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Conjugations on 6-Manifolds

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Abstract

Conjugation spaces are spaces with involution such that the fixed point set of the involution has $\mathbb{Z}_2$-cohomology isomorphic to the $\mathbb{Z}_2$-cohomology of the space itself, with the little difference that all degrees are divided by two (e.g. $\mathbb{C}\mathrm{P}^n$ with the complex conjugation). One also requires that a certain conjugation equation is fulfilled.

I give a new characterization of conjugation spaces and apply it to the following realization question: given M, a closed orientable 3-manifold, is there a 6-manifold X (with certain additional properties) containing M as submanifold such that M is the fixed point set of an orientation reversing involution on X? My main result is that for every such 3-manifold M there exists a simply connected conjugation 6-manifold X with fixed point set M.

Konjugationsräume sind Räume mit Involution, so dass der Raum selbst und die Fixpunktmenge der Involution isomorphe $\mathbb{Z}_2$-Kohomologie aufweisen, mit dem Unterschied, dass alle Grade durch zwei geteilt werden müssen (z.B. $\mathbb{C}\mathrm{P}^n$ mit der komplexen Konjugation). Für die genaue Definition verlangt man, dass zusätzlich eine sogenannte Konjugationsgleichung erfüllt ist.

Ich zeige zunächst eine alternative, einfachere Charakterisierung von Konjugationsräumen und wende diese dann auf die folgende Realisierungsfrage an: Gegeben sei eine geschlossene orientierbare 3-Mannigfaltigkeit M. Gibt es eine 6-Mannigfaltigkeit X (mit gewissen zusätzlichen Eigenschaften), die M als Untermannigfaltigkeit besitzt, und eine Involution auf X, deren Fixpunktmenge genau M ist? Mein Hauptresultat ist, dass für jede solche 3-Mannigfaltigkeit M eine einfachzusammenhängende Konjugations-6-Mannigfaltigkeit X mit Fixpunktmenge M existiert.

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Introduction

This thesis has its origin in the search for simply connected asymmetric manifolds. Asymmetric manifolds are manifolds not admitting any non-trivial finite group action. If we try to apply cohomological methods to show that a given manifold (for example a six-dimensional spin manifold) is asymmetric, we see that some $\mathbb{Z}_p$-actions are easier to exclude than others. V. Puppe's method excludes in certain cases all $\mathbb{Z}_p$-actions except orientation reversing involutions, since it says that the cohomology of the fixed point set of a $\mathbb{Z}_p$-action must be very similar to the cohomology of the manifold itself. In fact, Puppe's results show that such a manifold would have to be a "conjugation space".

Conjugation spaces are spaces with involution such that the fixed point set of the involution has $\mathbb{Z}_2$-cohomology isomorphic to the $\mathbb{Z}_2$-cohomology of the space itself, with the little difference that all degrees are divided by two (as is the case for the complex conjugation on $\mathbb{C}\mathrm{P}^n$ with fixed point set $\mathbb{R}\mathrm{P}^n$).

The actual definition requires an additional property that is more complicated: we need the existence of a "conjugation frame". This means that we also need to find a map of the ordinary cohomology of the space to its equivariant cohomology, such that a so-called "conjugation equation" in equivariant cohomology must be fulfilled.

The first important achievement in our work is to give an alternative definition for conjugation spaces, which is in some sense much easier to work with. Our main result is that for every closed oriented 3-manifold M there exists a simply connected conjugation 6-manifold X with fixed point set M.

The first chapter gives the necessary information about equivariant cohomology, defines conjugation spaces and mentions some of the theorems about conjugation spaces. It compares realization questions for conjugation spaces with the corresponding classical questions for real algebraic varieties. Finally we explain Puppe's approach that excludes on some simply-connected 6-dimensional manifolds every non-trivial finite group action except conjugations, and we discuss Kreck's result that in some cases, also conjugations are not possible.

The second chapter contains the new characterization of conjugation spaces:

Theorem 0.0.1 *Let X be a topological space equipped with an involution τ, such that $\dim_{\mathbb{Z}_2} H^i(X;\mathbb{Z}_2) < \infty$ for all i. Then X is a conjugation space if and only if the restriction map in equivariant cohomology*

$$r : H^*_C(X,Y;\mathbb{Z}_2) \to H^*_C(X^\tau, Y^\tau;\mathbb{Z}_2) \cong H^*(X^\tau, Y^\tau;\mathbb{Z}_2)[u]$$

induces an additive isomorphism

$$H^*_C(X,Y;\mathbb{Z}_2) \to H^*(X^\tau, Y^\tau;\mathbb{Z}_2)[u] / \bigoplus_{j>k} H^j(X^\tau, Y^\tau;\mathbb{Z}_2) \cdot u^k.$$

Furthermore, we give some applications of this theorem.

In chapter three, we ask the following question: given M, a closed orientable 3-manifold, is there a 6-manifold X that contains M as submanifold such that M is the fixed point set of an involution τ on X? Which additional properties may we impose on X? For example we may want X to be simply connected, and a spin manifold. Using the existence of a tubular neighbourhood of the fixed point set we look for a manifold with a decomposition of the form $X = M \times D^3 \cup V$. In fact we have to find such a manifold V with the right boundary, so we have to solve a certain bordism problem.

We will see that we can realize M as fixed point set of involutions on "simple" manifolds X:

Theorem 0.0.2 *Every closed orientable 3-manifold M is the fixed point set of an involution on a connected sum of $S^2 \times S^4$'s, and also the fixed point set of an involution on a connected sum of $S^3 \times S^3$'s.*

The fourth chapter contains the construction of 6-dimensional conjugation spaces. It begins with the direct application of theorem 0.0.1 to manifolds, which does not involve equivariant cohomology any more:

Theorem 0.0.3 *Let X be a closed $2n$-dimensional manifold, with a differentiable involution τ that has the n-dimensional submanifold M as fixed point* *set. Let ν be the normal bundle of M in X. Let $D(\nu)$, $S(\nu)$ and $P(\nu)$ denote respectively the disk bundle, sphere bundle and projective bundle of ν. Using the equivariant tubular neighbourhood theorem, write $X = D(\nu) \cup V$, such that $W = V/\tau$ is a manifold with boundary $P(\nu)$.*

Then X is a conjugation space iff

$$H^*(W; \mathbb{Z}_2) \to H^*(P(\nu); \mathbb{Z}_2)$$

induces an isomorphism:

$$H^*(W; \mathbb{Z}_2) \to H^*(P(\nu); \mathbb{Z}_2) / \bigoplus_{i>j} H^i(M; \mathbb{Z}_2) u^j$$

We continue by finding W such that this condition is fulfilled in small degrees, which comes down to a bordism problem, using surgery below the middle dimension. We apply the Atiyah-Hirzebruch spectral sequence and the Adams spectral sequence to compute the relevant bordism groups. Poincaré duality will suffice for high degrees, and finally in the middle dimension, we use again surgery theory.

Theorem 0.0.4 *For every orientable connected 3-manifold M, there exists a simply connected spin 6-manifold which is a conjugation space and has M as its fixed point set.*

In the appendix, we extend one of Puppe's results about non-existence of $\mathbb{Z}_p$-actions to a larger class of six-manifolds, and we prove a theorem about non-existence of codimension 1 fixed point sets.

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Chapter 1

Conjugation spaces, real algebraic varieties and asymmetric manifolds

1.1 Equivariant cohomology

Definition 1.1.1 *Let G be a topological (e.g. discrete) group. A G-space X is a topological space X together with a G-action on X, i.e. a continuous map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ such that for all $g, g' \in G$, $x \in X$, we have $(gg') \cdot x = g \cdot (g' \cdot x)$. If G and X are differentiable manifolds, we may require that the map $G \times X \to X$ is differentiable. X is then called a differentiable G-space.*

For each topological group G, there exists a classifying space BG which is unique up to homotopy equivalence, and is characterized by the fact that there is a natural bijection between homotopy classes of maps from a CW-complex to BG and principal G-bundles over this CW-complex. This bijection is given by pulling back a universal principal G-bundle over BG, whose total space is called universal free G-space EG. It follows that $BG = EG/G$. For the exact definition of a principal G-bundle and proofs of these results we refer to Switzer's book [Swi75]. We just recall the easiest construction of the universal free G-space EG and the classifying space BG for finite cyclic groups G (of order $p \in \mathbb{N}$):

For the groups $G = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$, one can define EG and BG in the following way: Let S^∞ be the union of $S^0 \subset S^1 \subset S^2 \subset \ldots$. We consider

S^1 as the unit sphere in $\mathbb{C}$, S^3 as the unit sphere in $\mathbb{C}^2$, ... and S^∞ as a subspace of $\mathbb{C}^\infty$. There is an action of $G = \mathbb{Z}_p$ on S^∞ given by:

$$(z_0, z_1, \ldots z_n, 0, 0, \ldots) \mapsto (e^{2\pi i/p} z_0, e^{2\pi i/p} z_1, \ldots e^{2\pi i/p} z_n, 0, 0, \ldots)$$

We can define $EG = S^\infty$ and $BG = S^\infty/(\mathbb{Z}_p)$. There is an obvious quotient map $EG \to BG$. For $G = \mathbb{Z}_2$, we have $BG = \mathbb{R}\mathrm{P}^\infty$, the real projective space; for odd primes, the spaces BG are infinite-dimensional lens spaces.

The Borel construction associates to a G-space X (with G-action on the right) the following fibre bundle:

$$X \xrightarrow{i} X_G = (EG \times X)/G \xrightarrow{pr_1} BG$$

(where G acts on $EG \times X$ by the diagonal action and pr_1 is the map induced by projection on the first factor.)

The idea behind this construction is that the complexity of the space X and of the action of G on X is described by the space X_G, and that the study of X_G and of the associated fibre bundle allows to draw conclusions about the action of G on X (for example about the fixed point set). We will be especially interested in the cohomology of X_G.

Example 1.1.2 *If F is a trivial G-space, i.e. $g \cdot x = x$ for all $g \in G$, $x \in F$, then $F_G = BG \times F$ is the trivial fibre bundle, i.e. the product of the base space and the fibre. For a non-trivial action, the fibre bundle is in general non-trivial, and could be described as a "product twisted by the G-action".*

Remark 1.1.3 *X_G may also be considered as a "homotopy quotient" of X, since for $G = \mathbb{Z}_p$, $X \times EG \to X_G$ is a covering space, and X is homotopy equivalent to $X \times EG$. This generalizes the following fact: If the action of $G = \mathbb{Z}_p$ on X is free, then X_G is homotopy equivalent to X/G since we have a fibre bundle $EG \to X_G \to X/G$ with contractible fibre.*

Definition 1.1.4 *Let us define $H^*_G(X) = H^*(X_G)$ (any coefficients), then the functors H^n_G form a G-equivariant cohomology theory, that is, a cohomology theory on the category of G-spaces. We simply call $H^*_G(X)$* ***the*** *equivariant cohomology of X. (Clearly, it depends on the G-space structure on X, i.e. on the G-action on X.)*

Remark 1.1.5 *If X is a G-space, then $H^*_G(X)$ is a ring, but also a module over $H^*_G(pt) = H^*(BG)$. If $f : X \to Y$ is a G-equivariant map between G-spaces, then $f^* : H^*_G(Y) \to H^*_G(X)$ is a $H^*(BG)$-module homomorphism (and a ring homomorphism). For $G \cong \mathbb{Z}_2$, we have $H^*(BG; \mathbb{Z}_2) \cong \mathbb{Z}_2[u]$, where u has degree one. In this case, equivariant cohomology with $\mathbb{Z}_2$-coefficients is a functor from the category of G-spaces and G-equivariant maps into the category of graded $\mathbb{Z}_2[u]$-modules and module homomorphisms.*

We mention at this point that for the computations of equivariant cohomology in the following sections, we will make use of the following two theorems:

- The Serre spectral sequence is a spectral sequence for any Serre fibration, hence we may apply it to the Borel construction. We get a spectral sequence (that involves in general cohomology with local coefficients):

$$E_2^{p,q} \cong H^p(BG; \underline{H^q(X)}) \Rightarrow H_G^{p+q}(X)$$

- The Localization theorem in equivariant cohomology gives (for example if the G-space X is compact) a relation between the equivariant cohomology of X and the equivariant cohomology of the fixed point set X^G of the G-action. We explain this in one of the next sections.

1.2 Conjugation spaces

The definition of a conjugation space [HHP05] has its origin in the observation that there are many examples of spaces where the following phenomenon appears:

Example 1.2.1 *Consider the complex projective space* $\mathbb{C}\mathrm{P}^n$*. Its cohomology ring (with* $\mathbb{Z}_2$*-coefficients) is* $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[a]/a^{n+1}$ *where* a *has degree* 2*. On* $\mathbb{C}\mathrm{P}^n$*, we have the complex conjugation, which is an involution* $\tau : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$*. We know the fixed point set of the involution: it is* $(\mathbb{C}\mathrm{P}^n)^\tau = \mathbb{R}\mathrm{P}^n$*. Now the cohomology ring of the fixed point set is* $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[b]/b^{n+1}$ *where* b *has degree* 1*.*

We see that the only change from the cohomology of the space to the cohomology of the fixed point set is that degrees are divided by two, i.e. there is an isomorphism from the cohomology of $\mathbb{C}\mathrm{P}^n$ *to the cohomology of* $(\mathbb{C}\mathrm{P}^n)^\tau = \mathbb{R}\mathrm{P}^n$ *dividing the degree by two.*

The same situation comes up for all complex Grassmannians $Gr_k(\mathbb{C}^n)$, where again the involution is a complex conjugation, the fixed point set is the real Grassmannian $Gr_k(\mathbb{R}^n)$, and there is a ring isomorphism between the cohomologies dividing degrees by two.

But there are other examples without a natural "complex structure", including natural involutions on smooth toric manifolds [DJ91], and on polygon spaces [HK98]. There is always a degree-halving isomorphism between the cohomologies of the space and its fixed point set.

In some examples from analytic geometry, already Borel and Haefliger discussed the significance of this isomorphism [BH61]. Hausmann, Holm and Puppe [HHP05] discovered that in all these examples, there is an even richer structure in the (equivariant) cohomology of the space itself and the space of fixed points of the involution. So, in the definition of a conjugation space, we require more than the existence of such an isomorphism:

Definition 1.2.2 *Let* X *be a topological space together with a continuous involution* τ*. We may view this as an action of the cyclic "conjugation group"* $C = \{id, \tau\} \cong \mathbb{Z}_2$ *on* X *and denote the natural restriction maps in equivariant cohomology by:*

$$\rho : H^*_C(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2),$$
$$r : H^*_C(X;\mathbb{Z}_2) \to H^*_C(X^\tau;\mathbb{Z}_2) \cong H^*(X^\tau;\mathbb{Z}_2)[u]$$

X is a conjugation space if the following conditions are satisfied:

- $H^{odd}(X;\mathbb{Z}_2) = 0$,
- *there exists an additive isomorphism $\kappa : H^{2*}(X;\mathbb{Z}_2) \to H^*(X^\tau;\mathbb{Z}_2)$ dividing the degrees in half,*
- *there exists an additive section $\sigma : H^*(X;\mathbb{Z}_2) \to H^*_C(X;\mathbb{Z}_2)$ of ρ,*
- *the so-called conjugation equation is fulfilled for all $x \in H^{2k}(X;\mathbb{Z}_2)$:*

$$r\sigma(x) = \kappa(x)u^k + \text{terms of lower degree in } u$$

(The definition generalizes immediately to pairs (X, Y), where Y is a τ-invariant subspace of X: we always consider the relative cohomology. (X, Y) is then called a conjugation pair.)

Theorem 1.2.3 *(Multiplicativity, naturality and uniqueness of the structure) [HHP05]*

- *κ and σ are automatically ring homomorphisms.*
- *κ and σ are natural for equivariant maps between conjugation spaces, that is, if X, Y are conjugation spaces with maps $\kappa_X, \sigma_X, \kappa_Y, \sigma_Y$, and $f : X \to Y$ is an equivariant map, then we have $f^*_C \circ \sigma_X = \sigma_Y \circ f^*$ and $(f^\tau)^* \circ \kappa_X = \kappa_Y \circ f^*$.*

  From this it follows that κ and σ are unique.

So κ and σ are really part of the conjugation space structure of X. This richer structure allows Hausmann, Holm and Puppe to prove several beautiful theorems about conjugation spaces (which are not true if we would just require the existence of an isomorphism between the ordinary cohomologies dividing degrees by two):

Theorem 1.2.4 *(Constructions that lead to conjugation spaces) [HHP05]*

- - *Direct limits,*
  - *products,*
  - *and (for manifolds) equivariant connected sums*

  of conjugation spaces are conjugation spaces.

- *Every "conjugation cell complex" is a conjugation space. A "conjugation cell complex" is a cell complex which has cells of even dimension* $2n$ *only, namely unit disks in* $\mathbb{C}^n$ *with the complex conjugation as involution, and such that the glueing maps are equivariant.*

There are two questions that we will discuss in the following sections:

1. In the definition of a conjugation space, we require the existence of the whole conjugation space structure. If we wanted to know whether a space X with $H^{odd}(X; \mathbb{Z}_2) = 0$ is a conjugation space, we would have to check for all possible pairs κ, σ whether they fulfill the conjugation equation. We will find a much simpler criterion that allows to decide whether a space is a conjugation space or not.

2. The core of this work: We ask whether there exist many "conjugation manifolds", that is, conjugation spaces that are differentiable manifolds with a differentiable conjugation. The examples we have seen yet are spheres of even dimensions (conjugation cell complexes with two cells), complex projective spaces, and equivariant connected sums and products of conjugation manifolds. (In the Hausmann, Holm, Puppe article, also other methods of obtaining conjugation manifolds are described.)

   We will concentrate on closed conjugation manifolds of small dimensions and try to give an answer to the question whether there exist many conjugation manifolds of small dimensions. Dimension 6 will be especially interesting.

1.3 "Real parts"

In the easiest examples of the complex Grassmannians, the involution of the conjugation space is really the complex conjugation. The fixed point space is the "real part" or "real locus" of the manifold.

We have a similar situation if we consider real projective varieties (or rather their complex points): That is, given finitely many homogeneous polynomials with real coefficients in $n+1$ variables, we consider their common zero set $X(\mathbb{C}) \subset \mathbb{C}\mathrm{P}^n$. Since the coefficients of the polynomials are real, $X(\mathbb{C})$ is invariant under complex conjugation, and we have (as fixed point set) the "real part" $X(\mathbb{R}) = X(\mathbb{C}) \cap \mathbb{R}\mathrm{P}^n$.

We may also ask if there are many real projective varieties $X(\mathbb{C})$. More precisely, if we are interested in differential topology, we may restrict to non-singular varieties - then $X(\mathbb{C})$ will be a manifold. The question we ask is: Are there many manifolds M that can be described as $X(\mathbb{C})$, where X is a real, smooth, projective variety ?

Actually the answer is that there are only "few" manifolds that can be written as $X(\mathbb{C})$. M must be a complex manifold, which is already a large restriction. In order to be projective, M must be a Hodge manifold, that is, M must have a Kähler structure with integral Kähler fundamental class (According to a theorem of Kodaira's, this is also sufficient.) There are further restrictions if we require X to be real. In another direction, most finitely presented groups can not be the fundamental group of a projective smooth variety.

The answer for conjugation spaces should be similar: Only even-dimensional manifolds with $H^{odd}(M;\mathbb{Z}_2) = 0$ might be given a conjugation space structure, and this is not the only restriction. It is a difficult problem to give further restrictions. In fact, it seems that conjugations are those non-trivial actions of a finite group that are most difficult to exclude, for example with further homological methods. We will discuss this question in the next section.

So the problem of realizing a given manifold as a conjugation space (or the proof that this is impossible) seems to be hard. If we go back to real projective varieties, there is another realization problem that is completely solved by work of Nash and Tognoli:

Theorem 1.3.1 *[Nas52], [Tog73] For every closed differentiable manifold M there is a real smooth projective variety X such that M is diffeomorphic to $X(\mathbb{R})$.*

Nash conjectured further that one could choose X rational (over $\mathbb{R}$), that is, X birationally equivalent to some projective space. This is false already in dimension 2. (A beautiful introduction to the question of realizing 3-manifolds as $X(\mathbb{R})$, where X is a variety that satisfies some of the properties above (rational, smooth, projective), is [Kol94].)

We want to ask the same question for conjugation spaces:

Is every closed manifold the "real part" of a conjugation space?

We restrict to manifolds of small dimension. In dimensions 0, 1 and 2, the remarks from the preceding sections indicate how to realize every closed manifold as fixed point set of an involution on a conjugation space (of twice the dimension). Indeed, from spheres and real projective spaces, using connected sums and products, one can construct all manifolds of dimension smaller than 3. We want to answer the question positively also for oriented three-dimensional manifolds, which will require a lot more work.

1.4 Twisted bordism

Let $\xi : E \to B$ be a real vector bundle of rank k. Define $\Omega_n^{SO}(B;\xi)$ to be the set of bordism classes of triples (X, f, o), where X is a closed n-manifold, $f : X \to B$ is a continuous map, and o is an orientation of the stable bundle $\nu_X - f^*\xi = \nu_X \oplus (f^*\xi)^{-1}$ where ν_X is the stable normal bundle of X, and the inverse is the K-theory inverse. (ξ and X do not have to be orientable.) One says that elements of $\Omega_n^{SO}(B;\xi)$ are "manifolds with an orientation twisted by the vector bundle ξ".

Let $\oplus : BO(k) \times BO \to BO$ denote the "sum" map, that is, the map that classifies $\eta(k) \boxplus \eta$, where $\eta(k)$ is the universal k-plane bundle over $BO(k)$, η the universal stable bundle over BO, and $\boxplus$ the outer direct sum of vector bundles.

Then $\Omega_n^{SO}(B;\xi)$ is the set of bordism classes of lifts of maps $\nu_X : X \to BO$ to $B \times BSO$, where the map $B \times BSO \to BO$ is the composition:

$$B \times BSO \xrightarrow{\xi \times Bi} BO(k) \times BO \xrightarrow{\oplus} BO$$

(Bi is the natural map from BSO to BO, which one obtains applying the functor B to the inclusion $SO \to O$. More exactly, in the above one has to replace $B \times BSO$ by a homotopy equivalent space such that the map to BO becomes a fibration.)

Here $(f : X \to B,\ o)$ corresponds to the lift $X \to B \times BSO$ whose first component is f and whose second component is given by o: a map to BSO up to homotopy corresponds to an oriented (stable) vector bundle. We must check that this is a lift (up to homotopy) of the normal bundle map. For this, we can compare the pull-back bundles of the universal stable vector bundle over BO. Under the normal bundle map, the universal bundle pulls back to the normal bundle of X. Under the map $B \times BSO \to BO$, the universal bundle pulls back to $\xi \boxplus \eta^{SO}$, where η^{SO} is the universal oriented bundle. Under the lift described above, the pull-back of the latter bundle is the stable bundle $\nu_X \oplus (f^*\xi)^{-1} \oplus f^*\xi = \nu_X$.

Applying the Pontrjagin-Thom isomorphism (between bordism groups and homotopy groups of the corresponding Thom spectrum) twice, we get:

$$\begin{aligned}\Omega_n^{SO}(B;\xi) \;&\cong\; \pi_n(M(\xi \boxplus \eta^{SO})) \;\cong\; \pi_n(T\xi \wedge \Sigma^{-k} MSO) \\ &\cong\; \pi_{n+k}(T\xi \wedge MSO) \;\cong\; \Omega_{n+k}^{SO}(D\xi, S\xi)\end{aligned}$$

Here $T\xi$ is the Thom space of ξ, M denotes the Thom spectrum, and Ω_*^{SO} is usual bordism of oriented manifolds. We have just proven a Thom isomorphism for oriented bordism (which holds for all bundles because we allow twisted coefficients):

$$\begin{array}{ccc} \Omega_{n+k}^{SO}(D\xi, S\xi) & \longrightarrow & \Omega_n^{SO}(B;\xi) \\ [f:(X,\partial X)\to(D\xi,S\xi)] & \longmapsto & [f^{-1}(B)\to B] \end{array}$$

(for maps f that are transversal to the 0-section B of ξ).

This allows us to compute $\Omega_n^{SO}(B;\xi)$ via the Atiyah-Hirzebruch spectral sequence: We use the homology Thom isomorphism (that also uses twisted coefficients in the general case) to write

$$\begin{tikzcd} H_{p+k}(D\xi, S\xi; \Omega_q^{SO}) \arrow[r, Rightarrow] \arrow[d, "\cong"] & \Omega_{p+q+k}^{SO}(D\xi, S\xi) \arrow[d, "\cong"] \\ H_p\left(B; \underline{\Omega_q^{SO}}\right) \arrow[r, Rightarrow] & \Omega_{p+q}^{SO}(B;\xi) \end{tikzcd}$$

Here the underlining means that the coefficients Ω_q^{SO} are local coefficients twisted by the first Stiefel-Whitney class $w_1(\xi)$, which we can consider as a homomorphism $\pi_1(B)\to\{\pm1\}$.

We can make the analogous definitions and constructions for twisted spin bordism instead of twisted oriented bordism, and get bordism groups $\Omega_q^{Spin}(B;\xi)$.

1.5 Asymmetric manifolds

We are also interested in the realization of six-dimensional conjugation manifolds because work of V. Puppe excludes on some simply-connected six-manifolds all possible (effective, nontrivial) actions of finite groups, except possibly conjugations. Furthermore, M. Kreck proved that some of these six-manifolds are not conjugation manifolds, hence asymmetric. In this section we discuss these ideas.

A theorem of Ebin [Ebi70] states that for every closed manifold M (of dimension > 1), the set of Riemannian metrics which do not admit any non-trivial isometry, is open and dense in the set of all Riemannian metrics on M. We ask whether the reason for the non-existence of symmetry is often already the topology of the manifold.

Definition 1.5.1 *A (closed, differentiable) manifold M is called asymmetric, if every (differentiable) action of any finite group on M is trivial.*

Remark 1.5.2 *Or, what is equivalent, we require that there is no non-trivial action of a compact Lie group on M. Or we require that there is no non-trivial $\mathbb{Z}_p$-action on M, where p runs through all prime numbers. Non-trivial $\mathbb{R}$-actions always exist: just take the flow generated by any vector field. There are even "more" $\mathbb{Z}$-actions: any self-diffeomorphism of M gives a $\mathbb{Z}$-action.*

Alternatively, an asymmetric manifold is a manifold such that for all Riemannian metrics, the isometry group is trivial.

If one considers non-simply-connected manifolds, one can find groups G and manifolds $K(G, 1)$ with no non-trivial action of a finite group. A result of Borel (see [Bor83] or [CRW72]) says that a manifold $K(G, 1)$ is asymmetric, if G has trivial center, and the group of outer automorphisms of G is torsion-free. (An action of a finite group on a manifold induces an action on its fundamental group, which is well-defined only up to inner automorphisms in general.) This result is known for more than 30 years, but the question if there exist simply-connected manifolds with no non-trivial action of a finite group remained open. The problem in finding asymmetric manifolds is that probably any explicit construction has a symmetry in it. Hence, a good idea is to use classification results, for example Wall's following theorem:

Theorem 1.5.3 *[Wal66] Diffeomorphism classes of six-dimensional manifolds which are simply-connected, spin, have free integer cohomology, and zero odd cohomology, correspond bijectively to isomorphism classes of the following invariants:*

1. *the rank m of the second cohomology with integer coefficients H^2,*
2. *a trilinear form μ on H^2 corresponding to the evaluation of triple cup products on the fundamental class,*
3. *and a linear form P_1 on H^2 corresponding to the Poincaré dual of the first Pontrjagin class,*

subject to the conditions:

$$\begin{aligned} \mu(x,x,y) &\equiv \mu(x,y,y) \text{ (mod 2) } \text{ for all } x,y \in H^2, \\ P_1(x) &\equiv 4\mu(x,x,x) \text{ (mod 24) } \text{ for all } x \in H^2. \end{aligned}$$

So let M be a simply-connected spin 6-manifold with free integer and zero odd cohomology, together with a (differentiable) $\mathbb{Z}_p$-action on it. There is a classical theorem that relates the equivariant cohomology of M with the equivariant cohomology of the fixed point set: For simplicity we state the theorem only for $p = 2$.

Theorem 1.5.4 ***(The Localization Theorem) [Hsi75]*** *Let M be a compact space with a $\mathbb{Z}_2$-action. Then*

$$H^*(B\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2[u] \quad \text{where } deg(u) = 2,$$

and the restriction map r to the fixed point set in equivariant cohomology becomes an isomorphism after localising away from u:

$$H^*_{\mathbb{Z}_2}(M;\mathbb{Z}_p)[u^{-1}] \xrightarrow{r} H^*_{\mathbb{Z}_2}(M^{\mathbb{Z}_2};\mathbb{Z}_2)[u^{-1}] \cong H^*(M^{\mathbb{Z}_2};\mathbb{Z}_p) \otimes \mathbb{Z}_2[u,u^{-1}]$$

The Serre spectral sequence gives a relation between the ordinary cohomology of M and the equivariant cohomology of M, we have:

$$H^p(B\mathbb{Z}_2; \underline{H^q(M;\mathbb{Z}_2)}) \Rightarrow H^{p+q}_{\mathbb{Z}_2}(M;\mathbb{Z}_2)$$

If we can assure the following two assumptions:

- The action of $\mathbb{Z}_2$ on M is homologically trivial.
- All the differentials in the Serre spectral sequence vanish.

then we know (using for example the Leray-Hirsch theorem) that as modules $H^*_{\mathbb{Z}_2}(M;\mathbb{Z}_2) \cong H^*(M;\mathbb{Z}_2)[u]$, but maybe with a different multiplication (a "deformation" of the standard multiplication).

If we further assure that no non-trivial deformations are possible, we can show [Pup88] that the cohomologies of M and of the fixed point set are isomorphic as non-graded algebras (we get the same result also for $p \neq 2$). We remark that it follows from this that the fixed point set is connected.

In most cases the cohomology of M can not be isomorphic to the cohomology algebra of a manifold of even dimension smaller than 6. Thus under all the assumptions made, M does not admit any non-trivial orientation preserving action of a finite group (because the codimension of the fixed point set is even in this case). The only possibly non-trivial (effective) actions that remain are orientation-reversing involutions. But the condition that the cohomologies of M and of the fixed point set are isomorphic as non-graded algebras excludes in most cases also fixed point sets of dimension 1 and 5 (see also Theorem A.1 in the appendix) and imply that if the action is non-trivial, there must be an isomorphism between the cohomologies of M and of the fixed point set dividing the degree by two.

Further investigation of the Serre spectral sequence and comparison with the Serre spectral sequence with integer coefficients leads to the following theorem. (Puppe uses for the proof the new characterisation of conjugation spaces given in the next chapter.)

Theorem 1.5.5 *[Pup06] Let M be a simply-connected spin 6-manifold with free integer and zero odd cohomology. Suppose that $H^*(M;\mathbb{Z}_2)$ is generated as an algebra by $H^2(M;\mathbb{Z}_2)$, and suppose the only non-trivial involution on $H^*(M;\mathbb{Z})$ is induced by multiplication with -1 on $H^2(M;\mathbb{Z})$. Suppose further that $H^*(M;\mathbb{Z}_2)$ does not admit non-trivial derivations of degree -2. (A derivation is a linear map $d : H^*(M;\mathbb{Z}_2) \to H^*(M;\mathbb{Z}_2)$ satisfying the Leibniz rule.) Let τ be an orientation-reversing involution on M. Then (M,τ) is a conjugation space.*

Puppe also gives examples of a manifold for which all the assumptions of the preceding theorems hold, i.e. such that the only non-trivial (effective) actions of a finite group are conjugations: All manifolds with the following invariants fulfill all the assumptions. (This gives infinitely many different manifolds, one for each possible choice of the first Pontrjagin class.) The invariants are $H^2 \cong \mathbb{Z}^6$, and the trilinear form μ which is determined by the following equation, where $e_1, \ldots, e_6$ is a basis for H^2, and $x = \sum_{i=1}^{6} x_i e_i$ (where $x_i \in \mathbb{Z}$) is any element of H^2:

$$\begin{aligned}\mu(x,x,x) \;=\; & 6(x_1x_4^2 - x_1^2x_4 + x_2x_4^2 + x_2^2x_4 - x_2^2x_5 + x_3^2x_4 - x_3x_4^2 \\ & + x_3^2x_6 + x_3x_6^2 + x_5^2x_6 + x_5x_6^2 + x_1x_2x_4 + x_1x_2x_5 \\ & + x_1x_3x_6 + x_2x_4x_6 + x_3x_5x_6 + x_4x_5x_6 + x_4^3 + x_6^3)\end{aligned}$$

Kreck [Kre06] has been able to exclude conjugations with a more geometrical approach: His first observation is that for a simply-connected 6-manifold M with free second homology group of rank k, involution τ on M inducing multiplication with -1 on $H^2(M;\mathbb{Z})$, and non-empty fixed point set, the second space in the Postnikov tower for the Borel construction M_C is $((\mathbb{C}\mathrm{P}^\infty)^k)_C$, where the involution on $(\mathbb{C}\mathrm{P}^\infty)^k$ is complex conjugation on each of the factors. We get the following diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \times S^\infty \arrow[r, "F_1"] \arrow[d] & (\mathbb{C}\mathrm{P}^\infty)^k \times S^\infty \arrow[r, "F_2"] \arrow[d] & \mathbb{C}\mathrm{P}^\infty \times S^\infty \arrow[d] \\ M/\tau & M_C \arrow[l, "p_1"'] \arrow[r, "f_1"] & ((\mathbb{C}\mathrm{P}^\infty)^k)_C \arrow[r, "f_2"] & (\mathbb{C}\mathrm{P}^\infty)_C \end{tikzcd}$$

Here p_1 is induced by projection on the first factor, f_1 is the map from the Postnikov tower, hence an isomorphism on π_2. And f_2 comes from an equivariant map $(\mathbb{C}\mathrm{P}^\infty)^k \to \mathbb{C}\mathrm{P}^\infty$, that may be any prescribed map on π_2. F_1 and F_2 are the maps of the universal covers corresponding to f_1 and f_2. We observe that we can realize by such a diagram any homomorphism $\pi_2(M) \to \pi_2(\mathbb{C}\mathrm{P}^\infty \times S^\infty)$, that is we may realize any element of $H^2(M;\mathbb{Z}) \cong [M, \mathbb{C}\mathrm{P}^\infty \times S^\infty]$ in such a diagram.

Now let us cheat and assume that τ was a free involution. Then the quotient space M/τ is again a manifold, and it has a spin double cover; this will eventually imply the existence of a spin structure twisted with a line bundle L on M/τ. Furthermore the map from M_C to M/τ is a homotopy equivalence and has a homotopy inverse. Now we get that the map $M \to \mathbb{C}\mathrm{P}^\infty \times S^\infty$ is (up to homotopy) a double cover of a map $M/\tau \to (\mathbb{C}\mathrm{P}^\infty)_C$. We use this information to formulate a statement about bordism groups: any element $x \in H^2(M;\mathbb{Z})$ gives a map $M \to \mathbb{C}\mathrm{P}^\infty \times S^\infty$, hence a bordism class in $\Omega_6^{Spin}(\mathbb{C}\mathrm{P}^\infty \times S^\infty)$ which is in the image of the transfer map:

$$\Omega_6^{Spin}((\mathbb{C}\mathrm{P}^\infty)_C; L) \to \Omega_6^{Spin}(\mathbb{C}\mathrm{P}^\infty \times S^\infty)$$

Now Kreck computes the image of the transfer map and concludes: Since the image of x is in the image of the transfer map, we must have for all $x \in H^2(M;\mathbb{Z})$:

$$\langle -\frac{1}{24}p_1(M)x + \frac{1}{6}x^3, [M]\rangle \equiv 0 \pmod 2$$

This is the restriction: some of the manifolds considered by Puppe do not satisfy this condition, hence they cannot be (smooth) conjugation manifolds. (Let us remind the reader that we cheated at some point: M/τ is not a manifold, and not homotopy equivalent to M_G. The difficult part in Kreck's work is the "excision" of the fixed point set, in order to obtain a free involution. If we excise a tubular neighbourhood N of the fixed point set, this introduces boundaries, and one has to involve special bordism groups of manifolds with boundaries (for example bordism groups with coefficients) in order to get a non-trivial image, but also such that M represents the same bordism class as $M \setminus N$.) So Kreck has proven the following theorem:

Theorem 1.5.6 *There exist infinitely many asymmetric simply-connected spin 6-manifolds.*

Chapter 2

Conjugation spaces without using conjugation equations

2.1 A definition of conjugation spaces without a conjugation equation

Let X be a topological space equipped with an involution τ, and let Y be a τ-invariant subspace. (The absolute case is given by setting $Y = \emptyset$.) We consider X and Y as C-spaces, where $C = \{1, \tau\} \cong \mathbb{Z}_2$. As a general assumption, suppose that pairs (X, Y) satisfy

$$\dim_{\mathbb{Z}_2} H^i(X, Y; \mathbb{Z}_2) < \infty \quad \text{for all } i.$$

We denote the restriction maps by:

$$\rho : H^*_C(X, Y; \mathbb{Z}_2) \to H^*(X, Y; \mathbb{Z}_2),$$
$$r : H^*_C(X, Y; \mathbb{Z}_2) \to H^*_C(X^\tau, Y^\tau; \mathbb{Z}_2) \cong H^*(X^\tau, Y^\tau; \mathbb{Z}_2)[u]$$

Our result is:

Proposition 2.1.1 *The following statements are equivalent:*

1. *(X, Y) is a conjugation pair, that is, $H^{odd}(X, Y; \mathbb{Z}_2) = 0$, there exist an additive section $\sigma : H^*(X, Y; \mathbb{Z}_2) \to H^*_C(X, Y; \mathbb{Z}_2)$ of ρ, and an additive isomorphism $\kappa : H^{2*}(X, Y; \mathbb{Z}_2) \to H^*(X^\tau, Y^\tau; \mathbb{Z}_2)$ dividing the degrees in half, such that the conjugation equation holds for all $x \in H^{2k}(X, Y; \mathbb{Z}_2)$:*

$$r\sigma(x) = \kappa(x)u^k + \text{terms of lower degree in } u$$

2. r induces an additive isomorphism

$$H_C^*(X,Y;\mathbb{Z}_2) \to H^*(X^\tau,Y^\tau;\mathbb{Z}_2)[u]/\bigoplus_{j>k} H^j(X^\tau,Y^\tau;\mathbb{Z}_2)\cdot u^k$$

3. The following composition is an additive isomorphism:

$$\bigoplus_{j\le k} H_j(X^\tau,Y^\tau;\mathbb{Z}_2)\otimes H_k(\mathbb{RP}^\infty;\mathbb{Z}_2) \hookrightarrow H_*^C(X^\tau,Y^\tau;\mathbb{Z}_2) \to H_*^C(X,Y;\mathbb{Z}_2)$$

Remark 2.1.2 *It follows that we can*

(for pairs (X,Y) satisfying $\dim_{\mathbb{Z}_2} H^i(X,Y;\mathbb{Z}_2)<\infty$ for all i)

define *a conjugation pair as a pair s.t. r induces an additive isomorphism:*

$$H_C^*(X,Y;\mathbb{Z}_2) \to H^*(X^\tau,Y^\tau;\mathbb{Z}_2)[u]/\bigoplus_{j>k} H^j(X^\tau,Y^\tau;\mathbb{Z}_2)\cdot u^k$$

This new definition is helpful because it says under which circumstances the pair (X,Y) has a "conjugation frame" (κ,σ), and because it does not a priori require the existence of such a structure in the (equivariant) cohomology. Nevertheless, one has to know whether a given map in equivariant cohomology is an isomorphism.

Proof:
1. $\Longrightarrow$ 2.:
By the Leray-Hirsch theorem, $H_C^*(X,Y;\mathbb{Z}_2)\cong H^*(X,Y;\mathbb{Z}_2)[u]$ as module over $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2)\cong\mathbb{Z}_2[u]$. A non-zero element in $H_C^i(X,Y;\mathbb{Z}_2)$ may be written as $y=\sum_{j\le j_0}\sigma(y_{i-j})u^j$, where $y_{i-j}\in H^{i-j}(X,Y;\mathbb{Z}_2)$ and $y_{i-j_0}\neq 0$, hence $r(y)=\kappa(y_{i-j_0})u^{(i+j_0)/2}$+terms of lower degree in u, which implies that r is injective and:

$$\operatorname{Im} r\cap\bigoplus_{j>k} H^j(X^\tau,Y^\tau;\mathbb{Z}_2)\cdot u^k=0$$

Now count dimensions to see that:

$$H_C^*(X^\tau,Y^\tau;\mathbb{Z}_2)\cong\operatorname{Im} r\oplus\bigoplus_{j>k} H^j(X^\tau,Y^\tau;\mathbb{Z}_2)\cdot u^k$$

2. $\Longrightarrow$ 1.:
Since r is injective, (X,Y) is totally nonhomologous to zero in (X_C, Y_C), that is, ρ is surjective (compare with Proposition 1.3.14 of [AP93]; for the part we use, one needs only the very weak general assumption from above); by the Leray-Hirsch theorem, $H^*_C(X,Y;\mathbb{Z}_2) \cong H^*(X,Y;\mathbb{Z}_2)[u]$ as $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$-module. Comparing dimensions for the isomorphism gives us the dimension equalities we need: From

$$\sum_{j=0}^{i} \dim_{\mathbb{Z}_2} H^j(X,Y;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^i_C(X,Y;\mathbb{Z}_2) = \sum_{j=0}^{[i/2]} \dim_{\mathbb{Z}_2} H^j(X^\tau, Y^\tau;\mathbb{Z}_2)$$

for all i we get $\dim_{\mathbb{Z}_2} H^i(X,Y;\mathbb{Z}_2) = 0$ if i is odd, and $\dim_{\mathbb{Z}_2} H^{2i}(X,Y;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^i(X^\tau,Y^\tau;\mathbb{Z}_2)$.

We will define the maps $\sigma : H^{2i}(X,Y;\mathbb{Z}_2) \to H^{2i}_C(X,Y;\mathbb{Z}_2)$ and $\kappa : H^{2i}(X,Y;\mathbb{Z}_2) \to H^i(X^\tau,Y^\tau;\mathbb{Z}_2)$ inductively for all i.
$i = 0$: ρ and r are isomorphisms in degree 0, and this defines σ and κ uniquely.
$i-1 \to i$: Since (X,Y) is totally nonhomologous to zero in (X_C,Y_C), there is some additive section $\sigma : H^{2i}(X,Y;\mathbb{Z}_2) \to H^{2i}_C(X,Y;\mathbb{Z}_2)$ of ρ; we will "improve" it by the following procedure: Given a basis $x_1,\ldots,x_n$ of $H^{2i}(X,Y;\mathbb{Z}_2)$, if we have $r\sigma(x_k) = au^{i+j} +$ terms of lower degree in u, and $j > 0$, then, by induction hypothesis, there is $y \in H^{2i-2j}(X,Y;\mathbb{Z}_2)$, such that $\kappa(y) = a$. Now define $\sigma_{new}(x_k) = \sigma_{old}(x_k) + \sigma_{old}(y)u^{2j}$, σ_{new} is a new section of ρ. Repeating this procedure if necessary, we can assume σ is such that $r\sigma(x) = \kappa(x)u^i +$terms of lower degree in u, where κ is defined by this equation. κ is injective since r maps to a complement of $\bigoplus_{j>k} H^j(X^\tau,Y^\tau;\mathbb{Z}_2)\cdot u^k$ and σ is injective, hence κ is an isomorphism since the dimensions are equal.
2. $\Longleftrightarrow$ 3.:
The third statement is the dual translation of the second to homology. **q.e.d.**

2.2 Applications

The new formulation has the advantage that some propositions are easier to prove, namely:

Proposition 2.2.1 *Let X be a C-space, with invariant subspaces $Z \subset Y$ of X (such that all pairs fulfill the general assumption from the beginning of section 2.1).*

a) Suppose that (X,Y) and (Y,Z) are conjugation pairs. Then (X,Z) is a conjugation pair.

b) Suppose that (X,Z) and (Y,Z) are conjugation pairs and that the map $H^*(X,Z;\mathbb{Z}_2) \to H^*(Y,Z;\mathbb{Z}_2)$ is surjective. Then (X,Y) is a conjugation pair.

c) Suppose that (X,Y) and (X,Z) are conjugation pairs and that the map $H^*(X,Y;\mathbb{Z}_2) \to H^*(X,Z;\mathbb{Z}_2)$ is injective. Then (Y,Z) is a conjugation pair.

Proof: Applying the 5-lemma to the diagram below (exact sequences of triples), we get the required isomorphism for part a).

$$\begin{tikzcd}
\bigoplus_{j\le k+1} H_j(X^\tau,Y^\tau;\mathbb{Z}_2)\otimes H_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \arrow[r] \arrow[d] & H_*^C(X,Y;\mathbb{Z}_2) \arrow[d] \\
\bigoplus_{j\le k} H_j(Y^\tau,Z^\tau;\mathbb{Z}_2)\otimes H_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \arrow[r] \arrow[d] & H_*^C(Y,Z;\mathbb{Z}_2) \arrow[d] \\
\bigoplus_{j\le k} H_j(X^\tau,Z^\tau;\mathbb{Z}_2)\otimes H_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \arrow[r] \arrow[d] & H_*^C(X,Z;\mathbb{Z}_2) \arrow[d] \\
\bigoplus_{j\le k} H_j(X^\tau,Y^\tau;\mathbb{Z}_2)\otimes H_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \arrow[r] \arrow[d] & H_*^C(X,Y;\mathbb{Z}_2) \arrow[d] \\
\bigoplus_{j\le k-1} H_j(Y^\tau,Z^\tau;\mathbb{Z}_2)\otimes H_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \arrow[r] & H_*^C(Y,Z;\mathbb{Z}_2)
\end{tikzcd}$$

For part b) the condition that $H^*(X,Z;\mathbb{Z}_2) \to H^*(Y,Z;\mathbb{Z}_2)$ is surjective is by naturality of κ equivalent to $H^*(X^\tau,Z^\tau;\mathbb{Z}_2) \to H^*(Y^\tau,Z^\tau;\mathbb{Z}_2)$ being surjective, which is equivalent to $H_*(Y^\tau, Z^\tau; \mathbb{Z}_2) \to H_*(X^\tau, Z^\tau; \mathbb{Z}_2)$ being injective.

Now apply the 5-lemma to a similar diagram as in a) (but form on the left side always the sum over all pairs (j, k) s.t. $j \leq k$). Proceed in the same way for part c). **q.e.d.**

The new criterion is also useful because it can be used together with all kinds of exact sequences: exact sequences of a triple, Mayer-Vietoris sequences, ... This will be an important tool in our construction of conjugation manifolds, so we will see another application of the criterion in Proposition 4.1.1.

Chapter 3

3-manifolds as fixed point sets of involutions on 6-manifolds

3.1 2-connected 6-manifolds

We want to answer the following question: Given M, a closed orientable 3-manifold, is there a 6-manifold X that contains M as submanifold such that M is the fixed point set of an involution τ on X?

Let us suppose there is an orientable 6-manifold X that contains the orientable 3-manifold M as submanifold with trivial normal bundle and as fixed point set of an involution. (We do not suppose X to be simply-connected, or spin.) By the equivariant tubular neighbourhood theorem, one may write

$$X = M \times D^3 \cup V,$$

where V is a 6-manifold with boundary $\partial V = M \times S^2$, and the involution restricts to a free involution τ on V s.t. $\tau|\partial V = (id, -id)$. Then $W := V/\tau$ is a 6-manifold with boundary $\partial W = M \times \mathbb{RP}^2$, and W has a double cover (namely V) with boundary $M \times S^2$.

Isomorphism classes of double covers of nice topological spaces correspond bijectively to line bundles over the same space, and are (thus) classified by maps to $\mathbb{RP}^\infty$, indeed they are isomorphic to the pullback of the double cover $S^\infty \to \mathbb{RP}^\infty$ by their classifying map.

A double cover of W with boundary $M \times S^2$ is given by the double cover corresponding to the line bundle $det(\nu_W)$, where ν_W is the (stable) normal bundle of W (corresponding to an embedding into $\mathbb{R}^n$ for some $n \gg 0$). (In the following we will always use the same name for a bundle and its classifying map.) Indeed the following diagram is commutative, and $S^\infty \to \mathbb{R}\mathrm{P}^\infty$ pulls back to $M \times S^2 \to M \times \mathbb{R}\mathrm{P}^2$:

$$\begin{tikzcd}
\partial W = M \times \mathbb{R}\mathrm{P}^2 \arrow[r] \arrow[d, "pr"] \arrow[dr, "det(\nu_{\partial W})"] & W \arrow[d, "det(\nu_W)"] \\
\mathbb{R}\mathrm{P}^2 \arrow[r] & \mathbb{R}\mathrm{P}^\infty
\end{tikzcd}$$

For any manifold N, the bundle $\nu_N - det(\nu_N)$ is an orientable bundle, since $w_1(\nu_N) = w_1(det(\nu_N))$. Hence if we consider $[M \times \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty]$ together with any orientation, as an element of $\Omega_5^{SO}(\mathbb{R}\mathrm{P}^\infty; L)$, it is zero if there exists such an X. Here L is the canonical line bundle over $\mathbb{R}\mathrm{P}^\infty$.

Let us now compute $\Omega_5^{SO}(\mathbb{R}\mathrm{P}^\infty; L)$: We compute the $E^2_{p,5-p}$-terms of the Atiyah-Hirzebruch spectral sequence:

$$H_p\left(\mathbb{R}\mathrm{P}^\infty; \underline{\Omega_q^{SO}}\right) \Rightarrow \Omega_{p+q}^{SO}(\mathbb{R}\mathrm{P}^\infty; L)$$

The coefficients Ω_q^{SO} for small q are [MS74]: $\Omega_0^{SO} = \mathbb{Z}$, $\Omega_1^{SO} = \Omega_2^{SO} = \Omega_3^{SO} = 0$, $\Omega_4^{SO} = \mathbb{Z}$, $\Omega_5^{SO} = \mathbb{Z}_2$. They are twisted by $w_1(L)$, i.e. $\mathbb{Z}_2 \cong \pi_1(\mathbb{R}\mathrm{P}^\infty)$ acts on the coefficients by multiplication with ± 1. We denote $\mathbb{Z}$ with this π_1-action by $\mathbb{Z}_-$. We obtain:

$$\begin{aligned}
H_5(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_-) &= 0 \\
H_4(\mathbb{R}\mathrm{P}^\infty; 0) &= 0 \\
H_3(\mathbb{R}\mathrm{P}^\infty; 0) &= 0 \\
H_2(\mathbb{R}\mathrm{P}^\infty; 0) &= 0 \\
H_1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_-) &= 0 \\
H_0(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) &= \mathbb{Z}_2
\end{aligned}$$

The $\mathbb{Z}_2$ on the diagonal corresponds to $[Y^5 \to pt \to \mathbb{R}\mathrm{P}^\infty]$, where Y^5 is orientable, and a generator of Ω_5^O, the unoriented bordism group. Hence this non-zero element of E^2 survives to E^∞, as it is, considered as an unoriented manifold, not null-bordant, and:

$$\Omega_5^{SO}(\mathbb{R}\mathrm{P}^\infty; L) = \mathbb{Z}_2\langle[Y^5 \to pt \to \mathbb{R}\mathrm{P}^\infty]\rangle$$

But $M \times \mathbb{R}\mathrm{P}^2$ is, considered as an unoriented manifold, not bordant to Y^5. This is because it is nullbordant, since $\Omega_3^O = 0$. Hence in $\Omega_5^{SO}(\mathbb{R}\mathrm{P}^\infty; L)$ one has:

$$[M \times \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty] \neq [Y^5 \to pt \to \mathbb{R}\mathrm{P}^\infty]$$

Hence

$$[M \times \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty] = 0 \in \Omega_5^{SO}(\mathbb{R}\mathrm{P}^\infty; L)$$

for all oriented 3-manifolds M.

This implies that for any M as before, there is a nullbordism W of $M \times \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty$, and one finds a double cover V of W with $\partial V = M \times S^2$ (it is the cover classified by $det(\nu_W)$, hence orientable) and a free involution on V given by covering transformations. Finally one gets $X = M \times D^3 \cup V$ and one extends the involution to X, such that the fixed point set is M. This answers our question.

We can even include a further condition on X: Suppose that we are given a nullbordism:

$$\begin{tikzcd}
 & & \mathbb{R}\mathrm{P}^\infty \times BSO \arrow[d, "\oplus\circ(id\times Bi)"] \\
\partial W = M \times \mathbb{R}\mathrm{P}^2 \arrow[r] & W \arrow[ur, "\overline{\nu}"] \arrow[r, "\nu"'] & BO
\end{tikzcd}$$

By surgery below the middle dimension, in the interior of W, one can replace the nullbordism and the lift by $(W, \overline{\nu})$ with equal boundary, such that $\overline{\nu} : W \to \mathbb{R}\mathrm{P}^\infty \times BSO$ is a 3-equivalence. Hence $\pi_1(W) \cong \mathbb{Z}_2$, and V, the double cover of W, will be the universal cover. Finally, using the theorem of Seifert-van Kampen, we get a simply-connected $X = M \times D^3 \cup V$.

Now let us see whether we can include that X is spinnable.

Lemma 3.1.1 *Let X be a closed spin 6-manifold, and M be a closed orientable 3-dimensional submanifold of X. Then M has trivial normal bundle in X.*

Proof: We first consider the tangent bundle of M. It has only one possibly non-zero Wu class v_1, since the k-th Wu class v_k of an n-dimensional manifold is defined by the identity $\langle Sq^k(x), [M]\rangle = \langle v_k \cdot x, [M]\rangle$ for all $x \in H^{n-k}(M; \mathbb{Z}_2)$ and $Sq^k(x) = 0$ for $k > n-k$. For the Stiefel-Whitney classes this implies: $0 = w_1 = v_1$ since M is orientable, $w_2 = v_1^2 = 0$, $w_3 = 0$.

We consider the fibration $SO(3) \to ESO(3) \to BSO(3)$ and ask whether the tangent bundle map $M \to BSO(3)$ lifts to $ESO(3)$. Then the tangential bundle must be trivial. We apply obstruction theory: the obstructions for a lift are cohomology classes in $H^{j+1}(M; \pi_j(SO(3)))$. Now $SO(3) \cong \mathbb{R}\mathrm{P}^3$ has fundamental group $\mathbb{Z}_2$ and the second homotopy group is trivial. Since all Stiefel-Whitney classes of M are zero, the map to $BSO(3)$ is trivial in $\mathbb{Z}_2$-cohomology. Thus the obstruction in $H^2(M; \pi_1(SO(3)))$ is zero by naturality. All other obstructions are zero because the group $H^2(M; \pi_1(SO(3)))$ is the zero group. So M has trivial normal bundle. Since the total Stiefel-Whitney class of a direct sum of vector bundles is the product of the total Stiefel-Whitney classes of its summands, we obtain that the normal bundle of M in X has $w_1 = w_2 = 0$. By the same argument as for the tangent bundle, the normal bundle of M in X is trivial. **q.e.d.**

So we suppose now that there is a 1-connected spin 6-manifold X that contains the given orientable 3-manifold M as submanifold with trivial normal bundle, and as fixed point set of an orientation-reversing involution. Again we write $X = M \times D^3 \cup V$. V is 1-connected spin if and only if X is, since M has codimension 3 and every spin structure on $M \times S^2$ extends to $M \times D^3$. The map $\pi : V \to V/\tau = W$ is the universal cover of W, which is classified by the unique (up to homotopy) non-trivial map $W \to \mathbb{R}\mathrm{P}^\infty$, since $\pi_1(W) = \mathbb{Z}_2$. We denote this map by w_1, since (when identifying H^1 with maps to $\mathbb{R}\mathrm{P}^\infty$) this is the first Stiefel-Whitney class, as $w_1(\partial W) \neq 0$.

The homotopy fiber of $w_1 : W \to \mathbb{R}\mathrm{P}^\infty$ is $\pi : V \to W$. From the corresponding Serre spectral sequence, we get an exact sequence (consider the terms on the second diagonal: $E_\infty^{2,0} \cong E_2^{2,0} \cong H^2(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ and $E_\infty^{0,2} \subset E_2^{0,2} \subset H^2(V; \mathbb{Z}_2)$):

$$\begin{array}{ccccc} 0 \to H^2(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) & \xrightarrow{w_1^*} & H^2(W; \mathbb{Z}_2) & \xrightarrow{\pi^*} & H^2(V; \mathbb{Z}_2) \\ & & w_2(\nu_W) = w_2(W) + w_1^2(W) & \mapsto & w_2(\nu_V) = 0 \end{array}$$

One has $w_2(\nu_V) = w_2(V) + w_1^2(V) = 0$ since V is spin. Since $w_2(\partial W) \neq 0$ and $w_1^2(\partial W) \neq 0$, we get $w_2(W) \neq 0$ and $w_1^2(W) \neq 0$, but their images under π are 0. Hence $w_2(W)$ and $w_1^2(W)$ are in the image of w_1^*, and necessarily $w_2(W) = w_1^2(W)$ since both are nonzero. This implies that $w_2(\nu_W) = 0$. For the total Stiefel-Whitney class, we get:

$$\begin{aligned} w(\nu_W) &= 1 + w_1(\nu_W) + 0 + w_3(\nu_W) + \ldots \\ w(\nu_W - det(\nu_W)) &= 1 + 0 + 0 + w_3(\nu_W) + \ldots \end{aligned}$$

Hence $\nu_W - det(\nu_W)$ is spinnable, and we obtain that

$$[M \times \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty] = 0 \in \Omega_5^{Spin}(\mathbb{R}\mathrm{P}^\infty; L)$$

if such a 1-connected spin X exists. One can try to compute $\Omega_5^{Spin}(\mathbb{R}\mathrm{P}^\infty; L)$ by the Atiyah-Hirzebruch spectral sequence. The coefficients Ω_q^{Spin} for small q are [Mil63]: $\Omega_0^{Spin} = \mathbb{Z}$, $\Omega_1^{Spin} = \Omega_2^{Spin} = \mathbb{Z}_2$, $\Omega_3^{Spin} = 0$, $\Omega_4^{Spin} = \mathbb{Z}$, $\Omega_5^{Spin} = \Omega_6^{Spin} = \Omega_7^{Spin} = 0$. They are twisted by $w_1(L)$, i.e. $\mathbb{Z}_2 \cong \pi_1(\mathbb{R}\mathrm{P}^\infty)$ acting on the coefficients by multiplication with ± 1. The following diagram describes the relevant part of the E^2-term.

q							
5		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
4	$\mathbb{Z}_2$		$\cdots$	$\cdots$	$\cdots$	$\cdots$	
3				$\cdots$	$\cdots$	$\cdots$	
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$	$\cdots$	
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$	
0	$\mathbb{Z}_2$		$\mathbb{Z}_2$		$\mathbb{Z}_2$		
	0	1	2	3	4	5	p

We use the following result of Kirby-Taylor [KT90]:

Theorem 3.1.2 $\Omega_4^{Spin}(\mathbb{R}\mathrm{P}^\infty; L) = 0$.

If one considers now the spectral sequence, one sees that all terms on the fifth diagonal of the E^2-term are needed to kill elements on the fourth diagonal, since on the fourth diagonal, we have four $\mathbb{Z}_2$'s, while on the third and fifth diagonal, there are two $\mathbb{Z}_2$'s respectively. Hence we get:

Theorem 3.1.3 $\Omega_5^{Spin}(\mathbb{R}\mathrm{P}^\infty; L) = 0$.

Again, $[M \times \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty] = 0 \in \Omega_5^{Spin}(\mathbb{R}\mathrm{P}^\infty; L)$ allows us to construct a suitable manifold X: Given a nullbordism

$$\begin{tikzcd} & & \mathbb{R}\mathrm{P}^\infty \times BSpin \arrow[d, "\oplus\circ(id\times Bp)"] \\ \partial W = M \times \mathbb{R}\mathrm{P}^2 \arrow[r] & W \arrow[ur, "\overline{\nu}"] \arrow[r, "\nu"] & BO \end{tikzcd}$$

one may replace it by a nullbordism $(W, \overline{\nu})$ such that $\overline{\nu} : W \to \mathbb{R}\mathrm{P}^\infty \times BSpin$ is a 3-equivalence. This implies that $\pi_1(W) \cong \mathbb{Z}_2$ and $\pi_2(W) = 0$. We construct V and X as in the orientable case. We obtain that V is spin: Since the map from W to $\mathbb{R}\mathrm{P}^\infty$ must be the first Stiefel-Whitney class, we get that V is the orientable cover, i.e. $det(\nu_V)$ is trivial. But $\nu_V - det(\nu_V)$ is the pullback of $\nu_W - det(\nu_W)$, and this implies that the first two Stiefel-Whitney classes of ν_V are zero. Furthermore $\pi_1(V) = \pi_2(V) = 0$. Finally also X is spin and $\pi_1(X) = \pi_2(X) = 0$ (since M has codimension 3 in X). Hence X is a 2-connected 6-manifold, which implies that X is a connected sum of copies of $S^3 \times S^3$.

So our theorem becomes very simple:

Theorem 3.1.4 *Every orientable* 3*-manifold M is the fixed point set of an orientation reversing involution on $\#_r S^3 \times S^3$ for some r.*

Remark 3.1.5 *One has $r \geq dim_{\mathbb{Z}_2} H^1(M; \mathbb{Z}_2)$ by a theorem of P.A.Smith [Bre62].*

3.2 6-manifolds with no odd cohomology

Our next goal is the following: Can we find X with all odd cohomology groups equal to zero? Then X has free integer homology. Our previous result considered 1-connected manifolds X, so $H_1(X;\mathbb{Z}) = 0$; we should try to find conditions that induce $H_3(X;\mathbb{Z}) = 0$. Actually there is an easier construction here: Since the bordism group of stably framed 3-manifolds is generated by the 3-sphere with various framings, we find a stably framed 4-manifold W' with boundary M. By framed surgery below the middle dimension, we can assume that W' is 1-connected. Now we take $W = W' \times \mathbb{R}\mathrm{P}^2$. On S^2, there is a stable framing that extends to D^3. This implies that if we glue in the right way, $X = M \times D^3 \cup W' \times S^2$ is a stably parallelizable 6-manifold, hence spin and with zero Pontrjagin class. By the theorem of Seifert-van Kampen: $\pi_1(X) = \pi_1(M) *_{\pi_1(M)} \pi_1(W') = 0$.

Let us consider the Mayer-Vietoris sequence:

$$\begin{aligned} &H_3(M \times S^2;\mathbb{Z}) \to H_3(M \times D^3;\mathbb{Z}) \oplus H_3(W' \times S^2;\mathbb{Z}) \to H_3(X;\mathbb{Z}) \\ &\to H_2(M \times S^2;\mathbb{Z}) \to H_2(M \times D^3;\mathbb{Z}) \oplus H_2(W' \times S^2;\mathbb{Z}) \to H_2(X;\mathbb{Z}) \to 0 \end{aligned}$$

The group $H_3(W' \times S^2;\mathbb{Z})$ is zero, since $H_1(W';\mathbb{Z}) = 0$ and $H_3(W';\mathbb{Z}) \cong H^1(W', M;\mathbb{Z}) = 0$ by Poincaré duality. Thus the map $H_3(M \times S^2;\mathbb{Z}) \to H_3(M \times D^3;\mathbb{Z}) \oplus H_3(W' \times S^2;\mathbb{Z})$ is surjective, and the map $H_2(M \times S^2;\mathbb{Z}) \to H_2(M \times D^3;\mathbb{Z}) \oplus H_2(W' \times S^2;\mathbb{Z})$ is injective, hence $H_3(X;\mathbb{Z}) = 0$. Hence the integer cohomology of X is concentrated in even degrees and a free abelian group (using the Universal Coefficient Theroem and Poincaré duality). The involution on X is orientation-reversing, so it is multiplication with -1 on $H^6(X;\mathbb{Z}) \cong \mathbb{Z}$. From the Mayer-Vietoris sequence above we see that $H_2(W';\mathbb{Z}) \cong H_2(X;\mathbb{Z})$, and this implies that the involution acts trivially on $H_2(X;\mathbb{Z})$ and also trivially on $H^2(X;\mathbb{Z})$. As a consequence, all triple cup products $H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to H^6(X;\mathbb{Z})$ are zero. Now we apply the Wall classification to see that X is diffeomorphic to a connected sum of $S^2 \times S^4$. We have proved:

Theorem 3.2.1 *Every orientable* 3*-manifold* M *is the fixed point set of an orientation reversing involution on* $\#_r S^2 \times S^4$ *for some* r*. Again, we have* $r \geq \dim_{\mathbb{Z}_2} H^1(M;\mathbb{Z}_2)$.

Chapter 4

Constructing conjugation 6-manifolds

4.1 When do we get a conjugation manifold?

Up to now, we saw that every orientable 3-manifold was realizable as fixed point set of an involution on a 6-manifold, but there was almost no relation between the 3-manifold and the 6-manifold. (One could take the same given 6-manifold for many different 3-manifolds.) Now we want to get closer to our goal: given M^3, find a 6-dimensional conjugation manifold X with fixed point set M^3. (X must be a spin manifold: Under the conjugation space isomorphism κ of $\mathbb{Z}_2$-cohomology algebras dividing the degree by two, the second Wu class of X will be mapped to the first Wu class of M, since we have $\kappa(Sq^{2k}(x)) = Sq^k(\kappa(x))$, see [FP05].)

We follow our basic approach: we make the ansatz $X = M \times D^3 \cup V$, $W = V/\tau$ and try to find W. Now we develop easily a necessary and sufficient condition for X to be a conjugation space. (Here our new definition of conjugation spaces shows its strength.)

We just consider the Mayer-Vietoris sequence in equivariant cohomology with $\mathbb{Z}_2$ coefficients. (Recall that on $M \times D^3$ the involution is $(id, -id)$ and on V it is free.)

$$\begin{tikzcd}
{} \arrow[r] & H^*_C(X;\mathbb{Z}_2) \arrow[r] \arrow[d, "\cong"] & H^*_C(M \times D^3;\mathbb{Z}_2) \oplus H^*_C(V;\mathbb{Z}_2) \arrow[r] \arrow[d, "\cong"] & H^*_C(M \times S^2;\mathbb{Z}_2) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r] & H^*_C(X;\mathbb{Z}_2) \arrow[r] & H^*(M \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \oplus H^*(W;\mathbb{Z}_2) \arrow[r] & H^*(M \times \mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) \arrow[r] & {}
\end{tikzcd}$$

The isomorphism $H^*_C(M \times D^3; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ comes from the equivariant deformation retraction $M \times D^3 \to M$ and example 1.1.2. Since the actions on the other spaces are free, the other isomorphisms follow from remark 1.1.3. Now in the lower row, the map $H^*(W; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$ is just the usual restriction to the boundary, and $H^*_C(X; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \cong H^*(M; \mathbb{Z}_2)[u]$ is the restriction to the fixed point set in equivariant cohomology. Now consider the map $p : H^*(M \times \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$. We have that $H^*(M; \mathbb{Z}_2)$ is mapped identically to $H^*(M; \mathbb{Z}_2)$ since all maps

$$M \times \mathbb{R}\mathrm{P}^\infty \leftarrow (M \times D^3)_C \to (M \times S^2)_C \to M \times \mathbb{R}\mathrm{P}^2$$

commute with the projection to M, and $u \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ is mapped to $u \in H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$ since all the above maps are isomorphisms on the fundamnetal group. So the map is given by dividing out the ideal generated by u^3. This is surjective, hence we have for all k a short exact sequence:

$$0 \to H^k_C(X; \mathbb{Z}_2) \to H^k(M \times \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \oplus H^k(W; \mathbb{Z}_2) \to H^k(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \to 0$$

Now we apply the new definition of a conjugation space. X is a conjugation space iff $r : H^*_C(X; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \cong H^*(M; \mathbb{Z}_2)[u]$ induces an isomorphism $\bar{r} : H^*_C(X; \mathbb{Z}_2) \to H^*(M; \mathbb{Z}_2)[u] / \bigoplus_{i>j} H^i(M; \mathbb{Z}_2)u^j$.

We combine this with the information from the short exact sequence: r is injective if and only if the map $j : H^*(W; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$ is injective, and $\mathrm{Im}(r) = p^{-1}(\mathrm{Im}(j))$. Hence j must be injective and the image must be a complement of $\bigoplus_{i>j} H^i(M; \mathbb{Z}_2)u^j$:

Theorem 4.1.1 *X is a conjugation space iff*

$$H^*(W; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$$

induces an isomorphism:

$$H^*(W; \mathbb{Z}_2) \to H^*(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) / \bigoplus_{i>j} H^i(M; \mathbb{Z}_2)u^j$$

Translated to homology this is equivalent to the condition that

$$H_*(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \to H_*(W; \mathbb{Z}_2)$$

induces an isomorphism:

$$\bigoplus_{i \le j} H_i(M; \mathbb{Z}_2) \otimes H_j(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \to H_*(W; \mathbb{Z}_2)$$

We can generalize this theorem to conjugation manifolds X of any even dimensions, and with fixed point submanifold M having any normal bundle in X (the proof is the same as above):

Theorem 4.1.2 *Let X be a $2n$-dimensional manifold, with a differentiable involution τ that has the n-dimensional submanifold M as fixed point set. Let ν be the normal bundle of M in X. Let $D(\nu)$, $S(\nu)$ and $P(\nu)$ denote respectively the disk bundle, sphere bundle and projective bundle of ν. Using the equivariant tubular neighbourhood theorem, write $X = D(\nu) \cup V$, such that $W = V/\tau$ is a manifold with boundary $P(\nu)$.*

Then X is a conjugation space iff

$$H^*(W;\mathbb{Z}_2) \to H^*(P(\nu);\mathbb{Z}_2)$$

induces an isomorphism:

$$H^*(W;\mathbb{Z}_2) \to H^*(P(\nu);\mathbb{Z}_2)/\bigoplus_{i>j} H^i(M;\mathbb{Z}_2)u^j$$

Here we use the theorem of the projective bundle / the construction of the Stiefel-Whitney classes described in [Hat02]: There is a class $u \in H^1(P(\nu);\mathbb{Z}_2)$ such that

$$\begin{aligned} H^*(P(\nu);\mathbb{Z}_2) &\cong H^*(M;\mathbb{Z}_2)[u]/\left(u^n + w_1(\nu)u^{n-1} + \cdots + w_n(\nu)\right) \\ &\cong \textstyle\bigoplus_{j\leq n-1} H^i(M;\mathbb{Z}_2)u^j \end{aligned}$$

Hence, in order to find conjugation manifolds with given fixed point set M of dimension n, one has to find manifolds W of dimension $2n$ with boundary $P(\nu)$, where ν is some n-dimensional vector bundle over M such that W satisfies the property from the theorem.

Let us now describe the strategy to find such manifolds W. (Let us assume the normal bundle of M in X shall be trivial.)

We start by trying to find a manifold W such that the isomorphism above holds below the middle dimension n. For this, we consider the $(n-1)$-type of W, denoted by $B^{n-1}(W)$. $B^{n-1}(W)$ is the $(n-1)$-th stage of a Moore-Postnikov factorization of the stable normal bundle map $W \to BO$. It follows that $W \to B^{n-1}(W)$ is n-connected, hence an isomorphism in cohomology for degrees smaller than n. Hence the composition $M \times \mathbb{R}\mathrm{P}^{n-1} \to W \to B^{n-1}(W)$ must induce an isomorphism

$$\bigoplus_{i \leq j, i+j=k} H_i(M;\mathbb{Z}_2) \otimes H_j(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2) \to H_k(B^{n-1}(W);\mathbb{Z}_2)$$

for all $k \leq n-1$. So the first step is to find a candidate for the $(n-1)$-type for W, and the second step, to find a candidate for the map $F : M \times \mathbb{R}\mathrm{P}^{n-1} \to B^{n-1}(W)$, that has this isomorphism property. In a third step, we prove that $[F] = 0 \in \Omega_{2n-1}^{B^{n-1}(W)}$, which is clearly necessary. If $[F] = 0$, then we find a nullbordism W_0 of $M \times \mathbb{R}\mathrm{P}^{n-1}$ together with a lift of its normal bundle to $B^{n-1}(W)$, and by surgery below the middle dimension we may assume that this map $W_0 \to B^{n-1}(W)$ is n-connected. Then it follows that below dimension n, we have the required isomorphism.

We will see that above the middle dimension, Poincaré duality does the job: we get also in dimension bigger than n the required isomorphism. Finally, in the middle dimension, we will use a surgery-theoretic argument. This step can in fact alter the $(n-1)$-type of W, but we still obtain the required isomorphism in $\mathbb{Z}_2$-cohomology, now in all degrees, and so, we end up with a conjugation manifold X with fixed point set M.

We work through this program step by step in the following sections.

4.2 The 2-type and the second stage of the Postnikov tower for W

We keep the condition that in the end, we want to obtain a simply-connected spin 6-manifold X. Then it remains true that W must carry a spin structure twisted by L, that is, the normal bundle map $W \to BO$ factors over $\mathbb{R}\mathrm{P}^\infty \times BSpin$. Since the action must be orientation-reversing, the first factor $W \to \mathbb{R}\mathrm{P}^\infty$ is the first stage of the Postnikov tower for W. Hence this map factors over the second stage of the Postnikov tower, $P_2(W)$. Thus the normal bundle map can be written as a composition $W \to P_2(W) \times BSpin \to BO$. Since BO is 3-connected, the map $W \to P_2(W) \times BSpin$ is 3-connected, and $P_2(W) \times BSpin \to BO$ is 3-coconnected. Hence the 2-type for W is $B^2(W) = P_2(W) \times BSpin$ (together with the fibration $P_2(W) \times BSpin \to BO$), and we are interested in $P_2(W)$.

Now let us make additional hypotheses:

1. Let us restrict to connected 3-manifolds M. (Clearly if M is non-connected, X must have one component for each component of M).
2. We would ideally want to obtain X with free integer homology.
3. We prescribe that the involution acts by -1 on $H_2(X;\mathbb{Z})$. Clearly the involution must induce a $\mathbb{Z}_2$-action on the integer homology that is multiplication with -1 on $H_6(X;\mathbb{Z})$. We choose the easiest involution on $H_2(X;\mathbb{Z})$ that could work in all cases.

It will turn out that these assumptions determine $P_2(W)$ (and so $B^2(W)$) uniquely. As we already mentioned, we will not be able to keep the assumptions in the final surgery step – we will see this later.

We consider the following Mayer-Vietoris sequence:

$$\begin{aligned}&\to H_4(X;\mathbb{Z}) \to H_3(M\times S^2;\mathbb{Z}) \to H_3(M\times D^3;\mathbb{Z}) \oplus H_3(V;\mathbb{Z})\\ &\to H_3(X;\mathbb{Z}) \to H_2(M\times S^2;\mathbb{Z}) \to H_2(M\times D^3;\mathbb{Z}) \oplus H_2(V;\mathbb{Z}) \to H_2(X;\mathbb{Z})\end{aligned}$$

In order to obtain $H_3(X;\mathbb{Z}) = 0$, the map

$$H_2(M\times S^2;\mathbb{Z}) \to H_2(M\times D^3;\mathbb{Z}) \oplus H_2(V;\mathbb{Z})$$

must be injective, which means that the map $g_* : H_2(S^2;\mathbb{Z}) \to H_2(V;\mathbb{Z})$ must be injective, where $g: S^2 \hookrightarrow M\times S^2 \hookrightarrow V$. V shall be 1-connected, so

$g_* = g_* : \pi_2(S^2) \to \pi_2(V)$ and this implies that $\mathbb{Z} \cong \pi_2(S^2) \to \pi_2(V)$ must be injective. The same holds then for the map $\mathbb{Z} \cong \pi_2(\mathbb{RP}^2) \to \pi_2(W)$. Actually the map should better be split injective, since its cokernel shall be the free group $H_2(X;\mathbb{Z})$. Then it also follows that $\pi_2(W) \cong H_2(V;\mathbb{Z}) \cong \mathbb{Z} \oplus H_2(X;\mathbb{Z})$ is free. The involution on $H_2(V;\mathbb{Z})$ is multiplication with -1 since this is true for both summands. Hence $\mathbb{Z}_2 \cong \pi_1(W)$ acts by multiplication with ± 1 on $\pi_2(W)$.

Consider now the first spaces of the Postnikov tower for W. We have $P_1(W) = \mathbb{RP}^\infty$ since $\pi_1(W) \cong \mathbb{Z}_2$. Furthermore $\pi_2(W) \cong \mathbb{Z}^m$ for some m, and $\pi_1(W)$ acts by multiplication with -1 on $\pi_2(W)$. We want to know $P_2(W)$. Following Baues [Bau77], such fibrations $(\mathbb{CP}^\infty)^m \xrightarrow{i} E \xrightarrow{p} \mathbb{RP}^\infty$ are given up to equivalence by their k-invariant

$$k(i,p) \in H^3(\mathbb{RP}^\infty; \pi_2((\mathbb{CP}^\infty)^m)_-) \cong \pi_2((\mathbb{CP}^\infty)^m) \otimes \mathbb{Z}_2 \cong (\mathbb{Z}_2)^m,$$

which is the obstruction for a section. The last isomorphism is given by the inclusions of the m factors - we will use it to identify a k-invariant with a vector in $(\mathbb{Z}_2)^m$.

Two fibrations $(\mathbb{CP}^\infty)^m \xrightarrow{i} E \xrightarrow{p} \mathbb{RP}^\infty$ and $(\mathbb{CP}^\infty)^m \xrightarrow{i'} E' \xrightarrow{p'} \mathbb{RP}^\infty$ are equivalent here, if one has a fibre homotopy equivalence $h : E \to E'$ such that hi is homotopic to i'.

Lemma 4.2.1 *There is a 3-dimensional vector bundle over $\mathbb{HP}^\infty$ whose sphere bundle is $S^2 \to \mathbb{CP}^\infty \to \mathbb{HP}^\infty$.*

Proof: We can consider S^∞ also as unit sphere in $\mathbb{H}^\infty$, and consider the right action of the group of unit quaternions on it. This group is isomorphic to $SU(2) \cong S^3$. The action of the subgroup S^1 can be interpreted as the action on the unit sphere of $\mathbb{C}^\infty$. The quotient spaces are $S^\infty/S^3 = \mathbb{HP}^\infty$ and $S^\infty/S^1 = \mathbb{CP}^\infty$, respectively. We get a fiber bundle $S^3/S^1 \to S^\infty/S^1 \to S^\infty/S^3$, which is exactly $S^2 \to \mathbb{CP}^\infty \to \mathbb{HP}^\infty$.

Now the fiber bundle $S^3/S^1 \to S^\infty/S^1 \to S^\infty/S^3$ can be seen as a bundle with structure group S^3: Over $\{q_i = z_i + J \cdot z_i' \neq 0\} \subset \mathbb{H}^\infty$, the bundle is trivial with section s_i mapping an element $q \in \mathbb{HP}^\infty$ with representative $[q_1 = z_1 + J \cdot z_1' : q_2 = z_2 + J \cdot z_2' : \ldots]$ such that $q_i \in \mathbb{R}_{>0}$ (there is a unique such representative since we consider only elements of norm 1 in $\mathbb{HP}^\infty$) to the corresponding element $[q_1 : q_2 : \ldots]$ or $[z_1 : z_1' : z_2 : z_2' : \ldots] \in \mathbb{CP}^\infty$ (depending on whether one considers S^∞/S^1 as quotient of $\mathbb{H}^\infty$ or $\mathbb{C}^\infty$). Over

$\{q_i \neq 0\} \cap \{q_j \neq 0\}$, the two sections s_i and s_j differ in the point $q \in \mathbb{H}\mathrm{P}^\infty$ by the action of $\frac{q_i \bar{q}_j}{\|q_i\| \cdot \|q_j\|}$, which gives a well-defined transition function to S^3.

Now under the action of S^3 on the fibers, -1 acts trivially, so we can also equip the fiber bundle with structure group $S^3/\{\pm 1\} \cong SO(3)$, and this implies that the fiber bundle is the sphere bundle of the corresponding (oriented) vector bundle with the same transition functions. **q.e.d.**

Denote the corresponding projective space bundle by $\mathbb{R}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^\infty/\tau \to \mathbb{H}\mathrm{P}^\infty$. Here τ is the free $\mathbb{Z}_2$-action on $\mathbb{C}\mathrm{P}^\infty$ which is multiplication with -1 on each fiber of the sphere bundle. We also know a non-free $\mathbb{Z}_2$-action on $\mathbb{C}\mathrm{P}^\infty$: the complex conjugation c. Now consider the spaces:

$$P_m = \left((\mathbb{C}\mathrm{P}^\infty)^m \times S^\infty \right) / (c^m, -1)$$

$$Q_m = \left((\mathbb{C}\mathrm{P}^\infty)^m \times S^\infty \right) / (\tau^m, -1)$$

They are the quotient spaces of the diagonal $\mathbb{Z}_2$-action which is τ (resp. c) on the first m factors, and multiplication with -1 on S^∞. There are maps to $P_1(W)$ induced by projection onto the last factor, and these are fibrations of the type described above. $(\mathbb{C}\mathrm{P}^\infty)^m \xrightarrow{i} P_m \xrightarrow{p} \mathbb{R}\mathrm{P}^\infty$, where i is induced by the inclusion into the first m components, corresponds to the k-invariant zero, since it has a section: map $x \in \mathbb{R}\mathrm{P}^\infty$ to the class of $(x, x, \ldots, x, \tilde{x})$, where $\tilde{x}$ is any preimage in S^∞. $(\mathbb{C}\mathrm{P}^\infty)^m \xrightarrow{i} Q_m \xrightarrow{q} \mathbb{R}\mathrm{P}^\infty$, where i is induced by the inclusion into the first m components, has a non-zero k-invariant. (This follows from the classification results and the fact that Q_m is not homotopy equivalent to P_m.)

Now $[(\mathbb{C}\mathrm{P}^\infty)^m, (\mathbb{C}\mathrm{P}^\infty)^m] \cong Mat(m \times m, \mathbb{Z})$; let $A \in GL(m, \mathbb{Z})$, denote a corresponding representative $(\mathbb{C}\mathrm{P}^\infty)^m \to (\mathbb{C}\mathrm{P}^\infty)^m$ again by A. Then the fibration $(\mathbb{C}\mathrm{P}^\infty)^m \xrightarrow{i \circ A} Q_m \to \mathbb{R}\mathrm{P}^\infty$ has k-invariant $k(i \circ A, q)$ such that $(A \bmod 2) \cdot k(i \circ A, q) = (k(i, q))$ since the modification corresponds to a map on the coefficients.

Let P be a permutation matrix. Then there is a fibre homotopy self-equivalence $h : Q_m \to Q_m$ such that $i \circ P$ is homotopic to $h \circ i$: we just permute the components in $Q_m = ((\mathbb{C}\mathrm{P}^\infty)^m \times S^\infty)/(\tau^m, -1)$ with the corresponding permutation. This implies that $k(i \circ P, q) = k(i, q)$, i.e. $P \cdot k(i, q) = k(i, q)$ for all permutation matrices P. It follows that $k(i, q) = (1, \ldots, 1) \in (\mathbb{Z}_2)^m$ and that, given $A \in GL(m, \mathbb{Z})$, there is a fibre homotopy self-equivalence

$h : Q_m \to Q_m$ such that $i \circ A$ is homotopic to $h \circ i$ if and only if $(A \bmod 2) \cdot (1, \ldots, 1) = (1, \ldots, 1)$.

Lemma 4.2.2 *$SL(m, \mathbb{Z}) \to GL(m, \mathbb{Z}_2)$ is surjective.*

Proof: Induction: for $m = 1$ this is obvious, and for an invertible $m \times m$-matrix A with $\mathbb{Z}_2$-coefficients we must find an integer lift with determinant 1. Now develop the determinant of A by the last column.

$$1 = \det\, A = \sum_{i=1}^{m} a_{in} \cdot \det\, A_{in}$$

Since the sum is 1, there must be an i such that $a_{in} = 1$ and $\det\, A_{in} = 1$. By induction, we find an integer lift B_{in} of A_{in} with determinant 1. Now let B the integer lift of A where one chooses the same lifts as in B_{in} of the entries that appear in A_{in}, one takes arbitrary integer lifts for the remaining entries, except a_{in}, and chooses b_{in} such that the matrix B has determinant 1 (i.e. $b_{in} = 1 - \sum_{j=1, j \neq i}^{m} b_{jn} \cdot \det\, B_{jn}$). **q.e.d.**

We apply this to our situation: Since the map $GL(m, \mathbb{Z}) \to GL(m, \mathbb{Z}_2)$ is surjective, we can realize every non-zero k-invariant by the fibration $Q_m \to \mathbb{R}\mathrm{P}^\infty$, with different inclusions of the fibre $(\mathbb{C}\mathrm{P}^\infty)^m$. If we forget about the inclusion of the fiber, there are just two fibrations $P_m \to \mathbb{R}\mathrm{P}^\infty$ and $Q_m \to \mathbb{R}\mathrm{P}^\infty$ that have the right properties.

Now consider the composition $\bar{g} : \mathbb{R}\mathrm{P}^2 \hookrightarrow W \to P_2(W)$. This is an isomorphism on π_1 and a split injection on π_2. We identify $\pi_2(P_m)$ and $\pi_2(Q_m)$ with $\pi_2((\mathbb{C}\mathrm{P}^\infty)^m) \cong \mathbb{Z}^m$ using the maps i.

There is a map $\mathbb{R}\mathrm{P}^2 \to P_m$ which is an isomorphism on π_1. In fact, compose the inclusion $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty$ with the section $\mathbb{R}\mathrm{P}^\infty \to P_m$. The generator of $\pi_2(\mathbb{R}\mathrm{P}^2)$ is mapped to $0 \in \mathbb{Z}^m \cong \pi_2(P_m)$, as one sees by considering universal coverings and the second homology group: it is a map $S^2 \to (\mathbb{C}\mathrm{P}^\infty)^m \times S^\infty$ such that the first m components factor through $\mathbb{R}\mathrm{P}^2$.

Obstruction theory tells us that for two lifts of $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^\infty$ to P_m, there is only one obstruction for a homotopy between them, and given by an element in $H^2(\mathbb{R}\mathrm{P}^2; \pi_2((\mathbb{C}\mathrm{P}^\infty)^m)_-)$. (This first obstruction is always realized.) The difference between the two maps is, up to homotopy, a difference of the maps restricted to the 2-cell of $\mathbb{R}\mathrm{P}^2$, namely the difference is some element $\phi \in C^2(\mathbb{R}\mathrm{P}^2; \pi_2((\mathbb{C}\mathrm{P}^\infty)^m)_-) \cong C^2_{\mathbb{Z}_2}(S^2; \pi_2((\mathbb{C}\mathrm{P}^\infty)^m))$, which means the equivariant cellular cochains, where S^2 has the cell decomposition with two cells in each dimension, and the involutions on S^2 and on $\pi_2((\mathbb{C}\mathrm{P}^\infty)^m)$ are multiplication with -1. Now the change in the map $\pi_2(S^2) \cong \pi_2(\mathbb{R}\mathrm{P}^2) \to \pi_2((\mathbb{C}\mathrm{P}^\infty)^m)$ is by $2x$, where x is the value of ϕ on one of the 2-cells of S^2. But this implies that every map from $\mathbb{R}\mathrm{P}^2$ to P_m inducing an isomorphism on π_1 maps the generator of $\pi_2(\mathbb{R}\mathrm{P}^2)$ to a class divisible by 2, so this map cannot be split injective.

Hence there does not exist a map $\mathbb{R}\mathrm{P}^2 \to P_m$ which satisfies all conditions. This implies that we must have $P_2(W) = Q_m$.

There is a map $f : \mathbb{R}\mathrm{P}^2 \to Q_m$ which is an isomorphism on π_1. Identify S^2 with $\mathbb{C}\mathrm{P}^1$, and map $[y] \in \mathbb{R}\mathrm{P}^2$ (where $y \in S^2 = \mathbb{C}\mathrm{P}^1$) to the class of $(y, \ldots, y, y)$. This works because under the identification $S^2 = \mathbb{C}\mathrm{P}^1$, multiplication with -1 is identified with τ.

The generator of $\pi_2(\mathbb{R}\mathrm{P}^2)$ is mapped to $(1, \ldots, 1) \in \mathbb{Z}^m \cong \pi_2(Q_m)$, as one sees by considering universal coverings and the second homology group. By obstruction theory (as above), homotopy classes of maps $\mathbb{R}\mathrm{P}^2 \to Q_m$ which are an isomorphism on π_1 correspond bijectively to elements of $\pi_2(Q_m)$ of the form $a = (1, \ldots, 1) + 2b$, by taking the image of the generator of $\pi_2(\mathbb{R}\mathrm{P}^2)$.

So there are several maps $\mathbb{R}\mathrm{P}^2 \to Q_m$ satisfying all conditions. But, given such a map $f' : \mathbb{R}\mathrm{P}^2 \to Q_m$ corresponding to a vector $a = (1, \ldots, 1) + 2b$ that is part of a basis $(a, v_2, \ldots v_m)$ of $\mathbb{Z}^m$, we find an automorphism A of $\mathbb{Z}^m$ sending a to $(1, \ldots, 1)$.

Using what we have said above, we can find a fiber homotopy self-equivalence $h : Q_m \to Q_m$ such that hf' is homotopic to f. Hence we may assume that the map $\mathbb{R}\mathrm{P}^2 \to P_2(W)$ is f.

4.3 The homology of Q_m

For Q_m, one has a homotopy equivalence $Q_1 \simeq \mathbb{C}\mathrm{P}^\infty/\tau$ and the bundles $\mathbb{R}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^\infty/\tau \to \mathbb{H}\mathrm{P}^\infty$ and $(\mathbb{C}\mathrm{P}^\infty)^{m-1} \to Q_m \to Q_1$, where the last map is induced by projection onto the first and last factor. In both cases the action of the fundamental group of the base space on the $\mathbb{Z}_2$-cohomology of the fiber is trivial. The Serre spectral sequences with $\mathbb{Z}_2$-coefficients collapse due to multiplicativity and degree reasons, and one may apply the Leray-Hirsch theorem, that is

$$H^*(Q_m;\mathbb{Z}_2) = \mathbb{Z}_2[q, x_1, \dots x_{m-1}, t]/t^3$$

where $\deg(q) = 4$, $\deg(x_i) = 2$ and $\deg(t) = 1$. (We do even get the full multiplicative structure.)

Let us now consider homology with integer and with $\mathbb{Z}_-$-coefficients. Q_m has the double cover $(\mathbb{C}\mathrm{P}^\infty)^m \times S^\infty$, and standard transfer arguments (i.e. we use the long exact sequences involving the transfer maps of integer and $\mathbb{Z}_-$-homology of Q_m to the integer homology of the double cover coming from the short exact sequences $\mathbb{Z}_+ \to \mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}_-$ and $\mathbb{Z}_- \to \mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}_+$, as well as the fact that composition of transfer and projection is multiplication with 2, while composition of projection and transfer is equal to $1 \pm \tau_*$, and the fact that the integer homology of the double cover is free over $\mathbb{Z}$) show that all torsion elements in $H_*(Q_m;\mathbb{Z})$ and in $H_*(Q_m;\mathbb{Z}_-)$ have order 2, and that:

$$\begin{aligned}
H_*(Q_m;\mathbb{Z})^{free} &\cong \begin{cases} H_*((\mathbb{C}\mathrm{P}^\infty)^m;\mathbb{Z}) & \text{if } * \equiv 0 \mod 4 \\ 0 & \text{else} \end{cases} \\
H_*(Q_m;\mathbb{Z}_-)^{free} &\cong \begin{cases} H_*((\mathbb{C}\mathrm{P}^\infty)^m;\mathbb{Z}) & \text{if } * \equiv 2 \mod 4 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Together with the universal coefficient theorem, and using the $\mathbb{Z}_2$-calculations, this suffices to compute the integer and $\mathbb{Z}_-$-homology of Q_m:

In low dimensions we get:

$*$	$H_*(Q_m;\mathbb{Z}_2)$	$H_*(Q_m;\mathbb{Z})$	$H_*(Q_m;\mathbb{Z}_-)$
0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
2	$(\mathbb{Z}_2)^m$	$(\mathbb{Z}_2)^{m-1}$	$\mathbb{Z}^m$
3	$(\mathbb{Z}_2)^{m-1}$	0	$(\mathbb{Z}_2)^{m-1}$
4	$(\mathbb{Z}_2)^{\binom{m+1}{2}}$	$\mathbb{Z}^{\binom{m+1}{2}}$	$(\mathbb{Z}_2)^{\binom{m}{2}+1}$
5	$(\mathbb{Z}_2)^{\binom{m}{2}+1}$	$(\mathbb{Z}_2)^{\binom{m}{2}+1}$	0
6	$(\mathbb{Z}_2)^{\binom{m+2}{3}}$	$(\mathbb{Z}_2)^{\binom{m+1}{3}+m-1}$	$\mathbb{Z}^{\binom{m+2}{3}}$
7	$(\mathbb{Z}_2)^{\binom{m+1}{3}+m-1}$	0	$(\mathbb{Z}_2)^{\binom{m+1}{3}+m-1}$

4.4 The map to Q_m

Now we are interested in an extension of f to $M \times \mathbb{R}\mathrm{P}^2$ such that the diagram below commutes. (If we had a nullbordism W of $M \times \mathbb{R}\mathrm{P}^2$ with 2-type Q_m, then $\partial W \to W \to P_2(W)$ would be such an extension.)

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^2 \arrow[r, "f"] \arrow[d, "i_2"] & Q_m \arrow[d] \\
M \times \mathbb{R}\mathrm{P}^2 \arrow[ur, "F"] \arrow[r, "i \circ pr_2"] & \mathbb{R}\mathrm{P}^\infty
\end{tikzcd}$$

And such an extension exists: just put $F = f \circ pr_2$.

But as we saw in Proposition 4.1.1, if we want to produce a conjugation manifold, we need an isomorphism $\bigoplus_{i \leq j} H_i(M;\mathbb{Z}_2) \otimes H_j(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) \to H_*(W;\mathbb{Z}_2)$. Since $W \to P_2(W) = Q_m$ is 3-connected, we need that F induces isomorphisms

$$\begin{aligned}
H_0(M;\mathbb{Z}_2) \otimes H_0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) &\to H_0(Q_m;\mathbb{Z}_2),\\
H_0(M;\mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) &\to H_1(Q_m;\mathbb{Z}_2),\\
H_0(M;\mathbb{Z}_2) \otimes H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) \oplus H_1(M;\mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) &\to H_2(Q_m;\mathbb{Z}_2).
\end{aligned}$$

(There are more conditions that we care about later.) Let us call an extension F with these properties a "good" extension. (The first two conditions are fulfilled by any extension F of f making the diagram commute.)

So we ask if there exists a good extension, and how many good extensions exist. The answer is given again by obstruction theory: without the additional properties, the extensions are classified up to homotopy by $H^2(M \times \mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2; (\pi_2(\mathbb{C}\mathrm{P}^\infty)^m)_-)$.

Lemma 4.4.1

$$H^2(M \times \mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2; \mathbb{Z}_-) \cong H^1(M;\mathbb{Z}_2) \otimes H^1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2)$$

and one finds cocycle representatives supported on cells "generating"

$$H_1(M;\mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2).$$

Proof: We calculate this group "by hand" using cellular decompositions. $C^2(M \times \mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2; \mathbb{Z}_-)$ is isomorphic to the group of equivariant cochains $C^2_{\mathbb{Z}_2}(M \times S^2, S^2)$, where the $\mathbb{Z}_2$-action on $\mathbb{Z}$ is multiplication with -1.

Now let us describe an equivariant cell decomposition of $M \times S^2$: we construct it as a product of some cell decomposition of M with the standard $\mathbb{Z}_2 = \{1, T\}$-equivariant cell decomposition of S^2. Let the two 0-, 1- and 2-cells of this cell decomposition of S^2 be denoted by a, Ta, b, Tb, and c, Tc respectively. (One has $\partial a = 0$, $\partial b = Ta - a$, $\partial c = Tb - b$.) We may find a cell decomposition of M with exactly one 0-cell d, 1-cells e_i, 2-cells f_i and 3-cells g_i.

Then $M \times S^2$ has 1-cells $d \times b$, $d \times Tb$, $e_i \times a$ and $e_i \times Ta$, 2-cells $d \times c$, $d \times Tc$, $e_i \times b$, $e_i \times Tb$, $f_i \times a$ and $f_i \times Ta$, and 3-cells $e_i \times c$, $e_i \times Tc$, $f_i \times b$, $f_i \times Tb$, $g_i \times a$ and $g_i \times Ta$.

$C^2_{\mathbb{Z}_2}(M \times S^2, S^2)$ consists of the functions that are equivariant and zero on $d \times c$, hence an element is determined by its value on the $e_i \times b$ and $f_i \times a$. Similar statements about $C^j_{\mathbb{Z}_2}$ hold for $j = 1$ and $j = 3$.

Let ϕ_2 be an equivariant 2-cocyle. Then by considering its coboundary one finds that it can have any prescribed value on the cells $e_i \times b$ such that for all i the value $\phi_2(\partial f_i \times b)$ is even, since this value must equal $2\phi_2(f_i \times a)$. So the values on $e_i \times b$ describe ϕ_2 uniquely. The coboundary of an equivariant 1-cochain ϕ_1 can be any prescribed function on the chains $e_i \times b$ with even values since $\delta\phi_1(e_i \times b) = 2\phi_1(e_i \times a)$.

We obtain that the group of equivariant 2-cocycles modulo equivariant coboundaries is isomorphic to the group of all integer-valued functions ϕ_2 on the cells $e_i \times b$ such that for all i the value $\phi_2(\partial f_i \times b)$ is even, modulo all such functions with even values.

So only the value of such a function modulo 2 is relevant, and the group is isomorphic to $H^1(M; \mathbb{Z}_2) \otimes H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$. Furthermore, one finds cocycle representatives corresponding to cocycle representatives of $H^1(M; \mathbb{Z}_2) \otimes H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$, hence supported on cells "generating" $H_1(M; \mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$. **q.e.d.**

It follows that

$$H^2(M \times \mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2; (\pi_2(\mathbb{C}\mathrm{P}^\infty)^m)_-)) \cong H^1(M; \mathbb{Z}_2) \otimes H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \otimes \pi_2(\mathbb{C}\mathrm{P}^\infty)^m$$

and that these different possibilities for the map F correspond to changes of the map on the 2-cells that generate $H_1(M; \mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$ by elements in $\pi_2((\mathbb{C}\mathrm{P}^\infty)^m) \otimes \mathbb{Z}_2 \cong H_2((\mathbb{C}\mathrm{P}^\infty)^m; \mathbb{Z}_2)$. Hence given any extension F of f, we may already change the map $H_1(M; \mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \to H_2(Q_m; \mathbb{Z}_2)$ by elements from the image of $H_2((\mathbb{C}\mathrm{P}^\infty)^m; \mathbb{Z}_2) \to H_2(Q_m; \mathbb{Z}_2)$, using a different extension F. We know that $H_2(Q_m; \mathbb{Z}_2)$ has rank m and generators

$t^2 := f_*([\mathbb{R}\mathrm{P}^2])$ and x_j, where $j = 1, \dots, m-1$, and x_j is in the image of $H_2((\mathbb{C}\mathrm{P}^\infty)^m; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^m \to H_2(Q_m; \mathbb{Z}_2)$. Under the map $Q_m \to \mathbb{R}\mathrm{P}^\infty$, $t^2 \mapsto t^2$, $x_j \mapsto 0$.

As a corollary we get that there are good extensions F: given a basis $e_1, \dots e_{m-1}$ of $H_1(M; \mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$, we can modify the extension $f \circ pr_2$ to an extension F such that $F_*(e_j) = x_j$, hence a good extension.

Now let F' be any other good extension. Then $F'_*(e_i)$ is a linear combination of the x_j, We find an integer matrix A with determinant ± 1 such that $A \cdot (1, \dots, 1) = (1, \dots, 1)$, and such that a fibre homotopy equivalence corresponding to A maps $F'_*(e_i)$ to x_i.

There is enough freedom in the choice of A to include also the changes on the cells generating $H_1(M; \mathbb{Z}_2) \otimes H_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \to H_2(Q_m; \mathbb{Z}_2)$ by elements from the kernel of of $\pi_2((\mathbb{C}\mathrm{P}^\infty)^m) \otimes \mathbb{Z}_2 \cong H_2((\mathbb{C}\mathrm{P}^\infty)^m; \mathbb{Z}_2) \to H_2(Q_m; \mathbb{Z}_2)$. The kernel is generated by $(1, \dots, 1) \in H_2((\mathbb{C}\mathrm{P}^\infty)^m; \mathbb{Z}_2)$ since this can be represented by a map from S^2 to $(\mathbb{C}\mathrm{P}^\infty)^m$ such that the composition with the covering map $(\mathbb{C}\mathrm{P}^\infty)^m \times S^\infty \to Q_m$ factors over $\mathbb{R}\mathrm{P}^2$, in fact the map will be the double cover of f. We change A such that it adds a given multiple of $(1, \dots, 1)$ to the image of an integer lift of $F'_*(e_i)$.

This implies that there is a fibre homotopy self equivalence $h : Q_m \to Q_m$ such that hf is homotopic to f and such that hF' is homotopic to F.

It follows (using the fact that the map $W \to P_2(W)$ is unique only modulo fiber homotopy self-equivalences of $P_2(W)$) that we have a unique map (modulo fiber homotopy self-equivalences) $F : M \times \mathbb{R}\mathrm{P}^2 \to Q_m$ that can be the map $\partial W \to W \to P_2(W)$.

Thus $[F : M \times \mathbb{R}\mathrm{P}^2 \to Q_m]$ should be the zero element in the bordism group $\Omega_5^{Spin}(Q_m; L)$, where L is the pullback of the canonical line bundle over $\mathbb{R}\mathrm{P}^\infty$. This is just another way of saying that we should now check whether:

$$[F \times (\nu - F^*L) : M \times \mathbb{R}\mathrm{P}^2 \to Q_m \times BSpin = B^2(W)] = 0 \in \Omega_5^{Q_m \times BSpin} \ ?$$

4.5 Computation of $\Omega_5^{Spin}(Q_1;L)$ and $\Omega_6^{Spin}(Q_1;L)$

We use two spectral sequences to compute $\Omega_5^{Spin}(Q_1;L)$ and $\Omega_6^{Spin}(Q_1;L)$: the Atiyah-Hirzebruch spectral sequence from section 1:

$$H_p\left(Q_1;\underline{\Omega_q^{Spin}}\right) \;\Rightarrow\; \Omega_{p+q}^{Spin}(Q_1;L)$$

and the Adams spectral sequence (we consider the prime 2 only, which will be justified later):

$$\begin{aligned} Ext_A^{s,t}(H^*(TL\wedge MSpin;\mathbb{Z}_2),\mathbb{Z}_2) \;\Rightarrow\;& \pi_{t-s}(TL\wedge MSpin)/\text{non-2-torsion} \\ &\cong \Omega_{t-s-1}^{Spin}(Q_1;L)/\text{non-2-torsion} \end{aligned}$$

where A is the mod 2 Steenrod algebra.

One can compute the relevant part of the E^2-term of the Atiyah-Hirzebruch spectral sequence directly from the (co)-homology of Q_1 (see 4.3) and the knowledge of the Spin-bordism groups.

The differential of the Atiyah-Hirzebruch spectral sequence $d^2: E^2_{p,1} \to E^2_{p-2,2}$ is the dual of $Sq^2 + w_1(L)Sq^1$, and the differential $d^2: E^2_{p,0} \to E^2_{p-2,1}$ is reduction mod 2 composed with the dual of $Sq^2 + w_1(L)Sq^1$, see for example [Tei93].

We can compute the action of the Steenrod squares on the generators of $H^*(Q_1;\mathbb{Z}_2)$: $Sq\, q = q + q^2$ by comparing with $\mathbb{HP}^\infty$, $Sq\, t = t + t^2$. So we see that the d^2-differential is zero in the lower diagonals.

So we have the following E^2-term, together with possible higher differentials:

q								
7		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
6			$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
5				$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
4	$\mathbb{Z}_2$		$\mathbb{Z}$		$\cdots$	$\cdots$	$\cdots$	$\cdots$
3						$\cdots$	$\cdots$	$\cdots$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$	$\cdots$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
0	$\mathbb{Z}_2$		$\mathbb{Z}$		$\mathbb{Z}_2$		$\mathbb{Z}$	
	0	1	2	3	4	5	6	7 p

The arrows denote the only possibly non-trivial differentials in the range $p + q \leq 5$. From the Atiyah-Hirzebruch spectral sequence we see that the only torsion appears at the prime 2. So it suffices to consider the Adams spectral sequence for the prime 2.

For the Adams spectral sequence, we first need to calculate $H^*(TL \wedge MSpin; \mathbb{Z}_2)$. Since we are interested only in the first nine columns, it suffices to compute $H^*(TL \wedge MSpin; \mathbb{Z}_2)$ for $* \leq 8$.

Let us start by computing the low degree part of $\tilde{H}^*(TL; \mathbb{Z}_2)$: Here we use the Thom isomorphism $\phi : H^*(Q_1; \mathbb{Z}_2) \cong \tilde{H}^{*+1}(TL; \mathbb{Z}_2)$ which is cup product with the Thom class $u_1 \in H^1(Q_1; \mathbb{Z}_2)$: Recall that $H^*(Q_1; \mathbb{Z}_2) \cong \mathbb{Z}_2[q,t]/t^3$. Since $u_1^2 = Sq^1(u_1) = w_1(L) \cup u_1 = t \cup u_1$ (this is one definition of the Stiefel-Whitney classes!), we get the following generators for $\tilde{H}^*(TL; \mathbb{Z}_2)$, $* \leq 8$:

$$u_1 = \phi(1), u_1^2 = \phi(t), u_1^3 = \phi(t^2), u_5 := \phi(q), u_1 u_5 = \phi(tq), u_1^2 u_5 = \phi(t^2 q)$$

We have

$$\begin{aligned} Sq(u_1) &= u_1 + u_1^2, & Sq(u_1^2) &= u_1^2, \\ Sq(u_1^3) &= u_1^3, & Sq(u_5) &= u_5 + u_1 u_5 + \ldots, \\ Sq(u_1 u_5) &= u_1 u_5 + \ldots, & Sq(u_1^2 u_5) &= u_1^2 u_5 + \ldots, \end{aligned}$$

where $\ldots$ denotes elements of degree > 8.

$\tilde{H}^*(MSpin; \mathbb{Z}_2)$ has the following generators for $* \leq 7$: v_0, the Thom class in degree 0, and v_4, v_6, v_7, the images of the corresponding Stiefel-Whitney classes under the Thom isomorphism. (The notation is chosen such that the index of an element always corresponds to its degree.) One has modulo elements of degree > 7:

$$\begin{array}{cc} Sq(v_0) = v_0 + v_4 + v_6 + v_7 + \dots, & Sq(v_4) = v_4 + v_6 + v_7 + \dots, \\ Sq(v_6) = v_6 + v_7 + \dots, & Sq(v_7) = v_7 + \dots. \end{array}$$

By the Künneth theorem, we get the following generators for $H^*(TL \wedge MSpin; \mathbb{Z}_2)$, $* \leq 8$:

$$\begin{gathered} a_1 := u_1 v_0, \quad Sq^1 a_1 = u_1^2 v_0, \quad a_3 := u_1^3 v_0, \quad a_5 := u_5 v_0, \\ Sq^4 a_1 = u_1 v_4, \quad Sq^5 a_1 = u_1^2 v_4, \quad Sq^1 a_5 = u_5 u_1 v_0, \\ a_7 := u_5 u_1^2 v_0, \quad Sq^4 a_3 = u_1^3 v_4, \quad Sq^6 a_1 = u_1 v_6, \\ Sq^{61} a_1 = u_1^2 v_6, \quad Sq^7 a_1 + Sq^{61} a_1 = u_1 v_7 \end{gathered}$$

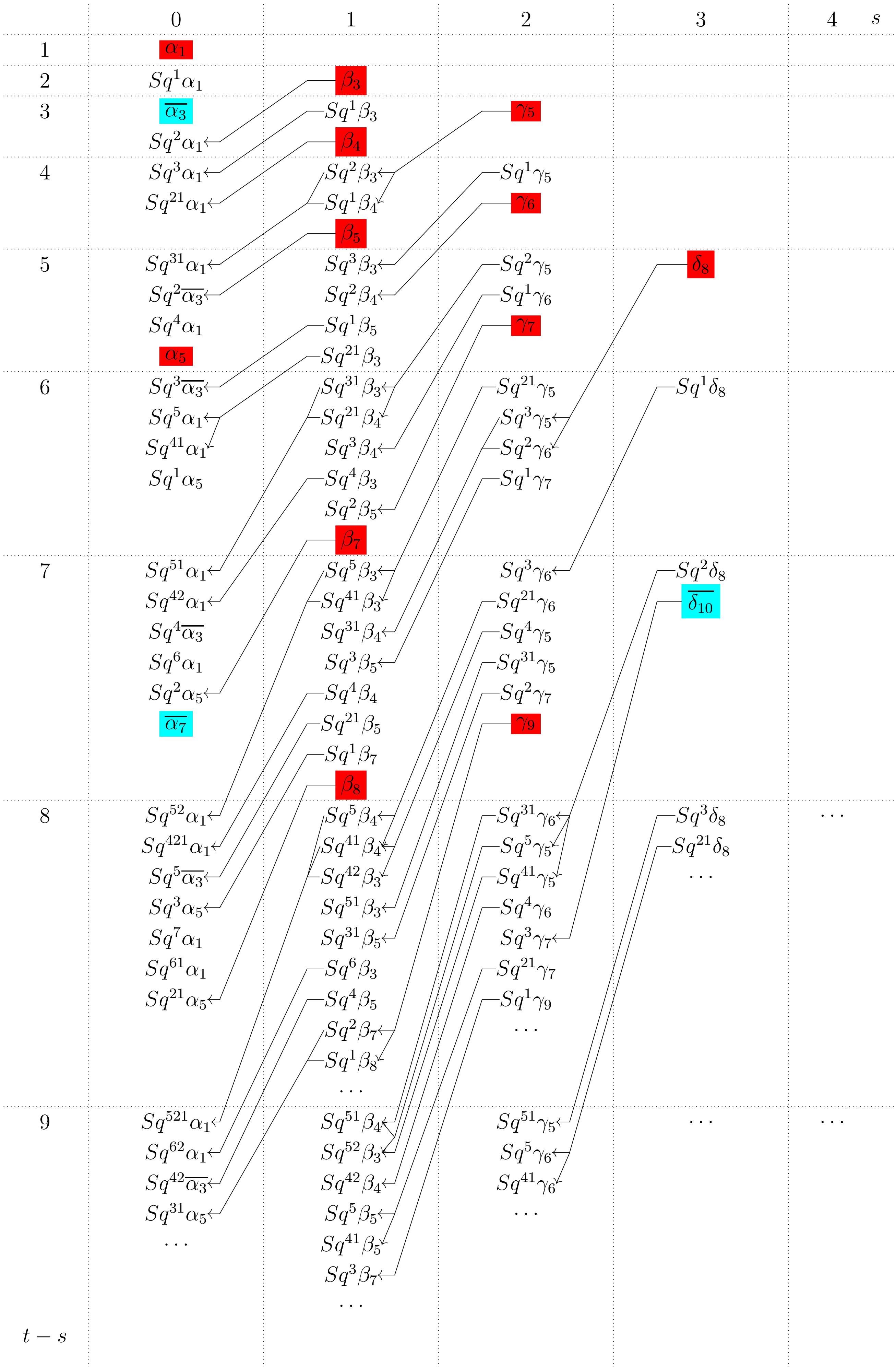
We can easily compute the Steenrod squares on these elements (modulo elements of degree > 8). As a module over the Steenrod algebra, we have in degrees ≤ 8 the generators a_1, a_3, a_5, a_7 and relations:

$$\begin{gathered} Sq^2 a_1 = Sq^3 a_1 = Sq^{21} a_1 = Sq^{31} a_1 = Sq^{51} a_1 = \\ = Sq^{42} a_1 = Sq^{52} a_1 = Sq^{421} a_1 = Sq^5 a_1 + Sq^{41} a_1 = \\ = Sq^1 a_3 = Sq^2 a_3 = Sq^3 a_3 = Sq^{21} a_3 = Sq^{31} a_3 = Sq^5 a_3 = \\ = Sq^{41} a_3 = Sq^2 a_5 = Sq^3 a_5 = Sq^{21} a_5 = Sq^1 a_7 = 0 \end{gathered}$$

By the procedure described in Hatcher [Hat04] and Stolz [Sto85], one can now compute the $Ext_A^{s,t}(H^*(TL \wedge MSpin; \mathbb{Z}_2), \mathbb{Z}_2)$ terms for $t - s \leq 7$ and $t - s = 8, s = 0$, together with parts of the multiplicative structure (or rather module structure over the Steenrod algebra).

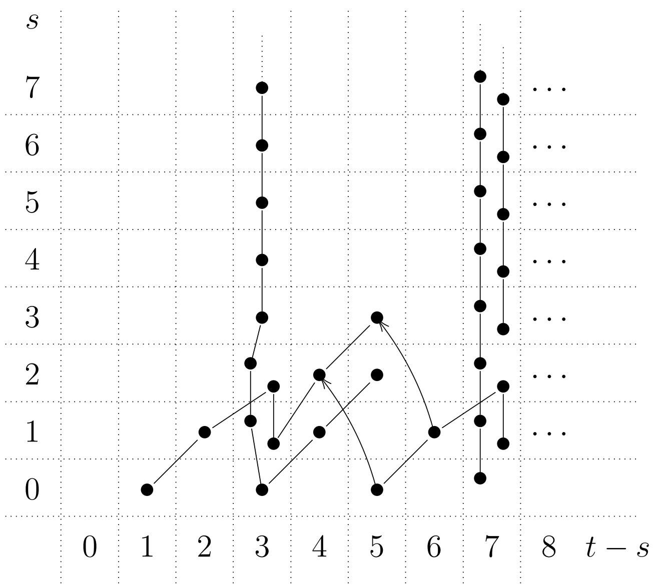
On the next page we display the diagram which describes the resolution used for the computation of the $Ext_A^{s,t}$-terms. It is the analogue of the diagram on page 24 of chapter 2 of [Hat04], but we use also the technique described by Stolz that recognizes "vertical columns", i.e. uses almost-free resolutions in order to get an algorithm terminating after finitely many steps.

We also checked our results using a computer program of Bruner's [Bru93], [Bru].



s
0
1
2
3
4
$t-s$
1
α_1
2
$Sq^1\alpha_1$
β_3
3
$\overline{\alpha_3}$
$Sq^2\alpha_1$
$Sq^1\beta_3$
β_4
γ_5
4
$Sq^3\alpha_1$
$Sq^{21}\alpha_1$
$Sq^2\beta_3$
$Sq^1\beta_4$
$Sq^1\gamma_5$
γ_6
5
$Sq^{31}\alpha_1$
$Sq^2\overline{\alpha_3}$
$Sq^4\alpha_1$
α_5
β_5
$Sq^3\beta_3$
$Sq^2\beta_4$
$Sq^1\beta_5$
$Sq^{21}\beta_3$
$Sq^2\gamma_5$
$Sq^1\gamma_6$
γ_7
δ_8
6
$Sq^3\overline{\alpha_3}$
$Sq^5\alpha_1$
$Sq^{41}\alpha_1$
$Sq^1\alpha_5$
$Sq^{31}\beta_3$
$Sq^{21}\beta_4$
$Sq^3\beta_4$
$Sq^4\beta_3$
$Sq^2\beta_5$
β_7
$Sq^{21}\gamma_5$
$Sq^3\gamma_5$
$Sq^2\gamma_6$
$Sq^1\gamma_7$
$Sq^1\delta_8$
7
$Sq^{51}\alpha_1$
$Sq^{42}\alpha_1$
$Sq^4\overline{\alpha_3}$
$Sq^6\alpha_1$
$Sq^2\alpha_5$
$\overline{\alpha_7}$
$Sq^5\beta_3$
$Sq^{41}\beta_3$
$Sq^{31}\beta_4$
$Sq^3\beta_5$
$Sq^4\beta_4$
$Sq^{21}\beta_5$
$Sq^1\beta_7$
β_8
$Sq^3\gamma_6$
$Sq^{21}\gamma_6$
$Sq^4\gamma_5$
$Sq^{31}\gamma_5$
$Sq^2\gamma_7$
γ_9
$Sq^2\delta_8$
$\overline{\delta_{10}}$
8
$Sq^{52}\alpha_1$
$Sq^{421}\alpha_1$
$Sq^5\overline{\alpha_3}$
$Sq^3\alpha_5$
$Sq^7\alpha_1$
$Sq^{61}\alpha_1$
$Sq^{21}\alpha_5$
$Sq^5\beta_4$
$Sq^{41}\beta_4$
$Sq^{42}\beta_3$
$Sq^{51}\beta_3$
$Sq^{31}\beta_5$
$Sq^6\beta_3$
$Sq^4\beta_5$
$Sq^2\beta_7$
$Sq^1\beta_8$
$\cdots$
$Sq^{31}\gamma_6$
$Sq^5\gamma_5$
$Sq^{41}\gamma_5$
$Sq^4\gamma_6$
$Sq^3\gamma_7$
$Sq^{21}\gamma_7$
$Sq^1\gamma_9$
$\cdots$
$Sq^3\delta_8$
$Sq^{21}\delta_8$
$\cdots$
$\cdots$
9
$Sq^{521}\alpha_1$
$Sq^{62}\alpha_1$
$Sq^{42}\overline{\alpha_3}$
$Sq^{31}\alpha_5$
$\cdots$
$Sq^{51}\beta_4$
$Sq^{52}\beta_3$
$Sq^{42}\beta_4$
$Sq^5\beta_5$
$Sq^{41}\beta_5$
$Sq^3\beta_7$
$\cdots$
$Sq^{51}\gamma_5$
$Sq^5\gamma_6$
$Sq^{41}\gamma_6$
$\cdots$
$\cdots$
$\cdots$

Hence the E^2-term of the Adams spectral sequence, together with the possible differentials, is in low degrees described by the following diagram:



Again, the arrows denote the only possibly non-trivial differentials with image in the range $t - s \leq 7$.

Now we compare both spectral sequences: In the Adams spectral sequence picture one sees that either none or both of the differentials are trivial. Therefore, the same holds in the Atiyah-Hirzebruch spectral sequence. The lower differential is non-trivial if and only if the edge homomorphism

$$\begin{array}{rcl} \Omega_4^{Spin}(Q_1; L) & \to & H_4(Q_1; \mathbb{Z}_-) \cong H_4(Q_1; \mathbb{Z}_2) \cong \mathbb{Z}_2 \\ [f : M \to Q_1] & \mapsto & f_*([M]_{\mathbb{Z}_2}) \end{array}$$

is trivial.

Now suppose there is an element $[f : M \to Q_1]$ such that $f_*([M]_{\mathbb{Z}_2}) \neq 0$. We may again use surgery below the middle dimension, and suppose that f is an isomorphism on π_1. Hence we may suppose $\pi_1(M) \cong \mathbb{Z}_2$, and M non-orientable, since the map $M \to Q_1 \to \mathbb{R}\mathrm{P}^\infty$ is an isomorphism on π_1 and describes the first Stiefel-Whitney class.

Now we use a result from [HKT94]: There are only very few possibilities for M, we have necessarily:

$$M = \mathbb{R}\mathrm{P}^2 \times S^2 \# k \cdot S^2 \times S^2$$

with $k \geq 0$. (One of the reasons for this is that the universal cover must be a 1-connected differentiable 4-manifold with zero signature, since it admits an orientation-reversing diffeomorphism.) Now since $\pi_3(Q_1) = 0$, we may apply surgery to replace the map f' from the connected sum by a bordant map from the disjoint sum,

$$f : \mathbb{R}\mathrm{P}^2 \times S^2 + k \cdot S^2 \times S^2 \to Q_1$$

such that:

$$f_*([\mathbb{R}\mathrm{P}^2]_{\mathbb{Z}_2}) + \sum_{i=1}^{k} f_*([(S^2 \times S^2)_i]_{\mathbb{Z}_2}) = f'_*([M]_{\mathbb{Z}_2}) \neq 0$$

But every map from $S^2 \times S^2 \to Q_1$ lifts to $\mathbb{C}\mathrm{P}^\infty \times S^\infty$, hence it is zero on H_4. (The map is zero on H^4 since the square of the generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}_2)$ is mapped to a square in $H^4(S^2 \times S^2; \mathbb{Z}_2)$.)

This means we may suppose $M = \mathbb{R}\mathrm{P}^2 \times S^2$. But every such map $f : \mathbb{R}\mathrm{P}^2 \times S^2 \to Q_1$ is a lift of the map $i \circ pr_1 : \mathbb{R}\mathrm{P}^2 \times S^2 \to \mathbb{R}\mathrm{P}^\infty$ and as such homotopic to the map $\mathbb{R}\mathrm{P}^2 \times S^2 \to \mathbb{R}\mathrm{P}^2 \to Q_1$, where the first map is projection onto the first factor and the second map is $x \mapsto f(x, (1,0,0))$. The reason for this is that the obstructions for a homotopy between these two lifts lie in $H^k(\mathbb{R}\mathrm{P}^2 \times S^2, \mathbb{R}\mathrm{P}^2; \underline{\pi_k(\mathbb{C}\mathrm{P}^\infty)})$, which is zero for all k. (The proof for $k = 2$ is Lemma 4.4.1, with minimal changes. All other obstruction groups are zero.) Since f factors over $\mathbb{R}\mathrm{P}^2$, it is trivial on the fourth homology.

So the edge homomorphism is zero, and both of the differentials in the spectral sequence are non-trivial, and we get the result:

$$\Omega_5^{Spin}(Q_1; L) = 0$$

For $\Omega_6^{Spin}(Q_1; L)$ we see that there cannot be any differentials in the first columns of the Adams spectral sequence, since $Ext^{8,0} = 0$, thus any differential from the eighth column to the seventh would have to kill a $\mathbb{Z}$ summand, which is impossible considered the E_2-term of the Atiyah-Hirzebruch spectral sequence. Hence:

$$\Omega_6^{Spin}(Q_1; L) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_4$$

As a result we also get that there is no non-zero differential ending or starting on the sixth diagonal of the Atiyah-Hirzebruch spectral sequence.

4.6 Computation of $\Omega_5^{Spin}(Q_m; L)$ and $\Omega_6^{Spin}(Q_m; L)$

We apply the Atiyah-Hirzebruch spectral sequence.

We start with the action of the Steenrod squares on the generators of $H^*(Q_m; \mathbb{Z}_2)$: $Sq\ q = q + q^2$ by comparing with $\mathbb{HP}^\infty$, $Sq\ t = t + t^2$, more difficult is the Sq^1-term in $Sq\ x_i = x_i + tx_i + x_i^2$. We know that Sq^1 is the Bockstein map - and from the corresponding long exact sequence and our computations of the integer and $\mathbb{Z}_2$-homology we see that Sq^1 maps the linear space spanned by the x_i isomorphically to $H^3(Q_m; \mathbb{Z}_2)$. Finally one has to compare (using the right projection) with Q_2, where we necessarily have $Sq^1\ x_1 = tx_1$.

The differential of the Atiyah-Hirzebruch spectral sequences $d^2 : E^2_{p,1} \to E^2_{p-2,2}$ is the dual of $Sq^2 + w_1(L)Sq^1$, and the differential $d^2 : E^2_{p,0} \to E^2_{p-2,1}$ is reduction mod 2 composed with the dual of $Sq^2 + w_1(L)Sq^1$, see for example [Tei93]. We compute $Sq^2 + tSq^1$ on $H^*(Q_m; \mathbb{Z}_2)$ in the relevant dimensions:
$t^2 \mapsto 0$, $x_i \mapsto x_i^2 + t^2 x_i$, $tx_i \mapsto tx_i^2$, $t^2 x_i \mapsto t^2 x_i^2$, $x_i x_j \mapsto x_i^2 x_j + x_i x_j^2 + t^2 x_i x_j$, $q \mapsto 0$, $tx_i x_j \mapsto tx_i^2 x_j + tx_i x_j^2$, $tq \mapsto 0$.

We get that
$d^2 : E^2_{4,1} \cong (\mathbb{Z}_2)^{\binom{m+1}{2}} \to E^2_{2,2} \cong (\mathbb{Z}_2)^m$ has rank $m-1$,
$d^2 : E^2_{5,1} \cong (\mathbb{Z}_2)^{\binom{m}{2}+1} \to E^2_{3,2} \cong (\mathbb{Z}_2)^{m-1}$ has rank $m-1$,
$d^2 : E^2_{6,1} \cong (\mathbb{Z}_2)^{\binom{m+2}{3}} \to E^2_{4,2} \cong (\mathbb{Z}_2)^{\binom{m+1}{2}}$ has rank $\binom{m}{2}$,
$d^2 : E^2_{6,0} \cong \mathbb{Z}^{\binom{m+2}{3}} \to E^2_{4,1} \cong (\mathbb{Z}_2)^{\binom{m+1}{2}}$ has rank $\binom{m}{2}$,
$d^2 : E^2_{7,0} \cong (\mathbb{Z}_2)^{\binom{m+1}{3}+m-1} \to E^2_{5,1} \cong (\mathbb{Z}_2)^{\binom{m}{2}+1}$ has rank $\binom{m-1}{2}$.

What remains after the d^2 - differential on the fifth diagonal is a single $\mathbb{Z}_2$ in $E^3_{4,1}$. This $\mathbb{Z}_2$ will kill the $\mathbb{Z}_2$ in $E^3_{0,4}$ by a d^4-differential by naturality of the spectral sequence, since this happens in the spectral sequence for Q_1, and there is a lift of the projection $Q_m \to Q_1$ to the first and the last factor: map the class of (x, y), where $x \in \mathbb{CP}^\infty$, $y \in S^\infty$ to the class of $(x, x, \ldots, x, y)$. So we have shown that $\Omega_5^{Spin}(Q_m; L) = 0$.

q								
7		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
6			$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
5				$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
4	$\mathbb{Z}_2$		$\mathbb{Z}^m$	$\mathbb{Z}_2^{m-1}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
3						$\cdots$	$\cdots$	$\cdots$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^m$	$\mathbb{Z}_2^{m-1}$	$\mathbb{Z}_2^{\binom{m+1}{2}}$	$\mathbb{Z}_2^{\binom{m}{2}+1}$	$\cdots$	$\cdots$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^m$	$\mathbb{Z}_2^{m-1}$	$\mathbb{Z}_2^{\binom{m+1}{2}}$	$\mathbb{Z}_2^{\binom{m}{2}+1}$	$\mathbb{Z}_2^{\binom{m+2}{3}}$	$\cdots$
0	$\mathbb{Z}_2$		$\mathbb{Z}^m$	$\mathbb{Z}_2^{m-1}$	$\mathbb{Z}_2^{\binom{m}{2}+1}$		$\mathbb{Z}^{\binom{m+2}{3}}$	$\mathbb{Z}_2^{\binom{m+1}{3}+m-1}$
	0	1	2	3	4	5	6	7 p

For $\Omega_6^{Spin}(Q_m; L)$, we use a similar Adams spectral sequence computation as for Q_1. We get that $E^{7+*,*}$ has one part that "would lead to a $\mathbb{Z}_4$ torsion" (as Q_1), and one part that "would lead to $\mathbb{Z}^{\binom{m+2}{3}+m}$". We see by naturality of the Adams spectral sequence that there may be no differential ending up in the "torsion part" of the seventh column, since for Q_1, $E_{0,8} = 0$. Again the comparison with the Atiyah-Hirzebruch spectral sequence implies that there may be no differential ending up in the "free part" of the seventh column.

So we get that $\Omega_6^{Spin}(Q_m; L) \cong \mathbb{Z}^{\binom{m+2}{3}+m} \oplus \mathbb{Z}_4$, where the free part has the same rank as the free group $\Omega_6^{Spin}(\tilde{Q}_{m,n} \cong (\mathbb{CP}^\infty)^{m+n})$.

4.7 The final step: Surgery

Now we know that $[F : M \times \mathbb{R}\mathrm{P}^2 \to Q_m] = 0 \in \Omega_5^{Spin}(Q_m; L) = 0$. Then one obtains a map $G : W \to Q_m$ with W a 6-manifold with $\nu(W) = G^*(L) \oplus \eta$, where η is some spin bundle, $\partial W = M \times \mathbb{R}\mathrm{P}^2$, and $G|_{\partial W} = F$. The diagram corresponding to our situation is:

$$\begin{tikzcd} & & Q_m \times BSpin \arrow[d, "\oplus\circ(\pi\times Bp)"] \\ \partial W = M \times \mathbb{R}\mathrm{P}^2 \arrow[r] & W \arrow[ur, "\overline{\nu}"] \arrow[r, "\nu"] & BO \end{tikzcd}$$

By surgery below the middle dimension, we may assume that $\overline{\nu}$ is a 3-equivalence, hence $\pi_1(W) \cong \mathbb{Z}_2$ and $\pi_2(W) \cong \mathbb{Z}^m$. Then we have assured that we really have $Q_m = P_2(W)$ and that we obtain the isomorphism of 4.1.1 below the middle dimension, i.e. for dimension ≤ 2.

Since $W \to Q_m$ is an isomorphism on π_2, and $\mathbb{R}\mathrm{P}^2 \to Q$ is split injective on π_2, $\mathbb{R}\mathrm{P}^2 \hookrightarrow W$ is split injective on π_2, and the same holds for $S^2 \hookrightarrow V$. Hence $H_2(S^2; \mathbb{Z}) \to H_2(V; \mathbb{Z})$ is split injective, and the map in the Mayer-Vietoris sequence starting in $H_3(X; \mathbb{Z})$ is the zero map.

Now we consider the exact sequence of the pair $(\partial W, W)$ in $\mathbb{Z}_2$-homology and cohomology. We have two isomorphisms to compare both sequences, given by the universal coefficient theorem and by Poincaré duality. Poincaré duality maps $H_i(M; \mathbb{Z}_2) \otimes H_j(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$ to $H^{3-i}(M; \mathbb{Z}_2) \otimes H^{2-j}(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$, the universal coefficient theorem maps $H_i(M; \mathbb{Z}_2) \otimes H_j(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$ to $(H^i(M; \mathbb{Z}_2) \otimes H^j(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2))^*$, and this is the usual translation from homology to cohomology that was already used in Theorem 4.1.1.

Using Poincaré duality, we see that isomorphisms

$$\bigoplus_{i+j=k, i\leq j} H_i(M; \mathbb{Z}_2) \otimes H_j(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \to H_k(W; \mathbb{Z}_2)$$

induce isomorphisms:

$$\bigoplus_{i+j=k, i\leq j} H^{3-i}(M; \mathbb{Z}_2) \otimes H^{2-j}(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \to H^{6-k}(W, \partial W; \mathbb{Z}_2)$$

$$H^{6-k}(W; \mathbb{Z}_2) \to H^{6-k}(M \times \mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \,/ \bigoplus_{i+j=k-1, i\leq j} H^{3-i}(M; \mathbb{Z}_2) \otimes H^{2-j}(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$$

Now we add the universal coefficient theorem and see that the isomorphisms

$$\begin{aligned}
H_0(M;\mathbb{Z}_2)\otimes H_0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) &\to H_0(W;\mathbb{Z}_2),\\
H_0(M;\mathbb{Z}_2)\otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) &\to H_1(W;\mathbb{Z}_2),\\
H_0(M;\mathbb{Z}_2)\otimes H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2)\oplus H_1(M;\mathbb{Z}_2)\otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) &\to H_2(W;\mathbb{Z}_2)
\end{aligned}$$

imply that there are isomorphisms

$$\begin{aligned}
0 &\to H_5(W;\mathbb{Z}_2),\\
H_2(M;\mathbb{Z}_2)\otimes H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) &\to H_4(W;\mathbb{Z}_2),
\end{aligned}$$

and

$$H_1(M;\mathbb{Z}_2)\otimes H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2)\to \mathrm{Im}(H_3(\partial W;\mathbb{Z}_2)\to H_3(W;\mathbb{Z}_2)).$$

The only problem that remains is that $H_3(W;\mathbb{Z}_2)$ can be too large: we have $H_3(W;\mathbb{Z}_2)\cong \mathrm{Im}(H_3(\partial W;\mathbb{Z}_2)\to H_3(W;\mathbb{Z}_2))\oplus C$ where C is the cokernel, which is also isomorphic to $H_3(W;\mathbb{Z}_2)/\mathrm{rad}$. Here we have divided out the radical of the $\mathbb{Z}_2$ intersection form. The $\mathbb{Z}_2$ intersection form on C is non-degenerated. Consider the following diagram:

$$\begin{tikzcd}
 & & H_1(M;\mathbb{Z})\otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \arrow[r] \arrow[d] & H_2(\partial W;\mathbb{Z}_2) \arrow[d] \\
 & H_3(\partial W;\mathbb{Z}_2) \arrow[r, "\tilde\beta"] \arrow[d] & H_2(\partial W;\mathbb{Z}) \arrow[ur, "red"] \arrow[d] & H_2(W;\mathbb{Z}_2) \\
H_3(W;\mathbb{Z}) \arrow[r, "red"] & H_3(W;\mathbb{Z}_2) \arrow[r, "\tilde\beta"] \arrow[d] & H_2(W;\mathbb{Z}) \arrow[ur, "red"] & \\
 & C & &
\end{tikzcd}$$

We want to show that the map $H_3(W;\mathbb{Z})\to C$ is surjective. The composition

$$H_1(M;\mathbb{Z})\otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z})\cong H_1(M;\mathbb{Z})\otimes\mathbb{Z}_2\to H_2(\partial W;\mathbb{Z})\to H_2(\partial W;\mathbb{Z}_2)$$

is injective, since the first map is split injective by the Künneth theorem, and the kernel of the second map consists of the classes divisible by 2.

As $\partial W \to Q_m$ was a good map, the composition of the above map with $H_2(\partial W;\mathbb{Z}_2) \to H_2(W;\mathbb{Z}_2)$ is still injective. Therefore the composition $H_1(M;\mathbb{Z}) \otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \to H_2(\partial W;\mathbb{Z}) \to H_2(W;\mathbb{Z})$ is injective. But

$$H_1(M;\mathbb{Z}) \otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) \otimes \mathbb{Z}_2 \cong H_1(M;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^{m-1}$$

and $H_2(W;\mathbb{Z}) \cong H_2(Q_m;\mathbb{Z}) \cong (\mathbb{Z}_2)^{m-1}$, so the map is a bijection. On $H_1(M;\mathbb{Z}) \otimes H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z})$, multiplication with 2 is the zero map, so all classes come from classes in $H_3(\partial W;\mathbb{Z}_2)$. Therefore $H_3(\partial W;\mathbb{Z}_2) \to H_2(W;\mathbb{Z})$ is surjective, and now an easy diagram chase proves that $H_3(W;\mathbb{Z}) \to C$ is surjective.

This implies that the intersection product $\Lambda(Y,Y)$ of any element in C with itself is zero, since the integer-valued intersection form on $H_3(W;\mathbb{Z})$ is skew-symmetric. Then it follows, that as a vector space with symmetric bilinear form, $C \cong H^r$, where the hyperbolic form H is given by $((\mathbb{Z}_2)^2;\Lambda)$, where $\Lambda(Y,Z) = Y_1Z_2 + Z_1Y_2$.

We have $\pi_3(W) \cong \pi_3(\tilde{W}) \twoheadrightarrow H_3(\tilde{W};\mathbb{Z})$ by the extended Hurewicz theorem which says that for an $(n-1)$-connected space, not only $\pi_n \cong H_n$, but also $\pi_{n+1} \twoheadrightarrow H_{n+1}$. Now we look at the long exact sequence in homology induced by the short exact sequence of coefficients $\mathbb{Z}_- \to \mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}$: we get:

$$\begin{tikzcd}
H_3(\tilde{W};\mathbb{Z}) \arrow[r] & H_3(W;\mathbb{Z}) \arrow[r] & H_2(W;\mathbb{Z}_-) \arrow[r] \arrow[d, "\cong"] & H_2(\tilde{W};\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H_2(W;\mathbb{Z}) \arrow[d, "\cong"] \\
 & & \mathbb{Z}^m \arrow[r] & \mathbb{Z}^m \arrow[r] & (\mathbb{Z}_2)^{m-1}
\end{tikzcd}$$

Since the rows are exact, the map $\mathbb{Z}^m \to \mathbb{Z}^m$ must have rank m, hence it is injective. Therefore $H_3(\tilde{W};\mathbb{Z}) \to H_3(W;\mathbb{Z})$ is surjective.

We conclude that we have a surjective composition:

$$p : \pi_3(W) \twoheadrightarrow H_3(W;\mathbb{Z}) \twoheadrightarrow C$$

If C is not zero, X will not be a conjugation space. But maybe we have just chosen a wrong nullbordism W. There are two “degrees of freedom” for W, if C is not zero: First, we may change W in its bordism class by surgery in its interior. Second, we may change the bordism class of W. The first "degree of freedom" will be used.

If we do not want to allow a different 2-type for W, then $W \to BSpin \times Q_m$ must be a three-connected map, i.e. a 2-smoothing. But if we just want that to produce a conjugation space, this might be a condition that is stronger than what we need. We could for example allow odd torsion in the second homology!

Following Wall [Wal99], we have on $\pi_3(W) = \mathrm{Ker}(\pi_3(W) \to \pi_3(BO) = 0)$ a (-1)-quadratic form $(\lambda, \tilde{\mu})$ over the ring $\mathbb{Z}[\mathbb{Z}_2]$ with the involution $\overline{a+bT} = a-bT$ given by intersections and self-intersections, compare also with [Kre99]:

$$\lambda : \pi_3(W) \times \pi_3(W) \to \mathbb{Z}[\mathbb{Z}_2]$$
$$\tilde{\mu} : \pi_3(W) \to \mathbb{Z}[\mathbb{Z}_2]/\mathbb{Z} \cdot 1 \cong \mathbb{Z} \cdot T$$

(We will identify the subgroup $\mathbb{Z} \cdot T$ with the quotient $\mathbb{Z}[\mathbb{Z}_2]/\mathbb{Z} \cdot 1$.)

We know that $\lambda(x, x) = \tilde{\mu}(x) - \overline{\tilde{\mu}(x)} \in \mathbb{Z}[\mathbb{Z}_2]$. Therefore, under the above identification, $\lambda(x, x) = 2\tilde{\mu}(x)$ and each one of $\lambda(x, x), \tilde{\mu}(x)$ determines the other one.

Since we count in both cases the intersections (but for C, we do not remember the sign and the element of the fundamental group corresponding to an intersection point), we get for $y, z \in \pi_3(W)$ with images $Y, Z \in C$:

$$\epsilon(\lambda(y, z)) = \Lambda(Y, Z),$$

where $\epsilon : \mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}_2$ maps $a + bT$ to $a + b$ modulo 2.

Suppose $C \cong (H^r, \Lambda)$, with basis $E_1, F_1, \ldots, E_r, F_r$, such that $\Lambda(E_i, F_j) = \delta_{ij}$, and $\Lambda(E_i, E_j) = \Lambda(F_i, F_j) = 0$.

We want to find lifts $e_i \in \pi_3(W)$ of E_i such that $\lambda(e_i, e_j) = 0$ (and therefore also $\tilde{\mu}(e_i) = 0$).

We will obtain this by the following procedure:
Choose any lifts $e_i^0, f_i^0 \in \pi_3(W)$.
Now do the following for $i = 1, \ldots, r$:

- Let $\lambda(e_i^{i-1}, e_i^{i-1}) = 2aT$, $\lambda(e_i^{i-1}, f_i^{i-1}) = b + cT$.
- Let $e_i = e_i^i = (-b + cT)e_i^{i-1} + (a - aT)f_i^{i-1}$.
  (This is still a lift for E_i since $p(e_i^i) = (-b + c)p(e_i^{i-1}) + (a - a)p(f_i^{i-1})$ and $-b + c$ is congruent to 1 modulo 2, since $\epsilon(b + cT) = \Lambda(E_i, F_i)$.

The advantage is that we have $\lambda(e_i^i, e_i^i) = (-b + cT)(-b - cT)2aT + (-b+cT)(a+aT)(b+cT)+(a-aT)(-b-cT)(-b+cT)+(a-aT)(a+aT)\lambda(f_i^{i-1}, f_i^{i-1}) = 0$.)

- Let $\lambda(e_i^i, f_i^{i-1}) = g + hT$.
- Let $f_i^i = (g + hT)f_i^{i-1}$.
  (This is still a lift for F_i since $g + h$ is congruent to 1 modulo 2.)
- Then we have $\lambda(e_i^i, f_i^i) = (g - hT)(g + hT) = g^2 - h^2 \in \mathbb{Z}$.
- For $j = i + 1, \ldots, r$:
  - Let $\lambda(e_i^i, e_j^{i-1}) = p + qT$, $\lambda(e_i^i, f_j^{i-1}) = r + sT$.
  - Let $e_j^i = (g^2 - h^2)e_j^{i-1} + (-p + qT)f_i^i$, and let $f_j^i = (g^2 - h^2)f_j^{i-1} + (-r + sT)f_i^i$. (These are still lifts for E_j, F_j. Then we have $\lambda(e_i^i, e_j^i) = (g^2 - h^2)(p + qT) + (-p - qT)(g^2 - h^2) = 0$ and $\lambda(e_i^i, f_j^i) = 0$. So e_i^i will also be orthogonal to all linear combinations of these elements, such that in the end, the e_i's will be pairwise orthogonal.)

Now we have lifts $e_i \in \pi_3(W)$ such that $\lambda(e_i, e_j) = 0$ and $\tilde{\mu}(e_i) = 0$. This implies that we can do surgery on those elements.

The result W' doesn't have to be a 3-smoothing for $Q_m \times BSpin$ any more. One cannot easily obtain its homotopy or integer homology groups, because we did not get lifts f_i of the elements F_i such that $\lambda(f_i, f_j) = 0$ and such that $\lambda(e_i, f_j) = \delta_{ij}$.

But we can compute its $\mathbb{Z}_2$-homology easily since we have the elements F_i. Namely, for example following Ranicki [Ran02], p.49, we get that the change in the $\mathbb{Z}_2$-homology between W and W' is exactly that we have killed C.

Therefore W' (although, let us repeat this, it doesn't have to be a 3-smoothing for $Q_m \times BSpin$ any more) fulfills the condition of proposition 4.1.1. At least the 3-surgeries didn't change the fundamental group, and the twisted spin structure.

Now we can exactly proceed as we wanted: we take the double cover, and glue in $M \times D^3$. We obtain a simply connected spin manifold, which is a conjugation space.

Theorem 4.7.1 *For every orientable connected 3-manifold M, there exists a simply connected spin 6-manifold which is a conjugation space and has M as its fixed point set.*

Remark 4.7.2 *The conjugation spaces (the conjugation is constructed from reflections and complex conjugations)*

- *pt with trivial involution,*
- *S^2 with fixed point set S^1,*
- *S^4 with S^2 as fixed point set,*
- *the connected sum of r copies of $S^2 \times S^2$ with fixed point set the connected sum of r copies of $S^1 \times S^1$,*
- *and the connected sum of r copies of $\mathbb{C}\mathrm{P}^2$ with fixed point set the connected sum of r copies of $\mathbb{R}\mathrm{P}^2$*

show that every closed manifold of dimensions 0, 1 and 2 is the fixed point set of a (simply-connected, and spin in the orientable case) conjugation space. Hence the above theorem holds also in smaller dimensions.

Appendix A

Fixed point sets of codimension 1

Theorem A.1 *Let M be a simply connected closed n-dimensional differentiable manifold, and suppose that $\mathbb{Z}_p$ acts smoothly on M such that the fixed point set contains a component of codimension 1. Then the action must be an orientation reversing involution, and there is a compact n-dimensional manifold M_1 with boundary F such that $M = M_1 \cup_F -M_1$, and the involution maps $x \in M_1$ to $x \in -M_1$ and inversely, with fixed point set F.*

Proof: By considering the differential of the action at a point in the codimension 1 fixed point component, one obtains that the action is an orientation reversing involution τ. Smith theory [Bre62] implies that F has only one component of codimension 1, say F_0. Let N be a small (open) tubular neighbourhood of F_0, and consider the Mayer-Vietoris sequence for $M = N \cup (M \setminus F_0)$:

$$\begin{aligned} \ldots \to H_n(N;\mathbb{Z}_2) \oplus H_n(M \setminus F_0;\mathbb{Z}_2) \to H_n(M;\mathbb{Z}_2) \to H_{n-1}(N \setminus F_0;\mathbb{Z}_2) \\ \to H_{n-1}(N;\mathbb{Z}_2) \oplus H_{n-1}(M \setminus F_0;\mathbb{Z}_2) \to H_{n-1}(M;\mathbb{Z}_2) \to \ldots \end{aligned}$$

N is homotopy equivalent to the $(n-1)$-dimensional manifold F_0. $M \setminus F_0$ is homotopy equivalent to $M \setminus N$, an n-dimensional manifold with boundary. This implies $H_n(N;\mathbb{Z}_2) = H_n(M \setminus F_0;\mathbb{Z}_2) = 0$ and $H_{n-1}(N;\mathbb{Z}_2) \cong \mathbb{Z}_2$. M is an n-dimensional closed simply connected manifold, hence $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and, by Poincaré duality, $H_{n-1}(M;\mathbb{Z}_2) = 0$. N is homeomorphic to a line bundle over F_0 (its normal bundle in M), so $N \setminus F_0$ is isomorphic to a line bundle over F_0 without the 0-section, hence homotopy equivalent to a double cover of F_0, hence to an $(n-1)$-dimensional closed manifold. By the Mayer-Vietoris sequence, $H_{n-1}(N \setminus F_0; \mathbb{Z}_2)$ has rank ≥ 2, hence the normal bundle of F_0 is trivial.

It follows that $M \setminus F_0$ is homotopy equivalent to $M \setminus N$, an n-dimensional manifold with boundary $F_0 \sqcup F_0$. Thus $H_0(M \setminus F_0; \mathbb{Z}_2) \cong H_n(M \setminus N, F_0 \sqcup F_0; \mathbb{Z}_2) \cong H_n(M, F_0; \mathbb{Z}_2)$ by Poincaré duality and since $(M \setminus N)/(F_0 \sqcup F_0) \cong M/F_0$. Consider the exact sequence of the pair (M, F_0):

$$\begin{aligned} \ldots \to H_n(F_0; \mathbb{Z}_2) \to H_n(M; \mathbb{Z}_2) \to H_n(M, F_0; \mathbb{Z}_2) \\ \to H_{n-1}(F_0; \mathbb{Z}_2) \to H_{n-1}(M; \mathbb{Z}_2) \to \ldots \end{aligned}$$

We obtain that the rank of $H_0(M \setminus F_0; \mathbb{Z}_2) \cong H_n(M, F_0; \mathbb{Z}_2)$ is 2, hence $M \setminus F_0$ is disconnected. But then the involution must exchange the components of $M \setminus F_0$, and the theorem follows. **q.e.d**

Denote $M_2 := -M_1$. We have a Mayer-Vietoris sequence for $M = M_1 \cup M_2$ and a Mayer-Vietoris sequence for $M = M_2 \cup M_1$ and a commutative diagram (for arbitrary coefficients):

$$\begin{tikzcd}
H^k(M) \arrow[r, "\binom{i_1^*}{i_2^*}"] \arrow[d, "\tau^*"] & H^k(M_1) \oplus H^k(M_2) \arrow[r, "(j_1^* \ -j_2^*)"] \arrow[d, "\tau^* \oplus \tau^*"] & H^k(F) \arrow[r, "d"] \arrow[d, "\tau^* = id"] & H^{k+1}(M) \arrow[d, "\tau^*"] \\
H^k(M) \arrow[r, "\binom{i_2^*}{i_1^*}"] & H^k(M_2) \oplus H^k(M_1) \arrow[r, "(j_2^* \ -j_1^*)"] \arrow[dr, "(-j_2^* \ j_1^*)"'] & H^k(F) \arrow[r, "d' = -d"] \arrow[d, "-1"] & H^{k+1}(M) \\
 & & H^k(F) \arrow[ur, "d"'] &
\end{tikzcd}$$

Hence, with the following action, the Mayer-Vietoris sequence is equivariant:

$$\begin{tikzcd}
H^k(M) \arrow[r] \arrow[loop below, "\tau^*"] & H^k(M_1) \oplus H^k(M_2) \arrow[r] \arrow[loop below, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & H^k(F) \arrow[r] \arrow[loop below, "-1"] & H^{k+1}(M) \arrow[loop below, "\tau^*"]
\end{tikzcd}$$

We want to apply our results to the following situation (that is, for example, to the manifolds described in the last section of [Pup95]):

Proposition A.2 *Let M be a 1-connected closed $(n = 4k+2)$-manifold s.t. $H^*(M; \mathbb{Z})$ is free, and $H^2(M; \mathbb{Z})$ generates $H^*(M; \mathbb{Z})$ as an algebra. Suppose* *the only graded non-trivial involution on $H^*(M;\mathbb{Z})$ is given by $(-1)^j$ on $H^{2j}(M;\mathbb{Z})$ (this involution always exists!). Then there is no involution on M with a codimension 1 fixed point component.*

Proof: Suppose there is such an involution, which is orientation reversing, hence in integer cohomology, it must be $(-1)^j$ on $H^{2j}(M;\mathbb{Z})$. We consider the above equivariant exact sequence with $\mathbb{Q}$-coefficients. Every rational representation of $\mathbb{Z}_2$ is a sum of representations of rank 1, namely $\mathbb{Q}_+$ and $\mathbb{Q}_-$ (where the action is by multiplication with $+1$ respectively -1).

Let $m_i = rk(H^i(M))$, $f_i = rk(H^i(F))$, $r_i = rk(H^i(M_1))$.

In the equivariant sequence we know that $H^{4j+1}(M) = H^{4j+3}(M) = 0$, $(H^{4j+2}(M), \tau^*) = \mathbb{Q}_-^{m_{4j+2}}$, $(H^{4j}(M), \tau^*) = \mathbb{Q}_+^{m_{4j}}$, $(H^i(F), -1) = \mathbb{Q}_-^{f_i}$ and $(H^i(M_1) \oplus H^i(M_2), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)) = \mathbb{Q}_+^{r_i} \oplus \mathbb{Q}_-^{r_i}$.

Now we can deduce a lot of relations: It follows that $r_{4j-1} = 0$, and since $H^{4j-1}(F) \to H^{4j}(M)$ must be the zero map, it follows that $f_{4j-1} = 0$. We get $m_{4j} = r_{4j} = f_{4j}$. We have that $r_{4j+2} = 0$, and hence $f_{4j+2} = 0$. Finally $r_{4j+1} = 0$, which implies $f_{4j+1} = m_{4j+2}$.

Now we consider $\mathbb{Z}_2$-cohomology, and we already know the dimensions of $H^i(M;\mathbb{Z}_2)$. $H^*(F;\mathbb{Z})$ has no 2-torsion by Smith theory (if $H^*(F;\mathbb{Z})$ had 2-torsion, one would obtain $rk(H^*(F;\mathbb{Z}_2)) > rk(H^*(M;\mathbb{Z}_2))$).

Hence we have $rk(H^*(F;\mathbb{Z}_2)) = rk(H^*(M;\mathbb{Z}_2))$, which implies that the Serre spectral sequence for the Borel construction $M \to M_G \to BG$ collapses and that $H^*(F;\mathbb{Z}_2)$ is a deformation of $H^*(M;\mathbb{Z}_2)$. But $H^*(M;\mathbb{Z}_2)$ is generated as an algebra by m_2 homogeneous elements, while $H^*(F;\mathbb{Z}_2)$ is not. This gives a contradiction. (For the whole last paragraph, we refer to [Pup78].) **q.e.d.**

Appendix B

“Most” manifolds in $\mathcal{N}$ do not admit orientation preserving $\mathbb{Z}_p$-actions

We are interested in the following class of differentiable manifolds:

$$\mathcal{N} = \{\text{closed 1-connected spin 6-manifolds with free integer cohomology}\}$$

(These manifolds have been classified by Wall in [Wal66], and this classification uses the result that for every $M \in \mathcal{N}$, we have that $rk(H^3(M)) = r$ is even and $M \cong M' \# \left(\#_{r/2} S^3 \times S^3\right)$, with $H^3(M') = 0$.)

Let us suppose that $G = \mathbb{Z}_2$ acts on $M \in \mathcal{N}$ preserving the orientation. (We will look at the differences for $\mathbb{Z}_p$-actions, p odd, at the end.) We consider the Borel construction, i.e. the fibration

$$M \to M_G = (M \times EG)/G \to BG$$

where EG is the universal free G-space S^∞ and BG the classifying space S^∞/G, where the action of G on $S^\infty \subset \mathbb{C}^\infty$ is by multiplication with a complex unit root.

We consider the Serre spectral sequence for this fibration, i.e.:

$$H^p(BG; \mathcal{H}^q(M)) \Longrightarrow H^{p+q}(M_G),$$

where $\mathcal{H}^q(M)$ means that we have to consider $H^q(M)$ as a local system, i.e. the left side is cohomology with local coefficients. This means we have to know the $\pi_1(BG) = G$-action on $H^q(M)$; this is just the induced action in cohomology of the action of G on M, as one verifies easily.

“Most” manifolds M are such that G acts trivially on $H^{even}(M)$, and such that $H^2(M)$ generates $H^{even}(M)$, see [Pup95]. Puppe has also treated the special case of our problem that G acts trivially on $H^*(M)$, see [Pup06]. Here we want to generalize one step further, that is, we don't want to “exclude” non-trivial actions of G on $H^3(M)$.

We will consider integer cohomology first. We have a complete classification result for integral G-representations, see for example [CR62], and what we will use here is that we can write

$$H^3(M;\mathbb{Z}) = H^3(M;\mathbb{Z})_+ \oplus H^3(M;\mathbb{Z})_0 \oplus H^3(M;\mathbb{Z})_-,$$

where the decomposition is such that G acts trivially on $H^3(M;\mathbb{Z})_+$, by multiplication with -1 on $H^3(M;\mathbb{Z})_-$, and $H^3(M;\mathbb{Z})_0$ is isomorphic to a direct sum of group rings $\mathbb{Z}[G]$. This decomposition is not unique. Let the **projection** of this direct sum decomposition under reduction mod 2 be

$$H^3(M;\mathbb{Z}_2) = H^3(M;\mathbb{Z}_2)_+ \oplus H^3(M;\mathbb{Z}_2)_0 \oplus H^3(M;\mathbb{Z}_2)_-.$$

We want to apply the localization theorem, see for example [Hsi75], which says that the restriction $H^*(M_G;\mathbb{Z}) \to H^*(F_G;\mathbb{Z})$, where F is the fixed point set, after localizing in $t \in \mathbb{Z}[t]/2t \cong H^*(BG;\mathbb{Z})$ (here $deg(t) = 2$), becomes an isomorphism of $H^*(BG;\mathbb{Z})[t^{-1}]$-modules. So we are interested in the $H^*(BG;\mathbb{Z})$-module structure of $H^*(M_G;\mathbb{Z})$.

The $H^*(BG;\mathbb{Z})$-module structure of the E_2-term of the Serre spectral sequence is

$$E_2^{*,*} \cong (\mathbb{Z}[t]/2t \otimes H^*(M;\mathbb{Z})_+) \oplus (s \cdot \mathbb{Z}_2[t] \otimes H^*(M;\mathbb{Z})_-) \oplus H^*(M;\mathbb{Z})_0^G$$

where $deg(s) = 1$, $deg(t) = 2$. (Here $H^*(M;\mathbb{Z})_0^G$ denotes the elements of $H^*(M;\mathbb{Z})_0$ fixed by the G-action.) For this, one has to compute the cohomology of BG with the different local coefficients (as a module over ordinary integer cohomology of BG).

A non-zero differential in the spectral sequence would indeed define a non-zero derivation of odd degree of $H^*(M;\mathbb{Z}_2)$, and this is not possible if $H^2(M;\mathbb{Z}_2)$ generates $H^{even}(M;\mathbb{Z}_2)$. Hence in most cases $E_2^{*,*} \cong E_\infty^{*,*}$, and the next thing we have to care about is the sequence of extensions of graded $\mathbb{Z}[t]/2t$-modules:

$$0 \to F_{q-1}(H^*(M_G;\mathbb{Z})) \to F_q(H^*(M_G;\mathbb{Z})) \to E_\infty^{*,q}[-q] \to 0$$

Lemma B.1 *All these extensions are split, hence the limit term $H^*(M_G;\mathbb{Z})$ has the same $H^*(BG;\mathbb{Z})$-module structure as the E_∞-term.*

Proof: We do induction over q. $E_\infty^{*,q}$ is for all q a direct sum of summands isomorphic to $\mathbb{Z}$, $\mathbb{Z}_2[t]$ or $\mathbb{Z}[t]/2t$. One can split this extension problem into extensions, where the quotient is always isomorphic to $\mathbb{Z}$, $\mathbb{Z}_2[t]$ or $\mathbb{Z}[t]/2t$. If the quotient is $\mathbb{Z}[t]/2t$, i.e. free, the extension splits. For the quotients $\mathbb{Z}$, $\mathbb{Z}_2[t]$, one can show by construction of a section, that as $\mathbb{Z}[t]/2t$-modules, all such extensions split. One only has to find a preimage of the generator of $\mathbb{Z}$ that is annihilated by t, respectively a preimage of the generator of $\mathbb{Z}_2[t]$ that is annihilated by 2. This is possible since by hypothesis, the left side is a direct sum of summands isomorphic to $\mathbb{Z}$, $\mathbb{Z}_2[t]$ or $\mathbb{Z}[t]/2t$, and (important!) that the maps are graded:

Let x be a preimage of $1 \in \mathbb{Z}$, then tx maps to zero, hence is the image of an element $y \in F_{q-1}(H^*(M_G;\mathbb{Z}))$. Since all maps are graded, y is divisible by t: we find z such that $y = tz$. Thus x minus the image of z maps to $1 \in \mathbb{Z}$ and is annihilated by t.

Let x be a preimage of $1 \in \mathbb{Z}_2[t]$, then $2x$ maps to zero, hence is the image of an element $y \in F_{q-1}(H^*(M_G;\mathbb{Z}))$. One has $ty = 0$ since its image is $2tx = 0$. Since all maps are graded, y is divisible by 2: we find z such that $y = 2z$. Thus x minus the image of z maps to $1 \in \mathbb{Z}_2[t]$ and is annihilated by 2. **q.e.d**

Remark B.2 *The proof of the lemma applies to the following general case: Let a $\mathbb{Z}_2$-action on X be given. If the integer cohomology of X is free over $\mathbb{Z}$, and all differentials in the Serre spectral sequence computing the equivariant integer cohomology of X are trivial, then also all extensions split, and $H_G^*(X;\mathbb{Z})$ has the same $H^*(BG;\mathbb{Z})$-module structure as the E_2-term of the spectral sequence.*

Hence our result is that as graded $\mathbb{Z}[t]/2t$-module, we have:

$$H^*(M_G;\mathbb{Z}) \cong \mathbb{Z}[t]/2t \otimes H^*(M;\mathbb{Z})_+ \oplus s \cdot \mathbb{Z}_2[t] \otimes H^*(M;\mathbb{Z})_- \oplus H^*(M;\mathbb{Z})_0^G$$

Here one may ask the question on which choices this isomorphism depends. Indeed, (in addition to the choice of decomposition of $H^*(M;\mathbb{Z})$) one has to choose a certain lift of $H^*(M_G;\mathbb{Z}) \to H^*(M;\mathbb{Z})^G$, and finally one needs some lift of $H^*(M_G;\mathbb{Z}) \to H^*(M_G;\mathbb{Z})/\left(\mathbb{Z}[t]/2t \otimes (H^*(M;\mathbb{Z})_+ \oplus H^*(M;\mathbb{Z})_0^G)\right)$.

Now we apply the localization theorem in the form of an exercise of [tDi87]: one has an isomorphism of $\mathbb{Z}_2$-graded algebras:

$$H^*(M_G;\mathbb{Z}) \otimes_{H^*(BG;\mathbb{Z})} \mathbb{Z}_2 \cong H^*(F;\mathbb{Z}_2)$$

where $\mathbb{Z}_2$ is a ($\mathbb{Z}_2$-graded) $\mathbb{Z}[t]/2t$-module by defining that t acts trivially on $\mathbb{Z}_2$. (For the proof, one shows using the localization theorem that the left hand side is isomorphic to $H^*(F \times BG;\mathbb{Z}) \otimes_{H^*(BG;\mathbb{Z})} \mathbb{Z}_2$ and shows that this fulfills all axioms of a cohomology theory, and that there is a multiplicative natural transformation to ordinary $\mathbb{Z}_2$-cohomology.)

Hence, as $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-vector spaces, we have:

$$H^*(F;\mathbb{Z}_2) \cong H^*(M;\mathbb{Z}_2)_+ \oplus H^*(M;\mathbb{Z}_2)_-[-1].$$

For the multiplicative structure, we look at the Serre spectral sequence with $\mathbb{Z}_2$ coefficients. We get that

$$H^*(M_G;\mathbb{Z}_2) \cong \mathbb{Z}_2[s] \otimes (H^*(M;\mathbb{Z}_2)_+ \oplus H^*(M;\mathbb{Z}_2)_-) \oplus H^*(M;\mathbb{Z}_2)_0^G.$$

As in the integer case, this identification depends on the corresponding lifts, that may be chosen such that they commute with the lifts in the integer case under reduction modulo 2, i.e. such that

$$\begin{array}{rcl} H^*(M_G;\mathbb{Z}) & \longrightarrow & H^*(M_G;\mathbb{Z}_2) \\ t^n \otimes m_+ & \longmapsto & s^{2n} \otimes \overline{m_+} \\ st^n \otimes m_- & \longmapsto & s^{2n+1} \otimes \overline{m_-} \\ m_0 & \longmapsto & \overline{m_0} \end{array}$$

This finally gives the result (with the same proof as in [Pup78], [Pup79]) that the multiplication in $H^*(F;\mathbb{Z}_2)$ corresponds to the multiplication in $H^*(M;\mathbb{Z}_2)_+ \oplus H^*(M;\mathbb{Z}_2)_-$ up to terms of lower degree. If we put together our results from the integer and the $\mathbb{Z}_2$-case, this implies that there is a filtration on $H^{even}(F;\mathbb{Z}_2)$ (namely the one corresponding to the filtration given by the degree on $H^{even}(M;\mathbb{Z}_2)_+ \oplus H^{odd}(M;\mathbb{Z}_2)_-$) such that the associated graded algebra is isomorphic to $H^{even}(M;\mathbb{Z}_2)_+ \oplus H^{odd}(M;\mathbb{Z}_2)_-$.

Since the action of G is orientation preserving, $H^{even}(F;\mathbb{Z}_2)$ is a product of Poincaré algebras, as all components of F thus must have even codimension. Hence, if the action of G on M is non-trivial, one can obtain $H^{even}(M;\mathbb{Z}_2)$ as the even part of an associated graded algebra to a multiplicative filtration on a product of even Poincaré algebras of formal dimension ≤ 4.

Given $m = rk(H^2(M;\mathbb{Z}))$, and $n = rk(H^3(M;\mathbb{Z})_-)$, one has

$$rk(H^{even}(F;\mathbb{Z}_2)) = 2 + 2m + n.$$

The algebra structure on $H^{even}(M;\mathbb{Z}_2)$ may be given by a trilinear form μ on $\mathbb{Z}_2^m$, such that $\mu(x,x,y) = \mu(x,y,y)$ for all $x,y \in \mathbb{Z}_2^m$ ([Wal66],[Pup95]). Given m and n, and using exactly the arguments from [Pup95], there are at most $2^{\binom{2m+n+3}{2}}$ possibilities for the algebra structure of $H^{even}(F;\mathbb{Z}_2)$, and at most $(2^{2m+n+1})^m \cdot (2^{m+n+1})^n \cdot 2^{m+1}$ filtrations on such an algebra, hence at most $2^{4m^2+4mn+\frac{3}{2}n^2+7m+\frac{7}{2}n+4}$ possibilities for $H^{even}(M;\mathbb{Z}_2)$ with a non-trivial action. Given $m = rk(H^2(M;\mathbb{Z}))$, and $r = rk(H^3(M;\mathbb{Z}))$, one forms the sum over all possibilities for the value of n and obtains that there are $\leq (r+1)\cdot 2^{4m^2+4mr+\frac{3}{2}r^2+7m+\frac{7}{2}r+4}$ possibilities for the even part of the cohomology algebra of a manifold $M \in \mathcal{N}$ with the given parameters that admits a non-trivial orientation preserving involution.

Now let $S^3(\mathbb{Z}^m;N)$ denote the set of symmetric 3-forms μ on $\mathbb{Z}^m$ such that all coefficients $\mu_{ijk} = \mu(e_i,e_j,e_k)$ satisfy $-N \leq \mu_{ijk} < N$, and let $R(m;N)$ denote the set of those 3-forms realisable by manifolds in $\mathcal{N}$ and with $rk(H^2(M;\mathbb{Z})) = m$ (i.e. the forms μ such that $\mu_{ijj} = \mu_{iij} \mod 2$; see [Wal66] for the relation to a classification of $\mathcal{N}$), and let $G_2(m;r)$ denote the subset of $S^3(\mathbb{Z}^m;N)$ which correspond to even algebras $H^{even}(M;\mathbb{Z}_2)$ that can be obtained as associated graded algebra to a filtration on a product of even Poincaré algebras of formal dimension ≤ 4, and of dimension $\leq 2+2m+r$ over $\mathbb{Z}_2$.

Then

$$\frac{\#(G_2(m;r)\cap R(m;N))}{\#(R(m;N))} \leq \frac{\#(G_2(m;r)\cap S^3(\mathbb{Z}^m;N))}{\#(R(m;N))}$$

$$\leq \frac{(r+1)\cdot 2^{4m^2+4mr+\frac{3}{2}r^2+7m+\frac{7}{2}r+4}}{2^{\binom{m+2}{3}-\binom{m}{2}}}$$

Hence, if $r = o(m^{\frac{3}{2}})$, then

$$\lim_{m\to\infty} \frac{\#(G_2(m;r)\cap R(m;N))}{\#(R(m;N))} = 0$$

for any N and in this sense, "most" manifolds in $\mathcal{N}$ do not admit orientation preserving involutions.

Now let us indicate the modifications for actions of $G = \mathbb{Z}_p$, where p is odd. The classification result is that

$$H^3(M;\mathbb{Z}) = H^3(M;\mathbb{Z})_+ \oplus H^3(M;\mathbb{Z})_0 \oplus H^3(M;\mathbb{Z})_-,$$

Here G acts trivially on $H^3(M;\mathbb{Z})_+$. $H^3(M;\mathbb{Z})_-$ is a direct sum of summands that are isomorphic to finitely generated $\mathbb{Z}[\theta]$-ideals A of $\mathbb{Q}[\theta]$, where θ is a p-th unit root and the action is given by $g\cdot a = \theta a$, where g is a generator of G. Every summand A is a free $\mathbb{Z}$-module of rank $p-1$. Let n be the number of summands, so $H^3(M;\mathbb{Z})_-$ is a free $\mathbb{Z}$-module of rank $n(p-1)$. $H^3(M;\mathbb{Z})_0$ is a direct sum of summands that are isomorphic to $A\oplus\mathbb{Z}\cdot y$ where A is as before, and the action is given by $g\cdot a = \theta a$ and $g\cdot y = y + a_0$ for some $a_0 \in A\setminus(\theta-1)A$, (that may be different in different summands). The cohomology computations become a little more difficult, but one still gets that:

$$\begin{array}{rll} E_2^{*,*} \cong & & \mathbb{Z}[t]/pt\otimes H^*(M;\mathbb{Z})_+ \\ & \oplus & s\cdot\mathbb{Z}_p[t]\otimes(H^*(M;\mathbb{Z})_-/(1-g^*)H^*(M;\mathbb{Z})_-) \\ & \oplus & H^*(M;\mathbb{Z})_0^G \end{array}$$

One has $H^3(M;\mathbb{Z})_-/(1-g^*)H^3(M;\mathbb{Z})_- \cong \mathbb{Z}_p^n$. The same arguments as above show that this is also the $\mathbb{Z}[t]/pt$-module structure of $H^*(M_G;\mathbb{Z})$, and that finally (an easier version of the exercise of [tDi87]) we have an isomorphism of $\mathbb{Z}_2$-graded vector spaces:

$$H^*(F;\mathbb{Z}_p) \cong H^*(M;\mathbb{Z}_p)_+ \oplus H^*(M;\mathbb{Z}_p)_-^G[-1].$$

Hence $H^{even}(F;\mathbb{Z}_p) \cong H^{even}(M;\mathbb{Z}_p)_+ \oplus H^{odd}(M;\mathbb{Z}_p)_-^G$. (Here the identification of the second factor (that is isomorphic to $\mathbb{Z}_p^n$) with $H^*(M;\mathbb{Z}_p)_-^G$ is not natural.) Now one can consider the spectral sequence with $\mathbb{Z}_p$-coefficients to get now the weaker result as in the $\mathbb{Z}_2$-case that still the degree on the right hand side defines a multiplicative filtration on the left hand side and that at least on the part corresponding to $H^{even}(M;\mathbb{Z}_p)_+$ multiplication corresponds to multiplication in $H^{even}(M;\mathbb{Z}_p)$ up to terms of lower degree. (But the whole is not a deformation any more.) Hence we still get that if the action of G on M is non-trivial, one can obtain $H^{even}(M;\mathbb{Z}_p)$ as the even part of an associated graded algebra to a multiplicative filtration on a product of even Poincaré algebras of formal dimension ≤ 4. So, with the analogous definition, we get:

$$\frac{\#(G_p(m;r)\cap R(m;N))}{\#(R(m;N))} \leq \frac{(r+1)\cdot p^{4m^2+4mr+\frac{3}{2}r^2+7m+\frac{7}{2}r+4}}{p^{\binom{m+2}{3}-\binom{m}{2}}}$$

if N is divisible by p, and a factor $(1+p/N)^{\binom{m+2}{3}-\binom{m}{2}}$ that does not disturb anything for general N, if N is sufficiently large. Hence, if $r = o(m^{\frac{3}{2}})$, then

$$\lim_{m\to\infty} \frac{\#(G_p(m;r)\cap R(m;N))}{\#(R(m;N))} = 0$$

for any sufficiently large N; and in this sense, “most” manifolds in $\mathcal{N}$ do not admit orientation preserving $\mathbb{Z}_p$-actions.

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