcal{M}^+(\mathbb{R}^+)$. □

The relationship between the narrow convergence on tight sets and the flat metric ρ formulated in Theorem 2.7 (i) and Remark 2.3 leads to the following corollary.

Corollary 4.4 *Suppose that* $F : \mathcal{M}(\mathbb{R}^+) \times [0,T] \longrightarrow \{(a,b,c) \in W^{1,\infty}(\mathbb{R}^+)^3 \mid b(0) > 0\}$ *satisfies*

(i) $\sup_{t \in [0,T]} \sup_{\nu \in \mathcal{M}(\mathbb{R}^+)} \|F(\nu,t)\|_{W^{1,\infty}} < \infty.$

(ii) $F : (\mathcal{M}(\mathbb{R}^+), \textit{narrow}) \times [0,T] \longrightarrow (W^{1,\infty}(\mathbb{R}^+)^3, \|\cdot\|_\infty)$ *is continuous.*

Then, for any initial measure $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$, *there exists a narrowly continuous weak solution* $\mu : [0,T[\longrightarrow \mathcal{M}(\mathbb{R}^+)$ *to the nonlinear population model (13) with* $\mu(0) = \nu_0$.
If, in addition, $\nu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ *and* $F_1(\nu,t)(\cdot) \geq 0$ *for every* $\nu \in \mathcal{M}^+(\mathbb{R}^+)$, $t \in [0,T]$, *then the solution* $\mu(\cdot)$ *has values in* $\mathcal{M}^+(\mathbb{R}^+)$.

Proof. Due to Proposition 3.11, the values of all possible Euler approximations belong to a tight compact subset $\mathcal{N}$ of the metric space $(\mathcal{M}(\mathbb{R}^+), \rho)$ with $\sup_{\nu \in \mathcal{N}} |\nu|(\mathbb{R}^+) < \infty$. In particular, $\mu(t) \in \mathcal{N}$ for each $t \in [0,T]$, since $\mu(t)$ is defined as the limit of a convergent subsequence in $\mathcal{N}$. Therefore, we can restrict further considerations to the compact metric space $(\mathcal{N}, \rho)$ instead of $(\mathcal{M}(\mathbb{R}^+), \rho)$. According to Theorem 2.7, $\mathcal{N}$ is tight, and thus the narrow topology on $\mathcal{N}$ is metrized by the flat metric ρ.
As a consequence, the restriction $F : (\mathcal{N}, \rho) \times [0,T] \longrightarrow (W^{1,\infty}(\mathbb{R}^+)^3, \|\cdot\|_\infty)$ is continuous. Similarly as in Theorem 4.3, we obtain the existence of a Lipschitz continuous function $\mu : [0,T] \longrightarrow (\mathcal{N}, \rho)$ that is a weak solution to the nonlinear population model (13). Finally, Theorem 2.7 guarantees that $\mu : [0,T] \longrightarrow \mathcal{N} \subset \mathcal{M}(\mathbb{R}^+)$ is narrowly continuous. □

Finally, we consider stability with respect to the initial measures and model coefficients. In the framework of mutational equations, the Cauchy-Lipschitz Theorem holds in the following sense: Lipschitz continuity of the right-hand side (with respect to state) implies uniqueness of solutions to each initial value problem. The distance between solutions to two nonautonomous mutational equations can be estimated if the right-hand side of at least one of these mutational equations is Lipschitz continuous with respect to state

(Proposition B.13).
Now we apply this estimate (with respect to the flat metric ρ) to weak solutions to the nonlinear model (13) that also solve the corresponding mutational equation. In combination with Corollary 4.1 (iv), we obtain

Theorem 4.5 (Stability)
Assume that for $F, G : \mathcal{M}(\mathbb{R}^+) \times [0,T] \longrightarrow \{(a,b,c) \in W^{1,\infty}(\mathbb{R}^+)^3 |\ b(0) > 0\}$,

$$(i)\quad M_F := \sup_{t\in[0,T]} \sup_{\mu\in\mathcal{M}(\mathbb{R}^+)} \|F(\mu,t)\|_{W^{1,\infty}} < \infty,$$
$$M_G := \sup_{t\in[0,T]} \sup_{\mu\in\mathcal{M}(\mathbb{R}^+)} \|G(\mu,t)\|_{W^{1,\infty}} < \infty,$$

(ii) *for any* $R > 0$, *there exists a constant* $L_R > 0$ *and a modulus of continuity* $\omega_R(\cdot)$,

with $\|F(\mu,s) - F(\nu,t)\|_\infty \leq L_R \cdot \rho(\mu,\nu) + \omega_R(|t-s|)$

for all $\mu,\nu \in \mathcal{M}(\mathbb{R}^+)$ *with* $|\mu|(\mathbb{R}^+),\ |\nu|(\mathbb{R}^+) \leq R$.

(iii) $G : (\mathcal{M}(\mathbb{R}^+),\rho) \times [0,T] \longrightarrow (W^{1,\infty}(\mathbb{R}^+)^3, \|\cdot\|_\infty)$ *is continuous.*

Let $\mu,\ \nu : [0,T[\longrightarrow (\mathcal{M}(\mathbb{R}^+),\rho)$ *denote solutions to the mutational equations* $\overset{\circ}{\mu}_t \ni F(\mu_t,t)$ *and* $\overset{\circ}{\nu}_t \ni G(\nu_t,t)$ *with given initial data* $\mu_0 \in \mathcal{M}(\mathbb{R}^+)$ *and* $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ *respectively.*

Then, for all $t \in [0,T[$, *it holds*

$$\rho(\mu_t,\nu_t) \leq \Big(\rho(\mu_0,\nu_0) + \mathrm{const}(M_F, M_G, |\mu_0|(\mathbb{R}^+), |\nu_0|(\mathbb{R}^+))\ \|F-G\|_\infty\ t\Big)\ e^{\ const(F)\ t}.$$

Finally, we show that in the class of Lipschitz continuous solutions with bounded total variations a weak solution is also a solution to the corresponding mutational equation and, therefore, based on Theorem 4.5, it is unique.

Theorem 4.6 (Uniqueness of the weak solutions)
Under the assumptions (i)–(iii) of Theorem 4.5, a Lipschitz continuous weak solution to equation (13) $\mu : [0,T[\longrightarrow (\mathcal{M}(\mathbb{R}^+),\rho)$ *with bounded total variation is unique.*

Proof. We show that every Lipschitz continuous weak solution with bounded total variation solves the mutational equation $\overset{\circ}{\mu}_t \ni F(\mu_t,t)$. Based on the definition of the weak solution we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^+} \psi\ d\Big(\mu_{t+h} - \vartheta_{F(\mu_t,t)}(h,\mu_t)\Big) = \\
&= \int_t^{t+h} \Big(\int_{\mathbb{R}^+} \psi(0)\Big(F_1(\mu_s,s) - F_1(\vartheta(s-t,\mu_t),s)\Big) + \partial_x\psi\Big(F_2(\mu_s,s) - F_2(\vartheta(s-t,\mu_t),s)\Big) \\
&\quad + \psi\Big(F_3(\mu_s,s) - F_3(\vartheta(s-t,\mu_t),s)\Big)\Big) d\mu_s ds + \int_t^{t+h}\int_{\mathbb{R}^+} \Big(\psi(0)F_1(\vartheta(s-t,\mu_t),s) \\
&\quad + \partial_x\psi F_2(\vartheta(s-t,\mu_t),s) + \psi F_3(\vartheta(s-t,\mu_t),s)\Big)\ d(\mu_s - \vartheta(s-t,\mu_t))\ ds,
\end{aligned}$$

where ϑ denotes $\vartheta_{F(\mu_t,t)}$. Using continuity of μ_t and Lipschitz continuity of F_i in the flat metric we conclude that $\frac{1}{h}\int\limits_{\mathbb{R}^+} \psi d\Big(\mu_{t+h} - \vartheta_{F(\mu_t,t)}(h,\mu_t)\Big)$ tends to zero as $h \downarrow 0$, uniformly for $\|\psi\|_{W^{1,\infty}} \leq 1$. Therefore, based on Definition B.7, every weak solution μ is a solution to the mutational equation $\overset{\circ}{\mu}_t \ni F(\mu_t, t)$. Then, Theorem 4.5 implies uniqueness. □

A Auxiliary results about semilinear differential equations and Volterra integral equations

Proof of Lemma 3.5 We start with the proof of the characterization (7). Notice that for any $t > 0$ fixed, $\widetilde{b} \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $\widetilde{c} \in W^{1,\infty}(\mathbb{R}^+)$ and $\widetilde{f} \in W^{1,\infty}(\mathbb{R}^+ \times [0,t])$ with $\widetilde{b}(0) < 0$ and every $\psi \in C^1(\mathbb{R}^+)$, the semilinear initial value problem

$$\begin{cases} \partial_\tau \xi(x,\tau) \ + \ \widetilde{b}(x)\partial_x \xi(x,\tau) \ + \ \widetilde{c}(x)\, \xi(x,\tau) \ + \ \widetilde{f}(x,\tau) = 0 \ \text{ in } \ \mathbb{R}^+ \times [0,t] \\ \hfill \xi(\cdot, 0) = \psi \ \text{ in } \ \mathbb{R}^+ \end{cases}$$

has a unique solution $\xi \in C^1(\mathbb{R}^+ \times [0,t])$ given explicitly by

$$\begin{aligned} \xi(x,\tau) = \quad & \psi\left(X_{-\widetilde{b}}(\tau,x)\right) \ e^{-\int_0^\tau \widetilde{c}(X_{-\widetilde{b}}(\tau-r,x))\, dr} \\ & - \int\limits_0^\tau \widetilde{f}\left(X_{-\widetilde{b}}(\tau-s,x),\ s\right) \ e^{-\int_s^\tau \widetilde{c}(X_{-\widetilde{b}}(\tau-r,x))\, dr}\, ds. \end{aligned}$$

This explicit representation of $\xi(x,\tau)$ results from the classical method of characteristics. It was presented in reference [8] for the corresponding problem in $\mathbb{R}^n$, instead of $\mathbb{R}^+$. Since $\widetilde{b}(0) < 0$, i.e., $\mathbb{R}^+$ is invariant under the characteristic flow of $-\widetilde{b}(\cdot)$, the expression obtained in [8] can be restricted to $\mathbb{R}^+$.
Substituting $\varphi(x,\tau) := \xi(x, t-\tau)$ yields the solution to the corresponding partial differential equation with an end-time condition and the coefficients $b(\cdot)$ and $c(\cdot)$ satisfying $b(0) > 0$. Indeed, let $t > 0$, $b \in C^1(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $c \in W^{1,\infty}(\mathbb{R}^+)$ and $f \in W^{1,\infty}(\mathbb{R}^+ \times [0,t])$ be arbitrary with $b(0) > 0$. For any function $\psi \in C^1(\mathbb{R}^+)$, the semilinear partial differential equation

$$\begin{cases} \partial_\tau \varphi(x,\tau) \ + \ b(x)\partial_x \varphi(x,\tau) \ + \ c(x)\, \varphi(x,\tau) \ + \ f(x,\tau) = 0 \ \text{ in } \ \mathbb{R}^+ \times [0,t], \\ \hfill \varphi(\cdot, t) = \psi \ \text{ in } \ \mathbb{R}^+, \end{cases}$$

has a unique solution $\varphi \in C^1(\mathbb{R}^+ \times [0,t])$ explicitly given by

$$\begin{aligned} \varphi(x,\tau) = \quad & \psi\left(X_b(t-\tau,x)\right) \ e^{\int_\tau^t c(X_b(r-\tau,x))\ dr} \\ & + \int\limits_\tau^t f\left(X_b(s-\tau,x),\ s\right) \ e^{\int_\tau^s c(X_b(r-\tau,x))\ dr}\, ds. \end{aligned}$$

Applying this result to $f(x,\tau) = a(x)\ \varphi(0,\tau)$, we obtain the equivalence between equations (6) and (7) for every $\varphi \in C^1(\mathbb{R}^+ \times [0,t])$ (with Lipschitz continuous $\varphi(0,\cdot)$).

We proceed with the proof of the items (i)–(v):

(i) Volterra equation (8) directly results from equation (7) by setting $x = 0$. The upper bound of $|\varphi(0,\cdot)|$, restricted to $[0,t]$, is a consequence of

$$|\varphi(0,\tau)|e^{\|c\|_\infty \tau} \leq \sup_{z\leq \|b\|_\infty t} |\psi(z)|e^{\|c\|_\infty t} + \|a\|_\infty \int_\tau^t |\varphi(0,s)|e^{\|c\|_\infty s} ds$$

and Gronwall's Lemma.
Moreover, the right-hand side of Volterra equation (8) is continuously differentiable with respect to τ and thus, $\varphi(0,\cdot) \in C^1([0,t])$. The product rule reveals that at every $\tau \in [0,t]$

$$\begin{aligned}\left|\frac{d}{d\tau}\varphi(0,\tau)\right| \leq\, & e^{\|c\|_\infty (t-\tau)}\Big(\|\partial_x\psi\|_\infty\|b\|_\infty + \|\psi\|_\infty\Big(\|c\|_\infty + (t-\tau)\|\partial_x c\|_\infty\|b\|_\infty\Big)\Big)\\ & + e^{\|c\|_\infty (t-\tau)}\Big(\|a\|_\infty\|\varphi(0,\cdot)\|_\infty + (t-\tau)\Big(\|\partial_x a\|_\infty\|b\|_\infty\|\varphi(0,\cdot)\|_\infty\\ & + \|a\|_\infty\|\varphi(0,\cdot)\|_\infty\Big(\|c\|_\infty + t\|\partial_x c\|_\infty\|b\|_\infty\Big)\Big)\Big).\end{aligned}$$

(ii) For a fixed arbitrary $x \in \mathbb{R}^+$, $\varphi(x,\cdot) : [0,t] \longrightarrow \mathbb{R}$ is continuously differentiable since it satisfies the integral equation (7) and $\varphi(0,\cdot)$ is continuous. The upper bound of the derivative of $\|\partial_\tau\varphi(x,\cdot)\|_\infty$ results from considerations similar to those concerning $\sup|\partial_\tau\varphi(0,\cdot)|$ in (i).

(iii) The upper bound of $\|\varphi(\cdot,\tau)\|_\infty$ directly results from the integral equation (7) and property (i)

$$\begin{aligned}\|\varphi(\cdot,\tau)\|_\infty &\leq \|\psi\|_\infty \left(e^{\|c\|_\infty t} + \int_0^t \|a\|_\infty(1+\|a\|_\infty s)e^{(\|a\|_\infty + \|c\|_\infty)\, s} e^{\|c\|_\infty s} ds\right)\\ &\leq \|\psi\|_\infty\Big(e^{\|c\|_\infty t} + \|a\|_\infty \int_0^t (1 + (\|a\|_\infty + 2\|c\|_\infty)s)e^{(\|a\|_\infty + 2\|c\|_\infty)\, s} ds\Big)\\ &= \|\psi\|_\infty\Big(e^{\|c\|_\infty t} + \|a\|_\infty t e^{(\|a\|_\infty + 2\|c\|_\infty)\, t}\Big)\\ &\leq \|\psi\|_\infty e^{(\|a\|_\infty + 2\|c\|_\infty)\, t}\Big(1 + \|a\|_\infty\, t\Big) \leq \|\psi\|_\infty e^{(2\|a\|_\infty + 2\|c\|_\infty)\, t}.\end{aligned}$$

The last inequality results from $1 + s \leq e^s$ for all $s \geq 0$. The form of the right-hand side of integral equation (7) ensures that $\varphi(\cdot,\tau) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is continuously differentiable for every $\tau \in [0,t]$. Furthermore, for every $x \in \mathbb{R}^+$, the chain rule and Lemma 3.3 imply

$$\begin{aligned}&\left|\tfrac{\partial}{\partial x}\,\varphi(x,\tau)\right| e^{\|c\|_\infty (\tau - t)} \leq\\ &\leq \|\partial_x\psi\|_\infty\|\partial_x X_b(t-\tau,\cdot)\|_\infty + \|\psi\|_\infty \int_\tau^t \|\partial_x c\|_\infty\|\partial_x X_b(r-\tau,\cdot)\|_\infty dr\\ &+ \int_\tau^t \Big(\|\partial_x a\|_\infty\|\partial_x X_b(s-\tau,\cdot)\|_\infty + \|a\|_\infty \int_\tau^s \|\partial_x c\|_\infty\, \|\partial_x X_b(r-\tau,\cdot)\|_\infty dr\Big)|\varphi(0,s)|ds,\end{aligned}$$

and thus due to property (i),

$$\|\partial_x\varphi\|_\infty \leq \|\partial_x\psi\|_\infty e^{(\|\partial_x b\|_\infty + \|c\|_\infty)\, t} + \|\psi\|_\infty\, \|\partial_x c\|_\infty\, e^{(\|\partial_x b\|_\infty + \|c\|_\infty)\, t}\, t$$

$$
\begin{aligned}
&+ \|\psi\|_\infty \, e^{(2\,\|a\|_\infty + \|\partial_x b\|_\infty + 2\,\|c\|_\infty)\,t} \Big(\|\partial_x a\|_\infty \, t + \|a\|_\infty \|\partial_x c\|_\infty \frac{t^2}{2} \Big) \\
\le\, & \max\{\|\partial_x \psi\|_\infty, 1\} e^{(2\,\|a\|_\infty + \|\partial_x b\|_\infty + 2\,\|c\|_\infty)\,t} \, \Big(1 + \|\psi\|_\infty \, (\|\partial_x c\|_\infty + \|\partial_x a\|_\infty) \, t \\
&+ \|\psi\|_\infty \, \|a\|_\infty \, \|\partial_x c\|_\infty \frac{t^2}{2} \Big) \\
\le\, & \max\{\|\partial_x \psi\|_\infty, 1\} \; e^{\max\{\|\psi\|_\infty, 1\} \;\, 3\,(\|a\|_{W^{1,\infty}} + \|\partial_x b\|_\infty + \|c\|_{W^{1,\infty}})\,t}.
\end{aligned}
$$

(iv) Volterra equation (8) has a unique continuous solution, since the integrand is Lipschitz continuous with respect to $\varphi(0, s)$ [24,26]. Therefore, the solution to the integral equation (7) is also the unique continuously differentiable solution to the equation (8).

Proof of Lemma 3.8 Similarly to Lemma 3.5, $\tau \longmapsto \varphi^\lambda(0, \tau)$ is a bounded and Lipschitz continuous solution to the following inhomogeneous Volterra equation of the second type

$$
\begin{aligned}
\varphi^\lambda(0, \tau) = \psi\Big|_{\left(\lambda X_b(t-\tau,0) + (1-\lambda) X_{\widetilde{b}}(t-\tau,0)\right)} & e^{\int_\tau^t \left(\lambda c(X_b(r-\tau,0)) \,+\, (1-\lambda)\widetilde{c}(X_{\widetilde{b}}(r-\tau,0))\right) \, dr} \\
+ \int_\tau^t & \Big(\lambda a\left(X_b(s-\tau, 0)\right) \;+\; (1-\lambda)\widetilde{a}\left(X_{\widetilde{b}}(s-\tau, 0)\right)\Big) \, \varphi^\lambda(0, s) \\
& e^{\int_\tau^s \left(\lambda c(X_b(r-\tau,0)) \,+\, (1-\lambda)\widetilde{c}(X_{\widetilde{b}}(r-\tau,0))\right) dr} ds.
\end{aligned}
$$

The bounds on the $\|\cdot\|_\infty$ norm and the Lipschitz constant mentioned in Lemma 3.5 (i) can be adapted by considering $\max\{\|a\|_{W^{1,\infty}}, \|\widetilde{a}\|_{W^{1,\infty}}\}$ instead of $\|a\|_{W^{1,\infty}}$ and so forth. According to reference [26] and [27], $\varphi^\lambda(0, \tau)$ depends on the parameter λ in a continuously differentiable way and, using the abbreviations $\widehat{a} := \max\{\|a\|_\infty, \ \|\widetilde{a}\|_\infty\}$, $\widehat{c} := \max\{\|c\|_\infty, \|\widetilde{c}\|_\infty\}$,

$$
\begin{aligned}
\left| \frac{\partial}{\partial\,\lambda} \varphi^\lambda(0, \tau) \right| \le\; & e^{\widehat{c}(t-\tau)} \, \Big(\|\partial_x \psi\|_\infty \, |X_b(t-\tau, 0) - X_{\widetilde{b}}(t-\tau, 0)| \\
&+ \|\psi\|_\infty (t-\tau) \, (\|c - \widetilde{c}\|_\infty + \|\partial_x c\|_\infty \sup_{[\tau,t]} \, |X_b(s-\tau, 0) - X_{\widetilde{b}}(s-\tau, 0)|) \Big) \\
&+ e^{\widehat{c}(t-\tau)} \int_\tau^t \Big(|\varphi^\lambda(0, s)| \, (\|a - \widetilde{a}\|_\infty + \|\partial_x a\|_\infty \, |X_b(s-\tau, 0) - X_{\widetilde{b}}(s-\tau, 0)|) \\
&+ |\partial_\lambda \, \varphi^\lambda(0, s)| \widehat{a} \;+\; |\varphi^\lambda(0, s)| \; \widehat{a} \; (s-\tau) (\|c - \widetilde{c}\|_\infty \\
&+ \|\partial_x c\|_\infty \sup_{[\tau,s]} \, |X_b(s-\tau, 0) - X_{\widetilde{b}}(s-\tau, 0)|) \Big) ds.
\end{aligned}
$$

Lemma 3.3 provides the estimate

$$
\|X_b(s, \cdot) - X_{\widetilde{b}}(s, \cdot)\|_\infty \;\le\; \|b - \widetilde{b}\|_\infty \; s \; e^{\|\partial_x b\|_\infty s}
$$

for all $s \ge 0$ and thus, Gronwall's Lemma implies the bound

$$
\left| \tfrac{\partial}{\partial\,\lambda} \; \varphi^\lambda(0, \tau) \right| \le C_0 \; \max\{\|\psi\|_\infty, \|\partial_x \psi\|_\infty, 1\} \; (\|a - \widetilde{a}\|_\infty + \|b - \widetilde{b}\|_\infty + \|c - \widetilde{c}\|_\infty) \; (t-\tau) \; e^{C_0 \, (t-\tau)}
$$

with a constant $C_0 = C_0(\|a\|_{W^{1,\infty}}, \ \|\widetilde{a}\|_{W^{1,\infty}}, \ \|b\|_{W^{1,\infty}}, \ \|\widetilde{b}\|_{W^{1,\infty}}, \ \|c\|_{W^{1,\infty}}, \ \|\widetilde{c}\|_{W^{1,\infty}})$. Integral equation (12) ensures that $\varphi^\lambda(x, \tau)$ is continuously differentiable with respect to the parameter λ. Similarly to the preceding estimate of $\left|\frac{\partial}{\partial\lambda}\,\varphi^\lambda(0,\tau)\right|$, the differentiation of equation (12) yields for all x, τ

$$\left|\frac{\partial}{\partial\lambda}\,\varphi^\lambda(x,\tau)\right| \;\le\; C\;\max\{\|\psi\|_\infty, \|\partial_x\psi\|_\infty, 1\}\;(\|a-\widetilde{a}\|_\infty+\|b-\widetilde{b}\|_\infty+\|c-\widetilde{c}\|_\infty)\;(t-\tau)\;\;e^{C\,(t-\tau)}$$

with a constant $C = C(\|a\|_{W^{1,\infty}},\ \|\widetilde{a}\|_{W^{1,\infty}},\ \|b\|_{W^{1,\infty}},\ \|\widetilde{b}\|_{W^{1,\infty}},\ \|c\|_{W^{1,\infty}},\ \|\widetilde{c}\|_{W^{1,\infty}})$. □

B Mutational equations in a metric space

This appendix provides a self-contained overview of mutational equations. The framework of mutational equations provides an abstract tool for bridging the gap between the linear problem (in Section 3) and the nonlinear problem (in Section 4). Mutational equations were introduced by Aubin [4,5] in the 1990s to extend ordinary differential equations to metric spaces. The fundamental idea is to replace the directional vector $v \in \mathbb{R}^N$ by the corresponding elementary deformation $(h,x) \longmapsto x+hv$. From the abstract point of view, this so-called transition ϑ is to specify the state $\vartheta(h,x)$ that the initial point x reaches at time h. Choosing suitable continuity assumptions about $(h,x) \longmapsto \vartheta(h,x)$, we proceed in the metric space (E,d) analogously to the case of ordinary differential equations. Using this method, Aubin proved the counterparts of the Cauchy-Lipschitz Theorem and the Invariance Theorem of Nagumo [4].

Our version is slightly more general than Aubin's original form (but not completely covered by earlier journal articles such as [15]). In brief, the parameters of transitions here may depend on the absolute value of the current element of E. The linear problem in Section 3 serves as an example that we cannot always expect the Lipschitz continuity of $t \mapsto \vartheta(t,x)$ to be uniform with respect to all initial states $x \in E$ (as Aubin did following his geometric motivation). For example, doubling the initial measure $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ in problem (3) leads to the doubling of the solution $\mu(\cdot) = \vartheta_{a,b,c}(\cdot,\nu_0) : [0,1] \longrightarrow \mathcal{M}(\mathbb{R}^+)$.

We introduce an analog of absolute value of elements in the metric space (E,d): $\lfloor\cdot\rfloor : E \longrightarrow [0,\infty[$. Parameters of transitions are now assumed uniform in all balls $\{x \in E \,|\, \lfloor x\rfloor \le r\}$ with radius $r > 0$. The proofs do not substantially change if we impose bound on $\lfloor\vartheta(\cdot,x)\rfloor$, for each initial element $x \in E$. After specifying the conditions on transitions, we use them to define time derivatives of curves $x : [0,T] \longrightarrow (E,d)$ as first-order approximation. The set of all transitions satisfying the first-order approximation condition at a given time t is called mutation $\overset{\circ}{x}(t)$.

Finally, we employ the notions of the Peano Theorem for ODEs and construct solutions to the corresponding initial value problem by the Euler method as well as suitable form of sequential compactness in E. Finally, the counterpart of the Cauchy-Lipschitz Theorem will be verified and sufficient conditions for the stability of the solutions will be provided.

Assumptions used in Appendix B E is a nonempty set and $d : E\times E \longrightarrow [0,\infty[$ is a metric on E. Furthermore, let $\lfloor\cdot\rfloor : (E,d) \longrightarrow [0,\infty[$ be an arbitrary lower semicontinuous function, which plays the role of a norm on E, but does not have to satisfy structural conditions such as homogeneity or the triangle inequality.

Now we specify the tools for describing deformations in the tuple $(E, d, \lfloor\cdot\rfloor)$. A map $\vartheta : [0,1] \times E \longrightarrow E$ defines which point $\vartheta(t,x) \in E$ is reached from the initial point $x \in E$ after time t. Of course, ϑ has to satisfy regularity conditions if it is to serve as a basis for a differential calculus.

Definition B.1 *A function $\vartheta : [0,1] \times E \longrightarrow E$ is called a transition on $(E, d, \lfloor\cdot\rfloor)$ if it satisfies the following conditions:*
(i) $\vartheta(0,\cdot) = \mathrm{Id}_E$,
(ii) $\lim_{h \downarrow 0} \frac{1}{h} d\left(\vartheta(t+h,x),\ \vartheta(h, \vartheta(t,x))\right) = 0$, *for all $x \in E$, $t \in [0,1[$,*
(iii) there exists a parameter function $\alpha(\vartheta;\ \cdot) : [0,\infty[\longrightarrow [0,\infty[$ such that

$$\limsup_{h \downarrow 0} \frac{d(\vartheta(h,x), \vartheta(h,y)) - d(x,y)}{h} \le \alpha(\vartheta; r) d(x,y),$$

for all $x, y \in E$, $r \ge 0$ with $\lfloor x\rfloor, \lfloor y\rfloor \le r$,
(iv) there exists a parameter function $\beta(\vartheta;\ \cdot) : [0,\infty[\longrightarrow [0,\infty[$ such that

$$d(\vartheta(s,x), \vartheta(t,x)) \ \le\ \beta(\vartheta;\ r)\ \cdot\ |t-s|$$

for all $x \in E$, $r \ge 0$, $s,t \in [0,1]$ with $\lfloor x\rfloor \le r$, and
(v) there exists a constant $\zeta(\vartheta) \in [0,\infty[$ such that

$$\lfloor \vartheta(h,x)\rfloor \le \lfloor x\rfloor e^{\zeta(\vartheta) h} + \zeta(\vartheta) h,$$

for all $x \in E$, $h \in [0,1]$.

Remark B.2 The first two conditions are motivated by the defining properties of semigroups, but in property (ii), the first-order change with respect to time is assumed to vanish.
Property (iii) imposes a special form of continuity with respect to the initial element. It implies that the initial distance of two points x, $y \in E$ may grow at most exponentially in time while evolving along the same transition ϑ and the corresponding exponent can be uniformly chosen on each ball $\{x \in E \mid \lfloor x\rfloor \le r\}$, $r \ge 0$.
Property (iv) guarantees Lipschitz continuity of $\vartheta(\cdot, x)$ for each initial point $x \in E$. Similarly as in property (iii), the Lipschitz constant may depend on $\lfloor x\rfloor$ and these dependencies are new in comparison to Aubin's original definition of transitions [4,5].
Finally, we need a bound on the absolute value $\lfloor \vartheta(h,x)\rfloor$ depending on both arguments. The combination of the exponential and linear growth has the advantage that for any continuous curve $x : [0,T[\longrightarrow E$ defined piecewise by finitely many transitions $\vartheta_1 \dots \vartheta_n$ with $\widehat{\zeta} := \sup_j \zeta(\vartheta_j) < \infty$ (as in the proof of Theorem B.10 further on), we conclude from Gronwall's Lemma that $\lfloor x(t)\rfloor \le \lfloor x(0)\rfloor\ e^{\widehat{\zeta}\, t} + \widehat{\zeta}\, t$, for all $t \in [0,T[$

Definition B.3 $\Theta(E, d, \lfloor\cdot\rfloor) \ne \emptyset$ *denotes a set of transitions on $(E, d, \lfloor\cdot\rfloor)$ assuming*

$$D(\vartheta, \tau;\ r) := \sup\ \left\{ \limsup_{h \downarrow 0} \tfrac{1}{h}\ d(\vartheta(h,x), \tau(h,x)) \ \middle|\ x \in E, \lfloor x\rfloor \le r \right\} < \infty$$

for any $\vartheta, \tau \in \Theta(E, d, \lfloor\cdot\rfloor)$ and $r \ge 0$. If $\{x \in E \mid \lfloor x\rfloor \le r\} = \emptyset$, set $D(\,\cdot\,,\,\cdot\,;\ r) := 0$.

Function $D(\cdot,\cdot; r) : \Theta(E, d, \lfloor\cdot\rfloor) \times \Theta(E, d, \lfloor\cdot\rfloor) \longrightarrow [0,\infty[$ is symmetric and it satisfies the triangle inequality for each $r \ge 0$. Moreover, it allows the basis for estimating the distance between two initial points $x, y \in E$ after evolving along two transitions $\vartheta, \tau \in \Theta(E, d, \lfloor\cdot\rfloor)$, respectively, for some time $h \in [0,1]$. Deriving this estimate from a form of Gronwall's Lemma is typical of mutational equations and will be used for similar inequalities later on (see Lemma B.11).

Lemma B.4 *Let $\vartheta, \tau \in \Theta(E, d, \lfloor\cdot\rfloor)$ be transitions. Then, for every time $h \in [0,1]$ and initial points $x, y \in E$ with $\lfloor x\rfloor, \lfloor y\rfloor \le r$, the distance between $\vartheta(h,x)$ and $\tau(h,y)$ satisfies*

$$d\left(\vartheta(h,x), \tau(h,y)\right) \;\le\; \Big(d(x,y) + h D(\vartheta, \tau;\, R)\Big) e^{\alpha(\tau; R) h},$$

where $R := r\ e^{\max\{\zeta(\vartheta),\, \zeta(\tau)\}} + \max\{\zeta(\vartheta),\ \zeta(\tau)\}$.

Proof. According to the property (v) (Definition B.1), $\lfloor x\rfloor, \lfloor y\rfloor \le r$ implies $\lfloor \vartheta(h,x)\rfloor \le R$ and $\lfloor \tau(h,y)\rfloor \le R$ for $h \in [0,1]$. The auxiliary function $\varphi : [0,1] \longrightarrow [0,\infty[, h \longmapsto d(\vartheta(h,x), \tau(h,y))$ is continuous due to property (iv) and the triangle inequality for d. Furthermore, we obtain for every $h \in [0,1[$ and $k \in [0, 1-t[$

$$\begin{array}{rlll} \varphi(h+k) \le & d(\vartheta(h+k,x), \vartheta(k, \vartheta(h,x))) & + & d(\vartheta(k, \vartheta(h,x)), \tau(k, \vartheta(h,x))) \\ + & d(\tau(k, \vartheta(h,x)), \tau(k, \tau(h,y))) & + & d(\tau(k, \tau(h,y)), \tau(h+k, y)) \\ \le & o(k) & + & D(\vartheta, \tau;\, R)\, k + o(k) \\ + & \varphi(h) + k\, \alpha(\tau; R) \varphi(h) + o(k) & + & o(k) \end{array}$$

and thus, $\limsup\limits_{k\downarrow 0} \frac{\varphi(h+k)-\varphi(h)}{k} \;\le\; \alpha(\tau; R)\ \varphi(h) + D(\vartheta, \tau;\, R)$.
The claimed estimate now results from Lemma B.5. □

Lemma B.5 (Lemma of Gronwall for upper Dini derivatives)
Let $\psi, f, g \in C^0([a,b[)$ satisfy $f(\cdot) \ge 0$ and

$$\limsup_{h\downarrow 0} \frac{\psi(t+h)-\psi(t)}{h} \le f(t)\ \psi(t) + g(t) \qquad \textit{for all } t \in]a,b[.$$

Then, for every $t \in [a,b[$, the function $\psi(\cdot)$ satisfies the inequality

$$\psi(t) \;\le\; \psi(a)\, e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)} g(s) ds$$

with $\mu(t) := \int_a^t f(s) ds$.

Proof. Let $\delta > 0$ be arbitrarily small. The proof is based on comparing ψ to the auxiliary function $\varphi_\delta : [a,b] \longrightarrow \mathbb{R}$ that includes $\psi(a) + \delta$ and $g(\cdot) + \delta$ instead of $\psi(a)$ and $g(\cdot)$:

$$\varphi_\delta(t) := (\psi(a) + \delta)\ e^{\mu(t)} + \int_a^t e^{\mu(t)-\mu(s)} (g(s) + \delta) ds.$$

Then, $\varphi_\delta'(t) = f(t)\varphi_\delta(t) + g(t) + \delta$ in $[a,b[$ and, $\varphi_\delta(t) > \psi(t)$ for all $t \in [a,b[$ close to a. Assume now that there exists a time $t_0 \in]a,b]$ with $\varphi_\delta(t_0) < \psi(t_0)$. Setting

$$t_1 := \inf\{t \in [a, t_0] | \varphi_\delta(t) < \psi(t)\},$$

we obtain $\varphi_\delta(t_1) = \psi(t_1)$ and $a < t_1 < t_0$ because

$$\varphi_\delta(t_1) = \lim_{h \downarrow 0} \ \varphi_\delta(t_1 - h) \ \geq \ \limsup_{h \downarrow 0} \ \psi(t_1 - h) \ \geq \ \psi(t_1),$$

$$\varphi_\delta(t_1) = \lim_{\substack{h \to 0 \\ h \geq 0}} \ \varphi_\delta(t_1 + h) \ \leq \ \limsup_{\substack{h \to 0 \\ h \geq 0}} \ \psi(t_1 + h) \ \leq \ \psi(t_1).$$

Thus, we conclude from the definition of t_1

$$\begin{aligned} \liminf_{h \downarrow 0} \ \tfrac{\varphi_\delta(t_1+h)-\varphi_\delta(t_1)}{h} &\leq \limsup_{h \downarrow 0} \ \tfrac{\psi(t_1+h)-\psi(t_1)}{h} \\ \varphi'_\delta(t_1) &\leq f(t_1)\,\psi(t_1) + g(t_1) \\ f(t_1)\,\varphi_\delta(t_1) + g(t_1) + \delta &\leq f(t_1)\,\varphi_\delta(t_1) + g(t_1) \qquad , \end{aligned}$$

which is a contradiction. Therefore $\varphi_\delta(\cdot) \geq \psi(\cdot)$ for any $\delta > 0$. □

A transition $\vartheta \in \Theta(E, d, \lfloor\cdot\rfloor)$ can be interpreted as an analog of the time derivative of curve $x(\cdot) : [0, T[\longrightarrow E$ at time $t \in [0, T[$ if it induces a first-order approximation, i.e., the evolution of $x(t)$ along ϑ differs from the curve $x(t + \cdot)$ up to order $1/h$:

$$\lim_{h \downarrow 0} \frac{1}{h} d\left(\vartheta(h, x(t)), x(t+h)\right) = 0.$$

This condition may be satisfied by more than one transition. We include all such transitions in the so-called mutation of $x(\cdot)$ at time t. The main notion of a mutational equation is to prescribe a transition in the mutation of the sought curve via a given function of the current state and time.

Definition B.6 *Let $x(\cdot) : [0, T[\longrightarrow E$ be a curve in E. The so-called mutation $\overset{\circ}{x}(t)$ of $x(\cdot)$ at time $t \in [0, T[$ consists of all transitions $\vartheta \in \Theta(E, d, \lfloor\cdot\rfloor)$ satisfying*

$$\limsup_{h \downarrow 0} \ \frac{1}{h} \ d\left(\vartheta(h, x(t)), x(t+h)\right) = 0.$$

Definition B.7 *For a given function $f : E \times [0, T[\longrightarrow \Theta(E, d, \lfloor\cdot\rfloor)$, a curve $x(\cdot) : [0, T[\longrightarrow E$ is called the solution to the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ in $[0, T[$ if it satisfies the following conditions:*
(i) for any $t \in [0, T[$, $f(x(t), t) \in \overset{\circ}{x}(t)$, i.e., $\lim_{h \downarrow 0} \frac{1}{h} \ d\left(f(x(t), t)(h, x(t)),\ x(t+h)\right) = 0$,
(ii) $x(\cdot)$ is continuous with respect to d,
(iii) $\sup_{0 \leq t < T} \lfloor x(t) \rfloor < \infty$.

Remark B.8 For any transition $\vartheta \in \Theta(E, d, \lfloor\cdot\rfloor)$ and initial element $x_0 \in E$, the curve $[0, 1[\longrightarrow E,\ h \longmapsto \vartheta(h, x)$ is a solution to the mutational equation $\overset{\circ}{x}(\cdot) \ni \vartheta$ in $[0, 1[$ with constant right-hand side. This results directly from the property (ii) of transitions in Definition B.1.

Similarly to the Peano Theorem for ordinary differential equations, the existence of solutions can be proved by means of the Euler method combined with sequential compactness.

Definition B.9 (Euler compactness) *The tuple $(E, d, \Theta(E, d, \lfloor\cdot\rfloor))$ is* Euler compact *if it satisfies the following condition for every initial element $x_0 \in E$, time $T \in]0, \infty[$ and*

bound $M > 0$.
Let $\mathcal{N} = \mathcal{N}(x_0, T, M) \subset C^0([0,T], E)$ denote the subset of all continuous curves $y(\cdot) : [0,T] \longrightarrow E$ constructed in the following piecewise way: Choosing an arbitrary equidistant partition $0 = t_0 < t_1 < \ldots < t_n = T$ of $[0,T]$ and transitions $\vartheta_1 \ldots \vartheta_n \in \Theta(E, d, \lfloor\cdot\rfloor)$ with $\sup_j \zeta(\vartheta_j) \leq M$, *define $y(\cdot) : [0,T] \longrightarrow E$ as*

$$y(0) := x_0, \qquad y(t) := \vartheta_j\,(t - t_j,\; y(t_{j-1})) \quad \textit{for } \; t \in \,]t_{j-1}, t_j],\; j = 1, 2 \ldots n.$$

Then the union of all images $\{y(t) \,|\, y(\cdot) \in \mathcal{N},\; t \in [0,T]\} \subset E$ is relatively compact in the metric space (E, d).

Proposition B.10 (Existence) *Let (E,d) be a metric space and $\lfloor\cdot\rfloor : E \longrightarrow [0, \infty[$ such that $(E, d, \Theta(E, d, \lfloor\cdot\rfloor))$ is Euler compact. Moreover suppose $f : (E,d) \times [0,T[\longrightarrow (\Theta(E, d, \lfloor\cdot\rfloor), D(\cdot,\cdot; r))$ is continuous with*

$$\begin{aligned} \widehat{\alpha}(r) &:= \sup_{x,t}\; \alpha(f(x,t); r) \;< \infty, \\ \widehat{\beta}(r) &:= \sup_{x,t}\; \beta(f(x,t); r) \;< \infty, \\ \widehat{\zeta} &:= \sup_{x,t}\; \zeta(f(x,t)) \qquad < \infty \end{aligned}$$

for each $r \geq 0$. Then for every initial element $x_0 \in E$, there exists a solution $x(\cdot) : [0,T[\longrightarrow E$ to the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ in $[0,T[$ with $x(0) = x_0$.

Proof. The proof is based on Euler approximations $x_n(\cdot) : [0,T] \longrightarrow E$ $(n \in \mathbb{N})$ combined with the Arzela–Ascoli theorem (see e.g. [2, Proposition 3.3.1]). Indeed, for each $n \in \mathbb{N}$, set

$$\begin{array}{llll} h_n & := \frac{T}{2^n}, & t_n^j := j\, h_n & \text{for } \; j = 0 \ldots 2^n, \\ x_n(0) & := x_0, & x_0(\cdot) := x_0, & \\ x_n(t) & := f(x_n(t_n^j),\, t_n^j)\Big(t - t_n^j,\;\; x_n(t_n^j)\Big) & & \text{for } \; t \in \,]t_n^j,\, t_n^{j+1}],\;\; j < 2^n. \end{array}$$

First, the piecewise construction of each $x_n(\cdot)$ implies $\lfloor x_n(t) \rfloor \;\leq\; \lfloor x_0 \rfloor e^{\widehat{\zeta}\, T} + \widehat{\zeta}\; T \;=:\; R$ for all $t \in [0,T]$, $n \in \mathbb{N}$. Second, due to Euler compactness, the union of all values $x_n(t)$ for $t \in [0,T]$, $n \in \mathbb{N}$ is contained in a compact subset $K \subset E$. Third, the triangle inequality ensures $d\,(x_n(s), x_n(t)) \leq \widehat{\beta}(R) |t - s|$ for all $s, t \in [0,T]$, $n \in \mathbb{N}$ and therefore, the sequence $(x_n(\cdot))_{n \in \mathbb{N}}$ is equicontinuous.
The Theorem of Arzela–Ascoli states that the set $\{x_n(\cdot) \,|n \in \mathbb{N}\} \subset C^0([0,T], K)$ is precompact with respect to uniform convergence and therefore, there exists a subsequence $\Big(x_{n_j}(\cdot)\Big)_{j \in \mathbb{N}}$ uniformly converging to a function $x(\cdot) \in C^0([0,T], K)$.
Finally, we verify that $x(\cdot)$ solves the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ in $[0,T[$. Indeed, $x(\cdot)$ is continuous with respect to d and it satisfies $\sup_t \lfloor x(t) \rfloor \leq R$ by virtue of its construction. Furthermore, using the notation $\delta_n := \sup_{[0,T]}\; d(x_n(\cdot), x(\cdot))$, we conclude, from Lemma B.11 further on, that for any $t \in [0,T[$, $h \in [0, T - t[$ and $n \in \mathbb{N}$

$$\begin{aligned} & d\,(f(x(t), t)\,(h,\; x(t)),\;\; x(t+h)) \\ & \leq d\Big(f(x(t), t)\,(h, x(t)),\; x_n(t+h)\Big) + d\,(x_n(t{+}h),\; x(t{+}h)) \\ & \leq \Big(\delta_n \;+\; h \sup_{\substack{-h_n \,\leq\, s \,\leq\, h \\ y:\, d(y, x(t+s)) \,\leq\, \delta_n}} \; D\,(f(x(t), t),\;\; f(y, t+s);\; R)\Big)\; e^{\widehat{\alpha}(R)\, h} + \delta_n. \end{aligned}$$

Due to the continuity of f with respect to $D(\cdot,\cdot;R)$, the limit as $n \longrightarrow \infty$ implies that $d\Big(f(x(t),t)\,(h,x(t)), x(t+h)\Big) \;\le\; h \sup_{0\le s\le h} D\left(f(x(t),t),\; f(x(t+s),t+s),\; R\right)\; e^{\widehat{\alpha}(R)\,h}$
and thus, $\limsup_{h\downarrow 0} \frac{1}{h}\, d\Big(f(x(t),t)\,(h,x(t)), x(t+h)\Big) \le 0.$ □

Lemma B.11 *Assume for $f,g : E\times[0,T[\longrightarrow \Theta(E,d,\lfloor\cdot\rfloor)$ and $x,y : [0,T[\longrightarrow E$ that $x(\cdot)$ is a solution to the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ in $[0,T[$ and $y(\cdot)$ is a solution to the mutational equation $\overset{\circ}{y}(\cdot) \ni g(y(\cdot),\cdot)$ in $[0,T[$. Furthermore, let $R>0$, $M>0$ and $\psi \in C^0([0,T[)$ satisfy for all $t\in[0,T[$*

$$\begin{cases} \lfloor x(t)\rfloor,\, \lfloor y(t)\rfloor \le R \\ \alpha(g(y(t),t);R) \le M \\ D\left(f(x(t),t), g(y(t),t);\; R\right) \le \psi(t). \end{cases}$$

Then, $d(x(t),y(t)) \le \Big(d(x(0),y(0)) + \int_0^t \psi(s)\; e^{-Ms} ds\Big)\; e^{Mt}$ for any $t\in[0,T[$.

Proof. The argument proceeds as in Lemma B.4 by comparing the evolutions along fixed transitions. The auxiliary function $\varphi : [0,1] \longrightarrow [0,\infty[,\ t\longmapsto d(x(t),y(t))$ is continuous due to the triangle inequality for d. Furthermore, we obtain for every $t\in[0,T[$ and $h\in[0,T-t[$

$$\begin{aligned}
\varphi(t+h) \le\;& d(x(t+h), f(x(t),t)(h,x(t))) &&+\; d(f(x(t),t)(h,x(t)), g(y(t),t)(h,x(t)))\\
&+ d(g(y(t),t)(h,x(t)), g(y(t),t)(h,y(t))) &&+\; d(g(y(t),t)(h,y(t)), y(t+h))\\
\le\;& o(h) &&+\; D(f(x(t),t), g(y(t),t);R)\; h + o(h)\\
&+ \varphi(t) + h\; M\varphi(t) + o(h) &&+\; o(h)
\end{aligned}$$

and thus, $\limsup_{h\downarrow 0} \frac{\varphi(t+h)-\varphi(t)}{h} \le M\;\varphi(h) + \psi(t)$. Therefore, the claim results from Gronwall's Lemma B.5. □

Proposition B.12 (Uniqueness) *Suppose $f : (E,d)\times[0,T[\longrightarrow (\Theta(E,d,\lfloor\cdot\rfloor), D(\cdot,\cdot;r))$ is λ_r–Lipschitz continuous in the first argument with $\widehat{\alpha}(r) := \sup_{x,t} \alpha(f(x,t);r) < \infty$ for any $r\ge 0$. Then for every initial element $x_0\in E$, the solution $x(\cdot) : [0,T[\longrightarrow E$ to the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ in $[0,T[$ with $x(0)=x_0$ is unique and it depends on x_0 in a locally Lipschitz way.*

Proof. The argument is based on the estimate in Lemma B.11. Let $x(\cdot), y(\cdot) : [0,T[\longrightarrow E$ be two solutions to the same mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot),\cdot)$ in $[0,T[$, generally not with the same initial element. Then, $R := \sup_t\{\lfloor x(t)\rfloor, \lfloor y(t)\rfloor\} < \infty$ due to the definition of solutions and $t\longmapsto d(x(t),y(t))$ is continuous. As a consequence of the inequality

$$D(f(x(t),t),\; f(y(t),t);\; R) \le \lambda_R\; d(x(t),y(t)),$$

Lemma B.11 implies for any $t \in [0, T[$

$$d(x(t), y(t)) \le d(x(0), y(0)) e^{\widehat{\alpha}(R)\, t} + \int_0^t \lambda_R \, d(x(s), y(s)) e^{\widehat{\alpha}(R)\,(t-s)} ds$$

and, Gronwall's Lemma in the integral form guarantees

$$d(x(t), y(t)) \le d(x(0), y(0)) \; e^{(\widehat{\alpha}(R)+\lambda_R)\, t} \qquad \text{for all } t \in [0, T[. \qquad \square$$

Proposition B.13 (Continuity w.r.t. initial data and the right-hand side) *For any $r \ge 0$, suppose $f : (E, d) \times [0, T[\longrightarrow (\Theta(E, d, \lfloor\cdot\rfloor),\ D(\cdot, \cdot; r))$ to be λ_r–Lipschitz continuous in the first argument with $\widehat{\alpha}(r) := \sup_{x,t} \alpha(f(x, t);\ r) < \infty$.*
Let $g : E \times [0, T[\longrightarrow \Theta(E, d, \lfloor\cdot\rfloor)$ fulfill $\sup_{z,s} \ D(f(z, s),\ g(z, s);\ r) < \infty$ for each $r \ge 0$.
(a) Let $y(\cdot) : [0, T[\longrightarrow E$ be a solution to the mutational equation $\overset{\circ}{y}(\cdot) \ni g(y(\cdot), \cdot)$. Then, every solution $x(\cdot) : [0, T[\longrightarrow E$ to the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ in $[0, T[$ satisfies the following inequality

$$d(x(t), y(t)) \le \Big(d(x(0), y(0)) + t \sup_{\substack{z : \lfloor z \rfloor \le R \\ 0 \le s < T}} D(f(z, s), g(z, s); R) \Big) e^{(\widehat{\alpha}(R)+\lambda_R) t},$$

for $t \in [0, T[$ and $R := \sup_t \{\lfloor x(t) \rfloor, \lfloor y(t) \rfloor\} < \infty$.
(b) Let $x(\cdot) : [0, T[\longrightarrow E$ be a solution to the mutational equation $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$. Then, every solution $y(\cdot) : [0, T[\longrightarrow E$ to the mutational equation $\overset{\circ}{y}(\cdot) \ni g(y(\cdot), \cdot)$ in $[0, T[$ satisfies

$$d(x(t), y(t)) \ \le \ \Big(d(x(0), y(0)) + t \sup_{\substack{z : \lfloor z \rfloor \le R \\ 0 \le s < T}} D(f(z, s), g(z, s); R) \Big) e^{(\widehat{\alpha}(R)+\lambda_R) t},$$

for $t \in [0, T[$ and $R := \sup_t \{\lfloor x(t) \rfloor, \lfloor y(t) \rfloor\} < \infty$.

Proof. The result follows from Lemma B.11 similarly to the preceding Proposition B.12, since $D(\cdot, \cdot; R)$ satisfies the triangle inequality and thus,

$$\begin{aligned} D\Big(f(x(t), t), g(y(t), t); R\Big) &\le D\Big(f(x(t), t), f(y(t), t); R\Big) + D\Big(f(y(t), t), g(y(t), t); R\Big) \\ &\le \lambda_R \, d(x(t), y(t)) + \sup_{\substack{z : \lfloor z \rfloor \le R \\ 0 \le s < T}} D\Big(f(z, s), g(z, s); R\Big). \end{aligned}$$

$\square$

Acknowledgements
The authors would like to thank Agnieszka Swierczewska-Gwiazda and Marek Kimmel for valuable discussions and help with the manuscript. TL is very grateful to Irina Surovtsova and Daniel Andrej for fruitful complementary discussions. PG was supported by Humboldt Fellowship. AMC was supported by WIN Kolleg of Heidelberg Academy of Sciences and Humanities.

References

[1] Ambrosio, L., Fusco, N. and Pallara, D. (2000): *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs

[2] Ambrosio, L., Gigli, N. and Savaré, G. (2005): *Gradient flows in metric spaces and in the space of probability measures*, Birkhäuser, ETH Lecture Notes in Mathematics

[3] Arino, O. and Kimmel, M. (1993): Comparison of Approaches to Modeling of Cell Population Dynamics, *SIAM J. Appl. Math.* **53**, pp.1480–1504.

[4] Aubin, J.-P. (1999): *Mutational and Morphological Analysis : Tools for Shape Evolution and Morphogenesis*, Birkhäuser-Verlag, Systems and Control: Foundations and Applications

[5] Aubin, J.-P. (1993): Mutational equations in metric spaces, *Set-Valued Analysis* **1**, pp. 3-46

[6] Bianchini, S., Colombo, R.M. (2002): On the stability of the standard Riemann semigroup. *Proc. Am. Math. Soc.* 130, No.7, 1961-1973.

[7] Bressan, A. (2000): *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*, Oxford University Press

[8] Conway, E.D. (1967): Generalized Solutions of Linear Differential Equations with Discontinuous Coefficients and the Uniqueness Question for Multidimensional Quasilinear Conservation Laws, *J. Math. Anal. and Appl.* **18**, pp. 238-2

[9] Diekmann O., Gyllenberg M, Huang H., Kirkilionis M., Metz J.A.J., Thieme H.R. (2001): On the formulation and analysis of general deterministic structured population models: II. Nonlinear theory, *J. Math. Biol.* **43**, pp. 157–189.

[10] Diekmann O. and Getto P. (2005): Boundedness, global existence and continuous dependence for nonlinear dynamical systems describing physiologically structured populations. *J. Differ. Equations*, **215**, pp. 268–319.

[11] Diekmann O., Gyllenberg M, Metz J.A.J., Thieme H.R. (1998): On the formulation and analysis of general deterministic structured population models: I. Linear theory, *J. Math. Biol.* **36**, pp. 349–388.

[12] DiPerna, R.J. and Lions, P.L. (1989): Ordinary differential equations, transport theory and Sobolev spaces, *Inventiones Mathematicae*, **98**, pp. 511–547

[13] Federer, H. (1969): *Geometric Measure Theory*, Springer-Verlag, Grundlehren der mathematischen Wissenschaften, Vol. 153

[14] Iannelli, M.: *Mathematical Theory of Age-Structured Population Dynamics*, Applied Math. Monographs, CNR, Giardini Editori e Stampatori in Pisa, 1995.

[15] Lorenz, T. (2007): Generalizing mutational equations for uniqueness of some nonlocal first-order geometric evolutions, to appear in *Set-Valued Analysis*, Preprint available online at `http://www.ub.uni-heidelberg.de/archiv/5580`

[16] Lorenz, T. (2007): Radon measures solving the Cauchy problem of the nonlinear transport equation, Preprint available online at `http://www.ub.uni-heidelberg.de/archiv/7252`

[17] Maniglia, St. (2007): Probabilistic representation and uniqueness results for measure-valued solutions of transport equations, *J. Math. Pures Appl.* (9) **87**, no. 6, pp.601-626

[18] Metz J.A.J. and Diekmann O. (1986): *The Dynamics of Physiologically Structured Populations*, Springer, Lecture Notes in Biomathematics, vol. 68

[19] Müller, S. and Ortiz, M. (2004): On the Γ-convergence of discrete dynamics and variational integrators, *J. Nonlinear Sci.*, **14**, No.3, pp. 276-296

[20] Panasyuk, A.I (1985): Quasidifferential equations in metric spaces, *Differ. Equations* **21**, pp.914–921

[21] Perthame B. (2007): Transport equations in biology. Birkhäuser.

[22] Poupaud, F. and Rascle, M. (1997): Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients. *Comm. Partial Differential Equations* **22**, No.1-2, pp. 337-358.

[23] Schwartz, L. (1973): *Radon measures*, Oxford University Press

[24] Smirnov, V. I. (1953): A Course in Higher Mathematics. Vol. 4

[25] Thieme, H. R. (2003): *Mathematics in population biology.* Woodstock Princeton university press.

[26] Walter, W. (1993): *Gewöhnliche Differentialgleichungen*, Springer

[27] Walter, W. (1970): *Differential and integral inequalities*, Springer

[28] Webb, G.F. (1985): *Nonlinear Age-Dependent Population Dynamics*, Dekker